Theoretical Economics 11 (2016), 735–756 1555-7561/20160735

Stability and incentives for college admissions with budget constraints

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We study two-sided matching where one side (colleges) can make monetary transfers (offer stipends) to the other (students). Colleges have fixed budgets and strict preferences over sets of students. One different feature of our model is that colleges value money only to the extent that it allows them to enroll better or additional students. A student can attend at most one college and receive a stipend from it. Each student has preferences over college–stipend bundles.

Conditions that are essential for most of the results in the literature fail in the presence of budget constraints. We define pairwise stability and show that a pairwise stable allocation always exists. We construct an algorithm that always selects a pairwise stable allocation. The rule defined through this algorithm is incentive compatible for students: no student should benefit from misrepresenting his preferences. Finally, we show that no incentive compatible rule selects a Pareto-undominated pairwise stable allocation.

Keywords. Pairwise stability, budget constraint, strategy-proofness, Pareto-undominated.

JEL classification. C78, D44.

1. Introduction

We study two-sided matching where one side can make monetary transfers to the other, subject to fixed budget constraints. Some real-world examples include assigning students to graduate programs when colleges can offer stipends to students and their budgets for admissions are fixed; assigning research assistants to faculty members each of whom has a fixed research fund from which he can pay salaries to the assistants he hires; assigning workers to different projects where each project has a fixed total benefit that can be distributed among the workers assigned to that project, etc. To fix ideas, we use graduate college admissions as a running example.

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I would like to thank William Thomson for his guidance, invaluable comments, and unending patience. I am grateful to Atila Abdulkadiroğlu, Mustafa Oğuz Afacan, Paulo Barelli, Siwei Chen, John Duggan, Umut Mert Dur, Lars Ehlers, Matthew Elliott, Srihari Govindan, Faruk Gül, Nigar Hashimzade, Çağatay Kayı, Bettina Klaus, Flip Klijn, Asen Kochov, Fuhito Kojima, Vikram Manjunath, Tayfun Sönmez, Utku Ünver, and Gabor Virag, anbd participants of Midwest Theory and Trade Meeting 2011 in Notre Dame, PET 2011 in Indiana, SED 2011 in Montreal, 22nd International GT Conference 2011 in Stony Brook, SING 8 in Budapest, GAMES 2012 in Istanbul, 11th Meeting of SSCW in New Delhi, Sabanci University Theory Workshop for their useful comments. I am also grateful to a co-editor and two anonymous referees for their useful comments.

DOI: 10.3982/TE1731

Every year, students intending to pursue graduate studies apply to graduate programs. For each program, there is a maximum number of students that can be admitted. At the beginning of the year, each college determines a budget for graduate admissions. The admission committee uses this budget for stipends. Each college also specifies a maximal stipend that the committee can offer.[1] Each program has minimum requirements for admission (e.g., a minimum score of 1200 on the Graduate Record Exam (GRE), a minimum graduation grade point average (GPA) of 3.00, etc.). If a student does not fulfill these requirements, his application is not considered. Each college has strict preferences over groups of students and values money only to the extent that it allows the college to enroll better or additional students. The preferences of the colleges over groups of students are "responsive to their preferences over individual students," which means (i) adding a student who fulfills its minimum requirements to any group of students, provided this does not result in exceeding its capacity, makes the college at least as well off as before, and (ii) adding one student to any group of students is at least as desirable as adding another student to the same group of students if and only if the college finds the first student at least as desirable as the second one.

Each student has outside options: staying home or getting a job. For simplicity, for the remainder of the paper, we refer to this option as staying home, and we normalize the monetary opportunity from staying home to zero. Each student has preferences over college–stipend bundles, including staying home and receiving no money.

An allocation for a problem specifies which student is assigned to which college and the stipend he receives from it. A rule is a systematic way to assign students to colleges and allocate the budgets of the colleges as stipends among the students so that the total stipend each college offers does not exceed its budget.

The central notion for two-sided matching problems is stability. We are interested in a notion of stability called *pairwise stability*, which says the following: let an allocation be selected for a problem. First, no student should prefer staying home to his assignment and no college should be assigned a student who does not fulfill its minimum requirements. Next, suppose that college K prefers student A to student B, and student B is assigned to K whereas student A is not. Now, consider the following deviation by college K and student A: suppose that using the stipend offered to B, college K can offer some stipend x to A so that A prefers college K with stipend x to his initial assignment. Then both college K and student A are better off from K admitting A with stipend x and releasing B.[2] For an allocation to be pairwise stable, there should be no such deviations by a pair of a college and a student.[3]

[1]Introducing maximal stipends is not a real restriction: a college can still set the maximal stipend equal to its budget. In real life, colleges offer admission with different packages: applicants are separated into different groups based on their qualifications, and different maximal stipends are set for each group. All our positive results can be extended to this more general model.

[2]Note that to be able to afford the stipend required to admit student A, college K can release more than one student, given that it still prefers the new assignment to the initial one.

[3]We consider pairwise deviations because they are the ones likely to occur in real life: a student who is not happy with his current assignment may contact a college he is not assigned to and propose that if this college offers him admission with an appropriate stipend, he would attend it. The college would consider whether it benefits from this proposal and make a decision accordingly.

Our first main result is that for each problem, a pairwise stable allocation always exists. We show existence by means of an algorithm we construct. This algorithm always results in a pairwise stable allocation. Moreover, the rule defined through this algorithm turns out to have another desirable property: it is incentive compatible for the students, i.e., no student can ever benefit from misrepresenting his preferences. A natural question then is whether the pairwise stable allocation selected by our rule is dominated by some other pairwise stable allocation. Unfortunately, the answer is yes. However, as our final result states, no rule that selects a Pareto-undominated pairwise stable allocation (even only for students) satisfies our strong incentive compatibility requirement. Therefore, if we want a rule to be Pareto-undominated pairwise stable, then it will not satisfy the strong incentive compatibility requirement. We also show that there might be no pairwise stable allocation that is the most desirable for every student among all the pairwise stable allocations.

The rest of the paper is organized as follows. In Section 2, we discuss related literature. In Section 3, we define the model and stability notion together with the incentive compatibility requirement. In Section 4, we state our main results, and we provide the proofs in the Appendix. In Section 5, we provide our concluding remarks.

2. Related literature

The college admissions problem was first studied by Gale and Shapley (1962) in a paper in which they propose the deferred-acceptance algorithm. When there are no monetary transfers and the preferences of colleges satisfy a certain condition called *responsiveness*, the deferred-acceptance algorithm selects a coalitional stable allocation, which is immune to deviations by a group of students and colleges.[4] Without any restriction on the preferences, the set of coalitional/pairwise stable allocations may be empty (Roth and Sotomayor 1990). When money is introduced, under certain restrictions on preferences, the set of competitive allocations, which coincides with the coalitional stable set and with the core, is nonempty (Shapley and Shubik 1971, Crawford and Knoer 1981, Kelso and Crawford 1982, Sotomayor 2003, Sun and Yang 2006).

Later, the matching problem with general "contracts" was introduced (Hatfield and Milgrom 2005). Sufficient conditions on preferences that are referred to as substitutes[5] and bilateral substitutes have been provided, guaranteeing the existence of coalitional stable[6] allocations (Hatfield and Milgrom 2005, Hatfield and Kojima 2008, 2010). These papers generalize most of the papers on college admissions when monetary transfers are or are not allowed. Recently, Echenique (2012) showed that under the "substitutes" condition, the matching with contracts model can be embedded within the Kelso and Crawford (1982) labor market model. He also noted that this embedding does not hold

[4] Two notions of stability—coalitional and pairwise—coincide under the responsiveness condition.

[5] Under the substitutes condition, the deferred acceptance algorithm becomes an application of Tarski's fixed-point theorem. Adachi (2000), Fleiner (2003), and Echenique and Oviedo (2004) used the fixed-point technique in lattice theory to relate to the deferred-acceptance algorithm in the absence of monetary transfers. We give the formal definition of this condition in the paper.

[6] When preferences satisfy the bilateral substitutes, pairwise stability and coalitional stability are equivalent.

under weaker conditions of "unilateral substitutes" and "bilateral substitutes" introduced by Hatfield and Kojima (2010). A new market design application of matching with contracts, cadet–branch matching, was introduced (Sönmez and Switzer 2013, Sönmez 2013) where the unilateral substitutes condition is satisfied but the substitutes condition is not. In that sense, Sönmez and Switzer (2013) is the first paper that provides a practical application of the matching with contracts model that cannot be embedded into the Kelso and Crawford labor market model.

Unlike the rest of the literature, we introduce fixed budget constraints. Another different feature of our model is that colleges value money only to the extent that it allows them to enroll better or additional students. This kind of assumption has been made in the literature for similar problems. One example is for hospital–doctor matching problems with application fees (Afacan 2013), where each intern has a limited budget to apply for hospitals and has no incentive to save money as long as it helps him to apply to some additional hospital. Another is course bidding problems (Sönmez and Ünver 2010), where each student is given a budget of fake money that they can use to bid on courses. Since the money left at the end of the process has no value for the students, they have no incentive to save.

In the terminology of matching with contracts literature, a contract is a triple consisting of a college, a student, and a stipend. We establish that in the presence of fixed budget (feasibility) constraints, the preferences of the colleges extended over contracts may fail to satisfy the unilateral and bilateral substitutes conditions, which are shown to be sufficient conditions for the existence of stable allocations (Hatfield and Kojima 2010, Sönmez and Switzer 2013). Thus, the results in the literature do not extend to our model.

Despite the failure of the unilateral and bilateral substitutes, we provide a possibility result related to stability. We show this result by constructing a modified version of the student-proposing deferred-acceptance algorithm (Gale and Shapley 1962). One additional feature of the rule defined through this algorithm is incentive compatibility: no student can ever benefit from misrepresenting his preferences. We know from the literature that in the absence of monetary transfers, there is no stable and incentive compatible rule (Roth 1982), even if we consider incentive compatibility only for colleges (Roth 1985). Therefore, our result on incentives is the best we can hope to obtain for our model.

