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Objective rationality and uncertainty averse preferences

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As in Gilboa et al. (2010), we consider a decision maker characterized by two binary relations: $\succsim^*$ and $\succsim^\wedge$. The first binary relation is a Bewley preference. It models the rankings for which the decision maker is sure. The second binary relation is an uncertainty averse preference, as defined by Cerreia-Vioglio et al. (2011c). It models the rankings that the decision maker expresses if he has to make a choice. We assume that $\succsim^\wedge$ is a completion of $\succsim^*$. We identify axioms under which the set of probabilities and the utility index representing $\succsim^*$ are the same as those representing $\succsim^\wedge$. In this way, we show that Bewley preferences and uncertainty averse preferences, two different approaches to modelling decision making under Knightian uncertainty, are complementary. As a by-product, we extend the main result of Gilboa et al. (2010), who restrict their attention to maxmin expected utility completions.

Keywords. Ambiguity, Bewley preferences, uncertainty averse preferences, preferences completion.

JEL classification. D81.

1. Introduction

In this paper, we consider two different approaches to modelling decision making under Knightian uncertainty: Bewley preferences and the class of uncertainty averse preferences of Cerreia-Vioglio et al. (2011c) (henceforth, CMMM). This latter class encompasses several models that appear in the literature: Gilboa–Schmeidler preferences (Gilboa and Schmeidler 1989), multiplier preferences (Hansen and Sargent 2001 and Strzalecki 2011), variational preferences (Maccheroni et al. 2006), and smooth ambiguity averse preferences (Klibanoff et al. 2005). Our goal is to show how these two different approaches are complementary. To achieve this, we model a decision maker (henceforth, DM) with preferences over Anscombe–Aumann acts by means of two binary relations $(\succsim^*, \succsim^\wedge)$, where the first relation, $\succsim^*$, is a Bewley preference and the second one, $\succsim^\wedge$, is

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an uncertainty averse preference that is a completion of $\succsim^*$. In doing so, we extend the findings of Gilboa et al. (2010) (henceforth, GMMS).

In GMMS, the first binary relation represents the part of the DM's rankings that appear uncontroversial to him (*objective rationality*). The second binary relation models the rankings that the DM expresses if he has to make a choice (*subjective rationality*). GMMS assume that the first binary relation is a Bewley preference, while the second binary relation is complete, transitive, and monotone, and it satisfies c-independence. The postulate of completeness justifies the role of $\succsim^\wedge$ as summarizing the rankings of the DM if he has to make a choice. Transitivity and monotonicity are basic rationality tenets. C-independence is a weakening of the standard notion of independence. It requires that for each $\alpha \in (0,1)$ and for each *constant* act h,

$$f \succsim^\wedge g \iff \alpha f + (1-\alpha)h \succsim^\wedge \alpha g + (1-\alpha)h. \tag{1}$$

Its justification rests on the fact that since h is constant, mixing symmetrically reduces the uncertainty relative to f and g. Thus, if f is weakly preferred to g, then $\alpha f + (1-\alpha)h$ should be weakly preferred to $\alpha g + (1-\alpha)h$, since h does not have any hedging effect. The decision theoretic structure of GMMS is capped by two axioms that impose some discipline on the relationship between $\succsim^*$ and $\succsim^\wedge$. The first axiom, dubbed *consistency*, states that given two acts f and g, $f \succsim^* g$ implies $f \succsim^\wedge g$. Formally, $\succsim^\wedge$ extends $\succsim^*$. Consistency means that the rankings for which the DM is sure are not reverted if he has to choose. Since $\succsim^\wedge$ is complete, we will also say that $\succsim^\wedge$ is a completion of $\succsim^*$. The second axiom, termed *caution*, states that given an act f and a constant act x, if $f \not\succsim^* x$, then $x \succsim^\wedge f$. This postulate imposes that $\succsim^\wedge$ models a rather uncertainty averse DM. In fact, whenever the DM cannot confidently declare an uncertain act f better than a certain act x, then if he has to choose, he weakly prefers the latter over the former.[1] The main result of Gilboa et al. (2010) is the following representation theorem: The binary relations $(\succsim^*, \succsim^\wedge)$ satisfy the aforementioned assumptions if and only if the following statements hold:

1. $\succsim^*$ can be represented à la Bewley (i.e., with a multi-expected utility representation) with an affine utility index u^* and a nonempty, closed, and convex set of probabilities C^*.

2. $\succsim^\wedge$ can be represented à la Gilboa and Schmeidler (i.e., with a maxmin expected utility representation) with an affine utility index u and a nonempty, closed, and convex set of probabilities C.

3. u is cardinally equivalent to u^* and $C = C^*$.[2]

Point 1 readily follows from the conditions imposed on $\succsim^*$. Consistency and caution, paired with the assumptions on $\succsim^\wedge$, imply that $\succsim^\wedge$ satisfies uncertainty aversion. Thus, point 2 is a consequence of Gilboa and Schmeidler (1989, Theorem 1). Finally,

[1]In Gilboa et al. (2010), $\succsim^\wedge$ is also assumed to be continuous and nontrivial.

[2]We say that the utility index u is cardinally equivalent to the utility index u^* if and only if the former is a positive affine transformation of the latter.

in point 3, consistency implies that the utility index used to evaluate consequences can be chosen to be the same for both binary relations—namely, we can set $u = u^*$—while caution implies that the relevant probabilities characterizing $\succsim^*$ are the same as those characterizing $\succsim^\wedge$.

Intuitively, the set C summarizes the set of probabilities that the DM deems plausible, while the utility index u represents his preferences over outcomes.[3] An act f is objectively/unambiguously better than an act g if and only if the expected utility of the first dominates that of the second for each probability in C. Nevertheless, when he has to make a choice, in evaluating an act f, the DM assigns to f the worst possible evaluation induced by the set C.

Thus, GMMS show how two seemingly unrelated approaches to address Ellsberg's critique (see Ellsberg 1961)—that of Bewley and that of Gilboa and Schmeidler—are connected and complementary, once modelled within a preference formation framework. In other words, any Gilboa–Schmeidler preference can be reinterpreted as a cautious completion of a Bewley preference.

In this paper, we extend the representation theorem of GMMS to a larger class of completions $\succsim^\wedge$ of $\succsim^*$ that goes beyond Gilboa–Schmeidler preferences. We maintain the same assumptions of GMMS on $\succsim^*$ while we assume that $\succsim^\wedge$ is an uncertainty averse preference. Finally, we also cap our decision theoretic structure with two extra axioms: consistency and a weakening of caution (dubbed weak caution). Our main result, Theorem 2, shows that $(\succsim^*, \succsim^\wedge)$ satisfy our assumptions if and only if the following statements hold:

1. $\succsim^*$ can be represented à la Bewley with an affine utility index u^* and a nonempty, closed, and convex set of probabilities C.

2. $\succsim^\wedge$ can be represented as in Cerreia-Vioglio et al. (2011c), that is,

$$f \succsim^\wedge g \iff \min_{p\in\Delta} G\left(\int u(f)\,dp, p\right) \geq \min_{p\in\Delta} G\left(\int u(g)\,dp, p\right), \tag{2}$$

   where u is an affine utility index, Δ is the set of all probabilities, and G can be interpreted as an index of uncertainty aversion (as shown by Cerreia-Vioglio et al. 2011c, Proposition 6).

3. u is cardinally equivalent to u^* and the set C representing the Bewley preference $\succsim^*$ is the same set characterizing $\succsim^\wedge$, that is, C is the smallest subset of Δ over which the min in (2) can be taken.

Point 3 is the contribution of our main result and it validates the interpretation that $(\succsim^*, \succsim^\wedge)$ capture different parts of the DM's rankings over acts. Indeed, consistency implies that the utility index used to evaluate consequences can be chosen to be the same for both binary relations while weak caution mainly implies that the relevant probabilities characterizing $\succsim^*$ are the same as those characterizing $\succsim^\wedge$.[4] The difference between

[3]For a similar interpretation, see also Cerreia-Vioglio et al. (2013, Sections 1 and 6).

