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Optimally constraining a bidder using a simple budget

Justin Burkett
Department of Economics, Wake Forest University

I study a principal's optimal choice of constraint for an agent participating in an auction (or auction-like allocation mechanism). I give necessary and sufficient conditions on the principal's beliefs about the value of the item for a simple budget constraint to be the optimal contract. The results link the observed use of budget constraints to their use in models incorporating budget-constrained bidders. Other implications of the model are that a general revenue equivalence result applies and that the optimal auction with budget-constrained bidders has a standard solution analogous to the one for classic models.

Keywords. Budget constraints, auctions, mechanism design.

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1. Introduction

A firm (principal) employs a manager (agent) to bid on an asset available to the firm via auction, because the manager is more capable of determining the asset's value than the board of directors. If the firm wants to constrain the bidding behavior, perhaps because they expect the manager to exaggerate the value of the asset,[1] how should it do so? It has been suggested that a common method used to constrain bidders is a simple budget constraint or a limit on the highest bid that may be placed (Cramton 1995, Bulow et al. 2009), and models incorporating financially constrained bidders have generally assumed that simple budget constraints are used, whether or not an explicit principal–agent relationship is modeled.[2]

One might argue that the prevalence of budget constraints is the result of practical considerations, such as enforceability and ease of implementation. In this paper, I show that budget constraints are also *optimal* in a general environment. Specifically, I provide sufficient conditions on the distribution of the principal's beliefs about the value of the asset to the firm that guarantee that a budget constraint is optimal for the principal under two different "regimes," which specify the feasible set of contracts. When the conditions fail, I show how to construct profitable deviations, proving necessity.

Justin Burkett: burketje@wfu.edu
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[1] Controlling the asset may privately benefit the manager beyond its contribution to firm profits (Jensen 1986).

[2] Early contributions to this literature were made by Che and Gale (1996, 1998). Che and Gale (1998) do consider an additional case where bidders may face an increasing cost of financing their bid. However, the literature since has focused on the simple budget-constraint case (see, for example, Che and Gale 2000, Pai and Vohra 2014, and the citations in Burkett 2015).

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Importantly, the conditions do not depend on the details of the mechanism used to allocate the good (the auction is modeled as a direct revelation mechanism), and hence the same conditions apply across a variety of auctions and auction-like situations in which a principal constrains an agent charged with obtaining a good that benefits the two of them.

The regimes specify the set of alternative contracts that the principal may choose, and both require that the contract be incentive compatible and individually rational for the agent. A contract specifies the reports the agent makes to the mechanism and side payments for each of the agent's signals. In the first regime, no conditional transfers between the principal and agent are allowed but the principal may conditionally require the agent to take a costly action that hurts both parties (i.e., only negative side payments are allowed). Although somewhat restrictive, this first regime includes any contract that does not specify side payments, which have been studied in the literature on optimal delegation (Alonso and Matouschek 2008, e.g.). In the second regime, the principal is allowed to use conditional transfers subject to a limited-liability condition.

These regimes allow for a variety of contractual arrangements, but budget constraints, which do not utilize either type of conditional payment, are shown to be optimal given simple conditions are satisfied. In fact, the budget chosen is the same in both regimes. Budget constraints would be classified as constrained delegation in the optimal delegation literature, so another interpretation of this result is that it provides conditions under which delegation is optimal despite the feasibility of a wide range of contracts that allow for conditional transfers.

These results provide a link between the observed use of budget constraints in auctions and their use in models that assume optimizing behavior. The fact that the optimality of the budget constraint does not depend on specifics of the mechanism used to allocate the good also implies several important and useful results. One consequence is a more general revenue equivalence result between auctions with budget-constrained bidders than the one given in Burkett (2015), from which the basic model of the principal–agent interaction is taken. Another is that the structure of the model here allows for budget constraints to be easily incorporated into more complex ones.

It is straightforward, for example, to use the results in this paper to incorporate budget-constrained bidders into a seller's revenue-maximization problem. In Section 5, I show that the revenue-maximizing auction with budget-constrained bidders is nearly identical to the one developed in Myerson (1981), after appropriately redefining some of the key terms. This result stands in contrast to the existing literature on optimal auctions with budget-constrained bidders, which among other results finds that the optimal auction is a modified all-pay auction (Pai and Vohra 2014). The primary distinction between the present model and the most common one in this literature is that I allow for an endogenous choice of budget constraint.

The structure of the paper is as follows. Section 2 discusses related literature. Section 3 introduces the model and states the principal's optimization problem. Section 4 begins by reporting the main results of the paper. Sections 4.1 and 4.2 give the formal arguments for the necessity and the sufficiency statements in the main results. Section 5 discusses an application to the optimal auction problem, and Section 6 concludes.

2. Related work

The effect of budget constraints on bidder behavior and auction outcomes was initiated by Che and Gale (1996, 1998), who show that when budget constraints are exogenously determined, the expected revenue is highest in the all-pay auction out of the all-pay, first-price, and second-price formats. The literature on the effect of exogenous budget constraints on auction outcomes is extensive. Several papers extend the model of Che and Gale (1998) to allow for affiliated values (Fang and Perreiras, 2002, 2003, Kotowski 2010). Brusco and Lopomo (2008), Dobzinski and Leme (2014), and Hafalir et al. (2012) study models that incorporate budget constraints into multiple unit auctions, while Ashlagi et al. (2010) is concerned with budget constraints in position auctions. Examples of papers that study mechanism design problems with exogenously constrained bidders include Maskin (2000), Laffont and Robert (1996), Che and Gale (2000), and Malakhov and Vohra (2008). Recent work by Pai and Vohra (2014) shows that a modified all-pay auction is the seller's revenue-maximizing choice when budget constraints are exogenous.

Benoît and Krishna (2001), who consider a complete information game where bidders can choose their budgets at zero marginal cost, is an early example of a paper considering endogenous budget choices.[3] Burkett (2015) introduced a model in which each bidder receives a budget from a principal prior to bidding in a first- or second-price auction. It is shown that the two auction formats are equivalent in terms of revenue and efficiency when values are independently distributed.

In this paper, I use a payoff structure analogous to the one used in Burkett (2015), but allow the principal more freedom in how the agent is constrained. With respect to Burkett (2015), this paper serves two purposes. One is to provide a theoretical justification for the use of a standard budget constraint in that paper, and the other is to extend the analysis beyond the first- and second-price auction rules.

In addition to the auction literature, this paper is a contribution to the study of agency problems and organizational decision making. The two regimes in this paper are instances of well studied problems in this literature.

In the delegation problem introduced by Holmström (1977), a principal delegates decision-making authority, without the use of transfers, to an agent who is better informed about the state of nature but is potentially biased toward picking suboptimal outcomes.[4] Alonso and Matouschek (2008) give conditions under which "interval delegation" is optimal in a setting with a biased agent with symmetric and single-peaked preferences (see this paper for citations to the delegation literature). Under interval delegation, the agent is allowed to choose her most preferred choice in an intermediate

[3] Zheng (2001), Rhodes-Kropf and Viswanathan (2005), and Hyde and Vercammen (2002) consider situations where bidders participate in a competitive financing market prior to the auction. Zheng (2010) considers the problem of a social planner who, in addition to auctioning a good, may offer financing for bidders who are cash-constrained.

[4] Several types of delegation problems have been studied that differ conceptually from the one considered here, including, for example, Armstrong and Vickers (2010), who study a delegation problem where the agent selects from a set of two-dimensional projects, but the principal does not know which projects are available.

range of states, but is restricted otherwise.[5] Interval delegation is analogous to a budget constraint with the additional constraint that low types are required to place higher bids than they would like.

Amador and Bagwell (2013) analyzes a model with fewer restrictions on the payoffs, providing conditions for the optimality of interval delegation.[6] They also consider an extension of the delegation problem where the agent may be required to take a costly action that is contingent on the state of nature and entails a loss of surplus by both parties. They refer to this case as *money burning*, and provide several interpretations and motivations for considering it.[7]

Despite the similarity of their model and results to the money-burning regime I consider here, I cannot simply apply their results in the first regime, primarily because their results depend on assumptions of concavity that I do not make. One way to interpret the results for the first regime is as an extension to Amador and Bagwell (2013) in which the assumptions of concavity are replaced by the weaker assumption that the direct revelation mechanism is incentive compatible (see Section 4.2 for further discussion of this point).

