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Maximal revenue with multiple goods: Nonmonotonicity and other observations

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Consider the problem of maximizing the revenue from selling a number of goods to a single buyer. We show that, unlike the case of one good, when the buyer's values for the goods increase, the seller's maximal revenue may well *decrease*. We then identify two circumstances where monotonicity does obtain: when optimal mechanisms are deterministic and symmetric, and when they have submodular prices. Next, through simple and transparent examples, we clarify the need for and the advantage of randomization when maximizing revenue in the multiple-good versus the one-good case. Finally, we consider "seller-favorable" mechanisms, the only ones that matter when maximizing revenue. They are essential for our positive monotonicity results, and they also circumvent well known non-differentiability issues.

Keywords. Revenue maximization, multidimensional types, nonmonotonicity, randomization, differentiability.

JEL classification. C6, C7, D4, D8.

1. Introduction

Consider the problem of a seller who wishes to maximize the revenue from selling multiple goods to a single buyer with private information about his value for the goods. In contrast to the one-good case where a complete solution has been known for years,[1] a general solution in the case of multiple goods remains elusive and, except under special circumstances,[2] very little is known about even the form of the solution or its properties.

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[1] See Myerson (1981), who deals also with multiple buyers.

[2] See, e.g., Armstrong (1996), Thanassoulis (2004), Pycia (2006), Manelli and Vincent (2006, 2007, 2012), Pavlov (2011), and Hart and Nisan (2014a, 2014b).

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The present paper highlights some important differences between the one-good and the multiple-good cases.

In Section 2, we exhibit the surprising phenomenon that the seller's maximal revenue may well *decrease* when the buyer's values for the goods *increase*.[3] This revenue nonmonotonicity can occur only when there is *more than one good*: revenue is easily shown to be nondecreasing in the buyer's value when there is only a single good. Thus, the seemingly clear intuition that the seller is able to extract more revenue from buyers whose valuations for the goods are higher turns out to be false in general.[4]

Under what circumstances will the seller's maximal revenue increase when the buyer's distribution of valuations increases? In Section 2.3, we identify two such circumstances: one when the mechanism is deterministic and symmetric, and the other when payments to the seller can be described by a submodular pricing function. These positive revenue-monotonicity results are both interesting in their own right and help explain why counterexamples to monotonicity (ours included) cannot be entirely transparent—they must involve randomizations or asymmetries, and their revenue function cannot be submodular.

In Section 3, we present a simple example where randomization is necessary for revenue maximization, and clarify why randomization is needed *only* when there are multiple goods.

In Section 1.2, we formally introduce "seller-favorable" mechanisms, where, when the buyer is indifferent, the tie is always broken in favor of the seller. These mechanisms are the only ones that matter when maximizing the seller's revenue, and also play an important role in the positive monotonicity results of Section 2.3. In the Appendix, we characterize the revenue of seller-favorable mechanisms using directional derivatives, which exist everywhere; this has the additional benefit of circumventing nondifferentiability issues that arise from incentive compatibility, which, while ultimately harmless, are often a distracting nuisance within the analysis.

The maximal-revenue problem has been shown to be significantly less well behaved when the values of the goods are not independent; see Hart and Nisan (2013, 2014a, 2014b).[5] It is, therefore, important also to obtain (inevitably more subtle) examples with independent, and even independent and identically distributed (i.i.d.), values. We do so both for revenue nonmonotonicity and for randomization.

[3]What we compare is the maximal revenue from two given distributions, one having higher values than the other (formally, this means first-order stochastic dominance).

[4]When the mechanism is held *fixed*, there are well known examples in which the seller's revenue unexpectedly falls; for instance, when the number of bidders increases (Matthews 1984, Menicucci 2009) and when the seller releases more information (Perry and Reny 1999). But in both of these cases, the seller's *maximal* revenue cannot fall since the seller can always choose to ignore the additional bidders and can always choose not to release new information. Adams and Yellen (1977) show that a multiple-good monopolist, when facing a buyer who can consume at most one good, can sometimes increase profits by using negative advertising to reduce a consumer's value for one of the goods. The constraint that buyers can consume at most one good is important for their example. See footnote 18 in Section 2.2 (we thank an anonymous referee for bringing the Adams and Yellen paper to our attention).

[5]It is shown there, for instance, that deterministic mechanisms always ensure at least one-half of the maximal revenue in the independent case, but only an arbitrarily small fraction in the general (correlated) case.

1.1 *Preliminaries*

The seller possesses $k \geq 1$ goods (or "items"), which are worth nothing to him (and there are no costs). The valuation of the goods to the buyer is given by a vector[6] $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}_+^k$, where $x_i \geq 0$ is his value for good i. The valuation is assumed to be additive over the goods: the buyer's value of a subset $L \subset \{1, 2, \ldots, k\}$ of goods is $\sum_{i \in L} x_i$. The buyer knows the valuation vector x, whereas the seller knows only that x is drawn from a given probability distribution $\mathcal{F}$ on $\mathbb{R}_+^k$. We make no further assumptions on $\mathcal{F}$. In particular, $\mathcal{F}$ may possess atoms, and its support may be finite or infinite and need not be convex or even connected. The seller and the buyer are risk-neutral and have quasilinear utilities.

A *(direct) mechanism* for selling the k goods is given by a pair of functions (q, s), where $q = (q_1, q_2, \ldots, q_k) : \mathbb{R}_+^k \to [0, 1]^k$ and $s : \mathbb{R}_+^k \to \mathbb{R}$. If the buyer reports that his valuation is x, then $q_i(x) \in [0, 1]$ is the probability that the buyer receives good[7] i (for $i = 1, \ldots, k$), and $s(x)$ is the payment that the seller receives from the buyer. We call q the *allocation function* and call s the *payment function*; the range $M := \{(q(x), s(x)) : x \in \mathbb{R}_+^k\} \subset [0, 1]^k \times \mathbb{R}$ of the mechanism is referred to as its *menu*.[8] When the buyer reports his valuation x truthfully, his payoff is $b(x) = \sum_{i=1}^k q_i(x) x_i - s(x) = q(x) \cdot x - s(x)$ and the seller's payoff is[9] $s(x)$. A mechanism (q, s) is *individually rational* (*IR*) if $b(x) \geq 0$ for all $x \in \mathbb{R}_+^k$, and it is *incentive compatible* (*IC*) if $b(x) \geq q(y) \cdot x - s(y)$ for all $x, y \in \mathbb{R}_+^k$. By the revelation principle, the *maximal revenue* from the distribution $\mathcal{F}$ is $\text{Rev}(\mathcal{F}) := \sup \mathbb{E}_{\mathcal{F}}[s(x)]$, where x is distributed according to $\mathcal{F}$, and the supremum is over all IC and IR mechanisms[10] (q, s).

1.2 *Seller-favorable mechanisms*

We now introduce the concept of *seller-favorable* mechanisms: these are incentive-compatible mechanisms for which it is not possible to increase the seller's payment function while leaving the buyer's payoff function unchanged, without violating incentive compatibility. Formally, the IC mechanism (q, s) is *seller-favorable* if there is no other IC mechanism $(\tilde{q}, \tilde{s})$ having the same buyer payoff function, i.e., $\tilde{q}(x) \cdot x - \tilde{s}(x) = b(x) = q(x) \cdot x - s(x)$ for all $x \in \mathbb{R}_+^k$, and a larger payment function, i.e., $\tilde{s}(x) \geq s(x)$ for every $x \in \mathbb{R}_+^k$, with strict inequality for some $x \in \mathbb{R}_+^k$. This implies, in particular, that when the buyer is indifferent, ties must be broken in favor of the seller, i.e., $q(y) \cdot x - s(y) = q(x) \cdot x - s(x)$ implies $s(y) \leq s(x)$.

[6] The variable $\mathbb{R}$ is the real line, $\mathbb{R}^k$ is the k-dimensional Euclidean space, and $\mathbb{R}_+^k = \{x \in \mathbb{R}^k : x \geq 0\}$ is its nonnegative orthant. We follow the standard assumption that valuations are nonnegative; in Section A.1, we deal with arbitrary valuations.

[7] The assumption of risk neutrality implies that only the marginal probabilities of getting each good matter.

[8] An interpretation is that the seller "posts" the menu and the buyer "chooses" from it.

[9] In the literature, this is called transfer, cost, price, or revenue, and is denoted by t, c, p, and so on. We hope that using the mnemonic s for the *s*eller's final payoff and b for the *b*uyer's final payoff will avoid confusion.

[10] Such that s is measurable. In Section A.1 (footnote 48), we will see that measurability is not an issue.

When maximizing revenue, these are the only mechanisms that matter. Moreover, the restriction to seller-favorable mechanisms simplifies the analysis (in particular, it circumvents nondifferentiability issues; see the Appendix) and, as we will see in Section 2.3, it is needed to obtain monotonicity results.

The characterization of IC mechanisms (q, s) as those whose allocation function, q, is a subgradient of the buyer's convex payoff function is well known (starting with Rochet 1985). It is an inconvenient and often technically annoying fact that the buyer's convex payoff function, while differentiable almost everywhere, need not be differentiable everywhere. Proofs that are otherwise simple and elegant often require detours through subgradient measurable selection arguments.[11]

Such detours can be avoided when one restricts attention to seller-favorable mechanisms. The reason is that the buyer's payoff function is not differentiable only when he is indifferent between a number of reports. But if the mechanism (q, s) is seller-favorable, the buyer's truthful report must maximize the seller's payoff among all of the buyer's optimal reports. As we will show in the Appendix, this implies that $q(x) \cdot x = b'(x; x)$, which denotes the directional derivative of b at x in the direction x (see the formal definitions after the proof of Lemma 13) for every buyer valuation x. Consequently, in a seller-favorable mechanism, the buyer's payoff function b completely determines the seller's payoff function s at *every* x, whether it is a point of differentiability of b or not, and $s(x) = b'(x; x) - b(x)$ for all x.

Seller-favorable mechanisms are relatively easy to construct from any IC mechanism; moreover, doing so while preserving the menu (up to closure) and so preserving certain useful properties (such as submodularity; see Section 2.3) turns out to be more subtle.

Proposition 1. *Let (q, s) be an IC mechanism, with buyer payoff function b and menu M. Then there exists a seller-favorable mechanism $(\tilde{q}, \tilde{s})$ with buyer payoff function $\tilde{b}$ and menu $\widetilde{M}$ such that $\tilde{b}(x) = b(x)$ and $\tilde{s}(x) \geq s(x)$ for all $x \in \mathbb{R}^k_+$, and*[12] $\widetilde{M} \subset \operatorname{cl} M$.

Proposition 1 is proved in the Appendix (see Proposition 16), together with a number of additional useful results.

2. Nonmonotonicity: Increasing values may decrease revenue

When the buyer's values for the goods increase, what happens to the seller's maximal revenue? It stands to reason that the revenue should also increase, as there is now more value for the seller to "extract." While this is easily shown to be true when there is one good (Section 2.1), it is perhaps a surprise that it no longer holds when there are multiple goods (Sections 2.2 and 2.4).

To see this, consider two situations: in the first, the valuation is given by the ($\mathbb{R}^k_+$-valued) random variable X_1 with distribution function $\mathcal{F}_1$; in the second, the valuation is given by the random variable X_2 with distribution function $\mathcal{F}_2$. Assume that $X_1 \leq X_2$

[11] For example, Lemma A.4 in Manelli and Vincent (2007); cf. footnote 56 in the Appendix.

[12] The notation $\operatorname{cl} M$ denotes the closure of the set M (in $[0, 1]^k \times \mathbb{R}$).

everywhere; i.e., in every state ω, the realization $X_1(\omega)$ of X_1 is less than or equal to (in all k coordinates) the realization $X_2(\omega)$ of X_2. What we will show is that the maximal revenue that is obtained from X_2 may well be *smaller* than the maximal revenue that is obtained from[13] X_1. In terms of distributions, the condition $X_1 \leq X_2$ amounts to *first-order stochastic domination* of $\mathcal{F}_1$ by[14] $\mathcal{F}_2$.

2.1 *Monotonicity for one good*

When there is only one good, i.e., $k = 1$, incentive compatibility (IC) implies that a buyer with a higher valuation pays no less than a buyer with a lower valuation. Thus, increasing the valuation of the buyer can only increase the revenue.

Proposition 2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two distributions on $\mathbb{R}_+$. If $\mathcal{F}_2$ first-order stochastically dominates $\mathcal{F}_1$, then* $\mathrm{Rev}(\mathcal{F}_1) \leq \mathrm{Rev}(\mathcal{F}_2)$.

Proof. First, we claim that every IC mechanism is monotonic in the sense that the seller's payoff increases weakly with the buyer's value: if $x > y \geq 0$, then $s(x) \geq s(y)$. Indeed, for all x, y, the IC inequalities at x and at y imply $(q(x) - q(y))x \geq s(x) - s(y) \geq (q(x) - q(y))y$, hence $(q(x) - q(y))(x - y) \geq 0$; when $x > y$, it follows that $q(x) - q(y) \geq 0$ and, thus, $s(x) - s(y) \geq 0$ (because $y \geq 0$).