3. Model

There is a finite set of *colleges* $C = \{c_1, c_2, \dots, c_n\}$ and a finite set of *students* $S = \{s_1, s_2, \dots, s_k\}$. Each student may attend a college or stay home. We denote "home" as c_0. Each college c has a *capacity* $q_c \in \mathbb{Z}_+$. There are no restrictions on how many students stay home. Let $q \equiv (q_c)_{c\in C}$ be the capacity profile. A college may leave some seats empty. Each college c has strict preferences P_c over all subsets of students, 2^S. That is, P_c is a total order. These preferences are *responsive* Roth (1985), i.e., (i) for each $s \in S$, each $\bar{S} \subseteq S \setminus \{s\}$ with $|\bar{S}| < q_c$, we have $\bar{S} \cup \{s\} \; P_c \; \bar{S}$ if and only if $\{s\} \; P_c \; \varnothing$, and (ii) for each pair $s, s' \in S$, each $\bar{S} \subseteq S \setminus \{s, s'\}$ with $|\bar{S}| < q_c$, we have $\bar{S} \cup \{s\} \; P_c \; \bar{S} \cup \{s'\}$ if and only if $\{s\} \; P_c \; \{s'\}$.

Let $\mathcal{P}$ be the set of all such preferences. Let $P_C \equiv (P_c)_{c\in C}$ be the *preference profile of the colleges*. Let $\mathcal{P}^C$ be the set of all such profiles.

Each college c has a fixed *budget* $b_c \in \mathbb{R}_+$ that it can distribute as stipends to the students it admits. Let $b \equiv (b_c)_{c\in C}$ be the budget profile. For each college $c \in C$ there is a *maximal stipend* $m_c \in \mathbb{R}_+$ that it can award to any student. To make our problem interesting, these maximal stipends are such that for each $c \in C$, we have $b_c \leq m_c q_c$. Otherwise, the excess budget $(b_c - m_c q_c > 0)$ is simply wasted. Let $m = (m_c)_{c\in C}$ be the maximal stipend profile.

Each student may attend at most one college and receive a stipend from it. An assignment for a student is a college–stipend bundle and his consumption space is $C \cup \{c_0\} \times \mathbb{R}_+$. Staying home is the bundle $(c_0, 0)$. Each student s has preferences R_s over college–stipend bundles and the option of staying home with no money, namely $C \cup \{c_0\} \times \mathbb{R}_+$. For each $s \in S$, let P_s denote the strict preference relation associated with R_s, and let I_s denote the indifference relation. For each student, each $c \in C$, and each $x \in \mathbb{R}_+$, bundle (c, x) means admission by college c with stipend x. By $(c, x)\ P_s\ (c', x')$ we mean that student s prefers (c, x) to (c', x'), by $(c, x)\ R_s\ (c', x')$ we mean that student s finds (c, x) at least as desirable as (c', x'), and by $(c, x)\ I_s\ (c', x')$ we mean that he is indifferent between these two bundles.

For each $s \in S$, the preferences R_s are *quasi-linear*, which means for each pair $c, c' \in C$, each $x, x', r \in \mathbb{R}_+$, $(c, x)\, I_s\, (c', x')$ implies that $(c, x + r)\, I_s\, (c', x' + r)$ and $(c, x)\, P_s\, (c', x')$ implies that $(c, x + r)\ P_s\ (c', x' + r)$. Note that for each $s \in S$, each preference relation of s, R_s, and each pair $c, c' \in C$, if $(c, x)\ I_s\ (c', x')$ for some $x, x' \in \mathbb{R}_+$, then for each $x'' \in \mathbb{R}_+$ with $x'' < x'$, we have $(c, x)\ P_s\ (c', x'')$, and each $x''' \in \mathbb{R}_+$ with $x''' > x'$, we have $(c', x''')\ P_s\ (c, x)$.[7] Let $\mathcal{R}$ be the set of all such preference relations. Let $R_S \equiv (R_s)_{s\in S}$ be the *preference profile of the students*. Let $\mathcal{R}^S$ be the set of all such profiles. For each $s \in S$, let $R_{-s} \equiv (R_{s'})_{s'\in S\setminus\{s\}}$.

A *problem* is a list $\pi \equiv (C, S, P_C, q, b, m, R_S)$. Let Π be the set of all problems. A *matching* is a function $\mu : S \cup C \to C \cup \{c_0\} \cup 2^S$, such that (i) for each $s \in S$, $\mu(s) \in C \cup \{c_0\}$ and $|\mu(s)| = 1$; (ii) for each $c \in C$, $\mu(c) \in 2^S$ and $|\mu(c)| \leq q_c$; and (iii) for each $s \in S$ and each $c \in C$, if $\mu(s) = c$, then $s \in \mu(c)$. For each $\pi \in \Pi$, let $M(\pi)$ be the set of all matchings for π. Let $x_s^c \in \mathbb{R}_+$ be the *stipend* offered by college c to student s. For each $\pi \in \Pi$, each $\mu \in M(\pi)$, let $x_S^\mu = (x_{s_1}^{\mu(s_1)}, x_{s_2}^{\mu(s_2)}, \ldots, x_{s_k}^{\mu(s_k)}) \in \mathbb{R}_+^{|S|}$ be the stipend list associated with μ. An *allocation* for π is a matching μ together with a stipend list x_S^μ, denoted as (μ, x_S^μ). For simplicity, we use (μ, x) to denote an allocation. Alternatively, an allocation (μ, x) can be represented as a list of triples $\{(s, \mu(s), x_s^{\mu(s)})_{s\in S}\}$.

An allocation (μ, x) is *feasible* if for each $c \in C$, $\sum_{s\in\mu(c)} x_s^{\mu(s)} \leq b_c$; for each $s \in S$ with $\mu(s) \in C$, we have $x_s^{\mu(s)} \leq m_{\mu(s)}$; and for each $s' \in S$ with $\mu(s') = c_0$, we have $x_{s'}^{\mu(s')} = 0$. Let $\mathcal{A}(\pi)$ denote the set of all feasible allocations for π. A *rule* $\varphi : \Pi \to \bigcup_{\pi\in\Pi} \mathcal{A}(\pi)$ associates with each problem an allocation for it.

Next, we introduce some properties for rules. Let φ be a rule and let $\pi \in \Pi$ be a problem. Student s is admissible for college c if he fulfills the minimum requirements

[7]We thank the co-editor for pointing out this issue of continuity of the preferences.

of college c, i.e., $\{s\}\, P_c\, \varnothing$. Bundle (c, x) is acceptable for student s if it is at least as desirable as staying home with no money, i.e., $(c, x)\, R_s\, (c_0, 0)$. Our first requirement is that colleges should enroll only admissible students, and students should find their assignments acceptable. Formally, an allocation (μ, x) is individually rational for π if for each $c \in C$, each $s \in \mu(c)$ is admissible for c, and for each $s \in S$, the bundle $(\mu(s), x_s^{\mu(s)})$ is acceptable for s. Let $\mathrm{IR}(\pi)$ be the set of all individually rational allocations for π. Rule φ is *individually rational* if for each $\pi \in \Pi$, we have $\varphi(\pi) \in \mathrm{IR}(\pi)$.

Next, we introduce two notions of blocking. Let an allocation be selected for a problem.

First, consider a college and a student that are not matched to each other. Suppose that there is a stipend that the college can afford—either by using the money left in hand or by releasing some of the students initially matched to it and using the money saved in this way plus whatever was left in hand as stipend—to offer to the student such that the student prefers this college with this stipend to his initial assignment, and the college prefers the updated class.[8] Formally, *a college–student pair* (c, s) *blocks allocation* $(\mu, x) \in \mathcal{A}(\pi)$ if $\mu(s) \neq c$, and there are $\bar{S} \in 2^{\mu(c)}$ and $x'_s \in \mathbb{R}_+$ such that

(a) $[(\mu(c) \setminus \bar{S}) \cup \{s\}]\, P_c\, \mu(c)$,

(b) $|\mu(c) \setminus \bar{S}| \leq q_c - 1$,

(c) $x'_s \leq \min\{m_c, b_c - \sum_{s' \in \mu(c) \setminus \bar{S}} x_{s'}^{\mu(s')}\}$,

(d) $(c, x'_s)\, P_s\, (\mu(s), x_s^{\mu(s)})$.

Next is our second notion of blocking. Consider a college and a group of students that are not matched to this college. Suppose that there is a stipend list that the college can afford—either by using the money left in hand or by releasing some of the students initially matched to it and using the money saved in this way plus whatever was left in hand as stipend—to offer to these students such that each student prefers this college with the stipend to his initial assignment, and the college prefers the updated class.[9] Formally, *a coalition of college and students* (c, S') *blocks allocation* $(\mu, x) \in \mathcal{A}(\pi)$ if for each $s \in S'$, we have $\mu(s) \neq c$, and there are $\bar{S} \in 2^{\mu(c)}$ and a list $(x'_s)_{s \in S'} \in \mathbb{R}_+^{|S'|}$ such that

(a) $[(\mu(c) \setminus \bar{S}) \cup S']\, P_c\, \mu(c)$,

(b) $|\mu(c) \setminus \bar{S}| \leq q_c - |S'|$,

(c) $\forall s \in S'$, $x'_s \leq m_c$,

(d) $\forall s \in S'$, $(c, x'_s)\, P_s\, (\mu(s), x_s^{\mu(s)})$,

(e) $\sum_{s \in S'} x'_s \leq b_c - \sum_{s'' \in \mu(c) \setminus \bar{S}} x_{s''}^{\mu(s'')}$.

[8] The class in which this student is admitted and those students whose stipends were used to compensate this student are released.

[9] The class in which this group of students are admitted and some of the initially admitted students are released.

Next we define our two stability requirements. An allocation is *pairwise stable for* π if it is individually rational and no college–student pair can block it. Let $\mathrm{PS}(\pi)$ be the set of all pairwise stable allocations for π. Rule φ is pairwise stable if for each $\pi \in \Pi$, we have $\varphi(\pi) \in \mathrm{PS}(\pi)$. An allocation is *coalitional stable* for π if it is individually rational and no coalition of a college and a group of students can block it. Let $\mathrm{CS}(\pi)$ be the set of all coalitional stable allocations for π. Rule φ is coalitional stable if for each $\pi \in \Pi$, we have $\varphi(\pi) \in \mathrm{CS}(\pi)$.[10]

Our final requirement is a strong incentive compatibility property. No student should ever benefit from misrepresenting his preferences. Formally, rule φ is *strategy-proof* (for students) if for each $\pi \in \Pi$, each $s \in S$, each $R'_s \in \mathcal{R}$ we have

$$\varphi_s(R_C, q, b, m, R_S)\, R_s\, \varphi(R_C, q, b, m, (R'_s, R_{-s})).$$

4. Main results

As we mentioned before, in the presence of budget constraints, the substitutes conditions, which guarantee the existence of (coalitional/pairwise) stable allocation, fail (Hatfield and Kojima 2010). As a result, there may be no coalitional stable allocation. Appendix B gives formal definitions of the substitutes conditions, and establishes these claims.

