Theoretical Economics 11 (2016), 357–380 1555-7561/20160357

Efficient networks in games with local complementarities

Mohamed Belhaj
Centrale Marseille (Aix-Marseille School of Economics), CNRS, and EHESS

Sebastian Bervoets
Aix-Marseille University (Aix-Marseille School of Economics), CNRS, and EHESS

Frédéric Deroïan
Aix-Marseille University (Aix-Marseille School of Economics), CNRS, and EHESS

We address the problem of a planner looking for the efficient network when agents play a network game with local complementarities and links are costly. We show that for general network cost functions, efficient networks belong to the class of *nested split graphs*. Next, we refine our results and find that, depending on the specification of the network cost function, complete networks, core–periphery networks, dominant group architectures, quasi-star networks, and quasi-complete networks can be efficient.

Keywords. Network games, strategic complementarity, nested split graphs.

JEL classification. C72, D85.

1. Introduction

Social networks play an important role in shaping agents' decisions. A natural concern for a planner is thus to find the network structure that maximizes social welfare. We examine this efficiency problem in a two-stage game: first, the planner designs a costly network and second, agents choose an effort level in a game where interactions are linear and neighbors' effort levels are strategic complements. In this setting, equilibrium utilities are proportional to the square of agents' Bonacich centralities (Ballester et al. 2006) and therefore are dependent on the network structure designed in the first stage. The cost of forming links is defined in a general way: agents have an individual linking type, which determines how costly it is to link them to each other, and the network cost is increasing in the sum of the value of all the links in the network. This general formulation includes the standard case of a constant cost per link, as well as many other cases described below.

Mohamed Belhaj: mbelhaj@ec-marseille.fr
Sebastian Bervoets: sebastian.bervoets@univ-amu.fr
Frédéric Deroïan: frederic.deroian@univ-amu.fr
The authors thank the French National Research Agency (ANR) for their support through the program ANR 13 JSH1 0009 01. We would like to thank participants in the CTN conference (January 2012) and in the LAGV conference (June 2012). In particular, we wish to thank N. Allouch, Y. Bramoullé, T. Hiller, M. Jackson, and Y. Zenou for helpful comments.

DOI: 10.3982/TE1742

Our general result (Theorem 1) is that efficient networks belong to the class of *nested split graphs* (NSG). We prove this by identifying a multiple-link reallocation between two agents that increases the sum of utilities. This reallocation, herein called a *neighborhood switch*, consists in deleting the links of the agent with lower centrality and recreating them for the agent with higher centrality. If the network cost increases with this reallocation, we combine it with the permutation of the two agents so as to guarantee that social welfare strictly increases. Our finding complements the results of König et al. (2014), who obtain the NSGs in a decentralized dynamic link formation process where costless linking opportunities appear at random. In other words, not only stable networks,[1] but also efficient networks are NSGs.

Next, we specify various network cost functions and single out specific members of the class of NSGs. We start by considering a case where individuals have the same linking type. When interaction intensity goes to zero, we show that for any network cost function, the efficient network is either empty, a *quasi-star* network, or a *quasi-complete* network (Proposition 1). A quasi-star network is built by forming as many central agents as possible with a given number of links, while a quasi-complete network is built by forming the largest possible complete component. When interaction intensity is general and the network cost is a concave or linear transformation of the number of links, the efficient network is either the empty or the complete network (Proposition 2). Under more general network costs, numerical simulations show that complex NSG structures can be efficient.

We then turn to heterogeneous individual linking types and consider two polar cases. When the cost of a link between two individuals depends on the lower individual linking type of the two, the efficient network is a *core–periphery* network (Proposition 3), which is a generalization of the star network with several central agents. Conversely, when the cost of a link between two agents depends on the higher individual linking type of the two, the efficient network is a *dominant group architecture* (Proposition 4), which is a family of networks consisting of a complete component of any size and isolated agents. Finally, we consider general cases of heterogeneity in individual linking types and illustrate again how efficient NSGs can have complex structures.

Related literature

The issue of finding efficient networks has been analyzed in the context of strategic network formation games. A pioneering literature addressed this question in a setting where agents derive utility from their connections (for seminal contributions, see Jackson and Wolinsky 1996 and Bala and Goyal 2000). However in this paper, agents derive utility from a chosen action.

Some recent papers have addressed efficiency in models of network formation with endogenous choice of action. In the context of research and development (R&D) networks, Goyal and Moraga-González (2001) restrict their attention to *regular* networks (i.e., where agents all have the same number k of links) and derive the number k that

[1]The stability notion we refer to is the one used in König et al. (2014), where agents dynamically revise their linking strategies. We refer the reader to their paper for more details.

maximizes social welfare. In a dynamical setting, König et al. (2012) show that the efficient network structure depends on the marginal cost of collaborations. When cost is high, it is asymmetric and has a nested structure. Westbrock (2010) and Billand et al. (2015) study efficiency in games with global spillovers à la Goyal and Joshi (2003), whereas we consider a network with bilateral influences. In the context of local public goods, Galeotti and Goyal (2010) focus on games where actions are strategic *substitutes* and show that either the star network or the empty network is efficient.

More closely related to our work, two papers analyze efficiency when the planner chooses both the network and the individual effort levels. Hiller (2014) explores nonlinear interactions in a game with pure local complementarities, and finds that the efficient network is either empty or complete, while König (2013) finds that the efficient network has a dominant group architecture in a model with both complements and substitute interactions. Our contribution differs in two principal respects. First, these papers deal with constant cost per link, whereas we use general network cost functions. Second, we assume that effort levels are endogenously chosen by the agents. When the cost per link is constant, we find that the efficient network is either empty or complete (see Proposition 2) and we also show (see Remark 1) that our result extends to the case where the planner additionally chooses effort levels. In König (2013), the substitution effect impedes the building of too large a group, explaining why dominant group architectures emerge. In Proposition 4 we also obtain that efficient networks are dominant group architectures, but the mechanism that leads to this class is different: they result from heterogeneous linking costs instead of global substitutabilities.

The contribution closest to ours is by Corbo et al. (2006), who analyze efficiency in the same game as we do, where the planner only chooses the network. However, they assume that the planner has an exogenous and fixed number of links to arrange and they restrict their attention to *connected graphs*. They show that when the level of interaction tends to its upper bound and when the number of links is equal to $n-1$, the star is the unique connected network maximizing social welfare. Our paper takes the analysis further: the number of links is not restricted to $n-1$, being determined endogenously; we do not restrict to connected graphs; and we also examine every possible level of interaction.

Finally, our finding complements König et al. (2014). In the decentralized framework described in that paper, an agent is picked at random at each step and is offered the opportunity to create a new link, while existing links vanish with time. The authors show that this process leads to the NSG class. This is because an agent's best decision is to create a link to an agent with high centrality, while the links that disappear first are those to agents with low centrality. This mechanism is therefore close to a single-link reallocation, where an agent can both delete and create a link at the same time. However, as we show through an example in Section 3, while a single-link reallocation increases the utility of the agent reallocating the link, it may decrease the sum of utilities due to negative externalities. In the efficiency problem addressed here, the multiple-link reallocation we identify can be thought of as a series of simultaneous single-link reallocations targeted on a specific group of agents. It is both this simultaneity and the targeting of a specific group that guarantee that the sum of utilities will increase. This parallel between the two mechanisms may explain why both stable and efficient networks are NSGs.

The rest of the paper is organized as follows. Section 2 introduces the model. In Section 3 we characterize efficient networks as NSGs, while in Section 4 we specify some cost functions and refine our results. We conclude in Section 5. All proofs can be found in the Appendix.

2. The model

We consider a fixed and finite set of agents $N = \{1, 2, \dots, n\}$ who interact on a network and choose some effort level. An agent's payoff is determined both by his effort and by the effort of the agents he is linked to.

2.1 *The network*

The networks we consider are collections of binary and symmetric relationships, represented by an adjacency matrix $G = (g_{ij})_{i,j\in(1,n)\times(1,n)}$, where $g_{ij} = 1$ when there is a link between agents i and j, and $g_{ij} = 0$ otherwise. By convention $g_{ii} = 0$. By abuse of notation, G will alternatively stand for the network and its adjacency matrix. When $g_{ij} = 1$ (resp. $g_{ij} = 0$) we will say $ij \in G$ (resp. $ij \notin G$). We let $N_i(G) = \{j \in N; g_{ij} = 1\}$ denote the set of neighbors of agent i in network G and we let $\deg(i, G) = \#N_i(G)$ denote the number of these neighbors (the degree).