The second regime is a departure from the delegation literature in that it allows for the principal to make conditional transfers directly to the agent. I impose a limited-liability condition, requiring that side payments to the agent be nonnegative. This corresponds to an extension of the Crawford and Sobel (1982) "cheap talk" model analyzed in Krishna and Morgan (2008).

Krishna and Morgan (2008) consider a scenario where the principal can fully commit to a contract specifying the choice of project (the project in their model corresponds to the value of the good in mine) and transfer, conditional on the agent's report, and one where the principal can only commit to the choice of transfer. I only consider an environment where the principal can fully commit. Krishna and Morgan (2008) describe some features of the optimal contract for the principal in this case, including that the principal never chooses to implement his most preferred contract (it is feasible to do so) and that the principal always sets the transfer to zero in the highest states. A budget constraint shares these features, because it never implements the principal's most preferred action with positive probability and never specifies a transfer.

Krishna and Morgan (2008) also provide a complete solution in the uniform-quadratic version of their model.[8] Although it shares the features mentioned, the solution they find in this case cannot be translated into a budget constraint, because it does specify nonzero transfers in some states. One important distinction between their

[5]For a precise definition, consider an agent who chooses an action from $[0, 1]$ based on a private observed signal, $s \in [0, 1]$. An example of interval delegation is a contract where the agent must select $a_l > 0$ when $0 < s < a_l$, can choose his most preferred action when $a_l < s < a_h$, and must select a_h when $a_h < s < 1$.

[6]In Proposition 3 of that paper, they consider a special case of the model where the optimal contract is actually a "budget constraint" as I use the term here.

[7]For example, they suggest that the principal may require the agent to undertake wasteful administrative tasks if the agent wants to take an "exceptional" action. See Amador and Bagwell (2013) for more examples.

[8]The uniform-quadratic model specifies that the principal's and agent's preferences are each quadratic functions of the state and a bias term, and that the principal's beliefs about the state are uniform. This is a common specification in this literature.

model and the one I analyze is that the bias between principal and agent can remain relatively large in their case when the realized state is small, whereas in the model I analyze, the magnitude of the bias shrinks with the value of the asset (see Section 3.2). It seems that this is the important difference driving the divergent results. Given the interpretation of the state as the value of an asset, the assumption that the bias is smaller when the value is smaller seems natural here.

3. Model

Burkett (2015) describes a model where principal–agent pairs compete in a sealed-bid auction for a single good. Each agent observes the value of the good to the pair and submits a bid for the good that is constrained by the budget the agent receives from the principal. The principal only has a noisy signal of the value and constrains the behavior of the agent because the agent derives more value from the good than the principal does.

The design of the model employed in this paper is motivated by this situation, but focuses on a single principal–agent pair, and abstracts away from the details of an auction. The basic idea is that the principal–agent pair will jointly decide on a report of a type to an incentive-compatible direct-revelation mechanism (the mechanism). The report determines a probability of receiving a good and an expected transfer. Models where multiple agents agree on how to manipulate their report to a central direct revelation mechanism have been used to study mechanism design in the presence of collusive agents (Laffont and Martimort 1997).

To decide on the report to the mechanism, the principal designs a second direct-revelation mechanism (the contract), to which the agent makes a report after learning the value of the good to the pair (the agent has the option of rejecting the contract and not participating in the mechanism). I consider two regimes governing the set of feasible contracts. I retain the assumptions that the principal and the agent are risk neutral, the agent has better information than the principal, and that the agent values the good more than the principal.

3.1 *Mechanism and contract*

The mechanism is proposed by a third party before any actions are taken and is composed of two functions, $P:[0,1]\to[0,1]$ and $T:[0,1]\to\mathbb{R}$, specifying the probability that the good is awarded as a function of the report t and the expected transfer made to the third party. The mechanism is assumed to be incentive compatible, which via standard arguments implies that $P(t)$ is nondecreasing and that $T(t)$ can be expressed as a function of $P(t)$ and $T(0)$. I also assume that the mechanism is individually rational, which requires that the payment made by the lowest report be nonpositive, $T(0)\leq 0$. Incentive compatibility is discussed in Section 4.1.

After the mechanism is announced, the principal proposes a second mechanism to the agent, which I refer to as the contract. The contract specifies as many as three functions (all functions of the agent's report), $\theta:[0,1]\to\{O\}\cup[0,1]$, and $\tau_i:[0,1]\to\mathbb{R}_+$, $i=p,a$. The term $\theta(t)$ is the report ultimately submitted to the mechanism and may specify that the agent and principal not participate (action O), while $\tau_a(t)$ ($\tau_p(t)$) is an additional transfer payment made by the agent (principal). I will assume throughout that $\tau_a(t) + \tau_p(t) \leq 0$, or that the contract is not subsidized by a third party.

The principal is assumed to be able to commit to the offered contract, and the agent cannot participate in the mechanism without a contract with the principal.

3.2 *Payoffs and information*

Ignoring any transfers specified by the contract, the principal and the agent earn a constant fraction of the difference between their perceived value of the good and the transfer made to the third party. For example, they may initially own equity shares in the firm or have some prior contractual arrangement equivalent to equity for this decision (e.g., a bonus tied to the firm's profit at the auction).

The agent's value of the good is given by his type, t, while the principal, and the firm if the principal is the owner, values the good according to $t - \beta(t)$. The function $\beta(t)$ captures the bias between the two valuations. I assume that the principal's valuation is positive, nondecreasing, and less than the agent's valuation. In the case where the bias is linear in the agent's type, $\beta(t) = \alpha t$, the assumption is that $\alpha \in (0, 1)$.

Conditional on a type-t agent reporting t' to the mechanism, the expected payoffs of the principal and the agent are proportional to $P(t')(t - \beta(t)) - T(t')$ and $P(t')t - T(t')$, respectively.[9] Incorporating transfers between the principal and the agent ($\tau_a(t)$ and $\tau_p(t)$) into the contract, the expected payoffs to the principal and the agent when a type-t agent reports t' are

$$\pi_p = P(\theta(t'))(t - \beta(t)) - T(\theta(t')) + \tau_p(t')$$
$$\pi_a = P(\theta(t'))t - T(\theta(t')) + \tau_a(t').$$

With risk neutrality, one can interpret $\tau_a(t)$ as the expected transfer specified by the contract (as opposed to an immediate transfer) similar to how $T(t)$ can be interpreted as the expected payment at an auction. This allows one to consider contracts where the agent only receives transfers if the item is won.[10]

When the contract is proposed by the principal, the principal only knows that t is distributed on $[0, 1]$ according to an absolutely continuous distribution, $F(t)$. The density is denoted by $f(t)$ and exists (a.e.). The agent learns the value of t after the contract is proposed but before she is required to make any decisions in the model.

3.3 *Timing*

Putting everything together, the timing of the game is as follows:

[9] They are proportional to these expressions to avoid double counting the auction "profits." One could multiply these payoffs by constant equity shares to avoid this conceptual difficulty, but I do not do so to simplify the notation.

[10] Given incentive-compatible choices of P, θ, and τ_a, there are incentive-compatible ex post transfers $\tau_a^{\mathrm{ex}}(t)$ satisfying $\tau_a(t) = P(\theta(t))\tau_a^{\mathrm{ex}}(t)$ that are only paid if the item is won.

1. The mechanism (P, T) is announced by a third party.
2. The principal proposes a contract (θ, τ_p, τ_a) to the agent.
3. The agent observes t and accepts or rejects the contract. If the agent rejects the contract, the game ends and the principal and agent each receive a zero payoff. If the agent accepts, the agent reports a type $t' \in \{O\} \cup [0, 1]$, which is translated into the report $\theta(t')$ to the mechanism.
4. The uncertainty is resolved and payoffs are realized.

