Career concerns with exponential learning

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This paper examines the interplay between career concerns and market structure. Ability and effort are complements: effort increases the probability that a skilled agent achieves a one-time breakthrough. Wages are based on assessed ability and on expected output. Effort levels at different times are strategic substitutes and, as a result, the unique equilibrium effort and wage paths are single-peaked with seniority. Moreover, for any wage profile, the agent works too little, too late. Commitment to wages by competing firms mitigates these inefficiencies. In that case, the optimal contract features piecewise constant wages and severance pay.

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1. Introduction

Career concerns are an important driver of incentives. This is particularly so in professional-service firms, such as law and consulting, but applies more broadly to environments in which creativity and originality are essential for success, including pharmaceutical companies, biotechnology research labs, and academia. Market structure differs across those industries, and so do labor market arrangements. Our goal is to understand how market structure (in particular, firms' commitment power) affects career concerns, and the resulting patterns of wages and performance. As we show, commitment leads to backloading of wages and effort, relative to what happens under spot contracting.

Our point of departure from nearly all the existing literature on career concerns, starting with Holmström (1982/1999), involves the news process. The ubiquitous assumption that rich measures of output are available throughout is at odds with the structure of learning in several industries. To capture the features of research-intensive and creative industries, we assume that success is rare and informative about the worker's skill. Breakthroughs are defining moments in a professional's career. In other words, information is coarse: either an agent reveals himself to be talented through a (first) breakthrough or he does not. Indeed, in many industries, there is growing evidence that the market rewards "star" professionals.[1]

Our assumption on the contracting environment follows the literature on career concerns. Explicit output-contingent contracts are not used. While theoretically attractive, innovation bonuses in research and development (R&D) firms are hard to implement due to complex attribution and timing problems. Junior associates in law and consulting firms receive fixed stipends. In the motion pictures industry, most contracts involve fixed payments rather than profit-sharing (see Chisholm 1997).[2]

In our model, information about ability is symmetric at the start.[3] Skill and output are binary, and effort is continuous. Furthermore, skill and effort are complements: only a skilled agent can achieve a high output, or *breakthrough*. The breakthrough time follows an exponential distribution, whose intensity increases with the worker's unobserved effort. Hence, effort increases not only expected output, but also the rate of learning, unlike in the additive Gaussian setup. When a breakthrough obtains, the market recognizes the agent's talent and that is reflected in future earnings. The focus is on the relationship until a breakthrough occurs.[4] Throughout, the market is competitive. We contrast the equilibrium when firms can commit to long-term wage policies and when they cannot.

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[1]Prominent examples include working a breakthrough case in law or consulting, signing a record deal, or acting in a blockbuster movie. See Gittelman and Kogut (2003) and Zucker and Darby (1996) for evidence on the impact of "star scientists," and Caves (2003) for a discussion of A-list vs. B-list actors and writers. For a different example, consider a local politician who seeks to establish a reputation for "getting things done" by securing congressional funding for a major public good.

[2]It is not our purpose to explain why explicit contracts are not used. As an example that satisfies both features, consider the biotechnology and pharmaceutical industries. Uncertainty and delay shroud profitability. Scientific breakthroughs are recognized quickly, commercial ones not so. Ethiraj and Zhao (2012) find a success rate of 1.7% for molecules developed in 1990. The annual report by the Pharmaceutical Research and Manufacturers of America (PhRMA) (2012) shows even higher attrition rates. Because the Food and Drug Administration (FDA) drug approval and molecule patenting are only delayed and noisy metrics of a drug's profitability, they are rarely tied explicitly to a scientist's compensation (see Cockburn et al. 2004). Likewise, few biotechnology companies offer variable pay in the form of stock options (see Stern 2004).

[3]One could also examine the consequences of overoptimism by the agent. In many applications, however, symmetrical ignorance appears to be the more plausible assumption. See Caves (2003).

[4]In Section 4.1, we turn to the optimal design of an up-or-out arrangement, i.e., a deadline. A probationary period is a hallmark of many occupations (law, accounting, and consulting firms, etc.). Though alternative theories have been put forth (e.g., tournament models), agency theory provides an appealing framework (see Fama 1980 or Fama and Jensen 1983). Gilson and Mnookin (1989) offer a vivid account of associate career patterns in law firms, and the relevance of the career concerns model as a possible explanation.

Spot contracts

Under spot contracts, the agent is paid his full expected marginal product at each moment in time. Therefore, the agent's compensation depends on the market's expectation about his talent and effort. In turn, this determines the value of incremental reputation, and hence his incentives to exert effort so as to establish himself as high skilled.[5] In equilibrium, both the agent's effort level and wage are single-peaked functions of time.

A key driver of the equilibrium properties is the *strategic substitutability* effect between incentives at different stages in a worker's career. Suppose the market expects the agent to exert high effort at some point. Failure to generate a success would lead the market to revise its belief downward and lower future wages. This provides strong incentives for effort. However, competitive wages at that time must reflect the resulting increased productivity. In turn, this depresses the agent's incentives and compensation at earlier stages, as future wages make staying on the current job relatively more attractive.

Strategic substitutability does not arise in Holmström's additively separable model or in the two-period model of Dewatripont et al. (1999a, 1999b), because career concerns disappear in the last period. Instead, the strategic complementarity between expected and realized effort in the first period generates equilibrium multiplicity.

Despite the complementarity between skill and effort, equilibrium is unique in our model, yielding robust yet sharp welfare predictions. In particular, *effort underprovision and delay* obtain very generally. Because output-contingent payments are impossible, competition among employers calls for positive flow wages even after prolonged failure. As a result, career concerns provide insufficient incentives for effort independently from any particular equilibrium notion. *For any* wage path, the total amount of effort exerted is inefficiently low. In addition, effort is exerted too late: a social planner constrained to the same total amount of effort exerts it sooner.[6] As we shall see, these properties also hold under competition with long-term contracts.

The most striking prediction of our model in terms of observable variables—single-peaked real wages—cannot be explained by the existing models (Holmström, and Dewatripont, Jewitt, and Tirole). Because successful agents are immediately promoted, single-peaked wages refer to wages *conditional on prolonged failure*. This prediction is borne out by the data in the two papers by Baker et al. (1994a, 1994b), arguably the most widely used data set on internal wage policy in the organizational economics literature.[7]

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[5]This is in contrast with Holmström, where the marginal benefit from effort is history-independent. This implies that reputation incentives evolve deterministically, decreasing over time as learning occurs, while wages decrease stochastically.

[6]Characterizing the agent's best reply to exogenous wages has implications for the agent's behavior following one of his own deviations. In particular, it helps clarify the effect of the agent's private beliefs.

[7]Baker et al. (1994a) describe the data as containing "personnel records for all management employees of a medium-sized U.S. firm in a service industry over the years 1969–1988." While the data set is confidential (and hence we cannot relate the properties of the industry to the parameters of our model), the population of employees is restricted to "exempt" management positions, i.e., those for whom career concerns are most salient.

Their data show that both the dynamics of real wages and the timing of promotions are heterogeneous across agents. Overall, Baker et al. (1994b) suggest that such heterogeneity is indicative of a "common underlying factor, such as ability, driving both wage increases and promotions." At the aggregate level, yearly real wages decrease on average, conditional on no promotion. However, this is not true at all tenure levels.[8] Baker et al. (1994b) provide more detail about the wage and promotion patterns of employees in the lowest two levels of the firm's hierarchy. Arguably, these are the employees with the strongest potential for establishing a reputation.

Baker et al. (1994b) show that the pattern of real wages is inverse-U-shaped for employees who are not promoted to the next level within 8 years. In other words, wages increase at first for all employees and then decline until the employee is promoted to the next level in the firm's hierarchy (if ever). In Section 3.4, we compare this pattern to our equilibrium wages.

There are, of course, alternative explanations: a combination of symmetric learning and accumulation of general human capital would suggest that, for a particular choice of technology, a worker's expected productivity may increase at first only to quickly decline in the absence of sufficiently positive signals. Yet our model matches the outcomes described in the data quite well, relying only on hidden talent and a lumpy output process.

Long-term contracts

The flexibility of our model allows us to study reputation incentives under market structures that are not tractable in the Gaussian framework. In particular, we consider long-term contracts. In many sectors, careers begin with a probationary period that leads to an up-or-out decision, and wages are markedly lower and more rigid before the tenure decision than after. Specifically, we allow firms to commit to a wage path, but the agent may leave at any time. To avoid "poaching," the contract must perpetually deliver a continuation payoff above what the agent can get on the market. This constraint implies that one must solve for the optimal contract in all possible continuation games, as the offers of competing firms must themselves be immune to further poaching.

Long-term contracts can mitigate the adverse consequences of output-independent wages because the timing of payments affects the optimal timing of the agent's effort. Because future wages paid in the event of persistent failure depress current incentives, it would be best to pay the worker his full marginal product ex ante. This payment being sunk, it would be equivalent, in terms of incentives, to no future payments for failure at all. Therefore, if the worker can commit to a *no-compete* clause, a simple signing bonus is optimal.

In most labor markets workers cannot commit to such clauses. It follows that firms do not offer signing bonuses, anticipating the worker's incentive to leave right after cashing them in. Lack of commitment on the worker's side prevents payments coming before the corresponding marginal product obtains. Surprisingly, as far as current incentives

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[8]See Table VI in Baker et al. (1994a).

are concerned, it is then best to pay him *as late as possible*. This follows from the value of learning: much later payments discriminate better than imminent ones between skilled and unskilled workers. Because effort and skill are complements, a skilled worker is likely to succeed by the end. Hence, payments made in the case of persistent failure are tied more to the agent's talent than to his effort. This mitigates the pernicious effect of future wages on current incentives. Backloading payments also softens the no-poaching constraint, as the worker has fewer reasons to quit.

To summarize, long-term contracts backload payments and frontload effort, relative to spot contracts. Backloading pay and severance payments (as well as signing bonuses and no-compete clauses) are anecdotally common in industries with one-sided commitment, such as law or consulting. In other words, several regularities observed in practice arise as optimal labor-market arrangements when firms compete with long-term contracts.