What makes this paper interesting is despite the failure of the substitutes conditions, we still obtain an existence result on stability. This is our first main result.

Theorem 1. *The pairwise stable set is nonempty.*

We prove this result by constructing a modified version of the student-proposing deferred-acceptance (MSDA) algorithm,[11] which always results in a *pairwise stable* allocation. Moreover, it turns out that the rule associated with the MSDA algorithm satisfies a strong incentive compatibility requirement as follows.

Theorem 2. *The MSDA rule is strategy-proof (for students).*

We define the algorithm below and defer the proofs of Theorems 1 and 2, together with the proofs of all the other results, to Appendix A.

Let $\pi \in \Pi$ be a problem. For each student $s \in S$, let $\succ_s$ be an order on $C \cup \{c_0\}$. For $s \in S$, let $\Gamma_s(\pi)$ be the set of all possible orders on $C \cup \{c_0\}$, such that for each $\succ_s \in \Gamma_s(\pi)$, each $c \in C$, we have $c \succ_s c_0$. For each $c \in C$, let $r_c \in \mathbb{Z}_+$ be maximal number of students who can receive m_c from c, i.e., $r_c \in \mathbb{Z}_+$ is such that $b_c - r_c m_c \geq 0$ and $b_c - (r_c + 1)m_c < 0$.

[10]By definition, pairwise stability is a weaker requirement than coalitional stability. The difference between the two requirements gets even wider in the presence of budget constraints.

[11]The algorithm has the flavor of both Gale and Shapley's student-proposing deferred-acceptance algorithm (Gale and Shapley 1962), and the cumulative offer algorithm (Hatfield and Milgrom 2005, Hatfield and Kojima 2010).

To define the algorithm, it is convenient to adopt the language of matching with contracts.[12] A *contract* is a triple $d \equiv (c, s, x) \in C \times S \times \mathbb{R}_+$ to be interpreted as college c enrolling student s with stipend x. Let $\mathcal{D}$ be the set of all contracts. For each $d \in \mathcal{D}$, let $c(d)$, $s(d)$, and $x(d)$ denote the college, student, and the stipend specified in d. For each $c \in C$, let $\mathcal{D}_c$ be the set of contracts that include college c, namely $\mathcal{D}_c \equiv \{d \in \mathcal{D} : c(d) = c\}$. For each $s \in S$, let $\mathcal{D}_s$ be the set of contracts that include student s, namely $\mathcal{D}_s \equiv \{d \in \mathcal{D} : s(d) = s\}$. For each set of contracts $D \in 2^{\mathcal{D}_c}$, let $A^c(D)$ be the set of affordable feasible sets of contracts in D for college c, namely $A^c(D) \equiv \{D' \subseteq D : \sum_{d \in D'} x(d) \leq b_c, |\bigcup_{d \in D'} s(d)| \leq q_c \text{ and } \forall d, d' \in D', s(d) \neq s(d')\}$. We define the preferences of students and colleges over contracts as follows: for each $s \in S$ and each pair $d, d' \in \mathcal{D}_s$, we have $d \, R_s \, d' \Leftrightarrow (c(d), x(d)) \, R_s \, (c(d'), x(d'))$. For each $c \in C$ and each pair $D', D'' \in 2^{\mathcal{D}_c}$, we have $D' \, P_c \, D'' \Leftrightarrow \bigcup_{d \in A^c(D')} s(d) \, P_c \bigcup_{d \in A^c(D'')} s(d)$.

For each $c \in C$, define choice function $\text{Ch}_c : 2^{\mathcal{D}_c} \to 2^{\mathcal{D}_c}$, such that for each $D \in 2^{\mathcal{D}_c}$, $\text{Ch}_c(D) \equiv \{\bar{D} \in A^c(D) : \forall D' \in A^c(D), \bar{D} \, P_c \, D'\}$.

Modified Student-proposing Deferred-Acceptance algorithm. For each $s \in S$, let $\succ_s \in \Gamma_s(\pi)$ be an order. Also, for each $s \in S$, define choice function $\text{Ch}_s : 2^{\mathcal{D}} \to \mathcal{D}$, such that for each $D \in 2^{\mathcal{D}}$,

$$\text{Ch}_s(D) \equiv \{d \in D : \forall d' \in D, d \, R_s \, d' \text{ and } \forall d'' \in D \text{ with } d \, I_s \, d'', c(d) \succ_s c(d'')\}.$$

For each $c \in C$, let $x_1^c \equiv m_c$, $x_2^c \equiv b_c - r_c m_c$, and $x_3^c \equiv 0$ be the three possible stipend amounts that college c may offer to a student.[13] That is, possible contracts for each college–student pair (c, s) are (s, c, x_1^c), (s, c, x_2^c), and (s, c, x_3^c). For each $s \in S$, let $D_s \equiv \bigcup_{c \in C} \bigcup_{i \in \{1,2,3\}} (s, c, x_i^c) \cup \{(s, c_0, 0)\}$ be the set of all available contracts for s.[14]

For each $D \in 2^{\mathcal{D}_c}$, let $\text{RA}^c(D) \equiv \{D \in A^c(D) : |\{d \in D : x(d) = x_2^c\}| \leq 1\}$ be the set of restrictively affordable feasible sets of contracts in D for college c. For each $c \in C$, define the *restricted choice function* $\text{RCh}_c : 2^{\mathcal{D}_c} \to 2^{\mathcal{D}_c}$, such that for each $D \in 2^{\mathcal{D}_c}$,

$$\text{RCh}_c(D) \equiv \{\bar{D} \in \text{RA}^c(D) : \forall D' \in \text{RA}^c(D), \bar{D} \, P_c \, D'\}.$$

Step 0. For each $s \in S$, let $D_0^s \equiv D_s$ be the set of contracts available for s.

Step 1. Each $s \in S$ chooses a contract from D_0^s using $\text{Ch}_s(\cdot)$. Let $d_1^s \equiv \text{Ch}_s(D_0^s)$ be the chosen contract. Let $D_1 \equiv \bigcup_{s \in S} \text{Ch}_s(D_0^s)$ be the set of all chosen contracts. Each student s proposes to college $c(d_1^s)$ with stipend $x(d_1^s)$. For each $c \in C$, let $D_1^c \equiv \{d_1^s \in D_1 : c(d_1^s) = c\}$ be the set of contracts proposed to c. Each college c chooses contracts from D_1^c using $\text{RCh}_c(\cdot)$.[15] A contract $d \in \bigcup_{s \in S} d_1^s$ is *rejected* if there is no $c \in C$ such that $d \in \text{RCh}_c(D_1^c)$. Let RD^1 be the set of rejected contracts in Step 1. If $\text{RD}^1 = \varnothing$, we stop. If

[12]Our previous version of the algorithm used stipend offers directly instead. We thank an anonymous referee for recommending using language of contracts, which simplified most of our proofs.

[13]Note that x_2^c might be equal to x_3^c, in which case there are only two possible stipend amounts that c may offer to a student.

[14]Note that for each college c with $x_2^c = x_3^c$, a set of contracts D_s includes only one of them.

[15]We use the restricted choice function $\text{RCh}_c(\cdot)$ so as to eliminate the possibility of the college choosing more than one contract with offer x_2^c. Without such a restriction a college c may choose two average

$\mathrm{RD}^1 \neq \varnothing$, for each $s \in S$, with $d_1^s \in \mathrm{RD}^1$, we let $D_1^s \equiv D_0^s \setminus \{d_1^s\}$ and for each $s \in S$, with $d_1^s \notin \mathrm{RD}^1$, we let $D_1^s \equiv D_0^s$, and proceed to Step 2.

Step $t = 2, 3, \ldots$. Each $s \in S$ chooses a contract from D_{t-1}^s using $\mathrm{Ch}_s(\cdot)$. Let $d_t^s \equiv \mathrm{Ch}_s(D_{t-1}^s)$ be the chosen contract. Let $D_t \equiv \bigcup_{s \in S} \mathrm{Ch}_s(D_{t-1}^s)$ be the set of all chosen contracts. Each student s proposes to college $c(d_t^s)$ with stipend $x(d_t^s)$. For each $c \in C$, let $D_t^c \equiv \{d_t^s \in D_t : c(d_t^s) = c\}$ be the set of contracts proposed to c. Each college c chooses contracts from D_t^c using $\mathrm{RCh}_c(\cdot)$. Let RD^t be the set of rejected contracts in step t. If $\mathrm{RD}^t = \varnothing$, we stop. If $\mathrm{RD}^t \neq \varnothing$, for each $s \in S$, with $d_t^s \in \mathrm{RD}^t$, we let $D_t^s \equiv D_{t-1}^s \setminus \{d_t^s\}$ and for each $s \in S$, with $d_t^s \notin \mathrm{RD}^t$, we let $D_t^s \equiv D_{t-1}^s$, and proceed to step $t+1$.

The algorithm continues in this way until step ℓ at which $\mathrm{RD}^\ell = \varnothing$. Then $\mathrm{MSDA}(\pi) \equiv (\mu, x)$ is the final allocation where for each $s \in S$, $\mu(s) = c(d_\ell^s)$ and $x_s^{\mu(s)} = x(d_\ell^s)$.

Since there are finitely many students, colleges, and possible contracts for each college–student pair, and since, at each step of the algorithm, each college is made at least as well off as it was at the previous step, the MSDA algorithm terminates in finitely many steps. Appendix C illustrates how the MSDA algorithm operates by means of an example.

By Theorems 1 and 2, we know that the MSDA rule is pairwise stable and strategy-proof. Is there a "better" pairwise stable allocation than the one selected by MSDA rule? Let us first explain what we mean by better. For each $\pi \in \Pi$, each pair (μ, x), $(\bar{\mu}, \bar{x}) \in \mathcal{A}(\pi)$, allocation (μ, x) *Pareto-dominates* $(\bar{\mu}, \bar{x})$ if there is no $s \in S$, such that $(\bar{\mu}(s), \bar{x}_s^{\bar{\mu}(s)}) \; P_s \; (\mu(s), x_s^{\mu(s)})$, and no $c \in C$, such that $\bar{\mu}(c) \; P_c \; \mu(c)$; and there is $s' \in S$ such that $(\mu(s'), x_{s'}^{\mu(s')}) \; P_{s'} \; (\bar{\mu}(s'), \bar{x}_{s'}^{\bar{\mu}(s')})$, or there is $c' \in C$ such that $\mu(c') \; P_{c'} \; \bar{\mu}(c')$. An allocation (μ, x) is *Pareto-undominated pairwise stable* for π if it is pairwise stable and there is no other pairwise stable allocation $(\mu', x') \in \mathrm{PS}(\pi)$ that Pareto-dominates it. For each $\pi \in \Pi$, let $\mathrm{PU}(\pi)$ be the set of all Pareto-undominated pairwise stable allocations for π. Rule φ is Pareto-undominated pairwise stable if for each $\pi \in \Pi$, we have $\varphi(\pi) \in \mathrm{PU}(\pi)$.

