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Implementation with interdependent valuations

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It is well known that the ability of the Vickrey–Clarke–Groves (VCG) mechanism to implement efficient outcomes for private value choice problems does not extend to interdependent value problems. When an agent's type affects other agents' utilities, it may not be incentive compatible for him to truthfully reveal his type when faced with VCG payments. We show that when agents are informationally small, there exist small modifications to the VCG transfers that restore incentive compatibility. We further show that truthful revelation is an approximate ex post equilibrium. Last, we show that in replicated settings, aggregate payments sufficient to induce truthful revelation go to zero.

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1. Introduction

There is a large literature aimed at characterizing the social choice functions that can be implemented in Bayes–Nash equilibria. This literature typically takes agents' information as exogenous and fixed throughout the analysis. For some problems this may be appropriate, but the assumption is problematic for others. A typical analysis, relying on the revelation principle, maximizes some objective function subject to an incentive compatibility constraint requiring that truthful revelation be a Bayes–Nash equilibrium. It is often the case that truthful revelation is not ex post incentive compatible, that is, for a given agent, there are some profiles of the other agents' types for which the agent may be better off by misreporting his type than by truthfully revealing it. Truthful revelation, of course, may still be a Bayes equilibrium, because agents announce their types without knowing other agents' types: choices must be made on the basis of their beliefs about other agents' types. The assumption that agents' information is exogenous can lead to a difficulty: if truthful revelation is not ex post incentive compatible, then agents

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have incentives to learn other agents' types. To the extent that an agent can, at some cost, learn something about the types of other agents, then agents' beliefs at the stage at which agents actually participate in the mechanism must be treated as endogenous: if an agent can engage in preplay activities that provide him with some information about other agents' types, then that agent's beliefs when he actually plays the game are the outcome of the preplay activity.[1]

A planner who designs a transfer scheme for which truthful revelation is ex post incentive compatible can legitimately ignore agents' incentives to engage in espionage to discover other agents' types and, consequently, ex post incentive compatibility is desirable. The Vickrey–Clarke–Groves mechanism (hereafter VCG)[2] for private values environments is a classic example of a mechanism for which truthful revelation is ex post incentive compatible. For this mechanism, each agent submits his or her valuation. The mechanism selects the outcome that maximizes the sum of the agents' submitted valuations and prescribes a transfer to each agent. These transfers can be constructed in such a way that it is a dominant strategy for each agent to reveal his valuation truthfully. Since dominant strategy incentive compatibility and ex post incentive compatibility coincide in a private values model, the classic VCG mechanism is ex post incentive compatible.

This paper is motivated by the following question: to what extent can the VCG mechanism be extended to an interdependent valuations scenario and still retain the ex post incentive compatibility property of the classic VCG mechanism in the private values model? In the context of auctions, this question has been addressed in Dasgupta and Maskin (2000), Perry and Reny (2002), Ausubel (1999), and McLean and Postlewaite (2004) (among others), who have proposed extensions of the Vickrey second price auction to the interdependent values setup in a way that assures an efficient outcome. Chung and Ely (2002) and Bergemann and Morris (2005) analyze the notion of ex post equilibrium more generally.

Most work on mechanism design or implementation in problems with interdependent valuations begins with payoff functions of the form $(c; t_i, t_{-i}) \mapsto u_i(c; t_i, t_{-i})$, where $c \in C$ is a possible outcome, $t_i \in T_i$ represents agent i's private information, and $t_{-i} \in T_{-i}$ is a vector representing other agents' private information. In the standard interpretation, u_i is a "reduced form" utility function that defines the utility of agent i for the outcome c under the particular circumstances likely to obtain given the agent's information. In the typical problem, the elements of each T_i are totally ordered and various "crossing" properties are imposed. In particular, it is typically assumed that each agent's types are ordered, and that agents' valuations are monotonic in any agent's type. Further, it is assumed that the utility function of each individual agent satisfies a classic single-crossing property and that, across agents, their utilities are linked by an "inter-agent crossing property." This latter property requires that a change in an agent's type

[1] Bikhchandani (2010) includes the acquisition of information regarding the types of the other agents as an endogenous strategic decision in the framework of a surplus extraction mechanism design problem. Obara (2008) and Bikhchandani and Obara (2012) similarly address the possibility that agents may acquire information about an unknown payoff-relevant state of nature in a surplus extraction mechanism design problem. In this paper, we study the problem of implementing a *given* social choice function where the notions of incentive compatibility may be interpreted as discouraging information acquisition.

[2] See Clarke (1971), Groves (1973), and Vickrey (1961).

from one type to a higher type causes his valuation to increase at least as much as any other agent's valuation. In the mechanism design context of the full surplus extraction problem, these same assumptions appear in Crémer and McLean (1985, 1988).

In this paper, we do not take the payoff functions $u_i : C \times T \to \mathbb{R}$ as the basic objects of study. Instead, we begin from more primitive data in which i has a payoff function $(c, \theta; t_i) \mapsto v_i(c, \theta; t_i)$, where θ is a complete description of the state of nature and t_i represents his private information. For a problem in which agents bid on an oil field, θ would include those things that affect i's value for the oil—the amount and composition of the oil, the demand for oil, etc. The relationship between agents' private information and the state is given by a probability distribution P over $\Theta \times T$. This formulation emphasizes the fact that the information possessed by other agents will affect agent i precisely to the extent that the information of others provides information about the state of nature. The reduced form utility function that is normally the starting point for mechanism design or implementation analysis can now be calculated from this more primitive structure: $u(c, t) \equiv \sum_\theta v_i(c, \theta; t_i)P(\theta|t)$. Consequently, this formulation extends the auction framework in McLean and Postlewaite (2004) to general implementation problems. We do not investigate the assumptions that v_i and $P_\Theta(\cdot|t)$ would need to satisfy for the reduced form $u(c, t) \equiv \sum_\theta v_i(c, \theta; t_i)P(\theta|t)$ to satisfy the various crossing conditions that appear in the extant literature. Instead, we take a complementary approach and make certain assumptions regarding the distribution P, but make no assumptions regarding the primitive valuation function v_i.

The main results of the paper highlight the interplay between our generalized VCG mechanism and the notion of informational size as formulated in McLean and Postlewaite (2002, 2004). First, we introduce the notion of *weak ε-ex post incentive compatibility*: a mechanism is weakly ε-ex post incentive compatible if truthful revelation is ex post incentive compatible with conditional probability at least $1 - \varepsilon$. If truthful revelation is weakly ε-ex post incentive compatible for a mechanism for small ε, then the incentive that agents have to collect information about other agents is small with high conditional probability. We show that the generalized VCG mechanism as we define it here exhibits a certain Lipschitz-like property (see Lemma A) and that, as a result, the mechanism is weakly ε-ex post incentive compatible when agents are informationally small. When agents have private information, the posterior probability distribution on the set of states of nature Θ will vary depending on a given agent's type. The informational size of agent i of type t_i corresponds roughly to the maximal expected change in the posterior on Θ as his type varies over types $t_i' \neq t_i$. We show that for any ε, there exists a δ such that if each agent's informational size is less than δ, then truthful reporting is a weak ε-ex post incentive compatible equilibrium in the generalized VCG mechanism.

Although this continuity result is useful, it only tells us that the generalized VCG mechanism is approximately ex post incentive compatible. In the second part of the paper, we modify the generalized VCG transfers with augmenting transfers that correspond to a spherical scoring rule. These augmenting transfers depend on an agent's own announcement and the announcements of others. If an agent's beliefs regarding the information of other agents satisfy a condition *implying* correlation of information, then the VCG mechanism with augmented transfers will be not only weakly ε-ex post incentive compatible, but also Bayesian incentive compatible. Most importantly, we show that as a result of the Lipschitz-like property described above, these augmenting transfers will be small if the informational size of the agents is sufficiently small relative to a very crude measure of the variation in an agent's beliefs (i.e., as t_i varies) regarding the information of other agents. Correlation of information is natural in our setting and we exploit this correlation in the construction of the augmenting transfers described above.[3] Conditions that imply correlation of information also play a significant role in the full surplus extraction problem in the mechanism design literature. The condition that we exploit differs from that typically employed in the full extraction literature. In Section 5, we provide a precise discussion of the relationship between our spherical scoring rule transfers and the transfer schemes employed in the full surplus extraction literature.

In the presence of many agents, each will typically be informationally small and, hence, the augmented VCG payment needed to elicit truthful revelation of any agent's private information will be small. However, the accumulation of a large number of small payments is potentially large. We show, however, that for a replica problem in which the number of agents goes to infinity, the agents' informational size goes to zero at an exponential rate and the aggregate payment needed to elicit the private information necessary to ensure efficient outcomes goes to zero.

We describe the model and provide definitions in Section 2. In Section 3, we introduce a generalized VCG mechanism, and in Lemma A, we identify a property of the generalized VCG mechanism that is fundamental to all of our results. We also present the relationship between ex post incentive compatibility and nonexclusive information. In Section 4, we extend these observations to the relationship between approximate ex post incentive compatibility and small informational size. In Section 5, we demonstrate that when agents have sufficiently small informational size, the generalized VCG mechanism can be modified by adding small positive transfers so as to induce truthful revelation as both an approximate ex post incentive compatible equilibrium and an exact Bayes–Nash equilibrium. This result shows that the additional transfers required to effect exact interim incentive compatibility are small when agents are informationally small, but does not address the size of the sum of these transfers. In Section 6, we show that in a "conditionally independent" informational framework, this sum becomes small as the number of agents increases. The paper concludes with the discussion in Section 7.

2. The model

Let $\Theta = \{\theta_1, \dots, \theta_m\}$ represent the finite set of states of nature and let T_i denote the finite set of types of player i. Let C denote the finite set of social alternatives. Agent i's payoff is represented by a nonnegative valued function $v_i : C \times \Theta \times T_i \to \mathbb{R}_+$. We will assume that there exists $c_0 \in C$ such that $v_i(c_0, \theta, t_i) = 0$ for all $(\theta, t_i) \in \Theta \times T_i$ and that there exists $M > 0$ such that $v_i(\cdot, \cdot, \cdot) \le M$ for each i. Since v_i takes on only nonnegative values, c_0 is the "uniformly worst" outcome for all agents. We will say that v_i satisfies the *pure common value property* if v_i depends only on $(c, \theta) \in C \times \Theta$ and satisfies the *pure private*

[3]See Jehiel et al. (2006) for the generic implications of interdependent valuations and multidimensional independent signals for ex post incentive compatible implementation.

value property if v_i depends only on $(c, t_i) \in C \times T_i$. Our notion of common value is more general than that typically found in the literature in that we do not require that all agents have the same value for a given decision. According to our definition of pure common value, an agent's "fundamental" valuation depends only on the state θ and not on any private information he may have.