A component is defined as a set of individuals such that there is a path between every pair of individuals belonging to the component, and there is no path between individuals inside the component and individuals outside the component. A component is said to be nontrivial if it contains strictly more than one agent. Let $\mathcal{G}^n$ denote the set of all networks with n agents and let $\mathcal{G}^n(l)$ denote the set of all networks in $\mathcal{G}^n$ with l links $(0 \le l \le n(n-1)/2)$. Finally denote by $\mu(G)$ the largest eigenvalue (or the *index*) of the adjacency matrix G.

2.2 *Bonacich centralities and social welfare*

Agent i chooses effort level $x_i \in \mathbb{R}_+$. Let $X \in \mathbb{R}^n_+$ be the profile of individual efforts and let x_{-i} denote the profile of the efforts of all agents other than i. We consider a standard linear quadratic utility function with synergies as in Ballester et al. (2006). It is formed of an idiosyncratic component resulting from own effort and a term reflecting strategic complementarities between neighbors.

$$u_i(x_i, x_{-i}, G, \delta) = x_i - \tfrac{1}{2}x_i^2 + \delta \sum_{j=1}^{n} g_{ij} x_i x_j,$$

where $\delta > 0$ measures the intensity of interactions between agents.

A (pure) Nash equilibrium X^* of this game satisfies the first order conditions

$$(I - \delta G)X^* = \mathbf{1}.$$

As shown in Ballester et al. (2006), an equilibrium exists if and only if $\delta\mu(G) < 1$. In that case, the inverse matrix $M = (I - \delta G)^{-1}$ is nonnegative and the linear system admits a unique solution. This solution coincides with the Bonacich centralities of agents (Bonacich 1987), i.e.,

$$X^* = B(G, \delta) = \sum_{k=0}^{+\infty} \delta^k G^k \mathbf{1}.$$

Agent i's Bonacich centrality can be interpreted as the weighted sum of paths of any length starting from agent i in network G. By denoting $B(G, \delta) = (b_i(G, \delta))_{i \in N}$, we have

$$b_i(G, \delta) = \sum_{k=0}^{+\infty} \delta^k P_k(i, G),$$

where $P_k(i, G)$ is the number of paths of length k, including loops, starting from agent i (formally, $P_k(i, G)$ is the ith component of $G^k\mathbf{1}$). Given the linear quadratic specification under consideration, agents' utility at equilibrium is given by

$$u_i(X^*, G, \delta) = \tfrac{1}{2} b_i^2(G, \delta).$$

We address the problem of a social planner looking for the network maximizing social welfare. When doing so, we implicitly assume that on any given network G and for any given level of interactions δ, agents exert their equilibrium effort. Because of the uniqueness of the equilibrium efforts, we drop X^* from the arguments of the utility function ($u_i(G, \delta) \equiv u_i(X^*, G, \delta)$). To guarantee that equilibrium exists on every network, we impose $\delta \in [0, 1/(n-1)[$.[2]

We assume that the total cost of a network (called the *network cost*) is given by

$$\Phi\left(\sum_{i,j \in N} g_{ij} \cdot c_{ij}\right),$$

where $\Phi(\cdot)$ is an increasing function, with $\Phi(0) = 0$, and where c_{ij} is a positive number characterizing the link between agents i and j, defined as a function of i and j's *individual linking types*: let $C = (c_i)_{i \in N}$ be a vector in $\mathbb{R}_+^n$ of *individual linking types*; then $c_{ij} = f(c_i, c_j)$, where $f(\cdot, \cdot)$ is positive, increasing in both arguments, and symmetric, i.e., $f(c_i, c_j) = f(c_j, c_i)$.

Individual linking types may reflect individuals' capacity for social life, ability to communicate with peers, etc. In turn, c_{ij} can be interpreted as the cost of forming a link between agents i and j when $\Phi(x) = x$. More general formulations of $\Phi(\cdot)$ indicate that the contribution of a link to the network cost depends on the structure of the current network.

[2]Indeed, $1/(n-1)$ is, among all possible networks with n agents, the largest possible index. It is associated with the complete network. If $\delta \geq 1/(n-1)$, the equilibrium efforts in the complete network are infinite, so that the problem at hand is trivial.

This formulation covers numerous situations. It includes the standard case where the network cost is proportional to the number of links it contains when $\Phi(x) = x$ and $c_{ij} = c$ for all i and j. Then the cost of a network with l links is simply $c \cdot l$.

It also allows for increasing transformations of the number of links (in some contexts any additional link may be more costly—or conversely, less costly—than the previous link), and it allows for more general situations where c_{ij} may be heterogeneous, such as $c_{ij} = c_i + c_j$, $c_{ij} = c_i \cdot c_j$, $c_{ij} = \mathrm{Min}\{c_i; c_j\}$, $c_{ij} = \mathrm{Max}\{c_i; c_j\}$, etc.

We define social welfare as a function of network, of interaction level, and of network cost:

$$\begin{aligned} W(G, \delta, C) &= \sum_{i \in N} u_i(G, \delta) - \Phi\Bigl(\sum_{i,j \in N} g_{ij} \cdot c_{ij}\Bigr) \\ &= \tfrac{1}{2} \sum_{i \in N} b_i^2(G, \delta) - \Phi\Bigl(\sum_{i,j \in N} g_{ij} \cdot c_{ij}\Bigr). \end{aligned} \tag{1}$$

A network $G \in \mathcal{G}^n$ is efficient whenever

$$W(G, \delta, C) \geq W(G', \delta, C) \quad \text{for all } G' \in \mathcal{G}^n.$$

Remark 1. We show in the Appendix that in a game where the planner chooses both the network and the effort levels, the sum of utilities is proportional to the sum of equilibrium efforts in a game with interaction intensity 2δ. Our proofs concerning the maximization of the sum of equilibrium utilities also cover the maximization of the sum of equilibrium efforts. Thus, all our results apply to this alternative problem.

2.3 *Some specific network structures*

Before turning to the analysis, we present some network structures that will play a prominent role: the class of nested split graphs (NSGs). NSGs were first introduced in graph theory by Chvátal and Hammer (1977) as *threshold graphs*. They propose several equivalent definitions, among which is the following.

Definition 1 (Nested split graph). A graph G is called a nested split graph if

$$[ij \in G \text{ and } \deg(k, G) \geq \deg(j, G)] \implies ik \in G.$$

This definition, while not standard in the graph theory literature, will be useful in our context. A nonempty NSG is a network with one nontrivial component, in which agents' neighborhoods are nested. A nonempty NSG can also have additional isolated agents. Agents in the nontrivial component can be partitioned into p classes, where agents in the same class have the same degree and agents in class i are linked to every agent in classes 1 to $p - i + 1$.[3]

[3]By convention we assume that isolated agents do not form a class of the NSG, as they will play a limited role in our analysis.

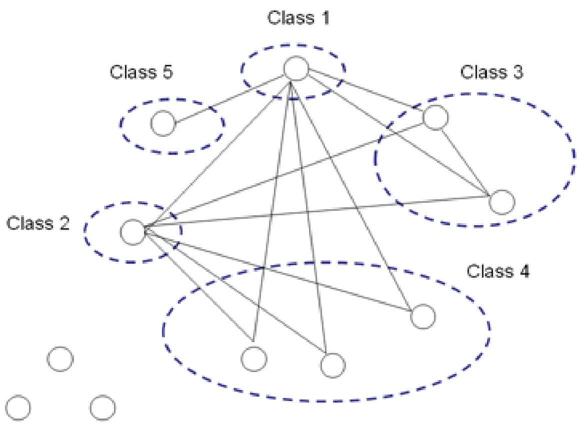

FIGURE 1. NSG with five classes and three isolated agents; $n = 11$, $l = 13$.

Out of their many interesting properties (see, for instance, König et al. (2014) for an extensive analysis of the properties of NSGs), the following four properties are particularly worthy of note. First, the nontrivial component of the NSG is of diameter 2. Second, agents are ordered by degree. Class 1 agents have degree $n_C - 1$ (where n_C is the size of the nontrivial component) and as the class index increases, degree strictly decreases. Thus in an NSG, $\deg(i, G) > \deg(j, G)$ implies that i is in a higher class than j. Third, the set of agents belonging to the first $E[(p+1)/2]$ classes forms a complete subgraph (also called a clique). Last, for every interaction level δ, the Bonacich centrality ranking of agents is aligned with their degree ranking. Figure 1 presents an NSG with 11 agents and 5 classes (dotted circles represent classes).

We now present some members of the class of NSGs that will play a role in the rest of the analysis. In addition to the standard complete network (a single class of agents with no isolated agents) or the star network (two classes, the central agent in class 1 and the peripherals in class 2), the NSG class contains other prominent networks. We present four subclasses of interest.

