Theoretical Economics 11 (2016), 937–969 1555-7561/20160937

Negotiation across multiple issues

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In the present work, agreement on allocation of payments from multiple issues requires unanimous consent of all parties involved. The agents are assumed to know the aggregate payoffs but do not know their decomposition by issues. This framework applies to many real-world problems, such as the formation of joint ventures. We present a novel solution concept to the problem, termed the multicore, wherein an agent consents to participate in the grand coalition if she can envision a decomposition of the proposed allocation for which each coalition to which she belongs derives greater benefit on each issue by cooperating with the grand coalition rather than operating alone. An allocation is in the multicore if all agents consent to participate in the grand coalition. We provide a theorem characterizing the nonemptiness of the multicore and show that the multicore generalizes the core. We prove that the approach of the multicore has the potential to increase cooperation among parties beyond that of solving issues independently. In addition, we establish that the multicore wherein agents take into account the specifics of the original issues is a refinement of the core of the sum of individual issues in which such information is ignored.

Keywords. Cooperative games, issue linkage, multi-issue bargaining, multi-core.
JEL classification. C71, C78, D74.

1. Introduction

A common practice for firms wishing to collaborate is to form a joint venture that entails establishing a new firm with sole responsibility for the joint activity, overriding the fact that the participating firms are the actual owners. When interested in collaborating on multiple projects, firms form either separate joint ventures for each project or a single joint venture that is responsible for all projects, thereby linking the projects.

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We are grateful to Faruk Gul and an anonymous referee for significantly improving this work. We also thank Shiri Alon, Pesach Ben Porat, Geoffroy de Clippel, Eddie Dekel, Ran Eilat, Chaim Fershtman, Itzhak Gilboa, Yoram Halevy, Ehud Lehrer, Mike Peters, Anna Rubinchik, Ariel Rubinstein, David Wettstein, and the participants of the 2014 Conference on Game Theory at Stony Brook University and various seminars in TAU, BIU and UBC for fruitful discussions and insightful comments.

DOI: 10.3982/TE1865

In the problem considered in this work, as in the example above, a group of agents aspire to solve several issues simultaneously. Agreement, when reached, is a single contract dividing the aggregate payoffs of all issues. The issues are independent a priori so the agents are aware of the potential gains to be derived from each one of them. Here we consider a specific type of issue linkage generated by agents' knowledge of the aggregate payoffs coupled with their ignorance of its specific breakdown by issues. We explore how such linkage of issues affects the set of acceptable aggregate allocations.

This model uses a reduced-form approach to bargaining by presenting a multi-issue problem as a set of cooperative games with transferable utility that we refer to as a *multi-game*. In contrast to the approach in noncooperative game theory that strongly depends on specific protocols, the setup of cooperative games allows us to concentrate on linkage of issues while at the same time removing those considerations limited to a particular context. This enables us to provide a general perspective on the problem.

In standard cooperative game theory an issue is represented by a characteristic function that assigns a value to each coalition. This value is typically interpreted as the available resources to be allocated among members of a coalition in the event that the issue is resolved without the cooperation of the other agents. The most prominent solution concept in a single-issue problem is the core. According to this concept an agent rejects a proposed allocation if there is a coalition to which she belongs that can attain more on its own relative to the total amount its members obtain according to that allocation. The core is the set of all efficient allocations that cannot be rejected by any agent. A trivial extension of the core to multiple issues is to require that a solution be a sum of the solutions in the cores of the individual issues. This approach solves each issue separately, forgoing the possibility of making use of issue linkage to enhance cooperation.

We propose a different extension of the core to multiple issues that we term the *multicore*,[1] wherein agents know the issues' characteristic functions and the agents' aggregate payoffs, but are uninformed of breakdown of payoffs by issues. Therefore, when considering to accept an aggregate allocation, agents will have to form a subjective belief on the decomposition of payoffs to issues that conforms to the available information. An agent will reject a proposed aggregate allocation if, for every possible decomposition of payoffs to issues, there is a coalition to which the agent belongs that would gain more operating alone relative to the total amount its members receive in some issue according to that decomposition. Alternatively, an agent will be able justify her acceptance of aggregate allocation if she is shown that each coalition to which she belongs is adequately compensated in each issue according to some decomposition of aggregate payoffs. An allocation is in the multicore if it can be justified by all agents. Our first result provides a characterization of the nonemptiness of the multicore by generalizing the balancing weights of Bondareva (1963) and Shapley (1967).

When an allocation can be decomposed into solutions for the individual issues, all agents can use the same justification to support the allocation. This implies that any solution that can be implemented by resolving each issue independently can also be

[1]The term multicore appears in Hwang and Liao (2011). However, apart from the name, there is no relation between their work and the solution concept suggested here.

obtained when issues are linked. Generally, however, a justification for one agent may not be a justification for another. This is because each agent's only concern is for full compensation for the coalitions to which she belongs, having no interest in the compensation awarded to other coalitions. Actually, it is precisely when agents have no common justification for supporting a given allocation that solving problems collectively is beneficial. In such cases, when resolving issues independently, there is a subset of agents who recognize that they are undercompensated, ruling out the possibility of reaching an agreement. When agents are only informed of aggregate payments, each agent still views the formation of the grand coalition as a win–lose situation, being aware that some agents are inadequately compensated, yet somehow believes that she is on the winning side.

A third extension of the core to solving multi-issue problems is the creation of a new single game wherein the values of each coalition in the individual games are summed together. It is well known that the core of the sum of individual games contains the sum of the cores of the individual games, thereby increasing cooperation. Bloch and de Clippel (2010) provide conditions for the coincidence of these two concepts.[2] Fernández et al. (2002, 2004) propose a solution to a game that is a weighted sum of the individual games where the weights are unknown.

In the sum of individual games, issues are linked under the presumption that agents are unaware that the problem is composed of multiple issues. On the other hand, one may interpret this as a situation in which agents do know that there are multiple issues, but must operate with the same subset of agents on all issues.[3] An implicit assumption in the current framework is that agents may choose to cooperate on a subset of issues, implying that the multicore is a refinement of the core of the sum of individual games.

2. The model

2.1 *Preliminaries*

The problem under consideration is that of a group of agents $N = \{1, 2, \dots, n\}$ who are trying to reach unanimous consent on m issues. The aim is to understand when all agents would agree to cooperate on all issues, thereby forming the grand coalition.

This setting is explored in the framework of cooperative games with transferable utility. A single game, $G = (N; V)$ is defined by a set of agents N and a single characteristic function V, which assigns a real number to every nonempty coalition. Typically, $V(S)$ is interpreted as the value attained by coalition S when operating independently. We extend this definition to our setting of multiple issues by defining the multigame as a set of agents and a set of characteristic functions.

[2] Kalai (1977) and Ponsati and Watson (1997) address a similar question in the context of Nash bargaining.

[3] Nax (2014) studies an adaptation of the core to an environment where there are externalities between the issues. Diamantoudi et al. (2013) explore the Shapley value in a similar environment. Assa et al. (2014) study an environment where every agent must participate in exactly one game.

Definition 1. An m-issue multigame $\bar{G}$ is a pair $\bar{G} = (N; \bar{V})$, where $\bar{V}$ is a set of characteristic functions $\bar{V} = \{V_1, V_2, \dots, V_m\}$ such that for every $j \in \{1, \dots, m\}$, $V_j : 2^N \to \mathbb{R}$ and $V_j(\varnothing) = 0$.[4]

As in standard cooperative game theory, an efficient aggregate payoff vector of $\bar{V}$ allocates all available surplus among the agents.

Definition 2. The allocation $x \in \mathbb{R}^n$ is an efficient aggregate payoff vector of $\bar{V}$ if $\sum_{i=1}^n x_i = \sum_{V_j \in \bar{V}} V_j(N)$.

A payoff vector x is in the core of a single game, $x \in C(V)$, if it is efficient and $\forall S \in 2^N$, $\sum_{i \in S} x_i \geq V(S)$. The standard interpretation is that when the payment of a coalition is strictly lower than its worth ($\sum_{i \in S} x_i < V(S)$), its members will operate alone. In those cases, dividing the coalition's worth among themselves allows them to be strictly better off relative to receiving their payments from operating within the grand coalition.

We present an extension of this well known solution to the multigame setup. The agents consider payoff vector x that describes their total payoff on all issues. The breakdown by issue, however, is unspecified. Not knowing the decomposition of a proposed aggregate payoff vector gives rise to many possible beliefs as to how x could be decomposed. The following describes the set of possible breakdowns:

Definition 3. The set of efficient decomposition matrices of an aggregate payoff vector x is

$$\hat{Y}(\bar{V}, x) = \left\{ y \in \mathbb{R}^{n \times m} \,\middle|\, \forall i \in N : \sum_{j \in \{1,\dots,m\}} y_{i,j} = x_i, \forall V_j \in \bar{V} : \sum_{k=1}^n y_{k,j} = V_j(N) \right\}.$$

Definition 3 ensures that the decomposition matrices envisioned by the agents conform to the available information. It guarantees that the decomposition of payoffs adds up to proposed vector x. In addition, the structure of the multigame entails the exhaustion of resources in the decomposition of payoffs in each issue and the inability to shift resources from one issue to another. Since all agents share the same information, their set of efficient decomposition matrices is the same.

An efficient aggregate payoff vector is in the multicore if every agent has an efficient decomposition matrix that justifies her participation in the grand coalition.

Definition 4. An efficient aggregate payoff vector x is in the multicore, $x \in M(\bar{V})$, if for every agent i there exists an efficient decomposition matrix $y^i \in \hat{Y}(\bar{V}, x)$ such that $\forall V_j \in \bar{V}$, $\forall S \in \{T \cup \{i\} \mid T \subseteq N \setminus \{i\}\}$: $\sum_{k \in S} y^i_{k,j} \geq V_j(S)$. We refer to y^i as a justification matrix of agent i with regard to the payoff vector x.

[4]When no confusion arises, we abuse notation by equating a multigame $\bar{G} = (N; \bar{V})$ with its set of characteristic functions $\bar{V}$.

Definition 4 certifies that y^i is sufficient justification for agent i to consent to x if any coalition that includes agent i has no reason to block the formation of the grand coalition on any one of the issues. While many efficient decomposition matrices exist, only one need satisfy the condition in Definition 4 for each agent. Furthermore, justification matrices may differ among agents since each agent considers only coalitions in which she participates and disregards all others.

Example 1. Let $\bar{V}$ be the following two-issue multigame with three agents:

$$V_1(S)=\begin{cases}0 & \text{if } |S|=1\\ \frac{3}{4} & \text{if } |S|=2\\ 1 & \text{if } |S|=3\end{cases};\qquad V_2(S)=\begin{cases}0 & \text{if } |S|=1\\ 0 & \text{if } |S|=2\\ 1 & \text{if } |S|=3.\end{cases}$$

The core of V_1 is empty[5] and, therefore, can be interpreted as a difficult problem to resolve. The core of V_2 includes every nonnegative payoff vector whose elements add up to 1 and, therefore, can be interpreted as a problem that is easy to solve. While it is impossible to reach unanimous agreement on all issues when they are solved independently, agreement can be reached by linking the issues, since the multicore of $\bar{V}$ is nonempty. For example, a payoff vector in which every agent gets $\frac{2}{3}$ is in the multicore.[6] The following decomposition matrices, one for every agent, support such aggregate payoffs:

$$y^1=\begin{pmatrix}\frac{2}{3} & 0\\ \frac{1}{6} & \frac{1}{2}\\ \frac{1}{6} & \frac{1}{2}\end{pmatrix};\qquad y^2=\begin{pmatrix}\frac{1}{6} & \frac{1}{2}\\ \frac{2}{3} & 0\\ \frac{1}{6} & \frac{1}{2}\end{pmatrix};\qquad y^3=\begin{pmatrix}\frac{1}{6} & \frac{1}{2}\\ \frac{1}{6} & \frac{1}{2}\\ \frac{2}{3} & 0\end{pmatrix}.$$

Each decomposition matrix y^i allocates a total of 1 to each issue so that they all are efficient. Moreover, in both issues, according to y^i, every coalition S to which agent i belongs achieves its worth (and possibly more).

Notice that neither agent 2 nor agent 3 would be convinced by y^1 to consent to the proposed payoff vector since their payoffs in V_1 are too low (agents 2 and 3 together receive $\frac{1}{3}$ according to y^1 while they can obtain $\frac{3}{4}$ by acting independently). $\Diamond$

2.2 *The story of the multicore*

In the interpretation of the multicore, not knowing the decomposition of x by issues, the agents must form a belief regarding its decomposition. This belief is subjective and may differ across agents. Based on this subjective belief, each agent goes over all the issues independently, choosing whether or not to deviate from the grand coalition on any specific issue. If the agent does deviate on some issue, she receives a certain share of the worth of the deviating coalition, whereas if she does not, she expects payment according to her decomposition of x.