Let $(\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n)$ be an $(n+1)$-dimensional random vector taking values in $\Theta \times T$ ($T \equiv T_1 \times \cdots \times T_n$) with associated distribution P, where

$$P(\theta, t_1, \dots, t_n) = \text{Prob}\{\tilde{\theta} = \theta, \tilde{t}_1 = t_1, \dots, \tilde{t}_n = t_n\}.$$

We will make the following full support assumptions regarding the marginal distributions: $P(\theta) = \text{Prob}\{\tilde{\theta} = \theta\} > 0$ for each $\theta \in \Theta$ and $P(t_i) = \text{Prob}\{\tilde{t}_i = t_i\} > 0$ for each $t_i \in T_i$. If K is a finite set, let Δ_K denote the set of probability measures on K. The set of probability measures in $\Delta_{\Theta \times T}$ satisfying the full support conditions will be denoted $\Delta^*_{\Theta \times T}$. If $P \in \Delta^*_{\Theta \times T}$, let $T^* := \{t \in T | P(t) > 0\}$. (The set T^* depends on P, but we will suppress this dependence to keep the notation lighter.)

In many problems with differential information, it is standard to assume that agents have utility functions $u_i : C \times T \to \mathbb{R}_+$ that depend on other agents' types. It is worthwhile noting that, while our formulation takes on a different form, it is equivalent. Given a problem as formulated in this paper, we can define $u_i(c, t_{-i}, t_i) = \sum_{\theta \in \Theta} [v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i)]$. Alternatively, given utility functions $u_i : C \times T \to \mathbb{R}_+$, we can define $\Theta \equiv T$ and define $v_i(c, t, t'_i) = u_i(c, t_{-i}, t'_i)$. Our formulation will be useful in that it highlights the nature of the interdependence: agents care about other agents' types to the extent that they provide additional information about the state θ. Because of the separation of an agent's fundamental valuation function from other agents' information, this formulation allows an analysis of the effects of changing the information structure while keeping an agent's fundamental valuation function fixed.[4]

A *social choice problem* is a collection $(v_1, \dots, v_n, P)$, where $P \in \Delta^*_{\Theta \times T}$. An *outcome function* is a mapping $q : T \to C$ that specifies an outcome in C for each profile of announced types. We will assume that $q(t) = c_0$ if $t \notin T^*$, where c_0 can be interpreted as a status quo point. A *mechanism* is a collection $(q, x_1, \dots, x_n)$ (written simply as $(q, (x_i))$), where $q : T \to C$ is an outcome function and the functions $x_i : T \to \mathbb{R}$ are transfer functions. For any profile of types $t \in T^*$, let

$$\hat{v}_i(c; t) = \hat{v}_i(c; t_{-i}, t_i) = \sum_{\theta \in \Theta} v_i(c, \theta, t_i) P(\theta | t_{-i}, t_i).$$

Although $\hat{v}$ depends on P, we suppress this dependence for notational simplicity as well. Finally, we make the simple but useful observation that the pure private value model is mathematically identical to a model in which $|\Theta| = 1$, where $|K|$ denotes the cardinality of a finite set K. Throughout the paper, $\| \cdot \|_2$ will denote the 2-norm and, for notational

[4]Our formulation also encompasses a valuation function of the form $v_i : C \times \Theta \times T \to \mathbb{R}_+$ since this can be rewritten as $\tilde{v}_i : C \times \tilde{\Theta} \times T_i \to \mathbb{R}_+$, where $\tilde{\Theta} = \Theta \times T$ and $\tilde{v}_i(c, \tilde{\theta}, t_i) = v_i(c, \theta, t)$. Of course when we move from a state space Θ to $\tilde{\Theta}$, the effect of a change in the agent's type will have a different effect on the conditional distribution on the state space $\tilde{\Theta}$.

simplicity, $\|\cdot\|$ will denote the 1-norm. The real vector spaces on which these norms are defined will be clear from the context.

DEFINITION 1. Let $(v_1, \dots, v_n, P)$ be a social choice problem. A mechanism $(q, (x_i))$ has the following qualities:

It is *ex post incentive compatible* if truthful revelation is an *ex post Nash equilibrium*: for all $i \in N$, all $t_i, t_i' \in T_i$, and all $t_{-i} \in T_{-i}$ such that $(t_{-i}, t_i) \in T^*$,

$$\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i) \geq \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i').$$

It is *interim incentive compatible* if truthful revelation is a *Bayes–Nash equilibrium*: for each $i \in N$ and all $t_i, t_i' \in T_i$

$$\sum_{\substack{t_{-i}\in T_{-i}:\\(t_{-i},t_i)\in T^*}} [\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + x_i(t_{-i}, t_i)]P_{T_{-i}}(t_{-i}|t_i)$$

$$\geq \sum_{\substack{t_{-i}\in T_{-i}:\\(t_{-i},t_i)\in T^*}} [\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + x_i(t_{-i}, t_i')]P_{T_{-i}}(t_{-i}|t_i).$$

It is *ex post individually rational* if

$$\hat{v}_i(q(t); t) + x_i(t) \geq 0 \quad \text{for all } i \text{ and all } t \in T^*.$$

It is *feasible* if for each $t \in T^*$,

$$\sum_{j\in N} x_j(t) \leq 0.$$

It is *outcome efficient* if for each $t \in T^*$,

$$q(t) \in \arg\max_{c\in C} \sum_{j\in N} \hat{v}_j(c; t).$$

In our framework, ex post means ex post the realization of the agents' information profile. All activity takes place after players learn their private information but before the realization of θ is known. If, for all i, $\hat{v}_i(c; t)$ does not depend on t_{-i}, then the notions of ex post Nash equilibrium and dominant strategy equilibrium coincide. There is, of course, a definition of dominant strategy equilibrium that is appropriate for the actual Bayesian game. This (interim) equilibrium concept is weaker than ex post dominant strategy equilibrium and stronger than Bayes–Nash equilibrium, but is not logically nested with respect to ex post Nash equilibrium. For a discussion of the relationship between ex post dominant strategy equilibrium, dominant strategy equilibrium, ex post Nash equilibrium, and Bayes–Nash equilibrium, see Crémer and McLean (1985) (henceforth, CM 1985).

3. The generalized VCG mechanism and ex post incentive compatibility

Let q be an outcome function and define transfers as

$$\begin{aligned}\alpha_i^q(t) &= \sum_{j\in N\setminus i} \hat{v}_j(q(t);t) - \max_{c\in C}\Bigl[\sum_{j\in N\setminus i}\hat{v}_j(c;t)\Bigr] \quad \text{if } t\in T^* \\ &= 0 \quad \text{if } t\notin T^*.\end{aligned}$$

Note that $\alpha_i^q(t)\leq 0$ for each i and t. The resulting mechanism $(q,(\alpha_i^q))$ is the *generalized VCG mechanism with interdependent valuations* (GVCG for short.) (Ausubel 1999 and Chung and Ely 2002 use the term generalized Vickrey mechanisms, but for different classes of mechanisms.) It is straightforward to show that the GVCG mechanism is ex post individually rational and feasible. If $\hat{v}_i$ depends only on t_i (as in the pure private value case, where $|\Theta|=1$ or, more generally, when $\tilde{\theta}$ and $\tilde{t}$ are stochastically independent), then the GVCG mechanism reduces to the classical VCG mechanism for private value problems, and it is well known that, in this case, the VCG mechanism satisfies ex post incentive compatibility (IC). In general, however, the GVCG mechanism will *not even* satisfy interim IC. However, we will show that the GVCG mechanism is ex post IC when P satisfies a property called nonexclusive information (Postlewaite and Schmeidler 1986).

The GVCG mechanism is a generalization of the VCG mechanism for private value problems in which agent i pays the cost that he imposes on other agents assuming that they have access to his information even though he is not present. However, an alternative generalization is possible. In particular, we could maximize the total payoff of the players in $N\setminus i$ using only the information of the agents in $N\setminus i$ and define the associated transfer as

$$\sum_{j\in N\setminus i}\hat{v}_j(q(t);t) - \max_{c\in C}\sum_{j\in N\setminus i}\Bigl[\sum_{\theta\in\Theta} v_j(c,\theta,t_j)P(\theta|t_{-i})\Bigr].$$

These two transfer schemes induce different games in the case of interdependent values. We are interested in the GVCG transfers that use agent i's information when calculating the cost that he imposes on other agents. One can think of the designer's problem as encompassing two stages. In the first stage, the designer elicits the agents' information to determine the posterior probability distribution over the states and makes that probability distribution available to the agents. The second stage consists of a VCG mechanism where the agents' values are computed with respect to the probability distribution from the first stage. If the designer has elicited truthful revelation in the first stage, the problem in the second stage is a private values problem, and truthful revelation is a dominant strategy. The interdependence of agents matters only for the first stage; our method is to show how the designer can extract the information needed to compute the probability distribution over the states, following which the problem becomes a private value problem. In this private value problem, the first payment scheme mimics the standard VCG mechanism.

We next identify a special "gain-bounded" property of the GVCG mechanism that is key to our results. (All proofs are relegated to the Appendix.)

Lemma A. *Suppose that* $q: T \to C$ *is outcome efficient for the problem* $(v_1, \ldots, v_n, P)$. *If* $(t_{-i}, t_i), (t_{-i}, t'_i) \in T^*$, *then*

$$\hat{v}_i(q(t_{-i}, t'_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t'_i) - \left[\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i)\right] \leq 2M(n-1)\|P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)\|.$$

In the case of the GVCG mechanism, Lemma A provides an upper bound on the "ex post gain" to agent i when i's true type is t_i but i announces t'_i and others announce truthfully. An important implication of Lemma A is that an agent's gain by misreporting his type is essentially bounded by the degree to which his type affects the posterior probability distribution on Θ; we return to this below.

If v_i does not depend on θ, then (letting $|\Theta| = 1$) we recover Vickrey's classic dominant strategy result for the VCG mechanism in the pure private values case as a special case of Lemma A. We can use Lemma A to extend the classic private values result to a special class of problems with interdependent valuations in which ex post Nash equilibrium replaces dominant strategy equilibrium. These are the problems in which P exhibits *nonexclusive information*.