The *core–periphery networks* are a generalization of the star network with several central players. They are defined and explored in Galeotti and Goyal (2010) among others. We provide the following equivalent definition.

DEFINITION 2 (Core–periphery networks). A network is a core–periphery if and only if it is an NSG with at most two classes of agents and no isolated agents.

Another prominent subclass of NSGs is the *dominant group architecture*, introduced by Goyal and Joshi (2003), which consists of a complete component and isolated agents. We provide the following definition in terms of classes.

DEFINITION 3 (Dominant group architecture). A network is a dominant group architecture if and only if it is an NSG with one class of agents and possibly some isolated agents.

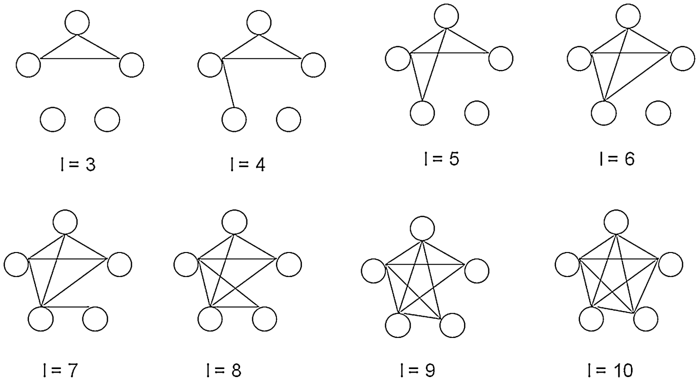

FIGURE 2. Quasi-complete graphs with $n = 5$.

So as to define the next member of the class, let K_p denote a complete subgraph with p agents.

DEFINITION 4 (Quasi-complete graph). A graph $G \in \mathcal{G}^n(l)$ is called a quasi-complete graph, denoted $QC(l)$, if it contains the complete subgraph K_p with $p(p-1)/2 \leq l < p(p+1)/2$, and the remaining $l - p(p-1)/2$ links are set between one other agent and agents in K_p.

A quasi-complete graph is an NSG with either a single class of agents when $l = p(p-1)/2$, or with three. Figure 2 presents some quasi-complete graphs. Note that $\mathrm{QC}(3) = K_3$, $\mathrm{QC}(6) = K_4$, and $\mathrm{QC}(10) = K_5$.

DEFINITION 5 (Quasi-star graph). A graph $G \in \mathcal{G}^n(l)$ is called a quasi-star graph, denoted $\mathrm{QS}(l)$, if it has a set of p central agents with $n-1$ links, and the remaining $l - p(n-1)$ links are set so as to construct another central agent.

A quasi-star graph contains one class of agents if it is the complete network; otherwise it contains either two, three, or four classes of agents. Figure 3 presents some quasi-star graphs, where, for instance, $\mathrm{QS}(4)$ contains two classes, $\mathrm{QS}(5)$ contains three classes, and $\mathrm{QS}(6)$ contains four classes.

3. EFFICIENT NETWORKS

First we identify a specific procedure of link reallocation that increases the sum of utilities without changing the number of links. A natural way to tackle our problem would be to delete an existing link between a pair of agents i and j in the network and replace it by a link between agent i and another agent k that has a higher Bonacich centrality than agent j. Unfortunately, these intuitive single-link reallocations may not be enough to increase social welfare. This is illustrated in Figure 4, where we present a network that is never the efficient network, whatever the cost function, but where such a link reallocation would result in a strict decrease of social welfare. In this example,

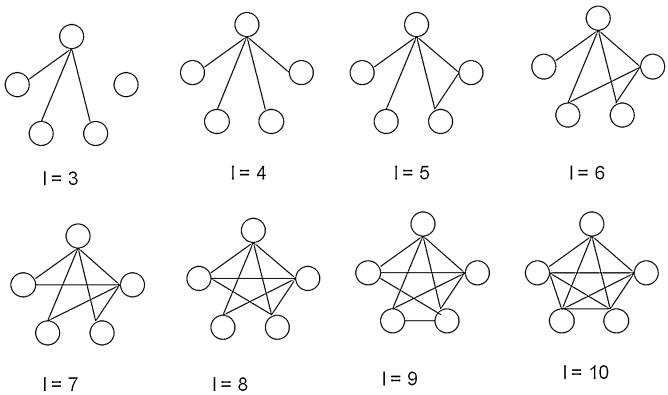

FIGURE 3. Quasi-star graphs with $n = 5$.

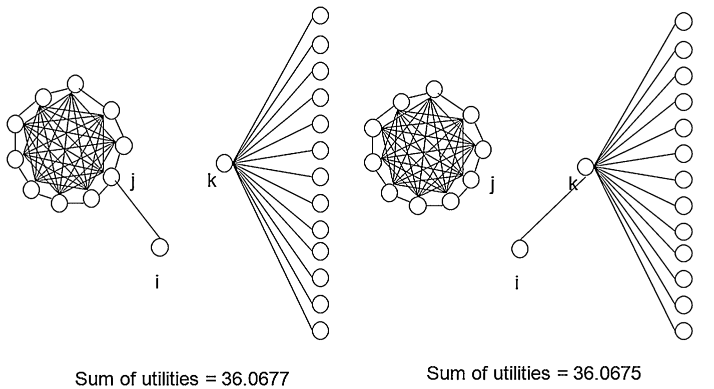

FIGURE 4. A single-link reallocation that decreases social welfare. An example with $n = 25$, $l = 59$, and $\delta = 0.03$.

$b_k(G, \delta) - b_j(G, \delta) = 0.005$ for $\delta = 0.03$. A single-link reallocation from ij to ik decreases social welfare. This is due to the fact that after the reallocation, although all the agents in the star are better off, all the agents in the complete component are less well off. Despite the increased sum of efforts after the reallocation, in this example the losses in terms of utility are not compensated by the gains.

We thus investigate specific multiple-link reallocations: Consider agents j and k, and define $N_{j\setminus k}(G)$ as the set of neighbors of j who are not neighbors of k: $N_{j\setminus k}(G) = \{i \in N; g_{ij} = 1 \text{ and } g_{ik} = 0\}$. We define a *neighborhood switch from j to k*, called hereafter an $N_{(j,k)}$ switch, as a reallocation of the links between j and $N_{j\setminus k}(G)$ to links between k and $N_{j\setminus k}(G)$.

Let $A_l^{j\setminus k} = (a_{im})_{i,m\in(1,n)\times(1,n)}$ be such that $a_{im} = a_{mi} = 1$ if $i \in N_{j\setminus k}(G)$ and $m = l$, $a_{im} = 0$ otherwise. This matrix contains a 1 between agent l and all of agent j's neighbors who are not neighbors of agent k, and it contains 0 otherwise.

DEFINITION 6 ($N_{(j,k)}$ switch). Consider a network G. An $N_{(j,k)}$ switch is a multiple-link reallocation leading to the network G', where $G' = G + A_k^{j\setminus k} - A_j^{j\setminus k}$.

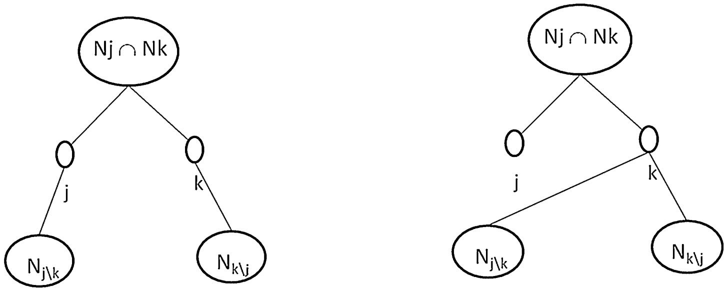

FIGURE 5. An $N_{(j,k)}$ switch.

Figure 5 illustrates an $N_{(j,k)}$ switch. Note that agent j could be isolated after the switch. An N switch has the following interesting properties in terms of aggregate utilities.

LEMMA 1. *Consider a network G with agents j and k such that $b_j(G,\delta) \leq b_k(G,\delta)$ and $N_{j\setminus k} \neq \varnothing$. Let $G' = G + A_k^{j\setminus k} - A_j^{j\setminus k}$. Then for any $1/(n-1) > \delta > 0$, $\sum_{i\in N} u_i(G',\delta) > \sum_{i\in N} u_i(G,\delta)$.*

To prove Lemma 1, we first determine the equilibrium efforts in the network before the $N_{(j,k)}$ switch. Then we implement the $N_{(j,k)}$ switch and we track how equilibrium efforts are modified. The special feature of this type of reallocation is that the only agent who may suffer from the reallocation is j, while all others benefit from k's higher centrality. This is its main difference from the single-link reallocation illustrated above, where all of j's neighbors may also suffer from j's decreased centrality. The use of a simultaneous best-response algorithm (SBRA) then allows us to conclude that the aggregate gains of others are higher than j's loss.