Related literature

The most closely related papers are Holmström (1982/1999), as mentioned, as well as Dewatripont et al. (1999a, 1999b). In Holmström, skill and effort enter linearly and additively into the mean of the output that is drawn in every period according to a normal distribution. Wages are as in our baseline model: the worker is paid upfront the expected value of the output. Our model shares with the two-period model of Dewatripont, Jewitt, and Tirole some features that are absent from Holmström's. In particular, effort and talent are complements. We shall discuss the relationship between the three models at length.

Our paper can also be viewed as combining career concerns and experimentation. As such, it relates to Holmström's original contribution in the same way as the exponential-bandits approach of Keller et al. (2005) does to the strategic experimentation framework introduced by Bolton and Harris (1999).

As in Gibbons and Murphy (1992), our paper examines the interplay of implicit incentives (career concerns) and explicit incentives (termination penalty). It shares with Prendergast and Stole (1996) the existence of a finite horizon, and thus, of complex dynamics related to seniority. See also Bar-Isaac (2003) for reputational incentives in a model in which survival depends on reputation. The continuous-time model of Cisternas (2016) extends the Gaussian framework to nonlinear environments, but maintains the additive separability of talent and action. Ferrer (2015) studies the effect of lawyers' career concerns on litigation in a model with complementarity between effort and talent. Jovanovic (1979) and Murphy (1986) provide models of career concerns that are less closely related: the former abstracts from moral hazard and focuses on turnover when agents' types are match-specific; the latter studies executives' experience–earnings profiles in a model in which firms control the level of capital assigned to them over time. Finally, Klein and Mylovanov (2016) analyze a career-concerns model of advice, and provide conditions under which reputational incentives over a long horizon restore the efficiency of the equilibrium.

The binary setup is reminiscent of Mailath and Samuelson (2001), Bergemann and Hege (2005), Board and Meyer-ter-Vehn (2013), and Atkeson et al. (2015). However, in

Bergemann and Hege (2005), the effort choice is binary and wages are not based on the agent's reputation, while Board and Meyer-ter-Vehn (2013) let a privately informed agent control the evolution of his type through his effort. In (the regulation model with reputation of) Atkeson et al. (2015), firms control their types through a one-time initial investment. Here instead, information is symmetric and types are fixed.[9]

Finally, several papers have already developed theories of wage rigidity and backloading with long-term contracts, none based on career concerns (incentive provision under incomplete information), as far as we know. These are discussed in Section 5.

Structure

The paper is organized as follows: Section 2 describes the model; Section 3 analyzes the case of spot contracts; Section 4 introduces probationary periods and discusses what happens when the agent is impatient or when even a low-ability agent might succeed with positive probability; Section 5 explores long-term contracts; Section 6 discusses the case of observable effort; and Section 7 briefly describes other extensions.

2. The model

2.1 Setup

We consider the incentives of an agent (or *worker*) to exert hidden effort (or *work*). Time is continuous, and the horizon (or *deadline*) is finite: $t \in [0, T]$, $T > 0$. Most results carry over to the case $T = \infty$, as shall be discussed.

There is a binary state of the world. If the state is $\omega = 0$, the agent is bound to fail, no matter how much effort he exerts. If the state is $\omega = 1$, a success (or *breakthrough*) arrives at a time that is exponentially distributed, with an intensity that increases in the instantaneous level of effort exerted by the agent. The state can be interpreted as the agent's ability, or skill. We will refer to the agent as a high- (resp., low-) ability agent if the state is 1 (resp., 0). The prior probability of state $\omega = 1$ is $p^0 \in (0, 1)$.

Effort is a nonnegative measurable function of time. If a high-ability agent exerts effort u_t over the time interval $[t, t + dt)$, the probability of a success over that time interval is $(\lambda + u_t) \, dt$. Formally, the instantaneous arrival rate of a breakthrough at time t is given by $\omega \cdot (\lambda + u_t)$, with $\lambda \geq 0$. Note that, unlike in Holmström's model, but as in the model of Dewatripont, Jewitt, and Tirole, work and talent are complements.

The parameter λ can be interpreted as the luck of a talented agent. Alternatively, it measures the minimum effort level that the principal can force upon the agent by direct oversight, i.e., the degree of contractibility of the worker's effort. Either interpretation might require some adjustment: as minimum effort, it is then important to interpret the flow cost of effort as net of this baseline effort, and to be aware of circumstances in which it would not be in either party's interest to exert this minimum level.[10] Fortunately, this

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[9]Board and Meyer-ter-Vehn (2014) study the Markov-perfect equilibria of a game in which effort affects the evolution of the player's type both under symmetric and asymmetric information.

[10]As a referee pointed out, too low an effort level may be undesirable from the agent's point of view, whose effort "bliss point" need not be zero.

issue does not arise for the equilibrium analysis under either commitment assumption. As for the luck interpretation, it is then arguably extreme to assume that luck correlates with ability, so that a low-ability agent is bound to fail. In Section 4.2 we provide a discussion and numerical simulations of what happens when an agent of low ability is less likely to succeed for a given effort level, yet might do so nonetheless.

As long as a breakthrough has not occurred, the agent receives a flow wage w_t. For now, think of this wage as an exogenous (integrable, nonnegative) function of time. Later on, equilibrium constraints will be imposed on this function, and this wage will reflect the market's expectations of the agent's effort and ability, given that the market values a success. The value of a success is normalized to 1.

In addition to receiving this wage, the agent incurs a cost of effort: exerting effort level u_t over the time interval $[t, t + dt)$ entails a flow cost $c(u_t)\, dt$. We assume that c is increasing, thrice differentiable, and convex, with $c(0) = 0$, $\lim_{u \to 0} c'(u) = 0$ and $\lim_{u \to \infty} c'(u) = \infty$.

After a breakthrough occurs, the agent is known to be of high ability, and can expect a flow outside wage of $v > 0$ until the end of the game at T. In line with the interpretation of wages throughout the paper, we think of this wage as the (equilibrium) marginal product of the agent in the activity in which he engages after a breakthrough. Thus the wage v is a (flow) opportunity cost that is incurred as long as no success has been achieved, which must be accounted for, not only in the worker's objective function, but also in the objective of the social planner.[11] Note that this flow opportunity cost lasts only as long as the game does, so that its impact fades away as time runs out.

The outside option of the low-ability agent is normalized to 0. There is no discounting.[12] We discuss the robustness of our results to the introduction of discounting in Section 4.2.

The worker's problem can then be stated as follows: to choose $u : [0, T] \to \mathbb{R}_+$, measurable, to maximize his expected sum of rewards, net of the outside wage v,

$$\mathbb{E}_u \left[\int_0^{T \wedge \tau} [w_t - v \chi_{\omega=1} - c(u_t)]\, dt \right],^{13}$$

where $\mathbb{E}_u$ is the expectation conditional on the worker's strategy u and τ is the time at which a success occurs (a random time that is exponentially distributed, with instantaneous intensity at time t equal to 0 if the state is 0 and to $\lambda + u_t$ otherwise). The indicator of event A is denoted by χ_A. Ignoring discounting is analytically convenient, but there is no discontinuity.

Of course, at time t effort is only exerted and the wage w_t collected, conditional on the event that no success has been achieved. We shall omit saying so explicitly, as those

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[11] In many applications, there is an inherent value in employing a "star," a possible interpretation for v, as suggested by a referee. Another natural case is that in which v equals the flow value of success conditional on $\omega = 1$ and no effort by the agent. In Section 3.1, we establish that if successes worth 1 arrive at rate $\lambda \geq 0$, then $v = \lambda$. In Section 3.2, we show that stronger results obtain for the case $v = \lambda$; other results can be generalized, but not interior effort and equilibrium uniqueness.

[12] At the beginning of the Appendix, we explain how to derive the objective function from its discounted version as discounting vanishes. Values and optimal policies converge pointwise.

[13] Stating the objective as a net payoff ensures that the program is well defined even when $T = \infty$.

histories are the only nontrivial ones. Given his past effort choices, the agent can compute his belief p_t that he is of high ability by using Bayes' rule. It is standard to show that, in this continuous-time environment, Bayes' rule reduces to the ordinary differential equation

$$\dot{p}_t = -p_t(1 - p_t)(\lambda + u_t), \qquad p_0 = p^0. \tag{1}$$

Observing that

$$\mathbb{P}[\tau \geq t] = \frac{\mathbb{P}[\omega = 0 \cap \tau \geq t]}{\mathbb{P}[\omega = 0 | \tau \geq t]} = \frac{\mathbb{P}[\omega = 0]}{\mathbb{P}[\omega = 0 | \tau \geq t]} = \frac{1 - p_0}{1 - p_t}, \tag{2}$$

the problem simplifies to the maximization of

$$\int_0^T \frac{1 - p_0}{1 - p_t}[w_t - c(u_t) - v]\, dt,^{14} \tag{3}$$

given w, over all measurable $u : [0, T] \to \mathbb{R}_+$, subject to (1).

Consider this last maximization. If the wage falls short of the outside option (i.e., if $w_t - v$ is negative), the agent has an incentive to exert high effort to stop incurring this flow deficit. Achieving this is more realistic when the belief p is high, so that this incentive should be strongest early on, when he is still sanguine about his talent. This suggests an effort pattern that is a decreasing function of time, as in Holmström. However, this ignores that, in equilibrium, the wage reflects the agent's expected effort. As a result, we shall show that this intuition is incorrect: equilibrium effort might be increasing, and in general is a single-peaked function of time.

2.2 *The social planner*

Before solving the agent's problem, we start by analyzing the simpler problem faced by a social planner. Recall that the value of a realized breakthrough is normalized to 1. But a breakthrough only arrives with instantaneous probability $p_t(\lambda + u_t)$, as it occurs at rate $\lambda + u_t$ only if $\omega = 1$. Furthermore, the planner internalizes the flow opportunity cost v incurred by the agent as long as no breakthrough is realized. Therefore, the planner maximizes

$$\int_0^T \frac{1 - p_0}{1 - p_t}[p_t(\lambda + u_t) - v - c(u_t)]\, dt, \tag{4}$$

over all measurable $u : [0, T] \to \mathbb{R}_+$, given (1). As for most of the optimization programs considered in this paper, we apply Pontryagin's maximum principle to get a characterization. The proof of the next lemma and of all formal results can be found in the Appendix.