Definition 2. A measure $P \in \Delta^*_{\Theta \times T}$ satisfies *nonexclusive information* (NEI) if

$$t \in T^* \quad \Rightarrow \quad P_\Theta(\cdot|t) = P_\Theta(\cdot|t_{-i}) \quad \text{for all } i \in N$$

or, equivalently, if

$$[(t_{-i}, t_i) \in T^* \text{ and } (t_{-i}, t'_i) \in T^*] \quad \Rightarrow \quad P_\Theta(\cdot|t_{-i}, t_i) = P_\Theta(\cdot|t_{-i}, t'_i) \quad \text{for all } i \in N.$$

As an aside, we note that nonexclusive information in this paper is more general than the notion defined by Postlewaite and Schmeidler. Nonexclusive information means that the conditional probability distribution on Θ given a type profile t is the same as the conditional probability distribution on Θ given t_{-i}. That is, the private information held by agent i regarding the state θ is redundant in the presence of the information jointly held by the other agents. However, the agents other than agent i cannot generally determine i's type only from the knowledge of t_{-i} unless, e.g., each agent's information is modeled as a partition of the state space as in the original definition of Postlewaite and Schmeidler.

As an immediate application of Lemma A, we have the following result.

Proposition 1. *Let* $\{v_1, \ldots, v_n\}$ *be a collection of payoff functions. If* $P \in \Delta^*_{\Theta \times T}$ *exhibits nonexclusive information and if* $q: T \to C$ *is outcome efficient for the problem* $(v_1, \ldots, v_n, P)$, *then the GVCG mechanism* (q, α_i^q) *is ex post IC and ex post individually rational (IR).*

If agents have "zero informational size," that is, if P exhibits nonexclusive information, then $\|P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t'_i)\| = 0$ if $(t_{-i}, t_i), (t_{-i}, t'_i) \in T^*$. Hence, truth is an ex post Nash equilibrium as a consequence of Proposition 1. Note that the private values problem in which v_i does not depend on θ is the special case of NEI where $|\Theta| = 1$. Since ex post Nash equilibrium coincides with dominant strategy equilibrium in the private values case, we conclude that Proposition 1 implies Vickrey's classic dominant strategy result for the VCG mechanism in the pure private values case.

Nonexclusive information, while subsuming the private values model, is a strong assumption. Our goal in this paper is to identify conditions under which we can modify the GVCG payments so that the new mechanism is interim IC. We begin by presenting a continuity result that is motivated by Proposition 1. If we (informally) think of NEI as meaning that an agent has no effect on the posterior distribution on Θ in the presence of the information of other agents, then we can interpret Proposition 1 as follows: if each agent has no information effect on the posterior on Θ, then the GVCG is exactly ex post incentive compatible. We will prove the following continuity result: if each agent has a small information effect on the posterior on Θ, then the GVCG is approximately ex post incentive compatible. Of course, this result requires that the notions of "small informational effect" and "approximate ex post incentive compatibility" be formalized, and to accomplish this, we introduce the notions of informational size and weak ε-ex post Nash equilibrium in the next section.

4. Approximate ex post incentive compatibility and small informational size

4.1 *Informational size*

If $t \in T^*$, recall that $P_\Theta(\cdot|t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ. A natural notion of an agent's informational size is one that measures the degree to which he can alter the best estimate of the state θ when other agents are announcing truthfully. In our setup, that estimate is the conditional probability distribution on Θ given a profile of types t. Any profile of agents' types $t = (t_{-i}, t_i) \in T^*$ induces a conditional distribution on Θ, and if agent i unilaterally changes his announced type from t_i to t_i', this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a small probability that he can induce a large change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t_i'. This is formalized in the following definition.

Definition 3. Let

$$I^i_\varepsilon(t_i', t_i) = \{t_{-i} \in T_{-i} | (t_{-i}, t_i) \in T^*, (t_{-i}, t_i') \in T^* \text{ and } \|P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')\| > \varepsilon\}.$$

The *informational size* of agent i is defined as

$$\nu_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \min\{\varepsilon \geq 0 | \operatorname{Prob}\{\tilde{t}_{-i} \in I^i_\varepsilon(t_i', t_i) | \tilde{t}_i = t_i\} \leq \varepsilon\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν_i^P is small. If agent i receives signal t_i but reports $t_i' \neq t_i$, the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t_i')$. If $\hat{t}_{-i} \in I^i_\varepsilon(t_i', t_i)$, then this change is "large" in the sense that

$\|P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t_i')\| > \varepsilon$. Therefore, $\text{Prob}\{\tilde{t}_{-i} \in I^i_\varepsilon(t_i', t_i)|\tilde{t}_i = t_i\}$ is the probability that i can have a large influence on the conditional distribution on Θ by reporting t_i' instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a large influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type. Informational size is closely related to the notion of nonexclusive information: if all agents have zero informational size, then P must satisfy NEI. In fact, we have the following easily demonstrated result: $P \in \Delta^*_{\Theta\times T}$ satisfies NEI if and only if $\nu_i^P = 0$ for each $i \in N$. If $T^* = T$, then ν^P is the *Ky Fan distance* between the random variables (r.v.s) $P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ and $P_\Theta(\cdot|\tilde{t}_{-i}, t_i')$ with respect to the probability measure $P_{T_{-i}}(\cdot|t_i)$ (see, e.g., Dudley 2002, Section 9.2).[5]

4.2 *Approximate ex post incentive compatibility*

Definition 4. Let $\varepsilon \geq 0$. A mechanism $(q, (x_i))$ is *weakly ε-ex post incentive compatible* if for all i and all $t_i, t_i' \in T_i$,

$$\text{Prob}\big\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } \hat{v}_i(q(\tilde{t}_{-i}, t_i'); \tilde{t}_{-i}, t_i) + x_i(\tilde{t}_{-i}, t_i') > \hat{v}_i(q(\tilde{t}_{-i}, t_i); \tilde{t}_{-i}, t_i) + x_i(\tilde{t}_{-i}, t_i) + \varepsilon|\tilde{t}_i = t_i\big\} \leq \varepsilon.$$

Note that $(q, (x_i))$ is a weakly 0-ex post incentive compatible mechanism if and only if $(q, (x_i))$ is an ex post incentive compatible mechanism.

4.3 *The result*

Theorem 1. *Suppose that $q: T \to C$ is outcome efficient for the problem $(v_1, \ldots, v_n, P)$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\nu_i^P < \delta$ for each agent i, then the GVCG mechanism $(q, (\alpha_i^q))$ is weakly ε-ex post incentive compatible.*

To explain Theorem 1, note that Lemma A provides an upper bound on the "ex post gain" to agent i when i's true type is t_i but i announces t_i' and others announce truthfully. If agent i is informationally small, then (informally) we can deduce that

$$\text{Prob}\big\{\|P_\Theta(\cdot|\tilde{t}_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}_{-i}, t_i')\| \approx 0|\tilde{t}_i = t_i\big\} \approx 1,$$

so truth is an approximate ex post equilibrium for the GVCG in the sense that

$$\text{Prob}\big\{\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i) - \big[\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i')\big] \underset{\approx}{>} 0|\tilde{t}_i = t_i\big\} \approx 1.$$

Consequently, we obtain the following continuity result embodied in Theorem 1: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that truth will be a weak ε-ex post Nash equilibrium whenever $\nu_i^P < \delta$ for each i.

[5]If X and Y are random variables defined on a probability space $(\Omega, \mathcal{F}, \mu)$ taking values in a separable metric space (S, d), then the Ky Fan distance is defined as $\min[\varepsilon \geq 0 : \mu\{d(X, Y) > \varepsilon\} \leq \varepsilon]$. If $T^* = T$, then ν_i^P is the Ky Fan distance between the r.v.s $X = P_\Theta(\cdot|\tilde{t}_{-i}, t_i)$ and $Y = P_\Theta(\cdot|\tilde{t}_{-i}, t_i')$ with respect to the probability measure $\mu = P_{T_{-i}}(\cdot|t_i)$.

5. Bayesian incentive compatibility and augmented mechanisms

Theorem 1 leaves two important questions unanswered. First, we would like to identify conditions under which agents are informationally small so that an outcome efficient social choice function is weakly ε-ex post implementable for small ε. It is reasonable to conjecture that this will be the case, inter alia, when there are many agents, and we provide a precise analysis of this case in Section 6 below. The second question concerns the possibility of modifying a mechanism via the introduction of small transfers so that the resulting modified mechanism is *exactly*, rather than *approximately*, interim incentive compatible when agents are informationally small. Given Theorem 1, the existence of such a mechanism is at least plausible since an agent's ex post gain from lying, i.e., his ex post informational rent, is small with high probability when the agent is informationally small. Consequently, his expected informational rent conditional on his type is small and truth will be an approximate Bayes–Nash equilibrium when agents are informationally small. In this section, we provide conditions under which a modified GVCG mechanism is approximately ex post incentive compatible and (exactly) Bayesian incentive compatible and the sum of the agents' ex post transfers is bounded by a number close to 0 when agents are informationally small.

Whether an agent i can be given incentives to reveal his information will depend on the magnitude of the difference between $P_{T_{-i}}(\cdot|t_i)$ and $P_{T_{-i}}(\cdot|t_i')$, the conditional distributions on T_{-i} given different types t_i and t_i' for agent i. If $P \in \Delta^*_{\Theta\times T}$, recall that $P_{T_{-i}}(\cdot|t_i) \in \Delta_{T_{-i}}$ is the conditional distribution on T_{-i} given that i receives signal t_i and define

$$\Lambda_i^P = \min_{t_i\in T_i} \min_{t_i'\in T_i\setminus\{t_i\}} \left\| \frac{P_{T_{-i}}(\cdot|t_i)}{\|P_{T_{-i}}(\cdot|t_i)\|_2} - \frac{P_{T_{-i}}(\cdot|t_i')}{\|P_{T_{-i}}(\cdot|t_i')\|_2} \right\|_2^2,$$

where $\|\cdot\|_2$ denotes the 2-norm on $\mathbb{R}^{|T_{-i}|}$. This is a measure of the "variability" of the conditional distribution $P_{T_{-i}}(\cdot|t_i)$ as a function of t_i. Note that agents' beliefs cannot be independent if $\Lambda_i^P > 0$ and we will say more about this below.

To state our main result, we need the notion of an augmented mechanism.

Definition 5. Let $(z_i)_{i\in N}$ be an n-tuple of functions $z_i : T \to \mathbb{R}_+$ each of which assigns to each $t \in T$ a nonnegative number, interpreted as a "reward" to agent i. If $(q, x_1, \dots, x_n)$ is a mechanism, then the associated *augmented* mechanism is defined as $(q, x_1 + z_1, \dots, x_n + z_n)$ and will be written simply as $(q, (x_i + z_i))$.