Lemma 1 identifies an operation that increases the sum of utilities. At the same time, the $N_{(j,k)}$ switch also changes the network cost. If $c_k \leq c_j$, the switch decreases the network cost, while it could increase it if $c_j < c_k$. In that case, we resort to a permutation of agents j and k at the same time as we implement the $N_{(j,k)}$ *switch*, to guarantee that the network cost does not increase. We can now state our main result.

THEOREM 1. *An efficient network G is a nested split graph. Moreover, if $c_i < c_j$, then* $\deg(i,G) \geq \deg(j,G)$.

The proof of this theorem heavily relies on Lemma 1 and on the very fact that the only networks with no N switches are the nested split graphs.

Complementarities, together with the convexity of equilibrium utilities, encourage accumulation around a subset of agents. This is what NSGs do, because agents in the first class of an NSG are connected to everyone else in the single component, turning them into (partially) central agents. This accumulation leads to very high centrality for these agents and to short distances that will guarantee strong feedback effects on other agents. Furthermore, the agents with the lowest individual linking types are more central because they have more links. Indeed, in NSGs, centrality and degree coincide.

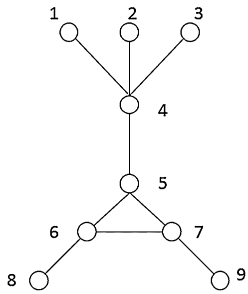

Figure 6. N-switches and key player policies.

Remark 2. Because N switches potentially exclude agents from the network, they might bring to mind key player policies (see Ballester et al. 2006). In our setting, there are two possible analogies with the key player. First, the key player could be the agent contributing most to aggregate efforts, and as such, should be the agent receiving the reallocated links. Alternatively, we could define the "inverse key player" as the player contributing least to aggregate efforts and consider that this agent should be the one excluded from the network.

However, we show with the counterexample in Figure 6 that neither of these analogies is correct. This network has 9 agents and 9 links. For $\delta = 0.19$, the player contributing most (the key player) is agent 5, while the players contributing least (the inverse key players) are agents 8 and 9. However, the best N switch to be implemented on this network is the $N_{(5,4)}$ switch (consisting of reallocating links 56 and 57 to 46 and 47). Thus, on the one hand it is better to maintain the agents contributing least in the network (agents 8 and 9) rather than disconnecting them, while on the other hand it is better to partly disconnect the key player (agent 5) despite his high contribution to aggregate outcomes.

The problem of discriminating between different NSGs so as to find the efficient network is difficult to tackle analytically, principally because in an NSG, no link reallocations are possible from an agent with low centrality to another with higher centrality. This implies that any improvement on a given NSG is obtained by reallocating links toward agents with lower centralities. However, we are able to refine our results with specific cost functions.

4. Specific cost functions

In this section, we examine some cost function specifications that include standard cases. Before proceeding, we present a process of adding links that guarantees that the gains in aggregate utility are increasing as links are added.

Consider an arbitrary network G. Let G^- be the network in which links from k to $N_{k\setminus j}(G)$ are severed, together with the links from j to $N_{j\setminus k}(G)$: $G^- = G - A_k^{k\setminus j} - A_j^{j\setminus k}$; let G^+ be the network in which links from j to $N_{k\setminus j}(G)$ are added, together with the links from k to $N_{j\setminus k}(G)$: $G^+ = G + A_j^{k\setminus j} + A_k^{j\setminus k}$. The three different networks are illustrated in Figure 7.

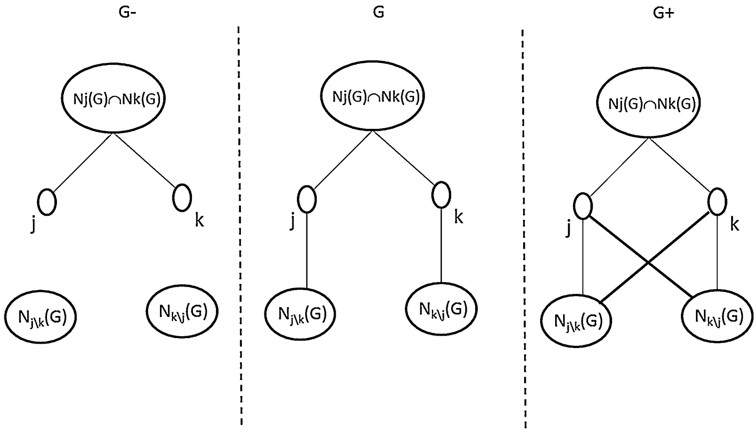

Figure 7. Networks G^- and G^+ are, respectively, left and right of the initial network G.

We have the following lemma.

Lemma 2. *If* $N_{k\setminus j}(G) \neq \varnothing$, *then*

$$\sum_{i\in N}(u_i(G^+,\delta)-u_i(G,\delta)) > \sum_{i\in N}(u_i(G,\delta)-u_i(G^-,\delta)).$$

To prove our result, we decompose the effect of both link additions (from G^- to G and from G to G^+) on agents j and k on the one hand and on the other agents on the other hand. The effect on the other agents is unambiguous: the second link addition increases their utilities more than the first link addition does. For agents j and k, the effects are not straightforward. We show that the effect on agent k's utility (respectively, j's utility) of the second link addition is greater than the effect of the first link addition on agent j's utility (respectively, k's utility).[4]

4.1 *Homogeneous individual linking types*

Assume $c_{ij} = c$ for any pair of agents i and j. Before turning to general levels of interaction intensity, we start by examining the case of low levels.

- *Low levels of interaction*

When interactions go to 0, the effects of complementarities vanish faster as they transit along longer paths. Short paths contribute more to centralities and the efficient networks are those maximizing the number of short paths.

Proposition 1. *Assume* $c_{ij} = c$ *for every* i *and* j. *When* δ *tends to* 0, *the network maximizing social welfare is either empty, a quasi-star network, or a quasi-complete network.*

[4]The process just considered consists of adding several links at once. This is a necessary condition. Indeed, we have constructed an example in which the addition of any single link increases aggregate utility less than the contribution of any existing link. The example is available upon request.

The sketch of the proof goes as follow. First, we note that when δ goes to 0, the problems of maximizing the sum of the Bonacich centralities or the sum of their squares are equivalent. Then we decompose the sum of Bonacich centralities into the weighted sum of paths of lengths 0, 1, and 2 and the weighted sum of all paths of length greater than 3. We show that when δ goes to 0, this last sum is negligible. We are then left with three terms, of which the first two are shown to be constant over the whole set of networks with n agents and a fixed number l of links. Finding the network maximizing the sum of Bonacich centralities thus boils down to finding the network that maximizes the number of paths of length 2. Observing that the number of such paths is equal to the sum of squares of the degrees in a network, we rely on Ábrego et al. (2009), who identify $\mathrm{QS}(l)$ and $\mathrm{QC}(l)$ as well as other networks identified for a few very specific values of n and l. All the networks belong to the NSG class. Where there is equality between different networks, we turn to paths of length 3 to discriminate. All the networks they identify are beaten at the third order either by $\mathrm{QS}(l)$ or $\mathrm{QC}(l)$.

Exploiting the results of Ábrego et al. (2009), we can go a little further: if $l \le \frac{1}{2}\binom{n}{2} - \frac{1}{2}n$, the network with l links that maximizes social welfare is $\mathrm{QS}(l)$, while if $l \ge \frac{1}{2}\binom{n}{2} + \frac{1}{2}n$, it is $\mathrm{QC}(l)$. This implies that the efficient network's structure is not unique, as it depends on the optimal number of links. When this number is low, $\mathrm{QS}(l)$ performs better, whereas $\mathrm{QC}(l)$ performs better when this number is high. This condition reveals the trade-off between having a hub linking a large number of agents, all of whom transit through the hub to generate many short paths with few links, or building many triangles with a complete subgraph.

Interestingly, the principles behind accumulation of links on a subset of agents is still what drives the result, but we clearly see here that there are two typical ways of accumulating. Accumulation can either be built around the largest possible subset of agents (as in $\mathrm{QC}(l)$) or it can be achieved by increasing the centrality of one central agent as much as possible before trying to include another central agent (as in $\mathrm{QS}(l)$). Which type of accumulation is best depends on the number of links in the network.

- *General levels of interaction*

For general levels of interaction, which network is efficient depends on whether $\Phi(\cdot)$ is concave, linear, or convex. We start with the concave and linear cases.