[5]Each pair must receive at least $\frac{3}{4}$, but the total payoffs of the three agents is less than $\frac{9}{8}$.
[6]The multicore is $M(\bar{V})=\{x\in[\frac{1}{2},1]^3 \mid x_1+x_2+x_3=2\}$.

To understand Definition 4, let $y \in \hat{Y}(\bar{V}, x)$ be Agent i's belief regarding the decomposition of aggregate payoff vector x. Suppose that there is a coalition S that includes agent i, such that on issue V_j its total payment by y is strictly lower than its worth when operating alone. Then agent i believes that members of S could each profit by jointly deviating from the grand coalition. In this case agent i's anticipated aggregate payoff is the sum of her share of $V_j(S)$ and her payments as specified by y on the remaining issues, which is greater than x_i.[7] The anticipated aggregate payment of members of S in case they deviate on issue V_j is $V_j(S) + \sum_{V_k \in \bar{V} \setminus \{V_j\}} \sum_{l \in S} y_{l,k} > \sum_{l \in S} x_l$. Moreover, agent i could reason that the members of S will go along with her plan to deviate on issue V_j as they too have a belief for which deviation is profitable (e.g., if they all share her belief y). Hence, given such a belief y, agent i will not comply with the grand coalition on all issues. If the same is true for every possible belief of agent i, x cannot be justified by agent i and, therefore, $x \notin M(\bar{V})$.[8]

Next, suppose that for every issue V_j and every coalition S, which includes agent i, the total payment entailed in belief y is at least as high as its worth. Then y serves as a justification for agent i to consent to x since any coalition that includes agent i has no reason to block the formation of the grand coalition on any one of the issues. When x belongs to the multicore, while agent i may believe that some coalition S (not including herself) would be better off deviating on some issue V_j, she will be able to rationalize these agents' consent to x, as they too have a subjective belief (y^l) concerning the decomposition of x that supports their agreement. As a result, when the conditions in Definition 4 apply for x, each agent has a justification for supporting x and is able to reason that x will be accepted unanimously. Therefore, $x \in M(\bar{V})$.

3. Nonemptiness of the multicore

The celebrated Bondareva–Shapley theorem (Bondareva 1963 and Shapley 1967) presents a necessary and sufficient condition for the nonemptiness of the core of a standard cooperative game. In this section, we provide a similar characterization for the multicore.

3.1 *Bondareva–Shapley theorem*

For all $S \in 2^N$, let $\chi^S \in \{0, 1\}^n$ denote the characteristic vector of S, so that $\chi_i^S = 1$ if $i \in S$ and $\chi_i^S = 0$ otherwise.

Definition 5. A function $\delta : 2^N \to \mathbb{R}_+$ is a system of balancing weights if $\sum_{S \in 2^N} \delta(S) \chi^S = \chi^N$.

[7] Agents are assumed to be compensated according to y for issues on which they do not deviate. Therefore, when an agent can profit from deviating on more than one issue, she can also profit from deviating on only one. Thus, we can concentrate on deviations concerning a single issue.

[8] This line of reasoning is similar to that of rationalizability proposed by Pearce (1984) and Bernheim (1984), whereby a player can rationalize an action of another player if that action is a best response to some belief that the other player may hold.

An interpretation of δ is that each agent is endowed with one unit of time that can be divided among the different coalitions to which she belongs. A system of balancing weights is an allocation of agents' time among the different coalitions, where $\delta(S)$ is the fraction of time devoted to coalition S. Then $V(S)$ is the value of production when members of S devote their entire time to S and $\delta(S)V(S)$ is the proportional amount when S's members devote only $\delta(S)$ of their time to it.

Theorem 1 (Bondareva–Shapley theorem). *The core of V is nonempty if and only if every system of balancing weights, $\delta(S)$, satisfies $V(N) \geq \sum_{S\in 2^N} \delta(S)V(S)$.*

Thus, when the core is nonempty, a planner trying to maximize production will instruct all agents to devote their entire time to the grand coalition. However, when the core is empty, the planner prefers a different allocation of agents' time whose payoff is greater than $V(N)$.

3.2 *Systems of balancing multiweights*

The following definition adapts Definition 5 to the multigame framework.

Definition 6. A function $\tilde{\delta} : 2^N \times N \times \bar{V} \to \mathbb{R}_+$ is a system of balancing multiweights if it satisfies the following requirements:

1. Zero to Nonmembers: For all $V_j \in \bar{V}$, $\forall i \in N$, $\forall S \in 2^{N\setminus\{i\}}$: $\tilde{\delta}(S, i, V_j) = 0$.
2. Resource Exhaustion: For all $V_j \in \bar{V}$: $\sum_{i\in N} \sum_{S\in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \chi^N$.
3. Constant Shares: For all $i \in N$, $\forall V_j, V_{j'} \in \bar{V}$: $\sum_{S\in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \sum_{S\in 2^N} \tilde{\delta}(S, i, V_{j'})\chi^S$.

Denote the set of all systems of balancing multiweights by Δ.

Returning to the context of production, wherein each agent is endowed with one unit of time per issue, for every issue $V_j \in \bar{V}$, planner j is in charge of allocating agents' time among themselves, denoted by $\{\alpha_{1j}, \dots, \alpha_{nj}\}$. The vector $\alpha_{ij} \in [0, 1]^n$ contains the fractions of time of all agents operating under the jurisdiction of agent i in issue V_j. Agent i then chooses the amount of time, $\tilde{\delta}(S, i, V_j)$, to be devoted to the various coalitions S in issue V_j (implicitly assuming that the time assigned to the various coalitions by agent i exhausts the time allocated to her by each planner j so that $\alpha_{ij} = \sum_{S\in 2^N} \tilde{\delta}(S, i, V_j)\chi^S$).

Zero to Nonmembers entails that each agent assigns time only to coalitions in which that agent participates. Resource Exhaustion ensures that the allocation of resources among agents by planner j adds up to the amount of resources under his responsibility. These two conditions allow planners to choose their allocations independently. The Constant Shares condition links issues by restricting the planners' allocations to be the same, namely, for every agent i and for every pair of planners j and j', $\alpha_{ij} = \alpha_{ij'}$.[9]

[9] See the Supplementary Appendix (available in a supplementary file on the journal website, http://econtheory.org/supp/1865/supplement.pdf) for a detailed example of systems of balancing multiweights.

3.3 *Nonemptiness of the multicore*

Next, we present a characterization of multigames with nonempty multicores.

Theorem 2. *The multicore of $\bar{V}$ is nonempty if and only if every $\tilde{\delta} \in \Delta$ satisfies*

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S).$$

Within the production interpretation, Theorem 2 affirms that the multicore is nonempty if and only if production is maximized when agents devote the entire time that is allocated to them to the grand coalition on all issues.

In the proof, the problem of nonemptiness of the multicore is translated into a linear program that minimizes the total amount of payoffs subject to two types of constraints: one guaranteeing that each agent decomposes the aggregate payoffs correctly and the other that coalitional rationality holds. Issue efficiency follows from $\sum_{V_j \in \bar{V}} V_j(N)$ creating the upper boundary of the total payments with coalitional rationality with respect to the grand coalition creating the lower boundary. The multicore is nonempty if and only if the minimal amount of payoffs needed to solve this problem is no greater than the total amount of resources available (the sum of values of the grand coalition across all issues).

Originally, the objective function is defined on the aggregate payoffs while the constraints are defined on the elements of the decomposition matrices. In the proof we use the constraints on the decomposition to obtain an equivalent problem for which both the objective function and the constraints are defined over the decomposition matrices. The dual program is the maximization of the weighted sum of the values of the coalitions subject to a set of constraints on these weights. We prove that the set of weights that satisfy these constraints is equivalent to the set of systems of balancing multiweights. This concludes the proof, since the duality theorem implies that the multicore is nonempty if and only if the value of the objective function of the dual linear program is no greater than the sum of the values of the grand coalition across all issues.

3.4 *The multicore when agents are homogeneous*

In this section we restrict our attention to the nonemptiness of the multicore when agents are homogeneous. In this special case, the value attained by any coalition depends only on its size and, therefore, the characteristic functions are symmetric.[10] A multigame is symmetric if for every $V_j \in \bar{V}$, V_j is symmetric.

Definition 7 characterizes a subset of balancing multiweights that depend only on the coalition's size as the agents are homogeneous.

Definition 7. A function $\tilde{\delta} : 2^N \times N \times \bar{V} \to \mathbb{R}_+$ is a system of homogeneous balancing multiweights if it is a system of balancing multiweights that satisfies the homogeneity

[10] An issue V_j is symmetric if $\forall S, S' \in 2^N$ such that $|S| = |S'|$, V_j satisfies $V_j(S) = V_j(S')$.

requirement $\forall V_j \in \bar{V}$ and $\forall S, S' \in 2^N$ such that $|S| = |S'|$ for every pair $i \in S$ and $i' \in S'$, $\tilde{\delta}(S, i, V_j) = \tilde{\delta}(S', i', V_j)$. Denote the set of all systems of homogeneous balancing multiweights by Δ_a.

According to this definition, one can restrict attention to a representative agent, as all agents experience the same considerations. Proposition 1 establishes that the multicore of a symmetric multigame is nonempty if and only if production is maximized when the representative agent devotes the entire time that is allocated to her to the grand coalition on all issues.

Proposition 1. *The multicore of a symmetric $\bar{V}$ is nonempty if and only if every $\tilde{\delta} \in \Delta_a$ satisfies*

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S).$$

The proof rests on the observation that when the agents are homogeneous, the equal allocation must belong to the multicore whenever it is nonempty. Thus, the problem of the nonemptiness of the multicore is equivalent to the problem of the inclusion of the equal allocation in the multicore. We characterize a generic justification matrix that supports the equal allocation and show that the set of weights that emerges from the dual problem is equivalent to Δ_a.

4. Properties of the multicore

The solutions to the unlinked multigame are the sum over the solutions in the cores of the single issues, $\sum_{V_j \in \bar{V}} C(V_j) = \{\sum_{V_j \in \bar{V}} x^j \mid x^j \in C(V_j)\}$.[11] In unlinked multigames, agents act as if issues are resolved separately and, therefore, they must know the payoffs of each issue.

A solution to the totally linked multigame is the core of the game obtained by summing the characteristic functions in $\bar{V}$, $C(\sum_{V_j \in \bar{V}} V_j)$. In this game agents are uninformed as to the structure of the multigame and so must act as if they only face just one single issue.

The following theorem establishes that the multicore falls between the sum of the cores of the individual games and the core of the sum of the individual games.[12]

Theorem 3. *For every multigame $\bar{V}$: $\sum_{V_j \in \bar{V}} C(V_j) \subseteq M(\bar{V}) \subseteq C(\sum_{V_j \in \bar{V}} V_j)$.*

The inclusions in Theorem 3 are a reflection of the underlying assumption regarding the amount of information that agents possess in each solution concept. When considering the sum of the cores of the individual games, the agents are aware of both the

[11] Moreover, if $\exists V_j \in \bar{V}$ such that $C(V_j) = \varnothing$ then $\sum_{V_j \in \bar{V}} C(V_j) = \varnothing$.

[12] See the Supplementary Appendix, available in supplementary file on the journal website, http://econtheory.org/supp/1865/supplement.pdf, for a characterization of the nonemptiness of both $\sum_{V_j \in \bar{V}} C(V_j)$ and $C(\sum_{V_j \in \bar{V}} V_j)$ in terms of balancing multiweights.

structure and the payoff vector of each individual issue. In the game obtained by the summation of issues, the agents are assumed to know only the aggregate structure and the aggregate payoffs, but have no knowledge of their breakdown by issues. The multicore represents a hybrid information structure in which agents, though aware of the characteristic functions of the individual games have only a subjective assessment as to the payoff vectors that are attached to each game.

The first inclusion, $\sum_{V_j\in\bar{V}} C(V_j) \subseteq M(\bar{V})$, relies on the observation that a matrix whose columns are allocations in the cores of the corresponding issues can serve as a justification matrix for all agents. For the second inclusion, take $x \in M(\bar{V})$. Then, for any coalition S and agent $i \in S$, the sum of the total payments to the agents in S according to agent i's decomposition matrix is weakly greater than $\sum_{V_j\in\bar{V}} V_j(S)$ and thus $x \in C(\sum_{V_j\in\bar{V}} V_j)$.[13]

4.1 *The sum of the cores of the individual games*

Theorem 3 posits that the set of allocations that can be agreed upon when issues are linked as in the multicore is larger than that of issues considered independently. Within the multicore, the allocations that can be supported by a common justification matrix are exactly those that can be achieved when issues are solved independently. Hence, the manifestation of the rent for linkage is the additional allocations for which there is no common justification. Example 1 is a special case of such a rent where none of the allocations in the multicore can be supported by a common justification matrix since the core of one of the individual games is empty.