Theorem 2. *Let $(v_1, \dots, v_n)$ be a collection of payoff functions.*

(i) Suppose that $P \in \Delta^*_{\Theta\times T}$ satisfies $\Lambda_i^P > 0$ for each i and suppose that $q : T \to C$ is outcome efficient for the problem $(v_1, \dots, v_n, P)$. Then there exists an augmented GVCG mechanism $(q, \alpha_i^q + z_i)$ for the social choice problem $(v_1, \dots, v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \le \delta \min_i \Lambda_i^P$$

and whenever $q: T \to C$ *is outcome efficient for the problem* $\{v_1, \ldots, v_n, P\}$, *there exists an augmented GVCG mechanism* $(q, (\alpha_i^q + z_i))$ *with* $0 \leq z_i(t) \leq \varepsilon$ *for every* i *and* t *satisfying ex post IR, interim IC, and weak* (2ε)*-ex post IC. Furthermore,* $\sum_i (\alpha_i^q + z_i) \leq n\varepsilon$.

Part (i) of Theorem 2 states that as long as $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t_i')$ whenever $t_i \neq t_i'$, then irrespective of the agents' informational sizes, the augmenting transfers can be chosen so that the augmented mechanism satisfies Bayesian incentive compatibility. However, the required augmenting transfers will be large if the agents have large informational size. Part (ii) states that the augmenting transfers will be small if the agents have informational size that is small enough relative to our measure of variability in the agents' beliefs.

To understand this condition, we first note that if agent i is informationally small, then truth is an approximate Bayes–Nash equilibrium in the GVCG mechanism, so the mechanism is approximately interim incentive compatible. More precisely, we can deduce from Lemma A that the interim expected gain from misreporting one's type is essentially bounded from above by one's informational size. If we want the mechanism to be exactly interim incentive compatible, then we must alter the mechanism (specifically, construct an augmented GVCG mechanism) so as to provide the correct incentives for truthful behavior. The number Λ_i^P quantifies the reward for honest reporting of one's type and the condition of part (ii) of Theorem 2 simply requires that the gain to lying quantified in terms of informational size be outweighed by the gain to honest reporting as quantified by Λ_i^P.

As we have mentioned previously, the agents' beliefs cannot be independent if $\Lambda_i^P > 0$. Correlated information also plays a significant role in the full surplus extraction problem in the mechanism design literature (see Crémer and McLean 1985, 1988.) Those papers (and subsequent work by McAfee and Reny 1992) demonstrated how one can use correlation to fully extract the surplus in certain mechanism design problems.

The problems, however, are quite different. Surplus extraction is a mechanism design problem, while our problem is an implementation problem. We do not look for transfers and an allocation scheme that solves a mechanism design problem of the type presented in, for example, Myerson (1981) or Crémer and McLean (1985, 1988). Instead, we study the problem of implementation of a given efficient social choice function and it is important to explicate the differences.

A (very) informal but misleading statement of the surplus extraction problem for (e.g.) auctions is "if agents' types are correlated, then there exists an incentive compatible auction mechanism whose associated revenue to the seller is the same as the revenue that would be attainable with complete information." Based on this statement, one might then conjecture that if types are simply correlated, then we could similarly design transfers to induce truthful announcement of types, which could then be used to implement the desired outcome. Full extraction of surplus, however, requires more than correlation. It requires that the set of conditional probability distributions $\{P_{T_{-i}}(\cdot|t_i) : t_i \in T_i\}$ indexed by a given agent's types satisfy a cone condition (or a stronger full rank condition if one wants the extracting mechanism to be dominant strategy incentive compatible).[6]

While these conditions for full surplus extraction imply that $P_{T_{-i}}(\cdot|t_i) \neq P_{T_{-i}}(\cdot|t_i')$ if $t_i \neq t_i'$ and, therefore, that $\Lambda_i^P > 0$, the actual (positive) size of Λ_i^P is not relevant in the Cremer–McLean constructions, and full extraction will be possible.

On the other hand, we do not assume in this paper that the collection $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is linearly independent (or satisfies the weaker cone condition in Crémer and McLean (1988)). However, the "closeness" of the members of $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ is an important issue. It can be shown that for each i, there exists a collection of numbers $\zeta_i(t)$ satisfying $0 \leq \zeta_i(t) \leq 1$ and

$$\sum_{t_{-i} \in T_{-i}} [\zeta_i(t_{-i}, t_i) - \zeta_i(t_{-i}, t_i')] P_{T_{-i}}(t_{-i}|t_i) > 0$$

for each $t_i, t_i' \in T_i$ if and only if $\Lambda_i^P > 0$. Indeed, we define

$$\zeta_i(t_{-i}, t_i) = \frac{P_{T_{-i}}(t_{-i}|t_i)}{\|P_{T_{-i}}(\cdot|t_i)\|_2}$$

so that the transfers ζ_i correspond to a spherical scoring rule. The elements of the collection $\{\zeta_i(t)\}_{i \in I, t \in T}$ can be thought of as "incentive payments" made to the agents as inducements to reveal their information. If the posteriors $\{P_{T_{-i}}(\cdot|t_i)\}_{t_i \in T_i}$ are all distinct, then the incentive compatibility inequalities associated with our scoring rule above are strict. However, the positive incentive for honest reporting decreases as $\Lambda_i^P \to 0$. Hence, the difference in the expected reward from a truthful report and from a false report will be very small if the conditional posteriors are very close to each other. Theorem 2(i) requires that $\Lambda_i^P > 0$ while Theorem 2(ii) requires that informational size be small relative to Λ_i^P.

An analogous theorem can be proven in a more general model in which Λ_i^P may be equal to 0.[7] While the VCG mechanism alone suffices for implementation in private value problems, the GVCG mechanism is not generally adequate for interdependent value problems except in the special case of nonexclusive information. The interdependence in our framework stems from the fact that all agents may have information about θ, which can affect an agent's utility for the social outcome. We could convert an interdependent value problem into a private value problem in this framework if we could eliminate the asymmetry of information about θ. The augmenting z_i's make it incentive compatible for agents to reveal their information about θ. They may reveal other information as well but, once all information that is correlated with θ has been truthfully elicited, the residual problem is a private value problem that the VCG mechanism handles well. A more general framework could model agents as having types that consist of two parts: a signal about θ and a personal characteristic that is independent of θ. It is truthful announcement of the signals regarding θ that is needed to solve the

[6] A related identifiability condition is used in Kosenok and Severinov (2008) in their study of ex post budget balanced Bayesian mechanisms.

[7] McLean and Postlewaite (2004) describe such an extended model for the special case of auctions.

interdependent value problem, and a "positive variability" condition of the type that we assume in this paper (i.e., that $\Lambda_i^P > 0$ for all i) would be sufficient to construct transfers analogous to our z_i's. This example illustrates an important difference between our implementation problem and the surplus extraction problem. The complex feature of our implementation problem with interdependent valuations results from the need to extract the agents' information that underlies the interdependence, i.e., that part of the agents' information that is relevant for predicting θ. Agents may have other personal information that is important for both implementation and surplus extraction. If $\Lambda_i^P > 0$ for all i, then our augmented GVCG mechanism will elicit all of the information required for implementation with interdependent valuations, even in situations when full extraction of the surplus is not possible.

Miller et al. (2007) provide an implementation result for problems in which the type spaces are continuua. In a differentiable framework, they use a different type of scoring rule to construct ε-interim incentive compatible mechanisms that implement smooth (though not necessarily efficient) social choice functions with balanced transfers. Their construction utilizes a condition (called stochastic relevance) that is stronger than a requirement that $\Lambda_i^P > 0$ for each i and they obtain a result similar to our Theorem 2(i) that may require large transfers. Using a strengthened version of stochastic relevance, they obtain an exact interim implementation result with balanced transfers that again may require large transfers. Informational size does not play a role in any of their results.

6. Asymptotic results

Informally, an agent is informationally small when the probability that he can affect the posterior distribution on Θ is small. One would expect, in general, that agents will be informationally small in the presence of many agents. For example, if agents receive conditionally independent signals regarding the state θ, then the announcement of one of many agents is unlikely to significantly alter the posterior distribution on Θ. Hence, it is reasonable to conjecture that (under suitable assumptions) an agent's informational size goes to zero in a sequence of models with an increasing number of agents. Consequently, the required rewards z_i that induce honest reporting will also go to zero as the number of agents grows. We will show below that this is, in fact, the case. Of greater interest, however, is the behavior of the aggregate reward necessary to induce truthful revelation. The argument sketched above only suggests that each individual's z_i becomes small as the number of agents goes to infinity, but does not address the asymptotic behavior of the sum of the z_i's. Roughly speaking, the size of the z_i that is necessary to induce agent i to reveal truthfully is of the order of magnitude of his informational size. Hence, the issue concerns the speed with which agents' informational size goes to zero as the number of agents increases. We will demonstrate below that under reasonably general conditions, agents' informational size goes to zero at an exponential rate and that the total reward $\sum_{i\in N} z_i$ goes to zero as the number of agents increases.

We will assume that all agents have the same finite signal set $T_i = A$. Let $J_r = \{1, 2, \dots, r\}$. For each $i \in J_r$, let $v_i^r : C \times \Theta \times A \to \mathbb{R}_+$ denote the payoff to agent i. For any positive integer r, let $T^r = A \times \cdots \times A$ denote the r-fold Cartesian product and let $t^r = (t_1^r, \dots, t_r^r)$ denote a generic element of T^r.

Definition 6. A sequence of probability measures $\{P^r\}_{r=1}^{\infty}$ with $P^r \in \Delta^*_{\Theta \times T^r}$ is a *conditionally independent sequence* if there exists $P \in \Delta^*_{\Theta \times A}$ such that the following statements hold:

(a) We have $P(\theta, t) > 0$ for all $(\theta, t) \in \Theta \times A$ and for every θ, $\hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in A$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

(b) For each r and each $(\theta, t_1, \dots, t_r) \in \Theta \times T^r$,

$$P^r(t_1, \dots, t_r|\theta) = \text{Prob}\{\tilde{t}_1^r = t_1, \tilde{t}_2^r = t_2, \dots, \tilde{t}_r^r = t_r|\tilde{\theta} = \theta\} = \prod_{i=1}^{r} P(t_i|\theta).$$

Because of the symmetry in the objects defining a conditionally independent sequence, it follows that for fixed r, the informational size of each $i \in J_r$ is the same. In the remainder of this section, we will drop the subscript i and will write ν^{P^r} for the value of the informational size of agents in J_r.