3.4 *Statement of the principal's problem*

Given the mechanism (P, T) proposed by the third party, the principal's problem is to design a contract (θ, τ_p, τ_a) to maximize his expected payoff, subject to the contract being incentive compatible (IC) and individually rational (IR) for the agent. In addition, I will separately consider two types of constraints on (τ_p, τ_a), labeled (R1) and (R2) below. Formally, the set of problems considered can be described as

$$\underset{\theta,\tau_p,\tau_a}{\text{maximize}}\, \mathrm{E}_t\big[P(\theta(t))(t-\beta(t)) - T(\theta(t)) + \tau_p(t)\big]$$

$$\text{subject to} \quad t \in \underset{t' \in [0,1]}{\arg\max}\, P(\theta(t'))t - T(\theta(t')) + \tau_a(t') \qquad \text{(IC)}$$

$$P(\theta(t))t - T(\theta(t)) + \tau_a(t) \geq 0 \qquad \text{(IR)}$$

$$\tau_a(t) \leq 0, \qquad \tau_p(t) = \tau_a(t), \quad \forall t \in [0, 1] \qquad \text{(R1)}$$

$$\tau_a(t) \geq 0, \qquad \tau_p(t) = -\tau_a(t), \quad \forall t \in [0, 1]. \qquad \text{(R2)}$$

The (IR) constraint is imposed after t is revealed to the agent. This assumption can be justified by supposing that the agent knows t when she decides whether or not to accept the contract. It may also be that the agent is able to irrevocably cancel the contract (by quitting the firm perhaps), so that acceptance of the contract is not complete until the agent learns t.

Regime 1 (R1) allows for the principal to impose a state-contingent cost on the agent that affects both parties equally, such as a wasteful administrative process. This is the money burning referred to in the Introduction. Note that optimality under (R1) implies optimality under the stricter condition $\tau_a(t) = \tau_p(t) = 0$. This would be the restriction if I were separately considering the delegation problem (see the discussion in Section 2).

Regime 2 (R2) allows for transfers between the principal and the agent in a proper sense, but imposes a limited-liability constraint on the principal, restricting the principal to providing positive transfers.

4. Results

The main results give necessary and sufficient conditions for budget-constraint contracts to be optimal in each regime. This section begins by discussing the optimal contract, the conditions required for a budget constraint to be optimal, and the main results.

The remainder of the section presents the formal analysis. The proofs of necessity and sufficiency are distinct and are presented separately, starting with necessity. Some intuition for the results is presented along with the analysis of necessary conditions.

I define a budget-constraint contract as one in which the principal sets a cap on the highest type the agent may report to the mechanism, leaves the agent unconstrained otherwise, and involves no conditional transfers. Formally, a budget constraint is defined for a given $\hat{t}$ as

$$\theta^{\mathrm{BC}}(t) = \min(t, \hat{t})$$
$$\tau_i^{\mathrm{BC}}(t) = 0, \quad i = a, p.$$

Although defined as a constraint on the reported type, this contract can be translated into a more traditional budget constraint. In an auction, for example, this contract caps the expected payment made to the auctioneer at $T(\hat{t})$. If the auction only requires that the winner makes a payment, the cap on the ex post payment is $T^{\mathrm{ex}}(\hat{t}) = T(\hat{t})/P(\hat{t})$. The optimal $\hat{t}$ will not depend on the details of the mechanism, (P, T), but note that this does not mean that the cap on expected payments (or ex post payments) is fixed across mechanisms, as $T(\hat{t})$ could certainly vary.

Burkett (2015) studies the first- and second-price auctions directly and shows how the choice of $\hat{t}$ translates into a constraint on the amount bid in those settings. In that paper, the prescription for the principal is to choose a $\hat{t}$ equal to the expected value of the asset to the principal given that the agent is constrained. The choice of $\hat{t}$ is shown to be independent of whether the auction uses first- or second-price rules. One consequence of the first result in this paper is that this fact generalizes to other auction formats and to both regimes given some restrictions on the distribution of the agent's types and the relation between the principal's and the agent's valuations.

The optimal choice for $\hat{t}$ in these cases is

$$\hat{t} \equiv \inf\left\{t \;\middle|\; \int_{t'}^{1} (1 - F(x) - \beta(x) f(x))\, dx \le 0, \forall t' > t\right\},^{11} \tag{1}$$

so $\hat{t}$ is the lowest type for which the integral is negative for all $t > \hat{t}$. When $\hat{t} \in (0, 1)$, it must be that $\int_{\hat{t}}^{1} 1 - F(x) - \beta(x) f(x)\, dx = 0$, by continuity of the integral. Integrating by parts and rearranging this expression yields

$$\int_{\hat{t}}^{1} 1 - F(x) - \beta(x) f(x)\, dx = 0$$
$$-(1 - F(\hat{t}))\hat{t} + \int_{\hat{t}}^{1} (x - \beta(x)) f(x)\, dx = 0$$
$$\frac{1}{1 - F(\hat{t})} \int_{\hat{t}}^{1} (x - \beta(x)) f(x)\, dx = \hat{t},$$

[11] This definition is an adaptation of the one provided by Amador et al. (2006).

which is equivalent to the value for $\hat{t}$ in Burkett (2015). The principal sets $\hat{t}$ equal to her expected valuation given that the budget binds. This assumes that $\hat{t}$ is an interior solution, which turns out to be the only relevant case.[12]

There is no reference to the mechanism, (P, T), in the definition for the budget-constraint contract. This is also true of the necessary and the sufficient conditions for the optimality of $\hat{t}$, which are drawn from the following three assumptions.

Assumption 1. *We have that* $F_\Lambda(t) \equiv F(t) + \beta(t)f(t)$ *is nondecreasing for* $t \in [0, \hat{t}]$.

Assumption 2. *We have that* $F_\Gamma(t) \equiv F(t) - \beta(t)f(t)$ *is nondecreasing for* $t \in [0, \hat{t}]$.

Assumption 3. *We have that* $2F(t) - (t - \hat{t})^{-1} \int_{\hat{t}}^{t} F_\Lambda(x)\,dx \geq F_\Gamma(\hat{t})$ *for* $t \in [\hat{t}, 1]$.

Assumptions 1 and 2 put lower and upper bounds on how quickly $\beta(t)f(t)$ can change in relation to $F(t)$ for $t < \hat{t}$. The term $\beta(t)f(t)$ is the weighted magnitude of the bias in payments, and can be thought of as measuring the importance of a given type's bias to the principal's problem (see Section 4.1). The first two assumptions are easy to check for a given $F(t)$ and $\beta(t)$, but to get a sense of how restrictive they are, first consider the case when t is uniformly distributed. In this case, Assumption 1 holds if, for example, the bias between valuations increases in t, while Assumption 2 is implied by the assumption that the principal's valuation, $t - \beta(t)$, is nondecreasing. Alternatively, if the bias is linear in t (e.g., $\beta(t) = \alpha t$, $\alpha \in (0, 1)$), one can verify that the first two assumptions are satisfied by common families of distribution functions on subsets of their parameter spaces.[13]

Assumption 3 is a technical assumption required for the proof of sufficiency in the transfer case. It is implied by the simpler assumption that $F_\Gamma(t)$ is nondecreasing on $[0, 1]$,[14] but it is useful in this weaker form as Example 1 illustrates.

The main results are divided into two propositions. The first gives the necessary conditions for optimality in each regime, while the second gives the sufficient conditions. In the money-burning regime (R1), Assumption 1 is all that is required, while in the transfer regime (R2), all three assumptions are required. The third assumption is a technical assumption used in the proof of sufficiency but not necessity.

[12]We have $\hat{t} > 0$, because $\hat{t} = 0$ would imply that $\int_0^1 (x - \beta(x))f(x)\,dx \leq 0$, a contradiction given $x \geq \beta(x)$. Suppose instead that $\hat{t} = 0$ and $\int_0^1 (x - \beta(x))f(x)\,dx > 0$. Then continuity would imply that $-(1 - F(\varepsilon))\varepsilon + \int_\varepsilon^1 (x - \beta(x))f(x)\,dx > 0$ for an arbitrarily small $\varepsilon > 0$, contradicting that $\hat{t} = 0$. That $\hat{t} < 1$ follows from $-(1 - F(t))t + \int_t^1 (x - \beta(x))f(x)\,dx \leq (1 - \beta(1) - t)(1 - F(t))$. The last term is less than 0 when $1 - \beta(1) < t$.