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[14]We have replaced $p_t v$ by the simpler v in the bracketed term inside the integrand. This is because

$$\int_0^T \frac{p_t}{1 - p_t} v\, dt = \int_0^T \frac{v}{1 - p_t}\, dt - vT,$$

and we can ignore the constant vT.

LEMMA 1. *The optimum exists. At any optimum, effort u is monotone in t. It is decreasing if and only if the deadline exceeds some finite length.*

Because the belief p is decreasing over time, note that the marginal product is decreasing whenever effort is decreasing, but the converse need not hold (as we will see in equilibrium). Monotonicity of effort can be roughly understood as follows. There are two reasons why effort can be valuable: because it helps reduce the time over which the waiting cost v is incurred and because it helps avoid reaching the deadline without a breakthrough. The first effect encourages early effort, and the second effect encourages later effort. When the deadline is short (and the final belief is high), terminal effort is high, and the efficient effort level is increasing throughout.

The effort level exerted at the deadline depends on how pessimistic the planner is at that point. By standard arguments (see Appendix A.1), the social planner exerts an effort level that solves

$$p_T = c'(u_T). \tag{5}$$

This states that the expected marginal social gains from effort (i.e., success) should equal the marginal cost. Note that the flow loss v no longer plays a role at that time.

2.3 *Exogenous wages*

We now consider the agent's best response to an entirely exogenous wage path. This allows us to provide an analysis of reputation incentives that is not tied to any particular equilibrium notion. In addition, it will guide our analysis of off-path behavior.

Consider an arbitrary *exogenous* (integrable) wage path $w : [0, T] \to \mathbb{R}_+$. The agent's problem given by (3) differs from the social planner's in two respects: the agent disregards the expected value of a success (in particular, at the deadline), which increases with effort, and he takes into account future wages, which are less likely to be pocketed if more effort is exerted. We start with a technical result stating there is a unique solution (uniqueness is stated in terms of the state variable p, from which it follows that the control u is also essentially unique, e.g., up to a zero measure set of times).

LEMMA 2. *There exists a unique trajectory p that solves the maximization problem (3).*

What determines the instantaneous *level* of effort? Transversality implies that, at the deadline, the agent exerts no effort:

$$c'(u_T) = 0.$$

Relative to the social planner's trade-off in (5), the agent does not take into account the lump-sum value of success. Hence his effort level is nil for any p_T.

It follows from Pontryagin's theorem that the amount of effort exerted at time t solves

$$c'(u_t) = \max\left\{ -\int_t^T (1 - p_t) \frac{p_s}{1 - p_s} [w_s - c(u_s) - v] \, ds, 0 \right\}. \tag{6}$$

The left-hand side is the instantaneous marginal cost of effort. The marginal benefit (right-hand side) can be understood as follows. Conditioning throughout on reaching time t, the expected flow utility over some interval ds at time $s \in (t, T)$ is

$$\mathbb{P}[\tau \geq s](w_s - c(u_s) - v) \, ds.$$

From (2), recall that

$$\mathbb{P}[\tau \geq s] = \frac{1 - p_t}{1 - p_s} = (1 - p_t)\left(1 + \frac{p_s}{1 - p_s}\right);$$

that is, effort at time t affects the probability that time s is reached only through the likelihood ratio $p_s/(1 - p_s)$. From (1), we obtain

$$\frac{p_s}{1 - p_s} = \frac{p_t}{1 - p_t} e^{-\int_t^s (\lambda + u_\tau) \, d\tau},$$

and so a slight increase in u_t decreases the likelihood ratio at time s precisely by $-p_s/(1 - p_s)$. Combining, the marginal impact of u_t on the expected flow utility at time s is given by

$$-(1 - p_t)\frac{p_s}{1 - p_s}[w_s - c(u_s) - v] \, ds,$$

and integrating over s yields the result.

Equation (6) establishes that increasing the wedge between the future rewards from success and failure $(v - w_s)$ encourages high effort, ceteris paribus. Higher wages in the future depress incentives to exert effort today, as they reduce this wedge. In particular, when future wages are very high, the agent may prefer not to exert any effort, in which case the corner solution $u_t = 0$ applies. That being said, throughout this section we restrict attention to wage functions w_t for which the agent's first-order condition holds. In Section 3, we shall establish that the corner solution $u_t = 0$ is never played in equilibrium.

The trade-off captured by (6) illustrates a key feature of career concerns in this model. Because information is coarse (either a success is observed or it is not), the agent can only affect the probability that the relationship terminates. It is then intuitive that incentives for effort depend on future wage prospects (with and without a breakthrough).[15] This is a key difference with Holmström's model, where future wages adjust linearly in output and incentives are therefore independent of the wage level itself. In our model (see Section 3), the level of future compensation does affect incentives to exert effort in equilibrium.

In particular, higher wages throughout reduce the agent's instantaneous effort level pointwise, because the prospect of foregoing higher future wages depresses incentives at all times. However, the relationship between the *timing* of wages and the agent's optimal effort is more subtle. In particular, as we shall see in Section 5, it is not true that

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[15]Note also that, although learning is valuable, the value of information cannot be read off first-order condition (6) directly: the maximum principle is an "envelope theorem," and as such does not explicitly reflect how future behavior adjusts to current information.

pushing wages back, holding the total wage bill constant, necessarily depresses total effort. As is clear from (6), higher wages far in the future have a smaller effect on current-period incentives for two reasons: the relationship is less likely to last until then and, conditional on reaching these times, the agent's effort is less likely to be productive (as the probability of a high type then is very low).

To understand how effort is allocated over time, let us differentiate (6). (See the proof of Proposition 1 for the formal argument.) We obtain

$$
p_t \cdot \underbrace{c(u_{t+dt})}_{\text{cost saved}} + \underbrace{p_t(v - w_t)}_{\text{wage premium}} + \underbrace{c''(u_t)\dot{u}_t}_{\text{cost smoothing}} = \underbrace{p_t(\lambda + u_t)}_{\text{pr. of success at } t} \cdot c'(u_t). \tag{7}
$$

The right-hand side captures the gains from shifting an effort increment du from the time interval $[t, t+dt)$ to $[t+dt, t+2\,dt)$ (*backloading*): the agent saves the marginal cost of this increment $c'(u_t)\,du$ with instantaneous probability $p_t(\lambda + u_t)\,dt$, i.e., the probability with which this additional effort will not have to be carried out. The left-hand side measures the gains from exerting this increment early instead (*frontloading*): the agent increases by $p_t\,du$ the probability that the cost of tomorrow's effort $c(u_{t+dt})\,dt$ is saved. He also increases at that rate the probability of getting the "premium" $(v - w_t)\,dt$ an instant earlier. Last, if effort increases at time t, frontloading improves the workload balance, which is worth $c''(u)\,du\,dt$. This yields the arbitrage equation (7) that is instructive about effort dynamics.[16] The next proposition formalizes this discussion.

PROPOSITION 1. *If w is decreasing, u is a quasi-concave function of time; if w is nondecreasing, u is strictly decreasing.*

Hence, even when wages are monotone, the worker's incentives need not be so. Not surprisingly then, equilibrium wages, as determined in Section 3, will not be either.

2.4 *Comparison with the social planner*

The social planner's arbitrage condition would coincide with the agent's if there were no wages, although the social planner internalizes the value of possible success at future times. This is because the corresponding term in (4) can be "integrated out,"

$$
\int_0^T \frac{1 - p_0}{1 - p_t} p_t(\lambda + u_t)\,dt = -(1 - p_0) \int_0^T \frac{\dot{p}_t}{(1 - p_t)^2}\,dt = (1 - p_0) \ln \frac{1 - p_T}{1 - p_0},
$$

so that it only affects the final belief, and hence the transversality condition. But the agent's and the social planner's transversality conditions do not coincide, even when $w_s = 0$. As mentioned, the agent fails to take into account the value of a success at the last instant. Hence, his incentives at T, and hence his strategy for the entire horizon, differ from the social planner's. The agent works too little, too late.

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[16]Note that all these terms are "second-order" terms. Indeed, to the first order, it does not matter whether effort is slightly higher over $[t, t + dt)$ or $[t + dt, t + 2\,dt)$. Similarly, while doing such a comparison, we can ignore the impact of the change on later payoffs, which is the same under both scenarios.

Proposition 2 formalizes this discussion. Given w, let p^* denote the belief trajectory solving the agent's problem, and let p^{FB} denote the corresponding trajectory for the social planner.

PROPOSITION 2. *Fix $T > 0$ and $w > 0$.*

(i) *The agent's aggregate effort is lower than the planner's, i.e., $p_T^* > p_T^{FB}$. Furthermore, instantaneous effort at any t is lower than the planner's, given the current belief p_t^*.*

(ii) *Suppose the planner's aggregate effort is constrained so that $p_T = p_T^*$. Then the planner's optimal trajectory p lies below the agent's trajectory, i.e., for all $t \in (0, T)$, $p_t^* > p_t$.*

The first part states that both aggregate effort and instantaneous effort are too low, *given* the agent's belief. Nonetheless, as a function of calendar time, effort might be higher for the agent at some dates, because the agent is more optimistic than the social planner at that point. Figure 1 illustrates this phenomenon.

The second part of the proposition states that, even fixing the aggregate effort, this effort is allocated too late relative to the first-best: the prospect of collecting future wages encourages "procrastination."[17]

3. EQUILIBRIUM

This section "closes" the model by considering equilibrium wages. Suppose that the wage is set by a principal (or *market*) without commitment power. This is the type of contracts considered in the literature on career concerns. The market does not observe the agent's past effort, only the lack of success. Noncommitment motivates the assumption that wage equals expected marginal product, i.e.,

$$w_t = \mathbb{E}_t[p_t(\lambda + u_t)],$$

where p_t and u_t are the agent's belief and effort, respectively, at time t, given his private history of past effort (as long as he has had no successes so far), and the expectation reflects the principal's beliefs regarding the agent's history (in case the agent mixes).[18] Given Lemma 2, the agent will not use a chattering control (i.e., a distribution over measurable functions u), but rather a single function. Therefore, we may write

$$w_t = \hat{p}_t(\lambda_t + \hat{u}_t), \tag{8}$$

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[17]Procrastination might be reminiscent of Bonatti and Hörner (2011), but the driving forces have little in common: in Bonatti and Hörner (2011), procrastination is due to the agent's incentives to free-ride on the effort of other agents. Here, there is only one agent.