[4]The paper studies completions of Bewley preferences $\succsim^*$ that use all the set of probabilities characterizing $\succsim^*$. Of course, there exist completions that use a smaller set of probabilities. Nevertheless, we can only

our result and those of GMMS is that the DM's response to the "objectively" specified ambiguity of the set C might not be as extreme as in Gilboa et al. (2010). In this way, we allow for a more permissive view on "subjective rationality" that is not prejudiced by a specific model.

We take two main departures from Gilboa et al. (2010): (a) While we maintain the same assumptions on $\succsim^*$, we impose less stringent conditions on $\succsim^\wedge$. In particular, we still maintain that $\succsim^\wedge$ is complete, transitive, and monotone, but we assume risk independence in place of c-independence, and we *explicitly* assume that $\succsim^\wedge$ satisfies uncertainty aversion.[5] This allows us to consider preferences $\succsim^\wedge$ that are variational, as in Maccheroni et al. (2006), or, more generally, are uncertainty averse as in Cerreia-Vioglio et al. (2011c). (b) We weaken the assumption of caution to *weak caution*. The axiom of weak caution states that, for each constant act x, there exists a weakly better constant act y such that, for each f,

$$f \not\succsim^* x \quad \Longrightarrow \quad y \succsim^\wedge f.$$

In words, weak caution amounts to imposing that, for any given constant act x, there exists a common bound y for *all* acts f that are not unambiguously preferred to x. In the paper of GMMS, this assumption is trivially satisfied; in fact, the bound y for $\succsim^\wedge$ is assumed to be x itself. There are three reasons for these changes:

(i) GMMS argue that c-independence is a suitable principle for subjective rationality. To encompass more general forms of subjective rationality, in this paper, we adopt one of the weakest forms of independence available: risk independence. Risk independence is the assumption of independence restricted to constant acts, that is, (1) when acts f, g, and h are all constant.

(ii) C-independence in conjunction with weak caution implies that $(\succsim^*, \succsim^\wedge)$ satisfy caution and $\succsim^\wedge$ is a Gilboa–Schmeidler preference (see Proposition 3 and Theorem 3). Thus, to extend the result of GMMS, we need to relax both c-independence and caution.

(iii) In Gilboa et al. (2010), $\succsim^\wedge$ turns out to be a Gilboa–Schmeidler preference and, in particular, it also satisfies uncertainty aversion. Since we want to relax caution and to preserve uncertainty aversion, we directly assume that $\succsim^\wedge$ satisfies the latter.

Finally, in Proposition 2, we derive the same result of GMMS under caution but with risk independence in place of c-independence. This further clarifies that a weakening of caution is needed to allow for less uncertainty averse forms of subjective rationality.

identify when the two sets coincide, and our axioms allow us to identify when this is the case. We thank a referee for making this point.

[5]As in Gilboa et al. (2010), $\succsim^\wedge$ is also assumed to be continuous and nontrivial. In addition, we assume that at least one of them satisfies unboundedness.

2. Preliminaries

We consider a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a set X of *consequences*. We denote by $\mathcal{F}$ the set of all *(simple) acts*: functions $f: S \to X$ that are Σ-measurable and take finitely many values. With the usual slight abuse of notation, given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. We thus identify X with the subset of constant acts in $\mathcal{F}$.

We assume additionally that X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of outcomes, as it happens in the classic setting of Anscombe and Aumann (1963). Using the linear structure of X, we define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$.

Given a binary relation $\succsim$ on $\mathcal{F}$, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$, respectively.

We denote by $B_0(\Sigma)$ the set of all real-valued Σ-measurable simple functions endowed with the supnorm. Thus, we have that $u(f) \in B_0(\Sigma)$ whenever $u: X \to \mathbb{R}$ is affine and $f \in \mathcal{F}$. Given an affine function $u: X \to \mathbb{R}$, we denote by $B_0(\Sigma, u(X))$ the set of all real-valued Σ-measurable simple functions that take values in $u(X)$.

It is well known that the norm dual of $B_0(\Sigma)$ can be identified with the set $\mathrm{ba}(\Sigma)$ of all bounded finitely additive measures on (S, Σ). The set of probabilities in $\mathrm{ba}(\Sigma)$ is denoted by Δ, and it is a weak* compact and convex subset of $\mathrm{ba}(\Sigma)$. The set Δ is endowed with the topology inherited from the weak* topology. The set $\mathbb{R}$ is endowed with the usual topology. The set $\mathbb{R} \times \Delta$ is endowed with the product topology. Elements of Δ are denoted by p and q.

Functions of the form $G: \mathbb{R} \times \Delta \to (-\infty, \infty]$ play a key role in CMMM's results and ours. We denote by $\mathrm{dom}_\Delta\, G$ the set

$$\{p \in \Delta : G(t, p) < \infty \text{ for some } t \in \mathbb{R}\}.$$

Borrowing and modifying the notation of Cerreia-Vioglio et al. (2011b), we denote by $\mathcal{L}_n(\mathbb{R} \times \Delta)$ the class of such functions that satisfy the following requirements:

(i) G is quasiconvex and lower semicontinuous on $\mathbb{R} \times \Delta$.

(ii) $G(\cdot, p)$ is increasing for all $p \in \Delta$.

(iii) $\min_{p \in \Delta} G(t, p) = t$ for all $t \in \mathbb{R}$.

Finally, let $\mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ denote the subset of $\mathcal{L}_n(\mathbb{R} \times \Delta)$ consisting of functions G that satisfy the following additional requirement:

(iv) $\sup_{p \in \mathrm{dom}_\Delta G} G(t, p) < \infty$ for all $t \in \mathbb{R}$.

3. The axiomatic framework

We consider a DM characterized by two different binary relations, $\succsim^*$ and $\succsim^\wedge$. The first binary relation captures the rankings that appear to the DM as uncontroversial, and it is potentially incomplete. The second binary relation captures the rankings of the DM if he has to make a choice or express a preference. We next list the assumptions that we impose on these two binary relations; in Section 4, we compare these axioms to those of Gilboa et al. (2010).

We start by listing the axioms that we impose on both $\succsim^*$ and $\succsim^\wedge$. We state them for a generic binary relation $\succsim$ on $\mathcal{F}$.

Basic conditions.

Preorder: *$\succsim$ is reflexive, transitive, and nontrivial.*

Monotonicity: *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Mixture continuity: *If $f, g, h \in \mathcal{F}$, then the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda)g\}$ are closed in $[0,1]$.*

Unboundedness. *For each x and y in X such that $x \succ y$ there are $z, z' \in X$ such that*

$$\tfrac{1}{2}z + \tfrac{1}{2}y \succsim x \succ y \succsim \tfrac{1}{2}x + \tfrac{1}{2}z'.$$

Preorder and monotonicity are standard rationality assumptions. Mixture continuity and unboundedness are technical assumptions. The latter means that there are arbitrarily good and arbitrarily bad consequences. We refer the interested reader to Gilboa et al. (2010) for a more complete discussion of preorder and monotonicity as basic tenets of rationality.

Next, we list the assumptions that are specific to $\succsim^*$.

C-completeness. *If $x, y \in X$, then either $x \succsim^* y$ or $y \succsim^* x$.*

Independence. *If $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, then*

$$f \succsim^* g \iff \alpha f + (1-\alpha)h \succsim^* \alpha g + (1-\alpha)h.$$

These assumptions, paired with the basic conditions, imply that the DM has complete preferences over the set of consequences, and that his preferences on X are represented by a nonconstant and affine utility index $u : X \to \mathbb{R}$. In the original Anscombe and Aumann setting, this is equivalent to saying that when he faces objective probabilities, the DM behaves as a standard expected utility DM. At the same time, under the basic conditions, it follows that $\succsim^*$ admits a representation à la Bewley (2002).

Definition 1. Let $\succsim^*$ be a binary relation on $\mathcal{F}$. $\succsim^*$ is a Bewley preference if and only if it satisfies the basic conditions, c-completeness, and independence.

The next three assumptions are specific to $\succsim^\wedge$.