The following example demonstrates that the multicore can bring about cooperation even if such cooperation cannot be achieved on any single issue when considered independently (as in Example 1, where the core of the second issue is nonempty).

Example 2. Consider the four-agent–two-issue multigame

$$V_1(S) = \begin{cases} 9 & \text{if } S \in \{S \subset N \mid \{1,2\} \subseteq S\} \\ 10 & \text{if } |S| = N \\ 1 & \text{otherwise;} \end{cases} \qquad V_2(S) = \begin{cases} 9 & \text{if } S \in \{S \subset N \mid \{3,4\} \subseteq S\} \\ 10 & \text{if } |S| = N \\ 1 & \text{otherwise.} \end{cases}$$

The cores of both issues are empty.[14] Nevertheless, the allocation $x = (5,5,5,5)'$ is in the multicore since it is supported by the justification matrices

$$y^1 = y^3 = \begin{pmatrix} 4 & 1 \\ 5 & 0 \\ 1 & 4 \\ 0 & 5 \end{pmatrix}; \qquad y^2 = y^4 = \begin{pmatrix} 5 & 0 \\ 4 & 1 \\ 0 & 5 \\ 1 & 4 \end{pmatrix}.$$

[13] A solution in the multicore assumes that for every coalition, all members are convinced that the coalition is not better off operating on its own in each issue. For the second inclusion in Theorem 3 to hold, this condition may be weakened by requiring each coalition to have one such member.

[14] The core of issue V_1 is empty as agents 3 and 4 must get at least 1 each, and agents 1 and 2 must get at least 9 together, adding up to more than the value of the grand coalition, which is 10. From symmetry, the core of issue V_2 is also empty.

An alternative to the multicore would be to apply an issue-by-issue solution to the subset of all issues for which stand-alone cooperation is possible (i.e., $\hat{V} = \{V_j \in \bar{V} : C(V_j) \neq \varnothing\}$). One case in which this approach might be considered is when $M(\bar{V}) = \varnothing$. However, Theorem 3 implies that for every such subset $\sum_{V_j \in \hat{V}} C(V_j) \subseteq M(\hat{V})$. Moreover, there is a subset $\hat{V} \subseteq \tilde{V} \subseteq V$ such that $M(\tilde{V}) \neq \varnothing$. Thus, even if cooperation cannot be achieved on all issues, linkage enhances cooperation relative to the unlinked case. Whether linkage can strictly increase cooperation will depend on the composition of issues in the multigame. $\diamondsuit$

4.2 *The effectiveness of the multicore*

In the previous section it was shown that the set of allocations in the multicore contains those allocations that can be achieved when issues are solved independently. We say that the multicore is effective when the $M(\bar{V})$ is strictly larger than $\sum_{V_j \in \bar{V}} C(V_j)$ and is ineffective when they are identical. In this section we study some cases where the multicore is ineffective.

4.2.1 *Convex games* Convex games[15] are an important class of games whose cores are nonempty. The next proposition shows that in cases where the multigame is composed of issues that are all convex, the multicore is ineffective.

Proposition 2. *Let $\bar{V}$ be a multigame where for every $V_j \in \bar{V}$, V_j is convex. The multicore of $\bar{V}$ is ineffective.*

Proposition 2 determines that when all issues are easy to solve, there is no rent for linkage. Intuitively, when a game is convex, an agent has a higher incentive to join a coalition the larger is the coalition,[16] making it relatively easy to support the formation of a grand coalition.

The proof follows from an implication of Bloch and de Clippel (2010) that provides a condition that is both necessary and sufficient for the additivity of the core.[17] In particular it implies that for convex games, the core is additive.[18] Then, by Theorem 3 the multicore is ineffective, thereby completing the proof.

4.2.2 *Superadditive games* In superadditive games, when two disjoint coalitions merge, the value of the new coalition is no less than the sum of the values of the separate coalitions.[19] Superadditivity is likely to hold in situations where the merged coalition has the same options as did the separate coalitions before the merger (and possibly even more). We show that if three agents participate in any number of superadditive games

[15] A game V is convex if it satisfies $V(S) + V(T) \leq V(S \cup T) + V(S \cap T)$ for every pair of coalitions $S, T \subseteq N$.

[16] It can be shown that a cooperative game is convex if and only if for every $i \in N$ and for every $S \subseteq T \subseteq N$, $V(S \cup \{i\}) - V(S) \leq V(T \cup \{i\}) - V(T)$.

[17] The core of a multigame is additive if $\sum_{V_j \in V} C(V_j) = C(\sum_{V_j \in V} V_j)$.

[18] The result in Bloch and de Clippel (2010) applies to the case of two games (see also Dragan et al. 1989). However, it can be easily extended to any number of issues by induction on the number of issues.

[19] Formally, a game is superadditive if for every $S \cap T = \varnothing$, $V(S) + V(T) \leq V(S \cup T)$.

that have nonempty cores, any solution in the multicore can be obtained by solving the issues separately.[20]

Proposition 3. *Let $N = \{1, 2, 3\}$. If every $V_j \in \bar{V}$ is superadditive with a nonempty core, then the multicore of $\bar{V}$ is ineffective.*

By Theorem 3, $\sum_{V_j \in V} C(V_j) \subseteq M(\bar{V})$. For the opposite inclusion, let $T^w_{V_j}$ be the maximal achievable production given a general time endowment vector w (not necessarily one unit per agent) in issue V_j. A lemma from Gayer et al. (2015)[21] shows that payoff vector x can be decomposed into elements in the cores of the individual issues of $\bar{V}$ if and only if $w'x \geq \sum_{V_j \in \bar{V}} T^w_{V_j}$ for any endowment vector w. In our context, this condition would be satisfied if there exists a decomposition matrix y of x, such that for every issue V_j, $w'y_j \geq T^w_{V_j}$ (where y_j is the jth column of the y). These conditions would be satisfied by an agent's justification matrix if this agent belongs to all the producing coalitions in a maximal production plan (such a matrix exists since $x \in M(\bar{V})$). The proof constructs such a maximal production plan, where the coalitions with positive weights have at least one agent in common, for every w (see Lemma 3).[22]

This result does not extend to the case of four agents as demonstrated in the following example of two superadditive games with nonempty cores:

$$V_1(S) = \begin{cases} 0 & \text{if } |S| \leq 2, S \notin \{\{2,4\},\{3,4\}\} \\ \frac{1}{2} & \text{if } S \in \{\{2,4\},\{3,4\}\} \\ \frac{1}{2} & \text{if } |S| = 3, S \neq \{1,2,3\} \\ 1 & \text{if } S \in \{\{1,2,3\},\{1,2,3,4\}\}; \end{cases} \qquad V_2(S) = \begin{cases} 0 & \text{if } S \notin \{\{2,3,4\},\{1,2,3,4\}\} \\ \frac{3}{4} & \text{if } S = \{2,3,4\} \\ 1 & \text{if } |S| = 4. \end{cases}$$

The core of the first issue includes only the vector $(0, \frac{1}{2}, \frac{1}{2}, 0)'$.[23] The core of the second issue includes many elements, all restricting agent 1's payoff to no more than $\frac{1}{4}$ since agents 2, 3, and 4 must get at least $\frac{3}{4}$ together. Thus, the payoff vectors that can be represented as a sum of the elements in the cores of the first and the second issues are characterized by allocating no more than $\frac{1}{4}$ to agent 1. Nevertheless, the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})'$ that divides the payoffs equally among the agents is in the multicore.[24] This example shows that the multicore may offer additional desirable solutions even when

[20] Example 1 demonstrates that this result does not apply to the case where the core of at least one of the games is empty.

[21] Definition 8 and the proof of Proposition 3 in the Appendix are self-contained.

[22] Lemma 3 constructs a maximal production plan based on Lemma 4 that uses superadditivity, and Lemma 5 that uses balancedness and there being only three agents.

[23] The value of x_4 must be zero since $V_1(\{1,2,3,4\}) = V_1(\{1,2,3\}) = 1$. Moreover, the values of x_2 and x_3 must be $\frac{1}{2}$ each since $V_1(\{2,4\}) = V_1(\{3,4\}) = \frac{1}{2}$, showing that $C(V_1) = (0, \frac{1}{2}, \frac{1}{2}, 0)'$.

[24] This is confirmed by the justification matrices

$$y^1 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}; \qquad y^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}; \qquad y^3 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; \qquad y^4 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{8} \\ \frac{1}{2} & 0 \end{pmatrix}$$

all games are superadditive and their cores are nonempty. In fact, both games in the example are totally balanced[25] so that a slightly stronger result is obtained: the multicore may be effective even if the games are totally balanced (provided that the issues involve more than three agents).

4.3 *The core of the sum of individual games*

Theorem 3 proves that it is possible to achieve more cooperation when agents are ignorant of the fact that the multigame is composed of several issues. Information on the structure of the original games gives agents reason to reject allocations that otherwise they might have been willing to accept.

To see this, observe that in Example 1 the sum of games is such that a single agent gets 0 by herself, while a pair of agents gains $\frac{3}{4}$, and the grand coalition gets 2. The core of this game consists of all nonnegative payoff vectors in which all elements are at most $\frac{5}{4}$ and add up to 2. Thus, $M(\bar{V}) \subset C(\sum_{V_j \in \bar{V}} V_j)$ (see footnote 6). Consider the aggregate payoff wherein agents 1 and 2 each get 1, while agent 3 gets 0. According to the multicore solution concept, agent 3 realizes that at least one of the coalitions consisting of another agent and herself is not sufficiently rewarded with regard to the first issue and, therefore, she cannot justify this aggregate payoff.[26] Considerations of this sort regarding the original structure of issues are totally absent in the solution of the core of the sum of the individual games. Indeed, $(1, 1, 0)' \in C(\sum_{V_j \in \bar{V}} V_j)$.

A specific implication of Theorem 3 is that emptiness of the core of the sum of the individual games is a sufficient, but not a necessary condition for the emptiness of the multicore.[27] However, whether cooperation, as implied by the core of the sum of the individual games, can be achieved depends on the information that agents possess regarding the structure of the original issues.

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[25]A subgame of $G = (N; V)$ is a game $G_T(T; V^T)$, where $\varnothing \neq T \subseteq N$ and $V^T(S) = V(S)$ for all $S \subseteq T$. A coalitional game G is totally balanced if every subgame of G has a nonempty core. The set of totally balanced games is a subset of the set of balanced superadditive games and a superset of the set of convex games.

[26]Coalition rationality for agent 3 in the first issue entails that both $y^3_{1,1} + y^3_{3,1} \geq \frac{3}{4}$ and $y^3_{2,1} + y^3_{3,1} \geq \frac{3}{4}$, and together with issue efficiency, this implies that $y^3_{3,1} \geq \frac{1}{2}$.

[27] To see that it is not a necessary condition, consider the following four-agent–two-issue multigame:

$$V_1(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1,2\},\{1,3\},\{2,3\}\} \\ \frac{2}{3} & \text{if } S \in \{\{1,4\},\{2,4\},\{3,4\}\} \text{ or } |S| = 3 \\ 1 & \text{if } |S| = 4; \end{cases} \qquad V_2(S) = \begin{cases} 0 & \text{if } |S| = 1, |S| = 2 \\ \frac{5}{6} & \text{if } |S| = 3 \\ 1 & \text{if } |S| = 4 \end{cases}$$

$$(V_1 + V_2)(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1,2\},\{1,3\},\{2,3\}\} \\ \frac{2}{3} & \text{if } S \in \{\{1,4\},\{2,4\},\{3,4\}\} \\ \frac{3}{2} & \text{if } |S| = 3 \\ 2 & \text{if } |S| = 4. \end{cases}$$

Note that $C(V_1 + V_2)$ contains the single payoff vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})'$, which is not in the multicore (since agent 4 requires at least half in each issue). Therefore, by Theorem 3, $M(\{V_1, V_2\}) = \varnothing$.