Proposition 2. *Assume $c_{ij} = c$ for every i and j and that $\Phi(\cdot)$ is concave or linear. If $n/(2(1-\delta(n-1))^2) > \Phi(\frac{1}{2}n(n-1)\cdot c)$, then the efficient network is the complete network, otherwise it is the empty network.*

Remark 3. In the alternative problem where effort levels are chosen by the planner, the above condition becomes $n/(1-2\delta(n-1)) > \Phi(\frac{1}{2}n(n-1)\cdot c)$.

When the cost function is concave or linear, it can be shown that the link contributing most to social welfare is the last link that is added when the complete network is formed. This is no longer true if $\Phi(\cdot)$ is not concave. In that case, accumulating links until the complete network is reached may not be efficient because the last links could

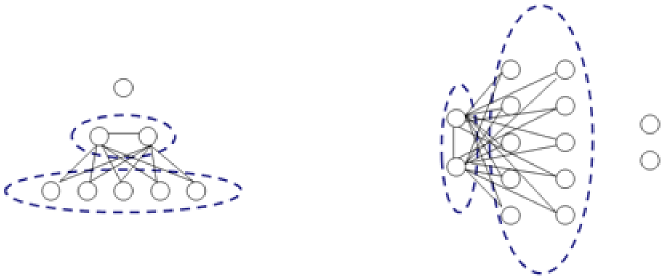

FIGURE 8. QS-like networks.

be too costly. To find the efficient network for a given level of interactions, the number of links has to be fixed and the corresponding network maximizing aggregate utilities have to be found. Then the social welfare for every possible number of links can be compared, identifying the best number of links l^*, together with the corresponding network structure.

However, the problem of finding, for a fixed number of links, the network maximizing the sum of utilities as a function of δ remains unsolved. We ran simulations to compare nested split graphs and check whether any pattern emerged. We tested for values of n between 3 and 24, and for every n, we varied l from 3 to $\frac{1}{2}n(n-1)$. For every n and l pair, we simulated 10^4 values of δ. Once n, l, and δ were fixed, we computed the sum of utilities in every possible nested split graph so as to find the best one.[5]

Our simulations reveal two main features. First, for a low density of links, only two networks appear to be candidates for efficiency: QC(l) and QS(l). The quasi-star network emerges as best for low values of δ, while the quasi-complete network is best for high values. In turn, when the density of links is high, the QC(l) appears to be the best, whatever the value of δ.

Second, for intermediate densities, other complex NSGs can be efficient. We illustrate in Figures 8–10 some typical structures that might emerge. Figure 8 shows two examples of "QS-like" networks formed of isolated agents and a quasi-star network with the remaining agents. The one on the left ($n = 8$) is efficient for $\delta \in [0.001, 0.025]$ when the network cost function $\Phi(\cdot)$ is such that the optimal number of links l^* is 11. The one on the right is efficient for $n = 14$, $l^* = 21$, and $\delta \in [0.029, 0.038]$. Figure 9 shows structures that are hybrids of QC and QS for $n = 9$ and $l^* = 11$ (efficient for $\delta \in [0.001, 0.04]$) as well as for $n = 14$ and $l^* = 39$ (efficient for $\delta \in [0.001, 0.011]$). We also found other structures that do not fit into the first two categories. In Figure 10, we illustrate two examples, for $n = 11$ and $l^* = 25$ (efficient for $\delta \in [0.001, 0.012]$) and for $n = 18$ and $l^* = 30$ (efficient for $\delta \in [0.021, 0.053]$).

[5]The generation of all NSGs is made possible by a mapping between the set of NSGs with n agents and the integers between 0 and 2^{n-1}, according to the following rule: every agent is a bit in a binary number, with state either 0 or 1. Every agent in state 1 is linked to all his predecessors while agents in state 0 are not. The last agent, having no predecessor, does not count. For instance, with $n = 4$, all NSGs are mapped by numbers from 0 to 7, with respective binary sequences 000 to 111. Sequence 000 is the empty network, sequence 111 is the complete network, sequence 001 is the star, and sequence 101, for instance, is the kite. The reader interested in the computational details of constructing NSGs can refer to Hagberg et al. (2006).

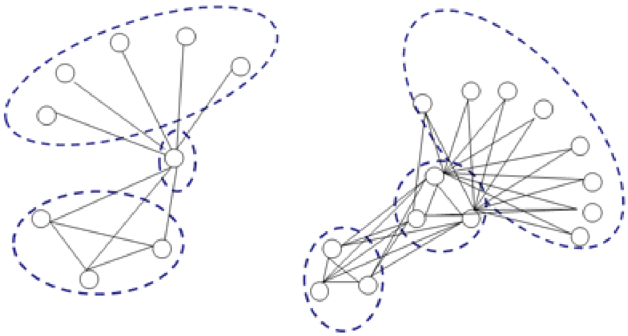

FIGURE 9. Hybrids.

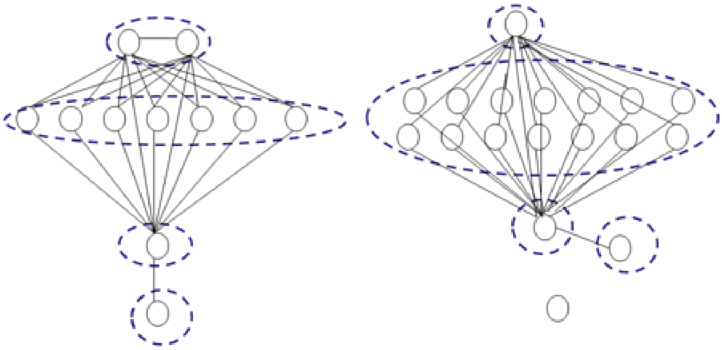

FIGURE 10. Other networks.

4.2 *Heterogeneous individual linking types with linear network cost function*

In this part, we assume that $\Phi(x) = x$ and allow for heterogeneity across individual linking types. We start by examining two polar cases where $c_{ij} \equiv \text{Min}\{c_i; c_j\}$ and $c_{ij} \equiv \text{Max}\{c_i; c_j\}$.

PROPOSITION 3. *Suppose that* $\Phi(x) = x$ *and* $c_{ij} \equiv \text{Min}\{c_i; c_j\}$. *The efficient network is a core–periphery network.*

This result drastically reduces the class of efficient networks to NSGs with one or two classes and no isolated agents. The intuition of this result relies on the observation that as soon as one agent is connected to another, any other agent should be connected to the least costly of these two agents.

PROPOSITION 4. *Suppose that* $\Phi(x) = x$ *and* $c_{ij} \equiv \text{Max}\{c_i; c_j\}$. *The efficient network is a dominant group architecture.*

The difference from the Min function is the following: When a pair is formed, every agent with an individual linking type lower than the most costly of the two agents in the pair should be connected to him. But he should also be connected to the other member of the pair, who has a lower linking type. This is why the component is complete and the NSG only contains one class. Contrary to the previous case, we can end up with some isolated agents, those with too high an individual linking type.

Remark 4. Assume that $\Phi(x) = x$ and $c_{ij} = \alpha c_i + \beta c_j$, where $\alpha, \beta \geq 0$. Propositions 3 and 4 combined give us some bounds on the efficient network: it contains the dominant group architecture maximizing social welfare when the cost function is $c_{ij} = (\alpha + \beta)\,\mathrm{Max}\{c_i; c_j\}$ and it is contained by the core–periphery network maximizing social welfare when the cost function is $c_{ij} = (\alpha + \beta)\,\mathrm{Min}\{c_i; c_j\}$.

With other cost functions, NSGs with more than two classes can be efficient, for the same reasons as with a convex function $\Phi(\cdot)$: some links may be worth creating while others may not. Indeed, assume, for instance, that $c_{ij} = c_i \cdot c_j$ and take as an example the complex network on the left side of Figure 10, with 11 agents and 4 classes of agent. By setting $c_1 = 0.001$, $c_2 = c_3 = 0.1$, $c_4 = \cdots = c_{10} = 0.3$, and $c_{11} = 10$, the four-class NSG is efficient. Accordingly, the same network would be efficient if the linking type function were $c_{ij} = c_i + c_j$ and we set $c_1 = 0.001$, $c_2 = c_3 = 0.015$, $c_4 = \cdots = c_{10} = 0.03$, and $c_{11} = 0.045$.

5. Conclusion

We have examined the problem of finding the efficient structure in a network game where interactions are linear, neighbors' efforts are pure strategic complements, and network formation is costly. We focus on the role played by the class of nested split graphs and some specific members of this class. This work complements König et al. (2014), who examine strategic network formation in a dynamic setting and show that stable networks are NSGs.