Completeness. *If $f, g \in \mathcal{F}$, then either $f \succsim^\wedge g$ or $g \succsim^\wedge f$.*

Risk independence. *If $x, y, z \in X$ and $\alpha \in (0,1)$, then*

$$x \succsim^{\wedge} y \iff \alpha x + (1-\alpha) z \succsim^{\wedge} \alpha y + (1-\alpha) z.$$

Uncertainty aversion. *If $f, g \in \mathcal{F}$ are such that $f \sim^{\wedge} g$, then $\alpha f + (1-\alpha) g \succsim^{\wedge} f$ for all $\alpha \in (0,1)$.*

Completeness amounts to imposing that the DM is always able to rank acts if he has to make a choice. Alternatively, in a problem of choice under Knightian uncertainty, uncertainty aversion means that hedging does not make the DM worse off.[6] Risk independence is the assumption of independence restricted to constant acts, where Knightian uncertainty has no bite.[7] Finally, given the basic conditions, completeness, and risk independence, we can conclude that $\succsim^{\wedge}$, on X, is represented by an affine utility index $u : X \to \mathbb{R}$.

Definition 2. Let $\succsim^{\wedge}$ be a binary relation on $\mathcal{F}$. $\succsim^{\wedge}$ is an uncertainty averse preference if and only if it satisfies the basic conditions, completeness, risk independence, and uncertainty aversion.

Theorem 1 (CMMM, Theorems 3 and 5). *Let $\succsim^{\wedge}$ be a binary relation on $\mathcal{F}$. $\succsim^{\wedge}$ is an uncertainty averse preference that satisfies unboundedness if and only if there exist an onto and affine function $u : X \to \mathbb{R}$ and a linearly continuous[8] $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that (u, G) represent $\succsim^{\wedge}$ as in (2). Moreover, u is cardinally unique and, given u, G is unique.*

Examples of uncertainty averse preferences are variational preferences of Maccheroni et al. (2006) and, in particular, Gilboa–Schmeidler preferences. Variational preferences are characterized by imposing weak c-independence, while Gilboa–Schmeidler preferences are characterized by imposing c-independence.[9] Variational preferences are characterized by an additively separable function G (see Theorem 1), that is,

$$G(t, p) = t + c(p) \quad \forall (t, p) \in \mathbb{R} \times \Delta, \tag{3}$$

where $c : \Delta \to [0, \infty]$ is the cost function of Maccheroni et al. (2006), which is grounded,[10] lower semicontinuous, and convex. Gilboa–Schmeidler preferences are characterized as

[6] For a similar interpretation, see also Debreu (1959), Schmeidler (1989), and Cerreia-Vioglio et al. (2011c).

[7] For the sake of generality, we could have equivalently imposed a weaker form of risk independence, as in Cerreia-Vioglio et al. (2011c). The actual formulation allows for an easier comparison with the corresponding independence axiom imposed on $\succsim^*$.

[8] A function $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is said to be *linearly continuous* if and only if the map

$$\varphi \mapsto \inf_{p \in \Delta} G\Big(\int \varphi \, dp, p\Big)$$

from $B_0(\Sigma)$ to $[-\infty, \infty]$ is extended-valued continuous.

[9] See Maccheroni et al. (2006, Axiom A.2) for the definition of weak c-independence and the section below for the definition of c-independence.

[10] The function c is grounded if and only if $\min_{p \in \Delta} c(p) = 0$.

having a function G as in (3) with c such that $c(p) = 0$ if $p \in C$ and $c(p) = \infty$ otherwise, where C is a nonempty, closed, and convex subset of Δ.

The next two assumptions connect our two binary relations.

Consistency. *If $f \succsim^* g$, then $f \succsim^\wedge g$.*

Weak caution. *For each $x \in X$ there exists $y \in X$ such that $y \succsim^\wedge x$ and for each $f \in \mathcal{F}$,*

$$f \not\succsim^* x \quad \Longrightarrow \quad y \succsim^\wedge f.$$

Consistency means that $\succsim^*$ is a subrelation of $\succsim^\wedge$. Together with completeness, consistency implies that $\succsim^\wedge$ is a completion of $\succsim^*$. Intuitively, if f is clearly/objectively weakly better than g, then the DM should deem f weakly better than g when he has to make a choice. Weak caution amounts to imposing that, for any given x in X, there exists a common bound y in X for *all* acts f that are not unambiguously preferred to x.

Weak caution provides the main axiomatic departure of our work from Gilboa et al. (2010). In the work of GMMS, this assumption is trivially satisfied; under caution, for each x, the bound y is x itself. In their case, consistency and caution are the two key conditions implying that $\succsim^\wedge$ satisfies

$$f \sim^\wedge \underline{x}_f \quad \forall f \in \mathcal{F},$$

where $\underline{x}_f \sim^* \sup\{x \in X : f \succsim^* x\}$ (see Proposition 2 below). In other words, caution allows for only a very restrictive completion of $\succsim^*$, namely the completion that to each act f associates the lowest possible evaluation in the interval objectively specified by $\succsim^*$. Alternatively, weak caution allows us to consider and axiomatize less restrictive completions, and therefore more general forms of subjective rationality.

As the proof of Theorem 2 shows, weak caution has bite only in a context where unboundedness from above is satisfied. In the classic Anscombe and Aumann setting, with X being the set of simple positive monetary lotteries, unboundedness from above is satisfied, for example, when the DM's risk attitude is represented by a constant relative risk aversion (CRRA) utility index.

4. Uncertainty averse completions

We can now state the main result of our paper. It shows that Bewley preferences and uncertainty averse preferences are connected via a completion procedure.

Theorem 2. *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$ and let one of them satisfy unboundedness. The following conditions are equivalent:*

(i) $\succsim^*$ satisfies the basic conditions, c-completeness, and independence; $\succsim^\wedge$ satisfies the basic conditions, completeness, risk independence, and uncertainty aversion; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency and weak caution.

(ii) There exist an onto and affine function $u: X \to \mathbb{R}$, a linearly continuous function $G \in \mathcal{L}_{\text{bd}}(\mathbb{R} \times \Delta)$, and a nonempty, closed, and convex set $C \subseteq \Delta$ such that $\operatorname{dom}_\Delta G = C$ *and for each f and g,*

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \quad \forall p \in C \tag{4}$$

and

$$f \succsim^\wedge g \iff \min_{p \in C} G\left(\int u(f)\,dp, p\right) \geq \min_{p \in C} G\left(\int u(g)\,dp, p\right). \tag{5}$$

Moreover, C is unique, u is cardinally unique, and, given u, G is unique.

Theorem 2 follows from the following arguments. The axioms on $\succsim^*$ imply that $\succsim^*$ is represented according to the unanimity rule of Bewley with a set of probabilities C and a utility index u^*. The axioms on $\succsim^\wedge$ imply that $\succsim^\wedge$ is an uncertainty averse preference. Thus, by Cerreia-Vioglio et al. (2011c), there exist a nonconstant and affine $u: X \to \mathbb{R}$ and a linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that $V: \mathcal{F} \to \mathbb{R}$, defined by

$$V(f) = \min_{p \in \Delta} G\left(\int u(f)\,dp, p\right) \quad \forall f \in \mathcal{F}, \tag{6}$$

represents $\succsim^\wedge$. Consistency delivers the fact that u^* can be chosen to be equal to u, while weak caution implies that G belongs to $\mathcal{L}_{\text{bd}}(\mathbb{R} \times \Delta)$ and the min in (6) can be taken over $C = \operatorname{dom}_\Delta G$. Given this equality, it follows that C is the smallest closed and convex set over which the min in (6) can be taken. This latter fact confirms that the set of probabilities characterizing $\succsim^\wedge$ is the same one characterizing $\succsim^*$. Moreover, in the Appendix, we show that C also characterizes the *unambiguous preference relation* of Ghirardato et al. (2004) for $\succsim^\wedge$.