[18]In discrete time, if $T < \infty$, noncommitment *implies* that wage is equal to marginal product in equilibrium. This follows from a backward induction argument, assuming that the agent and the principal share the same prior. Alternatively, this is the outcome if a sequence of short-run principals (at least two at every instant), whose information is symmetric and no worse than the agent's, compete through wages for the agent's services. We shall follow the literature by directly *assuming* that wage is equal to marginal product.

where $\hat{p}_t$ and $\hat{u}_t$ denote the belief and anticipated effort at time t, as viewed from the market.

In equilibrium, expected effort must coincide with actual effort. Note that if the agent deviates, the market will typically hold incorrect beliefs.

DEFINITION 1. An equilibrium is a measurable function u and a wage path w such that the following statements hold:

(i) Effort u is a best reply to wages w given the agent's private belief p, which he updates according to (1).

(ii) The wage equals the marginal product, i.e., (8) holds for all t.

(iii) Beliefs are correct on the equilibrium path, that is, for every t,

$$\hat{u}_t = u_t,$$

and therefore, also, $\hat{p}_t = p_t$ at all $t \in [0, T]$.

3.1 *The continuation game*

Here we start by briefly discussing the continuation game in which information is complete so that the belief is identically 1. We have so far assumed that the agent receives an exogenous wage v until the end of the game at T. However, there is no particular reason to assume a bounded horizon for the agent's career, before or after a success. For instance, in Section 4.1, we consider up-or-out arrangements where the agent earns a perpetual wage v if he obtains a breakthrough before the end of the probationary period T. The agent's objective (3) is unchanged. Moreover, the unique equilibrium payoff v in the continuation game does not depend on the interpretation for our model. While there might good reasons to treat this continuation payoff after a success as an exogenous parameter (after all, the agent may be assigned to another type of task once he has proved himself), it is easy to endogenize it by solving for the continuation equilibrium.

LEMMA 3. *Both in the finite and in the infinite continuation game when $\omega = 1$ is common knowledge, the unique equilibrium payoff is $v = \lambda$, and the agent exerts no effort.*

The result is clear with a fixed horizon, since the only solution consistent with backward induction specifies no effort throughout. It is more surprising that no effort is possible in equilibrium with an infinite continuation, even without restricting attention to Markov (or indeed public) equilibria. As we show in the Appendix, this reflects both the fact that the market behaves myopically and that, in continuous time, the likelihood ratio of the signal that must be used as a trigger for punishment is insensitive to effort.[19]

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[19]Yet this is not an artifact of continuous time: the same holds in discrete time if the frequency is high enough, as an immediate application of Fudenberg and Levine's (1994) algorithm. Conversely, if it is low enough, multiple equilibria can be constructed. The same holds in Holmström's (1982/1999) model, but this issue is somewhat obfuscated by his implicit focus on Markov equilibria.

This result is reminiscent of Abreu et al. (1991), but the relationship is superficial, as their result relies on the lack of identifiability in the monitoring. Here, all signals reflect the worker's effort only. Closer is the analysis of Faingold and Sannikov (2011) under complete information, where volatility in the Brownian noise prevents any effort to be sustained in their model.

3.2 *The equilibrium with incomplete information*

We now return to the game in which the agent's type is unknown, and use v as a continuation payoff, if only to distinguish it from the arrival rate of a breakthrough. To understand the structure of equilibria, consider the following example, illustrated in Figure 1. Suppose that the principal expects the agent to put in the efficient amount of effort, which decreases over time in this example. Accordingly, the wage paid by the firm decreases as well. The agent's best-reply, then, is quasi-concave: effort first increases, and then decreases (see left panel). The agent puts in little effort at the start, as he has no incentive "to kill the golden goose." Once wages come down, effort becomes more attractive, so that the agent increases his effort, before fading out as pessimism sets in. The market's expectation does not bear out: marginal product is single-peaked. In fact, it would decrease at the beginning if effort was sufficiently flat.

Eventually the agent exerts more effort than the social planner would, because the agent is more optimistic at those times, having worked less in the past (see right panel). Effort is always too low given the actual belief of the agent, but not always given calendar time.

As this example makes clear, effort, let alone wage, is not monotone in general. However, it turns out that the equilibrium structure, illustrated in Figure 2, remains simple enough.

THEOREM 1. *Assume $\lambda > 0$.*
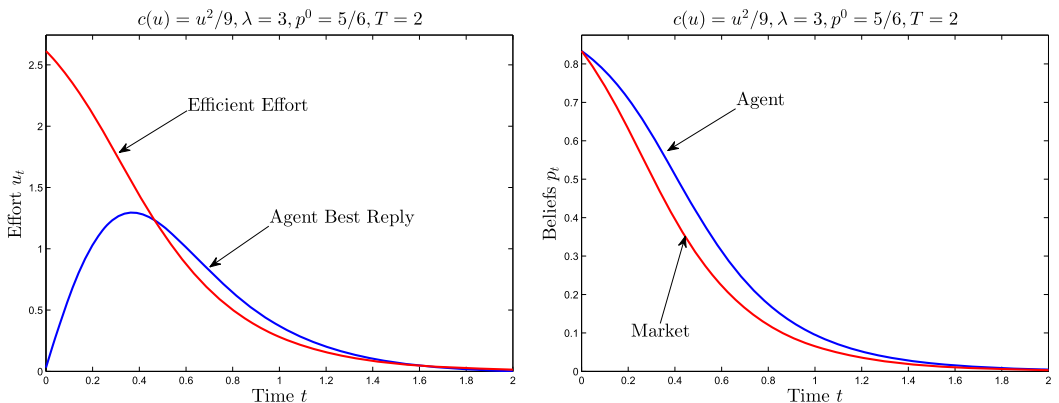
(i) A unique equilibrium exists.

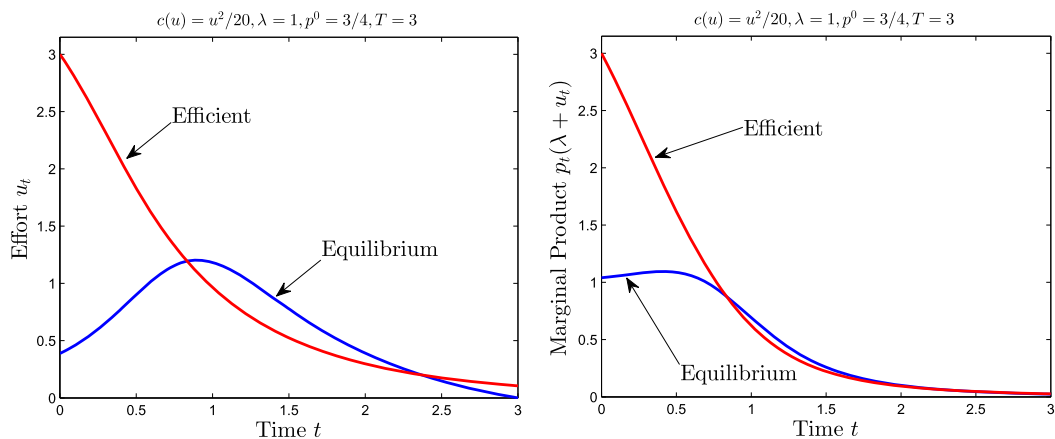FIGURE 1. Agent's best reply and beliefs.

FIGURE 2. Effort and wages with convex costs.

(ii) *Equilibrium (on path) effort u_t is strictly positive for all $t \in [0, T)$, continuous, and single-peaked.*

(iii) *If $3c''(u) + c'''(u) \geq 0$ for all u, the equilibrium wage is single-peaked.*

The proof is given in Appendix A.2. A sketch for uniqueness is as follows: From Lemma 2, the agent's best reply to any wage yields a unique path p. *Given* the value of the belief p_T, we argue there is a unique path of effort and beliefs consistent with the equilibrium restriction on wages. We then look at the time it takes, along the equilibrium path, to drive beliefs from p^0 to p_T and show that it is strictly increasing. Thus, given T, there exists a unique value of p_T that can be reached in equilibrium.[20]

How does the structure depend on parameters? Fixing all other parameters, if initial effort is increasing for some prior p^0, then it is increasing for higher priors. Thus, growing pessimism plays a role in turning increasing into decreasing effort.

Numerical simulations suggest that the payoff is single-peaked (with possibly interior mode) in p^0. This is a recurrent theme in the literature on reputation: uncertainty is the lever for reputational incentives. (Recall however that the payoff is net of the outside option, which is not independent of p^0; otherwise, it is increasing in p^0.)

Finally, note that we have not specified the worker's equilibrium strategy entirely, as we have not described his behavior following his own (unobservable) deviations. The worker's optimal behavior off path is the solution of the optimization problem studied before, for the belief that results from the agent's history, given the wage path. In particular, it can be shown that (a) the agent never reverts to the equilibrium path of beliefs, (b) long deviations can be profitable, but (c) "diverging" deviations are never profitable, i.e., the agent's belief eventually returns closer to the equilibrium path p_t^*.

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[20]The uniqueness result contrasts with the multiplicity found in Dewatripont, Jewitt, and Tirole. Although Holmström does not discuss uniqueness in his model, his model admits multiple equilibria.

3.3 *Discussion*

The key driver behind the equilibrium structure, as described in Theorem 1, is the strategic substitutability between effort at different dates, conditional on lack of success. If more effort is expected tomorrow, wages "tomorrow" will be higher in equilibrium, which depresses incentives, and hence effort "today." Substitutability between effort at different dates is also a feature of the social planner's solution, because higher effort tomorrow makes effort today less useful, but wages create an additional channel.