So, let us restate our earlier question: Is MSDA a Pareto-undominated pairwise stable rule? The answer is no. In fact, our next result says more.

Theorem 3. *No Pareto-undominated pairwise stable rule is strategy-proof.*

Therefore, if we want a rule to be Pareto-undominated pairwise stable, then we have to give up strategy-proofness. To weaken this requirement we focus only on students. For each $\pi \in \Pi$, each pair $(\mu, x), (\bar{\mu}, \bar{x}) \in \mathcal{A}(\pi)$, allocation (μ, x) *student Pareto-dominates* $(\bar{\mu}, \bar{x})$ if for each $s \in S$, we have $(\mu(s), x_s^{\mu(s)}) \; R_s \; (\bar{\mu}(s), \bar{x}_s^{\bar{\mu}(s)})$, and there is $s' \in S$ such that $(\mu(s'), x_{s'}^{\mu(s')}) \; P_{s'} \; (\bar{\mu}(s'), \bar{x}_{s'}^{\bar{\mu}(s')})$. An allocation (μ, x) is Pareto-undominated pairwise stable for students if it is pairwise stable and there is no other pairwise stable allocation $(\mu', x') \in \mathrm{PS}(\pi)$ that student Pareto-dominates it. For each $\pi \in \Pi$, let

students with stipends x_2^c instead of one good student with stipend x_1^c (given that two average students together are better than that one good student). Making such a choice is basically trying to move in the direction of obtaining coalitional stable allocation, which we know is a dead end. Therefore, using the restricted choice function for colleges is crucial for the MSDA algorithm to result in pairwise stable allocation.

$PUs(\pi)$ be the set of all Pareto-undominated pairwise stable for students allocations for π. Rule φ is Pareto-undominated pairwise stable for students if for each $\pi \in \Pi$, we have $\varphi(\pi) \in PUs(\pi)$.

The next corollary states that even if we restrict attention only to students, the results of Theorem 3 still hold.

Corollary 1. *No Pareto-undominated pairwise stable for students rule is strategy-proof.*

Finally, since the pairwise stable set is nonempty, a natural follow-up question concerns the structure of this set? For each $\pi \in \Pi$, an allocation (μ, x) is *student-optimal pairwise stable* if for each $s \in S$, and each $(\mu', x') \in PS(\pi)$, we have (μ, x) student Pareto-dominates (μ', x'). We know from the literature that when monetary transfers are not allowed, the student-proposing deferred-acceptance algorithm always selects student-optimal stable allocation. Unfortunately, for our model, student-optimal pairwise stable allocation may not exist.[16]

Proposition 1. *There are problems for which no student-optimal pairwise stable allocation exists.*

5. Concluding remarks

We study two-sided matching problems where one-way monetary transfers are allowed. Differently from the earlier literature, in this paper, we specify a fixed budget for each college. Another different feature of our model is that colleges value money only to the extent that it allows them to enroll better or additional students. We show that introducing budget constraints results in the failure of the substitutes conditions (Proposition 2 in Appendix B), which are essential for most of the results in the literature. Despite the failure of these conditions, we obtain a positive result: a pairwise stable allocation always exists (Theorem 1). We define a rule through an algorithm we construct that not only selects a pairwise stable allocation, but also satisfies the strong incentive compatibility requirement that no student can ever benefit from misrepresenting his preferences (Theorem 2). Our final result states that we cannot find a better (in terms of efficiency) pairwise stable rule, unless we give up the incentives condition (Theorem 3 and Corollary 1).

We realize that in real life there is no centralized system and each student independently applies to graduate programs. Therefore, it is highly likely that the resulting allocations are not stable and there is the possibility for deviations. The main motivation for this paper was to first address the question, “Is there a way to assign students to colleges and allocate the budgets of the colleges among students so that no student and

[16]This kind of phenomenon has been observed in the literature as well (Kominers and Sönmez 2016). Some may view this result as one of the reasons for the negative results in Theorem 3 and Corollary 1. Moreover, for each problem π, if we define the preferences of the students as an order on the $\mathcal{A}(\pi)$, then by Proposition 1, the set of pairwise stable allocations does not have lattice structure.

college will benefit from deviating?" We answer this question in the positive and we also provide a rule that not only selects a stable allocation, but also satisfies other desirable properties.

One can also construct a modification of the college-proposing deferred-acceptance algorithm (MCDA) with some adjustments due to budget constraints, where colleges make offers starting from the maximal stipend.[17] Although the MCDA rule is pairwise stable, it is not strategy-proof (for students).

Appendix A: Omitted proofs

Proof of Theorem 1. Suppose by contradiction that there is a problem π with $\mathrm{MSDA}(\pi) = (\mu, x)$, a college–student pair (c, s) with $\mu(s) \neq c$, $\bar{S} \in 2^{\mu(c)}$, and $x'_s \in \mathbb{R}_+$, such that

(a) $[(\mu(c) \setminus \bar{S}) \cup \{s\}]\, P_c\, \mu(c)$,

(b) $|\mu(c) \setminus \bar{S}| \leq q_c - 1$,

(c) $x'_s \leq \min\{m_c, b_c - \sum_{s' \in \mu(c) \setminus \bar{S}} x_{s'}^{\mu(s')}\}$,

(d) $(c, x'_s)\, P_s\, (\mu(s), x_s^{\mu(s)})$.

First, note that for each college c, if $x_2^c \neq x_3^c$, then only one student receives stipend x_2^c from it. While blocking allocation (μ, x), college c releases students in $\bar{S}$ for one or both of the following reasons: (i) to obtain an empty seat for s and/or (ii) to obtain a stipend x'_s for s. We argue that college c can achieve both (i) and (ii) by releasing at most one student

Claim 1. *If there is $\bar{S}$ with $|\bar{S}| > 1$ satisfying requirements (a), (b), (c), and (d) above, then there is $\hat{S} \subset \bar{S}$, with $|\hat{S}| \leq 1$ satisfying these requirements.*

Proof. We consider three cases:

Case (i). Let $x'_s \in (x_2^c, x_1^c]$. Since only one student receives stipend x_2^c, there is $s' \in \bar{S}$ with $x_{s'}^c = x_1^c = m_c$. Then, so as to offer x'_s to s, it suffices to release student s'.

Case (ii). Let $x'_s \in (x_3^c, x_2^c]$. (If $x_2^c = x_3^c$, this case is redundant.) Since only one student receives stipend x_2^c, either he or some student with a higher stipend is in $\bar{S}$. Then, so as to offer x'_s to s, it suffices to release the student with stipend x_2^c.

Case (iii). Let $x'_s = x_3^c = 0$. Since s requires no stipend, it suffices to release any student. ◁

Once we know that to block and offer x'_s to s college c needs to release at most one student, we differentiate three cases:

[17] When money is a finitely divisible good, the MCDA algorithm can also be defined in the spirit of the "salary-adjustment" rule offered by Kelso and Crawford (1982), where stipend offers start from zero and increase upon rejection by the student. But due to the budget constraints, such an algorithm may not converge, i.e., there might be cycles within the algorithm.

Case 1: $x'_s \in (x^c_2, x^c_1]$. This implies that at some step t, student s applied to college c with x^c_1 (i.e., with contract (s, c, x^c_1)). Since $s \notin \mu(c)$, college c rejected s at some later step t'. In other words, at step t', there are at least r_c students who applied to c with stipend x^c_1 and are better than s for c. Since college c does not get worse off at any later step, there are at least r_c students in $\mu(c)$ with stipends x^c_1 who are better than s for c. Therefore, there is no student in $\mu(c)$ whom college c can release so as to deviate and offer x'_s to s.

Case 2: $x'_s \in (x^c_3, x^c_2]$ *and* $x^c_2 \neq x^c_3$. This case is similar to Case 1.

Case 3: $x'_s = 0$. Once again this case is similar to Case 1. □

Before providing the proof of Theorem 2, we introduce a notion and a lemma. For each $\pi \in \Pi$, if for each $c \in C$ we have $b_c = m_c q_c$, then budget constraints b are not binding for π. Let $\text{NB} \subseteq \Pi$ be the set of all problems where no budget constraint is binding.

Lemma 1. *Let* $\pi \equiv (C, S, P_C, q, b, m, R_S) \in \text{NB}$. *We define the associated problem* $\text{CA}(\pi) = (\hat{C}, \hat{S}, \hat{P}_C, \hat{q}, \hat{P}_S)$, *where* $\hat{C} = C$, $\hat{S} = S$, $\hat{q} = q$, *and* $\hat{P}_C = P_C$. *Preferences of student* $s \in \hat{S}$, $\hat{P}_s$, *over* $\hat{C}$ *are defined as follows: for each pair* $c, c' \in \hat{C}$, $c\,\hat{R}_s\,c'$ *if and only if for each* $x \in \mathbb{R}_+$, $(c, x)\,R_s\,(c', x)$. *To clarify,* $\text{CA}(\pi)$ *is the college admission problem without money that we obtain from* π. *Let* $(\mu, x) \equiv \text{MSDA}(\pi)$ *and* $\mu' \equiv \text{DA}(\text{CA}(\pi))$, *where DA is the usual student-proposing deferred-acceptance algorithm, formally defined in Gale and Shapley (1962) for the college admissions problem without monetary transfers. Then, for each* $s \in S$, *we have* $\mu(s) = \mu'(s)$ *and* $x^{\mu(s)}_s = m_c$.

Proof. Since the budgets in b are not binding for π, for each $s \in S$, $x^{\mu(s)}_s = x^{\mu(s)}_1 = m_{\mu(s)}$. The rest follows from the way $\hat{P}_C$ and $\hat{P}_S$ are defined. □

Proof of Theorem 2. Let $\pi \in \Pi$. Without loss of generality, for each $c \in C$, let $x^c_2 \neq x^c_3$. The proof trivially extends to the case when for some (or all) colleges equality holds. Let $\text{MSDA}(\pi) = (\mu, x)$. Note the following properties of (μ, x):

- If there is $c \in C$ with $|\mu(c)| > r_c + 1$, then r_c students in $\mu(c)$ obtain m_c, one student obtains $b_c - r_c m_c$, and $q_c - r_c - 1$ students obtain 0.
- If there is $c \in C$ with $|\mu(c)| = r_c + 1$, then r_c students in $\mu(c)$ obtain m_c and one student obtains $b_c - r_c m_c$.
- If there is $c \in C$ with $|\mu(c)| \leq r_c$, then all students in $\mu(c)$ obtain m_c.