Our DM acts as if, in forming his preferences, he first identifies the set of relevant and plausible probabilities C and a utility index u. These two objects characterize the rankings, as in (4), that the DM deems uncontroversial. For example, in the classic Ellsberg two-color urn experiment, C could be the convex hull of all possible urn compositions of the unknown urn, and u could be a utility index over all the objective urns. Nevertheless, C and u are not enough for the DM to be able to always rank acts. For this reason, to complete his preferences, he then selects an index of uncertainty aversion G that is also bounded on C. This allows him to consider certain probabilistic scenarios in C more plausible than others. Finally, he uses these three objects to consistently form his preferences $\succsim^\wedge$ according to the cautious rule in (5) and thus he only uses the probabilities in C.

The condition that G belongs to $\mathcal{L}_{\text{bd}}(\mathbb{R} \times \Delta)$ amounts to imposing that, in completing his preferences $\succsim^*$, the DM might not be willing to consider all the probabilistic scenarios in C to be equivalent, as in the Gilboa–Schmeidler case, but, at the same time, he does not want to penalize these alternative probabilities in an arbitrarily unbounded fashion.

The class of functions $\mathcal{L}_{\text{bd}}(\mathbb{R} \times \Delta)$ characterizes a subset of uncertainty averse preferences that we call *effectively bounded* uncertainty averse preferences.

Definition 3. Let $\succsim^\wedge$ be a binary relation on $\mathcal{F}$. $\succsim^\wedge$ is an *effectively bounded* uncertainty averse preference if and only if there exist an onto and affine function $u : X \to \mathbb{R}$ and a linearly continuous $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ such that (u, G) represent $\succsim^\wedge$ as in (2).

The intersection between effectively bounded uncertainty averse preferences and variational preferences is easily characterized and contains Gilboa–Schmeidler preferences. If the DM's preferences $\succsim^\wedge$ are variational, then G is as in (3). If we further impose that $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$, the condition

$$\infty > \sup_{p \in \mathrm{dom}_\Delta G} G(t, p) = \sup_{p \in \mathrm{dom}(c)} \{t + c(p)\} \quad \forall t \in \mathbb{R} \tag{7}$$

is equivalent to c being bounded over $\mathrm{dom}_\Delta G = \mathrm{dom}(c)$, where

$$\mathrm{dom}(c) = \{p \in \Delta : c(p) < \infty\}$$

is the effective domain of c. Thus, it is also immediate to verify that the function G characterizing a Gilboa–Schmeidler preference satisfies condition (7) and is an element of $\mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$.

It follows that our main result provides a foundation for the larger class of effectively bounded uncertainty averse preferences.[11]

As already mentioned, in Theorem 2, $\succsim^\wedge$ turns out to belong to the special class of effectively bounded uncertainty averse preferences. The next result shows that this latter class is "dense" in the class of uncertainty averse preferences, proving that effectively bounded uncertainty averse preferences are a "topologically" large subset of the set formed by uncertainty averse preferences.

Proposition 1. *Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies unboundedness. If $\succsim$ is an uncertainty averse preference, then there exists a sequence of effectively bounded uncertainty averse preferences $\{\succsim_n\}_{n \in \mathbb{N}}$ such that*

$$\lim_n V_n(f) = V(f) \quad \forall f \in \mathcal{F},$$

where $V, V_n : \mathcal{F} \to \mathbb{R}$ represent $\succsim$ and $\succsim_n$ as in (6) and for all $n \in \mathbb{N}$.

We conclude by formally discussing the relationship between our main result and the work of GMMS. We start by listing two assumptions that play a major role in Gilboa et al. (2010).

C-independence. *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0, 1)$, then*

$$f \succsim^\wedge g \iff \alpha f + (1 - \alpha)x \succsim^\wedge \alpha g + (1 - \alpha)x.$$

[11] It is also possible, within the single preference framework adopted by Cerreia-Vioglio et al. (2011c), to provide a foundation for the class of effectively bounded uncertainty averse preferences. This can be achieved by requiring the unambiguous preference relation of Ghirardato et al. (2004) to satisfy weak caution.

CAUTION. *If $f \in \mathcal{F}$ and $x \in X$, then*

$$f \not\succsim^* x \quad \Longrightarrow \quad x \succsim^\wedge f.$$

THEOREM 3 (GMMS, Theorem 3). *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$. The following conditions are equivalent:*

(i) $\succsim^*$ satisfies the basic conditions, c-completeness, and independence; $\succsim^\wedge$ satisfies the basic conditions, completeness, and c-independence; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency and caution.

(ii) There exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a nonempty, closed, and convex set $C \subseteq \Delta$ such that for each f and g,

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\, dp \geq \int u(g)\, dp \quad \forall p \in C$$

and

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad \min_{p \in C} \int u(f)\, dp \geq \min_{p \in C} \int u(g)\, dp.$$

Moreover, C is unique and u is cardinally unique.

Our work departs from Gilboa et al. (2010) in three ways. The first departure consists in restricting attention to binary relations that satisfy unboundedness, that is, preferences for which there are arbitrarily good and arbitrarily bad consequences. For example, this is the case if $X = \mathbb{R}$ and the DM satisfies the basic conditions as well as c-completeness and risk independence.[12] The second departure consists in weakening c-independence to risk independence and in explicitly assuming uncertainty aversion for $\succsim^\wedge$.[13] This allows us to consider preferences $\succsim^\wedge$ that are variational, as in Maccheroni et al. (2006), or, more generally, uncertainty averse as in Cerreia-Vioglio et al. (2011c).[14] Finally, we impose on $(\succsim^*, \succsim^\wedge)$ a weaker form of caution, termed weak caution. It is immediate to see that weak caution is a weakening of caution.

To explore the extent of the assumption of caution, we first propose a stronger version of Gilboa et al. (2010, Theorem 3) (see also Gilboa et al. 2010, Theorem 4). In comparison to Gilboa et al. (2010, Theorem 3), here we only weaken c-independence to risk independence in (i), but we still obtain the same functional characterization in (ii).

PROPOSITION 2. *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$. The following conditions are equivalent:*

[12] In this case, the DM can also be interpreted as being risk neutral.

[13] Binary relations that satisfy the basic conditions, completeness, and c-independence are studied and defined as invariant biseparable preferences by Ghirardato et al. (2004); see also Ghirardato and Marinacci (2001 and 2002).

[14] See also Maccheroni et al. (2006, p. 1454) for a positive/normative discussion justifying an axiomatic departure from c-independence.

(i) *$\succsim^*$ satisfies the basic conditions, c-completeness, and independence; $\succsim^\wedge$ satisfies the basic conditions, completeness, and risk independence; and jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency and caution.*

(ii) *There exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a nonempty, closed, and convex set $C \subseteq \Delta$ such that for each f and g,*

$$f \succsim^* g \iff \int u(f)\,dp \geq \int u(g)\,dp \quad \forall p \in C$$

and

$$f \succsim^\wedge g \iff \min_{p\in C} \int u(f)\,dp \geq \min_{p\in C} \int u(g)\,dp.$$

Moreover, C is unique and u is cardinally unique.

Thus, in GMMS, weakening c-independence to risk independence has no effect. Note also that in the previous proposition, we did not make any assumption on $\succsim^\wedge$ regarding attitudes toward uncertainty or independence involving uncertain acts.[15] Thus, as also emerges from the proof of Proposition 2, it is primarily caution that drives the representation of the completion $\succsim^\wedge$ of $\succsim^*$ to be maxmin expected utility. The next proposition shows that *under unboundedness*, weakening caution to weak caution also has no effect.

Proposition 3. *Let $(\succsim^*, \succsim^\wedge)$ be two binary relations on $\mathcal{F}$ and let one of them satisfy unboundedness. Moreover, let $\succsim^*$ satisfy the basic conditions, c-completeness, and independence; let $\succsim^\wedge$ satisfy the basic conditions, completeness, and c-independence; and jointly let $(\succsim^*, \succsim^\wedge)$ satisfy consistency. The following conditions are equivalent:*

(i) *Jointly $(\succsim^*, \succsim^\wedge)$ satisfy weak caution.*

(ii) *Jointly $(\succsim^*, \succsim^\wedge)$ satisfy caution.*

As a corollary to this result, we can prove Gilboa et al. (2010, Theorem 3) again. In fact, we can replace caution with weak caution and retain c-independence, and still obtain the same representation result via Proposition 3 and Theorem 3.