Second, first-order stochastic dominance implies that $\mathbb{E}_{\mathcal{F}_1}[s(x)] \leq \mathbb{E}_{\mathcal{F}_2}[s(x)]$ for every IC mechanism, since s is a nondecreasing function. □

Remark. Proposition 2 also follows easily from Myerson's (1981) characterization of the optimal revenue when there is one good as $\mathrm{Rev}(\mathcal{F}) = \sup_{p \geq 0} p \cdot (1 - \mathcal{F}(p))$. However, the proof above shows that revenue monotonicity holds not only for optimal mechanisms, but for *any* incentive-compatible mechanism.

2.2 *Nonmonotonicity for multiple goods*

Surprisingly, Proposition 2 does not hold when there is more than one good. That is, increasing the buyer's valuations need *not* yield higher revenue to the seller.

When there are multiple goods, one can easily construct examples of IR and IC mechanisms that are not monotonic.[15] Take, for instance, the mechanism where the buyer is offered a choice from the following menu of four outcomes: get nothing and pay nothing (with payoff $= 0$); or get good 1 for price 1 (with payoff $= x_1 - 1$); or get good 2

[13] Thus, in applications where the distribution of X_2 is not precisely known, and only a certain lower bound X_1 is given, the optimal revenue from X_1 does *not* necessarily provide a lower bound for the optimal revenue from X_2.

[14] Formally, $\mathcal{F}_2$ *first-order stochastically dominates* $\mathcal{F}_1$ if and only if $\mathbb{E}_{\mathcal{F}_1}[u(X)] \leq \mathbb{E}_{\mathcal{F}_2}[u(X)]$ for every nondecreasing function $u : \mathbb{R}^k \to \mathbb{R}$. As is well known, this is equivalent to having two random variables X_1 and X_2 with distributions $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, that are defined on the same probability space and satisfy $X_1 \leq X_2$ pointwise (this is called *coupling*). A comprehensive treatment of stochastic dominance can be found in Shaked and Shanthikumar (2007).

[15] Our first such example was constructed together with Noam Nisan.

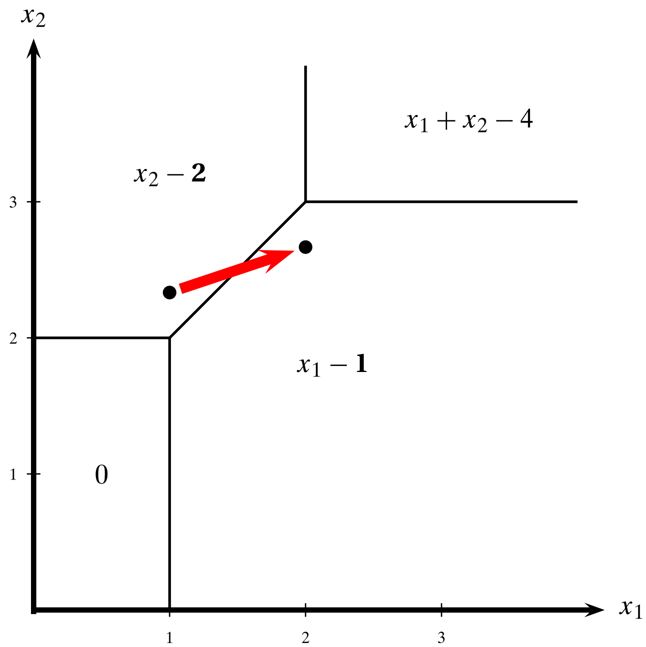

FIGURE 1. The nonmonotonic mechanism (1).

for price 2 (with payoff $= x_2 - 2$); or get both goods for price 4 (with payoff $= x_1 + x_2 - 4$); thus, the buyer's payoff is

$$b(x_1, x_2) = \max\{0, x_1 - 1, x_2 - 2, x_1 + x_2 - 4\}. \tag{1}$$

See Figure 1 for the regions in the buyer's valuation space where each outcome is chosen. If the valuation of the buyer is, say, $x = (1, \frac{7}{3})$, then his optimal choice is to pay 2 for good 2 (so $q(x) = (0, 1)$ and $s(x) = 2$), whereas if his valuation increases to $x' = (2, \frac{7}{3})$ (where the first good is worth more) or even to $x'' = (2, \frac{8}{3})$ (where both goods are worth more), then his optimal choice is to pay 1 for good 1 (so $q(x') = q(x'') = (1, 0)$ and $s(x') = s(x'') = 1$). Thus the seller receives a lower payment (1 instead of 2) when the buyer's values increase.[16] What is happening is that the initially unchosen good 1 is acting as an outside option for the buyer. When the buyer's value for this outside option increases sufficiently, he switches away from good 2 toward good 1, which, because it happens to be cheaper than good 2, causes the seller's revenue to fall.

The above example, while insightful, ignores the fact that the seller optimally sets prices. In particular, the seller can optimally adjust prices in response to changes in

[16]When the value of one good goes up, the probability of getting it cannot go down (i.e., $q_i(x)$ is nondecreasing in x_i for each good i; this follows from the convexity of the buyer payoff function b). However, at the same time, the probabilities of getting other goods may well go down, and in a such a way, moreover, that the allocation is worth *less* to the buyer—and so, by incentive compatibility, the seller's payment *goes down*. In our example, for $x = (1, \frac{7}{3})$ and $x' = (2, \frac{7}{3})$, we have $q(x) = (0, 1)$ and $q(x') = (1, 0)$, and so $q(x) \cdot x > q(x') \cdot x'$ and $s(x) > s(x')$.

the buyer's value distribution. The difficult question then is whether this nonmonotonicity can also occur for the *maximal* revenue. We provide two examples: a simpler one (below), where the unique optimal mechanism is exactly the mechanism described above,[17] and a more complicated one (Section 2.4), where the valuations of the two goods are independent and identically distributed.

One reason that such examples are subtle is because the buyer can consume multiple goods (in fact, there is always a buyer type who gets the bundle of both goods in optimal mechanisms). Thus, when the buyer's value of an inexpensive good that he is not purchasing increases (as in the above example), the seller may find it optimal to change the prices so that the buyer purchases a bundle of goods that includes the goods he originally purchased as well as the inexpensive good whose value increased. Consequently, the substitution away from an expensive good toward a cheaper one—which causes revenue to fall in the above example—might never occur (cf. our results in Section 2.3 below which give sufficient conditions for monotonicity). Thus, having the option to optimally price bundles of goods makes it more likely that the seller's maximal revenue will not fall after an increase in the buyer's values, and, hence, makes it more difficult to find an example in which they, in fact, do fall.[18]

Example 1. For every $0 \le \alpha \le \frac{1}{4}$, let $\mathcal{F}_\alpha$ be the distribution on $\mathbb{R}^2$:

$$\mathcal{F}_\alpha = \begin{cases} (1,1) & \text{with probability } \frac{1}{4} \\ (1,2) & \text{with probability } \frac{1}{4} - \alpha \\ (2,2) & \text{with probability } \alpha \\ (2,3) & \text{with probability } \frac{1}{2}. \end{cases}$$

As α increases, probability mass is moved from $(1,2)$ to $(2,2)$, and so $\mathcal{F}_\alpha$ *first-order stochastically dominates* $\mathcal{F}_{\alpha'}$ when $\alpha > \alpha'$. Nevertheless, the maximal revenue $\mathrm{Rev}(\mathcal{F}_\alpha)$ *decreases* with α (in the region $0 \le \alpha \le \frac{1}{12}$). ◊

Proposition 3. *In Example 1, for every* $0 \le \alpha \le \frac{1}{12}$,

$$\mathrm{Rev}(\mathcal{F}_\alpha) = \frac{11}{4} - \alpha.$$

[17] This explains the reason for including the outcome $x_1 + x_2 - 4$ in the mechanism.

[18] In contrast, when the consumer can purchase at most one good—the "unit-demand" setup—the seller's maximal revenue can easily fall. For example, suppose the buyer's valuation is equally likely to be $(1,1)$ or $(1,3)$, and that he can consume at most one of the two goods. It is then optimal for the seller to set prices $p_1 = 1$ and $p_2 = 3$, generating revenue $1 + 3 = 4$. However, if the buyer's valuation of $(1,3)$ increases to $(2,3)$, while the valuation $(1,1)$ does not change, it is now optimal to set prices $p_1 = 1$ and $p_2 = 2$, generating less revenue, namely $1 + 2 = 3$ (prices $p_1 \ge 2$ and $p_2 = 3$ also generate revenue 3). This example is a simplification of an example in Adams and Yellen (1977; Figure 9), who consider a buyer who can consume at most one of two brands (goods) of a single product. In the Adams and Yellen example, the maximal revenue decreases after the seller, through advertising, optimally modifies some values of some buyer types. While this is not a first-order stochastic dominance change in values, by considering first the decrease in some values and then the increase in other values, an example of the kind we have provided here emerges. A caveat with the Adams and Yellen analysis is that they restrict attention to pricing mechanisms and so they do not show that their mechanisms are fully optimal. However, it can be shown that the pricing mechanisms in the simplified example that we have given here are fully optimal among all mechanisms.

Valuation $x=(x_1,x_2)$	Outcome $q(x)=(q_1(x),q_2(x))$	$s(x)$
$(1,1)$	$(1,0)$	1
$(2,2)$	$(1,0)$	1
$(1,2)$	$(0,1)$	2
$(2,3)$	$(1,1)$	4

TABLE 1. The optimal mechanism for Example 1.

PROOF. First, revenue of $\frac{11}{4}-\alpha$ is achieved by the mechanism given in Table 1 (whose buyer payoff function is (1)). Indeed, revenue is $\frac{1}{4}\cdot 1+(\frac{1}{4}-\alpha)\cdot 2+\alpha\cdot 1+\frac{1}{2}\cdot 4=\frac{11}{4}-\alpha$.

Second, we show that a higher revenue cannot be obtained. Consider the inequalities

$$\begin{array}{rl|l} q_1^{11}+q_2^{11}-s^{11}\geq 0 & & 1 \\ q_1^{12}+2q_2^{12}-s^{12}\geq q_1^{11}+2q_2^{11}-s^{11} & & \frac{1}{2} \\ 2q_1^{22}+2q_2^{22}-s^{22}\geq 2q_1^{11}+2q_2^{11}-s^{11} & & 3\alpha \\ 2q_1^{23}+3q_2^{23}-s^{23}\geq 2q_1^{11}+3q_2^{11}-s^{11} & & \frac{1}{4}-3\alpha \\ 2q_1^{23}+3q_2^{23}-s^{23}\geq 2q_1^{12}+3q_2^{12}-s^{12} & & \frac{1}{4}+\alpha \\ 2q_1^{23}+3q_2^{23}-s^{23}\geq 2q_1^{22}+3q_2^{22}-s^{22} & & 2\alpha \end{array} \tag{2}$$

(the first inequality is IR at $(1,1)$ and the others are various IC constraints). Multiplying these inequalities by the multipliers on the right (which are all nonnegative when $0\leq\alpha\leq\frac{1}{12}$) and then adding them up yields

$$-\Big(\frac{3}{4}-3\alpha\Big)q_2^{11}-2\alpha q_1^{12}+\Big(\frac{1}{4}-3\alpha\Big)q_2^{12}+2\alpha q_1^{22}+q_1^{23}+\frac{3}{2}q_2^{23} \geq \frac{1}{4}s^{11}+\Big(\frac{1}{4}-\alpha\Big)s^{12}+\alpha s^{22}+\frac{1}{2}s^{23}.$$

The right-hand side is precisely the expected revenue at $\mathcal{F}_\alpha$, and the left-hand side is at most $0+0+(\frac{1}{4}-3\alpha)+2\alpha+1+\frac{3}{2}=\frac{11}{4}-\alpha$ (since $q_2^{11},q_1^{12}\geq 0$ and $q_2^{12},q_1^{22},q_1^{23}$, $q_2^{23}\leq 1$). Therefore, the revenue cannot exceed $\frac{11}{4}-\alpha$, and so the revenue of $\frac{11}{4}-\alpha$ achieved by Table 1 is indeed maximal. □

Once again, what is happening here is that the value of a good that is unchosen by one type of buyer—i.e., good 1 for the buyer with valuation $(1,2)$—increases with positive probability, to $(2,2)$. This leads that buyer to switch away from the good he is currently purchasing, namely good 2, toward the previously unchosen good 1. Since good 1 is cheaper than good 2 and because the optimal prices do not change in this example, the seller's maximal revenue falls.[19]

[19]In fact, for mechanisms (such as those in the examples above) that assign each of the goods with probabilities 0 or 1 only (called *deterministic* mechanisms below), a necessary condition for nonmonotonicity of the seller's maximal revenue is that the value of a good to a buyer to whom the good is not assigned must rise (with positive probability).

Remarks. (a) The mechanism Table 1 is the *unique* optimal mechanism at each $\mathcal{F}_\alpha$ with $0 \le \alpha < \frac{1}{12}$; indeed, to get revenue $\frac{11}{4} - \alpha$, one needs all relevant inequalities to become equalities (thus $q_2^{11} = q_1^{12} = 0$ and $q_2^{12} = q_1^{23} = q_1^{23} = q_2^{23} = 1$, which together with (2) as equalities can be easily shown to yield $q_1^{11} = 1$, $q_2^{22} = 0$, $s^{11} = 1$, $s^{12} = 2$, $s^{22} = 1$, $s^{23} = 4$, which is precisely Table 1).