From each $c \in C$, we create three "clones" of college c:

- Clone c^1 is a *type 1* clone: it has capacity $\bar{q}_{c^1} = r_c$, budget $\bar{b}_{c^1} = r_c m_c$, and maximal stipend $\bar{m}_{c^1} = m_c$.
- Clone c^2 is a *type 2* clone: it has capacity $\bar{q}_{c^2} = 1$, budget $\bar{b}_{c^2} = b_c - r_c m_c$, and maximal stipend $\bar{m}_{c^2} = b_c - r_c m_c$.
- Clone c^3 is a *type 3* clone: it has capacity $\bar{q}_{c^3} = q_c - r_c - 1$, budget $\bar{b}_{c^3} = 0$, and maximal stipend $\bar{m}_{c^3} = 0$.

Note that if there is $c \in C$ with $q_c = r_c$, then c only has type 1 clones, and if $q_c = r_c + 1$, then c has only type 1 and type 2 clones. Also note that for each $c \in C$, $b_c = \sum_{i=1}^{3} \bar{b}_{c^i}$ and $q_c = \sum_{i=1}^{3} \bar{q}_{c^i}$.

For each $c \in C$, let $C(c) \equiv \{c^1, c^2, c^3\}$ be the set of clones of c. Let $C' \equiv \bigcup_{c \in C} C(c)$. Let $\bar{b} \equiv (\bar{b}_c)_{c^i \in C'}$ be the budget profile, let $\bar{q} \equiv (\bar{q}_c)_{c^i \in C'}$ be the capacity profile, and let $\bar{m} \equiv (\bar{m}_c)_{c^i \in C'}$ be the maximal stipend profile. For each $c \in C$ and each $i \in \{1, 2, 3\}$, let $\bar{P}_{c^i}$ be the preferences of c^i over S obtained from P_c as follows: for each pair $s, s' \in S$, we have $s \, \bar{P}_{c^i} \, s'$ if and only if $s \, P_c \, s'$. For each $s \in S$, let $\bar{R}_s$ be the preferences of student s over colleges C' extended from R_s as follows: for each pair $c, \bar{c} \in C$, each $i \in \{1, 2, 3\}$, and each $x \in \mathbb{R}_+$, we have $(c, x) \, R_s \, (\bar{c}, x)$ if and only if $(c^i, x) \, \bar{R}_s \, (\bar{c}^i, x)$. For each $c \in C$, let preferences P_{c^1}, P_{c^2}, and P_{c^3} be identical to P_c. For each $c \in C$, let the maximal stipend of each clone of c be the same as the maximal stipend of c, i.e., $m_{c^1} = m_{c^2} = m_{c^3} = m_c$.

Let $\bar{\pi} = (C', S, (\bar{P}_c)_{c \in C'}, \bar{q}, \bar{b}, \bar{m}, \bar{R}_S)$. Let $(\bar{\mu}, \bar{x}) \equiv \mathrm{MSDA}(\pi')$. Note that for each $c \in C$, we have $\mu(c) \equiv \bigcup_{i \in \{1,2,3\}} \mu(c^i)$, and for each $s \in \mu(c)$, we have $x_s^{\mu(s)} = \bar{x}_s^{\bar{\mu}(s)}$. Since for each $c \in C$ and each $i \in \{1, 2, 3\}$, we have $\bar{b}_{c^i} = \bar{q}_{c^i} \bar{m}_{c^i}$, the budget constraints $\bar{b}$ are not binding for $\bar{\pi}$. Then by Lemma 1, and results in Dubins and Freedman (1981) and Roth (1982) that for the college admissions problem when monetary transfers are not allowed, the student-proposing deferred-acceptance algorithm is strategy-proof for students, we obtain our result. □

Proof of Theorem 3. Let φ be a Pareto-undominated pairwise stable rule. Let $\pi \in \Pi$ be such that $C = \{c_1, c_2, c_3\}$ and $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. Let $B = (7, 7, 7)$, $m = (7, 7, 7)$, and $q = (2, 2, 2)$.

Let preferences of students be

$$(c_1, 6) \, I_{s_1} \, (c_2, 8) \, I_{s_1} \, (c_3, 0) \, I_{s_1} \, (c_0, 9) \qquad (c_1, 0) \, I_{s_4} \, (c_2, 4 - \epsilon) \, I_{s_4} \, (c_3, 8) \, I_{s_4} \, (c_0, 12)$$

$$(c_1, 1) \, I_{s_2} \, (c_2, 6) \, I_{s_2} \, (c_3, 8) \, I_{s_2} \, (c_0, 0) \qquad (c_1, 8) \, I_{s_5} \, (c_2, 3 - \epsilon) \, I_{s_5} \, (c_3, 0) \, I_{s_5} \, (c_0, 1.99 - \epsilon)$$

$$(c_1, 0) \, I_{s_3} \, (c_2, 8) \, I_{s_3} \, (c_3, 6) \, I_{s_3} \, (c_0, 9) \qquad (c_1, 8) \, I_{s_6} \, (c_2, 0) \, I_{s_6} \, (c_3, 2) \, I_{s_6} \, (c_0, 1 - \epsilon).$$

Let the preferences of the colleges be

$$P_{c_1} : \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_1\}, \{s_3, s_4\}, \{s_2\}, \{s_3\}, \{s_4\}, \varnothing$$

$$P_{c_2} : \{s_2, s_4\}, \{s_2, s_5\}, \{s_4, s_5\}, \{s_2, s_6\}, \{s_4, s_6\}, \{s_5, s_6\}, \{s_2\}, \{s_4\}, \{s_5\}, \{s_6\}, \varnothing$$

$$P_{c_3} : \{s_1, s_3\}, \{s_3, s_6\}, \{s_1, s_6\}, \{s_3, s_5\}, \{s_1, s_5\}, \{s_3\}, \{s_1\}, \{s_5, s_6\}, \{s_6\}, \{s_5\}, \varnothing.$$

One can easily check that the outcome of the MSDA rule is $\mathrm{MSDA}(\pi) \equiv (\mu, x) \equiv \{(s_1, c_1, 7), (s_2, c_2, 7), (s_3, c_3, 7), (s_4, c_1, 0), (s_5, c_3, 0), (s_6, c_2, 0)\}$.

By Theorems 1 and 2, the MSDA rule is pairwise stable and strategy-proof. Unfortunately, MSDA is not a Pareto-undominated pairwise stable rule. To prove the theorem, all we need to do is to identify the pairwise stable allocations that Pareto-dominate $\mathrm{MSDA}(\pi) = (\mu, x)$ and show that no strategy-proof rule selects any of these allocations. Let $\mathrm{PD}(\mu, x)$ be the set of allocations (not necessarily pairwise stable) that Pareto-dominate (μ, x). For each $(\bar{\mu}, \bar{x}) \in \mathrm{PD}(\mu, x)$, the matching $\bar{\mu}$ should be one of the following two scenarios:

(i) We have $\bar{\mu} = \mu'$, where μ' is such that $\mu'(c_1) = \{s_1, s_4\}$, $\mu'(c_2) = \{s_2, s_5\}$, and $\mu'(c_3) = \{s_3, s_6\}$.

(ii) We have $\bar{\mu} = \mu''$, where μ'' is such that $\mu''(c_1) = \{s_2, s_3\}$, $\mu''(c_2) = \{s_4, s_5\}$, and $\mu''(c_3) = \{s_1, s_6\}$.

As we show next, the allocations obtained with matching μ' are not pairwise stable.

Claim 2. $(\mu', x') \notin \mathrm{PS}(\pi)$. *Let us write the allocation as* $(\mu', x') \equiv \{(s_1, c_1, x'_1), (s_2, c_2, x'_2), (s_3, c_3, x'_3), (s_4, c_1, x'_4), (s_5, c_2, x'_5), (s_6, c_3, x'_6)\}$.

Proof. Suppose by contradiction that $(\mu', x') \in \mathrm{PS}(\pi)$. Then, by individual rationality, $x'_5 \geq 1 + \epsilon$ (otherwise, student s_5 prefers $(c_0, 0)$ to (c_2, x'_5)). This implies that $x'_2 = 7 - x'_5 < 6$. But then, since $s_2\, P_{c_1}\, s_4$ and $(c_1, 0)\, P_{s_2}\, (c_2, x'_2)$, pair (c_1, s_2) blocks (μ', x') with stipend 0. $\lhd$

Therefore, for $(\bar{\mu}, \bar{x}) \in \mathrm{PD}(\mu, x)$ we have $\bar{\mu} = \mu''$. Let us write the allocation as $(\mu'', x'') \equiv \{(s_1, c_3, x''_1), (s_2, c_1, x''_2), (s_3, c_1, x''_3), (s_4, c_2, x''_4), (s_5, c_2, x''_5), (s_6, c_3, x''_6)\}$.

Next, we define some constraints on stipends to guarantee (μ'', x'') being pairwise stable:

- We have $x''_2 \geq x''_5 - 5$: otherwise, (c_2, s_2) forms a blocking pair with stipend x''_5.
- We have $x''_3 \geq x''_6 - 6$: otherwise, (c_3, s_3) forms a blocking pair with stipend x''_6.
- We have $x''_1 \geq x''_3 - 6$: otherwise, (c_1, s_1) forms a blocking pair with stipend x''_1.
- We have $x''_1 \geq x''_2 - 6$: otherwise, (c_1, s_1) forms a blocking pair with stipend x''_1.

One can easily identify that if $x''_2 + x''_3 = x''_5 - 5 + x''_6 - 6 = 3$, then no blocking pair includes either s_2 or s_3. Thus, for any pairwise stable allocation that matches both s_2 and s_3 with c_1, the following conditions should be satisfied: $x''_2 \geq x''_5 - 5$, $x''_3 \geq x''_6 - 6$, and $x''_2 + x''_3 \geq 3$. If (μ'', x'') is Pareto-undominated pairwise stable, then $x''_2 + x''_3 \geq 7$.

To make the proof clearer, we will, without loss of generality, fix values for $x''_5 = 7$, $x''_6 = 7$, and $x''_1 = 0$ (note that for given numbers, all the constraints above are satisfied). Thus, for (μ'', x'') to be Pareto-undominated pairwise stable, our constraints simplify to $x''_2 \in [2, 6]$, $x''_3 \in [1, 5]$, and $x''_2 + x''_3 = 7$.