Appendix

Given a binary relation $\succsim^\wedge$ on $\mathcal{F}$, we define $\succsim^\circ$ by

$$f \succsim^\circ g \iff \lambda f + (1-\lambda)h \succsim^\wedge \lambda g + (1-\lambda)h \quad \forall \lambda \in (0,1], \forall h \in \mathcal{F}.$$

The binary relation $\succsim^\circ$ is the revealed unambiguous preference relation of Ghirardato et al. (2004). In the sequel, with a small abuse of notation, given $k \in \mathbb{R}$, we will denote by k both the real number and the constant function on S that takes value k.

[15]Binary relations that satisfy the basic conditions, completeness, and risk independence are called rational preferences and are studied in Cerreia-Vioglio et al. (2011a).

In the rest of the paper, we will invoke some of the results of GMMS. Even though all the results in Gilboa et al. (2010) were derived under the hypothesis that X is the set of all simple lotteries over a generic outcome space, their extension to the case when X is a generic convex set is straightforward.

Before proving the main results, we need some extra notation and an ancillary proposition. Given a functional $I : B_0(\Sigma) \to \mathbb{R}$, we define $\succcurlyeq$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq \psi \quad \Longleftrightarrow \quad I(\varphi) \geq I(\psi).$$

We define $\succcurlyeq^\circ$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad I(\lambda\varphi + (1-\lambda)\phi) \geq I(\lambda\psi + (1-\lambda)\phi) \quad \forall \lambda \in (0, 1], \forall \phi \in B_0(\Sigma). \tag{8}$$

Given $C \subseteq \Delta$, we define $\succcurlyeq_C$ to be the binary relation on $B_0(\Sigma)$ such that

$$\varphi \succcurlyeq_C \psi \quad \Longleftrightarrow \quad \int \varphi\, dp \geq \int \psi\, dp \quad \forall p \in C.$$

Given C and I, we say that I is consistent with C if and only if

$$\varphi \succcurlyeq_C \psi \quad \Longrightarrow \quad I(\varphi) \geq I(\psi).$$

Finally, a function $G : \mathbb{R}\times\Delta \to (-\infty, \infty]$ is said to be *linearly continuous* if and only if the map

$$\varphi \mapsto \inf_{p\in\Delta} G\left(\int \varphi\, dp, p\right)$$

from $B_0(\Sigma)$ to $[-\infty, \infty]$ is extended-valued continuous. For example, the function G defined in (3) is linearly continuous.

Proposition 4. *Let I be a functional from $B_0(\Sigma)$ to $\mathbb{R}$ and let C be a nonempty, closed, and convex subset of Δ. The following conditions are equivalent:*

(i) I is normalized, monotone, continuous, quasiconcave, consistent with C, and such that for each $k \in \mathbb{R}$, there exists $h \geq k$ such that

$$\varphi \not\succcurlyeq_C k \quad \Longrightarrow \quad h \geq I(\varphi). \tag{9}$$

(ii) There exists a unique linearly continuous $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ such that

$$I(\varphi) = \min_{p\in\Delta} G\left(\int \varphi\, dp, p\right) \quad \forall \varphi \in B_0(\Sigma)$$

and $\mathrm{dom}_\Delta\, G = C$.

Proof. (i) implies (ii). We proceed by steps. But, first, by construction, observe that $\succcurlyeq$, $\succcurlyeq_C$, and $\succcurlyeq^\circ$ are binary relations over acts in an Anscombe and Aumann setting where S is the state space, Σ is the algebra, and $X = \mathbb{R}$.

Step 1. *$\succcurlyeq$ satisfies the basic conditions, completeness, risk independence, and uncertainty aversion. Moreover, $\succcurlyeq$ restricted to $\mathbb{R}$ is represented by the identity.*

Proof. Since I is normalized, $\succcurlyeq$ restricted to $\mathbb{R}$ is represented by the identity. By Cerreia-Vioglio et al. (2011c, Lemma 57) and since I is normalized, monotone, continuous, and quasiconcave, the statement follows. $\lhd$

Step 2. *There exists a nonempty, closed, and convex set $C^\circ \subseteq \Delta$ such that for each φ and ψ in $B_0(\Sigma)$,*

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad \int \varphi\, dp \geq \int \psi\, dp \quad \forall p \in C^\circ$$

and

$$\varphi \succcurlyeq^\circ \psi \quad \Longrightarrow \quad I(\varphi) \geq I(\psi).$$

Moreover, C° is unique and $\succcurlyeq^\circ = \succcurlyeq_{C^\circ}$.

Proof. By definition of $\succcurlyeq^\circ$ and $\succcurlyeq$, we have that

$$\varphi \succcurlyeq^\circ \psi \quad \Longleftrightarrow \quad \lambda\varphi + (1-\lambda)\phi \succcurlyeq \lambda\psi + (1-\lambda)\phi \quad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma). \tag{10}$$

By Step 1 and Cerreia-Vioglio et al. (2011a, Proposition 2), the first part of the statement follows as well as the uniqueness of C° and $\succcurlyeq^\circ = \succcurlyeq_{C^\circ}$. By taking $\lambda = 1$ in (10) and by definition of $\succcurlyeq$, the second part follows as well. $\lhd$

Step 3. *We have $C^\circ \subseteq C$.*

Proof. By the definition of $\succcurlyeq_C$ and $\succcurlyeq^\circ$ and Step 2, and since I is consistent with C, we have that

$$\begin{aligned} \varphi \succcurlyeq_C \psi \quad &\Longrightarrow \quad \lambda\varphi + (1-\lambda)\phi \succcurlyeq_C \lambda\psi + (1-\lambda)\phi \quad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma) \\ &\Longrightarrow \quad I(\lambda\varphi + (1-\lambda)\phi) \geq I(\lambda\psi + (1-\lambda)\phi) \quad \forall \lambda \in (0,1], \forall \phi \in B_0(\Sigma) \\ &\Longrightarrow \quad \varphi \succcurlyeq^\circ \psi. \end{aligned}$$

By Step 2 and Ghirardato et al. (2004, Proposition A.1), this implies that $C^\circ \subseteq C$. $\lhd$

Step 4. *There exists a unique linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that*

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi\, dp, p\right) \quad \forall \varphi \in B_0(\Sigma). \tag{11}$$

Moreover, for each $(t, p) \in \mathbb{R} \times \Delta$,

$$G(t, p) = \sup\left\{I(\varphi) : \int \varphi\, dp \leq t\right\} \tag{12}$$

and $\mathrm{cl}(\mathrm{dom}_\Delta\, G) = C^\circ$.

Proof. By Cerreia-Vioglio et al. (2011c) (see also Cerreia-Vioglio et al. 2011b) and Step 1, and since I is normalized, monotone, continuous, and quasiconcave, there exists a unique linearly continuous $G \in \mathcal{L}_n(\mathbb{R} \times \Delta)$ such that (11) and (12) hold. By Cerreia-Vioglio et al. (2011c, Theorem 10), we also have that $\mathrm{cl}(\mathrm{dom}_\Delta G) = C^\circ$. ◁

Step 5. *We have* $C^\circ = C$.