(b) Any small enough perturbation of the example—such as having full support on, say $[0, 4]^2$, or strictly increasing *all* valuations as α—will not affect the nonmonotonicity, since the inequality $\mathrm{Rev}(\mathcal{F}_0) > \mathrm{Rev}(\mathcal{F}_{1/12})$ is strict. This also implies that nonmonotonicity occurs in similar models, for instance, when the seller has nonzero costs of producing the goods (consider small enough such costs).

(c) There are special cases where increasing values yields monotonicity of the maximal revenue. When the valuations are increased *uniformly*—i.e., each $x \in \mathbb{R}_+^k$ is replaced by $x + d$ for fixed $d \in \mathbb{R}_+^k$—then the optimal revenue cannot decrease. Indeed, for every mechanism (q, s), let $(\tilde{q}, \tilde{s})$ be the mechanism given by $\tilde{q}(x + d) := q(x)$ and $\tilde{s}(x + d) := s(x) + q(x) \cdot d$; it is immediate to check that if (q, s) is IC, then so is $(\tilde{q}, \tilde{s})$. Let X be an $\mathbb{R}_+^k$-valued random variable; then[20] $\mathrm{Rev}(X + d) \ge \mathbb{E}[\tilde{s}(X + d)] = \mathbb{E}[s(X)] + \mathbb{E}[q(X)] \cdot d \ge \mathbb{E}[s(X)]$; this holds for every IC mechanism (q, s), and so $\mathrm{Rev}(X + d) \ge \mathrm{Rev}(X)$.

When the valuations are proportionately increased *uniformly*, i.e., each $x \in \mathbb{R}_+^k$ is replaced by $c * x = (c_1 x_1, c_2 x_2, \ldots, c_k x_k)$ for fixed $c_1, c_2, \ldots, c_k \ge 1$, then, again, the optimal revenue cannot decrease. To see this, for every mechanism (q, s), let $(\tilde{q}, \tilde{s})$ be given by $\tilde{q}_i(c * x) := q_i(x)/c_i$, $i = 1, \ldots, k$, and $s(c * x) := s(x)$; if (q, s) is IC, then so is $(\tilde{q}, \tilde{s})$ (since $\tilde{q}(c * y) \cdot (c * x) - s(c * y) = q(y) \cdot x - s(y)$ for every $x, y \in \mathbb{R}_+^k$). Therefore, $\mathrm{Rev}(c * X) \ge \mathbb{E}[\tilde{s}(c * X)] = \mathbb{E}[s(X)]$ and so[21] $\mathrm{Rev}(c * X) \ge \mathrm{Rev}(X)$.

To conclude this section, observe that Example 1, whose optimal mechanism is deterministic, shows that nonmonotonicity for multiple goods is *not* due to the need for lotteries in this case (see Section 3).

2.3 *Some classes of monotonic mechanisms*

In this section, we identify two natural classes of mechanisms for which monotonicity is guaranteed to hold in the multiple-good case: mechanisms that are symmetric and deterministic, and mechanisms whose pricing is submodular.[22]

[20] We slightly abuse the notation and write $\mathrm{Rev}(X)$ instead of $\mathrm{Rev}(\mathcal{F})$ when X is a random variable with distribution $\mathcal{F}$.

[21] One cannot get the opposite inequality by interchanging X and $c * X$, since $c_i \ge 1$ is needed so as to guarantee that $\tilde{q}_i \le 1$. However, if all the c_i are equal, say $c_i = \gamma$, then taking $\hat{q}(\gamma x) := q(x)$ and $\hat{s}(\gamma x) := \gamma s(x)$ (instead of $(\tilde{q}, \tilde{s})$) yields $\mathrm{Rev}(\gamma X) = \gamma \mathrm{Rev}(X)$ for *every* $\gamma \ge 0$. Combining all these, we get

$$\left(\min_i c_i\right) \mathrm{Rev}(X) \le \mathrm{Rev}(c * X + d) \le \left(\max_i c_i\right) \mathrm{Rev}(X) + \sum_{i=1}^k d_i$$

for every fixed $c_i, d_i \ge 0$ (use $c_i = c_i' \gamma$ where $\gamma = \min_i c_i$ and $c_i' \ge 1$). We thank a referee, whose comment led us to consider these affine transformations.

[22] While mechanisms in one or the other of these two classes arise as optimal mechanisms in various examples within the literature, general conditions on the distributions that yield such optimal mechanisms are not known. See the concluding paragraph of this section.

Recall Example 1. The optimal mechanism there, which is deterministic, is not symmetric across the goods, since the price of good 1 is different from the price of good 2. Also, it is not subadditive (or submodular), since the price of the bundle is larger than the sum of the prices of the separate goods. The results of this section will imply that any counterexample to nonmonotonicity *must* satisfy these conditions.

A mechanism (q,s) is *deterministic* if the range $Q := \{q(x) : x \in \mathbb{R}^k_+\}$ of the allocation function q is included in $\{0,1\}^k$ (rather than $[0,1]^k$; recall Section 1.1). Incentive compatibility implies that to each allocation that appears in the mechanism, i.e., each $g \in Q$, there is a unique payment: if $q(x) = q(y)$, then $s(x) = s(y)$. Thus the menu $M = \{(q(x), s(x)) : x \in \mathbb{R}^k_+\}$ of a deterministic mechanism is finite (there are at most 2^k allocation outcomes), and such a mechanism can be represented by specifying the payment—or "price"—p_I for each subset I of goods, i.e., for each[23] $I \subset K := \{1, 2, \ldots, k\}$: thus, $b(x) = \max_{I \subset K}\{\sum_{i \in I} x_i - p_I\}$. A deterministic mechanism is *symmetric* if p_I depends only on the size $|I|$ of I, i.e., the price depends only on the number of goods (and so the function b is symmetric, i.e., invariant under permutations of the coordinates $x_1, \ldots, x_k$).

THEOREM 4. *Let $\mathcal{F}_1$ be a distribution on $\mathbb{R}^k_+$ such that there is a deterministic and symmetric IC mechanism that is optimal for $\mathcal{F}_1$. Then* $\mathrm{Rev}(\mathcal{F}_1) \le \mathrm{Rev}(\mathcal{F}_2)$ *for every distribution $\mathcal{F}_2$ on $\mathbb{R}^k_+$ that first-order stochastically dominates $\mathcal{F}_1$.*

The proof makes use of the notion of "seller-favorable" mechanisms introduced in Section 1.2 (see the Appendix for details). We show, first, that deterministic and symmetric seller-favorable mechanisms have monotonic payment functions, and, second, that from any optimal deterministic and symmetric mechanism, one obtains a similar mechanism that is, in addition, seller-favorable.[24]

PROPOSITION 5. *Let (q,s) be a deterministic and symmetric seller-favorable IC mechanism on $\mathbb{R}^k_+$. Then the payment function s is nondecreasing.*

PROOF. With a slight abuse of notation, we will write p_i instead of p_I for all $I \subset K$ with $|I| = i$. For $x \in \mathbb{R}^k_+$, let $x^{\#}$ denote the vector obtained by permuting the coordinates of x so that they are in nonincreasing order, i.e., $x^{\#}_1 \ge x^{\#}_2 \ge \cdots \ge x^{\#}_k$; symmetry implies that $s(x) = s(x^{\#})$. Let $y, z \in \mathbb{R}^k_+$ be such that $y \le z$; then it easily follows that $y^{\#} \le z^{\#}$ and so we need to show that $s(y^{\#}) \le s(z^{\#})$. Put $\eta_i := y^{\#}_1 + \cdots + y^{\#}_i - p_i$ and $\zeta_i := z^{\#}_1 + \cdots + z^{\#}_i - p_i$; then $b(y^{\#}) = \max_{1 \le i \le k} \eta_i$ and $b(z^{\#}) = \max_{1 \le i \le k} \zeta_i$ (since, for the same price, the buyer always prefers the highest-valued goods). Let $s(y^{\#}) = p_j$ and $s(z^{\#}) = p_\ell$; then these maxima are attained at j and ℓ, respectively, and, in particular, we have $\eta_j \ge \eta_\ell$ and $\zeta_\ell \ge \zeta_j$. If $\ell \ge j$, then $p_\ell - p_j = (\eta_j - \eta_\ell) + (y^{\#}_{j+1} + \cdots + y^{\#}_\ell) \ge 0$ (since $y^{\#} \ge 0$) and so $p_\ell \ge p_j$. If $\ell < j$, then $\zeta_j - \zeta_\ell = (\eta_j - \eta_\ell) + (z^{\#}_{\ell+1} - y^{\#}_{\ell+1}) + \cdots + (z^{\#}_j - y^{\#}_j) \ge 0$ (since $z^{\#} \ge y^{\#}$); but $\zeta_\ell \ge \zeta_j$, and so $\zeta_\ell = \zeta_j$. Therefore, ζ_j is also a maximizer for $b(z^{\#})$; since $s(z^{\#}) = p_\ell$, the seller-favorable condition implies that $p_\ell \ge p_j$, which completes the proof. □

[23] If I is not a possible outcome of the mechanism, put $p_I = \infty$.

[24] While there is always a seller-favorable optimal mechanism, it is not immediately clear that there is a deterministic and symmetric such mechanism.

Corollary 6. *Let* DSRev *denote the optimal revenue obtained when restricted to deterministic and symmetric mechanisms. If* $\mathcal{F}_2$ *first-order stochastically dominates* $\mathcal{F}_1$*, then* $\text{DSRev}(\mathcal{F}_1) \le \text{DSRev}(\mathcal{F}_2)$.

Proof. Let (q, s) be a deterministic and symmetric mechanism, and let M be its menu, which is finite. Apply Proposition 16 in Section A.1 to get a seller-favorable IC mechanism $(\tilde{q}, \tilde{s})$ that is symmetric (use the construction in the proof there for x with $x_1 \ge x_2 \ge \cdots \ge x_k$, and then extend to all x by $(\tilde{q}(x), \tilde{s}(x)) = (\tilde{q}(x^\#), \tilde{s}(x^\#))$; that is, break ties so that symmetry is preserved). Since the menu $\widetilde{M}$ of $(\tilde{q}, \tilde{s})$ satisfies $\widetilde{M} \subset \text{cl}\, M = M$ (as M is a finite set), the new mechanism is also deterministic. Thus $(\tilde{q}, \tilde{s})$ is deterministic and symmetric. Because $\tilde{s}(x) \ge s(x)$ for all x, it follows that DSRev is, in fact, the optimal revenue over deterministic, symmetric, *and* seller-favorable mechanisms. Proposition 5 says that the payment function $\tilde{s}$ of such a mechanism is monotonic, which implies that $\mathbb{E}_{\mathcal{F}_1}[\tilde{s}(x)] \le \mathbb{E}_{\mathcal{F}_2}[\tilde{s}(x)]$ by stochastic dominance, and, hence, our result. □

Proof of Theorem 4. We have

$$\text{Rev}(\mathcal{F}_1) = \text{DSRev}(\mathcal{F}_1) \le \text{DSRev}(\mathcal{F}_2) \le \text{Rev}(\mathcal{F}_2)$$

(the first equality by assumption, the second by Corollary 6, and the third since DSRev considers only a subclass of mechanisms). □

Remark. Each of the three conditions in Proposition 5—deterministic, symmetric, and seller-favorable—is indispensable: dropping any one of them allows mechanisms with nonmonotonic s. Indeed, Example 2 in Section 2.4 below is not deterministic, Example 1 in Section 2.2 is not symmetric, and the mechanism with buyer payoff function $b(x_1, x_2) = \max\{0, x_1 - 2, x_2 - 2, x_1 + x_2 - 3\}$ and $q(2, 1) = (1, 1)$, $s(2, 1) = 3$, $q(3, 1) = (1, 0)$, and $s(3, 1) = 2$—thus, nonmonotonic s—is not seller-favorable.[25]

A deterministic mechanism is *submodular* (Manelli and Vincent 2006) if the payments satisfy $p_I + p_J \ge p_{I\cup J} + p_{I\cap J}$ for all subsets of goods $I, J \subset K := \{1, 2, \ldots, k\}$ (this is a generalization of subadditivity, which obtains when I and J are disjoint); alternatively, $p_{I\cup L} - p_I$ is monotonically decreasing in $I \subset K \setminus L$, which is a standard decreasing marginal price property. We now generalize this condition to arbitrary (i.e., nondeterministic) mechanisms.