This analysis could be extended to more general utility functions. Indeed, one can show that the N switch increases the sum of utilities as soon as utilities are convex in Bonacich centralities. However, not much is known for more general utility functions. It would therefore be interesting to characterize the class of network games for which efficient networks are NSGs.

Appendix: Proofs

Proof of Remark 1. Once a network G is fixed, the planner maximizes

$$\sum_{i\in N} u_i(X, G, \delta) = \sum_{i\in N}\left(x_i - \tfrac{1}{2}x_i^2\right) + \sum_{i\in N}\delta \sum_{j\in N} g_{ij}x_i x_j. \tag{2}$$

The first order conditions (FOCs) give

$$x_i = 1 + 2\delta \sum_{j\in N} g_{ij}x_j$$

and the solution $\hat{X}$ is given by

$$\hat{x}_i = b_i(G, 2\delta),$$

which is equal to the equilibrium efforts in the game with interaction intensity $\delta' = 2\delta$. Plugging the FOC into (2), we get

$$\begin{aligned}\sum_{i\in N} u_i(\hat{X}, G, \delta) &= \sum_{i\in N} \hat{x}_i\Bigl(1 - \tfrac{1}{2}\hat{x}_i + \delta \sum_{j\in N} g_{ij}\hat{x}_j\Bigr) \\ &= \tfrac{1}{2}\sum_{i\in N} \hat{x}_i.\end{aligned}$$

Thus a planner choosing both efforts and the network is maximizing

$$V(G, \delta, C) = \tfrac{1}{2}\sum_{i\in N} b_i(G, 2\delta) - \Phi\Bigl(\sum_{i,j\in N} g_{ij}\cdot c_{ij}\Bigr). \tag{3}$$

Note that (3) is obtained by replacing $b_i^2(G, \delta)$ by $b_i(G, 2\delta)$ in (1). □

Proof of Lemma 1. We show that following an $N_{(j,k)}$ switch, a sequence of simultaneous myopic individual best responses leads to an increase of both the sum of efforts and the sum of utilities. Consider an initial network G with equilibrium efforts $X^*(G)$. There are two cases.

Case 1: $jk \notin G$. We modify network G by implementing an $N_{(j,k)}$ switch so as to obtain the network $G' = G + A_k^{j\backslash k} - A_j^{j\backslash k}$ and initiate a simultaneous best-response algorithm (SBRA) on the modified network G'. We denote by $X^{(t)}$ the vector of efforts of agents at the end of period t. We start with initial conditions $X^{(0)} = X^*(G)$ that satisfy

$$\begin{cases} x_j^{(0)} = 1 + \delta \sum_{c\in N_j(G)\cap N_k(G)} x_c^{(0)} + \delta \sum_{s\in N_{j\backslash k}(G)} x_s^{(0)} \\ x_k^{(0)} = 1 + \delta \sum_{c\in N_j(G)\cap N_k(G)} x_c^{(0)} + \delta \sum_{p\in N_{k\backslash j}(G)} x_p^{(0)}, \end{cases}$$

and agent i's best-response updating process at period $t+1$ is given by $x_i^{(t+1)} = 1 + \delta \sum_{j\in N_i(G)} x_j^{(t)}$.

This SBRA will converge to the unique effort equilibrium on the network G' by standard contraction properties (see, for instance, Milgrom and Roberts 1990, for convergence in games with strategic complementarities).

At step 1 of the algorithm we get

$$\begin{cases} x_j^{(1)} = x_j^{(0)} - \delta \sum_{s\in N_{j\backslash k}(G)} x_s^{(0)} \\ x_k^{(1)} = x_k^{(0)} + \delta \sum_{s\in N_{j\backslash k}(G)} x_s^{(0)} \\ x_s^{(1)} = x_s^{(0)} + \delta(x_k^{(0)} - x_j^{(0)}) \quad \text{for all } s \in N_{j\backslash k}(G) \\ x_q^{(1)} = x_q^{(0)} \quad \text{for all } q \neq j, k;\ q \notin N_{j\backslash k}(G). \end{cases}$$

Since, by hypothesis, $x_k^{(0)} \geq x_j^{(0)}$, all efforts weakly increase except that of agent j. But agent k's increase compensates agent j's decrease. Further, utilities being quadratic in effort, the increase in utility of agent k is larger than the decrease in utility of agent j by convexity (given that $x_k^{(0)} \geq x_j^{(0)}$). It follows that the sum of utilities at $X^{(1)}$ is greater than at $X^{(0)}$. However, agent j's loss could have feedback effects on other agents in future steps of the SBRA. We examine step 2:

$$\begin{cases} x_j^{(2)} = x_j^{(1)} \\ x_k^{(2)} = x_k^{(1)} + \delta \sum_{s \in N_{j\setminus k}(G)} (x_s^{(1)} - x_s^{(0)}) \\ x_s^{(2)} = x_s^{(1)} + \delta (x_k^{(1)} - x_k^{(0)}) \\ x_q^{(2)} \geq x_q^{(1)} \quad \text{for all } q \neq j, k;\, q \notin N_{j\setminus k}(G) \end{cases}$$

with $x_q^{(2)} \geq x_q^{(1)}$ because of complementarities. Therefore, $X^{(2)} \geq X^{(1)}$. By complementarities, from step 2 onward, the efforts will increase at each step of the SBRA and converge to $X^{(\infty)} = X^*(G') \geq X^{(1)}$.

Case 2: $jk \in G$. The process needs to be decomposed into two sequential SBRA on network G'. First, we take $X^{(0)} = X^*(G)$ as the initial efforts on G', and we restrict the SBRA to agents j and k, keeping all other efforts fixed. This process converges to $X^{(\infty)}$, where only agents j and k have changed their effort. Second, we take $Y^{(0)} = X^{(\infty)}$ as the initial efforts and apply a SBRA to all agents. This process will converge to $Y^{(\infty)} = X^*(G')$, which is the equilibrium in the modified network.

As $jk \in G$, $X^{(0)}$ now satisfies

$$\begin{cases} x_j^{(0)} = 1 + \delta \sum_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum_{s \in N_{j\setminus k}(G)} x_s^{(0)} + \delta x_k^{(0)} \\ x_k^{(0)} = 1 + \delta \sum_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum_{p \in N_{k\setminus j}(G)} x_p^{(0)} + \delta x_j^{(0)}. \end{cases} \tag{4}$$

Thus we get

$$\begin{cases} y_j^{(0)} = 1 + \delta \sum_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta y_k^{(0)} \\ y_k^{(0)} = 1 + \delta \sum_{c \in N_j(G) \cap N_k(G)} x_c^{(0)} + \delta \sum_{s \in N_{j\setminus k}(G)} x_s^{(0)} + \delta \sum_{p \in N_{k\setminus j}(G)} x_p^{(0)} + \delta y_j^{(0)}. \end{cases} \tag{5}$$

Using (4) and (5), we find

$$x_j^{(0)} + x_k^{(0)} = y_j^{(0)} + y_k^{(0)}. \tag{6}$$

Noticing by (4) and (5) that $y_k^{(0)} - x_k^{(0)} = \delta \sum_{s \in N_{j\setminus k}(G)} x_s^{(0)} + \delta (y_j^{(0)} - x_j^{(0)})$ and by (6), that $y_k^{(0)} - x_k^{(0)} = x_j^{(0)} - y_j^{(0)}$, we get

$$y_k^{(0)} - x_k^{(0)} = \frac{\delta}{1+\delta} \sum_{s \in N_{j\setminus k}(G)} x_s^{(0)} > 0.$$

Altogether, the first SBRA, restricted to agents j and k, leads to

$$\begin{cases} y_j^{(0)} + y_k^{(0)} = x_j^{(0)} + x_k^{(0)} \\ y_k^{(0)} > x_k^{(0)}. \end{cases}$$

Now apply a second SBRA on network G', with $Y^{(0)}$ as initial efforts. Let us describe the modified efforts after the first step for every type of agent. Agents in $N_{j\setminus k}(G)$ increase their effort because $y_k^{(0)} > x_j^{(0)}$. Agents in $N_{k\setminus j}(G)$ also increase their effort because $y_k^{(0)} > x_k^{(0)}$. Agents in $N_j(G) \cap N_k(G)$ do not modify their effort because $x_j^{(0)} + x_k^{(0)} = y_j^{(0)} + y_k^{(0)}$. Finally, agents j and k do not modify their effort because they are at the equilibrium effort level of the first SBRA.

At the end of the first step, $Y^{(1)} \geq Y^{(0)}$, and by complementarities, $X^*(G') = Y^{(\infty)} \geq Y^{(0)}$. In profile $Y^{(0)}$, the sum of utilities exceeds the sum of utilities in profile $X^{(0)}$; therefore our conclusion holds. □

Proof of Theorem 1. We first need to prove a lemma that uses the following definition.