5. Variations on the multicore

In the multicore the justifications matrices of the participating agents are tied together only through knowledge of the structure of the multigame and the proposed aggregate payoff vector.[28] In this section different constraints linking issues together are considered.[29]

In the multicore, members of a coalition S are permitted to have very different views regarding the total amount that the coalition receives from issue V_j. Eliminating such incompatibilities requires that the views of agents in the same coalition regarding the coalition's total payment be the same for each issue.[30] This type of imposition naturally emerges when agents' justifications are suggested by a mediator who needs to avoid any inconsistency in the event that members of the same coalition compare proposals on a coalitional level. The set of allocations that satisfies this restriction strictly lies between the multicore and the sum of solutions in the individual cores.[31]

Instead, the incompatibilities embedded in the multicore may be restricted by confining the justification matrices of some agents to being identical. This would certainly be the case if a group of agents authorizes a single representative to approve allocations on its behalf. Once again this restriction results in a subset of allocations that are located strictly between the multicore and the sum of solutions in the individual cores.[32]

Alternatively, the multicore may be weakened by assuming that deviating with coalition S from the grand coalition requires unanimous consent of all members of S. Thus, if even a single member of a coalition is subjectively satisfied with the compensation offered to that coalition, no deviation is possible. If this satisfied agent is the same across issues, then the set of allocations that can be supported by this approach falls between the multicore and the core of the sum of the individual issues (see footnote 13). In the more general case, where the satisfied agent differs across issues, the supported allocations may not even belong to the core of the sum of the individual issues.[33]

[28]Let $\tilde{Y}_i(\bar{V}, x)$ be the set of efficient decomposition matrices that justify the aggregate payoff vector x for agent i in the multigame $\bar{V}$ and let $\hat{Y}(\bar{V}, x) = \times_{i=1}^{n} \tilde{Y}_i(\bar{V}, x)$. The multicore includes every x such that $\hat{Y}(\bar{V}, x) \neq \varnothing$. This notation will become useful in describing the following variants of the multicore.

[29]We thank the anonymous referee for suggesting the first two extensions mentioned below.

[30]Formally, $\bar{Y} = \{\{y^1, \dots, y^n\} \in \hat{Y} \mid \forall j \in \{1, \dots, m\}, \forall S \in 2^N, \forall i, i' \in S : \sum_{k\in S} y^i_{k,j} = \sum_{k\in S} y^{i'}_{k,j}\}$.

[31]Example 2 shows that the first inclusion is strict since there are no justification matrices for $(5, 5, 5, 5)'$ that satisfy the restriction. Example 1 demonstrates that the second inclusion is strict.

[32]Whether this restriction affects the allocations the agents could agree upon depends on the specific sets of agent that are imposed to have the same justification. For instance, in Example 2, the allocation $(5, 5, 5, 5)'$ cannot be supported by identical justifications of agents 1 and 2, but can be supported by identical justifications of agents 1 and 3.

[33]The paper is supplemented with a Matlab package that implements all the solution concepts discussed in this work. The package can be downloaded from http://www.tau.ac.il/~persitzd/research.html. The operating instructions appear in the Readme.txt file of the package.

Appendix

Proof of Theorem 2. Consider the following linear program that minimizes the sum of aggregate payoffs subject to each agent's having a decomposition matrix in which all coalitions to which she belongs have no incentive to deviate on any one of the issues:

$$\min_{x\in\mathbb{R}^n}\sum_{i=1}^n x_i$$
$$\text{subject to:}\quad \forall i,l\in N:\sum_{j\in\{1,\dots,m\}} y^i_{l,j}=x_l$$
$$\forall i\in N,\forall V_j\in\bar{V},\forall S\in\{T\cup\{i\}\mid T\subseteq N\setminus\{i\}\}:\sum_{l\in S}y^i_{l,j}\geq V_j(S).$$

The constraints include n^2 equalities and $n\times m\times 2^{n-1}$ inequalities. There exists $x\in\mathbb{R}^n$ that satisfies the constraints and since the objective function is linear and bounded from below, there exists a solution to the problem, which we denote by $\bar{x}$. Most importantly, due to the efficiency requirement, the multicore is nonempty if and only if $\sum_{i=1}^n\bar{x}_i\leq\sum_{V_j\in\bar{V}}V_j(N)$.

The n equalities of the justification matrix of agent 1 are substituted into the objective function. The other n^2-n equalities of the other agents are used to isolate the values ascribed by their justification matrices to the payoff vector in V_m. These values are then substituted into the corresponding inequalities, leading to the linear problem

$$\min_{y^1\in\mathbb{R}^{n\times m}}\sum_{l=1}^n\sum_{j\in\{1,\dots,m\}}y^1_{l,j}$$

subject to:

$$\forall i\in N,\forall V_j\in\bar{V}\setminus\{V_m\},\forall S\in\{T\cup\{i\}\mid T\subseteq N\setminus\{i\}\}:\sum_{l\in S}y^i_{l,j}\geq V_j(S)$$
$$\forall S\in\{T\cup\{1\}\mid T\subseteq N\setminus\{1\}\}:\sum_{l\in S}y^1_{l,m}\geq V_m(S)$$
$$\forall i\in N\setminus\{1\},\forall S\in\{T\cup\{i\}\mid T\subseteq N\setminus\{i\}\}:\sum_{l\in S}\sum_{j\in\{1,\dots,m\}}y^1_{l,j}-\sum_{l\in S}\sum_{j\in\{1,\dots,m-1\}}y^i_{l,j}\geq V_m(S).$$

This problem in matrix form becomes,

$$\min_{y\in\mathbb{R}^p}c'y$$
$$\text{subject to:}\quad Ay\geq b,$$

where y and c are column vectors of length $p=nm+(n-1)[n(m-1)]$. The first nm elements of y are obtained by converting y^1 into a vector by stacking its m columns (issues) one on top of the other, an operation called vectorization. The next $n(m-1)$ elements of y are obtained by vectorizing the first $m-1$ columns of y^2 followed by vectorization of the first $m-1$ columns of y^3 and so on. To preserve the previous objective function, c is defined such that the first nm cells all have a value of 1 while the other $(n-1)[n(m-1)]$ cells all have a value of 0. Therefore, $c'y = \sum_{l=1}^{n} \sum_{j\in\{1,\dots,m\}} y^1_{l,j}$.

Let L^i be an $2^{n-1} \times n$ matrix whose rows contain the characteristic vectors corresponding to the coalitions that include agent i. Let $\mu_i(S)$ be an ordering on these characteristic vectors. Thus, the $\mu_i(S)$ row of L^i consists of the characteristic vector of coalition S ($\mu_i(S) = 0$ for all S such that $i \notin S$).[34] We also use the function $\mu_i^{-1}(l)$ ($l \in \{1, \dots, 2^{n-1}\}$), which is the coalition in the lth place in the ordering for agent i. Note that for every $l \in \{1, \dots, 2^{n-1}\}$, $\mu_i(\mu_i^{-1}(l)) = l$. Let BL^i be a block matrix of size $(m-1)2^{n-1} \times (m-1)n$, where there are $(m-1) \times (m-1)$ blocks, each of size $2^{n-1} \times n$, such that $m-1$ blocks of L^i occupy the diagonal and the rest of the cells in the matrix contain zeros.[35] For agent 1, FL is a block matrix of size $m2^{n-1} \times p$, obtained from the concatenation of two matrices. On the left are $m \times m$ blocks of $2^{n-1} \times n$, where m blocks of L^1 occupy the diagonal and the other $m^2 - m$ blocks are zeros. On the right is an $m2^{n-1} \times (n-1)[n(m-1)]$ matrix of zeros.[36] For the other agents ($i \in \{2, \dots, n\}$), let ZBL^i be an $(m-1)2^{n-1} \times p$ block matrix

[34] The choice of the specific ordering is inconsequential to the rest of the proof. For example, we can order the row vectors by their binary values. Hence, if $N = \{1, 2, 3\}$, then

$$L^1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}; \qquad L^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}; \qquad L^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

For instance, $\mu_1(\{1,3\}) = 2$, $\mu_2(\{1,3\}) = 0$, and $\mu_3(\{1,3\}) = 3$.

[35] For example, if there are three issues:

$$BL^1 = \begin{pmatrix} L^1 & 0 \\ 0 & L^1 \end{pmatrix}; \qquad BL^2 = \begin{pmatrix} L^2 & 0 \\ 0 & L^2 \end{pmatrix}; \qquad BL^3 = \begin{pmatrix} L^3 & 0 \\ 0 & L^3 \end{pmatrix}.$$

[36] For example, if there are three agents and three issues, FL is the following 12×21 matrix:

$$FL = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

that has BL^i starting at the $nm+(i-2)n(m-1)+1$ column and zeros elsewhere.[37] Let ML^i be an $2^{n-1}\times p$ block matrix of blocks of size $2^{n-1}\times n$, such that there are m blocks of L^i in the first m blocks, $m-1$ blocks of $-L^i$ starting from the $m+(i-2)(m-1)+1$th block, and zeros elsewhere.[38] Finally, let A be an $nm2^{n-1}\times p$ block matrix where FL occupies the first $m2^{n-1}$ rows, followed by ZBL^2 and ML^2, and so on to ZBL^n and ML^n. Then, Ay is the left hand side of the inequality constraints in the linear programming problem.

To complete the program, let b be an $nm2^{n-1}$ length vector where the first 2^{n-1} elements are the values of the coalitions that include agent 1 in issue V_1 ordered by $\mu_1(S)$, the next 2^{n-1} elements are the values of the coalitions that include agent 1 in issue V_2 ordered by $\mu_1(S)$, and so on, so that the 2^{n-1} elements starting from place $[(i-1)m+(j-1)]\times 2^{n-1}+1$ are the values of the coalitions that include agent i in issue V_j ordered by $\mu_i(S)$. Formally, $b[k]=V_j(\mu_i^{-1}(l))$, where $i=\lceil k/(m2^{n-1})\rceil$, $j=\lceil (k-(i-1)m2^{n-1})/2^{n-1}\rceil$, and $l=k-(i-1)m2^{n-1}-(j-1)2^{n-1}$. This completes the matrix notation for the linear program. The multicore is nonempty if and only if $c'\bar{y}\le\sum_{V_j\in\bar{V}}V_j(N)$, where $\bar{y}$ is the solution to the linear program.

The asymmetric dual problem is

$$\max_{z\in\mathbb{R}^{nm2^{n-1}}} b'z$$
$$\text{subject to:}\quad A'z=c,\ z\ge 0.$$

The strong duality theorem states that in a primal–dual pair of linear programs, if either the primal or the dual problem has a feasible optimal solution, then so does the other and the two optimal objective values are equal. Since the primary problem, in this case, has a solution, so does its asymmetric dual problem, denoted by $\bar{z}$. Moreover, $b'\bar{z}=c'\bar{y}$. Thus, the multicore is nonempty if and only if $b'\bar{z}\le\sum_{V_j\in\bar{V}}V_j(N)$. Equivalently, the multicore is nonempty if and only if every $z\in\mathbb{R}_+^{nm2^{n-1}}$ such that $A'z=c$ satisfies $b'z\le\sum_{V_j\in\bar{V}}V_j(N)$.

[37] For example, if there are three agents and three issues, ZBL^2 is the following 8×21 matrix:

$$ZBL^2=\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0
\end{pmatrix}.$$

[38] For example, if there are three agents and three issues, then

$$ML^2=\begin{pmatrix}L^2 & L^2 & L^2 & -L^2 & -L^2 & 0 & 0\end{pmatrix}$$
$$ML^3=\begin{pmatrix}L^3 & L^3 & L^3 & 0 & 0 & -L^3 & -L^3\end{pmatrix}.$$

Next we characterize the set $Z = \{z \in \mathbb{R}_+^{nm2^{n-1}} \mid A'z = c\}$. The first step is to add the corresponding rows of all agents (except agent 1) to those of agent 1, for every $V_j \in \bar{V}$ (except V_m).[39] Since this is an elementary row operation on A', and since for every $i > nm$, $c[i] = 0$, the solutions set for the linear equations system continues to be $Z = \{z \in \mathbb{R}_+^{nm2^{n-1}} \mid \tilde{A}'z = c\}$.[40]

We denote the kth element of z by $z[k]$. Let us define the function $\tilde{\delta}(S, i, V_j)$ in the following manner: if $i \notin S$, then $\tilde{\delta}(S, i, V_j) = 0$, and if $i \in S$, then $\tilde{\delta}(S, i, V_j) = z[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)]$.[41] By Lemma 1 below, for every $z \in Z$, $\tilde{\delta}(S, i, V_j)$ is a system of balancing multiweights as defined in Definition 6. Moreover, together with Lemma 2, this construction facilitates a one-to-one and onto correspondence between Z and Δ.