Let (q, s) be an IC mechanism with buyer payoff function $b : \mathbb{R}^k_+ \to \mathbb{R}$. A *pricing function* $p : [0, 1]^k \to \mathbb{R} \cup \{\infty\}$ associates a price $p(g)$ to every lottery $g \in [0, 1]^k$ (we allow infinite prices for lotteries that are not used). Given a pricing function p, consider offering to the buyer the choices $(g, p(g))$ for all possible lotteries $g \in [0, 1]^k$; the resulting payoff to the buyer is $b_p(x) := \sup_{g\in[0,1]^k}(g \cdot x - p(g))$. We say that p is *a pricing function for the mechanism* (q, s) if it yields the same choices and payoffs, i.e., $p(q(x)) = s(x)$ and $b(x) = b_p(x)$ for every $x \in \mathbb{R}^k_+$. Thus,

$$b(x) = q(x) \cdot x - s(x) = q(x) \cdot x - p(q(x)) = \sup_{g\in[0,1]^k} (g \cdot x - p(g)) \tag{3}$$

[25]This example, whose payments are subadditive, shows that seller favorability is indispensable also for the result of Proposition 8 below.

for every $x \in \mathbb{R}_+^k$. While the price of each lottery $q(x)$ that is used in (q, s) is determined,[26] there is freedom in choosing the prices of the other lotteries. However, as we will show in Section A.2, there is a *canonical* pricing function for (q, s), namely, the lowest one possible; it is given by the function p_0, defined as[27] $p_0(g) := \sup_{x\in\mathbb{R}_+^k}(g\cdot x - b(x))$.

We say that the mechanism (q, s) is *submodular* if it has a pricing function p that satisfies

$$p(g) + p(g') \geq p(g \vee g') + p(g \wedge g') \tag{4}$$

for all g and g' in the range, i.e., $g, g' \in Q := q(\mathbb{R}_+^k)$, where $g \vee g'$ and $g \wedge g'$ denote, respectively, the coordinatewise maximum and minimum of the two vectors g and g'. In Appendix A.2, we will show that this is equivalent to requiring the canonical pricing function p_0 to satisfy (4).

Theorem 7. *Let $\mathcal{F}_1$ be a distribution on $\mathbb{R}_+^k$ such that there is a submodular IC mechanism that is optimal for $\mathcal{F}_1$. Then* $\mathrm{Rev}(\mathcal{F}_1) \leq \mathrm{Rev}(\mathcal{F}_2)$ *for every distribution $\mathcal{F}_2$ on $\mathbb{R}_+^k$ that first-order stochastically dominates $\mathcal{F}_1$.*

The proof will use the following proposition.

Proposition 8. *Let (q, s) be a submodular seller-favorable IC mechanism. Then the payment function s is nondecreasing.*

Proof. Let p be a pricing function for (q, s) (thus (3) holds for every $x \in \mathbb{R}_+^k$) that satisfies (4) for every $g, g' \in Q = q(\mathbb{R}_+^k)$. Let $y, z \in \mathbb{R}_+^k$ be such that $y \leq z$. We will show that $s(y) \leq s(z)$.

Put $g_y := q(y)$ and $g_z := q(z)$. Then $s(y) = p(g_y)$ and $s(z) = p(g_z)$. From (3), for y we get, in particular,

$$g_y \cdot y - p(g_y) \geq (g_y \wedge g_z)\cdot y - p(g_y \wedge g_z), \tag{5}$$

and from (3), for z we get in particular

$$g_z \cdot z - p(g_z) \geq (g_y \vee g_z)\cdot z - p(g_y \vee g_z). \tag{6}$$

Subtracting $g_z \cdot (z - y)$ from the left-hand side of (6) and subtracting the larger $(g_y \vee g_z)\cdot(z - y)$ from its right-hand side (it is larger since $g_y \vee g_z \geq g_z$ and $z - y \geq 0$) gives

$$g_z \cdot y - p(g_z) \geq (g_y \vee g_z)\cdot y - p(g_y \vee g_z). \tag{7}$$

Adding (5) and (7), and using $g_y + g_z = g_y \vee g_z + g_y \wedge g_z$ (this equality holds for each coordinate) implies

$$0 \geq p(g_y) + p(g_z) - p(g_y \vee g_z) - p(g_y \wedge g_z),$$

[26] The graph of p restricted to $Q := q(\mathbb{R}_+^k)$ is precisely the menu $M = (q, s)(\mathbb{R}_+^k)$.

[27] Thus p_0 is the *Fenchel conjugate* of the function b (with the usual convention that $b(x) = \infty$ for x outside the domain, i.e., $x \notin \mathbb{R}_+^k$; or, as in Section A.1, extend b to all $\mathbb{R}^k$).

which must, therefore, be an equality by the submodularity condition (4). Therefore, we must have equalities throughout, in particular in (6):

$$b(z) = g_z \cdot z - p(g_z) = (g_y \vee g_z) \cdot z - p(g_y \vee g_z). \tag{8}$$

Hence, the maximum in $b(z)$ is attained also at $g = g_y \vee g_z$. Hence, the mechanism (q', s') that is identical to (q, s) except at z, where we put $(q'(z), s'(z)) := (g_y \vee g_z, p(g_y \vee g_z))$, is IC and has the same buyer payoff function b (by (3) and (8)). Since (q, s) is seller-favorable, it follows that $s'(z) \leq s(z)$ or

$$p(g_y \vee g_z) \leq p(g_z). \tag{9}$$

Finally, $g_y \cdot y - p(g_y) \geq (g_y \vee g_z) \cdot y - p(g_y \vee g_z) \geq g_y \cdot y - p(g_y \vee g_z)$ (the first inequality is from (3) for y, and the second is since $g_y \vee g_z \geq g_y$ and $y \geq 0$); hence,

$$p(g_y \vee g_z) \geq p(g_y). \tag{10}$$

Combining (9) and (10) yields $s(y) = p(g_y) \leq p(g_z) = s(z)$. □

Remark. The proof of Proposition 8 shows that $p(g_z) \geq p(g_y \vee g_z)$ and $b(z) = g_z \cdot z - p(g_z) = (g_y \vee g_z) \cdot z - p(g_y \vee g_z)$ (recall that $g_y = q(y)$ and $g_z = q(z)$), from which it follows that $(g_y \vee g_z) \cdot z \leq g_z \cdot z$. Therefore, we have equality (since $g_y \vee g_z \geq g_z$ and $z \geq 0$), which implies that $(g_y \vee g_z)_i = (g_z)_i$, i.e., $q_i(y) \leq q_i(z)$, for every coordinate i such that $z_i > 0$. Thus, q is coordinatewise nondecreasing on $\mathbb{R}^k_{++}$. Moreover, q can be adjusted on the boundary of $\mathbb{R}^k_+$ without changing b or s, while maintaining IC, and such that the redefined q is coordinatewise nondecreasing on all of[28] $\mathbb{R}^k_+$.

Corollary 9. *Let* SbRev *denote the optimal revenue obtained when restricted to submodular mechanisms. If $\mathcal{F}_2$ first-order stochastically dominates $\mathcal{F}_1$, then* $\mathrm{SbRev}(\mathcal{F}_1) \leq \mathrm{SbRev}(\mathcal{F}_2)$.

Proof. Given any submodular IC mechanism (q, s), Corollary 20 (with $D = \mathbb{R}^k_+$) yields a submodular seller-favorable IC mechanism $(\tilde{q}, \tilde{s})$ with $\tilde{s}(x) \geq s(x)$ for every $x \in \mathbb{R}^k_+$. Therefore, SbRev is, in fact, the optimal revenue over submodular seller-favorable mechanisms. Since the payment function $\tilde{s}$ for such mechanisms is monotonic by applying Proposition 8 to $(\tilde{q}, \tilde{s})$, the result follows. □

Proof of Theorem 7. Similar to the proof of Theorem 4, using Corollary 9,

$$\mathrm{Rev}(\mathcal{F}_1) = \mathrm{SbRev}(\mathcal{F}_1) \leq \mathrm{SbRev}(\mathcal{F}_2) \leq \mathrm{Rev}(\mathcal{F}_2). \qquad \square$$

[28]Redefine $q(y)$ for y on the boundary of $\mathbb{R}^k_+$ by making its ith coordinate zero if $y_i = 0$. Hence, for $y \leq z$, if $z_i = 0$, then $y_i = 0$ and so $q_i(y) = 0 \leq q_i(z)$. Therefore, $q_i(y) \leq q_i(z)$ holds now for all coordinates i when y, $z \in \mathbb{R}^k_+$ are such that $y \leq z$. Since the probabilities $q_i(y)$ have only gone down and $q(y) \cdot y$ has not been affected, the redefined mechanism (q, s) remains IC on $\mathbb{R}^k_+$, and the buyer's payoff function is unchanged.

The results of this section imply that monotonicity holds in every case where one can prove that there are optimal mechanisms that are either deterministic and symmetric, or submodular. Thus, $\mathrm{Rev}(\mathcal{F}_2) \geq \mathrm{Rev}(\mathcal{F}_1)$ when $\mathcal{F}_2$ first-order stochastically dominates $\mathcal{F}_1$, and $\mathcal{F}_1 = F \times F$ is the distribution of two i.i.d. goods with F that is (i) uniform (where the optimal mechanism is deterministic, symmetric, and submodular; Manelli and Vincent 2006); (ii) Pareto with index $\geq \frac{1}{2}$ (where bundling, which is a deterministic and symmetric mechanism, is optimal; Hart and Nisan 2014a, Theorem 28; or 2014b, Theorem 6); (iii) the conditions of Theorems 2 and 4 in Manelli and Vincent (2006) are satisfied (yielding a submodular deterministic optimal mechanism).

2.4 *Nonmonotonicity for independent and identically distributed goods*

We now provide an example of nonmonotonicity where two goods are independent and identically distributed, and we increase the values of *both* goods; that is, we compare the optimal revenue of $F_1 \times F_1$ with that of $F_2 \times F_2$, where F_1 and F_2 are one-dimensional distributions such that F_2 first-order stochastically dominates F_1.

The example must be more complicated than Example 1, for two main reasons. First, its optimal mechanisms cannot be submodular (by Theorem 7), and if it is unique,[29] it cannot be deterministic (since an i.i.d. distribution is symmetric and, thus, there always exists a symmetric optimal mechanism, to which we apply Theorem 4). Second, because the two distributions are i.i.d., it is not enough to find an optimal mechanism whose payment function s is nonmonotonic at a single point, since, by independence, a change in a value of one good applies to all values of the other good. To understand this last point, assume that we increase a value $\bar{x}_1$ of good 1 to $\bar{x}_1 + \delta$ for some $\delta > 0$; the $(\bar{x}_1, x_2)$ valuations now become $(\bar{x}_1 + \delta, x_2)$ for *all* x_2. While s may be nonmonotonic at *some* $(\bar{x}_1, x_2)$, it need not be nonmonotonic at *all* $(\bar{x}_1, x_2)$; what is needed is that the *expectation* of $s(\bar{x}_1, x_2)$ over x_2 be nonmonotonic in[30] $\bar{x}_1$.

Example 2. Let F_1 and F_2 be the one-dimensional distributions

$$F_1 = \begin{cases} 10 & \text{with probability } \frac{4}{15} \\ 46 & \text{with probability } \frac{1}{90} \\ 47 & \text{with probability } \frac{1}{3} \\ 80 & \text{with probability } \frac{7}{30} \\ 100 & \text{with probability } \frac{7}{45} \end{cases} \qquad F_2 = \begin{cases} 10 & \text{with probability } \frac{2{,}399}{9{,}000} \\ 13 & \text{with probability } \frac{1}{9{,}000} \\ 46 & \text{with probability } \frac{1}{90} \\ 47 & \text{with probability } \frac{1}{3} \\ 80 & \text{with probability } \frac{7}{30} \\ 100 & \text{with probability } \frac{7}{45}. \end{cases}$$

Clearly F_2 first-order stochastically dominates F_1 (since F_2 is obtained from F_1 by moving a probability mass of $\frac{1}{9{,}000}$ from 10 to 13), which of course implies that $F_2 \times F_2$ first-order stochastically dominates $F_1 \times F_1$. However, the optimal revenue from $F_1 \times F_1$ turns out to be *higher* than the optimal revenue from $F_2 \times F_2$. ◊

[29] Which, for discrete distributions, is generically true (i.e., for small perturbations of the distribution, the optimal mechanism is unique).

[30] This "accounting" is not entirely correct, as we also need to consider the valuations (x_2, x_1^0) for all x_2, and then (x_1^0, x_1^0) is counted twice.

Valuations x	Outcome $q(x)$	$s(x)$
(10, 10), (10, 13), (13, 10), (13, 13), (10, 46), (46, 10), (13, 46), (46, 13), (46, 46), (13, 47), (47, 13), (10, 47), (47, 10)	(0, 0)	0
(46, 47)	$(\frac{32}{1{,}187}, \frac{384}{13{,}057})$	$\frac{34{,}240}{13{,}057} \approx 2.6$
(47, 46)	$(\frac{384}{13{,}057}, \frac{32}{1{,}187})$	$\frac{34{,}240}{13{,}057} \approx 2.6$
(47, 47)	$(\frac{35}{1{,}187}, \frac{35}{1{,}187})$	$\frac{3{,}258}{1{,}187} \approx 2.7$
(13, 80)	$(\frac{32}{1{,}187}, \frac{5{,}647}{5{,}935})$	$\frac{90{,}672}{1{,}187} \approx 76.4$
(80, 13)	$(\frac{5{,}647}{5{,}935}, \frac{32}{1{,}187})$	$\frac{90{,}672}{1{,}187} \approx 76.4$
(46, 80)	$(\frac{35}{1{,}187}, \frac{5{,}647}{5{,}935})$	$\frac{90{,}810}{1{,}187} \approx 76.5$
(80, 46)	$(\frac{5{,}647}{5{,}935}, \frac{35}{1{,}187})$	$\frac{90{,}810}{1{,}187} \approx 76.5$
(10, 80), (10, 100), (13, 100)	(0, 1)	80
(80, 10), (100, 10), (100, 13)	(1, 0)	80
(46, 100), (100, 46), (47, 80), (80, 47), (47, 100), (100, 47), (80, 80), (80, 100), (100, 80), (100, 100)	(1, 1)	126

TABLE 2. The unique optimal mechanism for $F_2 \times F_2$.