Definition 7. Let $G_{(j,k)} = G + A_k^{j\setminus k} - A_j^{j\setminus k} + A_j^{k\setminus j} - A_k^{k\setminus j}$ denote the network G in which agents j and k are permuted.

Lemma 3. *Consider a network G and agents j and k such that $b_j(G,\delta) \leq b_k(G,\delta)$ and $N_{j\setminus k} \neq \varnothing$. If $c_k \leq c_j$, the $N_{(j,k)}$ switch strictly increases social welfare. If $c_j < c_k$, then permuting agents j and k and implementing the $N_{(k,j)}$ switch on network $G_{(j,k)}$ strictly increases social welfare.*

Note that when $c_j < c_k$, the N switch and the agent permutation have to be implemented together. Indeed, either the permutation alone or the switch alone might result in increased network cost.

Proof of Lemma 3. We examine the two cases and use the following observation: because $f(\cdot,\cdot)$ is increasing in both arguments, we have $[c_i < c_j \Rightarrow f(c_i, c_k) \leq f(c_j, c_k)]$ for all $k \in N \setminus \{i, j\}$.

Case 1: If $c_k \leq c_j$, then every link that is reallocated from agent j to agent k is weakly less costly. As the number of links in the network is constant with the reallocation, Lemma 1 guarantees that social welfare strictly increases after the switch.

Case 2: If $c_j < c_k$, consider network $G_{(j,k)}$ in which agents j and k exchange their position. Because this permutation does not affect the network structure, we have $b_j(G,\delta) = b_k(G_{(j,k)},\delta) \leq b_j(G_{(j,k)},\delta) = b_k(G,\delta)$. We now implement an $N_{(k,j)}$ switch on network $G_{(j,k)}$ and we denote by G' the resulting network.

In G', the cost of every link that does not involve j or k is the same as the cost in the initial network G. This is also true for every link in $N_j(G') \cap N_k(G')$ as well as for every link that has been reallocated (because agents j and k exchanged positions before the reallocation). Finally, the links between k and $N_{k\setminus j}(G)$ in the original network G have become links between j and $N_{k\setminus j}(G)$. Therefore, their cost is lower in G' than in G. $\triangleleft$

We can now proceed with the proof of Theorem 1. We show that a network that is not an NSG always offers the possibility of an N switch, i.e., there is a pair j, k such that $N_{j\setminus k}(G) \neq \varnothing$ and $b_k(G, \delta) \geq b_j(G, \delta)$. Assume G is not an NSG. Then, by definition, there are three agents i, j, and k such that $ij \in G$, $\deg(k, G) \geq \deg(j, G)$, and $ik \notin G$. Therefore, $i \in N_{j\setminus k}(G)$. Assume now $b_k(G, \delta) < b_j(G, \delta)$. If $N_{k\setminus j} \neq \varnothing$, then we can implement the $N_{(k,j)}$ switch, a contradiction. Hence, $N_{k\setminus j} = \varnothing$ and, therefore, $N_k(G) \subseteq N_j(G)$. Together with the condition that $\deg(k, G) \geq \deg(j, G)$, we obtain $\deg(k, G) = \deg(j, G)$ which is a contradiction to the fact that $i \in N_{j\setminus k}(G)$. An efficient network is necessarily an NSG.

Next, assume G is an NSG in which $\deg(j, G) > \deg(k, G)$ and $c_j > c_k$. Then we can increase social welfare by permuting agents j and k. $\square$

Proof of Lemma 2. If, for a fixed δ, we denote by X (resp. Y and Z) the vector of equilibrium efforts in network G (resp. G^- and G^+), we have

$$\sum_{i\in N}(u_i(G^+, \delta) - u_i(G, \delta)) > \sum_{i\in N}(u_i(G, \delta) - u_i(G^-, \delta))$$
$$\iff \quad \sum_{i\neq j,k}(z_i^2 - x_i^2) + (z_j^2 - x_j^2) + (z_k^2 - x_k^2) > \sum_{i\neq j,k}(x_i^2 - y_i^2) + (x_j^2 - y_j^2) + (x_k^2 - y_k^2).$$

Let $A^+ = G^+ - G$ and $A^- = G - G^-$. Then using the fact that $(I - \delta G)X = (I - \delta G^-)Y = (I - \delta G^+)Z = \mathbf{1}$, we get

$$\begin{cases} X - Y = \delta M^- A^- X \\ Z - X = \delta M A^+ Z, \end{cases}$$

where $M = (I - \delta G)^{-1}$ and $M^- = (I - \delta G^-)^{-1}$. Hence,

$$\begin{cases} x_i - y_i = (\sum_{l\in N_{k\setminus j}} \delta m_{il}^-)x_k + \delta m_{ik}^-(\sum_{l\in N_{k\setminus j}} x_l) \\ z_i - x_i = (\sum_{l\in N_{k\setminus j}} \delta m_{il})z_j + \delta m_{ij}(\sum_{l\in N_{k\setminus j}} z_l). \end{cases} \tag{7}$$

Step 1. We show that for all $i \neq j, k$,

$$z_i^2 - x_i^2 > x_i^2 - y_i^2. \tag{8}$$

Note that $G^- \subsetneq G \subsetneq G^+$ implies $Y < X < Z$, as well as $M^- \leq M$ (this inequality holds term by term). Next, agents j and k have symmetric positions in G^- and G^+, so $z_j = z_k$ and $y_j = y_k$, and for all l, $m_{lk}^- = m_{lj}^-$ and $m_{lk}^+ = m_{lj}^+$. Putting these observations together, we get $z_i - x_i > x_i - y_i$. Because $z_i + x_i > x_i + y_i$, we get (8).

Step 2. We show that

$$\begin{cases} z_k^2 - x_k^2 > x_j^2 - y_j^2 \\ z_j^2 - x_j^2 > x_k^2 - y_k^2. \end{cases}$$

Since $G^- \subset G \subset G^+$, we have $x_k > y_k$, and as $y_k = y_j$, we get $x_k > y_j$; we also have $z_j > x_j$, and as $z_k = z_j$, we get $z_k > x_j$. Altogether,

$$z_k + x_k > x_j + y_j. \tag{9}$$

We now use (7), applied to j for the first equality and to k for the second equality. Because $m^-_{lj} = m^-_{lk}$ and $M^- \leq M$, we get $m^-_{lj} \leq m_{lk}$ for all $l \in N_{k\setminus j}$; we also have $x_k < z_k = z_j$ and $m^-_{jk} \leq m_{jk}$, so that

$$z_k - x_k > x_j - y_j. \tag{10}$$

Putting (9) and (10) together, we obtain

$$z_k^2 - x_k^2 > x_j^2 - y_j^2.$$

Reproducing the same steps, we also obtain

$$z_j^2 - x_j^2 > x_k^2 - y_k^2$$

and the desired conclusion follows. □

Proof of Proposition 1. Let $P_k(G)$ be the number of paths of length k in network G (including loops). Then

$$\sum_i b_i(G, \delta) = n + \delta 2l + \delta^2 P_2(G) + \sum_{k=3}^{+\infty} \delta^k P_k(G).$$

The first two terms of the sum are constant across any network in $\mathcal{G}(l)$. Furthermore, if G^c is the complete network with n agents (i.e., the network with $\binom{n}{2}$ links), then

$$\sum_{k=3}^{+\infty} \delta^k P_k(G) \leq \sum_{k=3}^{+\infty} \delta^k P_k(G^c) = \sum_{k=3}^{+\infty} \delta^k n(n-1)^k,$$

while

$$\delta^2 P_2(G) \geq \delta^2.$$

Therefore,

$$\frac{\sum_{k=3}^{+\infty} \delta^k P_k(G)}{\delta^2 P_2(G)} \leq \frac{\delta^3 n(n-1)^3 [\sum_{j=0}^{+\infty} (\delta(n-1))^j]}{\delta^2}.$$

As $\sum_{j=0}^{+\infty} (\delta(n-1))^j = 1/(1-\delta(n-1))$, we get

$$\frac{\sum_{k=3}^{+\infty} \delta^k P_k(G)}{\delta^2 P_2(G)} \leq \frac{\delta n(n-1)^3}{1-\delta(n-1)},$$

which implies

$$\lim_{\delta \to 0} \frac{\sum_{k=3}^{+\infty} \delta^k P_k(G)}{\delta^2 P_2(G)} = 0.$$

Hence, when $\delta \to 0$, the network maximizing the sum of Bonacich centralities is the network maximizing the number of paths of length 2. Simple algebra leads to the conclusion that maximizing the sum of the squares of Bonacich centralities is also equivalent to maximizing the number of paths of length 2.