Proof. We start by giving a definition. Given $\phi \in B_0(\Sigma)$, we define $k_\phi = \min_{p \in C} \int \phi \, dp$. By contradiction, suppose that $C^\circ \neq C$. By Steps 3 and 4, we know that this implies that there exists $q \in C \setminus C^\circ$ and $q \notin \mathrm{dom}_\Delta G$. By Rudin (1991, Theorem 3.4) and since C° is closed and convex, there exists $\psi \in B_0(\Sigma)$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\min_{p \in C} \int \psi \, dp \leq \int \psi \, dq \leq \alpha - \varepsilon < \alpha + \varepsilon \leq \min_{p \in C^\circ} \int \psi \, dp. \tag{13}$$

Without loss of generality, we can assume that ψ is such that $k_\psi \leq -\varepsilon < 0$ and $\min_{p \in C^\circ} \int \psi \, dp \geq \varepsilon > 0$. By (13), if we define the sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subseteq B_0(\Sigma)$ to be such that $\varphi_n = n\psi$ for all $n \in \mathbb{N}$, then it follows that

$$k_{\varphi_n} < 0 \quad \text{and} \quad \min_{p \in C^\circ} \int \varphi_n \, dp = \min_{p \in C^\circ} \int n\psi \, dp = n \min_{p \in C^\circ} \int \psi \, dp \geq n\varepsilon > 0 \quad \forall n \in \mathbb{N}. \tag{14}$$

Recall that I satisfies (9), that is, for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\geq_C k \quad \Longrightarrow \quad h \geq I(\varphi).$$

Take $k = 0$ and h as in (9). By (14), it follows that there exists $\bar{n} \in \mathbb{N}$ such that

$$k_{\varphi_{\bar{n}}} < 0 = k \quad \text{and} \quad \int \varphi_{\bar{n}} \, dp' \geq \min_{p \in C^\circ} \int \varphi_{\bar{n}} \, dp > h + 1 \quad \forall p' \in C^\circ. \tag{15}$$

By Step 2, I is consistent with C°. Thus, the first part of (15) yields that $\varphi_{\bar{n}} \not\geq_C k$ while the second part delivers that $I(\varphi_{\bar{n}}) > h$, a contradiction. ◁

Step 6. *We have* $\sup_{p \in \mathrm{dom}_\Delta G} G(t, p) < \infty$ *for all* $t \in \mathbb{R}$*, that is,* $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$.

Proof. Before starting recall that by Step 4, G satisfies (12), that is,

$$G(t, p) = \sup\left\{I(\varphi) : \int \varphi \, dp \leq t\right\} \quad \forall (t, p) \in \mathbb{R} \times \Delta.$$

By contradiction, suppose that $\sup_{p \in \mathrm{dom}_\Delta G} G(\bar{t}, p) = \infty$ for some $\bar{t}$ in $\mathbb{R}$. By working hypothesis, there exists a sequence $\{p_n\}_{n \in \mathbb{N}} \subseteq \mathrm{dom}_\Delta G$ such that $G(\bar{t}, p_n) \geq n$ for all $n \in \mathbb{N}$. By (12) and since $C = C^\circ = \mathrm{cl}(\mathrm{dom}_\Delta G)$, this implies that for each $n \in \mathbb{N}$ there exists $\varphi_n \in B_0(\Sigma)$ such that

$$\min_{p \in C} \int \varphi_n \, dp \leq \int \varphi_n \, dp_n \leq \bar{t} < \bar{t} + 1 \quad \text{and} \quad I(\varphi_n) \geq \frac{n}{2}. \tag{16}$$

Since I satisfies (9), consider $k = \bar{t} + 1$ and fix $h \geq k$ to satisfy (9). From the first part of (16), we have that $\varphi_n \not\geq_C k$ for all $n \in \mathbb{N}$. At the same time, by the second part of (16), it is immediate to see that there exists $\bar{n} \in \mathbb{N}$ such that $I(\varphi_{\bar{n}}) \geq \bar{n}/2 \geq h$, a contradiction with I satisfying (9). $\triangleleft$

Step 7. *We have* $\mathrm{cl}(\mathrm{dom}_\Delta G) = \mathrm{dom}_\Delta G$.

Proof. It is enough to prove that given a generic net $\{p_\alpha\}_{\alpha \in A} \subseteq \mathrm{dom}_\Delta G$ such that $p_\alpha \to \bar{p}$, then $\bar{p} \in \mathrm{dom}_\Delta G$. Fix a generic $t \in \mathbb{R}$. By Step 6, it follows that $G(t, p_\alpha) \leq \sup_{p \in \mathrm{dom}_\Delta G} G(t, p) < \infty$ for all $\alpha \in A$. Since $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$, we have that

$$\infty > \sup_{p \in \mathrm{dom}_\Delta G} G(t, p) \geq \liminf_\alpha G(t, p_\alpha) \geq G(t, \bar{p}).$$

Hence, $\bar{p} \in \mathrm{dom}_\Delta G$. $\triangleleft$

Steps 4 and 6 imply the first part of (ii), while Steps 4, 5, and 7 imply that $C = C^\circ = \mathrm{cl}(\mathrm{dom}_\Delta G) = \mathrm{dom}_\Delta G$.

(ii) implies (i). By assumption, we have that there exists a linearly continuous $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ such that

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi\, dp, p\right) \quad \forall \varphi \in B_0(\Sigma).$$

By Cerreia-Vioglio et al. (2011b), it follows that I is normalized, monotone, and quasiconcave. Since G is linearly continuous, I is continuous. Next, by definition of $\mathrm{dom}_\Delta G$, we have that

$$I(\varphi) = \min_{p \in \mathrm{dom}_\Delta G} G\left(\int \varphi\, dp, p\right) \quad \forall \varphi \in B_0(\Sigma). \tag{17}$$

Since G is increasing in the first component and $\mathrm{dom}_\Delta G = C$, it follows that I is consistent with C. Finally, we show that I satisfies (9). We proceed by arguing by contradiction. Suppose that there exists $k \in \mathbb{R}$ such that for each $h \geq k$ in $\mathbb{R}$, we can find $\varphi_h \in B_0(\Sigma)$ such that $\varphi_h \not\geq_C k$ and $I(\varphi_h) > h$. It follows that for each $n \in \{\lfloor k \rfloor + 1, \ldots, \lfloor k \rfloor + m, \ldots\}$ there exists $\varphi_n \in B_0(\Sigma)$ such that $\varphi_n \not\geq_C k$ and $I(\varphi_n) > n$.[16] Thus, for each $n \in \{\lfloor k \rfloor + 1, \ldots, \lfloor k \rfloor + m, \ldots\}$ there exists $p_n \in C = \mathrm{dom}_\Delta G$ such that $\int \varphi_n\, dp_n < k$. By (17) and since $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$, it follows that

$$\sup_{p \in \mathrm{dom}_\Delta G} G(k, p) \geq G(k, p_n) \geq G\left(\int \varphi_n\, dp_n, p_n\right) \geq I(\varphi_n) > n$$

for all $n \in \{\lfloor k \rfloor + 1, \ldots, \lfloor k \rfloor + m, \ldots\}$, a contradiction with $\sup_{p \in \mathrm{dom}_\Delta G} G(t, p) < \infty$ for all $t \in \mathbb{R}$. $\square$

Proof of Theorem 2. (i) implies (ii). We again proceed by steps.

[16] Given $k \in \mathbb{R}$, we denote by $\lfloor k \rfloor$ the floor of k, that is, the largest integer not greater than k.

Step 1. *$\succsim^\wedge$ coincides with $\succsim^*$ on X.*

Proof. Notice that $\succsim^*$ and $\succsim^\wedge$ restricted to X satisfy c-completeness, mixture continuity, and risk independence. By Herstein and Milnor (1953) and since $\succsim^*$ and $\succsim^\wedge$ satisfy the basic conditions, it follows that there exist two nonconstant and affine functions, u^* and $u^\wedge$, from X to $\mathbb{R}$ that represent $\succsim^*$ and $\succsim^\wedge$ on X, respectively. Since jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency, it follows that for each $x, y \in X$,

$$u^*(x) \geq u^*(y) \quad \Longrightarrow \quad u^\wedge(x) \geq u^\wedge(y).$$

By Ghirardato et al. (2004, Corollary B.3), it follows that u^* and $u^\wedge$ are equal up to an affine and positive transformation, hence the statement. ◁

Step 2. *There exist an onto and affine function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that*

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u^*(f)\, dp \geq \int u^*(g)\, dp \quad \forall p \in C. \tag{18}$$

Moreover, C is unique.

Proof. By assumption, $\succsim^*$ satisfies the basic conditions, c-completeness, and independence. By Gilboa et al. (2010, Theorem 1) and since, by Step 1 and the premises of Theorem 2, $\succsim^*$ satisfies unboundedness, the statement follows. ◁

Step 3. *There exist an onto and affine function $u^\wedge : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\Sigma) \to \mathbb{R}$ such that*

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad I(u^\wedge(f)) \geq I(u^\wedge(g)).$$

Moreover, $u^\wedge$ is cardinally unique and, given $u^\wedge$, I is unique.