PROPOSITION 10. *In Example 2, F_2 first-order stochastically dominates F_1 and*

$$\text{Rev}(F_2 \times F_2) < \text{Rev}(F_1 \times F_1).$$

PROOF. Maximizing revenue for a distribution with finite support is a linear programming problem (the unknowns are the $q_i(x)$ and $s(x)$ for all x in the support, the constraints are the IR and IC inequalities, and the objective function is the expected revenue). Using Maple yields the following situation.[31]

The (unique[32]) optimal mechanism for $F_2 \times F_2$ consists of 11 outcomes; see Table 2 (the outcomes are ordered according to increasing payment to the seller s). For $F_1 \times F_1$, the same mechanism is optimal; however, the fifth and sixth outcomes are not used (the value 13 has probability 0) and may be dropped. This yields

$$\text{Rev}(F_1 \times F_1) = \frac{408{,}189{,}937}{5{,}875{,}650} \quad \text{and} \quad \text{Rev}(F_2 \times F_2) = \frac{30{,}614{,}162{,}731}{440{,}673{,}750},$$

and so indeed $\text{Rev}(F_1 \times F_1) > \text{Rev}(F_2 \times F_2)$ (these revenues are $69.47145\ldots$ and $69.47126\ldots$). The nonmonotonicity of the payments can be seen at[33] $s(10, 80) > s(13, 80), s(46, 80)$ and $s(80, 10) > s(80, 13), s(80, 46)$. □

[31] The fractions appearing in the solutions below are *exact*.

[32] Uniqueness is proved using the dual linear progamming problem as in Section 2.2.

[33] Recall, however, that s being a nonmonotonic function, while necessary, is not sufficient; cf. the second paragraph of this section.

3. Lotteries and revenue

To maximize revenue in the one-good case (i.e., $k = 1$), it suffices to consider deterministic mechanisms (specifically, "posted-price" mechanisms; see Myerson 1981). That is *not* so in the multiple-good case. Examples where the optimal mechanism requires randomization (i.e., in some of the outcomes the probability of getting a good is strictly between 0 and 1) have been provided by Thanassoulis (2004) (in the slightly different context of "unit demand," where the buyer is limited to one good), Pycia (2006), Manelli and Vincent (2006, 2007), Briest et al. (2010) (for unit demand), and Pavlov (2011, Example 3(ii)). However, most of these examples are relatively complicated and require nontrivial computations, and it is not clear how and why randomization helps only when there are multiple goods.

We will provide two examples that are simple and transparent enough that the need for randomization becomes clear. In the first, the values of the two goods are correlated; in the second, the values are independent and identically distributed.[34]

3.1 *Lotteries for multiple goods*

Consider the following example with two goods and three possible valuations[35] (the values of the two goods are correlated).

Example 3. Let $\mathcal{F}$ be the two-dimensional probability distribution

$$\mathcal{F} = \begin{cases} (1,0) & \text{with probability } \frac{1}{3} \\ (0,2) & \text{with probability } \frac{1}{3} \\ (3,3) & \text{with probability } \frac{1}{3}. \end{cases} \qquad \diamondsuit$$

Proposition 11. *The mechanism* (q, s) *defined by Table 3 with buyer payoff function*

$$b(x_1, x_2) = \max\left\{0, \tfrac{1}{2}x_1 - \tfrac{1}{2}, x_2 - 2, x_1 + x_2 - 5\right\} \tag{11}$$

is the unique *revenue-maximizing IC and IR mechanism for* $\mathcal{F}$ *of Example 3.*

Thus, the buyer can get both goods for price 5, or get good 2 for price 2, or *get good* 1 *with probability* $\frac{1}{2}$ for price $\frac{1}{2}$; the optimal revenue is $\frac{5}{2} = 2.5$. If the seller were restricted

[34] Manelli and Vincent (2007) provide an example (Example 1) of an "undominated mechanism" that uses lotteries. While they prove that an undominated mechanism is optimal for some distribution $\mathcal{F}$, it is also claimed there (Theorem 9) that any undominated mechanism is optimal for some distribution with *independent* goods (i.e., a product distribution). However, there is an error in the proof of Theorem 9, as the set of product distributions (specifically, the set G in their proof) is not convex. See the corrigendum in Manelli and Vincent (2012).

[35] Pycia (2006) solves the seller's problem when there are exactly two valuations and shows that randomization may be needed. For instance, when the valuations are $(2, 3)$ and $(6, 1)$ with equal probabilities, the *unique* optimal mechanism gives the $(2, 3)$ buyer good 2 and a $\frac{1}{2}$ chance of getting good 1, for the total price of 4, and gives the $(6, 1)$ buyer both goods for the total price of 7. However, we have found that Example 3, with three possible valuations, provides more transparent insights (as there is a clearer separation between the IC and IR constraints).

Valuation x	Outcome $q(x)$	$s(x)$
$(1,0)$	$(\frac{1}{2},0)$	$\frac{1}{2}$
$(0,2)$	$(0,1)$	2
$(3,3)$	$(1,1)$	5

TABLE 3. The unique optimal mechanism for Example 3.

to deterministic mechanisms (where each q_i is either 0 or 1), then the optimal revenue would decrease to $\frac{7}{3} = 2.33\ldots$ (attained, for instance, by selling separately, at the optimal single-good prices of 3 for good 1 and 2 for good 2; see below). A detailed explanation of the role of randomization and why it is needed only when there are multiple goods, follows the proof below.

PROOF OF PROPOSITION 11. Let $\langle(\alpha_1,\beta_1);\sigma_1\rangle$, $\langle(\alpha_2,\beta_2);\sigma_2\rangle$, and $\langle(\alpha_3,\beta_3);\sigma_3\rangle$ be the outcome $\langle(q_1(x),q_2(x));s(x)\rangle$ at $x=(1,0),(0,2)$, and $(3,3)$, respectively (thus $\alpha_i,\beta_i \in [0,1]$). The objective function is $S := \sigma_1+\sigma_2+\sigma_3$ (this is three times the revenue). Consider the relaxed problem of maximizing S subject only to the individual-rationality constraints at $(1,0)$ and $(0,2)$, and to the two incentive-compatibility constraints at $(3,3)$, i.e.,

$$\alpha_1-\sigma_1 \geq 0$$
$$2\beta_2-\sigma_2 \geq 0$$
$$3\alpha_3+3\beta_3-\sigma_3 \geq 3\alpha_1+3\beta_1-\sigma_1$$
$$3\alpha_3+3\beta_3-\sigma_3 \geq 3\alpha_2+3\beta_2-\sigma_2.$$

These inequalities can be rewritten as

$$\sigma_3+3\alpha_1+3\beta_1-3\alpha_3-3\beta_3 \leq \sigma_1 \leq \alpha_1$$
$$\sigma_3+3\alpha_2+3\beta_2-3\alpha_3-3\beta_3 \leq \sigma_2 \leq 2\beta_2.$$

Therefore, so as to maximize $S=\sigma_1+\sigma_2+\sigma_3$, we must take $\sigma_1=\alpha_1$ and $\sigma_2=2\beta_2$, which gives

$$\sigma_3 \leq 3\alpha_3+3\beta_3-2\alpha_1-3\beta_1$$
$$\sigma_3 \leq 3\alpha_3+3\beta_3-3\alpha_2-\beta_2.$$

Thus, we must take $\alpha_3=\beta_3=1$, $\beta_1=\alpha_2=0$, and then $\sigma_3=\min\{6-2\alpha_1, 6-\beta_2\}$, and so $S=\alpha_1+2\beta_2+\min\{6-2\alpha_1,6-\beta_2\}=\min\{2\beta_2-\alpha_1,\beta_2+\alpha_1\}+6$. Since S is increasing in β_2, we must take $\beta_2=1$, and then $S=\min\{2-\alpha_1,1+\alpha_1\}+6$ is maximized at[36] $\alpha_1=\frac{1}{2}$.

[36]For deterministic mechanisms (i.e., $\alpha_i,\beta_i \in \{0,1\}$), everything is the same up to this point, but now S is maximized at both $\alpha_1=0$ and $\alpha_1=1$; the optimal revenue for deterministic mechanisms is thus $S/3=\frac{7}{3}$.

x	$q(x)$	$s(x)$	$q^{(1)}(x)$	$s^{(1)}(x)$	$q^{(2)}(x)$	$s^{(2)}(x)$
$(1, 0)$	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	$(1, 0)$	1	$(0, 0)$	0
$(0, 2)$	$(0, 1)$	2	$(0, 1)$	2	$(0, 1)$	2
$(3, 3)$	$(1, 1)$	$\underline{\mathbf{5}}$	$(1, 1)$	4	$(1, 1)$	$\underline{\mathbf{5}}$

TABLE 4. Replacing a lottery outcome when there are *two* goods.

This is precisely the mechanism in Table 3, which is easily seen to satisfy also all the other IR and IC constraints. □

To understand the use of randomization, consider the outcome $\langle(\frac{1}{2}, 0); \frac{1}{2}\rangle$ at $x = (1, 0)$ in Table 3: it is a lottery ticket that costs $\frac{1}{2}$ and gives a $\frac{1}{2}$ probability of getting good 1; alternatively,[37] it is a $\frac{1}{2} - \frac{1}{2}$ lottery between getting good 1 for the price 1 (i.e., $\langle(1, 0); 1\rangle$) and getting nothing, and paying nothing (i.e., $\langle(0, 0); 0\rangle$). It is thus the average of these two deterministic outcomes, and we now consider what happens when we replace the lottery by either one of them (see Table 4). It turns out that in *both* cases, the *revenue strictly decreases*. In the first case, replacing $\langle(\frac{1}{2}, 0); \frac{1}{2}\rangle$ by $\langle(1, 0); 1\rangle$ forces the price of the bundle to decrease to 4 (otherwise, the $(3, 3)$ buyer would switch from paying 5 for the bundle to paying 1 for good 1); therefore, the net change in the revenue is $\frac{1}{3} \cdot (1 - \frac{1}{2}) + \frac{1}{3} \cdot (4 - 5)$, which is negative.[38] In the second case, replacing $\langle(\frac{1}{2}, 0); \frac{1}{2}\rangle$ by $\langle(0, 0); 0\rangle$ results in the loss of the revenue from the $(1, 0)$ buyer, without, however, increasing the revenue from the $(3, 3)$ buyer: indeed, if we were to increase the bundle price, then $(3, 3)$ would switch to $\langle(0, 1); 2\rangle$, i.e., would get good 2 for price 2 (and, if we were to drop this outcome $\langle(0, 1); 2\rangle$ altogether so as to increase the bundle price to 6, the total revenue would again decrease).[39]

It is instructive to compare this with a similar example, but with a single good. Assume the values are $x = 1, 0, 3$, with equal probabilities of $\frac{1}{3}$ each (just like good 1 in Example 3). Take the mechanism with outcomes $\langle\frac{1}{2}; \frac{1}{2}\rangle$, $\langle 0; 0\rangle$, $\langle 1; 2\rangle$ (see Table 5); it is easy to see that it is IC and IR, and its revenue is $\frac{5}{6}$. The lottery outcome $\langle\frac{1}{2}; \frac{1}{2}\rangle$—getting the good with probability $\frac{1}{2}$ for price $\frac{1}{2}$—is the average of $\langle 0; 0\rangle$ and $\langle 1; 1\rangle$. Replacing the lottery $\langle\frac{1}{2}; \frac{1}{2}\rangle$ with $\langle 1; 1\rangle$ lowers the revenue to $\frac{2}{3}$: the 3 buyer switches to $\langle 1; 1\rangle$. Replacing the lottery $\langle\frac{1}{2}; \frac{1}{2}\rangle$ with $\langle 0; 0\rangle$ *increases* the revenue to 1: the 3 buyer is now offered, and chooses, $\langle 1; 3\rangle$. The revenue of $\frac{5}{6}$ of the original mechanism, which used the lottery outcome, is precisely the average of the revenues from these two resulting mechanisms, $\frac{2}{3}$ and 1 (this averaging property holds at each valuation x).

This is a general phenomenon when there is only one good: the revenue from a mechanism that includes an outcome that is a probabilistic mixture of two outcomes (a "lottery outcome") is the *average* of the revenues obtained by replacing the lottery

[37] Because of risk neutrality.

[38] The buyer's payoff function in this mechanism is $b^{(1)}(x) = \max\{x_1 - 1, x_2 - 2, x_1 + x_2 - 4\}$.