Recall that we have shown that the multicore is nonempty if and only if every $z \in Z$ satisfies $b'z \leq \sum_{V_j \in \bar{V}} V_j(N)$ or, explicitly, the multicore is nonempty if and only if every $z \in Z$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in \{T \cup \{i\} \mid T \subseteq N \setminus \{i\}\}} z[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)] V_j(S).$$

[39] Formally, recall that A' is a $p \times nm2^{n-1}$ matrix, where $p = nm + (n-1)[n(m-1)]$. For every $k \in \{0, \ldots, m + (n-1)(m-1) - 1\}$, we denote the block consisting of the n rows from $nk+1$ to $n(k+1)$ in A' by CC_k. For every $j \in \{1, \ldots, m-1\}$, let $D(j) = \sum_{k \geq m \mid k \bmod m-1=j} CC_k$ be the sum of the blocks corresponding to issue V_j over all agents $i \in N \setminus \{1\}$ (for simplicity let $km \bmod m = m$ instead of $km \bmod m = 0$). Now, let $\tilde{A}'$ be the matrix that results from replacing, for every $j \in \{1, \ldots, m-1\}$, CC_{j-1} by $CC_{j-1} + D(j)$.

[40] To illustrate, if there are three agents and three issues, then

$$A' = \begin{pmatrix} L^{1'} & 0 & 0 & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & L^{1'} & 0 & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & 0 & L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & 0 & 0 & L^{2'} & 0 & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L^{2'} & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & 0 & -L^{3'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & -L^{3'} \end{pmatrix}$$

$$(CC_0 \to CC_0 + CC_3 + CC_5) \qquad \Downarrow \qquad (CC_1 \to CC_1 + CC_4 + CC_6)$$

$$\tilde{A}' = \begin{pmatrix} L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} & 0 & 0 \\ 0 & L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} & 0 \\ 0 & 0 & L^{1'} & 0 & 0 & L^{2'} & 0 & 0 & L^{3'} \\ 0 & 0 & 0 & L^{2'} & 0 & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L^{2'} & -L^{2'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & 0 & -L^{3'} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L^{3'} & -L^{3'} \end{pmatrix}.$$

[41] The function $\tilde{\delta}(S, i, V_j)$ is well defined as it is defined for every combination of $S \in 2^N$, $V_j \in \bar{V}$, and $i \in N$ and the index of z does not exceed its length.

Therefore, the multicore is nonempty if and only if every system of balancing multi-weights satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \tilde{\delta}(S, i, V_j) V_j(S),$$

and since $\tilde{\delta}(S, i, V_j) = 0$ if $i \notin S$, then $\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$. □

Lemma 1. *Let $z \in Z$ and set $\tilde{\delta}(S, i, V_j)$ to zero if $i \notin S$ and $\tilde{\delta}(S, i, V_j) = z[(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S)]$ otherwise. Then $\tilde{\delta} \in \Delta$.*

Proof. Since $i \notin S$ implies $\tilde{\delta}(S, i, V_j) = 0$, "Zero to Nonmembers" is satisfied. Consider a typical equation in the first nm rows of $\tilde{A}'z = c$. Given an agent i and issue V_j,

$$\sum_{l=1}^{n} \sum_{q \in \{1,\dots,m\}} \sum_{S \in \{T \cup \{l\} | T \subseteq N \setminus \{l\}\}} \big[\tilde{A}'[(j-1)n + i, (l-1)m2^{n-1} + (q-1)2^{n-1} + \mu_l(S)] \\ \times z[(l-1)m2^{n-1} + (q-1)2^{n-1} + \mu_l(S)]\big] = 1.$$

Note that for every $q \neq j$ the values of $\tilde{A}'$ in row $(j-1)n + i$ are zeros. Therefore,

$$\sum_{l=1}^{n} \sum_{S \in \{T \cup \{l\} | T \subseteq N \setminus \{l\}\}} \big[\tilde{A}'[(j-1)n + i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S)] \\ \times z[(l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S)]\big] = 1.$$

By the definition above,

$$\sum_{l=1}^{n} \sum_{S \in \{T \cup \{l\} | T \subseteq N \setminus \{l\}\}} \big[\tilde{A}'[(j-1)n + i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S)] \times \tilde{\delta}(S, l, V_j)\big] = 1.$$

Extending the summation to the entire collection of coalitions,

$$\sum_{l=1}^{n} \sum_{S \in 2^N} \big[\tilde{A}'[(j-1)n + i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S)] \times \tilde{\delta}(S, l, V_j)\big] = 1.$$

Given an agent l and a coalition S such that $l \in S$, by the definition of L^l,

$$\tilde{A}'[(j-1)n + i, (l-1)m2^{n-1} + (j-1)2^{n-1} + \mu_l(S)] = 1 \quad \text{iff} \quad i \in S$$

and, therefore, can be substituted by χ_i^S. Moreover, if $l \notin S$, then $\tilde{\delta}(S, l, V_j) = 0$. Hence, for a given agent i and issue V_j, $\sum_{l=1}^{n} \sum_{S \in 2^N} [\chi_i^S \times \tilde{\delta}(S, l, V_j)] = 1$. Since this is true for every agent i and issue V_j, $\tilde{\delta}(S, i, V_j)$ satisfies "Resources Exhaustion."

Consider a typical equation in rows $nm+1$ to p of $\tilde{A}'z=c$. Given agent $i>1$, agent l, and issue $V_j \neq V_m$,

$$\begin{aligned}&\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}} \tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}\\&\qquad+(j-1)2^{n-1}+\mu_i(S)]\times z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]\\&=\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}} \tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}\\&\qquad+(m-1)2^{n-1}+\mu_i(S)]\times z[(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)].\end{aligned}$$

By the definition of $\tilde{\delta}(S,i,V_j)$,

$$\begin{aligned}&\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}} \tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}\\&\qquad+(j-1)2^{n-1}+\mu_i(S)]\times\tilde{\delta}(S,i,V_j)\\&=\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}} \tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}\\&\qquad+(m-1)2^{n-1}+\mu_i(S)]\times\tilde{\delta}(S,i,V_m).\end{aligned}$$

Given an agent i and a coalition S such that $i\in S$, by the definition of L^i,

$$\begin{aligned}&\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,\\&\qquad(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]=1\quad\text{iff}\quad l\in S\end{aligned}$$

and, therefore, can be substituted by χ_l^S. Hence, for a given agent $i>1$, agent l, and issue $V_j\neq V_m$, $\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\chi_l^S\tilde{\delta}(S,i,V_j)=\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\chi_l^S\tilde{\delta}(S,i,V_m)$. Since $\tilde{\delta}(S,i,V_j)=0$ if $i\notin S$, then for every agent $i>1$, agent l, and issue $V_j\neq V_m$,

$$\sum_{S\in 2^N}\chi_l^S\tilde{\delta}(S,i,V_j)=\sum_{S\in 2^N}\chi_l^S\tilde{\delta}(S,i,V_m).$$

Finally, since $\tilde{\delta}(S,i,V_j)$ satisfies “Resource Exhaustion,” this equality is satisfied also for agent 1. Hence, $\tilde{\delta}(S,i,V_j)$ satisfies “Constant Shares.” □

Lemma 2. *Let $\tilde{\delta}\in\Delta$ and set $z[k]=\tilde{\delta}(\bar{S}(k),\bar{i}(k),V_{\bar{j}(k)})$, where $\bar{i}(k)=\lceil k/(m2^{n-1})\rceil$, $\bar{j}(k)=\lceil (k-(\bar{i}(k)-1)m2^{n-1})/2^{n-1}\rceil$ and $\bar{S}(k)=\mu_{\bar{i}(k)}^{-1}(k-(\bar{i}(k)-1)m2^{n-1}-(\bar{j}(k)-1)2^{n-1})$. Then $\tilde{A}'z=c$.*

Proof. By the “Resource Exhaustion” requirement, for every agent l and every issue $V_j\in\bar{V}$, we have $\sum_{i\in N}\sum_{S\in 2^N}\chi_l^S\times\tilde{\delta}(S,i,V_j)=1$. By the “Zero for Nonmembers” requirement, $\sum_{i\in N}\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\chi_l^S\times\tilde{\delta}(S,i,V_j)=1$. For every pair of agents i and l, and for every issue V_j, by the definition of L^i, χ_l^S can be substituted by $\tilde{A}'[(j-1)n+l,$ $(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$. Hence,

$$\sum_{i\in N}\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\tilde{A}'[(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]\times\tilde{\delta}(S,i,V_j)=1.$$

Let $k=(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)$. By the construction of z, $\bar{i}(k)=i$, $\bar{j}(k)=j$, and $\bar{S}(k)=S$, and $z[k]=\tilde{\delta}(S,i,V_j)$.[42] Therefore, for every agent i, every issue $V_j\in\bar{V}$, and every coalition S such that $i\in S$, $\tilde{\delta}(S,i,V_j)$ can be replaced by $z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$.

$$\sum_{i\in N}\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\tilde{A}'[(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$$
$$\times z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]=1.$$

Note that every entry is of the type $\tilde{A}'[(j-1)n+l,(i-1)m2^{n-1}+(h-1)2^{n-1}+\mu_i(S)]$, where $j\neq h=0$. Also, for every row $r\leq nm$, $c_r=1$. Therefore, for every row $r\leq nm$, the z constructed satisfies $\tilde{A}'z=c$.

Next, since $\tilde{\delta}\in\Delta$, for every two agents i and l and issue $V_j\in\bar{V}$, $\sum_{S\in2^N}\chi_l^S\times\tilde{\delta}(S,i,V_j)=\sum_{S\in2^N}\chi_l^S\times\tilde{\delta}(S,i,V_m)$. By the "Zero for Nonmembers" condition, for $i\notin S$, $\tilde{\delta}(S,i,V_j)=0$, and, therefore, $\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\chi_l^S\times\tilde{\delta}(S,i,V_j)=\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\chi_l^S\times\tilde{\delta}(S,i,V_m)$.

For every pair of agents $i>1$ and l and for every issue $V_j\neq V_m$, by the definition of L^i, χ_l^S can be substituted by $\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$ and for issue V_m, χ_l^S can be substituted by $-\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)]$. Then,

$$\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\bigl\{\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,$$
$$(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$$
$$\times\tilde{\delta}(S,i,V_j)+\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}$$
$$+(m-1)2^{n-1}+\mu_i(S)]\times\tilde{\delta}(S,i,V_m)\bigr\}=0.$$

As was shown earlier, $z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]=\tilde{\delta}(S,i,V_j)$ and $z[(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)]=\tilde{\delta}(S,i,V_m)$.

Then

$$\sum_{S\in\{T\cup\{i\}|T\subseteq N\setminus\{i\}\}}\bigl\{\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}$$
$$+(j-1)2^{n-1}+\mu_i(S)]\times z[(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)]$$
$$+\tilde{A}'[nm+(i-2)(m-1)n+(j-1)n+l,(i-1)m2^{n-1}$$
$$+(m-1)2^{n-1}+\mu_i(S)]\times z[(i-1)m2^{n-1}+(m-1)2^{n-1}+\mu_i(S)]\bigr\}=0.$$

[42] It can be easily seen that $k=(i-1)m2^{n-1}+(j-1)2^{n-1}+\mu_i(S)$ is a one-to-one and onto correspondence between $\{1,\dots,nm2^{n-1}\}$ and $N\times\bar{V}\times\{T\cup\{i\}\mid T\subseteq N\setminus\{i\}\}$.

For every coalition S, entries of type $\tilde{A}'[nm + (i-2)(m-1)n + (j-1)n + l, x]$, where $x \notin \{(i-1)m2^{n-1} + (j-1)2^{n-1} + \mu_i(S), (i-1)m2^{n-1} + (m-1)2^{n-1} + \mu_i(S)\}$, equal zero. Also, for every row $r > nm$, $c_r = 0$. Therefore, for every row $r > nm$, the z constructed satisfies $\tilde{A}'z = c$, which concludes the proof. □

Proof of Proposition 1. The multicore is a solution of a linear program (see the proof to Theorem 2) and, therefore, it is a convex set. Consider the case where $M(\bar{V}) \neq \varnothing$ and let $x \in M(\bar{V})$. Then, since the agents are homogeneous, every permutation of x is also a member of $M(\bar{V})$. Having that the multicore is convex, then every mixture of these permutations is also a member of the multicore. In particular, the equal allocation is in the multicore (weight each permutation by $1/(n!)$). Therefore, $M(\bar{V}) \neq \varnothing$ if and only if $((\sum_{V_j \in \bar{V}} V_j(N))/n, \dots, (\sum_{V_j \in \bar{V}} V_j(N))/n)' \in M(\bar{V})$. Hence, the problem of nonemptiness of the multicore is equivalent to the problem of justifying the equal allocation.