Proof. By assumption, Step 1, and the premises of Theorem 2, $\succsim^\wedge$ satisfies the basic conditions, completeness, risk independence, uncertainty aversion, and unboundedness. By Cerreia-Vioglio et al. (2011c, Lemma 57 and Lemma 59), the statement follows. ◁

Notice that, by Step 1, we can assume without loss of generality that $u^* = u^\wedge = u$.

Step 4. *I is consistent with C.*

Proof. Consider $\varphi, \psi \in B_0(\Sigma)$ and assume that $\varphi \succcurlyeq_C \psi$. It is immediate to see that there exist $f, g \in \mathcal{F}$ such that $\varphi = u(f)$, $\psi = u(g)$, and $f \succsim^* g$. By Steps 2 and 3 and since jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency, we have that

$$\begin{aligned} \varphi \succcurlyeq_C \psi \quad &\Longrightarrow \quad f \succsim^* g \quad \Longrightarrow \quad f \succsim^\wedge g \quad \Longrightarrow \quad I(u(f)) \geq I(u(g)) \\ &\Longrightarrow \quad I(\varphi) \geq I(\psi), \end{aligned}$$

proving the statement. ◁

STEP 5. *I satisfies (9).*

PROOF. We need to show that for each $k \in \mathbb{R}$ there exists $h \geq k$ such that

$$\varphi \not\succsim_C k \quad \Longrightarrow \quad h \geq I(\varphi).$$

Fix a generic $k \in \mathbb{R}$. Since $\succsim^\wedge$ satisfies unboundedness, there exists $x \in X$ such that $k = u(x)$. Define $h = u(y)$, where $y \in X$ is such that $y \succsim^\wedge x$ and

$$f \not\succsim^* x \quad \Longrightarrow \quad y \succsim^\wedge f.$$

Next, consider $\varphi \in B_0(\Sigma)$ such that $\varphi \not\succsim_C k$. Given (18), it is immediate to see that there exists $f \in \mathcal{F}$ such that $\varphi = u(f)$ and $f \not\succsim^* x$. Since jointly $(\succsim^*, \succsim^\wedge)$ satisfy weak caution, it follows that $y \succsim^\wedge f$. By Step 3, this implies that $h = u(y) = I(u(y)) \geq I(u(f)) = I(\varphi)$, hence the statement. ◁

STEP 6. *There exist an onto and affine function $u : X \to \mathbb{R}$, a linearly continuous function $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$, and a nonempty, closed, and convex set $C \subseteq \Delta$ such that* $\mathrm{dom}_\Delta\, G = C$ *and for each f and g,*

$$f \succsim^* g \quad \Longleftrightarrow \quad \int u(f)\, dp \geq \int u(g)\, dp \quad \forall p \in C \tag{19}$$

and

$$f \succsim^\wedge g \quad \Longleftrightarrow \quad \min_{p \in C} G\left(\int u(f)\, dp, p\right) \geq \min_{p \in C} G\left(\int u(g)\, dp, p\right). \tag{20}$$

PROOF. Define $V : \mathcal{F} \to \mathbb{R}$ by $V(f) = I(u(f))$ for all $f \in \mathcal{F}$, where $u = u^\wedge$ and I are as in Step 3. It is immediate to see that V represents $\succsim^\wedge$. By Steps 2, 3, 4, and 5 and Proposition 4, it follows that there exists a linearly continuous $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ such that $\mathrm{dom}_\Delta\, G = C$, where C is nonempty, closed, and convex and

$$V(f) = \min_{p \in \Delta} G\left(\int u(f)\, dp, p\right) = \min_{p \in C} G\left(\int u(f)\, dp, p\right) \quad \forall f \in \mathcal{F},$$

proving that (20) holds. By Step 2 and since $u^* = u$, (19) holds. ◁

(ii) implies (i). Consider a nonempty, closed, and convex set $C \subseteq \Delta$, an onto and affine function $u : X \to \mathbb{R}$, and a linearly continuous $G \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ such that $C = \mathrm{dom}_\Delta\, G$. Suppose also that C and (u, G) satisfy (4) and (5). By Gilboa et al. (2010, Theorem 1), it follows that $\succsim^*$ satisfies the basic conditions, c-completeness, and independence. By Cerreia-Vioglio et al. (2011c, Theorem 3), $\succsim^\wedge$ satisfies the basic conditions, completeness, risk independence, and uncertainty aversion (as well as unboundedness). Define $I : B_0(\Sigma) \to \mathbb{R}$ by

$$I(\varphi) = \min_{p \in \Delta} G\left(\int \varphi\, dp, p\right) \quad \forall \varphi \in B_0(\Sigma).$$

By Proposition 4, it follows that I is consistent with C and satisfies (9). Since I composed with u represents $\succsim^{\wedge}$, this implies that jointly $(\succsim^*, \succsim^{\wedge})$ satisfy consistency and weak caution.

The uniqueness part of the statement follows from routine arguments (see Gilboa et al. 2010 and Cerreia-Vioglio et al. 2011c). □

Proof of Proposition 1. Let $\succsim$ be a binary relation on $\mathcal{F}$ that satisfies unboundedness and assume $\succsim$ is an uncertainty averse preference. By Cerreia-Vioglio et al. (2011c, Lemma 57 and Lemma 59), there exist an onto and affine function $u : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\Sigma) \to \mathbb{R}$ such that $f \succsim g$ if and only if $V(f) \geq V(g)$, where $V(f) = I(u(f))$ for all $f \in \mathcal{F}$. For each $n \in \mathbb{N}$, define $J_n : B_0(\Sigma) \to \mathbb{R}$ by $\varphi \mapsto \min_{s\in S} \varphi(s) + n$ and $I_n : B_0(\Sigma) \to \mathbb{R}$ by

$$I_n(\varphi) = \min\{I(\varphi), J_n(\varphi)\} \quad \forall \varphi \in B_0(\Sigma).$$

It is immediate to verify that I_n is a normalized, monotone, continuous, and quasiconcave functional for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $\succcurlyeq_n^{\circ}$ as in (8). It follows that there exists a nonempty, closed, and convex set C_n of Δ such that

$$\varphi \succcurlyeq_n^{\circ} \psi \quad \Longleftrightarrow \quad \varphi \succcurlyeq_{C_n} \psi$$

and

$$\varphi \succcurlyeq_{C_n} \psi \quad \Longrightarrow \quad I_n(\varphi) \geq I_n(\psi).$$

We next show that I_n satisfies (9) for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. First, given $k \in \mathbb{R}$ define $h_k = k + n$. Next, consider $\varphi \in B_0(\Sigma)$ such that $\varphi \not\succcurlyeq_{C_n} k$. This implies that $\min_{s\in S} \varphi(s) < k$. It follows that

$$I_n(\varphi) = \min\{I(\varphi), J_n(\varphi)\} \leq J_n(\varphi) < k + n = h_k,$$

proving that I_n satisfies (9). By Proposition 4, it follows that for each $n \in \mathbb{N}$, there exists a unique linearly continuous $G_n \in \mathcal{L}_{\mathrm{bd}}(\mathbb{R} \times \Delta)$ such that

$$I_n(\varphi) = \min_{p\in\Delta} G_n\left(\int \varphi\, dp, p\right) \quad \forall \varphi \in B_0(\Sigma).$$

Moreover, note that

$$\lim_n I_n(\varphi) = I(\varphi) \quad \forall \varphi \in B_0(\Sigma). \tag{21}$$

For each $n \in \mathbb{N}$, define $V_n : \mathcal{F} \to \mathbb{R}$ and $\succsim_n$ to be such that

$$V_n(f) = \min_{p\in\Delta} G_n\left(\int u(f)\, dp, p\right) \quad \forall f \in \mathcal{F}$$

and

$$f \succsim_n g \quad \Longleftrightarrow \quad V_n(f) \geq V_n(g).$$

It follows that $\succsim_n$ is an effectively bounded uncertainty averse preference for all $n \in \mathbb{N}$. By (21), we also have that

$$\lim_n V_n(f) = V(f) \quad \forall f \in \mathcal{F},$$

proving the statement. □

Proof of Proposition 2. (i) implies (ii). By Gilboa et al. (2010, Theorem 1) and since $\succsim^*$ satisfies the basic conditions, c-completeness, and independence, there exist a nonconstant and affine function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that

$$f \succsim^* g \iff \int u^*(f)\, dp \geq \int u^*(g)\, dp \quad \forall p \in C. \tag{22}$$