[13]The following assumptions guarantee that $F_\Lambda(t)$ and $F_\Gamma(t)$ are nondecreasing on $[0, 1]$ with linear $\beta(t)$. The uniform distribution clearly satisfies the assumptions, but so do all Gamma distributions where the shape (k) and scale (θ) parameters satisfy $k < 1/\alpha + 1/\theta$. Special cases of the Gamma distributions are the exponential distribution $(k = 1, \theta = 1/\lambda)$, which always satisfies this inequality, and the χ^2 distribution $(k = v/2, \theta = 2)$, which satisfies the inequality for any α if $v \leq 3$ and for all degrees of freedom (v) up to an α-dependent upper bound. For more examples, there are subsets of parameters for the beta (see Example 1), normal, and Pareto densities that satisfy both assumptions with arbitrary α.

[14]With $F_\Gamma(t)$ nondecreasing, $2F(t) - (t - \hat{t})^{-1} \int_{\hat{t}}^{t} F_\Lambda(x)\,dx = 2(t - \hat{t})^{-1} \int_{\hat{t}}^{t} (F(t) - F(x))\,dx + (t - \hat{t})^{-1} \times \int_{\hat{t}}^{t} F_\Gamma(x)\,dx \geq F_\Gamma(\hat{t})$.

Proposition 1 (Necessity). *(i) Assumption 1 is a necessary condition for optimality when the constraints on the principal are (IC), (IR), and (R1). (ii) Assumptions 1 and 2 are necessary conditions when the constraints on the principal are (IC), (IR), and (R2).*

Proposition 2 (Sufficiency). *(i) Under Assumption 1, a budget-constraint contract is optimal when the constraints on the principal are (IC), (IR), and (R1). (ii) Under Assumptions 1, 2, and 3, a budget-constraint contract is optimal when the constraints on the principal are (IC), (IR), and (R2).*

To illustrate the relationship between the definition of $\hat{t}$ and Assumptions 1–3, consider the following family of beta distributions that yield explicit solutions for $\hat{t}$.

Example 1. Assume that $\beta(t) = \alpha t$ with $\alpha \in (0, 1)$ and $F(t; b) = 1 - (1-t)^b$ with $b > 0$ (this is a beta distribution with parameters 1 and b). The mean of this distribution is $(1+b)^{-1}$, and it includes the uniform distribution as a special case ($b = 1$). I calculate that $\hat{t} = (1-\alpha)/(1+\alpha b)$. Assumption 1 requires that $t(1+\alpha b) \leq 1+\alpha$ for $t \in [0, \hat{t}]$, and Assumption 2 requires that $t(1-\alpha b) \leq 1-\alpha$ for $t \in [0, \hat{t}]$. For $t \leq \hat{t}$, both inequalities hold. After simplifying, $(t-\hat{t})^{-1} \int_{\hat{t}}^{t} F_\Lambda(x)\,dx = 1 - (1+\alpha b)(1-t)^b/(1+b)$, from which it follows that Assumption 3 holds as well. Note that $F_\Gamma(t) = 1 - (1-t)^b - \alpha bt(1-t)^{b-1}$ decreases for values of $t > (1-\alpha)/(1-\alpha b)$ with $b < 1$, so the simpler version of Assumption 3 mentioned above would not hold here. The solution for $\hat{t}$ shows that the principal tightens the constraint if either the bias increases or the types become more concentrated at the lower end of the interval (b decreases). ◊

The remainder of this section presents the formal analysis. The first subsection addresses the necessity proposition after some preliminary analysis, while the second addresses the sufficiency proposition.

4.1 *Necessity*

By assumption, the mechanism offered to the principal–agent pair, (P, T), is incentive compatible (i.e., a report of t maximizes the payoff from the mechanism). Following standard arguments, this requires that $P(t)$ be nondecreasing and that the payoff from a report of t with a value of t is $U(t,t) = P(t)t - T(t) = \int_0^t P(x)\,dx - T(0)$. The payoff from a report of $\theta(t)$ is, therefore,

$$U(\theta(t), t) = P(\theta(t))(t - \theta(t)) + \int_0^{\theta(t)} P(x)\,dx - T(0).$$

It will be useful to observe that ignoring τ_a and τ_p, the principal's payoff from a report of $\theta(t)$ is $U(\theta(t), t) - \beta(t)P(\theta(t)) = U(\theta(t), t - \beta(t))$.

The incentive compatibility of the composition of the contract and the mechanism implies (using an analogous envelope theorem argument) that

$$\int_0^t P(\theta(x))\,dx = U(\theta(t), t) + \tau_a(t) - \tau_a(0) - U_0(\theta_0), \tag{2}$$

where $U_0(\theta_0) = -P(\theta(0))\theta(0) + \int_0^{\theta(0)} P(x)\,dx - T(0)$.

It also must be the case that $P(\theta(t))$ is a nondecreasing function of t, which given the assumption on P implies that $\theta(t)$ is nondecreasing. Together, these two conditions completely characterize the incentive-compatibility constraint on the principal.

To get a sense for why the necessary conditions should hold, it is helpful to first consider the principal's expected payoff when there are no transfers between the principal and agent ($\tau_i(t) = 0$) and $T(0) = 0$. For such a contract, $\int_0^t P(\theta(z))\, dz = U(\theta(t), t)$, and the principal's expected payoff is

$$\int_0^1 \big\{U(\theta(t), t) - \beta(t)P(\theta(t))\big\}\, dF(t) + U_0(\theta_0) \\ = \int_0^1 P(\theta(t))\{1 - F(t) - \beta(t)f(t)\}\, dt + U_0(\theta_0). \tag{3}$$

The basic trade-off the principal faces in choosing θ is between capturing the information rents the agent and principal receive from the mechanism designer and mitigating the bias in payoffs from the agent's report. In (3), the weights given to each concern are $1 - F(t)$ and $\beta(t)f(t)$, respectively.[15] The first weight measures the impact on the rents earned of increasing $\theta(t)$ for all higher types, while the second measures the importance of agent t's bias to the principal's payoff.

For the highest types, the benefits to increases in θ diminish, and the second concern must eventually overwhelm the first (the argument in footnote 12 implies that the term in brackets must become negative). The motivation is similar to that behind the "no distortion at the top" principle, as the principal is no longer interested in distorting the report (relative to her payoffs) for the highest types. Note, however, that it is not in general true that $\theta(1) = 1 - \beta(1)$ (see Example 1). The optimal choice of $\hat{t}$ is the highest type for which the expected rents of all $t > \hat{t}$ compensate for the expected bias of those types.

For types $t < \hat{t}$, the budget-constraint contract allows the agent to make his preferred report ($\theta = t$). This report maximizes the rents from the mechanism for these types, as the mechanism is incentive compatible, but incurs a cost of $\beta(t)f(t)$ for the principal. To consider potential deviations from this contract, it is important to recognize that without transfers, the principal is quite limited. For example, without transfers, incentive compatibility requires that $\theta(t)$ have a zero slope off of the diagonal.

A possible deviation that satisfies incentive compatibility and does not require transfers is shown for the interval $[t_1, t_2]$ in Figure 1. For such a deviation to be profitable, the reduction in expected bias must compensate for the loss in rents from the mechanism. The condition required to rule this out is exactly Assumption 1. Intuitively, when this assumption is violated, $\beta(t)f(t)$ decreases quickly over an interval and the expected bias contributed by the lower types in the interval is much more important than that of the higher types. This leads to the conclusion that it is not too costly to have the

[15] Factoring out $f(t)$ would make the bracketed expression $(1 - F(t))/f(t) - \beta(t)$, resembling a type of virtual surplus. The typical increasing hazard rate assumption along with the assumption that $\beta(t)$ is nondecreasing would imply that this term is decreasing, and although this assumption is related to the assumptions required here, one can show that they are not equivalent.

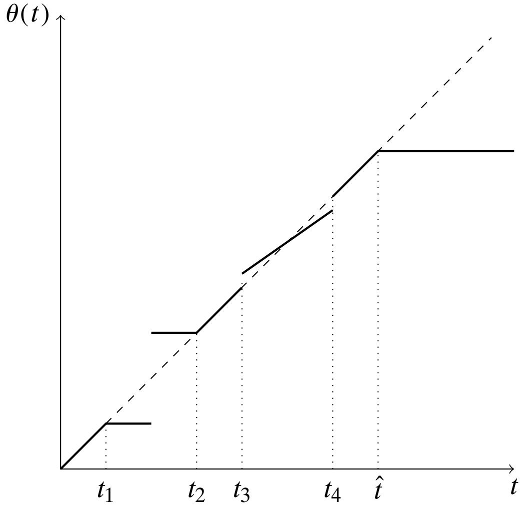

Figure 1. Possible deviations.

higher types in the interval overreport their types, which leads to a higher probability of winning for them. Lemma 1 shows that one must only consider deviations of this type to prove that Assumption 1 is a necessary condition for optimality of the budget constraint. Because the deviation constructed for $t \in [t_1, t_2]$ involves no transfers, it is feasible in both regimes and, consequently, Assumption 1 is a necessary condition in both cases.