Lemma B. *Suppose that* $\{P^r\}_{r=1}^{\infty}$ *is a conditionally independent sequence. For every* $\varepsilon > 0$ *and every positive integer* k*, there exists an* $\hat{r}$ *such that*

$$r^k \nu^{P^r} \leq \varepsilon$$

whenever $r > \hat{r}$.

The proof is provided in the Appendix and is an application of a classic large deviations result due to Hoeffding (1963). With this lemma, we can prove the following asymptotic result.

Theorem 3. *Suppose that* $\{P^r\}_{r=1}^{\infty}$ *is a conditionally independent sequence associated with* $P \in \Delta^*_{\Theta \times A}$*. Let* M *and* ε *be positive numbers. Let* $\{(v_1^r, \dots, v_r^r)\}_{r \geq 1}$ *be a sequence of payoff function profiles and for each* r*, let* $\{q^r(r), \alpha_1^r(r), \dots, \alpha_r^r(r)\}$ *denote the GVCG mechanism for the social choice problem (SCP)* $(v_1^r, \dots, v_r^r, P^r)$*. Suppose that* $0 \leq v_i^r(\cdot, \cdot, \cdot) \leq M$ *for all* r *and* $i \in J_r$*, and that* P *satisfies the following condition: for each pair* t*,* t' *in* A *with* $t \neq t'$*, there exists an* $s \in A$ *such that*[8]

$$\sum_{\theta} P(s|\theta)P(\theta|t) \neq \sum_{\theta} P(s|\theta)P(\theta|t').$$

Then for every $\varepsilon > 0$*, there exists an* $\hat{r}$ *such that for all* $r > \hat{r}$*, there exists an augmented GVCG mechanism* $(q^r, \alpha_1^r + z_1^r, \dots, \alpha_r^r + z_r^r)$ *for the social choice problem* $(v_1^r, \dots, v_r^r, P^r)$ *satisfying ex post IR, interim IC, and weak* (2ε)*-ex post IC. Furthermore, for each* $i \in J_r$ *and each* $t^r \in T^r$*,* $z_i^r(t^r) \geq 0$ *and* $\sum_{i \in J_r} z_i^r(t^r) \leq \varepsilon$.

[8]The condition can be restated in a different way: for each pair t, t' in A with $t \neq t'$, an agent's beliefs regarding the type of another single agent must be different.

7. Discussion

7.1 *Gain-bounded mechanisms*

In a typical implementation or mechanism design problem, one computes the mechanism for each instance of the data that define the social choice problem. Therefore, in most cases of interest, the mechanism is parametrized by the valuation functions and probability structure that define the social choice problem. If we fix a profile $(v_1, \ldots, v_n)$ of payoff functions, then we can analyze the parametric dependence of the mechanism on the probability distribution P and this dependence can be modeled as a mapping that associates a mechanism with each $P \in \Delta^*_{\Theta\times T}$. We will denote this mapping $P \mapsto (q^P, x_1^P, \ldots, x_n^P)$. For example, the mapping naturally associated with the GVCG mechanism is defined by

$$q^P(t) \in \arg\max_{c\in C} \sum_{j\in N}\sum_{\theta\in\Theta} v_i(c, \theta, t_i)P(\theta|t_{-i}, t_i) \quad \text{if } t \in T^*$$

$$q^P(t) = c_0 \quad \text{if } t \notin T^*$$

and

$$x_i^P(t) = \sum_{j\in N\setminus i}\sum_{\theta\in\Theta} v_i(q^P(t), \theta, t_i)P(\theta|t_{-i}, t_i) - \max_{c\in C}\left[\sum_{j\in N\setminus i}\sum_{\theta\in\Theta} v_i(c, \theta, t_i)P(\theta|t_{-i}, t_i)\right] \quad \text{if } t \in T^*$$
$$= 0 \quad \text{if } t \notin T^*.$$

Definition 7. Let $(v_1, \ldots, v_n)$ be a profile of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$, let $(q^P, x_1^P, \ldots, x_n^P)$ be a mechanism for the social choice problem $(v_1, \ldots, v_n, P)$. We will say that the mapping $P \mapsto (q^P, x_1^P, \ldots, x_n^P)$ is *gain-bounded with respect to conditional probabilities*, or simply *gain-bounded*, if there exists a $K > 0$ such that for all $P \in \Delta^*_{\Theta\times T}$,

$$\hat{v}_i(q^P(t_{-i}, t_i'); t_{-i}, t_i) + x_i^P(t_{-i}, t_i') - \left[\hat{v}_i(q^P(t_{-i}, t_i); t_{-i}, t_i) + x_i^P(t_{-i}, t_i)\right] \leq K\|P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot|t_{-i}, t_i')\|$$

whenever $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$.

Note that in Definition 7 above, the social choice function need not be outcome efficient. Lemma A shows that the GVCG mechanism is gain-bounded with $K = 2M(n-1)$, and this is the essential property of the GVCG mechanism that drives Theorems 1, 2, and 3. In fact, using the same proof, an important extension of Theorem 2 holds for any gain-bounded mechanism.

Theorem 4. *Let $(v_1, \ldots, v_n)$ be a collection of payoff functions and suppose that $P \mapsto (q^P, x_1^P, \ldots, x_n^P)$ is gain-bounded.*

(i) If $\Lambda_i^P > 0$ for each i, then there exists an augmented mechanism $(q^P, (x_i^P + z_i^P))$ for the social choice problem $(v_1, \ldots, v_n, P)$ satisfying ex post IR and interim IC.

(ii) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $P \in \Delta^*_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leq \delta \min_i \Lambda_i^P,$$

there exists an augmented mechanism $(q^P, (x_i^P + z_i^P))$ with $0 \leq z_i^P(t) \leq \varepsilon$ for every i and t satisfying ex post IR, interim IC, and weak ε-ex post IC.

Gain-bounded mechanisms are of interest in that they identify an important feature of GVCG mechanisms that makes implementation possible. When an agent misreports his type, he changes the posterior distribution on Θ, leading to a change in the mechanism's outcome and, consequently, his gain from misreporting. The Lipschitz-like continuity of an agent's gain from misreporting with respect to the change in probability the agent can induce assures that an agent's misreporting gain is of the same order as the change in the distribution that he can induce. Thus, the rewards needed to assure incentive compatibility are of the order of informational size.

There are trivial gain-bounded mechanisms, for example, a constant mechanism. We will show next that GVCG is not the only interesting gain-bounded mechanism however. We present next an example of a balanced gain-bounded mechanism for pure common value models that is quite different from the GVCG mechanism. Let $(v_1, \dots, v_n)$ be a collection of payoff functions. For each $P \in \Delta^*_{\Theta\times T}$, suppose that $q^P : T \to C$ is a social choice function for the problem $(v_1, \dots, v_n, P)$ and define transfer payments associated with q^P as

$$\beta_i^P(t) = \frac{1}{n}\sum_j \hat{v}_j(q^P(t), t) - \hat{v}_i(q^P(t), t).$$

In this simple scheme, agent i receives money if his individual payoff is less than the average payoff and he pays out money if his individual payoff is greater than the average payoff. Furthermore, note that

$$\sum_i \beta_i^P(t) = 0$$

so that the mechanism $(q^P, (\beta_i^P))$ is balanced for each $P \in \Delta^*_{\Theta\times T}$.

If q^P is outcome efficient for the problem $(v_1, \dots, v_n, P)$, then the associated mechanism with transfer payments $(\beta_i^P)_{i\in N}$ is gain-bounded in pure common value problems (though not for general problems).

7.2 *Relation to the previous literature*

There is a resemblance between Theorem 1 in McLean and Postlewaite (2002) and Theorem 2 in this paper. The difference is that Theorem 1 in McLean and Postlewaite (2002) shows that one can implement an approximately efficient outcome in a pure exchange model with asymmetric information without any external infusion of resources when agents are informationally small. In addition, the basic utility functions in McLean and Postlewaite (2002) from which the interdependent payoffs are derived depend on consumption bundles and the state of nature θ, but do not depend on an agent's private information t_i. Theorem 2 in this paper provides conditions under which a given efficient social choice function can be implemented, though the efficiency of the outcome in this paper, of course, comes at the cost of the positive payments z_i. In our 2002 paper, we could have posed a question analogous to that addressed here: when can we find an IC, ex post IR mechanism that implements an efficient outcome in the asymmetric information economy and for which any outside infusion of resources (if required) is small? Alternatively, one could ask in the framework of the current paper a question analogous to that posed in our 2002 piece: when can we find an approximately outcome efficient choice function that can be implemented using IC, ex post IR transfers that require no external infusion of money?

Our goal in this paper is to investigate the role of the GVCG transfers in ex post implementation of efficient outcomes in quasilinear environments with interdependent valuations. These GVCG transfers result in a mechanism satisfying the "gain boundedness" property embodied in Lemma A, a property on which we have elaborated above. Our 2002 paper has certain conceptual similarities to the work presented here, but the techniques in that paper are quite different from those employed here, precisely because there is no analogue of the GVCG mechanism in that framework.

McLean and Postlewaite (2004) is closer to the present work. In that paper, we study the Vickrey second price auction with interdependent values with a particular emphasis on general information structures that allow for a bidder's type to consist, for example, of a component that is independent of the state θ and an informationally relevant component that is correlated with θ. If the independent component is not present, then Corollary 1 in McLean and Postlewaite (2004) is a special case of Theorem 2 in this paper. The methods of McLean and Postlewaite (2004) can be extended to the more general implementation setup of this paper in a straightforward way. More importantly, however, we identify and exploit in this paper the gain boundedness condition described in Lemma A that is satisfied by the GVCG mechanism, something that was not apparent in the simpler auction framework of McLean and Postlewaite (2004). In addition, we obtain the asymptotic result of Theorem 3 while McLean and Postlewaite (2004) do not address asymptotic issues at all.

Appendix: Proofs

We begin with a simple result regarding Lipschitz continuity of the optimal value function.