The number of paths of length 2 in a network G is given by the sum of all the elements of G^2: $P_2(G) = \mathbf{1}^T G^2 \mathbf{1}$. Because $G = G^T$, we get $P_2(G) = (G\mathbf{1})^T(G\mathbf{1})$, where $G\mathbf{1}$ is the vector of degrees in network G. Therefore, $P_2(G) = \sum_i d_i^2$. We then refer to Ábrego et al. (2009) to conclude. □

Proof of Proposition 2. Theorem 1 tells us that the efficient network G is an NSG. Assume that G is not empty and that it contains at least two classes of agents. Pick an agent j in the second class and another agent k in the first class. Then $N_j(G) \subset N_k(G)$ and Lemma 2 ensures that adding the links between j and agents that are in $N_{k\setminus j}(G)$ will increase aggregate utilities more than the contribution to aggregate utilities of all existing links between k and $N_{k\setminus j}(G)$. Because $\Phi(\cdot)$ is concave or linear, the cost of adding these new links to G is weakly lower than the cost of the existing links between k and $N_{k\setminus j}(G)$. This contradicts the fact that G is efficient and implies that an efficient NSG that is nonempty contains only one class.

Next, an efficient network has no isolated agents: if an agent i is isolated, we can apply the same reasoning as above by replacing j by i and get the same contradiction. The only NSG containing all agents in one class is the complete network.

Finally, the social welfare of the empty network being 0, the complete network becomes the efficient network once it induces positive social welfare. The Bonacich centrality of every agent in the complete network with n agents is $1/(1-\delta(n-1))$, and the sum of utilities is $n/(2(1-\delta(n-1))^2)$. There are $\frac{1}{2}n(n-1)$ links in the complete network, so the cost of the complete network is $\Phi(\frac{1}{2}n(n-1)\cdot c)$ □

Proof of Proposition 3. Consider a nonempty NSG G and assume it is efficient. We first show that it has no isolated agents. Assume agent j is isolated, and pick any agent k in the nontrivial component of the NSG. Then $N_{k\setminus j}(G) = N_k(G)$ and $N_{j\setminus k}(G) = \varnothing$. By Lemma 2, adding links between j and $N_k(G)$ is profitable in terms of aggregate utilities. Furthermore, Theorem 1 tells us that $c_j \geq c_i$ for every agent i in the component. Therefore, $c_{ij} = c_i$ for any i in the component, and linking j to the network by forming links between j and $N_k(G)$ increases social welfare.

Next we show that a nonempty efficient network has at most two classes. Assume G has p classes of agents ($p > 2$). We consider two cases.

- The case when $p \geq 4$. Then pick agent j in the class p (the last one) and k in class $p-1$. By definition of an NSG, j is connected to every agent of class 1, while k is connected to everyone in class 1 as well as to agents in class 2. By Lemma 2,

$$\sum_{i\in N}(u_i(G^+,\delta) - u_i(G,\delta)) > \sum_{i\in N}(u_i(G,\delta) - u_i(G^-,\delta)),$$

  where $G^+ = G + A_j^{k\setminus j}$ and $G^- = G - A_k^{k\setminus j}$. Moreover, by Theorem 1, $c_i \leq c_j$ for all $i \in N_{k\setminus j}(G)$, so that every link in $G^+ - G$ has a cost of $\mathrm{Min}\{c_i; c_j\} = c_i$, which is exactly the same cost as for the links in $G - G^-$. Taking benefits and costs together, the addition of links from G to G^+ strictly increases social welfare.

- The case when $p = 3$. Pick agent j in the last class (class 3) and agent k as the agent in class 2 with the highest individual linking type. By definition, j is linked to every agent in class 1 while k is also linked to agents of his own class. Therefore, $N_{k\setminus j}(G)$ is the set of all agents in class 2, except k. Applying Lemma 2, it is clear that aggregate utilities strictly increase.

Because k is the agent with the highest linking type in class 2, we have $\mathrm{Min}\{c_i; c_k\} = c_i$ for all links in $G - G^-$. Also, $\mathrm{Min}\{c_i; c_j\} = c_i$ for all new links in $G^+ - G$, so that $c_{ij} = c_{ik}$ for all i in class 2 other than k.

Again, taking benefits and costs together, the addition of links from G to G^+ strictly increases social welfare. □

Proof of Proposition 4. Consider a nonempty NSG G, with at least two classes of agents and assume it is efficient. Pick agent k in the first class (the higher one) and pick the agent j with the highest individual linking type c_j. By Theorem 1, agent j has to be in the last class (the lower one).

Lemma 2 says that

$$\sum_{i\in N}(u_i(G^+, \delta) - u_i(G, \delta)) > \sum_{i\in N}(u_i(G, \delta) - u_i(G^-, \delta)),$$

where $G^+ = G + A_j^{k\setminus j}$ and $G^- = G - A_k^{k\setminus j}$.

Now, because $c_j \geq c_i$ for all i, we have $\mathrm{Max}\{c_j; c_k\} = c_j = \mathrm{Max}\{c_j; c_l\}$ for all $l \in N_{k\setminus j}(G)$ so that

$$\sum_{i,j\in N} c_{ij}(g_{ij}^+ - g_{ij}) = \sum_{i,j\in N} c_{ij}(g_{ij} - g_{ij}^-).$$

Combining the two, we get

$$W(G^+, \delta, C) - W(G, \delta, C) > W(G, \delta, C) - W(G^-, \delta, C),$$

which contradicts the fact that G is efficient. □

References

Ábrego, Bernardo M., Silvia Fernández-Merchant, Michael G. Neubauer, and William Watkins (2009), "Sum of squares of degrees in a graph." *Journal of Inequalities in Pure and Applied Mathematics*, 10, 64. [369, 378]

Bala, Venkatesh and Sanjeev Goyal (2000), "A noncooperative model of network formation." *Econometrica*, 68, 1181–1229. [358]

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou (2006), "Who's who in networks. Wanted: The key player." *Econometrica*, 74, 1403–1417. [357, 360, 361, 367]

Billand, Pascal, Christophe Bravard, Jacques Durieu, and Sudipta Sarangi (2015), "Efficient networks for a class of games with global spillovers." *Journal of Mathematical Economics*, 61, 203–210. [359]

Bonacich, Philip (1987), "Power and centrality: A family of measures." *American Journal of Sociology*, 92, 1170–1182. [361]

Chvátal, Václav and Peter L. Hammer (1977), "Aggregation of inequalities in integer programming." In *Studies in Integer Programming*, volume 1 of Annals of Discrete Mathematics, 145–162, North-Holland, Amsterdam. [362]

Corbo, Jacomo, Antoni Calvó-Armengol, and David C. Parkes (2006), "A study of Nash equilibrium in contribution games for peer-to-peer networks." *Operating Systems Review*, 40, 61–66. [359]

Galeotti, Andrea and Sanjeev Goyal (2010), "The law of the few." *American Economic Review*, 100, 1468–1492. [359, 363]

Goyal, Sanjeev and Sumit Joshi (2003), "Networks of collaboration in oligopoly." *Games and Economic Behavior*, 43, 57–85. [359, 363]

Goyal, Sanjeev and José L. Moraga-González (2001), "R&D networks." *RAND Journal of Economics*, 32, 686–707. [358]

Hagberg, Aric, Pieter J. Swart, and Daniel A. Schult (2006), "Designing threshold networks with given structural and dynamical properties." *Physical Review E*, 74, 056116. [370]

Hiller, Timo (2014), "Peer effects in endogenous networks." Working paper, University of Bristol. [359]

Jackson, Matthew O. and Asher Wolinsky (1996), "A strategic model of social and economic networks." *Journal of Economic Theory*, 71, 44–74. [358]

König, Michael D. (2013), "Dynamic R&D networks." Working Paper 109, Department of Economics, University of Zurich. [359]

König, Michael D., Stefano Battiston, Mauro Napoletano, and Frank Schweitzer (2012), "The efficiency and stability of R&D networks." *Games and Economic Behavior*, 75, 694–713. [359]

König, Michael D., Claudio Tessone, and Yves Zenou (2014), "Nestedness in networks: A theoretical model and some applications." *Theoretical Economics*, 9, 695–752. [358, 359, 363, 372]

Milgrom, Paul and John Roberts (1990), "Rationalizability, learning, and equilibrium in games with strategic complementarities." *Econometrica*, 58, 1255–1277. [373]

Westbrock, Bastian (2010), "Natural concentration in industrial research collaboration." *RAND Journal of Economics*, 41, 351–371. [359]

Submitted 2013-12-14. Final version accepted 2015-2-17. Available online 2015-2-17.