By Cerreia-Vioglio et al. (2011a, Proposition 1) and since $\succsim^\wedge$ satisfies the basic conditions, completeness, and risk independence, there exist a nonconstant and affine function $u^\wedge : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, u^\wedge(X)) \to \mathbb{R}$ such that

$$f \succsim^\wedge g \iff I(u^\wedge(f)) \geq I(u^\wedge(g)).$$

Moreover, by Cerreia-Vioglio et al. (2011a, Proposition 2), it follows that there exists a nonempty, closed, and convex set C° such that

$$f \succsim^\circ g \iff \int u^\wedge(f)\, dp \geq \int u^\wedge(g)\, dp \quad \forall p \in C^\circ.$$

Since $(\succsim^*, \succsim^\wedge)$ jointly satisfy consistency, it follows that for each $x, y \in X$,

$$u^*(x) \geq u^*(y) \implies u^\wedge(x) \geq u^\wedge(y).$$

By Ghirardato et al. (2004, Corollary B.3), it follows that u^* is a positive affine transformation of $u^\wedge$. Without loss of generality, we can assume that $u^\wedge = u^* = u$. By (22), we have that if $f \succsim^* g$, then $\lambda f + (1-\lambda)h \succsim^* \lambda g + (1-\lambda)h$ for all $\lambda \in (0,1]$ and all $h \in \mathcal{F}$. Since $(\succsim^*, \succsim^\wedge)$ jointly satisfy consistency, it follows that

$$\lambda f + (1-\lambda)h \succsim^\wedge \lambda g + (1-\lambda)h \quad \forall \lambda \in (0,1], \forall h \in \mathcal{F},$$

which in turn yields $f \succsim^\circ g$. In other words, we have that if $f \succsim^* g$, then $f \succsim^\circ g$. Since $B_0(\Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$ and by Ghirardato et al. (2004, Proposition A.1), this implies that $C^\circ \subseteq C$. By Cerreia-Vioglio et al. (2011a, Corollary 3), we have that

$$\min_{p \in C} \int u(f)\, dp \leq \min_{p \in C^\circ} \int u(f)\, dp \leq I(u(f)) \quad \forall f \in \mathcal{F}. \tag{23}$$

Conversely, fix $f \in \mathcal{F}$ and define $k = \min_{p \in C} \int u(f)\, dp$. Since u is affine and $C \subseteq \Delta$, we have that $k \in u(X)$. Thus, there exists $x \in X$ such that $u(x) = k$. We have two cases:

1. We have $x \succsim^\wedge y$ for all $y \in X$. By monotonicity, this implies that $x \succsim^\wedge f$, that is,
$$I(u(f)) \le I(u(x)) = u(x) = \min_{p \in C} \int u(f)\, dp.$$
2. There exists $y \in X$ such that $y \succ^\wedge x$. Define $x_\varepsilon = \varepsilon y + (1-\varepsilon)x$ for all $\varepsilon \in (0,1)$. Since u is affine and represents $\succsim^\wedge$ on X, we have that
$$u(x_\varepsilon) > u(x) \quad \forall \varepsilon \in (0,1).$$
This implies that $f \not\succsim^* x_\varepsilon$ for all $\varepsilon \in (0,1)$. Since $(\succsim^*, \succsim^\wedge)$ jointly satisfy caution, it follows that $x_\varepsilon \succsim^\wedge f$ for all $\varepsilon \in (0,1)$, that is,
$$I(u(f)) \le I(u(x_\varepsilon)) = u(x_\varepsilon) = \varepsilon u(y) + (1-\varepsilon)u(x) \quad \forall \varepsilon \in (0,1).$$
This implies that $I(u(f)) \le u(x) = \min_{p \in C} \int u(f)\, dp$.

In both cases and by (23), we obtain that $I(u(f)) = \min_{p \in C} \int u(f)\, dp$, proving the statement since f was chosen to be generic.

(ii) implies (i). This follows from Gilboa et al. (2010, Theorem 3).

The uniqueness part of the statement follows from routine arguments. □

Proof of Proposition 3. (i) implies (ii). By contradiction, suppose that jointly $(\succsim^*, \succsim^\wedge)$ do not satisfy caution. Therefore, there exist $\bar{x} \in X$ and $\bar{f} \in \mathcal{F}$ such that $\bar{f} \not\succsim^* \bar{x}$ and $\bar{f} \succ^\wedge \bar{x}$. By the premises and Gilboa et al. (2010, Theorem 1), it follows that there exist an affine and nonconstant function $u^* : X \to \mathbb{R}$ and a nonempty, closed, and convex set C such that
$$f \succsim^* g \quad \Longleftrightarrow \quad \int u^*(f)\, dp \ge \int u^*(g)\, dp \quad \forall p \in C.$$
By the premises and Herstein and Milnor (1953), it follows that there exists an affine and nonconstant function $u : X \to \mathbb{R}$ such that
$$x \succsim^\wedge y \quad \Longleftrightarrow \quad u(x) \ge u(y).$$
By Ghirardato et al. (2004, Corollary B.3), and since jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency and one binary relation between $\succsim^*$ and $\succsim^\wedge$ satisfies unboundedness, we can assume that $u^* = u$, $u(\bar{x}) = 0$, and $u(X) = \mathbb{R}$.

By the premises and Ghirardato et al. (2004, Lemma 1) there exists a normalized and positively homogeneous functional $I : B_0(\Sigma) \to \mathbb{R}$ such that
$$f \succsim^\wedge g \quad \Longleftrightarrow \quad I(u(f)) \ge I(u(g)).$$
Moreover, since jointly $(\succsim^*, \succsim^\wedge)$ satisfy consistency, we have that $f \succsim^* g$ implies $I(u(f)) \ge I(u(g))$. Define $x_a, x_b \in X$ to be such that
$$u(x_a) = I(u(\bar{f})) \quad \text{and} \quad u(x_b) = \min_{p \in C} \int u(\bar{f})\, dp.$$

Since $\bar{f} \not\succsim^* \bar{x}$ and $\bar{f} \succ^\wedge \bar{x}$, it follows that $u(x_a) > 0$ and $u(x_b) < 0$. Define now $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ to be such that for each $n \in \mathbb{N}$,

$$u(f_n) = nu(\bar{f}) \quad \text{and} \quad u(x_n) = nu(x_a).$$

This implies that for each $n \in \mathbb{N}$,

$$\min_{p\in C} \int u(f_n)\, dp = \min_{p\in C} \int nu(\bar{f})\, dp = n \min_{p\in C} \int u(\bar{f})\, dp = nu(x_b) < 0 = u(\bar{x})$$

and

$$I(u(f_n)) = I(nu(\bar{f})) = nI(u(\bar{f})) = nu(x_a) = u(x_n).$$

That is, we have that $f_n \not\succsim^* \bar{x}$ and $f_n \succsim^\wedge x_n$ for all $n \in \mathbb{N}$. Finally, observe that jointly $(\succsim^*, \succsim^\wedge)$ satisfy weak caution. Therefore, it follows that there exists $\bar{y} \succsim^\wedge \bar{x}$ such that

$$f \not\succsim^* \bar{x} \quad \Longrightarrow \quad \bar{y} \succsim^\wedge f.$$

Consider then $\bar{n} \in \mathbb{N}$ such that $u(x_{\bar{n}}) = \bar{n}u(x_a) > u(\bar{y})$. By construction, it follows that $f_{\bar{n}} \not\succsim^* \bar{x}$, but $f_{\bar{n}} \succsim^\wedge x_{\bar{n}} \succ^\wedge \bar{y}$, a contradiction.

(ii) implies (i). This is trivial. □

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