Regime 2 allows the principal to make state-contingent transfers to the agent, giving her more freedom in the types of deviations she can implement. For example, with transfers, incentive compatible $\theta(t)$ can have a nonzero slope. Consequently, it becomes possible to deviate from the budget constraint in a way that reduces the bias for the higher types of agents in an interval at the expense of having the lower types overreport their types. One such deviation is shown for the interval $[t_3, t_4]$ in Figure 1. This is implemented by paying the lower type agents to overreport, adjusting the transfer across the interval to maintain incentive compatibility, and finally removing the transfer outside of the interval. This deviation becomes profitable exactly when Assumption 2 is violated (Lemma 2). This assumption mirrors Assumption 1 in the sense that it requires that the expected bias contributed by higher types in an interval not be too great relative to the lower types in an interval.

The remainder of the section formalizes these ideas, proving Proposition 1. The proposition is broken down into two lemmas, which together imply the proposition. Each lemma uses the deviations shown in Figure 1 in its proof.

Lemma 1. *If $F_\Lambda(t)$ decreases on $[t_1, t_2]$ with $t_2 \le \hat{t}$, there is a profitable deviation from the budget constraint that requires no transfers and, hence, is feasible in both regimes.*

Proof. Modify the budget-constraint contract to have the agent report $\theta(t) = t_1 \le t$ for $t_1 \le t \le t_m$ and $\theta(t) = t_2 \ge t$ for $t_m < t \le t_2$. The contract is unchanged otherwise. The term $\theta(t)$ is clearly still nondecreasing. Incentive compatibility without transfers also requires that the agent be indifferent between reporting t_1 and t_2 when $t = t_m$.[16] This means that t_m must satisfy

$$P(t_1)(t_m - t_1) + \int_0^{t_1} P(x)\,dx = P(t_2)(t_m - t_2) + \int_0^{t_2} P(x)\,dx$$
$$\int_{t_1}^{t_m} (P(x) - P(t_1))\,dx = \int_{t_m}^{t_2} (P(t_2) - P(x))\,dx.$$

Next consider $\pi_D - \pi_{\text{BC}}$, the difference between the principal's payoff with the deviation and that with the budget constraint. It is easy to check that with this deviation $\int_0^t P(\theta(x))\,dx = U(\theta(t), t)$, so the deviation involves no transfers. Since the contracts are identical outside $[t_1, t_2]$, using (3) this difference can be written as

$$\begin{aligned}\int_{t_1}^{t_m} (1 - F_\Lambda(x))(P(t_1) - P(x))\,dx + \int_{t_m}^{t_2} (1 - F_\Lambda(x))(P(t_2) - P(x))\,dx \\ > (1 - F_\Lambda(t_m))\left(\int_{t_1}^{t_m} (P(t_1) - P(x))\,dx + \int_{t_m}^{t_2} (P(t_2) - P(x))\,dx\right) = 0.\end{aligned}$$

The inequality follows because $1 - F_\Lambda(t)$ increases by the assumption of the lemma. □

The next lemma shows the necessity of Assumption 2 in the transfer case. The proof of this proposition constructs the linear deviation shown in Figure 1 over $[t_3, t_4]$ in a way that does not affect the payoff of the principal for types outside of the interval in question. To do this, the principal must make a positive transfer to the agent to overreport his type initially. This transfer is adjusted over the interval to preserve incentive compatibility, and is eventually removed at the end of the interval.

Lemma 2. *If $F_\Gamma(t)$ decreases on $[t_3, t_4]$ with $t_4 \le \hat{t}$ (i.e., Assumption 2 does not hold), then a profitable deviation from the budget-constraint contract exists in the transfer case (R2).*

Proof. It will be sufficient to consider a linear deviation on $[t_3, t_4]$, given by $\theta(t, \alpha) = \alpha t + (1 - \alpha) t_m(\alpha)$ with $0 < \alpha < 1$ and $t_m(\alpha)$ chosen so that $\tau(t_3, \alpha) = \tau(t_4, \alpha)$. This deviation will require that the principal pay some transfer to get the agent to overreport his type at t_3, gradually increase the transfer across $[t_3, t_m]$, and then gradually decrease until t_4. The adjustment in the transfer must satisfy $\tau_t(t, \alpha) = -P'(\theta(t, \alpha))\theta_t(t, \alpha) \times (t - \theta(t, \alpha)) = P'(\theta(t, \alpha))\alpha(1 - \alpha)(t - t_m(\alpha))$. Define $t_m(\alpha)$ implicitly as the solution to

$$\tau(t_3, \alpha) - \tau(t_4, \alpha) = -\alpha(1 - \alpha)\int_{t_3}^{t_4} P'(\theta(t, \alpha))(t - t_m(\alpha))\,dt = 0.$$

The middle term is continuous in $t_m(\alpha)$, less than zero at $t_m(\alpha) = t_3$, and greater than zero at $t_m(\alpha) = t_4$. So it has a solution. This contract is feasible because the transfers

[16]This follows from the continuity of $\int_0^t P(\theta(x))\,dx$.

are smallest at $\tau(t_3, \alpha)$ and $\tau(t_4, \alpha)$, which are both positive ($\tau(t_3, \alpha)$, for example, is chosen so that the agent is indifferent between reporting $\theta(t_3, \alpha)$ and receiving $\tau(t_3, \alpha)$, and reporting t_3 with no transfer).

Evaluating the principal's payoff (over the interval $[t_3, t_4]$) from this deviation, then differentiating with respect to α and evaluating at $\alpha = 1$, I get

$$\begin{aligned}
\pi_D(\alpha) &= \int_{t_3}^{t_4} 2U(\theta(t,\alpha),t)f(t) - P(\theta(t,\alpha))(1-F_\Gamma(t))\,dt \\
\pi_D'(\alpha) &= \int_{t_3}^{t_4} 2U_\theta(\theta(t,\alpha),t)\theta_\alpha(t,\alpha)f(t) - P'(\theta(t,\alpha))\theta_\alpha(t,\alpha)(1-F_\Gamma(t))\,dt \\
\pi_D'(1) &= -\int_{t_3}^{t_4} P'(t)(t-t_m(1))(1-F_\Gamma(t))\,dt \\
&< -(1-F_\Gamma(t_m))\int_{t_3}^{t_4} P'(t)(t-t_m(1))\,dt = 0.
\end{aligned}$$

The inequality follows from the assumption of the proposition that $1 - F_\Gamma(t)$ is increasing, so that $\int_{t_3}^{t_m} P'(t)(t - t_m(1))(1 - F_\Gamma(t))\,dt > (1 - F_\Gamma(t_m)) \int_{t_3}^{t_m} P'(t)(t - t_m(1))\,dt$ and $\int_{t_m}^{t_4} P'(t)(t-t_m(1))(1-F_\Gamma(t))\,dt > (1-F_\Gamma(t_m)) \int_{t_m}^{t_4} P'(t)(t-t_m(1))\,dt$. Therefore, for some $\alpha < 1$, the payoff can be made greater and the proposition follows. □

4.2 *Sufficiency*

To prove the sufficiency statements in Proposition 2, I employ the following modified version of Luenberger's sufficiency theorem (Luenberger 1969, Theorem 1, p. 220) given by Amador and Bagwell (2013). The theorem provides sufficient conditions for the solution of a general constrained optimization problem on vector spaces.