Next we consider two cases:

Case 1: $x''_2 < 6$. Let $\hat{R}_{s_2}$ be such that $(c_1, 6)\, \hat{I}_{s_2}\, (c_2, 6)\, \hat{I}_{s_2}\, (c_3, 8)\, \hat{I}_{s_2}\, (c_0, 0)$. Let $\hat{\pi}$ be obtained from π by only changing the preferences of student s_2 to $\hat{R}_{s_2}$ instead of R_{s_2}. Then the set of Pareto-undominated pairwise stable allocations for $\hat{\pi}$ is $(\hat{\mu}, \hat{x}) = \{(s_1, c_3, \hat{x}_1), (s_2, c_1, \hat{x}_2), (s_3, c_1, \hat{x}_3), (s_4, c_2, \hat{x}_4), (s_5, c_2, \hat{x}_5), (s_6, c_3, \hat{x}_6)\}$, with constraints including $\hat{x}_1 + 6 \geq \hat{x}_2 \geq \max\{\hat{x}_5, 6\} = 6$. Otherwise, when $\hat{x}_2 < 6$, the allocation is not individually rational; when $\hat{x}_5 > \hat{x}_2 > 6$, pair (c_2, s_2) forms a blocking pair with stipend $\hat{x}_5$ and when $\hat{x}_2 > \hat{x}_1 + 6$, pair (c_1, s_1) forms a blocking pair with stipend $\hat{x}_2$. Thus, $\hat{x}_2 \geq 6$. Since $(c_1, \hat{x}_2)\, P_{s_2}\, (c_1, x''_2)$, student s_2 benefits from misreporting his preferences as $\hat{R}_{s_2}$.

Case 2: $x''_3 < 5$. Let $\hat{R}_{s_3}$ be such that $(c_1, 5)\ \hat{I}_{s_3}\ (c_2, 8)\ \hat{I}_{s_3}\ (c_3, 6)\ \hat{I}_{s_2}\ (c_0, 9)$. Let $\hat{\pi}$ be obtained from π by only changing the preferences of student s_3 to $\hat{R}_{s_3}$ instead of R_{s_3}. Then the set of Pareto-undominated pairwise stable allocations for $\hat{\pi}$ is $(\hat{\mu}, \hat{x}) = \{(s_1, c_3, \hat{x}_1), (s_2, c_1, \hat{x}_2), (s_3, c_1, \hat{x}_3), (s_4, c_2, \hat{x}_4), (s_5, c_2, \hat{x}_5), (s_6, c_3, \hat{x}_6)\}$, with constraints including $\hat{x}_1 + 6 \geq \hat{x}_3 \geq \max\{\hat{x}_6 - 1, 5\} = 5$. Otherwise, when $\hat{x}_2 < 5$, allocation is not individually rational; when $\hat{x}_6 - 1 > \hat{x}_2 > 5$, pair (c_3, s_3) forms a blocking pair with stipend $\hat{x}_6$, and when $\hat{x}_3 > \hat{x}_1 + 6$, pair (c_1, s_1) forms a blocking pair with stipend $\hat{x}_3$. Thus, $\hat{x}_3 \geq 5$. Since $(c_1, \hat{x}_3)\ P_{s_3}\ (c_1, x''_3)$, student s_3 benefits from misreporting his preferences as $\hat{R}_{s_3}$.

Therefore, by strategy-proofness, $x''_2 = 6$ and $x''_2 = 5$, and thus, $x''_2 + x''_3 = 11 > 7 = B_{c_1}$, which is a contradiction. □

Proof of Corollary 1. We know from the proof of Theorem 3 that the MSDA rule is not Pareto-undominated pairwise stable for students. Consider the problem in the proof of Theorem 3. Pick a rule that is Pareto-undominated pairwise stable for students. Note that s_2 and s_3 can be made better than they are at (μ, x), if only they are matched to c_1, with stipends greater than 1 and 2, respectively. If either one but not the other is matched, then the allocation is not pairwise stable. If neither is matched, then by Pareto-undominated pairwise stability for students, the matching has to be μ' (since at least one student should be better off), which we know is not pairwise stable. If both of them are matched to c_1, then by pairwise stability, the matching has to be μ'', for which we know that no strategy-proof rule selects such an allocation. Therefore, no rule that is Pareto-undominated pairwise stable for students is strategy-proof. □

Proof of Proposition 1. Let π be a problem with $S = \{s, s'\}$ and $C = \{c\}$. The preferences of c are $P_c : \{s, s'\}, \{s\}, \{s'\}, \varnothing_S$. Let $b_c = 13$, $m_c = 7$, and $q_c = 2$. Let $(\mu, x) \equiv \{(s, c, 7), (s', c, 6)\}$ and $(\mu', x') \equiv \{(s, c, 6), (s', c, 7)\}$ be two pairwise stable allocations for π. Student s prefers (μ, x) to (μ', x') and student s' prefers (μ', x') to (μ, x). For an allocation (μ^*, x^*) to be the student optimal pairwise stable allocation, we should have $x^*_s \geq 7$ and $x^*_{s'} \geq 7$, which is not feasible, since $x^*_s + x^*_{s'} \geq 14 > 13 = b_c$. Thus, there is no student optimal pairwise stable allocation for this problem. □

Appendix B: Substitutes conditions and coalitional stability

For the matching with contracts model, several conditions on the preferences of the colleges have been introduced that guarantee the existence of stable allocations (Hatfield and Milgrom 2005, Hatfield and Kojima 2008, 2010, Sönmez and Switzer 2013). We show that in the presence of fixed budget constraints, the preferences of the colleges over contracts do not necessarily satisfy these conditions.

In words, contracts are substitutes for a college if addition of a contract to the choice set never induces a college to choose a contract it previously rejected. Differently from the substitutes condition, the bilateral substitutes condition requires that the student in previously rejected and newly added contracts should not be in any other contract in the choice set. Formally, for each $c \in C$, contracts are *substitutes* for college c if there is no pair of contracts $d, d' \in \mathcal{D}_c$ such that $s(d) \neq s(d')$ and a set of contracts $D \subseteq \mathcal{D}_c \setminus \{d, d'\}$ such that $d \notin \mathrm{Ch}_c(D \cup \{d\})$ and $d \in \mathrm{Ch}(D \cup \{d, d'\})$. Similarly, for each $c \in C$, contracts are *bilateral substitutes* for college c if there is no pair of contracts $d, d' \in \mathcal{D}_c$ such that $s(d) \neq s(d')$ and a set of contracts $D \subseteq \mathcal{D}_c \setminus \{d, d'\}$ such that for each $d'' \in D$, $s(d'') \neq s(d')$ and $s(d'') \neq s(d')$, for which $d \notin \mathrm{Ch}_c(D \cup \{d\})$ and $d \in \mathrm{Ch}_c(D \cup \{d, d'\})$.

Next, we show that due to the presence of budget constraints, the preferences of the colleges fail to satisfy the bilateral substitutes condition.

Proposition 2. *Preferences of colleges over contracts violate the bilateral substitutes condition.*

Proof. Let π be a problem. Let $S = \{s, s', s''\}$. Consider a college c with the preferences $P_c : \{s, s', s''\}, \{s, s'\}, \{s, s''\}, \{s', s''\}, \{s\}, \{s'\}, \{s''\}$. Let $b_c = 8$, $m_c = 8$, and $q_c = 3$. Let $D \equiv \{(c, s, 6), (c, s', 4)\}$. College c chooses contract $\mathrm{Ch}_c(D) = (c, s, 6)$ and rejects $(c, s', 4)$. Now let us add another contract $(c, s'', 4)$ to D. Now, college c chooses contracts $\mathrm{Ch}_c(D \cup \{(c, s'', 4)\}) = \{(c, s', 4), (c, s'', 4)\}$ and rejects $(c, s, 6)$. This is a violation of the bilateral substitutes condition (and thus of the substitutes condition): adding contract $(c, s'', 4)$ to D results in the college choosing contract $(c, s', 4)$ that it previously rejected.[18] $\square$

Note that in the proof neither the maximal stipend nor the capacity constraint are binding. Therefore, the failure of the bilateral substitutes condition is entirely due to the budget constraint.

Given the failure of the bilateral substitutes condition, which is a sufficient condition for the existence of stable allocation, our next result may not seem so surprising: there may be no coalitionally stable allocation. To prove this, it suffices to focus on coalitions consisting of one college and at most two students. We use a modified version of the example in Mongell and Roth (1986).

Proposition 3. *There is a problem $\pi \in \Pi$ with $\mathrm{CS}(\pi) = \varnothing$.*

Proof. Let $C = \{c_1, c_2\}$ and $S = \{s_1, s_2, s_3\}$. Let $b = (5, 11)$, $m = (5, 11)$, and $q = (2, 2)$.

The preferences of students are

$$(c_1, 5)\, I_{s_1}\, (c_2, 10)\, I_{s_1}\, (c_0, 0) \qquad (c_1, 10)\, I_{s_2}\, (c_2, 8)\, I_{s_2}\, (c_0, 0) \qquad (c_1, 3.5)\, I_{s_3}\, (c_2, 2)\, I_{s_3}\, (c_0, 0).$$

The preferences of colleges are

$$P_{c_1} : \{s_2, s_3\}, \{s_3\}, \{s_2\}, \varnothing$$

$$P_{c_2} : \{s_1, s_3\}, \{s_1, s_2\}, \{s_2, s_3\}, \{s_1\}, \{s_3\}, \{s_2\}, \varnothing.$$

For each individually rational allocation, the following conditions should hold:

IR_1: College c_1 offers stipends $x_2 \in (4, 5]$ to s_2 and $x_3 \in (4.5, 5]$ to s_3.[19]

IR_2: College c_2 offers stipends $x'_1 \in (10, 11]$ to s_1, $x'_2 \in (8, 11]$ to s_2, and $x'_3 \in (2, 11]$ to s_3.

[18] Note that student s'' in the new contract is not in any previously available contracts.

[19] Since s_1 is not admissible for c_1, to satisfy individually rationality, s_1 should not be assigned to c_1.

Since $x'_1 + x'_2 > 11 = b_{c_2}$ and $x'_1 + x'_3 > 11 = b_{c_2}$, there is no individually rational allocation at which c_2 is assigned student s_1 together with some other student. Similarly, since $x_2 + x_3 > 5 = b_{c_1}$, there is no individually rational allocation at which c_1 is assigned students s_2 and s_3 simultaneously.