[39] The buyer's payoff function in this mechanism is $b^{(2)}(x) = \max\{0, x_2 - 2, x_1 + x_2 - 5\}$.

x	q	s	$q^{(1)}$	$s^{(1)}$	$q^{(2)}$	$s^{(2)}$
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0
0	0	0	0	0	0	0
3	1	$\underline{\mathbf{2}}$	1	1	1	$\underline{\mathbf{3}}$

Table 5. Replacing a lottery outcome when there is *one* good.

with each one of these two outcomes and then adapting the remaining outcomes.[40] Formally, this is the counterpart of expressing the corresponding buyer payoff function b as an average of two such functions; in the example above, $b(x) = \max\{0, x/2 - \frac{1}{2}, x - 2\}$ is the $\frac{1}{2} - \frac{1}{2}$ average of $b^{(1)}(x) = \max\{0, x - 1\}$ and $b^{(2)}(x) = \max\{0, x - 3\}$ (i.e., $b(x) = (b^{(1)}(x) + b^{(2)}(x))/2$ for all x). Thus, lotteries are indeed not needed when there is only one good.

Example 3 illustrates why this is *not* the case for multiple goods: replacing the lottery outcome with $\langle(0, 0); 0\rangle$ yields the mechanism $(q^{(2)}, s^{(2)})$, whose revenue is *lower* than that of (q, s) (whereas replacing $\langle\frac{1}{2}; \frac{1}{2}\rangle$ with $\langle 0; 0\rangle$ yields a *higher* revenue). In fact, the function b of (11) is an extreme point in the set of buyer payoff functions (in particular, it is *not* the average of the buyer functions in footnotes 38 and 39).

This is exactly where having more than one good matters. In the case of *one* good, there is only *one* binding constraint per value x, namely, the outcome chosen by the next lower value. Consequently, removing an outcome (such as a lottery outcome) that is chosen by x enables the seller to increase the revenue obtained from all higher-valuation buyers (i.e., with values $y > x$), as they can no longer switch to the outcome that has been removed and they strictly prefer their own outcome to any of the outcomes chosen by values below x. In contrast, when there are *multiple goods*, such an increase in revenue may not be possible because there may be *multiple* binding constraints for each valuation x (in our example, buyer $(3, 3)$ is indifferent between reporting truthfully and reporting either $(1, 0)$ or $(0, 2)$). These buyer types may switch to other outcomes that involve other goods, and so the total revenue may well decrease.

Next, how does a lottery outcome increase revenue? The seller would like to earn positive revenue from selling good 1 to the $(1, 0)$ buyer, but without jeopardizing the higher revenue obtained from selling the bundle of both goods to the $(3, 3)$ buyer (and, as we have seen, he cannot increase the price of the bundle because of the "good 2 for price 2" alternative, i.e., $\langle(0, 1); 2\rangle$). If the price of good 1 is above 1, then $(1, 0)$ will not buy it; if it is below 1, then $(3, 3)$ will switch from buying the bundle to buying good 1 (since his payoff will increase from 1 to 2 or more).[41] Thus, selling good 1 does not help. What does help is selling only a *fractional part* of good 1, which has the effect of making this option less attractive to the high-valuation buyer $(3, 3)$ (since his possible gain

[40] This statement, which is easily proved in general—even when the two outcomes that are averaged are not necessarily deterministic—provides another proof of Myerson's result that in the one-good case, it suffices to consider deterministic mechanisms (use this "local decomposition" repeatedly).

[41] As we saw above, lowering the price of the bundle to 4 (while keeping the price of good 1 at 1) will not help either, because the total revenue decreases.

Valuations x	Outcome $q(x)$	$s(x)$
$(1,1)$	$(0,0)$	0
$(2,1)$	$(\frac{1}{2},0)$	1
$(1,2)$	$(0,\frac{1}{2})$	1
$(1,4),(4,1),(2,2),(2,4),(4,2),(4,4)$	$(1,1)$	4

TABLE 6. The unique optimal mechanism for Example 4.

is smaller: it is only *that fraction* of the difference in values). Thus, the two conflicting desiderata—getting some revenue from a low-valuation buyer and not jeopardizing the higher revenue from a higher-valuation buyer—are reconciled by offering to sell fractions of the goods, i.e., lotteries. In the present example, that optimal fraction turns out to be $\frac{1}{2}$; it comes from balancing the incentives between the two goods (specifically, $\frac{1}{2}$ is the ratio of two value differences, $3-2$ for good 2 and $3-1$ for good 1; see the Proof of Proposition 11 above).[42]

Finally, we note that mechanism design is a sequential game, with the seller moving first. In such games, the use of randomization may, in general, be strictly advantageous to the first mover (take, for instance, the sequential matching pennies game). Thus, the surprising fact here is not that randomization can increase revenue (when there are multiple goods), but that it *cannot* do so when there is only one good.[43,44]

3.2 *Lotteries for independent and identically distributed goods*

We now provide a simple example where lotteries are necessary to achieve the maximal revenue for two goods that are independent and identically distributed.

EXAMPLE 4. Let F be the one-dimensional probability distribution

$$F=\begin{cases} 1 & \text{with probability } \frac{1}{6} \\ 2 & \text{with probability } \frac{1}{2} \\ 4 & \text{with probability } \frac{1}{3}, \end{cases}$$

and take two independent F-distributed goods, i.e., $\mathcal{F}=F\times F$. ◊

PROPOSITION 12. *The mechanism* (q,s) *defined by Table 6 with buyer payoff function*

$$b(x_1,x_2)=\max\left\{0,\tfrac{1}{2}x_1-1,\tfrac{1}{2}x_2-1,x_1+x_2-4\right\}$$

is the unique *optimal mechanism for* $\mathcal{F}=F\times F$ *of Example 4.*

[42] Thus, one can easily get other probabilities by changing the values. Moreover, the example is highly robust: it has a large neighborhood of distributions for which any optimal mechanism requires lotteries.

[43] We thank Bob Aumann for this comment.

[44] Pycia (2006) shows how, in the multiple-goods case, nondeterministic mechanisms are generically needed to maximize revenue.

Proof. First, the revenue from the mechanism in Table 6 is easily computed: it is $\frac{61}{18}$.

Second, consider the following inequalities, which are various individual rationality and incentive-compatibility constraints:[45]

$$\begin{array}{r|l}
q_1^{11}+q_2^{11}-s^{11}\geq 0 & 3\\
q_1^{12}+2q_2^{12}-s^{12}\geq 0 & 8\\
2q_1^{21}+q_2^{21}-s^{21}\geq 0 & 8\\
2q_1^{22}+2q_2^{22}-s^{22}\geq 0 & 17\\
q_1^{12}+2q_2^{12}-s^{12}\geq q_1^{11}+2q_2^{11}-s^{11} & 1\\
2q_1^{21}+q_2^{21}-s^{21}\geq 2q_1^{11}+q_2^{11}-s^{11} & 1\\
2q_1^{22}+2q_2^{22}-s^{22}\geq 2q_1^{12}+2q_2^{12}-s^{12} & 3\\
2q_1^{22}+2q_2^{22}-s^{22}\geq 2q_1^{21}+2q_2^{21}-s^{21} & 3\\
q_1^{14}+4q_2^{14}-s^{14}\geq q_1^{12}+4q_2^{12}-s^{12} & 3\\
4q_1^{41}+q_2^{41}-s^{41}\geq 4q_1^{21}+q_2^{21}-s^{21} & 3\\
2q_1^{22}+2q_2^{22}-s^{22}\geq 2q_1^{14}+2q_2^{14}-s^{14} & 1\\
2q_1^{22}+2q_2^{22}-s^{22}\geq 2q_1^{41}+2q_2^{41}-s^{41} & 1\\
2q_1^{24}+4q_2^{24}-s^{24}\geq 2q_1^{22}+4q_2^{22}-s^{22} & 8\\
4q_1^{42}+2q_2^{42}-s^{42}\geq 4q_1^{22}+2q_2^{22}-s^{22} & 8\\
4q_1^{44}+4q_2^{44}-s^{44}\geq 4q_1^{24}+4q_2^{24}-s^{24} & 2\\
4q_1^{44}+4q_2^{44}-s^{44}\geq 4q_1^{42}+4q_2^{42}-s^{42} & 2
\end{array} \tag{12}$$

Multiplying each inequality by the weight on the right and adding up yields

$$\begin{aligned}
&s^{11}+3s^{12}+3s^{21}+9s^{22}+2s^{14}+2s^{41}+6s^{24}+6s^{42}+4s^{44}\\
&\quad\leq 2q_1^{22}+q_1^{14}+10q_1^{41}+8q_1^{24}+24q_1^{42}+16q_1^{44}\\
&\qquad\qquad +2q_2^{22}+10q_2^{14}+q_2^{41}+24q_2^{24}+8q_2^{42}+16q_2^{44}.
\end{aligned} \tag{13}$$

The left-hand side turns out to be precisely 36 times the expected revenue of the seller for the distribution $\mathcal{F}=F\times F$, i.e., $36\mathbb{E}_{\mathcal{F}}[s(x)]$, and the right-hand side is bounded from above by 122 (replace all q_1 and q_2 there by their upper bound of 1). Therefore, $\mathbb{E}_{\mathcal{F}}[s(x)]\leq \frac{122}{36}=\frac{61}{18}$. Recalling that $\frac{61}{18}$ is precisely the revenue of the mechanism in Table 6 shows that Table 6 is optimal.

Finally, to see that Table 6 is the only optimal mechanism, by the proof above, for the maximal revenue of $\frac{61}{18}$ to be achieved, all the inequalities must become equalities. First, all the q_1 and q_2 appearing on the right-hand side of (13) must equal 1:

$$\begin{aligned}
1&=q_1^{22}=q_1^{14}=q_1^{41}=q_1^{24}=q_1^{42}=q_1^{44}\\
&=q_2^{22}=q_2^{14}=q_2^{41}=q_2^{24}=q_2^{42}=q_2^{44}.
\end{aligned} \tag{14}$$

[45]These specific inequalities and their corresponding multipliers below were obtained by solving the dual of the linear programming problem of maximizing revenue.

Second, the inequalities in (12), which are now equalities, yield, after substituting (14),

$$s^{44} = s^{24} = s^{42} = s^{22} = s^{14} = s^{41} = 4, \qquad s^{12} = s^{21} = 1, \qquad s^{11} = 0,$$
$$q_1^{11} = q_2^{11} = q_1^{12} = q_2^{21} = 0, \qquad q_1^{21} = q_2^{12} = \tfrac{1}{2}.$$

Together with (14) this yields precisely the mechanism in Table 6. □

It can be checked that the maximal revenue achievable by a deterministic mechanism is $\frac{10}{3}$ (obtained by the mechanism with price 2 for each good).

Appendix

A.1 *Seller-favorable mechanisms*

This appendix deals with incentive-compatible and seller-favorable mechanisms, introduced in Section 1.2. The main results are collected in Theorem 17; see also Remarks (a) and (b) after Corollary 18 for a discussion of implementation issues.

To be as general as possible, we will work here with an arbitrary domain $D \subset \mathbb{R}^k$ of valuations; D could be $\mathbb{R}^k_+$, or it may be finite or infinite, and, in general, need not be convex or even connected. A mechanism is thus $(q, s): D \to [0,1]^k \times \mathbb{R}$ and the buyer's payoff function is $b(x) = q(x) \cdot x - s(x)$. The range $M := (q, s)(D) = \{(q(x), s(x)) : x \in D\}$ of the mechanism, also called the *menu* of the mechanism, consists of all those combinations of allocations $g \in [0,1]^k$ and payments $t \in \mathbb{R}$ that are used (M is a subset of $[0,1]^k \times \mathbb{R}$). The mechanism is incentive-compatible (IC) if $b(x) = \max_{y \in D}(q(y) \cdot x - s(y)) = \max_{(g,t) \in M}(g \cdot x - t)$ for every $x \in D$. It is *seller-favorable* if there is no other incentive-compatible mechanism $(\tilde{q}, \tilde{s})$ on D having the same buyer payoff function, i.e., $\tilde{q}(x) \cdot x - \tilde{s}(x) = b(x) = q(x) \cdot x - s(x)$ for all $x \in D$, and a larger payment function, i.e., $\tilde{s}(x) \geq s(x)$ for every $x \in D$, with strict inequality for some $x \in D$. This implies, in particular, that when the buyer is indifferent, ties must be broken in favor of the seller, i.e., $q(y) \cdot x - s(y) = q(x) \cdot x - s(x)$ implies $s(y) \leq s(x)$.

The first lemma shows that one may extend any IC mechanism to a larger domain, even all $\mathbb{R}^k$, and without increasing the menu M beyond its closure, which we denote by $\operatorname{cl} M$.