Theorem 1 (Amador and Bagwell 2013, Theorem 1). *Let f_0 be a real-valued functional defined on a subset Ω of a linear space X. Let G be a mapping from Ω into the normed space Z having nonempty positive cone Z_+. Suppose that (i) there exists a linear functional $S : Z \to \mathbb{R}$ such that $S(z) \geq 0$ for all $z \in Z_+$, (ii) there is an element $x_0 \in \Omega$ such that for all $x \in \Omega$,*

$$f_0(x_0) + S(G(x_0)) \leq f_0(x) + S(G(x)),$$

(iii) $-G(x_0) \in Z_+$, and (iv) $S(G(x_0)) = 0$. Then x_0 solves

$$\textit{minimize } f_0(x) \quad \textit{subject to} \quad -G(x) \in Z_+, \quad x \in \Omega.$$

The functional $\mathcal{L}(x) = f_0(x) + S(G(x))$ plays the role of a Lagrangian function with $S(x)$ being constructed from Lagrange multipliers in a manner to be described shortly. As explained in Amador et al. (2006) and Amador and Bagwell (2013), one advantage to using this theorem is that monotonicity constraints on the choice of function, x, can be embedded in the description of Ω, instead of being described by the functional $G(x)$.

The basic strategy used for proving optimality of budget constraints is as follows. First, I eliminate τ_p from the objective using the equation resulting from the incentive-compatibility constraints, forming f_0 in the theorem above. In combination with defining Ω to be the subset of increasing functions on $[0, 1]$, this takes care of the (IC) constraint. Then $G(x)$ can be used to incorporate the restrictions imposed by (R1) or (R2). This leaves (IR), which can safely be ignored and checked at the end.

The theorem is applied by first constructing the functional $S(x)$ and then proving directly that $\mathcal{L}(x)$ is minimized by the budget-constraint contract. This approach, which relies on the assumption of the incentive compatibility of the mechanism to prove optimality, differs from the one used in Amador et al. (2006) and Amador and Bagwell (2013), which uses the concavity of the principal's objective to prove optimality.[17] Specifically, they use the concavity of the principal's objective to guarantee that the first-order conditions of the principal's problem are sufficient for optimality. In my case, the budget constraint is shown to be optimal without utilizing first-order conditions. Instead, I show in the proofs of Proposition 2 that the incentive compatibility of the mechanism implies that the budget constraint maximizes the principal's objective pointwise given the sufficient conditions.

One way to think of the result from the money-burning regime is as an extension of the results from Amador and Bagwell (2013) to an environment that potentially involves a nonconcave objective (there is no analog to the transfer regime in Amador and Bagwell 2013). Section 5 shows how to use the results of this paper to consider questions involving mechanism design with budget constraints, and in that context it is useful to not restrict the space of feasible mechanisms to the set of concave mechanisms. In that respect, it is important to show that a budget-constraint contract remains optimal when all that is known is that the mechanism is incentive compatible (and not necessarily concave).

The $S(x)$ functional I use is determined by the selection of a nondecreasing function $\Lambda(t)$ as

$$S(x) = \int_0^1 x(t)\, d\Lambda(t).$$

The requirement that $\Lambda(t)$ be nondecreasing comes from the requirement that $S(z) \geq 0$ for all $z \in Z_+$.

The two subsections that follow give separate proofs for statements (i) and (ii) made in Proposition 2.

4.2.1 *Proof of Proposition 2(i): Money-burning regime* This subsection proves the statement made in Proposition 2(i). When money burning is allowed, the constraints on the principal are (IC), (IR), and (R1). The budget-constraint contract is feasible for the problem, because $\theta^{\mathrm{BC}}(t)$ is weakly increasing, it involves no transfers, and it satisfies (IR) given the assumption that (P, T) is individually rational.

[17]Assumption 1 in Amador and Bagwell (2013) provides the assumptions used on the principal's and agent's objectives in that paper. The principal's objective is $w(\gamma, \boldsymbol{\pi}(\gamma)) - \mathbf{t}(\gamma)$, where γ is the agent's type, $\boldsymbol{\pi}(\gamma)$ plays the role of $\theta(t)$ in this paper, and $\mathbf{t}(\gamma)$ is the money-burning transfer. The agent's problem is to choose $\tilde{\gamma}$ to $\max \gamma\boldsymbol{\pi}(\tilde{\gamma}) + b(\boldsymbol{\pi}(\tilde{\gamma})) - \mathbf{t}(\tilde{\gamma})$. The concavity assumptions made in Assumption 1 are that $w(\gamma_0, \cdot)$ is concave for any γ_0 and that $b(\cdot)$ is strictly concave. Fitting the current model into this framework would require making analogous assumptions about the concavity of the corresponding terms, which I do not do.

The function $F_\Lambda(t)$ will be the basis for the construction of the multiplier function ($\Lambda(t)$), and the requirements put on $F_\Lambda(t)$ are directly derived from requirements put on $\Lambda(t)$ by Theorem 1. That the two should be related becomes clear after forming the Lagrangian for this problem. Using (2) to replace $\tau_a(t)$ in the principal's objective,

$$\begin{aligned}
-\mathcal{L}(\theta) &= \int_0^1 \left\{ U(\theta(t), t-\beta(t)) + \int_0^t P(\theta(x))\,dx - U(\theta(t), t) \right\} dF(t) \\
&\qquad + \int_0^1 \left\{ U(\theta(t), t) - \int_0^t P(\theta(x))\,dx \right\} d\Lambda(t) \\
&\qquad - (\Lambda(1) - \Lambda(0) - 1)(\tau_a(0) + U_0(\theta_0)) \\
&= \int_0^1 P(\theta(t))(1 - F_\Lambda(t) - \Lambda(1) + \Lambda(t))\,dt + \int_0^1 U(\theta(t), t)\,d\Lambda(t) \\
&\qquad - (\Lambda(1) - \Lambda(0) - 1)(\tau_a(0) + U_0(\theta_0)).
\end{aligned}$$

The second equation follows from integration by parts and the definitions of $U(\theta(t), t)$ and $F_\Lambda(t)$. Note that I form the negative of the Lagrangian, because Theorem 1 refers to a minimization problem.

Consider the multiplier,

$$\Lambda(t) = \begin{cases} F_\Lambda(t) & \text{if } t \le \hat{t} \\ \sup_{t' \in [\hat{t}, t]} \{ \frac{1}{t'-\hat{t}} \int_{\hat{t}}^{t'} F_\Lambda(x)\,dx \} & \text{if } t > \hat{t}. \end{cases}$$

Assumption 1 guarantees that $\Lambda(t)$ is nondecreasing for $t \le \hat{t}$ and the definition guarantees that it is nondecreasing for $t > \hat{t}$. Clearly, $\Lambda(0) = 0$. It is continuous at $\hat{t}$, and it must be that $\Lambda(t) \le 1$[18] and that $\Lambda(1) = 1$.[19]

Combined, these properties make $\Lambda(t)$ a distribution function and imply that it is differentiable (a.e.). I next show that a budget-constraint contract is optimal. Incorporating the multiplier, write the Lagrangian as

$$\begin{aligned}
-\mathcal{L}(\theta) &= \int_0^{\hat{t}} U(\theta(t), t)\,d\Lambda(t) + \int_{\hat{t}}^1 U(\theta(t), t)\,d\Lambda(t) - \int_{\hat{t}}^1 (F_\Lambda(t) - \Lambda(t))P(\theta(t))\,dt \\
&= \int_0^{\hat{t}} U(\theta(t), t)\,d\Lambda(t) + \int_{\hat{t}}^1 U(\theta(t), \hat{t})\,d\Lambda(t) + \int_{\hat{t}}^1 P(\theta(t))(t - \hat{t})\,d\Lambda(t) \\
&\qquad + \int_{\hat{t}}^1 P(\theta(t))(\Lambda(t) - F_\Lambda(t))\,dt.
\end{aligned}$$

[18]To see that this is true, suppose that $\Lambda(t) > 1$ for some $t \in [\hat{t}, 1]$. Then for some $t' \in [\hat{t}, t]$,

$$1 - \frac{1}{t'-\hat{t}} \int_{\hat{t}}^{t'} F_\Lambda(x)\,dx < 0 \quad \Leftrightarrow \quad \int_{\hat{t}}^{t'} (1 - F_\Lambda(x))\,dx < 0 \quad \Leftrightarrow \quad \int_{t'}^1 (1 - F_\Lambda(x))\,dx > 0,$$

where the last implication follows from $\int_{\hat{t}}^1 (1 - F_\Lambda(x))\,dx = 0$. This contradicts the definition of $\hat{t}$, so $\Lambda(t) \le 1$.