Lemma 1. *For each $S \subseteq N$ and for each $p \in \Delta(\Theta)$, let*

$$F_S(p) = \max_{\hat{c} \in C} \sum_{\theta \in \Theta} \sum_{i \in S} v_i(\hat{c}, \theta, t_i) p(\theta).$$

Then for each $p, p' \in \Delta(\Theta)$,

$$|F_S(p) - F_S(p')| \leq |S| M \| p - p' \|.$$

Proof. Choose $S \subseteq N$ and $p, p' \in \Delta(\Theta)$. Choose c and c' so that

$$\sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)p(\theta) = \max_{\hat{c}\in C}\sum_{\theta\in\Theta}\sum_{i\in S} v_i(\hat{c},\theta,t_i)p(\theta)$$

$$\sum_{\theta\in\Theta}\sum_{i\in S} v_i(c',\theta,t_i)p'(\theta) = \max_{\hat{c}\in C}\sum_{\theta\in\Theta}\sum_{i\in S} v_i(\hat{c},\theta,t_i)p'(\theta).$$

Then

$$\begin{aligned} F_S(p) - F_S(p') &= \sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)[p(\theta)-p'(\theta)] + \sum_{\theta\in\Theta}\sum_{i\in S}[v_i(c,\theta,t_i) - v_i(c',\theta,t_i)]p'(\theta) \\ &\le \sum_{\theta\in\Theta}\sum_{i\in S} v_i(c,\theta,t_i)[p(\theta)-p'(\theta)] \\ &\le |S|M\|p-p'\|. \end{aligned}$$

Reversing the roles of p and p' yields the result. □

A.1 *Proof of Lemma A*

Choose $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$. Then

$$\begin{aligned} &\hat{v}_i(q(t_{-i},t_i); t_{-i},t_i) + \alpha_i(t_{-i},t_i) \\ &\quad = \hat{v}_i(q(t_{-i},t_i); t_{-i},t_i) + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i); t_{-i},t_i) - \max_{c\in C}\Bigg[\sum_{j\in N\setminus i}\hat{v}_j(c; t_{-i},t_i)\Bigg] \end{aligned}$$

and

$$\begin{aligned} &\hat{v}_i(q(t_{-i},t_i'); t_{-i},t_i) + \alpha_i(t_{-i},t_i') \\ &\quad = \hat{v}_i(q(t_{-i},t_i'); t_{-i},t_i) + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i'); t_{-i},t_i) \\ &\qquad\qquad - \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i'); t_{-i},t_i) \\ &\qquad\qquad + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i'); t_{-i},t_i') - \max_{c\in C}\Bigg[\sum_{j\in N\setminus i}\hat{v}_j(c; t_{-i},t_i')\Bigg]. \end{aligned}$$

Since

$$\begin{aligned} &\hat{v}_i(q(t_{-i},t_i); t_{-i},t_i) + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i); t_{-i},t_i) \\ &\qquad \ge \hat{v}_i(q(t_{-i},t_i'); t_{-i},t_i) + \sum_{j\in N\setminus i}\hat{v}_j(q(t_{-i},t_i'); t_{-i},t_i), \end{aligned}$$

we conclude that

$$\begin{aligned}&\big(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i(t_{-i},t_i)\big)-\big(\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i(t_{-i},t_i')\big)\\&\quad\geq\max_{c\in C}\Bigg[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i')\Bigg]-\max_{c\in C}\Bigg[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i)\Bigg]\\&\qquad\qquad-\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')+\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i).\end{aligned}$$

Lemma 1 implies that

$$\max_{c\in C}\Bigg[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i')\Bigg]-\max_{c\in C}\Bigg[\sum_{j\in N\backslash i}\hat{v}_j(c;t_{-i},t_i)\Bigg]\geq-(n-1)M\big\|P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')\big\|,$$

so the result follows from the observation that

$$\begin{aligned}&\Bigg|\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i)-\sum_{j\in N\backslash i}\hat{v}_j(q(t_{-i},t_i');t_{-i},t_i')\Bigg|\\&\qquad\qquad\leq(n-1)M\|P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')\|.\end{aligned}$$

A.2 *Proof of Theorem 1*

Suppose that $(t_{-i},t_i)\in T^*$ and define

$$U_i(t_i'|t_{-i},t_i)=\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)+\alpha_i^q(t_{-i},t_i').$$

If $(t_{-i},t_i')\notin T^*$, then

$$\begin{aligned}U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i)&=(\hat{v}_i(c_0;t_{-i},t_i)+0)-\big(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)\big)\\&=-\big(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)\big)\\&\leq 0\end{aligned}$$

and we conclude that

$$U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i)>2M(n-1)\nu_i^P\quad\text{implies that }(t_{-i},t_i')\in T^*.$$

Applying Lemma A, we observe that

$$\begin{aligned}&\big\{t_{-i}|(t_{-i},t_i)\in T^*\text{ and }U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i)>2M(n-1)\nu_i^P\big\}\\&\quad=\{t_{-i}|(t_{-i},t_i)\in T^*,(t_{-i},t_i')\in T^*,\text{and }U_i(t_i'|\tilde{t}_{-i},t_i)-U_i(t_i|\tilde{t}_{-i},t_i)>2M(n-1)\nu_i^P\}\\&\quad\subseteq\big\{t_{-i}\in T_{-i}|(t_{-i},t_i)\in T^*,(t_{-i},t_i')\in T^*,\|P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')\|>\hat{\nu}_i^P\big\}.\end{aligned}$$

If $\varepsilon>0$, then choosing

$$0<\delta<\min\bigg\{\frac{\varepsilon}{2M(n-1)},\varepsilon\bigg\}$$

and $\hat{\nu}_i^P < \delta$ yields

$$\begin{aligned}
&\text{Prob}\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) > \varepsilon | \tilde{t}_i = t_i\} \\
&\quad \le \text{Prob}\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) > 2M(n-1)\nu_i^P | \tilde{t}_i = t_i\} \\
&\quad \le \text{Prob}\big\{(\tilde{t}_{-i}, t_i) \in T^*, (\tilde{t}_{-i}, t_i') \in T^*, \|P_\Theta(\cdot|\tilde{t}_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}_{-i}, t_i')\| > \hat{\nu}_i^P | \tilde{t}_i = t_i\big\} \\
&\quad \le \hat{\nu}_i^P \\
&\quad \le \varepsilon,
\end{aligned}$$

and the proof is complete.

A.3 *Proof of Theorem 2*

We prove part (ii) first. Choose $\varepsilon > 0$. Recall that $0 \le v_i(\cdot, \cdot, \cdot) \le M$ for each i and let $|T|$ denote the cardinality of T. Choose δ so that

$$0 < \delta < \min\left\{\frac{\varepsilon}{4M(n+1)\sqrt{|T|}}, \frac{\varepsilon}{2}\right\}.$$

Suppose that $P \in \Delta^*_{\Theta \times T}$ satisfies

$$\max_i \nu_i^P \le \delta \min_i \Lambda_i^P.$$

Define $\hat{\nu}^P = \max_i \nu_i^P$ and $\Lambda^P = \min_i \Lambda_i^P$. Therefore, $\hat{\nu}^P \le \delta \Lambda^P$.

Now we define an augmented GVCG mechanism. For each $t \in T$, define

$$z_i(t_{-i}, t_i) = \varepsilon \frac{P_{T_{-i}}(t_{-i}|t_i)}{\|P_{T_{-i}}(\cdot|t_i)\|_2}.$$

Since $0 \le P_{T_{-i}}(t_{-i}|t_i)/\|P_{T_{-i}}(\cdot|t_i)\|_2 \le 1$, it follows that

$$0 \le z_i(t_{-i}, t_i) \le \varepsilon$$

for all i, t_{-i}, and t_i. For each $(t_{-i}, t_i) \in T^*$, define

$$U_i(t_i'|t_{-i}, t_i) = \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i^q(t_{-i}, t_i').$$

The augmented VCG mechanism $\{q, \alpha_i^q + z_i\}_{i \in N}$ is clearly ex post efficient. Individual rationality follows from the observations that

$$\hat{v}_i(q(t); t) + \alpha_i^q(t) \ge 0$$

and

$$z_i(t) \ge 0.$$

Claim 1. *For i and for each $t_i, t_i' \in T_i$,*

$$\sum_{t_{-i}:(t_{-i},t_i)\in T^*} (z_i(t_{-i},t_i) - z_i(t_{-i},t_i'))P_{T_{-i}}(t_{-i}|t_i) = \sum_{t_{-i}} (z_i(t_{-i},t_i) - z_i(t_{-i},t_i'))P_{T_{-i}}(t_{-i}|t_i) \geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda^P.$$

Proof. We have

$$\begin{aligned}\sum_{t_{-i}}(z_i(t_{-i}|t_i) - z_i(t_{-i},t_i'))P_{T_{-i}}(t_{-i}|t_i) &= \sum_{t_{-i}} \varepsilon\left[\frac{P_{T_{-i}}(t_{-i}|t_i)}{\|P_{T_{-i}}(\cdot|t_i)\|_2} - \frac{P_{T_{-i}}(t_{-i}|t_i')}{\|P_{T_{-i}}(\cdot|t_i')\|_2}\right]P_{T_{-i}}(t_{-i}|t_i) \\ &= \frac{\varepsilon\|P_{T_{-i}}(\cdot|t_i)\|_2}{2}\left\|\frac{P_{T_{-i}}(\cdot|t_i)}{\|P_{T_{-i}}(\cdot|t_i)\|_2} - \frac{P_{T_{-i}}(\cdot|t_i')}{\|P_{T_{-i}}(\cdot|t_i')\|_2}\right\|^2 \\ &\geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P \\ &\geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda^P.\end{aligned}$$

This completes the proof of Claim 1. □

Claim 2. *For each i and for each $t_i, t_i' \in T_i$,*

$$\sum_{t_{-i}:(t_{-i},t_i)\in T^*} [U_i(t_i|t_{-i},t_i) - U_i(t_i'|t_{-i},t_i)]P_{T_{-i}}(t_{-i}|t_i) \geq -(n+1)2M\hat{\nu}^P.$$

Proof. Define

$$A_i(t_i',t_i) = \left\{t_{-i}\in T_{-i}|(t_{-i},t_i)\in T^*, (t_{-i},t_i')\in T^*, \|P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i},t_i')\| > \hat{\nu}^P\right\}$$
$$B_i(t_i',t_i) = \left\{t_{-i}\in T_{-i}|(t_{-i},t_i)\in T^*, (t_{-i},t_i')\in T^*, \|P_\Theta(\cdot|t_{-i},t_i) - P_\Theta(\cdot|t_{-i},t_i')\| \leq \hat{\nu}^P\right\}$$

and

$$C_i(t_i',t_i) = \{t_{-i}\in T_{-i}|(t_{-i},t_i)\in T^*, (t_{-i},t_i')\notin T^*\}.$$

Since $\nu_i^P \leq \hat{\nu}^P$, we conclude that

$$\text{Prob}\{\tilde{t}_{-i}\in A_i(t_i',t_i)|\tilde{t}_i = t_i\} \leq \nu_i^P \leq \hat{\nu}^P.$$