This substitutability appears to be new to the literature on career concerns. As we have mentioned, in the model of Holmström, the optimal choices of effort today and tomorrow are entirely independent, and because the variance of posterior beliefs is deterministic with Gaussian signals, the optimal choice of effort is deterministic as well. Dewatripont, Jewitt, and Tirole emphasize the complementarity between expected effort and incentives for effort (at the same date): if the agent is expected to work hard, failure to achieve a high signal will be particularly detrimental to tomorrow's reputation, which provides a boost to incentives today. Substitutability between effort today and tomorrow does not appear in their model, because it is primarily focused on two periods, and at least three are required for this effect to appear. With two periods only, there are no reputation-based incentives to exert effort in the second (and final) period anyhow.

Conversely, complementarity between expected and actual effort at a given time is not discernible in our model, because time is continuous. It does, however, appear in discrete time versions of it, and three-period examples can be constructed that illustrate both contemporaneous complementarity and intertemporal substitutability of effort levels.

As a result of this novel effect, effort and wage dynamics display original features. Both in Holmström's and in Dewatripont, Jewitt, and Tirole's models, the wage is a supermartingale. Here instead, effort and wages can be first increasing, then decreasing. These dynamics are not driven by the horizon length.[21] Neither are they driven by the fact that with two types, the variance of the public belief need not be monotone.[22] The same pattern emerges in examples with an infinite horizon, and a prior $p^0 < \frac{1}{2}$ that guarantees that this variance only decreases over time.

As (6) makes clear, the provision of effort is tied to the capital gain that the agent obtains if he breaks through. Viewed as an integral, this capital gain is too low early on, it increases over time, and then declines again, for a completely different reason. Indeed, this wedge depends on two components: the wage gap, and the impact of effort on the (expected) arrival rate of a success. High initial wages depress the former component, and hence kill incentives to exert effort early on. The latter component declines over time, so that eventually effort fades out.[23]

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[21]This is unlike for the social planner, for whom we have seen that effort is nonincreasing with an infinite horizon, while it is monotone (and possibly increasing) with a finite horizon.

[22]Recall that in Holmström's model, this variance decreases (deterministically) over time, which plays an important role in his results.

[23]We have assumed—as is usually done in the literature—that the agent does not know his own skill. The analysis of the game in which the agent is informed is simple, as there is no scope for signaling. An agent who knows that his ability is low has no reason to exert any effort, so we focus on the high-skilled agent. Because of the market's declining belief, the same dynamics arise, and this agent's effort is single-peaked.
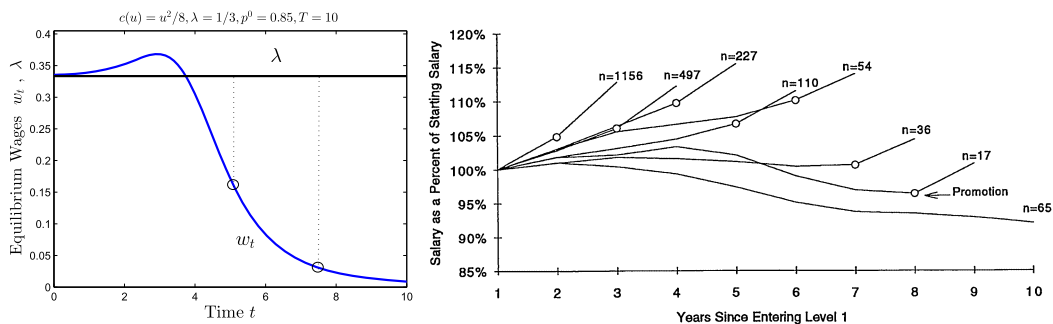
FIGURE 3. Equilibrium wages and Figure IV in Baker et al. (1994b).

Similarly, one might wonder whether the possibility of nonincreasing wages in this model is driven by the fact that effort and wage paths are truly conditional paths, inasmuch as they assume that the agent has not succeeded. Yet it is not hard to provide numerical examples that illustrate the same phenomenon for the unconditional flow payoff (v in case of a past success), although the increasing cumulative probability that a success has occurred by a given time leads to higher payoffs (at least if $w_t < v$) and dampens the downward tendency.

3.4 *Evidence*

Cases abound in which compensation suffers from recurrent failure.[24] In Figure 3 we contrast the equilibrium wage dynamics of our model with the data from Baker et al. (1994a, 1994b) on managerial employees at one firm. These data uncover similar patterns in the wages of unsuccessful employees. In particular, the wages of those employees who receive a promotion late in their career (if ever) are nonmonotone over time.

Our model is too stylized to capture the nuanced properties of the observed wage patterns. For one, in equilibrium, wages need not increase upon a promotion. This is because all career concerns cease after the first breakthrough. A richer model where the agent has incentives to exert some positive effort $\tilde{u}$ after a breakthrough (so that $w = \lambda + \tilde{u}$) would clearly fit the data better. Second, as pointed out by Baker, Gibbs, and Holmström, wages tend to increase even before a promotion, suggesting that information is not as coarse as modeled. Finally, human-capital accumulation and explicit incentive payments tend to increase wages over time. Nevertheless, our model offers a parsimonious mechanism through which nonmonotone patterns emerge upon recurrent failure.

4. ROBUSTNESS

Undoubtedly, our model has stylized features: careers have a fixed length, the low-ability agent cannot achieve a breakthrough, and there is no discounting. In this section, we

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[24]For instance, the income of junior analysts is proportional to the number of job assignments they are being given, so that struggling analysts end up earning lower wages. Similar patterns arise in the art, entertainment, and sports industries.

briefly consider several variations of our baseline model, and we argue that none of these features is critical to our main findings.

4.1 *Probationary periods*

A probationary period is a distinguishing feature of several professional-services industries. To capture the effect of up-or-out arrangements on reputational incentives, we allow for a fixed penalty of $k \geq 0$ for not achieving a success by the deadline T. This might represent diminished future career opportunities to workers with such poor records. Alternatively, consider an interpretation of our model with two phases of the agent's career: one preceding and one following the "clock" T, which lasts until an exogenous retirement time $T' > T$. Thus, the penalty k represents the difference between the wage the agent would have earned had he succeeded, and the wage he will receive until his eventual retirement. Our baseline model with no penalty ($k = 0$) constitutes a special case.

Given a wage function w_t, the agent's problem simplifies to the maximization of

$$\int_0^T \frac{1-p_0}{1-p_t}[w_t - c(u_t) - v]\,dt - \frac{1-p_0}{1-p_T}k. \qquad (9)$$

Considering the maximization problem (9), there appear to be two drivers of the worker's effort. First, if the wage falls short of the outside option (i.e., if $w_t - v$ is negative), he has an incentive to exert high effort to stop incurring this flow deficit. As in the baseline model, this is more realistic when the belief p is high. Second, there is an incentive to succeed so as to avoid paying the penalty. This incentive should be most acute when the deadline looms close, as success becomes unlikely to arrive without effort. Taken together, this suggests an effort pattern that is a convex function of time. However, this ignores that, in equilibrium, the wage reflects the agent's expected effort. Equilibrium analysis shows that the worker's effort pattern is, in fact, the exact opposite of what this first intuition suggests.

In particular, the transversality condition now reads

$$c'(u_T) = p_T \cdot k,$$

as the agent is motivated to avoid the penalty and will exert positive effort until the deadline. However, the arbitrage equation (7) is unaffected by the transversality condition. Hence, equilibrium effort is single-peaked for the same reasons as in Theorem 1. Clearly, for a high enough penalty k, equilibrium effort may be increasing throughout. One might wonder whether the penalty k really hurts the worker. After all, it endows him with some commitment. It is, in fact, possible to construct examples in which the optimal (i.e., payoff-maximizing) termination penalty is strictly positive.

4.2 *Additive technology and positive discounting*

Here, we allow a low-skilled agent to succeed. We do not provide a complete analysis, limiting ourselves to the interaction before the first breakthrough, and considering a
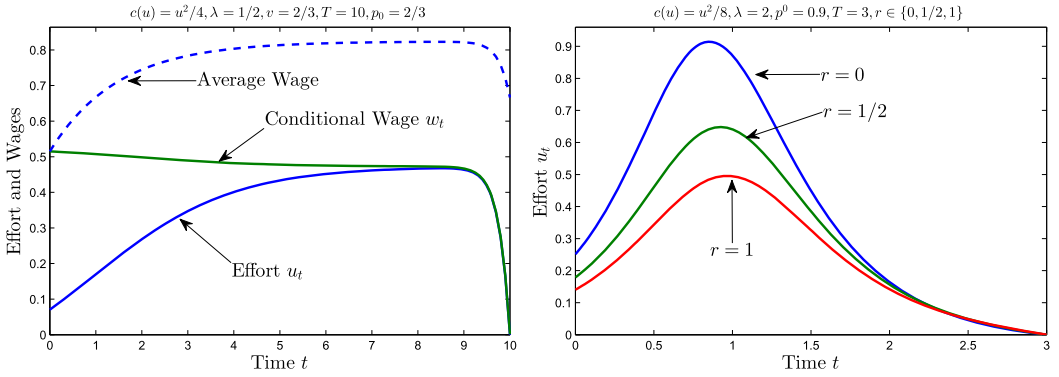
FIGURE 4. Additive and discounted cases.

fixed continuation wage v. We assume the arrival rate of a breakthrough is given by

$$u + \lambda \cdot \omega.$$

This technology eliminates the contemporaneous complementarity between antici-pated and actual effort, and allows us to focus on the intertemporal trade-offs.[25] In particular, future wages drive a wedge in the agent's capital gains from achieving a break-through. As a result, the equilibrium effort level is again single-peaked. Despite a qual-itatively similar equilibrium effort pattern under spot contracts, the additive and multi-plicative technologies have different implications vis-à-vis the effect of wages on effort at different times. We shall return to the role of complementarity between talent and effort when characterizing the effect of long-term contracts in Section 5.

To conclude, the single-peaked patterns of effort and wages are not artifacts of our undiscounted model. In other words, the quasi-concavity property of u and w is strict, and all the conclusions of Theorem 1 hold for sufficiently low discount rates r. Figure 4 shows the equilibrium effort and wage for the additive case and for several values of the discount rate.