[19]This follows from the definition of $\hat{t}$ and that it must be interior (see footnote 12). From (1), $\int_{\hat{t}}^1 F_\Lambda(x)\,dx = 1 - \hat{t}$.

The first two terms are maximized by choosing $\theta(t) = \theta^{\mathrm{BC}}(t)$, so if the last term is also maximized by this choice, it is optimal. Using the identities $P(\theta(t)) = P(\hat{t}) + \int_{\hat{t}}^{t} dP(\theta(x))$ and $\int_a^b (t-a)\, d\Lambda(t) = \int_a^b (\Lambda(b) - \Lambda(t))\, dt$, the third term becomes

$$
\begin{aligned}
P(\hat{t}) \int_{\hat{t}}^{1} (1 - F_\Lambda(t))\, dt + \int_{\hat{t}}^{1} \left\{ \int_x^1 (t - \hat{t})\, d\Lambda(t) + \int_x^1 (\Lambda(t) - F_\Lambda(t))\, dt \right\} dP(\theta(x)) \\
= \int_{\hat{t}}^{1} \left\{ \int_x^1 (1 - F_\Lambda(t))\, dt + (x - \hat{t})(1 - \Lambda(x)) \right\} dP(\theta(x)) \\
= \int_{\hat{t}}^{1} \left\{ \frac{-1}{x - \hat{t}} \int_{\hat{t}}^{x} (1 - F_\Lambda(t))\, dt + 1 - \Lambda(x) \right\} (x - \hat{t})\, dP(\theta(x)) \\
= \int_{\hat{t}}^{1} \left\{ \frac{1}{x - \hat{t}} \int_{\hat{t}}^{x} F_\Lambda(t)\, dt - \Lambda(x) \right\} (x - \hat{t})\, dP(\theta(x)).
\end{aligned}
$$

The definition of $\Lambda(t)$ implies that the term inside the brackets is nonpositive, so this term is maximized by setting $dP(\theta(x)) = 0$, which is the case with the budget-constraint contract.

To formally apply Theorem 1, I follow Amador and Bagwell (2013) in letting $X \equiv \{\theta \mid \theta : [0,1] \to [0,1]\}$, $Z \equiv \{z \mid z : [0,1] \to \mathbb{R}$ and z integrable$\}$, Ω be the set of nondecreasing functions in X, and Z_+ be the positive functions in Z (i.e., $Z_+ = \{z \mid z \in Z$ and $z(t) \geq 0, \forall t\}$). Condition (i) is clearly satisfied, and condition (ii) follows from the previous discussion. Finally, since $\Lambda(t)$ is nondecreasing and the constraint (R1) binds everywhere ($\tau_a(t) = 0, \forall t$), conditions (iii) and (iv) are also satisfied.

As mentioned above, optimality of budget constraints in the delegation problem, which maybe defined in this context as the stronger constraint that $\tau_a(t) = \tau_p(t) = 0$, is implied by the optimality of the budget constraint under the weaker restriction (R1). That is, under Assumption 1, budget-constraint contracts are necessarily optimal in the delegation problem. This follows from the observation that budget-constraint contracts remain feasible in the problem with tighter constraints.

4.2.2 *Proof of Proposition 2(ii): Transfer regime* This case allows the principal to pay nonnegative transfers to the agent conditional on the realized signal, so the constraints are (IC), (IR), and (R2). Again, the budget-constraint contract is feasible for the problem, because $\theta^{\mathrm{BC}}(t)$ is weakly increasing, it involves no transfers, and it satisfies (IR). From the statement of the proposition, I assume that Assumptions 1, 2, and 3 all hold for this proof.

Again, by using (2) to replace $\tau_p(t)$ in the principal's objective, the Lagrangian where $\Gamma(t)$ is the multiplier function is

$$
\begin{aligned}
-\mathcal{L}(\theta) = \int_0^1 \left\{ U(\theta(t), t - \beta(t)) + U(\theta(t), t) - \int_0^t P(\theta(x))\, dx \right\} dF(t) \\
+ \int_0^1 \left\{ \int_0^t P(\theta(x))\, dx - U(\theta(t), t) \right\} d\Gamma(t) \\
+ (\Gamma(1) - \Gamma(0) - 1)(\tau_a(0) + U_0(\theta_0))
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 \{2U(\theta(t), t) - \beta(t)P(\theta(t))\} f(t)\,dt \\
&\qquad + \int_0^1 (\Gamma(1) - \Gamma(t) - 1 + F(t))P(\theta(t))\,dt - \int_0^1 U(\theta(t), t)\,d\Gamma(t) \\
&\qquad + (\Gamma(1) - \Gamma(0) - 1)(\tau_a(0) + U_0(\theta_0)) \\
&= \int_0^1 U(\theta(t), t)\,d[2F(t) - \Gamma(t)] - \int_0^1 P(\theta(t))(1 - F_\Gamma(t) - \Gamma(1) + \Gamma(t))\,dt \\
&\qquad + (\Gamma(1) - \Gamma(0) - 1)(\tau_a(0) + U_0(\theta_0)).
\end{aligned}
$$

The first equation follows from integration by parts and the definition of $U(\theta(t), t)$. The second uses the definition of $F_\Gamma(t)$.

The proposed multiplier is

$$
\Gamma(t) = \begin{cases} 2F(t) - \Lambda(t) & \text{if } t \leq \hat{t} \\ \inf_{t' \in [t,1]}\{2F(t') - \Lambda(t')\} & \text{if } t > \hat{t}. \end{cases}
$$

Note that $2F(t) - \Lambda(t) = F_\Gamma(t)$ for $t \leq \hat{t}$. Assumption 2 and the definition of $\Gamma(t)$ for $t > \hat{t}$ guarantee that $\Gamma(t)$ is nondecreasing. As with $\Lambda(t)$, $\Gamma(0) = 0$ and $\Gamma(1) = 1$.[20] The equality $\Gamma(1) = 1$ implies that $\Gamma(t) \leq 1$. Assumption 3 guarantees that $\Gamma(t) \geq \Gamma(\hat{t})$ for $t > \hat{t}$.[21] Therefore, $\Gamma(t)$ is a distribution function and is differentiable (a.e.).

Following the idea of the proof in Section 4.2.1 and incorporating the new multiplier, the Lagrangian for the problem becomes

$$
\begin{aligned}
-\mathcal{L}(\theta) &= \int_0^{\hat{t}} U(\theta(t), t)\,d[2F(t) - \Gamma(t)] + \int_{\hat{t}}^1 U(\theta(t), \hat{t})\,d[2F(t) - \Gamma(t)] \\
&\qquad + \int_{\hat{t}}^1 P(\theta(t))(F_\Gamma(t) - \Gamma(t))\,dt + \int_{\hat{t}}^1 P(\theta(t))(t - \hat{t})\,d[2F(t) - \Gamma(t)] \\
&= \int_0^{\hat{t}} U(\theta(t), t)\,d\Lambda(t) + \int_{\hat{t}}^1 U(\theta(t), \hat{t})\,d[2F(t) - \Gamma(t)] \\
&\qquad + \int_{\hat{t}}^1 P(\theta(t))(F_\Gamma(t) - \Gamma(t))\,dt + \int_{\hat{t}}^1 P(\theta(t))(t - \hat{t})\,d[2F(t) - \Gamma(t)].
\end{aligned}
$$

The choice $\theta(t) = \theta^{\mathrm{BC}}(t)$ maximizes the first term pointwise, because $\Lambda(t)$ is nondecreasing. The definition of $\Gamma(t)$ for $t > \hat{t}$ implies that either $\Gamma(t)$ is constant or that $\Gamma(t) = 2F(t) - \Lambda(t)$. In either case, it is nondecreasing, so the second term is also maximized pointwise by the budget constraint. This leaves the third and fourth terms, which

[20] Since $\Lambda(1) = 1$, $\Gamma(1) = 2 - 1$.