Since the agents are homogeneous, a representative agent can justify the equal allocation if and only if the multicore is nonempty. The justification problem of such a representative agent[43] is a linear program that minimizes the aggregate payoff that can be justified in a symmetric multigame when y_j and w_j are the payments in issue V_j to the representative agent and to any other agent, respectively:

$$\begin{aligned} &\min_{x \in \mathbb{R}} x \\ &\text{subject to:} \quad \sum_{j=1}^{m} y_j = x \\ &\qquad\qquad\qquad \sum_{j=1}^{m} w_j = x \\ &\qquad\qquad\qquad \forall V_j \in \bar{V}, \forall k \in \{1, 2, \dots, n\} : y_j + (k-1)w_j \geq V_j(k). \end{aligned}$$

The constraints include two equalities and $n \times m$ inequalities. There exists $x \in \mathbb{R}$ that satisfies the constraints, and since the objective function is linear and bounded from below, there exists a solution to the problem, which we denote by $\bar{x}$. The multicore is nonempty if and only if $\bar{x} \leq (\sum_{V_j \in \bar{V}} V_j(n))/n$.

The program can be written in the manner

$$\begin{aligned} &\min_{y \in \mathbb{R}^m} \sum_{j=1}^{m} y_j \\ &\text{subject to:} \quad \forall V_j \in \bar{V} \setminus \{V_m\}, \forall k \in \{1, 2, \dots, n\} : y_j + (k-1)w_j \geq V_j(k) \\ &\qquad\qquad\quad \forall k \in \{1, 2, \dots, n\} : k y_m + (k-1)\sum_{j=1}^{m-1} y_j - (k-1)\sum_{j=1}^{m-1} w_j \geq V_m(k). \end{aligned}$$

[43] We simplify notation by replacing the argument to V_j to be the size of the coalition instead of the coalition itself.

In this program there are $n \times m$ inequalities and we denote its solution by $\bar{y}$. The multicore is nonempty if and only if $\sum_{j=1}^{m} \bar{y}_j \leq (\sum_{V_j \in \bar{V}} V_j(n))/n$.

This problem in matrix form becomes

$$\min_{u \in \mathbb{R}^{2m-1}} c'u$$
$$\text{subject to:} \quad Au \geq b,$$

where u and c are column vectors of length $2m-1$. The first m elements of u are y and the next $m-1$ are w (excluding w_m). To preserve the previous objective function, c is defined such that the first m cells all have a value of 1 while the other $m-1$ cells all have a value of 0. Therefore, $c'u = \sum_{j=1}^{m} y_j$.

The matrix A has $n \times m$ rows and $2m-1$ columns. Each of the first $n \times (m-1)$ rows corresponds to an inequality of the form $y_j + (k-1)w_j \geq V_j(k)$. Thus, for every $j \in \{1, \ldots, m-1\}$ and $k \in \{1, \ldots, n\}$, the $(n \times (j-1) + k)$th row has 1 in the jth column, $k-1$ in the $(m+j)$th column, and zeros elsewhere. Each of the bottom n rows corresponds to an inequality of the form $ky_m + (k-1)\sum_{j=1}^{m-1} y_j - (k-1)\sum_{j=1}^{m-1} w_j \geq V_m(k)$. Hence, for every $k \in \{1, \ldots, n\}$, the $[n(m-1)+k]$th row has $k-1$ in the first $m-1$ columns, k in the mth column, and $-(k-1)$ elsewhere.

To complete the program let b be an nm length vector where the n elements starting from $b[(j-1)n+1]$ are the values of issue V_j ordered by coalition size. Formally, $b[l] = V_j(k)$, where $j = \lceil l/n \rceil$ and $k = l - (j-1)n$.[44]

Denote the solution of the program by $\bar{u}$. Then the multicore is nonempty if and only if $c'\bar{u} \leq (\sum_{V_j \in \bar{V}} V_j(n))/n$.

The asymmetric dual problem is,

$$\max_{z \in \mathbb{R}^{nm}} b'z$$
$$\text{subject to:} \quad A'z = c,\ z \geq 0.$$

By the strong duality theorem the problem has a solution denoted by $\bar{z}$. Moreover, $b'\bar{z} = c'\bar{u}$. Thus, the multicore is nonempty if and only if $b'\bar{z} \leq (\sum_{V_j \in \bar{V}} V_j(n))/n$. Equivalently, denote $Z = \{z \in \mathbb{R}_+^{nm} \mid A'z = c\}$. Then the multicore is nonempty if and only if every $z \in Z$ satisfies $b'z \leq (\sum_{V_j \in \bar{V}} V_j(n))/n$.

[44] For example, if there are three agents and three issues,

$$u = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ w_1 \\ w_2 \end{pmatrix}; \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & -1 & -1 \\ 2 & 2 & 3 & -2 & -2 \end{pmatrix}; \quad b = \begin{pmatrix} V_1(1) \\ V_1(2) \\ V_1(3) \\ V_2(1) \\ V_2(2) \\ V_2(3) \\ V_3(1) \\ V_3(2) \\ V_3(3) \end{pmatrix}.$$

Construct $\tilde{A}'$ from A' by adding the $(m+j)$th row to the jth row for every $j \in \{1, \dots, m-1\}$. Since these are elementary row operations on A', and since for every $l > m$, $c[l] = 0$, $Z = \{z \in \mathbb{R}_+^{nm} \mid \tilde{A}'z = c\}$. In addition, construct $\hat{A}'$ from $\tilde{A}'$ by subtracting the jth row and adding the mth row to the $(m+j)$th row, and multiply the result by -1 for every $j \in \{1, \dots, m-1\}$. Again, since these are elementary row operations on $\tilde{A}'$ and since for every $k \in \{1, \dots, m\}$, we have $c[k] = c[m]$, $Z = \{z \in \mathbb{R}_+^{nm} \mid \hat{A}'z = c\}$.[45]

Next we show that every $z \in Z$ corresponds to a system of homogeneous balancing multiweights. For every $z \in Z$, define $\tilde{\delta}(S, i, V_j)$ as follows: if $i \notin S$, then $\tilde{\delta}(S, i, V_j) = 0$, while if $i \in S$, then $\tilde{\delta}(S, i, V_j) = z[(j-1)n + |S|]/\binom{n-1}{|S|-1}$. Since $i \notin S$ implies $\tilde{\delta}(S, i, V_j) = 0$, $\tilde{\delta}(S, i, V_j)$ satisfies "Zero to Nonmembers." In addition, by construction, $\tilde{\delta}(S, i, V_j)$ does not depend on i and on the identity of the members of S; therefore $\forall V_j \in \bar{V}$ and $\forall S, S' \in 2^N$ such that $|S| = |S'|$ for every pair $i \in S$ and $i' \in S'$, $\tilde{\delta}(S, i, V_j) = \tilde{\delta}(S', i', V_j)$, meaning that $\tilde{\delta}(S, i, V_j)$ satisfies "Homogeneity."

Consider a typical equation in the first m rows of $\hat{A}'z = c$. For every $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} \big[k \times z[(j-1)n + k]\big] = 1.$$

Since each agent is a member of $\binom{n-1}{|S|-1}$ coalitions of size $|S|$, we can write for every issue $V_j \in \bar{V}$ and for every agent $i' \in N$,

$$\sum_{k=1}^{n} \sum_{S \in \{T \cup \{i'\} | T \subseteq N \setminus \{i'\}, |T| = k-1\}} \left[|S| \times \frac{z[(j-1)n + |S|]}{\binom{n-1}{|S|-1}} \right] = 1.$$

[45]To illustrate, if there are three agents and three issues, then

$$A'|c = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & | & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & -1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & -2 & | & 0 \end{pmatrix}$$

$$(1 \to 1+4) \qquad \Downarrow \qquad (2 \to 2+5)$$

$$\tilde{A}'|c = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & | & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & -1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & -2 & | & 0 \end{pmatrix}$$

$$(4 \to -(4+3-1)) \qquad \Downarrow \qquad (5 \to -(5+3-2))$$

$$\hat{A}'|c = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & | & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & | & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & | & 0 \end{pmatrix}.$$

For every agent $i' \in N$ and every issue $V_j \in \bar{V}$, this can be written,

$$\sum_{S \in \{T \cup \{i'\} | T \subseteq N \setminus \{i'\}\}} \sum_{i \in S} \left[\frac{z[(j-1)n + |S|]}{\binom{n-1}{|S|-1}} \right] = 1.$$

By the definition of $\tilde{\delta}(S, i, V_j)$, for every agent $i' \in N$ and every issue $V_j \in \bar{V}$,

$$\sum_{S \in \{T \cup \{i'\} | T \subseteq N \setminus \{i'\}\}} \sum_{i \in S} \tilde{\delta}(S, i, V_j) = 1.$$

By "Zero to Nonmembers" and by the definition of a characteristic vector, for every agent $i' \in N$ and every issue $V_j \in \bar{V}$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi^S_{i'} = 1.$$

And thus, for every issue $V_j \in \bar{V}$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi^S = \chi^N,$$

meaning that $\tilde{\delta}(S, i, V_j)$ satisfies "Resource Exhaustion."

Consider a typical equation in the final $m-1$ rows of $\hat{A}'z = c$. For every agent $i \in N$ and for every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{k=1}^{n} z[(j-1)n + k] = \sum_{k=1}^{n} z[(j'-1)n + k].$$

As before, since each agent is a member of $\binom{n-1}{|S|-1}$ coalitions of size $|S|$, we can write for every agent $i \in N$ and for every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \frac{z[(j-1)n + |S|]}{\binom{n-1}{|S|-1}} = \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \frac{z[(j'-1)n + |S|]}{\binom{n-1}{|S|-1}}.$$

By the definition of $\tilde{\delta}(S, i, V_j)$ for every $i \in N$ and for every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi^S_i = \sum_{S \in 2^N} \tilde{\delta}(S, i, V_{j'}) \chi^S_i.$$

For every two issues $V_j, V_{j'} \in \bar{V}$, the following statements hold:

- The first m equations are $\sum_{k=1}^{n} k z[(j-1)n + k] = \sum_{k=1}^{n} k z[(j'-1)n + k]$.
- The final $m-1$ equations are $\sum_{k=1}^{n} z[(j-1)n + k] = \sum_{k=1}^{n} z[(j'-1)n + k]$.

Therefore,

$$\sum_{k=1}^{n} \big[(k-1) \times z[(j-1)n + k] \big] = \sum_{k=1}^{n} \big[(k-1) \times z[(j'-1)n + k] \big].$$

Since each agent is a member of $\binom{n-1}{|S|-1}$ coalitions of size $|S|$, we can write for every agent $i \in N$ and for every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{k=1}^{n} \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}, |T|=k-1\}} \left[(|S|-1) \times \frac{z[(j-1)n+|S|]}{\binom{n-1}{|S|-1}} \right] = \sum_{k=1}^{n} \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}, |T|=k-1\}} \left[(|S|-1) \times \frac{z[(j'-1)n+|S|]}{\binom{n-1}{|S|-1}} \right].$$

As before, for every agent $i \in N$ and every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \sum_{i' \in S \setminus \{i\}} \frac{z[(j-1)n+|S|]}{\binom{n-1}{|S|-1}} = \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \sum_{i' \in S \setminus \{i\}} \frac{z[(j'-1)n+|S|]}{\binom{n-1}{|S|-1}}.$$

By the definition of $\tilde{\delta}(S, i, V_j)$, for every $i \in N$ and for every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \sum_{i' \in S \setminus \{i\}} \tilde{\delta}(S, i, V_j) = \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \sum_{i' \in S \setminus \{i\}} \tilde{\delta}(S, i, V_j).$$

By the "Zero to Nonmembers" and by the definition of a characteristic vector, for every agent $i \in N$ and every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{i' \in N \setminus \{i\}} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi_{i'}^S = \sum_{i' \in N \setminus \{i\}} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi_{i'}^S.$$

Since for agent i all other agents are identical, for every agent $i' \in N \setminus \{i\}$ and every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi_{i'}^S = \sum_{S \in 2^N} \tilde{\delta}(S, i, V_{j'}) \chi_{i'}^S.$$

Since we showed above that $\sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi_i^S = \sum_{S \in 2^N} \tilde{\delta}(S, i, V_{j'}) \chi_i^S$, for every agent i and every two issues $V_j, V_{j'} \in \bar{V}$,

$$\sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi^S = \sum_{S \in 2^N} \tilde{\delta}(S, i, V_{j'}) \chi^S.$$

Hence, $\tilde{\delta}(S, i, V_j)$ satisfies "Constant Shares." Therefore, every $z \in Z$ corresponds to a system of homogeneous balancing multiweights.