4.3 *The linear case*

We now consider the case of linear cost of effort, $c(u) = \alpha \cdot u$, where $\alpha \in (0, 1)$ and $0 \leq u \leq \bar{u} < \infty$. While the linear case is not a special case of our baseline model, it yields similar results and allows for illustrations and sharper characterizations.

In particular, a more precise description of the overall structure of the equilibrium can be given in the case of linear cost. Effort is first nil, then interior, then maximum, and finally nil. Therefore, in line with the results on convex cost, effort is single-peaked. Depending on parameters, any of these time intervals might be empty. Figure 5 provides

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[25]It is immediate to see that the evolution of beliefs is exogenous, leading to the equilibrium wage

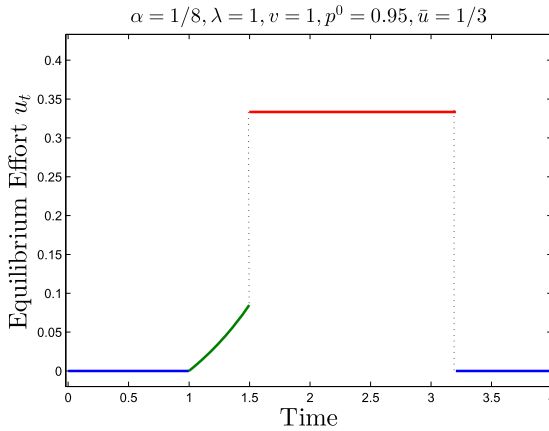$$w_t = \frac{p^0}{1 - p^0 + p^0 e^{\lambda t}} + u_t.$$

$\alpha = 1/8, \lambda = 1, v = 1, p^0 = 0.95, \bar{u} = 1/3$

FIGURE 5. Equilibrium effort with linear cost.

an illustration, and the reader is referred to the Appendix to a formal description (see Proposition 4).

We use the linear-cost specification in the two major extensions of Sections 5 and 6.

5. LONG-TERM CONTRACTS

So far, we have assumed spot contracts, i.e., the wage is equal to the marginal product. This is a reasonable premise in a number of industries in which lack of transparency or volatility in the firm's revenue stream might inhibit commitment by the firm to a particular wage scheme. Alternatively, it is sometimes argued that competition for the agent's services leads to a similar outcome. Our model does not substantiate such a claim: if the principal can commit to a wage path, matters change drastically, even under competition. Throughout this section, we maintain linear costs of effort and a positive termination penalty k, but we no longer constrain the continuation payoff v to be equal to λ.

In particular, if the principal could commit to a breakthrough-contingent wage scheme, the moral hazard problem would be solved: under competition, the principal would offer the agent the value of a breakthrough, 1, whenever a success occurs, and nothing otherwise.

If the principal could commit to a time-contingent wage scheme that involved payments after a breakthrough, the moral hazard would also be mitigated. This can be achieved, for example, through an escrow account, with payments depending on the agent staying with the firm, but not on the realization of output. If promised payments at time t in the case of no breakthrough are also made if a breakthrough has occurred, all disincentives due to wages are eliminated.

Here, we examine a weaker form of commitment. The agent cannot be forced to stay with a principal (he can leave at any time). Once a breakthrough occurs, the agent moves on (e.g., to a different industry or position), and the firm is unable to retain him in this event. The principal can commit to a wage path that is conditional on the agent working

for her firm. Thus, wages can only be paid in the continued absence of a breakthrough. Until a breakthrough occurs, other firms, who are symmetrically informed (they observe the wages paid by all past employers), compete by offering wage paths. The same deadline applies to all wage paths, i.e., the tenure clock is not reset. For instance, the deadline could represent the agent's retirement age, so that switching firms does not affect the horizon.

We write the principal's problem as that of maximizing the agent's *welfare* subject to constraints. We take it as an assumption that competition among principals leads to the most preferred outcome of the agent, subject to the firm breaking even.

Formally, we solve the following optimization problem $\mathcal{P}$.[26] The principal chooses $u : [0, T] \to [0, \bar{u}]$ and $w : [0, T] \to \mathbb{R}_+$, integrable, to maximize $W(0, p^0)$, where, for any $t \in [0, T]$,

$$W(t, p_t) := \max_{w,u} \int_t^T \frac{1 - p_t}{1 - p_s}(w_s - v - \alpha u_s)\, \mathrm{d}s - k\frac{1 - p_t}{1 - p_T},$$

such that, given w, the agent's effort is optimal,

$$u = \arg\max_u \int_t^T \frac{1 - p_t}{1 - p_s}(w_s - v - \alpha u_s)\, \mathrm{d}s - k\frac{1 - p_t}{1 - p_T},$$

and the principal offers as much to the agent at later times than the competition could offer at best, given the equilibrium belief,

$$\forall \tau \geq t: \quad \int_\tau^T \frac{1 - p_\tau}{1 - p_s}(w_s - v - \alpha u_s)\, \mathrm{d}s - k\frac{1 - p_\tau}{1 - p_T} \geq W(\tau, p_\tau); \tag{10}$$

finally, the firm's profit must be nonnegative,

$$0 \leq \int_t^T \frac{1 - p_t}{1 - p_s}(p_s(\lambda + u_s) - w_s)\, \mathrm{d}s.$$

Note that competing principals are subject to the same constraints as the principal under consideration: because the agent might ultimately leave them as well, they can offer no better than $W(\tau, p_\tau)$ at time τ, given belief p_τ. This leads to an "infinite regress" of constraints, with the value function appearing in the constraints themselves. To be clear, $W(\tau, p_\tau)$ is not the continuation payoff that results from the optimization problem, but the value of the optimization problem if it started at time τ.[27] Because of the constraints, the solution is not time consistent, and dynamic programming is of little

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[26]We are not claiming that this optimization problem yields the equilibrium of a formal game, in which the agent could deviate in his effort scheme, leave the firm, and competing firms would have to form beliefs about the agent's past effort choices, etc. Given the well known modeling difficulties that continuous time raises, we view this merely as a convenient shortcut. Among the assumptions that it encapsulates, note that there is no updating based on an off-path action (e.g., switching principals) by the agent.

[27]Harris and Holmström (1982) impose a similar condition in a model of wage dynamics under incomplete information. However, because their model abstracts from moral hazard, constraint (10) reduces to a nonpositive continuation profit condition.

help. Fortunately, this problem can be solved, as shown in Appendix A.3, at least as long as $\bar{u}$ and v are large enough. Formally, we assume that

$$\bar{u} \geq \left(\frac{v}{\alpha\lambda} - 1 \right) v - \lambda \quad \text{and} \quad v \geq \lambda(1 + k). \qquad (11)$$

Before describing its solution, let us provide some intuition. Future payments do depress incentives, but their timing is irrelevant if $\omega = 0$, as effort makes no difference in that event anyhow. Hence, the impact on incentives can be evaluated *conditional* on $\omega = 1$. However, in that event, remote payments are unlikely to be collected anyhow, as a success is likely to have occurred by then. This dampens the detrimental impact of such payments on incentives. The benefit from postponing future payments as much as possible comes from this distinction: the rate at which they are discounted from the principal's point of view is not the same as the one used by the agent when determining his optimal effort, because of this conditioning.

More precisely, recall the first-order condition (6) that determines the agent's effort. Clearly, the lower the future total wage bill, the stronger the agent's incentives to exert effort, which is inefficiently low in general. Now consider two times $t < t'$: to provide strong incentives at the later time t', it is best to frontload any promised payment to times before t', as such payments will no longer matter at that time. Ideally, the principal would pay what he owes up front, as a "signing bonus." Of course, this violates the constraint (10), as an agent left with no future payments would leave right after cashing in the signing bonus.

From the perspective of incentives at the earlier time t, however, backloading promised payments can be beneficial. To see this formally, note that the coefficient of the wage w_s, $s > t$, in (6) is the likelihood ratio $p_s/(1 - p_s)$, as explained before. Up to the factor $(1 - p_t)$, we obtain

$$(1 - p_t) \frac{p_s}{1 - p_s} = \mathbb{P}[\omega = 1 | \tau \geq s]\mathbb{P}[\omega = 1] = \mathbb{P}[\omega = 1 \cap \tau \geq s];$$

that is, effort at time t is affected by wage at time $s > t$ inasmuch as time s is reached and the state is 1: otherwise effort plays no role anyhow.

In terms of the firm's profit, the coefficient placed on the wage at time s (see (10)) is

$$\mathbb{P}[\tau \geq s],$$

i.e., the probability that this wage is paid (or collected). Thus, if the firm wishes to back-load payments and break even, the nominal value of those payments must be increased, as a breakthrough might occur until then, which would void them; but the probability that these payments must be made decreases in the same proportion. Thus, what matters for incentives is not the probability that time s is reached, but the fact that reaching those later times is indicative of state 0.

In particular, because players grow more pessimistic over time, the coefficient on future wages in the agent's first-order condition (6) decreases faster than in the firm's

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[28]We do not know whether these assumptions are necessary for the result.

profits. Hence, later payments depress current incentives less than earlier payments: backloading payments is good for incentives at time t.

To sum up: from the perspective of time t, backloading payments is useful; from the point of view of $t' > t$, it is detrimental, but frontloading is constrained by (10). Note that, as $T \to \infty$, the planner's solution tends to the agent's best response to a wage of $w = 0$. Hence, the firm can approach first best by promising a one-time payment arbitrarily far in the future (and wages equal to marginal product thereafter). This would be almost as if $w = 0$ for the agent's incentives, and would induce efficient effort. The lump-sum payment would then be essentially equal to $p^0/(1 - p^0)$.

Note finally that, given the focus on linear cost, there is no benefit in giving the agent any "slack" in his incentive constraint at time t; otherwise, by frontloading slightly future payments, incentives at time t would not be affected, while incentives at later times would be enhanced. Hence, the following result should come as no surprise.