[21] Suppose that $\Gamma(t) < F_\Gamma(\hat{t})$. Then for some $t' \leq t$, $F_\Gamma(\hat{t}) > 2F(t) - (t' - \hat{t})^{-1} \int_{\hat{t}}^{t'} F_\Lambda(x)\,dx \geq 2F(t') - (t' - \hat{t})^{-1} \int_{\hat{t}}^{t'} F_\Lambda(x)\,dx$, which contradicts the assumption.

can be rewritten as[22]

$$
\begin{aligned}
&P(\hat{t})\left\{\int_{\hat{t}}^{1}(F_\Gamma(t)-\Gamma(t))\,dt+\int_{\hat{t}}^{1}(t-\hat{t})\,d[2F(t)-\Gamma(t)]\right\}\\
&\qquad+\int_{\hat{t}}^{1}\left\{\int_{x}^{1}(F_\Gamma(t)-\Gamma(t))\,dt+\int_{x}^{1}(t-\hat{t})\,d[2F(t)-\Gamma(t)]\right\}dP(\theta(x))\\
&=P(\hat{t})\int_{\hat{t}}^{1}\{1-F_\Lambda(t)\}\,dt\\
&\qquad+\int_{\hat{t}}^{1}\left\{\int_{x}^{1}(1-F_\Lambda(t))\,dt+(x-\hat{t})(1-2F(x)+\Gamma(x))\right\}dP(\theta(x))\\
&=\int_{\hat{t}}^{1}\left\{\frac{-1}{x-\hat{t}}\int_{\hat{t}}^{x}(1-F_\Lambda(t))\,dt+1-2F(x)+\Gamma(x)\right\}(x-\hat{t})\,dP(\theta(x))\\
&=\int_{\hat{t}}^{1}\left\{\frac{1}{x-\hat{t}}\int_{\hat{t}}^{x}F_\Lambda(t)\,dt-2F(x)+\Gamma(x)\right\}(x-\hat{t})\,dP(\theta(x)).
\end{aligned}
$$

The term in brackets is nonpositive because $\Gamma(x)\le 2F(x)-\Lambda(x)\le 2F(x)-(x-\hat{t})^{-1}\times\int_{\hat{t}}^{x}F_\Lambda(t)\,dt$. This implies that a budget constraint also maximizes this third term, so it is optimal for the problem.

The remainder of the proof is exactly as it is in the proof in Section 4.2.1. The spaces are defined the same way, and the conditions hold for analogous reasons.

5. Application: Optimal auctions

Given a description of the bias between the principal's and the agent's payoffs and the principal's information about the agent's type, Propositions 1 and 2 give easily checked conditions that ensure optimality of a budget-constraint contract for the principal. The fact that these conditions do not restrict the mechanism (P, T) that is offered to the principal and agent means that it is straightforward to incorporate this model of budget constraints into a larger mechanism design problem, in which the mechanism is offered to many principal–agent pairs.

I discuss one way to incorporate the budget-constraint model into a larger mechanism design problem. Since the subsequent analysis follows almost immediately from the work of Myerson (1981), I omit formal proofs.

The first step is to extend the above model to allow for many, possibly asymmetric, principal–agent pairs. This extension is based on the model in Burkett (2015). Suppose that principal i (i will index the principal–agent pairs) privately observes a signal, s_i, before deciding on the agent's constraint that determines the distribution F_i. That is, let s_i be distributed on $[0, 1]$ according to some absolutely continuous distribution function $G_i(s_i)$ with density $g_i(s_i)$, and replace the distribution $F(t)$ used above with $F_i(t_i|s_i)$. One can allow the bias to depend on the identity of the pair as well with $\beta_i(t_i)$. If $F_i(t_i|s_i)$ and

[22]I use the same idea as in the proof in Section 4.2.1. The key observation is that $\int_x^1(t-\hat{t})\,d[2F(t)-\Gamma(t)]=\int_x^1(t-x)\,d[2F(t)-\Gamma(t)]+(x-\hat{t})(1-2F(x)+\Gamma(x))=\int_x^1(1-2F(t)+\Gamma(t))\,dt+(x-\hat{t})(1-2F(x)+\Gamma(x))$.

$\beta_i(t_i)$ satisfy Assumption 1 in the money burning case or Assumptions 1, 2, and 3 in the transfer case, for all s_i and all pairs i, then a budget-constraint contract is optimal. With a budget constraint, the agent reports $\min(t_i, \hat{t}_i(s_i))$ to any incentive-compatible and individually rational mechanism, where $\hat{t}_i(s_i)$ is defined as above.[23] I assume for the remainder of this section that the assumptions guaranteeing the optimality of budget constraints hold.

That the optimal auction problem should be standard follows from the observation that to any incentive-compatible and individually rational mechanism, each principal–agent pair reports $\min(t_i, \hat{t}_i(s_i))$. The mechanism design problem is consequently equivalent to a standard problem in which there are no budget constraints and the bidder's types are $v_i \equiv \min(t_i, \hat{t}_i(s_i))$. Define the distribution of v_i to be

$$F_i^v(x) \equiv \Pr(\min(t_i, \hat{t}_i(s_i)) \leq x) = \int_0^1 \int_0^1 \mathbb{1}\{\min(t_i, \hat{t}_i(s_i)) \leq x\} f(t_i|s_i) g(s_i)\, dt_i\, ds_i.$$

This observation implies that the general statement of the revenue equivalence theorem that follows from Myerson (1981) holds here: the expected payment of a bidder is completely determined by the allocation rule and the payment made by the lowest type. It is not true, however, that any two bidders with the same value make the same expected payment, because one may be constrained and the other not (depending on s_i).

The qualitative results of the optimal auction in Myerson (1981) carry over to this model when the bidder valuations are v_i. The revenue-maximizing auction allocates to the bidder with the largest virtual valuation, $v_i - (1 - F_i^v(v_i))/f_i^v(v_i)$, when this virtual valuation is increasing. As with the classic results, when the bidders are symmetric (now the principal–agent pairs need to be symmetric), the revenue-maximizing auction can be implemented as a first- or second-price auction with a reserve price.[24]

This solution to the optimal auction problem with budget-constrained bidders contrasts starkly with the existing literature on the optimal choice of auction with budget-constrained bidders, which considers the design of an optimal auction when the budget constraints are chosen for bidders exogenously (i.e., the distribution of budgets does not depend on the auction rules). In the exogenous budget-constraint setting, revenue can be increased by switching from a second-price to a first-price auction (Che and Gale 1998), and the optimal auction, which resembles an all-pay auction, can be quite complicated, involving pooling at the "top" and in the "middle" (Pai and Vohra 2014). In the setting considered here, that $\hat{t}$ is fixed across auction formats implies that the expected payment made by types $t > \hat{t}$ is determined by $T(\hat{t})$, so the cap on expected payments is endogenously determined by the choice of mechanisms and importantly does not vary across mechanisms with the same $T(\hat{t})$. Burkett (2015) shows that incorporating the budget as an endogenous choice made by a principal restores the revenue equivalence of the first- and second-price auctions, while this paper shows that this result extends to essentially all auctions with the same allocation rule. This section goes further to show

[23]It is now a function of s_i through the dependence of $F_i(t_i|s_i)$ on s_i.

[24]The optimal reserve price in the symmetric case is the solution to $v_i - (1 - F_i^v(v_i))/f_i^v(v_i) = 0$ when the virtual valuation is increasing (Myerson 1981).

that one can apply the logic of Myerson (1981) directly to a mechanism design problem with budget constraints.[25]

6. Conclusion

The simplicity of a budget constraint is highlighted by the fact that the choice of a budget constraint involves deciding on the value of a scalar parameter. This paper gives conditions, which are satisfied by "common" distribution functions, that guarantee this contract is optimal in an infinite-dimensional choice set. Intuitively, the conditions require that there be no abrupt changes in the weighted bias between the principal's and agent's payoffs (i.e., that $f(t)\beta(t)$ not change "too quickly").

The results of Che and Gale (1996, 1998) and subsequent papers suggest that budget constraints cannot be incorporated into auction models by simply redefining the valuation of a bidder as the minimum of the value and a budget. If this were the case, one could appeal to the classic results of auction theory. However, that conclusion is to some extent based on the model of budget constraints that those papers were based on, specifically, the assumption that budget constraints are exogenously specified. The results in Burkett (2015) and this paper show that when budget constraints are set endogenously in the model, there is a way to incorporate them that does not upset the classic auction theory results. Section 5 illustrates this point.

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[25] Of course, if there is reason to believe that budget constraints are exogenous, then the conclusions of the prior literature would apply.

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