Lemma 13. *Let (q, s) be an IC mechanism on a domain $D \subset \mathbb{R}^k$, with menu $M = (q, s)(D)$. Then (q, s) can be extended to an IC mechanism $(\bar{q}, \bar{s})$ on the whole space $\mathbb{R}^k$, with menu $\overline{M} = (\bar{q}, \bar{s})(\mathbb{R}^k)$ that satisfies $M \subset \overline{M} \subset \operatorname{cl} M$.*

Proof. Defining $b(x) := \sup_{(g,t) \in M}(g \cdot x - t)$ for every $x \in \mathbb{R}^k$ extends the buyer's payoff function from D to $\mathbb{R}^k$. The function s is bounded from below (since $b(x) = q(x) \cdot x - s(x)$ is finite for $x \in D$), and so there is τ such that $t \geq \tau$ for all $(g, t) \in M$. Fix some element $(g_0, t_0) \in M$; then, for every $x \in \mathbb{R}^k$, only those (g, t) in M with $t \leq \|x\|_1 + t_0$ matter for $b(x)$ (since $g \cdot x - t \geq g_0 \cdot x - t_0$ implies $t \leq (g - g_0) \cdot x + t_0 \leq \|x\|_1 + t_0$). Therefore, for every $x \in \mathbb{R}^k$, the supremum in the definition of $b(x)$ is attained, say at $(\bar{q}(x), \bar{s}(x)) \in \operatorname{cl} M$; for $x \in D$, we take $(\bar{q}(x), \bar{s}(x)) = (q(x), s(x))$. Thus $b(x) = \bar{q}(x) \cdot x - \bar{s}(x) =$

$\max_{(g,t)\in \mathrm{cl}\, M}(g\cdot x - t) = \max_{y\in\mathbb{R}^k}(\bar{q}(y)\cdot x - \bar{s}(y))$ for every $x \in \mathbb{R}^k$, which says that $(\bar{q}, \bar{s})$ is IC on $\mathbb{R}^k$. □

We shall thus consider without loss of generality mechanisms on $\mathbb{R}^k$. The buyer's payoff function b, as the pointwise supremum of affine functions, is a convex function on $\mathbb{R}^k$, finite everywhere (since, as we have seen, for every $x \in \mathbb{R}^k$, the supremum is attained).

We recall a few useful concepts for convex functions.[46] Let $f : \mathbb{R}^k \to \mathbb{R}$ be a real convex function defined on $\mathbb{R}^k$ (and so $\mathrm{dom}\, f = \mathbb{R}^k$). The *directional derivative of* f *at* $x \in \mathbb{R}^k$ *in the direction* $y \in \mathbb{R}^k$ is $f'(x; y) := \lim_{\delta\to 0^+}(f(x + \delta y) - f(x))/\delta$. Since f is convex, $f'(x; y)$ always exists. A vector $g \in \mathbb{R}^k$ is a *subgradient* of f at $x \in \mathbb{R}^k$ if $f(y) - f(x) \geq g \cdot (y - x)$ for all $y \in \mathbb{R}^k$. The set $\partial f(x)$ of subgradients of f at x is a nonempty compact set, and $f'(x; y) = \max\{g \cdot y : g \in \partial f(x)\}$ for every $x, y \in \mathbb{R}^k$. Finally, if $0 \leq f(x + z) - f(x) \leq \sum_{i=1}^k z_i$ holds for every $x, z \in \mathbb{R}^k$ with $z \geq 0$, then the function f is *nondecreasing* and *nonexpansive*.[47]

Let $\mathcal{B}^k$ be the collection of all real functions on $\mathbb{R}^k$ that are nondecreasing, nonexpansive, and convex.

Lemma 14. *Let (q, s) be an IC mechanism on $\mathbb{R}^k$. Then the buyer's payoff function b belongs to $\mathcal{B}^k$, and for every $x \in \mathbb{R}^k$, the vector $q(x)$ is a subgradient of b at x and $s(x) \leq b'(x; x) - b(x)$.*

Proof. Let $M = (q, s)(\mathbb{R}^k)$ be the menu of (q, s); then $b(x) = q(x)\cdot x - s(x) = \max_{(g,t)\in M}(g \cdot x - t)$, as a supremum of affine functions, is a convex function. For every $x, y \in \mathbb{R}^k$, we get

$$b(y) - b(x) \geq (q(x)\cdot y - s(x)) - (q(x)\cdot x - s(x)) = q(x)\cdot(y - x), \tag{15}$$

which says that $q(x)$ is a subgradient of b at x. Thus, $q(x)\cdot x \leq \sup\{g \cdot x : g \in \partial b(x)\} = b'(x; x)$ and so $s(x) = q(x)\cdot x - b(x) \leq b'(x; x) - b(x)$ for every $x \in \mathbb{R}^k$. Finally, taking $y = x + z$ with $z \geq 0$ in (15) implies that $0 \leq q(x)\cdot z \leq b(x + z) - b(x) \leq q(x + z)\cdot z \leq \sum_{i=1}^k z_i$, and so b is nondecreasing and nonexpansive. □

Lemma 15. *Let b be a function in $\mathcal{B}^k$. Then there exists an IC mechanism (q, s) on $\mathbb{R}^k$ such that the buyer's payoff function is b.*

Proof. Being nondecreasing, nonexpansive, and convex on $\mathbb{R}^k$, the function b satisfies $0 \leq b(x) - b(x - z) \leq g \cdot z \leq b(x + z) - b(x) \leq \sum_{i=1}^k z_i$ for every $x \in \mathbb{R}^k$, every $g \in \partial b(x)$, and every $z \in \mathbb{R}^k_+$. In particular, $\partial b(x) \subset [0, 1]^k$, and so we choose for each x some $q(x) \in \partial b(x)$ and put $s(x) := q(x)\cdot x - b(x)$. Then $q(y)\cdot y - s(y) = b(y) \geq b(x) + q(x)\cdot(y - x) = q(x)\cdot y - s(x)$ (the inequality since $q(x) \in \partial b(x)$) and so (q, s) is IC. □

[46] See Rockafellar (1970) for convex functions, their derivatives, and (sub)gradients.

[47] For convex f, this is equivalent to $0 \leq \partial f(x)/\partial x_i \leq 1$ for all i and all x, where the derivative exists (i.e., for a.e. x).

Proposition 16. *Let (q,s) be an IC mechanism on a domain $D \subset \mathbb{R}^k$, with buyer payoff function b and menu $M = (q,s)(D)$. Then there exists a seller-favorable IC mechanism $(\tilde{q}, \tilde{s})$ on $\mathbb{R}^k$ such that*[48]

(i) $\tilde{q}(x) \cdot x - \tilde{s}(x) = b(x) = q(x) \cdot x - s(x)$ *for every* $x \in D$

(ii) $\tilde{s}(x) = b'(x; x) - b(x) \geq s(x)$ *for every* $x \in D$

(iii) $\widetilde{M} := (\tilde{q}, \tilde{s})(\mathbb{R}^k) \subset \mathrm{cl}((q,s)(D)) = \mathrm{cl}\, M$.

Remark. Since $b'(x; x) = \max_{g \in \partial b(x)} g \cdot x$, choosing in the proof of Lemma 15 a $\tilde{q}(x)$ in $\partial b(x)$ where this maximum is attained yields $(\tilde{q}, \tilde{s})$ that satisfies (i) and (ii), and so it is seller-favorable (by Lemma 14). However, the additional conclusion (iii) that the menu does not change (up to closure) is needed so as to guarantee that certain properties of the mechanism, such as submodularity, are preserved by seller-favorability (as in Corollary 20 in Section A.2 below); (iii) requires a somewhat more elaborate proof.

Proof of Proposition 16. Applying Lemma 13 allows us to assume without loss of generality that the domain of (q,s) is the whole space, i.e., $D = \mathbb{R}^k$ (the result for the extended mechanism clearly implies the result for the original one; note that if $\overline{M}$ is the menu of the extended mechanism, then $\widetilde{M} \subset \mathrm{cl}\, \overline{M}$ implies $\widetilde{M} \subset \mathrm{cl}\, M$ because $\overline{M} \subset \mathrm{cl}\, M$).

For every $x \in \mathbb{R}^k$, define $(\tilde{q}(x), \tilde{s}(x))$ to be any limit point of the bounded sequence of points $(q(x_n), s(x_n)) \in [0,1]^k \times \mathbb{R}$, where $x_n := (1 + 1/n)x$ for each positive integer n (the sequence $s(x_n)$ is bounded because $s(x_n) = q(x_n) \cdot x_n - b(x_n)$ and b is continuous). Thus $(\tilde{q}, \tilde{s})$ satisfies (i) (again, because b is continuous) and (iii), and it is IC because (q,s) is IC. Now for any x and n, Lemma 14 implies that $\tilde{q}(x) \in \partial b(x)$ and $q(x_n) \in \partial b(x_n)$. In particular,[49] for every $g \in \partial b(x)$, we have $0 \leq (q(x_n) - g) \cdot (x_n - x) = (q(x_n) - g) \cdot x/n$. Multiplying by n and taking the limit gives $\tilde{q}(x) \cdot x \geq g \cdot x$, and so $\tilde{q}(x) \cdot x = \max_{g \in \partial b(x)} g \cdot x = b'(x; x)$. The equality in (ii) follows because $\tilde{s}(x) = \tilde{q}(x) \cdot x - b(x) = b'(x; x) - b(x)$ by (i), and the inequality in (ii) follows from Lemma 14, which also implies that $(\tilde{q}, \tilde{s})$ is seller-favorable. □

It is useful to gather the above results into one theorem.

Theorem 17. *Let $(q,s): D \to [0,1]^k \times \mathbb{R}$ be a mechanism defined on a domain $D \subset \mathbb{R}^k$, with menu $M := (q,s)(D) = \{(q(x), s(x)) : x \in D\}$ and buyer payoff function*[50] *$b: \mathbb{R}^k \to \mathbb{R}$ given by $b(x) := \sup_{y \in D}(q(y) \cdot x - s(y)) = \sup_{(g,t) \in M}(g \cdot x - t)$ for every $x \in \mathbb{R}^k$. Then the following statements hold:*

(i) The mechanism (q,s) is an IC mechanism if and only if it is the restriction to D of an IC mechanism $(\bar{q}, \bar{s})$ on $\mathbb{R}^k$ with the same buyer payoff function b and menu $\overline{M} := (\bar{q}, \bar{s})(\mathbb{R}^k)$ that satisfies $M \subset \overline{M} \subset \mathrm{cl}\, M$.

[48] As the proof below shows, (i) and (ii) hold, in fact, for all $x \in \mathbb{R}^k$, with b the buyer payoff function of the extension of (q,s) from D to $\mathbb{R}^k$ obtained by Lemma 13. The equality in (ii) implies, in particular, that the payment function $\tilde{s}$ of a seller-favorable mechanism is a Borel-measurable function on $\mathbb{R}^k$.

[49] If f is a convex function, then $(g - g') \cdot (x - y) \geq 0$ for all x, y, all $g \in \partial f(x)$, and $g' \in \partial f(y)$ (add the inequalities $f(y) - f(x) \geq g \cdot (y - x)$ and $f(x) - f(y) \geq g' \cdot (x - y)$).

[50] The buyer's payoff function b is always taken to be defined on the whole space $\mathbb{R}^k$.

(ii) *The mechanism* (q, s) *is an IC mechanism if and only if* $b \in \mathcal{B}^k$ *and, for every* $x \in D$, *we have* $q(x) \in \partial b(x)$ *and* $s(x) = q(x) \cdot x - b(x) \leq b'(x; x) - b(x)$.

(iii) *The mechanism* (q, s) *is, in addition, seller-favorable if and only if for every* $x \in D$, *we have* $s(x) = b'(x; x) - b(x)$.

(iv) *If* (q, s) *is IC, then there is a seller-favorable IC mechanism* $(\tilde{q}, \tilde{s})$ *on* $\mathbb{R}^k$ *with the same buyer payoff function* b, *with menu* $\widetilde{M} := (\tilde{q}, \tilde{s})(\mathbb{R}^k)$ *satisfying* $\widetilde{M} \subset \operatorname{cl} M$, *and such that* $\tilde{s}(x) = b'(x; x) - b(x) \geq s(x)$ *for all* $x \in D$.

Consider now the problem of maximizing the seller's expected revenue subject to individual rationality (IR) for the buyer (i.e., $b \geq 0$). Since, by Theorem 17(iv), it is without loss of generality to restrict attention to seller-favorable mechanisms, we get the following corollary.

Corollary 18. *The seller's maximal expected revenue is*

$$\operatorname{Rev}(\mathcal{F}) = \sup_{b \in \mathcal{B}^k, b \geq 0} \mathbb{E}_{\mathcal{F}}[b'(x; x) - b(x)].$$

Remarks. (a) *Full implementation of approximate seller-favorable payoffs.* Given an IC mechanism (q, s), the buyer may have more than one optimal choice at some valuation x in which case the seller's revenue may be above or below $s(x)$. However, we can modify the mechanism so that *all* optimal choices of the buyer yield (almost) the seller-favorable payment. The idea is simple: the seller gives a small fixed proportional discount on all payments. Consequently, whenever the buyer was indifferent, he will now strictly prefer the choice with the higher payment (where he gets the higher discount). Formally, consider an arbitrary IC mechanism (q, s) on a domain D, let $M := (q, s)(D)$ be its menu, and take a small $\varepsilon > 0$. Let $(q_\varepsilon, s_\varepsilon)$ be an IC mechanism that is obtained from the menu of $(q, (1 - \varepsilon)s)$, i.e., the buyer's payoff function is $b_\varepsilon(x) := \sup_{y \in D}(q(y) \cdot x - (1 - \varepsilon)s(y))$ and $(q_\varepsilon(x), s_\varepsilon(x)) \in \operatorname{cl}\{(q(y), (1 - \varepsilon)s(y)) : y \in D\}$ for every $x \in D$. We claim that such a mechanism $(q_\varepsilon, s_\varepsilon)$ guarantees to the seller, for *any* optimal choice of the buyer at x, a payment of at least $(1 - \varepsilon)(b'(x; x) - b(x))$, i.e., a $(1 - \varepsilon)$ proportion of the seller-favorable payoff, for every[51] $x \in D$. To see this, assume for simplicity that the menu M is already a closed set.[52] Then given $x \in D$, there is $z \in D$ such that $(q(z), s(z))$ is the seller-favorable choice $(\tilde{q}(x), \tilde{s}(x))$ at x, i.e., $b(x) = q(z) \cdot x - s(z)$ and $s(z) = b'(x; x) - b(x)$ (see Theorem 17(iv)). Then, in $(q_\varepsilon, s_\varepsilon)$, for *every* optimal choice $(q(y), (1 - \varepsilon)s(y))$ at x we have

$$\begin{aligned} q(y) \cdot x - (1 - \varepsilon)s(y) &\geq q(z) \cdot x - (1 - \varepsilon)s(z) = b(x) + \varepsilon s(z) \\ &\geq q(y) \cdot x - s(y) + \varepsilon s(z) \end{aligned}$$

[51] Thus, the tie-breaking rule in favor of the seller is obtained as the limit of *any* optimal behavior of the buyer in the perturbed mechanisms. Moreover, this result holds starting with *any* IC mechanism (q, s).