Next, note that

$$0 \leq \hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) + \alpha_i^q(t_{-i},t_i) \leq \hat{v}_i(q(t_{-i},t_i);t_{-i},t_i) \leq M$$

for all i, t_i, and t_{-i}. Therefore,

$$\begin{aligned}&|U_i(t_i'|t_{-i},t_i)| \\ &\quad = \left|\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i) - \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i') + \hat{v}_i(q(t_{-i},t_i');t_{-i},t_i') + \alpha_i^q(t_{-i},t_i')\right|\end{aligned}$$

$$\leq \left|\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i)-\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i')\right|+\left|\hat{v}_i(q(t_{-i},t_i');t_{-i},t_i')+\alpha_i^q(t_{-i},t_i')\right|$$
$$\leq 3M$$

for all i, t_i, t_i', and t_{-i}. Applying the definitions, it follows that

$$\sum_{t_{-i}\in A_i(t_i',t_i)}[U_i(t_i|t_{-i},t_i)-U_i(t_i'|t_{-i},t_i)]P_{T_{-i}}(t_{-i}|t_i)\geq -4M\sum_{t_{-i}\in A_i(t_i',t_i)}P_{T_{-i}}(t_{-i}|t_i)\geq -4M\hat{\nu}^P.$$

In addition,

$$\begin{aligned}\sum_{t_{-i}\in B_i(t_i',t_i)}&[U_i(t_i|t_{-i},t_i)-U_i(t_i'|t_{-i},t_i)]P_{T_{-i}}(t_{-i}|t_i)\\&\geq -2M(n-1)\sum_{t_{-i}\in B_i(t_i',t_i)}\|P_\Theta(\cdot|t_{-i},t_i)-P_\Theta(\cdot|t_{-i},t_i')\|P_{T_{-i}}(t_{-i}|t_i)\\&\geq -2M(n-1)\hat{\nu}^P\end{aligned}$$

and, finally,

$$\begin{aligned}\sum_{t_{-i}\in C_i(t_i',t_i)}&[U_i(t_i|t_{-i},t_i)-U_i(t_i'|t_{-i},t_i)]P_{T_{-i}}(t_{-i}|t_i)\\&=\sum_{t_{-i}\in C_i(t_i',t_i)}\left[(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i))-(\hat{v}_i(c_0;t_{-i},t_i)+0)\right]P_{T_{-i}}(t_{-i}|t_i)\\&=\sum_{t_{-i}\in C_i(t_i',t_i)}\left(\hat{v}_i(q(t_{-i},t_i);t_{-i},t_i)+\alpha_i^q(t_{-i},t_i)\right)P_{T_{-i}}(t_{-i}|t_i)\\&\geq 0.\end{aligned}$$

Combining these observations completes the proof of Claim 2. □

Applying Claims 1 and 2, it follows that

$$\begin{aligned}\sum_{t_{-i}:(t_{-i},t_i)\in T^*}&[U_i(t_i|t_{-i},t_i)-U_i(t_i'|t_{-i},t_i)]P_{T_{-i}}(t_{-i}|t_i)\\&+\sum_{t_{-i}:(t_{-i},t_i)\in T^*}(z_i(t_{-i},t_i)-z_i(t_{-i},t_i'))P_{T_{-i}}(t_{-i}|t_i)\\&\geq \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P-(n+1)2M\hat{\nu}^P\\&\geq 0\end{aligned}$$

and the mechanism is interim incentive compatible. If $(t_{-i},t_i)\in T^*$, then $\Lambda^P\leq 2$ implies that

$$U_i(t_i'|t_{-i},t_i)-U_i(t_i|t_{-i},t_i)\leq 2M(n-1)\nu^P\leq 2M(n-1)\frac{\varepsilon}{4M(n+1)\sqrt{|T|}}\Lambda^P\leq\varepsilon.$$

In addition, $\Lambda^P \le 2$ implies that $\nu^P \le (\varepsilon/2)\Lambda^P \le \varepsilon$. Applying the same argument used in the proof of Theorem 1, we conclude that

$$\begin{aligned}
&\mathrm{Prob}\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) + z_i(\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) - z_i(\tilde{t}_{-i}, t_i') > 2\varepsilon|\tilde{t}_i = t_i\} \\
&\quad \le \mathrm{Prob}\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) + z_i(\tilde{t}_{-i}, t_i) > 2\varepsilon|\tilde{t}_i = t_i\} \\
&\quad \le \mathrm{Prob}\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) > \varepsilon|\tilde{t}_i = t_i\} \\
&\quad \le \mathrm{Prob}\{(\tilde{t}_{-i}, t_i) \in T^* \text{ and } U_i(t_i'|\tilde{t}_{-i}, t_i) - U_i(t_i|\tilde{t}_{-i}, t_i) > 2M(n-1)\nu_i^P|\tilde{t}_i = t_i\} \\
&\quad \le \mathrm{Prob}\big\{(\tilde{t}_{-i}, t_i) \in T^*, (\tilde{t}_{-i}, t_i') \in T^*, \|P_\Theta(\cdot|\tilde{t}_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}_{-i}, t_i')\| > \nu_i^P|\tilde{t}_i = t_i\big\} \\
&\quad \le \nu_i^P \\
&\quad \le 2\varepsilon
\end{aligned}$$

and it follows that the mechanism is weakly (2ε)-ex post incentive compatible. This completes the proof of part (ii).

Part (i) follows from the computations in part (ii). In particular, Claims 1 and 2 of part (ii) show that for any positive number ε, there exists an augmented GVCG mechanism $\{q, \alpha_i^q + z_i\}_{i\in N}$ satisfying

$$\begin{aligned}
&\sum_{t_{-i}:(t_{-i},t_i)\in T^*} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)]P_{T_{-i}}(t_{-i}|t_i) \\
&\qquad\qquad + \sum_{t_{-i}:(t_{-i},t_i)\in T^*} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t_i'))P_{T_{-i}}(t_{-i}|t_i) \\
&\quad \ge \frac{\varepsilon}{2\sqrt{|T|}}\Lambda_i^P - (n+1)2M\hat{\nu}^P
\end{aligned}$$

for each i and each t_i, t_i'. If $\Lambda_i^P > 0$ for each i, then ε can be chosen large enough so that incentive compatibility is satisfied. This completes the proof of part (i).

A.4 *Proof of Lemma B*

For each $\theta \in \Theta$, let $P(\cdot|\theta)$ denote the conditional measure on A given $\theta \in \Theta$, and for each r, let $P_\Theta(\cdot|t^r)$ denote the conditional measure on Θ given $t^r \in T^r$. For each $\alpha \in A$, let $f_\alpha(t^r) = \#\{i \in J_r | t_i^r = \alpha\}$ and define $f(t^r) = (f_\alpha(t^r))_{\alpha\in A}$.

Step 1. If $s \in A$ and $t_{-i}^r \in T^{r-1}$, then

$$\left|\frac{f_\alpha(t_{-i}^r, s)}{r} - \frac{f_\alpha(t_{-i}^r)}{r-1}\right| = \left|\frac{f_\alpha(t_{-i}^r)+1}{r} - \frac{f_\alpha(t_{-i}^r)}{r-1}\right| = \frac{r-1-f_\alpha(t_{-i}^r)}{r(r-1)} \le \frac{1}{r} \quad \text{if } \alpha = s$$

$$\left|\frac{f_\alpha(t_{-i}^r, s)}{r} - \frac{f_\alpha(t_{-i}^r)}{r-1}\right| = \left|\frac{f_\alpha(t_{-i}^r)}{r} - \frac{f_\alpha(t_{-i}^r)}{r-1}\right| = f_\alpha(t_{-i}^r)\frac{1}{r(r-1)} \le \frac{1}{r} \quad \text{if } \alpha \ne s,$$

implying that

$$\left\|\frac{f(t_{-i}^r, s)}{r} - \frac{f(t_{-i}^r)}{r-1}\right\| \le \frac{|A|}{r}.$$

Step 2. For each θ, let

$$\mu(\theta) := \max_{\hat\theta\neq\theta} \prod_{\alpha\in A}\left[\frac{P(\alpha|\hat\theta)}{P(\alpha|\theta)}\right]^{P(\alpha|\theta)}$$

and let $R = \max_\theta \mu(\theta)$. Let $\chi_\theta \in \Delta_\Theta$ denote the Dirac measure with $\chi_\theta(\theta) = 1$ and let $\beta := \min_{\theta\in\Theta} P(\theta)$. There exists a $\delta > 0$ such that for each $\theta \in \Theta$ and each r,

$$\left\|\frac{f(t^r)}{r} - P(\cdot|\theta)\right\| < \delta \quad\Rightarrow\quad \|\chi_\theta - P_\Theta(\cdot|t^r)\| \leq \frac{2R^{r/2}}{\beta}.$$

To see this, fix θ and note that assumption (a) in the definition of conditionally independent sequence and the strict concavity of the function $\ln(\cdot)$ imply that $\mu(\theta) < 1$. Therefore, $R < 1$. Again by computing the logarithm, there exists a $\delta_\theta > 0$ such that

$$\prod_{\alpha\in A}\left[\frac{P(\alpha|\hat\theta)}{P(\alpha|\theta)}\right]^{f_\alpha(t^r)/r-P(\alpha|\theta)} \leq \frac{1}{\sqrt{\mu(\theta)}}$$

whenever $\hat\theta \neq \theta$ and $\|f(t^r)/r - P(\cdot|\theta)\| < \delta_\theta$. Letting $\delta = \min\delta_\theta$, we conclude that for each $\theta \in \Theta$, $\|f(t^r)/r - P(\cdot|\theta)\| < \delta$ implies that

$$\frac{P_\Theta(\hat\theta|t^r)P(\theta)}{P_\Theta(\theta|t^r)P(\hat\theta)} = \left[\prod_{\alpha\in A}\left[\frac{P(\alpha|\hat\theta)}{P(\alpha|\theta)}\right]^{P(\alpha|\theta)} \prod_{\alpha\in A}\left[\frac{P(\alpha|\hat\theta)}{P(\alpha|\theta)}\right]^{f_\alpha(t^r)/r-P(\alpha|\theta)}\right]^r \leq \left[\mu(\theta)\frac{1}{\sqrt{\mu(\theta)}}\right]^r \leq R^{r/2}$$

whenever $\hat\theta \neq \theta$. Therefore, $\|f(t^r)/r - P(\cdot|\theta)\| < \delta$ implies that

$$\|\chi_\theta - P_\Theta(\cdot|t^r)\| = 2\sum_{\hat\theta\neq\theta} P_\Theta(\hat\theta|t^r) \leq 2\sum_{\hat\theta\neq\theta}\frac{P(\hat\theta)}{P(\theta)}P_\Theta(\theta|t^r)R^{r/2} \leq \frac{2R^{r/2}}{\beta}.$$