Let $(\mu, x) \in \mathrm{IR}(\pi)$. We consider several cases:

Case 1: $\mu(c_2) = \{s_2, s_3\}$.[20] Since s_1 is not admissible for c_1, we have $\mu(c_1) = \varnothing$ and $\mu(s_1) = c_0$. Given budget constraint $x^{c_2}_{s_2} + x^{c_2}_{s_3} \leq 11$ and by IR_2, we know that $x^{c_2}_{s_2} \in (8, 9)$ and $x^{c_2}_{s_3} \in (2, 3)$. Note that for student s_3, bundle $(c_2, x^{c_2}_{s_3})$ is better than any offer c_1 can make. Alternatively, for each $x^{c_2}_{s_2} \in (8, 9)$, we have $(c_1, 5)\ P_{s_2}\ (c_2, x^{c_2}_2)$. Then pair c_1 and s_2 block (μ, x) with stipend 5.

Case 2: Let $\mu(c_2) = \{s_1\}$. By IR_2, we know that $x^{c_2}_{s_1} \in (10, 11]$. As we mentioned before, c_1 cannot be assigned students s_2 and s_3 simultaneously. Thus, it is assigned either one of these students or neither of them. We consider several subcases.

Subcase 2.1: $\mu(c_1) = \varnothing$. Since $\{s_2\}\ P_{c_1}\ \{s_3\}\ P_{c_1}\ \varnothing$, $(c_1, 5)\ P_{s_2}\ (c_0, 0)$, and $(c_1, 5)\ P_{s_3}\ (c_0, 0)$, the pair of c_1 with either of these students blocks (μ, x) with stipend 5.

Subcase 2.2: $\mu(c_1) = \{s_2\}$. By IR_1 we know that $x^{c_1}_{s_2} \in (4, 5]$. Since $\{s_3\}\ P_{c_1}\ \{s_2\}$ and $(c_1, 5)\ P_{s_2}\ (c_0, 0)$, the pair of c_1 and s_3 blocks (μ, x) with stipend 5.

Subcase 2.3: $\mu(c_1) = \{s_3\}$. By IR_1 we know that $x^{c_1}_{s_3} \in (4.5, 5]$. Since $\{s_2, s_3\}\ P_{c_2}\ \{s_1\}$, $(c_2, 2.6)\ P_{s_3}\ (c_1, x^{c_1}_{s_3})$, and $(c_2, 8.4)\ P_{s_3}\ (c_0, 0)$, the coalition of c_1, s_2, and s_3 blocks (μ, x) with stipends 2.6 and 8.4, respectively.

Case 3: $\mu(c_2) = \{s_3\}$. Note that $\mu(s_1) = c_0$. Since $\{s_1\}\ P_{c_2}\ \{s_3\}$ and $(c_2, 11)\ P_{s_1}\ (c_0, 0)$, the pair of c_2 and s_1 blocks (μ, x) with stipend 11.

Case 4: $\mu(c_2) = \{s_2\}$. Note that $\mu(s_1) = c_0$. Since $\{s_1\}\ P_{c_2}\ \{s_2\}$ and $(c_2, 11)\ P_{s_1}\ (c_0, 0)$, the pair of c_2 and s_1 blocks (μ, x) with stipend 11.

Case 5: $\mu(c_2) = \varnothing$. Note that $\mu(s_1) = c_0$. Since $\{s_1\}\ P_{c_2}\ \varnothing_s$ and $(c_2, 11)\ P_{s_1}\ (c_0, 0)$, the pair of c_2 and s_1 blocks (μ, x) with stipend 11.

Thus, there is no coalitional stable allocation. □

Appendix C: How MSDA algorithm operates

Example. Let $C = \{c_1, c_2\}$ and $S = \{s_1, s_2, s_3, s_4\}$. Let $c_1 \succ_{s_1} c_2$, $c_2 \succ_{s_2} c_1$, $c_1 \succ_{s_3} c_2$, and $c_1 \succ_{s_4} c_2$. For each $s \in S$ and each $c \in C$, $c \succ_s \varnothing$. Let $B = (7, 7)$, $m = (5, 6)$, and $q = (2, 2)$.

Preferences of students and the colleges are

$$(c_1, 5)\ I_{s_1}\ (c_2, 0)\ I_{s_1}\ (c_0, 6) \qquad (c_1, 0)\ I_{s_2}\ (c_2, 4)\ I_{s_2}\ (c_0, 0)$$

$$(c_1, 0)\ I_{s_3}\ (c_2, 2)\ I_{s_3}\ (c_0, 1) \qquad (c_1, 2)\ I_{s_4}\ (c_2, 1)\ I_{s_4}\ (c_0, 0)$$

$$P_{c_1}: \{s_1, s_3\}, \{s_1, s_2\}, \{s_1\}, \{s_2, s_3\}, \{s_3\}, \{s_2\}, \varnothing, \{s_4\}$$

$$P_{c_2}: \{s_2, s_3\}, \{s_2, s_4\}, \{s_1, s_2\}, \{s_3, s_4\}, \{s_1, s_3\}, \{s_2\}, \{s_3\}, \{s_1, s_4\}, \{s_4\}, \{s_1\}, \varnothing.$$

We have $x^{c_1}_1 = 5$, $x^{c_1}_2 = 2$, $x^{c_1}_3 = 0$ and $x^{c_2}_1 = 6$, $x^{c_2}_2 = 1$, $x^{c_2}_3 = 0$. For each $s \in S$, we have $D_s = \{(s, c_1, 5), (s, c_1, 2), (s, c_1, 0), (s, c_2, 6), (s, c_2, 1), (s, c_2, 0), (s, c_0, 0)\}$.

[20] Note that the best set of students college c_2 can obtain at any individually rational allocation is the pair of s_2 and s_3.

Step 0. For each $s \in S$, we have $D_0^s = D_s$.

Step 1. Each student $s \in S$ chooses the most preferred contract in D_0^s, namely,

s_1 chooses $\mathrm{Ch}_{s_1}(D_0^{s_1}) = (s_1, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_2 chooses $\mathrm{Ch}_{s_2}(D_0^{s_2}) = (s_2, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_3 chooses $\mathrm{Ch}_{s_3}(D_0^{s_3}) = (s_3, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_4 chooses $\mathrm{Ch}_{s_4}(D_0^{s_4}) = (s_4, c_2, 6)$ and thus proposes to c_2 with stipend 6.

We have $D_1^{c_1} = \{(s_2, c_1, 5), (s_3, c_1, 5)\}$ and $D_1^{c_2} = \{(s_1, c_2, 6), (s_4, c_2, 6)\}$. Each college c chooses its most preferred affordable set of contracts in D_1^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_1^{c_1}) = (s_3, c_1, 5)$ and rejects $(s_2, c_1, 5)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_1^{c_2}) = (s_4, c_2, 6)$ and rejects $(s_1, c_2, 6)$.

We have $D_1^{s_3} = D_0^{s_3}$, $D_1^{s_4} = D_0^{s_4}$,

$$D_1^{s_1} = D_0^{s_1} \setminus (s_1, c_2, 6) = \{(s_1, c_1, 5), (s_1, c_1, 2), (s_1, c_1, 0), (s_1, c_2, 1), (s_1, c_2, 0), (s_1, c_0, 0)\}$$
$$D_1^{s_2} = D_0^{s_2} \setminus (s_2, c_1, 5) = \{(s_2, c_1, 2), (s_2, c_1, 0), (s_2, c_2, 6), (s_2, c_2, 1), (s_2, c_2, 0), (s_2, c_0, 0)\}.$$

We proceed to Step 2.

Step 2. Each student $s \in S$ chooses the most preferred contract in D_1^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_1^{s_1}) = (s_1, c_2, 1)$ and thus proposes to c_2 with stipend 1

s_2 chooses $\mathrm{Ch}_{s_2}(D_1^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_1^{s_3}) = (s_3, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_4 chooses $\mathrm{Ch}_{s_4}(D_1^{s_4}) = (s_4, c_2, 6)$ and thus proposes to c_2 with stipend 6.

We have $D_2^{c_1} = \{(s_3, c_1, 5)\}$ and $D_2^{c_2} = \{(s_1, c_2, 1), (s_2, c_2, 6), (s_4, c_2, 6)\}$. Each college c chooses its most preferred affordable set of contracts in D_2^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_2^{c_1}) = (s_3, c_1, 5)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_2^{c_2}) = \{(s_1, c_2, 1), (s_2, c_2, 6)\}$ and rejects $(s_4, c_2, 6)$.

We have $D_2^{s_1} = D_1^{s_1}$, $D_2^{s_2} = D_1^{s_2}$, $D_2^{s_3} = D_1^{s_3}$,

$$D_2^{s_4} = D_1^{s_4} \setminus (s_4, c_2, 6) = \{(s_4, c_1, 5), (s_4, c_1, 2), (s_4, c_1, 0), (s_4, c_2, 1), (s_4, c_2, 0), (s_4, c_0, 0)\}.$$

We proceed to Step 3.

Step 3. Each student $s \in S$ chooses the most preferred contract in D_2^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_2^{s_1}) = (s_1, c_2, 1)$ and thus proposes to c_2 with stipend 1

s_2 chooses $\mathrm{Ch}_{s_2}(D_2^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_2^{s_3}) = (s_3, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_4 chooses $\mathrm{Ch}_{s_4}(D_2^{s_4}) = (s_4, c_1, 5)$ and thus proposes to c_1 with stipend 5.

We have $D_3^{c_1} = \{(s_3, c_1, 5), (s_4, c_1, 5)\}$ and $D_3^{c_2} = \{(s_1, c_2, 1), (s_2, c_2, 6)\}$. Each college c chooses its most preferred affordable set of contracts in D_3^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_3^{c_1}) = (s_3, c_1, 5)$ and rejects $(s_4, c_1, 5)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_3^{c_2}) = \{(s_1, c_2, 1), (s_2, c_2, 6)\}$.

We have $D_3^{s_1} = D_2^{s_1}$, $D_3^{s_2} = D_2^{s_2}$, $D_3^{s_3} = D_2^{s_3}$,

$$D_3^{s_4} = D_2^{s_4} \setminus (s_4, c_1, 5) = \{(s_4, c_1, 2), (s_4, c_1, 0), (s_4, c_2, 1), (s_4, c_2, 0), (s_4, c_0, 0)\}.$$

We proceed to Step 4.

Step 4. Each student $s \in S$ chooses the most preferred contract in D_3^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_3^{s_1}) = (s_1, c_2, 1)$ and thus proposes to c_2 with stipend 1

s_2 chooses $\mathrm{Ch}_{s_2}(D_3^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_3^{s_3}) = (s_3, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_4 chooses $\mathrm{Ch}_{s_4}(D_3^{s_4}) = (s_4, c_1, 2)$ and thus proposes to c_1 with stipend 2.