THEOREM 2. *The following statement is a solution to the optimization problem $\mathcal{P}$, for some $t \in [0, T]$. Maximum effort is exerted up to time t, and zero effort is exerted afterward. The wage is equal to $v - \alpha\lambda$ up to time t, so that the agent is indifferent between all levels of effort up to then, and it is 0 for all times $s \in (t, T)$; a lump sum is paid at time T.*[29]

The proof is given in Appendix A.3, and it involves several steps: we first conjecture a solution in which effort is first full (and the agent is indifferent), then nil; we relax the objective in program $\mathcal{P}$ to maximization of aggregate effort, and relax constraint (10) to a nonpositive continuation profit constraint; we verify that our conjecture solves the relaxed program, and finally that it also solves the original program. In the last step we show that (a) given the shape of our solution, maximizing total effort implies maximizing the agent's payoff, and (b) the competition constraint (10) is slack at all times $t > 0$.

Hence, under one-sided commitment, high effort might be exerted throughout. This happens if T is short and $k > 0$. When $\bar{u}$ is high enough (precisely, when (11) holds), the agent produces revenue that exceeds the flow wage collected as time proceeds: the liability recorded by the principal grows over time, shielding it from the threat of competition. This liability will eventually be settled via a lump-sum payment at time T that can be interpreted as severance pay. If the horizon is longer or $k = 0$, the lump sum wipes out incentives close to the deadline, and effort is zero in a terminal phase. Thus, a phase with no effort exists if and only if the deadline is long enough. The two cases are illustrated in Figure 6.

As mentioned, all these results are proved for the case of linear cost. It is all the more striking that the result obtains in this "linear" environment. Indeed, rigidity and severance pay are usually attributed to risk aversion by the agent (see Azariadis 1975, Holmström 1983, Harris and Holmström 1982, Thomas and Worrall 1988).[30] Our model

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[29]The wage path that solves the problem is not unique in general.

[30]In Thomas and Worrall (1988), there is neither incomplete information nor moral hazard. Instead, the spot market wage is treated as an exogenous independent and identically distributed (i.i.d.) process (though Thomas and Worrall do not require risk aversion; with market power and commitment, backloading improves incentives). Holmström (1983) is a two-period example. As mentioned above,
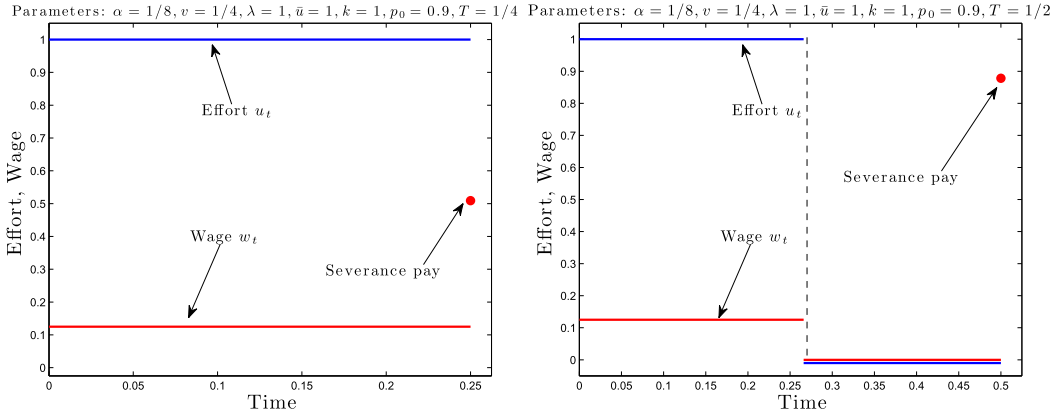
FIGURE 6. Wages and effort under commitment for two horizon lengths.

with commitment and competition provides an alternative rationale. To sum up, with a long horizon, commitment alleviates the unavoidable delay in effort provision that comes with compensating a worker for persistent failure by backloading compensation.

6. OBSERVABLE EFFORT

The inability to contract on effort might be attributable to the subjectivity in its measurement rather than to the impossibility of monitoring it. To understand the role of observability of effort, we assume here that effort is observed. Because effort is monitored, the firm and agent beliefs coincide on- and off-path. The flow wage is given by

$$w_t = p_t(\lambda + \hat{u}_t),$$

where p_t is the belief and $\hat{u}_t$ is expected effort. We assume linear cost. The agent maximizes

$$V(p_0, 0) := \int_0^T \frac{1 - p_0}{1 - p_t}[p_t(\lambda + \hat{u}_t) - \alpha u_t - v]\,dt - k\frac{1 - p_0}{1 - p_T}.$$

In contrast to (3), the firm's revenue is no longer a function of time only, as effort affects future beliefs, thus wages. Hence, effort is a function of both t and p. We focus on equilibria in Markov strategies

$$u : [0, 1] \times [0, T] \to [0, \bar{u}],$$

such that $u(p, t)$ is upper semicontinuous and the value function $V(p, t)$ is piecewise differentiable.[31]

Harris and Holmström (1982) has incomplete information about the worker's type, but no moral hazard. Hence, the logic of rigidity and backloading that appears in these models is very different from ours, driven by optimal incentive provision under incomplete information.

[31] That is, there exists a partition of $[0, 1] \times [0, T]$ into closed S_i with nonempty interior, such that V is differentiable on the interior of S_i, and the intersection of any S_i, S_j is either empty or a smooth one-dimensional manifold.

LEMMA 4. *Fix a Markov equilibrium. Suppose the agent exerts strictly positive effort at time t (given p). Then the equilibrium specifies strictly positive effort at all times $t' \in (t, T]$.*

Thus, in any equilibrium involving extremal effort levels only, there are at most two phases: the worker exerts no effort and then full effort. This is the opposite of the socially optimal policy, which frontloads effort (see Lemma 1). The agent can only be trusted to put in effort if his "back is to the wall," so that effort remains optimal at any later time, no matter what he does; if the market paid for effort, yet the agent was expected to let up later on, then he would gain by deviating to no effort, pocketing the high wage in the process; because such a deviation makes everyone more optimistic, it would only increase his incentives to exert effort (and so his wage) at later times.

This does not imply that the equilibrium is unique, as the next theorem establishes.

THEOREM 3. *Given $T > 0$, there exists continuous, distinct, nonincreasing $\underline{p}, \bar{p} : [0, T] \rightarrow [0, 1]$ (with $\underline{p}_t \leq \bar{p}_t$ with equality if $t = T$) such that the following statements hold:*

(i) *All Markov equilibria involve maximum effort if $p_t > \bar{p}_t$.*

(ii) *All Markov equilibria involve no effort if $p_t \leq \underline{p}_t$.*

(iii) *These bounds are tight: there exists a Markov equilibrium in which effort is either 0 or $\bar{u}$ if and only if p is below or above $\underline{p}$ (resp., $\bar{p}$).*

The proof of Theorem 3 provides a description of these belief boundaries. These boundaries might be as high as 1, in which case effort is never exerted at that time: indeed, there is t^* (independent of T) such that effort is zero at all times $t < T - t^*$ (if $T > t^*$). The threshold $\underline{p}$ is decreasing in the cost α, and increasing in v and k. Considering the equilibrium with maximum effort—the agent works more—the more desirable success is.[32] Figure 7 illustrates these dynamics. In any extremal equilibrium, wages decrease over time, except for an upward jump when effort jumps up to $\bar{u}$. In the interior-effort equilibrium described in the proof (in which effort is continuous throughout), wages decrease throughout. Further comparative statics are given in the Appendix.

Equilibrium multiplicity has a simple explanation. Because the firm expects effort only if the belief is high and the deadline is close, such states (belief and times) are desirable for the agent, as the higher wage more than outweighs the effort cost. Yet low effort is the best way to reach those states, as effort depresses beliefs: hence, if the firm expects the agent to shirk until a high boundary is reached (in (p, t) space), the agent has strong incentives to shirk to reach it; if the firm expects shirking until an even higher boundary, this would only reinforce this incentive.[33]

[32]While the equilibria achieving the result in part (iii) of Theorem 3 (say, $\underline{\sigma}$ and $\bar{\sigma}$) provide upper and lower bounds on equilibrium effort (in the sense of parts (i) and (ii)), these equilibria are not the only ones. Other equilibria exist that involve only extremal effort, with a switching boundary in between $\underline{p}$ and $\bar{p}$; there are also equilibria in which interior effort levels are exerted at some states.

[33]Non-Markov equilibria exist. Defining them in our environment is problematic, but it is clear that threatening the agent with reversion to the Markov equilibrium $\bar{\sigma}$ provides incentives for effort extending beyond the high-effort region defined by $\underline{\sigma}$—in fact, beyond the high-effort region in the unobservable case. The planner's solution remains out of reach, as punishments are restricted to beliefs below $\underline{p}$.
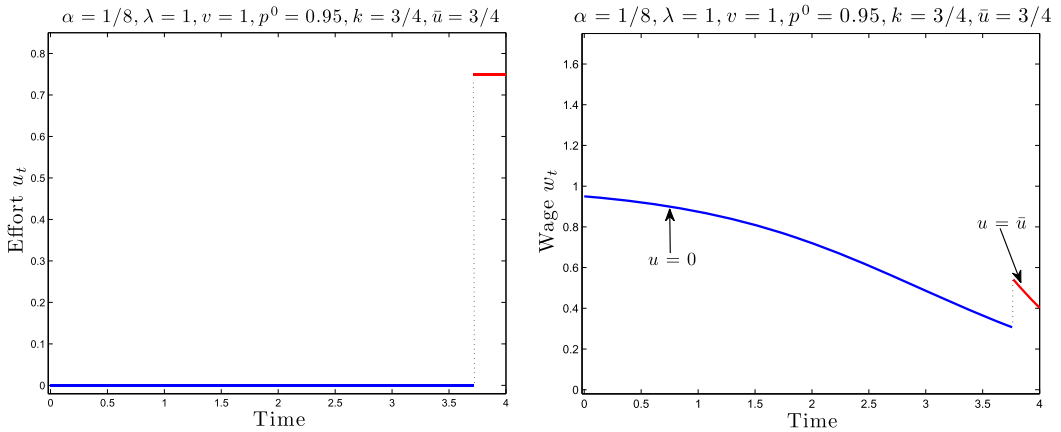
FIGURE 7. Effort and wages in the observable case.