[52] Otherwise one takes appropriate sequences $z_n \in D$ and $y_n \in D$ instead of z and y below.

(the last inequality by IC of (q,s)). Hence $s(y) \geq s(z)$ (subtract and divide by ε), and so the seller's payoff, $s_\varepsilon(x) = (1-\varepsilon)s(y)$, is indeed at least $(1-\varepsilon)s(z) = (1-\varepsilon)(b'(x;x) - b(x))$, i.e., $(1-\varepsilon)$ of the seller-maximal payoff at x.

(b) *Unique equilibrium.* Given any IC mechanism (q,s), there are various ways to eliminate, at arbitrarily small cost, the problem of the buyer having only weak incentives to report truthfully. For example, one can introduce an arbitrarily small positive probability that, after the buyer reports his valuation, the given (IC) mechanism is replaced by a random reserve price on each good (independently).[53]

(c) *Boundary points.* Let (q,s) be a seller-favorable mechanism on a domain $D \subset \mathbb{R}^k$ and let x be a boundary point of D. Let $h \neq 0$ belong to the normal cone to D at x, i.e., $h \cdot x \geq h \cdot y$ for every $y \in D$ (such an h always exists if D is convex or if x is a boundary point of the convex hull of D). We claim that if $h \cdot x > 0$, then $g := q(x)$ must be maximal in the direction h in $[0,1]^k$, i.e., $g_\lambda := g + \lambda h \notin [0,1]^k$ for every $\lambda > 0$. The reason is that if $g_\lambda \in [0,1]^k$, then the mechanism $(\hat{q},\hat{s})$ on D that is identical with (q,s) except at x, where we put $\hat{q}(x) := g_\lambda$ and $\hat{s}(x) := b(x) - g_\lambda \cdot x = s(x) - \lambda h \cdot x$, which is larger than $s(x)$ (because $h \cdot x > 0$), is IC (since for every $y \in D$, we have $\hat{q}(x) \cdot y - \hat{s}(x) \leq q(x) \cdot y - s(x)$ because $h \cdot y \leq h \cdot x$) and has the same buyer payoff function b, which contradicts the seller favorability of (q,s). If $h \cdot x = 0$, then $\hat{s}(x) = s(x)$ in the construction above, and so we can assume without loss of generality that g is maximal in the direction h.

In particular, we may conclude the following:

- Without loss of generality, $q_i(x) = 0$ when $x_i = 0$ (take $h = -e^{(i)}$, where $e^{(i)} \in \mathbb{R}^k$ is the ith unit vector).
- We have $q_i(x) = 1$ when $x_i = \max\{y_i : y \in D\} > 0$ (for instance, if $D = [0,1]^k$, then $q_i(x) = 1$ when $x_i = 1$; take $h = e^{(i)}$).
- We have $\max_i q_i(x) = 1$ when $\sum_i x_i = \max\{\sum_i y_i : y \in D\} > 0$ (for instance, when D is the unit simplex in $\mathbb{R}^k_+$; take $h = (1,1,\ldots,1)$).

(d) *Nonnegative revenue.* If $b(0) = 0$ (which, when maximizing revenue, can always be assumed when[54] $D \subset \mathbb{R}^k_+$), then[55] $s(x) \geq 0$ (i.e., there are no positive transfers from seller to buyer), and so $b'(x;x) \geq b(x)$.

(e) We have that $s(x) = b'(x;x) - b(x)$ is the right derivative of the function $t \to b(tx) - tb(x)$ at $t = 1$ (because $b'(x;x) = \lim_{\delta \to 0^+}(b((1+\delta)x) - b(x))/\delta$ is the right derivative of the function $t \to b(tx)$ at[56] $t = 1$; these functions relate to the local returns to scale of b).

[53] Thus, in the case of a single buyer, the possibility of multiple optimal reports for the buyer, which is sometimes described as problematic (see, for instance, footnote 3 in Manelli and Vincent 2007), in fact is not problematic.

[54] If $b(0) > 0$, then the revenue from $\tilde{b}(x) = b(x) - b(0)$ is higher by the amount $b(0)$ than the revenue from b.

[55] Since $0 = b(0) \geq b(x) + q(x) \cdot (0 - x) = -s(x)$ as $q(x) \in \partial b(x)$.

[56] In the one-dimensional case ($k = 1$), we have $b'(x;x) = xb'_+(x)$. A useful property (for any $k \geq 1$) is $\int_{t_1}^{t_2} b'(tx;x)\,\mathrm{d}t = \int_{t_1}^{t_2} -b'(tx;-x)\,\mathrm{d}t = b(t_2 x) - b(t_1 x)$ (take $f(t) := b(tx)$ and use Corollary 24.2.1 in Rockafellar 1970), from which we get $\int_{t_1}^{t_2} q(tx) \cdot x\,\mathrm{d}t = b(t_2 x) - b(t_1 x)$ for *any* IC mechanism (q,s) (recall Theorem 17(ii): $q(tx) \in \partial b(tx)$ and so $-b'(tx;-x) \leq q(tx) \cdot x \leq b'(tx;x)$). No measurability is needed here:

(f) *Nonnegative production costs.* Suppose it costs the seller $c_i \geq 0$ to produce good i and put $c := (c_1, \dots, c_k)$. The seller's expected profits are now $\pi(x) = s(x) - q(x) \cdot c$. Since, as we have seen, $q(x) \in \partial b(x)$ and $s(x) = q(x) \cdot x - b(x)$, it follows that $\pi(x) = q(x) \cdot (x - c) - b(x) \leq \max_{q \in \partial b(x)} q \cdot (x - c) - b(x) = b'(x; x - c) - b(x)$. Arguments analogous to those above establish that an IC mechanism with buyer payoff function b and seller costs c is seller favorable (where favorability is in terms of profits π not revenue s) if and only if $\pi(x) = b'(x; x - c) - b(x)$, which reduces to the characterization above when $c = 0$.

A.2 *Pricing functions and submodularity*

A function $p : [0,1]^k \to \mathbb{R} \cup \{\infty\}$ is a *pricing function for the IC mechanism* (q, s) on the domain D if for all $x \in D$, we have $p(q(x)) = s(x)$ and[57] $b(x) = q(x) \cdot x - p(q(x)) = \max_{g \in [0,1]^k} (g \cdot x - p(g))$; thus, the choice function (q, s) obtains from the menu consisting of all lotteries $g \in [0,1]^k$, each one priced at $p(g)$. A pricing function always exists: take $p(g) = s(x)$ for $g = q(x) \in Q := q(D)$ and $p(g) = \infty$ otherwise.

Lemma 19. *Let (q, s) be an IC mechanism on a domain $D \subset \mathbb{R}^k$. Then the function $p_0 : [0,1]^k \to \mathbb{R} \cup \{\infty\}$ given by $p_0(g) := \sup_{x \in D}(g \cdot x - b(x))$ for all $g \in [0,1]^k$ is the minimal pricing function for (q, s); i.e., p_0 is a pricing function for (q, s), and any pricing function p for (q, s) satisfies $p(g) \geq p_0(g)$ for all $g \in [0,1]^k$.*

Proof. The function p_0 is a pricing function for (q, s) since for $g = q(x)$ with $x \in D$, we have $g \cdot x - p(g) = b(x)$ and $g \cdot y - p(g) \leq b(y)$ for all $y \in D$, and so the sup in the definition of $p_0(g)$ is attained at x and equals $s(x)$. If p is any pricing function, then for each $g \in [0,1]^k$, we have $g \cdot x - p(g) \leq b(x)$ for every $x \in D$, and so $\sup_{x \in D}(g \cdot x - b(x)) \leq p(g)$, i.e., $p_0(g) \leq p(g)$. □

We refer to this minimal p_0 as the *canonical* pricing function of (q, s) (it is the Fenchel conjugate b^* of the buyer payoff function[58] $b : \mathbb{R}^k \to \mathbb{R}$), in particular in view of the following discussion.

The mechanism (q, s) is *submodular* on D if it has a pricing function p that satisfies

$$p(g) + p(g') \geq p(g \vee g') + p(g \wedge g') \tag{16}$$

for all $g, g' \in Q = q(D)$. In this case, the minimal pricing function p_0 also satisfies (16) (because, for p_0, the left-hand side is the same, and the right-hand side can only be

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the function q can be any selection from the subgradient correspondence (unlike Krishna and Maenner 2001 and Manelli and Vincent 2007).

[57] Recall that if $q(x) = q(y)$, then $s(x) = s(y)$ (by IC). Thus the menu $M = (q, s)(D)$ of (q, s) is precisely the graph of the restriction of p to $Q := q(D)$ (and the value $p(g)$ for $g \notin Q$ can be taken to be arbitrary in the range $p_0(g) \leq p(g) \leq \infty$, where p_0 is given in Lemma 19).

[58] It can be shown that p_0 is also the only closed convex pricing function p (since $p^* = b = p_0^*$). Convexity is a natural requirement on pricing functions, as a risk-neutral buyer can always randomize between choices and pay the expected price.

smaller by Lemma 19). Therefore, (q, s) is submodular if and only if its minimal pricing function p_0 satisfies (16), i.e.,

$$p_0(g) + p_0(g') \geq p_0(g \vee g') + p_0(g \wedge g') \tag{17}$$

for all $g, g' \in Q$.

We now show that submodularity is preserved for seller-favorable mechanisms with the same b.

Corollary 20. *Let (q, s) be a submodular IC mechanism on a domain $D \subset \mathbb{R}^k$. Then there exists a seller-favorable submodular IC mechanism $(\tilde{q}, \tilde{s})$ on D with the same buyer payoff function (and so $\tilde{s}(x) \geq s(x)$ for all $x \in D$).*

Proof. Let $(\bar{q}, \bar{s})$ be the extension of (q, s) to $\mathbb{R}^k$ given by Theorem 17(i), and apply Theorem 17(iv) to $(\bar{q}, \bar{s})$ to obtain a seller-favorable IC mechanism $(\tilde{q}, \tilde{s})$. Since the buyer's payoff function on D is the same for (q, s) and $(\tilde{q}, \tilde{s})$, the corresponding function p_0 is also the same. However, the set Q to which g and g' belong is now replaced by the set $\widetilde{Q} := \tilde{q}(D)$. We will now show that if p_0 satisfies (17) on Q, then it satisfies (17) also on $\widetilde{Q}$.

Let $\tilde{g} = \tilde{q}(x) \in \widetilde{Q}$. Theorem 17(iv) implies that there is a sequence $(g_n, t_n) \in (q, s)(D)$ converging to $(\tilde{g}, \tilde{s}(x))$, and so, by the definition of p_0, we have $p_0(g_n) = t_n$ and $p_0(\tilde{g}) \geq \tilde{g} \cdot x - b(x) = \tilde{s}(x) = \lim_n t_n = \lim_n p_0(g_n)$ (in fact, we have equality here, namely $p_0(\tilde{g}) = \lim p_0(g_n)$, since p_0 is a lower-semicontinuous function[59]). Therefore, given $\tilde{g}, \tilde{g}' \in \widetilde{Q}$, take appropriate sequences $g_n, g_n' \in Q$ such that $g_n \to \tilde{g}$, $g_n' \to \tilde{g}'$, and then we have

$$p_0(g_n) + p_0(g_n') \geq p_0(g_n \vee g_n') + p_0(g_n \wedge g_n')$$

for every n by (17) on Q. The limit of the left-hand side is (at most) $p_0(\tilde{g}) + p_0(\tilde{g}')$, and the lim inf of the right-hand side is at least $p_0(\tilde{g} \vee \tilde{g}') + p_0(\tilde{g} \wedge \tilde{g}')$ by the lower-semicontinuity of p_0, which proves that p_0 satisfies (17) on $\widetilde{Q}$. □

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[59]The function p_0, as the sup of a collection of affine—hence, continuous—functions, is lower-semicontinous: if $g_n \to g$, then $\liminf_n p_0(g_n) \geq p_0(g)$.

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