We have $D_4^{c_1} = \{(s_3, c_1, 5), (s_4, c_1, 2)\}$ and $D_4^{c_2} = \{(s_1, c_2, 1), (s_2, c_2, 6)\}$. Each college c chooses its most preferred affordable set of contracts in D_3^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_4^{c_1}) = (s_3, c_1, 5)$ and rejects $(s_4, c_1, 2)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_4^{c_2}) = \{(s_1, c_2, 1), (s_2, c_2, 6)\}$.

We have $D_4^{s_1} = D_3^{s_1}$, $D_4^{s_2} = D_3^{s_2}$, $D_4^{s_3} = D_3^{s_3}$,

$$D_4^{s_4} = D_3^{s_4} \setminus (s_4, c_1, 2) = \{(s_4, c_1, 0), (s_4, c_2, 1), (s_4, c_2, 0), (s_4, c_0, 0)\}.$$

We proceed to Step 5.

Step 5. Each student $s \in S$ chooses the most preferred contract in D_4^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_4^{s_1}) = (s_1, c_2, 1)$ and thus proposes to c_2 with stipend 1

s_2 chooses $\mathrm{Ch}_{s_2}(D_4^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_4^{s_3}) = (s_3, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_4 chooses $\mathrm{Ch}_{s_4}(D_4^{s_4}) = (s_4, c_2, 1)$ and thus proposes to c_2 with stipend 1.

We have $D_5^{c_1} = \{(s_3, c_1, 5)\}$ and $D_5^{c_2} = \{(s_1, c_2, 1), (s_2, c_2, 6), (s_4, c_2, 1)\}$. Each college c chooses its most preferred affordable set of contracts in D_5^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_5^{c_1}) = (s_3, c_1, 5)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_5^{c_2}) = \{(s_2, c_2, 6), (s_4, c_2, 1)\}$ and rejects $(s_1, c_2, 1)$.

We have $D_5^{s_2} = D_4^{s_2}$, $D_5^{s_3} = D_4^{s_3}$, $D_5^{s_4} = D_4^{s_4}$,

$$D_5^{s_1} = D_4^{s_1} \setminus (s_1, c_2, 1) = \{(s_1, c_1, 5), (s_1, c_1, 2), (s_1, c_1, 0), (s_1, c_2, 0), (s_1, c_0, 0)\}.$$

We proceed to Step 6.

Step 6. Each student $s \in S$ choose the most preferred contract in D_5^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_5^{s_1}) = (s_1, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_2 chooses $\mathrm{Ch}_{s_2}(D_5^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_5^{s_3}) = (s_3, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_4 chooses $\mathrm{Ch}_{s_4}(D_5^{s_4}) = (s_4, c_2, 1)$ and thus proposes to c_2 with stipend 1.

We have $D_6^{c_1} = \{(s_1, c_1, 5), (s_3, c_1, 5)\}$ and $D_6^{c_2} = \{(s_2, c_2, 6), (s_4, c_2, 1)\}$. Each college c chooses its most preferred affordable set of contracts in D_6^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_6^{c_1}) = (s_1, c_1, 5)$ and rejects $(s_3, c_1, 5)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_6^{c_2}) = \{(s_2, c_2, 6), (s_4, c_2, 1)\}$.

We have $D_6^{s_1} = D_5^{s_1}$, $D_6^{s_2} = D_5^{s_2}$, $D_6^{s_4} = D_5^{s_4}$,

$$D_6^{s_3} = D_5^{s_3} \setminus (s_3, c_1, 5) = \{(s_3, c_1, 2), (s_3, c_1, 0), (s_3, c_2, 6), (s_3, c_2, 1), (s_3, c_2, 0), (s_3, c_0, 0)\}.$$

We proceed to Step 7.

Step 7. Each student $s \in S$ chooses the most preferred contract in D_6^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_6^{s_1}) = (s_1, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_2 chooses $\mathrm{Ch}_{s_2}(D_6^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_6^{s_3}) = (s_3, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_4 chooses $\mathrm{Ch}_{s_4}(D_6^{s_4}) = (s_4, c_2, 1)$ and thus proposes to c_2 with stipend 1.

We have $D_7^{c_1} = \{(s_1, c_1, 5)\}$ and $D_7^{c_2} = \{(s_2, c_2, 6), (s_3, c_2, 6), (s_4, c_2, 1)\}$. Each college c chooses its most preferred affordable set of contracts in D_7^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_7^{c_1}) = (s_1, c_1, 5)$

c_2 chooses $\mathrm{RCh}_{c_2}(D_7^{c_2}) = \{(s_2, c_2, 6), (s_4, c_2, 1)\}$ and rejects $(s_3, c_2, 6)$.

We have $D_7^{s_1} = D_6^{s_1}$, $D_7^{s_2} = D_6^{s_2}$, $D_7^{s_4} = D_6^{s_4}$,

$$D_7^{s_3} = D_6^{s_3} \setminus (s_3, c_2, 6) = \{(s_3, c_1, 2), (s_3, c_1, 0), (s_3, c_2, 1), (s_3, c_2, 0), (s_3, c_0, 0)\}.$$

We proceed to Step 8.

Step 8. Each student $s \in S$ chooses the most preferred contract in D_7^s:

s_1 chooses $\mathrm{Ch}_{s_1}(D_7^{s_1}) = (s_1, c_1, 5)$ and thus proposes to c_1 with stipend 5

s_2 chooses $\mathrm{Ch}_{s_2}(D_7^{s_2}) = (s_2, c_2, 6)$ and thus proposes to c_2 with stipend 6

s_3 chooses $\mathrm{Ch}_{s_3}(D_7^{s_3}) = (s_3, c_1, 2)$ and thus proposes to c_1 with stipend 2

s_4 chooses $\mathrm{Ch}_{s_4}(D_7^{s_4}) = (s_4, c_2, 1)$ and thus proposes to c_2 with stipend 1.

We have $D_8^{c_1} = \{(s_1, c_1, 5), (s_3, c_1, 2)\}$ and $D_8^{c_2} = \{(s_2, c_2, 6), (s_4, c_2, 1)\}$. Each college c chooses its most preferred affordable set of contracts in D_8^c:

c_1 chooses $\mathrm{RCh}_{c_1}(D_8^{c_1}) = \{(s_1, c_1, 5), (s_3, c_1, 2)\}$

c_2 chooses $\mathrm{RCh}_{c_2}(D_8^{c_2}) = \{(s_2, c_2, 6), (s_4, c_2, 1)\}$.

Since no offer is rejected, the algorithm stops. The final allocation is

$$\mathrm{MSDA}(\pi) = \{(s_1, c_1, 5), (s_2, c_2, 6), (s_3, c_1, 2), (s_4, c_2, 1)\}. \qquad \Diamond$$

References

Adachi, Hiroyuki (2000), "On a characterization of stable matchings." *Economics Letters*, 68, 43–49. [737]

Afacan, Mustafa Ŏ. (2013), "Application fee manipulations in matching markets." *Journal of Mathematical Economics*, 49, 446–453. [738]

Crawford, Vincent P. and Elsie Marie Knoer (1981), "Job matching with heterogeneous firms and workers." *Econometrica*, 49, 437–450. [737]

Dubins, Lester E. and David A. Freedman (1981), "Machiavelli and the Gale–Shapley algorithm." *American Mathematical Monthly*, 88, 485–494. [747]

Echenique, Federico (2012), "Contracts versus salaries in matching." *American Economic Review*, 102, 594–601. [737]

Echenique, Federico and Jorge Oviedo (2004), "Core many-to-one matchings by fixed-point methods." *Journal of Economic Theory*, 115, 358–376. [737]

Fleiner, Tamás (2003), "A fixed point approach to stable matching and some applications." *Mathematics of Operations Research*, 28, 103–126. [737]

Gale, David and Lloyd S. Shapley (1962), "College admissions and the stability of marriage." *American Mathematical Monthly*, 69, 9–15. [737, 738, 741, 746]

Hatfield, John W. and Fuhito Kojima (2008), "Matching with contracts: Comment." *American Economic Review*, 98, 1189–1194. [737, 749]

Hatfield, John W. and Fuhito Kojima (2010), "Substitutes and stability for matching with contracts." *Journal of Economic Theory*, 145, 1704–1723. [737, 738, 741, 749]

Hatfield, John W. and Paul R. Milgrom (2005), "Matching with contracts." *American Economic Review*, 95, 913–935. [737, 741, 749]

Kelso, Alexander S. and Vincent P. Crawford (1982), "Job matching, coalition formation, and gross substitutes." *Econometrica*, 50, 1483–1504. [737, 745]

Kominers, Scott Duke and Tayfun Sönmez (2016), "Matching with slot-specific priorities: Theory." *Theoretical Economics*, 11, 683–710. [744]

Mongell, Susan J. and Alvin E. Roth (1986), "A note on job matching with budget constraints." *Economics Letters*, 21, 135–138. [750]

Roth, Alvin E. (1982), "The economics of matching: Stability and incentives." *Mathematics of Operations Research*, 7, 617–628. [738, 747]

Roth, Alvin E. (1985), "The college admissions problem is not equivalent to the marriage problem." *Journal of Economic Theory*, 36, 277–288. [738]

Roth, Alvin E. and Marilda A. Oliveira Sotomayor (1990), *Two-Sided Matching*. Cambridge University Press, Cambridge. [737]

Shapley, Lloyd S. and Martin Shubik (1971), "The assignment game I: The core." *International Journal of Game Theory*, 1, 111–130. [737]

Sönmez, Tayfun (2013), "Bidding for army career specialties: Improving the ROTC branching mechanism." *Journal of Political Economy*, 121, 186–219. [738]

Sönmez, Tayfun and Tobias B. Switzer (2013), "Matching with (branch-of-choice) contracts at the United States Military Academy." *Econometrica*, 81, 451–488. [738, 749]

Sönmez, Tayfun and M. Utku Ünver (2010), "Course bidding at business schools." *International Economic Review*, 51, 99–123. [738]

Sotomayor, Marilda (2003), "A labor market with heterogeneous firms and workers." *International Journal of Game Theory*, 31, 269–283. [737]

Sun, Ning and Zaifu Yang (2006), "Equilibria and indivisibilities: Gross substitutes and complements." *Econometrica*, 74, 1385–1402. [737]

Co-editor Faruk Gul handled this manuscript.

Submitted 2013-11-27. Final version accepted 2015-7-31. Available online 2015-8-5.