Step 3. To complete the argument, choose θ and $t_i, t_i' \in A$, and choose $\delta > 0$ as in Step 2 such that for each $\theta \in \Theta$ and each r,

$$\left\|\frac{f(t^r)}{r} - P(\cdot|\theta)\right\| < \delta \quad\Rightarrow\quad \|\chi_\theta - P_\Theta(\cdot|t^r)\| \leq \frac{2R^{r/2}}{\beta}.$$

Then for all $r > 2|A|/\delta$, we have

$$\begin{aligned}
&\|P_\Theta(\cdot|t^r_{-i}, t_i) - P_\Theta(\cdot|t^r_{-i}, t_i')\| > \frac{4R^{r/2}}{\beta}\\
&\qquad\Rightarrow\quad \exists s\in A: \|\chi_\theta - P_\Theta(\cdot|t^r_{-i}, s)\| > \frac{2R^{r/2}}{\beta}\\
&\qquad\Rightarrow\quad \exists s\in A: \left\|\frac{f(t^r_{-i}, s)}{r} - P(\cdot|\theta)\right\| \geq \delta\\
&\qquad\Rightarrow\quad \exists s\in A: \left\|\frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta)\right\| + \left\|\frac{f(t^r_{-i}, s)}{r} - \frac{f(t^r_{-i})}{r-1}\right\| \geq \delta
\end{aligned}$$

$$\Rightarrow \quad \left\| \frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta) \right\| \geq \delta - \frac{|A|}{r}$$

$$\Rightarrow \quad \left\| \frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta) \right\| \geq \delta/2.$$

Applying Hoeffding's inequality (see Hoeffding (1963)), we have

$$\text{Prob}\left\{ \left\| \frac{f(\tilde{t}^r_{-i})}{r-1} - P(\cdot|\theta) \right\| \geq \delta/2 \middle| \tilde{\theta} = \theta \right\} \leq 2|A| \exp\left[\frac{-(r-1)\delta^2}{2|A|^2} \right].$$

Using the conditional independence assumption, it follows that, for all sufficiently large r,

$$\begin{aligned}
&\text{Prob}\left\{ \|P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}^r_{-i}, t'_i)\| > \frac{4R^{r/2}}{\beta} \middle| \tilde{t}_i = t_i \right\} \\
&\quad = \sum_\theta \text{Prob}\left\{ \|P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}^r_{-i}, t'_i)\| > \frac{4R^{r/2}}{\beta}, \tilde{\theta} = \theta \middle| \tilde{t}_i = t_i \right\} \\
&\quad = \sum_\theta \text{Prob}\left\{ \|P_\Theta(\cdot|\tilde{t}^r_{-i}, t_i) - P_\Theta(\cdot|\tilde{t}^r_{-i}, t'_i)\| > \frac{4R^{r/2}}{\beta} \middle| \tilde{\theta} = \theta \right\} P(\theta|t_i) \\
&\quad \leq \sum_\theta \text{Prob}\left\{ \left\| \frac{f(t^r_{-i})}{r-1} - P(\cdot|\theta) \right\| \geq \delta/2 \middle| \tilde{\theta} = \theta \right\} P(\theta|t_i) \\
&\quad \leq 2|A| \exp\left[\frac{-(r-1)\delta^2}{2|A|^2} \right].
\end{aligned}$$

Hence, for all r sufficiently large,

$$\nu_i^{P^r} \leq \max\left\{ \frac{4R^{r/2}}{\beta}, 2|A| \exp\left[\frac{-(r-1)\delta^2}{2|A|^2} \right] \right\}.$$

A.5 *Proof of Theorem 3*

The proof is essentially identical to that of Theorem 2. First, note that $(T^r)^* = T^r$. For notational ease, we will write q_i, α_i, and z_i instead of q^r_i, α^r_i, and z^r_i, and write T, t, t_{-i}, and t_i instead of T^r, t^r, t^r_{-i}, and t^r_i. Choose $\varepsilon > 0$. Let M be the bound defined in the statement of the theorem. For each $\alpha, \beta \in A$, let

$$Q(\beta|\alpha) = \sum_\theta P(\beta|\theta) P(\theta|\alpha)$$

so that

$$\|Q(\cdot|\alpha)\|_2 = \left[\sum_{\beta \in A} Q(\beta|\alpha)^2 \right]^{1/2}.$$

For each i and $(t_1, \dots, t_r) \in T$, define

$$z_i(t_{-i}, t_i) = \frac{\varepsilon}{r}\frac{Q(t_{i+1}|t_i)}{\|Q(\cdot|t_i)\|_2} \quad \text{if } i = 1, \dots, r-1$$
$$= \frac{\varepsilon}{r}\frac{Q(t_1|t_r)}{\|Q(\cdot|t_r)\|_2} \quad \text{if } i = r.$$

Therefore,

$$0 \le z_i(t_{-i}, t_i) \le \frac{\varepsilon}{r}$$

for all i, t_{-i}, and t_i. Individual rationality of the augmented mechanism follows from the observations that $\hat{v}_i(q(t); t) + \alpha_i(t) \ge 0$ and $z_i(t) \ge 0$.

If $\alpha, \alpha' \in A$ with $\alpha \ne \alpha'$, then, by assumption, $Q(\cdot|\alpha) \ne Q(\cdot|\alpha')$ and, therefore,

$$\Lambda^* := \min_{\alpha \in A} \min_{\alpha' \in A\setminus\{\alpha\}} \left\| \frac{Q(\cdot|\alpha)}{\|Q(\cdot|\alpha)\|_2} - \frac{Q(\cdot|\alpha')}{\|Q(\cdot|\alpha')\|_2} \right\|_2^2 > 0.$$

Finally, note that

$$\sum_{s\in A}\left[\frac{Q(s|t_i)}{\|Q(\cdot|t_i)\|_2} - \frac{Q(s|t_i')}{\|Q(\cdot|t_i)\|_2}\right]Q(s|t_i) = \frac{\|Q(\cdot|t_i)\|_2}{2}\Lambda^* \ge \frac{\Lambda^*}{2\sqrt{|A|}}.$$

Claim 3. *Let $|A|$ denote the cardinality of A. Then*

$$\sum_{t_{-i}}(z_i(t_{-i}|t_i) - z_i(t_{-i}|t_i'))P^r(t_{-i}|t_i) \ge \frac{\varepsilon}{2r\sqrt{|A|}}\Lambda^*.$$

Proof. If $1 \le i \le r-1$, then

$$\sum_{t_{-i}}(z_i^r(t_{-i}|t_i) - z_i^r(t_{-i}|t_i'))P^r(t_{-i}|t_i) = \sum_{t_{i+1}}\sum_{t_{-\{i,i+1\}}}\frac{\varepsilon}{r}\left[\frac{Q(t_{i+1}|t_i)}{\|Q(\cdot|t_i)\|_2} - \frac{Q(t_{i+1}|t_i')}{\|Q(\cdot|t_i)\|_2}\right]P^r(t_{-i}|t_i)$$
$$= \sum_{t_{i+1}}\frac{\varepsilon}{r}\left[\frac{Q(t_{i+1}|t_i)}{\|Q(\cdot|t_i)\|_2} - \frac{Q(t_{i+1}|t_i')}{\|Q(\cdot|t_i)\|_2}\right]Q(t_{i+1}|t_i)$$
$$\ge \frac{\varepsilon}{2r\sqrt{|A|}}\Lambda^*.$$

A similar computation is applied when $i = r$ and this completes the proof of Claim 3. □

Claim 4. *We have*

$$\sum_{t_i}\left[\left(\hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i)\right) - \left(\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i')\right)\right]P^r(t_{-i}|t_i)$$
$$\ge -2M(r+1).$$

Proof. For each $(t_{-i}, t_i) \in T$, define

$$U_i(t_i'|t_{-i}, t_i) = \hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i').$$

As in the proof of Theorem 2, define

$$A_i(t_i', t_i) = \left\{t_{-i} \in T_{-i} | \|P^r_\Theta(\cdot|t_{-i}, t_i) - P^r_\Theta(\cdot|t_{-i}, t_i')\| > \hat{\nu}^{P^r}\right\}$$

and

$$B_i(t_i', t_i) = \left\{t_{-i} \in T_{-i} | \|P^r_\Theta(\cdot|t_{-i}, t_i) - P^r_\Theta(\cdot|t_{-i}, t_i')\| \le \hat{\nu}^{P^r}\right\}.$$

Using the arguments of Theorem 2, we conclude that

$$\text{Prob}\{\tilde{t}_{-i} \in A_i(t_i', t_i) | \tilde{t}_i = t_i\} \le \nu^{P^r}$$

$$0 \le \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i) \le \hat{v}_i(q(t_{-i}, t_i); t_{-i}, t_i) \le M$$

and

$$|\hat{v}_i(q(t_{-i}, t_i'); t_{-i}, t_i) + \alpha_i(t_{-i}, t_i')| \le 3M$$

for all i, t_i, t_i', and t_{-i}. Again using the arguments of Theorem 2, it follows that

$$\sum_{t_{-i} \in A_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)] P^r(t_{-i}|t_i) \ge -4M\nu^{P^r}$$

and that

$$\sum_{t_{-i} \in B_i(t_i', t_i)} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)] P^r(t_{-i}|t_i) \ge -2M(r-1)\nu^{P^r}.$$

Combining these observations completes the proof of the Claim 4. □

Applying Lemma B and Claims 3 and 4, it follows that, for sufficiently large r,

$$\begin{aligned}
&\sum_{t_{-i}} (U_i(t_i|t_{-i}, t_i) + z_i(t_{-i}, t_i)) P^r(t_{-i}|t_i) - \sum_{t_{-i}} (U_i(t_i'|t_{-i}, t_i) + z_i(t_{-i}, t_i')) P^r(t_{-i}|t_i) \\
&\quad = \sum_{t_{-i}} [U_i(t_i|t_{-i}, t_i) - U_i(t_i'|t_{-i}, t_i)] P^r(t_{-i}|t_i) + \sum_{t_{-i}} (z_i(t_{-i}, t_i) - z_i(t_{-i}, t_i')) P^r(t_{-i}|t_i) \\
&\quad \ge \frac{\varepsilon}{2r\sqrt{|A|}} \Lambda^* - 2M(r-1)\nu^{P^r} \\
&\quad = \frac{1}{r}\left[\frac{\varepsilon}{2\sqrt{|A|}} \Lambda^* - 2Mr(r-1)\nu^{P^r}\right] \\
&\quad \ge 0
\end{aligned}$$

and the proof of interim IC is complete.

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