It is left to show that every system of homogeneous balancing multiweights corresponds to some $z \in Z$. Since $\tilde{\delta}(S, i, V_j)$ satisfies "Homogeneity," every system of homogeneous balancing multiweights corresponds to a function $\bar{\delta}(|S|, V_j) : \{1, \dots, n\} \times \bar{V} \to \mathbb{R}_+$. Define $z \in \mathbb{R}_+^{nm}$ such that $z[l] = \binom{n-1}{\bar{s}-1} \times \bar{\delta}(\bar{s}, V_{\bar{j}})$, where $\bar{j} = \lceil l/n \rceil$ and $\bar{s} = l - (\bar{j}-1) \times n$.

By construction, for every issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} \left[k \times z[(j-1)n+k] \right] = \sum_{k=1}^{n} \left[k \times \binom{n-1}{k-1} \times \bar{\delta}(k, V_j) \right].$$

Since each agent is a member of $\binom{n-1}{|S|-1}$ coalitions of size $|S|$, we can write for every agent $i \in N$ and every issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} \big[k \times z[(j-1)n+k]\big] = \sum_{k=1}^{n} \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}, |T|=k-1\}} \big[|S| \times \bar{\delta}(|S|, V_j)\big],$$

or, equivalently, for every agent $i \in N$ and issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} \big[k \times z[(j-1)n+k]\big] = \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \sum_{i' \in S} \tilde{\delta}(S, i', V_j).$$

By "Zero to Nonmembers" and by the definition of a characteristic vector, for every agent $i \in N$ and every issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} \big[k \times z[(j-1)n+k]\big] = \sum_{i' \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i', V_j) \chi_i^S.$$

Since $\tilde{\delta}(S, i, V_j)$ satisfies "Resource Exhaustion," we get

$$\sum_{k=1}^{n} \big[k \times z[(j-1)n+k]\big] = 1,$$

meaning that the suggested z satisfies the first m equations.

By construction, for every issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} z[(j-1)n+k] - \sum_{k=1}^{n} z[(m-1)n+k] = \sum_{k=1}^{n} \binom{n-1}{k-1} \times [\bar{\delta}(k, V_j) - \bar{\delta}(k, V_m)].$$

Since each agent is a member of $\binom{n-1}{|S|-1}$ coalitions of size $|S|$, we can write for every agent $i' \in N$ and every issue $V_j \in \bar{V}$,

$$\begin{aligned}\sum_{k=1}^{n} z[(j-1)n+k] &- \sum_{k=1}^{n} z[(m-1)n+k] \\ &= \sum_{k=1}^{n} \sum_{S \in \{T \cup \{i'\} | T \subseteq N \setminus \{i'\}, |T|=k-1\}} \big[\bar{\delta}(|S|, V_j) - \bar{\delta}(|S|, V_m)\big],\end{aligned}$$

or, equivalently, for every pair of agents $i, i' \in N$ and issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} z[(j-1)n+k] - \sum_{k=1}^{n} z[(m-1)n+k] = \sum_{S \in \{T \cup \{i'\} | T \subseteq N \setminus \{i'\}\}} \tilde{\delta}(S, i, V_j) - \tilde{\delta}(S, i, V_m).$$

Hence, for every pair of agents $i, i' \in N$ and issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} z[(j-1)n+k] - \sum_{k=1}^{n} z[(m-1)n+k] = \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi_{i'}^S - \sum_{S \in 2^N} \tilde{\delta}(S, i, V_m) \chi_{i'}^S.$$

Since the same is true for every $i' \in N$, we can write for every agent i and every issue $V_j \in \bar{V}$,

$$\sum_{k=1}^{n} z[(j-1)n+k] - \sum_{k=1}^{n} z[(m-1)n+k] = \sum_{S\in 2^N} \tilde{\delta}(S,i,V_j)\chi^S - \sum_{S\in 2^N} \tilde{\delta}(S,i,V_m)\chi^S.$$

Since $\tilde{\delta}(S,i,V_j)$ satisfies "Constant Shares," we get

$$\sum_{k=1}^{n} z[(j-1)n+k] - \sum_{k=1}^{n} z[(m-1)n+k] = 0.$$

Therefore, every system of homogeneous balancing multiweights corresponds to some $z \in Z$ and the proof is completed. □

Proof of Theorem 3. First, we show that $\sum_{V_j\in\bar{V}} C(V_j) \subseteq M(\bar{V})$. Note that if $\sum_{V_j\in\bar{V}} C(V_j) = \varnothing$, the statement is vacuously true. Otherwise, let $x \in \sum_{V_j\in\bar{V}} C(V_j)$. Then, by definition, $x = \sum_{j\in\{1,\dots,m\}} x^j$, where $\forall V_j \in \bar{V}$: $x^j \in C(V_j)$. Consider the matrix $Y = [x^1, x^2, \dots, x^m]$. Since $\forall V_j \in \bar{V}$: $x^j \in C(V_j)$, Y is an efficient decomposition matrix. For the same reason, $\forall S \subseteq N$, $\forall V_j \in \bar{V}$: $\sum_{i\in S} x_i^j \geq V_j(S)$. Since this condition is satisfied for all coalitions, the coalitional rationality condition is satisfied for all agents. Hence, Y justifies x for every agent $i \in N$ and, therefore, $x \in M(\bar{V})$.

Next, we show that $M(\bar{V}) \subseteq C(\sum_{V_j\in\bar{V}} V_j)$. Again, note that if $M(\bar{V}) = \varnothing$, the statement is vacuously true. Otherwise, let $x \in M(\bar{V})$. Therefore, $\sum_{i=1}^{n} x_i = \sum_{V_j\in\bar{V}} V_j(N)$ so that x is an efficient payoff vector in the game that sums all the individual issues. Denote the justification matrix of agent i by y^i (such a matrix exists since $x \in M(\bar{V})$). Then, for every nonempty coalition $S \subseteq N$, every $i \in S$ satisfies (see footnote 13),

$$\sum_{k\in S} x_k = \sum_{k\in S}\sum_{j\in\{1,\dots,m\}} y^i_{k,j} = \sum_{j\in\{1,\dots,m\}}\sum_{k\in S} y^i_{k,j} \geq \sum_{V_j\in\bar{V}} V_j(S).$$

The first equality is due to y^i being a decomposition matrix and the inequality holds since y^i satisfies the coalitional rationality condition. Hence, $x \in C(\sum_{V_j\in\bar{V}} V_j)$. □

Definition 8 (taken from Gayer et al. 2015). Let $F : 2^N \to \mathbb{R}_+$ be a system of weights. Let $W^F = \sum_{S\in 2^N} F(S)\chi^S$ denote the agents' weights vector induced by F. We say that F_1 and F_2 are W-equivalent if $W^{F_1} = W^{F_2}$. Denote the set of all W-equivalence classes by Γ. For every class $\gamma \in \Gamma$, denote the agents' weights by W^γ. For every characteristic function V and $\gamma \in \Gamma$, denote $T_V^\gamma \equiv \max_{F\in\gamma} \sum_{S\in 2^N} F(S)V(S)$.

Proof of Proposition 3. Consider, with no loss of generality, an equivalence set γ such that $W^\gamma[1] \geq W^\gamma[2] \geq W^\gamma[3]$. By Lemma 3, for every $V_j \in \bar{V}$, there exists $F_j \in \gamma$

such that $\sum_{S\in 2^N} F_j(S)V_j(S) = T^{\gamma}_{V_j}$ and $F_j(\{2\}) = F_j(\{3\}) = F_j(\{2,3\}) = 0$. Alternatively, for every characteristic function $V_j \in \bar{V}$, there exists $F_j \in \gamma$ such that $\sum_{S\in\{T\cup\{1\}\mid T\subseteq N\setminus\{1\}\}} F_j(S)V_j(S) = T^{\gamma}_{V_j}$. Let $x \in M(\bar{V})$ and let y^1 be a justification matrix for agent 1. For every $V_j \in \bar{V}$ and for every $S \in \{T \cup \{1\} \mid T \subseteq N \setminus \{1\}\}$, $\sum_{i\in S} y^1_{i,j} \geq V_j(S)$. Then multiplying both sides of each inequality by the corresponding $F_j(S)$ and aggregating over all $S \in \{T \cup \{1\} \mid T \subseteq N \setminus \{1\}\}$ yields for every $V_j \in \bar{V}$, $\sum_{S\in\{T\cup\{1\}\mid T\subseteq N\setminus\{1\}\}} F_j(S)\sum_{i\in S} y^1_{i,j} \geq \sum_{S\in\{T\cup\{1\}\mid T\subseteq N\setminus\{1\}\}} F_j(S)V_j(S)$ or, equivalently, for every $V_j \in \bar{V}$, $\sum_{i\in N} y^1_{i,j} \sum_{S\in\{T\cup\{1,i\}\mid T\subseteq N\setminus\{1,i\}\}} F_j(S) \geq \sum_{S\in 2^N} F_j(S)V_j(S)$.

The inequality becomes $\sum_{i\in N} y^1_{i,j} W^{\gamma}[i] \geq T^{\gamma}_{V_j}$ for every $V_j \in \bar{V}$ since $\sum_{S\in\{T\cup\{1,i\}\mid T\subseteq N\setminus\{1,i\}\}} F_j(s) = W^{\gamma}[i]$ for every agent i and $\sum_{S\in 2^N} F_j(S)V_j(S) = T^{\gamma}_{V_j}$. By aggregating over all the issues, $\sum_{j\in\{1,\dots,m\}} \sum_{i\in N} y^1_{i,j} W^{\gamma}[i] \geq \sum_{V_j\in\bar{V}} T^{\gamma}_{V_j}$, and changing the order of summation $\sum_{i\in N} W^{\gamma}[i] \sum_{j\in\{1,\dots,m\}} y^1_{i,j} \geq \sum_{V_j\in\bar{V}} T^{\gamma}_{V_j}$ is obtained. The justification matrix y^1 decomposes x and, therefore, $\sum_{i\in N} W^{\gamma}[i]x_i \geq \sum_{V_j\in\bar{V}} T^{\gamma}_{V_j}$. Then by the decomposition lemma in Gayer et al. (2015), the aggregate payoff vector x can be decomposed into m vectors $\{x^1, \dots, x^m\}$ such that for every $V_j \in \bar{V}$, $x^j \in C(V_j)$ and $\sum_{j\in\{1,\dots,m\}} x^j = x$. Hence, $x \in M(\bar{V})$ implies $x \in \sum_{V_j\in\bar{V}} C(V_j)$. Together with Theorem 3 we conclude that the multicore is ineffective. $\square$

Lemma 3. *Let V be a three-agent superadditive cooperative game such that $C(V) \neq \varnothing$. Let γ be an equivalence class such that $W^{\gamma}[1] \geq W^{\gamma}[2] \geq W^{\gamma}[3]$. There exists a system of weights $F \in \gamma$ such that $\sum_{S\in 2^N} F(S)V(S) = T^{\gamma}_V$ and $F(\{2\}) = F(\{3\}) = F(\{2,3\}) = 0$.*

Proof. The set $A \in \{F \in \gamma \mid \sum_{S\in 2^N} F(S)V(S) = T^{\gamma}_V\}$ is nonempty since every equivalence class γ is closed and $\sum_{S\in 2^N} F(S)V(S)$ is a linear function on γ and is, therefore, continuous. The proof is constructive. We take any $F \in A$ and use it to construct $\bar{F} \in A$ such that $\bar{F}(\{2\}) = \bar{F}(\{3\}) = \bar{F}(\{2,3\}) = 0$ in three steps. The first step is to use F to construct $\tilde{F} \in A$ such that $\tilde{F}(\{2\}) = 0$. There are four cases.

1. If $F(\{2\}) = 0$, we are done by $\tilde{F} = F$.

2. If $F(\{2\}) \neq 0$ and $F(\{1\}) = 0$, since $W^{\gamma}[1] \geq W^{\gamma}[2]$, then $F(\{1\}) + F(\{1,2\}) + F(\{1,3\}) + F(\{1,2,3\}) \geq F(\{2\}) + F(\{1,2\}) + F(\{2,3\}) + F(\{1,2,3\})$ or $F(\{1,3\}) \geq F(\{2\}) + F(\{2,3\})$. Since $F(\{2,3\}) \geq 0$, we get $F(1,3) \geq F(2)$. By Lemma 4, we set $\tilde{F} = F_{\{1,3\},\{2\}}$ and get $\tilde{F} \in A$ and $\tilde{F}(\{2\}) = 0$.

3. If $F(\{2\}) \geq F(\{1\}) > 0$, then by Lemma 4, we set $\tilde{F} = F_{\{1\},\{2\}}$ and get $\tilde{F} \in A$ and $\tilde{F}(\{1\}) = 0$. Then we redo step one.

4. If $F(\{1\}) \geq F(\{2\}) > 0$, then by Lemma 4, we set $\tilde{F} = F_{\{1\},\{2\}}$ and get $\tilde{F} \in A$ and $\tilde{F}(\{2\}) = 0$.

The second step is to construct $\hat{F} \in A$ from $\tilde{F}$ such that $\hat{F}(\{3\}) = 0$, which is similar to the construction of the first step. The third and final step is to use $\hat{F}$ to construct $\bar{F} \in A$ such that $\bar{F}(\{2\}) = \bar{F}(\{3\}) = \bar{F}(\{2,3\}) = 0$. There are four cases.

1. If $\hat{F}(\{2,3\}) = 0$, we are done by $\bar{F} = \hat{F}$.
2. If $\hat{F}(\{2,3\}) > 0$ and $\hat{F}(\{1\}) = 0$, by Lemma 5, we set $\bar{F} = \hat{F}_{-23}$ and get $\bar{F} \in A$ and $\bar{F}(\{2,3\}) = 0$.
3. If $\hat{F}(\{2,3\}) \geq \hat{F}(\{1\}) > 0$, then by Lemma 4, we set $\bar{F} = \hat{F}_{\{1\},\{2,3\}}$ and get $\bar{F} \in A$ and $\bar{F}(\{1\}) = 0$. Then we redo step three.
4. If $0 < \hat{F}(\{2,3\}) \leq \hat{F}(\{1\})$, then by Lemma 4, $\bar{F} = \hat{F}_{\{1\},\{2,3\}}$, $\bar{F} \in A$, and $\bar{F}(\{2,3\}) = 0$. □

Lemma 4. *Let V be a superadditive cooperative game. Let $F \in \gamma$ and let S and S' be two disjoint coalitions ($S \cap S' = \varnothing$). Define for all $t \in 2^N$,*

$$F_{S,S'}(t) = \begin{cases} F(t) - \min\{F(S), F(S')\} & \text{if } t \in \{S, S'\} \\ F(t) + \min\{F(S), F(S')\} & \text{if } t = S \cup S' \\ F(t) & \text{otherwise.} \end{cases}$$

Then, $F_{S,S'} \in \gamma$ and $\sum_{t\in 2^N} F_{S,S'(t)} V(t) \geq \sum_{t\in 2^N} F(t)V(t)$.

Proof. We show first that $F_{S,S'} \in \gamma$ and then that $\sum_{t\in 2^N} F_{S,S'}(t)V(t) \geq \sum_{t\in 2^N} F(t)V(t)$:

$$\begin{aligned}
\sum_{t\in 2^N} F_{S,S'}(t)\chi^t &= \sum_{t\in 2^N\setminus\{S,S',S\cup S'\}} F_{S,S'}(t)\chi^t + F_{S,S'}(S)\chi^S \\
&\qquad + F_{S,S'}(S')\chi^{S'} + F_{S,S'}(S\cup S')\chi^{S\cup S'} \\
&= \sum_{t\in 2^N\setminus\{S,S',S\cup S'\}} F(t)\chi^t + \big[F(S) - \min\{F(S), F(S')\}\big]\chi^S \\
&\qquad + \big[F(S') - \min\{F(S), F(S')\}\big]\chi^{S'} \\
&\qquad + \big[F(S\cup S') + \min\{F(S), F(S')\}\big]\chi^{S\cup S'} \\
&= \sum_{t\in 2^N} F(t)\chi^t + \min\{F(S), F(S')\}[\chi^{S\cup S'} - \chi^S - \chi^{S'}] \\
&= \sum_{t\in 2^N} F(t)\chi^t = W^\gamma
\end{aligned}$$

$$\begin{aligned}
\sum_{t\in 2^N} F_{S,S'}(t)V(t) &= \sum_{t\in 2^N\setminus\{S,S',S\cup S'\}} F_{S,S'}(t)V(t) + F_{S,S'}(S)V(S) \\
&\qquad + F_{S,S'}(S')V(S') + F_{S,S'}(S\cup S')V(S\cup S') \\
&= \sum_{t\in 2^N\setminus\{S,S',S\cup S'\}} F(t)V(t) + \big[F(S) - \min\{F(S), F(S')\}\big]V(S) \\
&\qquad + \big[F(S') - \min\{F(S), F(S')\}\big]V(S') \\
&\qquad + \big[F(S\cup S') + \min\{F(S), F(S')\}\big]V(S\cup S')
\end{aligned}$$

$$= \sum_{t \in 2^N} F(t)V(t) + \min\{F(S), F(S')\}[V(S \cup S') - V(S) - V(S')]$$
$$\geq \sum_{t \in 2^N} F(t)V(t).$$
□

Lemma 5. *Let V be a three-agent superadditive cooperative game such that $C(V) \neq \varnothing$. Let γ be an equivalence class such that $W^\gamma[1] \geq W^\gamma[2] \geq W^\gamma[3]$ and let $F \in \gamma$ be such that $F(\{1\}) = 0$. Define for all $t \in 2^N$,*

$$F_{-23}(t) = \begin{cases} F(t) & \text{if } |t| = 1 \\ F(t) - F(\{2,3\}) & \text{if } |t| = 2 \\ F(t) + 2F(\{2,3\}) & \text{if } |t| = 3. \end{cases}$$

Then $F_{-23} \in \gamma$ and $\sum_{t \in 2^N} F_{-23}(t)V(t) \geq \sum_{t \in 2^N} F(t)V(t)$.

Proof. First we show that $F_{-23}(t) \geq 0$ for every coalition t, as required by Definition 8. Note that, $W^\gamma[1] \geq W^\gamma[2]$ if and only if $F(\{1\}) + F(\{1,2\}) + F(\{1,3\}) + F(\{1,2,3\}) \geq F(\{2\}) + F(\{1,2\}) + F(\{2,3\}) + F(\{1,2,3\})$, meaning that $W^\gamma[1] \geq W^\gamma[2]$ if and only if $F(\{1,3\}) \geq F(\{2\}) + F(\{2,3\})$. Therefore, if $W^\gamma[1] \geq W^\gamma[2]$, then $F(\{1,3\}) \geq F(\{2,3\})$. Similarly, if $W^\gamma[1] \geq W^\gamma[3]$, then $F(\{1,2\}) \geq F(\{2,3\})$. Therefore, for every coalition t of size 2, $F(t) - F(\{2,3\}) \geq 0$.

Next, we show that $F_{-23} \in \gamma$:

$$\begin{aligned}
\sum_{t \in 2^N} F_{-23}(t)\chi^t &= F_{-23}(\{1\})\chi^{\{1\}} + F_{-23}(\{2\})\chi^{\{2\}} + F_{-23}(\{3\})\chi^{\{3\}} + F_{-23}(\{1,2\})\chi^{\{1,2\}} \\
&\quad + F_{-23}(\{1,3\})\chi^{\{1,3\}} + F_{-23}(\{2,3\})\chi^{\{2,3\}} + F_{-23}(\{1,2,3\})\chi^{\{1,2,3\}} \\
&= F(\{1\})\chi^{\{1\}} + F(\{2\})\chi^{\{2\}} + F(\{3\})\chi^{\{3\}} + \big[F(\{1,2\}) - F(\{2,3\})\big]\chi^{\{1,2\}} \\
&\quad + \big[F(\{1,3\}) - F(\{2,3\})\big]\chi^{\{1,3\}} + \big[F(\{2,3\}) - F(\{2,3\})\big]\chi^{\{2,3\}} \\
&\quad + \big[F(\{1,2,3\}) + 2F(\{2,3\})\big]\chi^{\{1,2,3\}} \\
&= \sum_{t \in 2^N} [F(t)\chi^t] + F(\{2,3\})[2\chi^{\{1,2,3\}} - \chi^{\{1,2\}} - \chi^{\{1,3\}} - \chi^{\{2,3\}}] \\
&= \sum_{t \in 2^N} F(t)\chi^t = W^\gamma.
\end{aligned}$$

Finally, we show that $\sum_{t \in 2^N} F_{-23}(t)V(t) \geq \sum_{t \in 2^N} F(t)V(t)$:

$$\begin{aligned}
\sum_{t \in 2^N} F_{-23}(t)V(t) &= F_{-23}(\{1\})V(\{1\}) + F_{-23}(\{2\})V(\{2\}) + F_{-23}(\{3\})V(\{3\}) \\
&\quad + F_{-23}(\{1,2\})V(\{1,2\}) + F_{-23}(\{1,3\})V(\{1,3\}) \\
&\quad + F_{-23}(\{2,3\})V(\{2,3\}) + F_{-23}(\{1,2,3\})V(\{1,2,3\}) \\
&= F(\{1\})V(\{1\}) + F(\{2\})V(\{2\}) + F(\{3\})V(\{3\})
\end{aligned}$$

$$
\begin{aligned}
&+\big[F(\{1,2\})-F(\{2,3\})\big]V(\{1,2\})\\
&+\big[F(\{1,3\})-F(\{2,3\})\big]V(\{1,3\})\\
&+\big[F(\{2,3\})-F(\{2,3\})\big]V(\{2,3\})\\
&+\big[F(\{1,2,3\})+2F(\{2,3\})\big]V(\{1,2,3\})\\
=\sum_{t\in 2^N} F(t)V(t)&\\
&+F(\{2,3\})\big[2V(\{1,2,3\})-V(\{1,2\})-V(\{1,3\})-V(\{2,3\})\big].
\end{aligned}
$$

The inequality $2V(\{1,2,3\})\geq V(\{1,2\})+V(\{1,3\})+V(\{2,3\})$ follows from the fact that $C(V)$ is nonempty.[46] Thus, $\sum_{t\in 2^N} F_{-23}(t)V(t)\geq \sum_{t\in 2^N} F(t)V(t)$. $\square$

References

Assa, Hirbod, Sheridon Elliston, and Ehud Lehrer (2014), "Joint games and compatibility." Working paper. [939]

Bernheim, B. Douglas (1984), "Rationalizable strategic behavior." *Econometrica*, 52, 1007–1028. [942]

Bloch, Francis and Geoffroy de Clippel (2010), "Cores of combined games." *Journal of Economic Theory*, 145, 2424–2434. [939, 947]

Bondareva, Olga N. (1963), "Some applications of linear programming methods to the theory of cooperative games." *Problemy Kybernetiki*, 10, 119–139. (In Russian). [938, 942]

Diamantoudi, Effrosyni, Inés Macho-Stadler, David Pérez-Castrillo, and Licun Xue (2013), "Sharing the surplus in games with externalities within and across issues." Working paper. [939]

Dragan, Irinal, Jos Potters, and Stef Tijs (1989), "Superadditivity for solutions of coalitional games." *Libertas Mathematica*, 9, 101–110. [947]

Fernández, Francisco R., Miguel A. Hinojosa, and Justo Puerto (2002), "Core solutions in vector-valued games." *Journal of Optimization Theory and Applications*, 112, 331–360. [939]

Fernández, Francisco R., Miguel A. Hinojosa, and Justo Puerto (2004), "Set-valued TU-games." *European Journal of Operational Research*, 159, 181–195. [939]

Gayer, Gabrielle, Ehud Lehrer, and Dotan Persitz (2015), "Aggregating non additive beliefs." Working paper. [948, 964, 965]

Hwang, Yan-An and Yu-Hsien Liao (2011), "The multi-core, balancedness and axiomatizations of multi-choice games." *International Journal of Game Theory*, 40, 677–689. [938]

[46] Suppose that $x\in C(V)$. Then, since $x_1+x_2\geq V(\{1,2\})$, $x_1+x_3\geq V(\{1,3\})$ and $x_2+x_3\geq V(\{2,3\})$, we can assert that $x_1+x_2+x_3\geq \frac{1}{2}\times[V(\{1,2\})+V(\{1,3\})+V(\{2,3\})]$. By the efficiency of x, $2V(\{1,2,3\})\geq V(\{1,2\})+V(\{1,3\})+V(\{2,3\})$.

Kalai, Ehud (1977), “Proportional solutions to bargaining situations: Interpersonal utility comparisons.” *Econometrica*, 45, 1623–1630. [939]

Nax, Heinrich H. (2014), “A note on the core of TU-cooperative games with multiple membership externalities.” *Games*, 5, 191–203. [939]

Pearce, David G. (1984), “Rationalizable strategic behavior and the problem of perfection.” *Econometrica*, 52, 1029–1050. [942]

Ponsati, Clara and Joel Watson (1997), “Multiple-issue bargaining and axiomatic solutions.” *International Journal of Game Theory*, 26, 501–524. [939]

Shapley, Lloyd S. (1967), “On balanced sets and cores.” *Naval Research Logistics Quarterly*, 14, 453–460. [938, 942]

Co-editor Faruk Gul handled this manuscript.

Submitted 2014-5-20. Final version accepted 2015-10-6. Available online 2015-10-8.