Let us turn to a comparison with the case of nonobservable effort, as described in Section 3. Along the equilibrium path, the dynamics of effort look very different when one compares the social planner's solution to the agent's optimum under either observability assumption. Yet effort can be ranked across those cases. To do so, the key is to describe effort in terms of the state (p, t), i.e., the public belief and calendar time. In all equilibria, effort is lower under observability. In the following theorem, *maximum effort* region refers to the range of parameters (p, t) over which maximum effort is prescribed by some (Markov) equilibrium (as defined in Theorem 3), and similarly in the case in which effort is not observable (recall that the equilibrium is essentially unique then).

PROPOSITION 3. *The maximum effort region for the observable case is contained in the maximum effort region(s) for the nonobservable case.*

This confirms that observability depresses incentives: the highest-effort equilibrium with observability entails less effort than without. In addition, whatever effort is exerted in the observable case, it is exerted later than in the unobservable case. (Formally, the equilibrium belief paths p_t in the two cases cannot cross.) Because effort costs are linear, it follows that social surplus is lower with observability than without. In turn, recall from Proposition 2 that the aggregate equilibrium effort is lower than the social planner's.[34]

7. CONCLUDING REMARKS

Endogenous deadlines

Suppose the worker decides when to quit the profession and assume he has no commitment power. Upon quitting, the agent suffers a fixed penalty $k > 0$. The principal anticipates the quitting decision, and takes this into account while determining the agent's

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[34]The same deleterious effect of observability on incentives is present in Holmström (1982/1999). The comparison of aggregate effort in equilibrium and in the planner's solution extends to the case of linear cost. See the working paper (Bonatti and Hörner 2013) for details.

equilibrium effort and, therefore, the wage he should be paid. For simplicity, we adopt passive beliefs. That is, if the agent is supposed to drop out at some time but fails to do so, the principal does not revise his belief regarding past effort choices, ascribing the failure to quit to a mistake (i.e., he expects the agent to quit at the next instant).

It is easy to show that endogenous deadlines do not affect the pattern of effort and wage: the wage is decreasing at the end (but not necessarily at the beginning); effort and wages are single-peaked. Furthermore, the belief at the deadline is too high relative to the social planner's at the first-best deadline.[35] How about if the worker could commit to the deadline (but still not to effort levels)? The optimal deadline with commitment is set so as to increase aggregate effort and therefore wages. Numerical simulations suggest this requires decreasing the deadline, so as to provide stronger incentives through a larger penalty, i.e., to make high effort levels credible.

Multiple breakthroughs

Suppose that one success does not resolve all uncertainty. Specifically, there are three states and two consecutive projects. The first project can be completed if and only if the agent is *not bad* (i.e., if he is *good* or *great*). If the first project is completed, an observable event, the agent tackles the second, which can be completed only if the agent is *great*. Such an extension can be solved by backward induction. Once the first project is completed, the continuation game reduces to the game of Section 3. The value function of this problem then serves as a continuation payoff to the first stage. While this value function cannot be solved in closed form, it is easy to derive the solution numerically. The same pattern as in our baseline model emerges: effort is single-peaked, and, as a result, wages can be first decreasing and then single-peaked.

Furthermore, effort at the start of the second project is also single-peaked as a function of the time at which this project is started (the later it is started, the more pessimistic the agent at that stage, though his belief has obviously jumped up given the success).

Learning by doing

Memorylessness is a convenient but stark property of the exponential distribution. It implies that past effort plays no role in the probability of instantaneous breakthrough. In many applications, agents learn from the past not only about their skill levels, but about the best way to achieve a breakthrough. While a systematic analysis of learning by doing is beyond the scope of this paper, we can gain some intuition from numerical simulations. We assume the evolution of human capital is given by

$$\dot{z}_t = u_t - \delta z_t, \qquad z_0 = 0,$$

while its productivity is

$$h_t = u_t + \rho z_t^{\phi}.$$

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[35]Having the worker quit when it is best for him (without commitment to the deadline) reinforces our comparison between observable and nonobservable effort. See the working paper (Bonatti and Hörner 2013) for details.

Not surprisingly, simulations suggest that the main new feature is a spike of effort at the beginning, whose purpose is to build human capital. This spike might lead to decreasing initial effort, before it becomes single-peaked, though this need not be the case. Beyond this new twist, features from the baseline model appear quite robust.

<center>APPENDIX</center>

Throughout this appendix, we shall use the log-likelihood ratio

$$x_t := \ln \frac{1 - p_t}{p_t}$$

of state $\omega = 0$ vs. $\omega = 1$. We set $x^0 := \ln(1 - p^0)/p^0$. Note that x increases over time and, given u, follows the ordinary differential equation (O.D.E.)

$$\dot{x}_t = \lambda + u_t,$$

with $x_0 = x^0$. We shall also refer to x_t as the *belief*, hoping that this will create no confusion.

We start by explaining how the objective function can be derived as the limit of a discounted version of our problem. Suppose that W_t is the value of a success at time t, corresponding to earning the flow wage v until the end of the game at T. Given the discount rate r, we have

$$W_t = \frac{v}{r}(1 - e^{-r(T-t)}),$$

and hence the agent's ex ante payoff V_0 is given by

$$V_0 = \int_0^T e^{-rt} \frac{1 + e^{-x_t}}{1 + e^{-x_0}} \left(\frac{\dot{x}_t}{1 + e^{x_t}} W_t + w_t - c(u_t) \right) dt.$$

Integrating the first term by parts, we obtain

$$V_0 - \frac{1}{1 + e^{x_0}} \frac{v}{r}(1 - e^{-rT}) = \int_0^T e^{-rt} \frac{1 + e^{-x_t}}{1 + e^{-x_0}} \left(w_t - c(u_t) - \frac{v}{1 + e^{x_t}} \right) dt,$$

so that as $r \to 0$, we obtain

$$(1 + e^{-x_0})V_0 - e^{-x_0}vT = \int_0^T (1 + e^{-x_t}) \left(w_t - c(u_t) - \frac{v}{1 + e^{x_t}} \right) dt.$$

Similarly, one can show the social planner's payoff is given by

$$(1 + e^{-x_0})V_0 - e^{-x_0}(vT + 1) = -\int_0^T (1 + e^{-x_t}) \left(c(u_t) + \frac{v}{1 + e^{x_t}} \right) dt - e^{-x_T}.$$

A.1 *Proofs for Section 2*

PROOF OF LEMMA 1. Existence and uniqueness of a solution follow as a special case of Lemma 2, when $w = 0$ identically (the transversality condition must be adjusted). To see that the social planner's problem is equivalent to this, note that the "revenue" term of the social planner's objective satisfies

$$\int_0^T (1 + e^{-x_t}) \frac{\lambda + u_t}{1 + e^{x_t}} \, dt = \int_0^T \dot{x}_t e^{-x_t} \, dt = e^{-x^0} - e^{-x_T},$$

and so this revenue only affects the necessary conditions through the transversality condition at T.

The social planner maximizes

$$\int_0^T (1 + e^{-x_t}) \left(\frac{\lambda + u_t}{1 + e^{x_t}} - c(u_t) - v \right) dt, \quad \text{s.t.} \quad \dot{x}_t = \lambda + u_t.$$

We note that the maximization problem cannot be abnormal, since there is no restriction on the terminal value of the state variable. See Seierstad and Sydsæter (1987, Chapter 2, Note 5). The same holds for all optimization problems considered below. From now on, it will be understood that statements about derivatives only hold almost everywhere.

Let γ_t be the costate variable. The Hamiltonian for this problem is

$$H(x, u, \gamma, t) = e^{-x_t}(\lambda + u_t) - (1 + e^{-x_t})(v + c(u_t)) + \gamma_t(\lambda + u_t).$$

Applying Pontryagin's theorem (and replacing the revenue term by its expression in terms of x_t and x^0, as explained above) yields as necessary conditions

$$\dot{\gamma}_t = -e^{-x}(c(u) + v), \qquad \gamma_t = (1 + e^{-x_t})c'(u_t).$$

Differentiate the second expression with respect to time, and use the first one to obtain

$$\dot{u} = \frac{(\lambda + u)c'(u) - c(u) - v}{c''(u)(1 + e^x)}, \tag{12}$$

in addition to $\dot{x} = \lambda + u$ (time subscripts will often be dropped for brevity). Let

$$\phi(u) := (\lambda + u)c'(u) - c(u) - v.$$

Note that $\phi(0) = -v < 0$, and $\phi'(u) = (\lambda + u)c''(u) > 0$, and so ϕ is strictly increasing and convex. Let $u^* \geq 0$ be the unique solution to

$$\phi(u^*) = 0,$$

and so ϕ is negative on $[0, u^*]$ and positive on $[u^*, \infty)$. Accordingly, $u < u^* \Rightarrow \dot{u} < 0$, $u = u^* \Rightarrow \dot{u} = 0$, and $u > u^* \Rightarrow \dot{u} > 0$. Given the transversality condition

$$(1 + e^{x_T})c'(u_T) = 1,$$

we define

$$x_T(x^0) = \frac{1}{\lambda + u^*}\left[\ln\left(\frac{1}{c'(u^*)} - 1\right) - x^0\right],$$

and so effort is decreasing throughout if $x_T > x_T(x^0)$, increasing throughout if $x_T < x_T(x^0)$, and equal to u^* throughout otherwise.

Finally, we establish that the belief x_T at the deadline is increasing in T. Note that the necessary conditions define a vector field $(\dot{u}, \dot{x})$, with trajectories that only depend on the time left before the deadline and the current belief. Because trajectories do not cross (in the plane $(-\tau, x)$, where τ is time to go and x is the belief), and belief x can only increase with time, if we compare two trajectories starting at the same level x^0, the one that involves a longer deadline must necessarily involve as high a terminal belief x as the other (as the deadline expires).   □

PROOF OF LEMMA 2. We address the two claims in turn.

Existence. Note that the state equation is linear in the control u, while the objective's integrand is concave in u. Hence the set $N(x, U, t)$ is convex (see Seierstad and Sydsæter 1987, Chapter 2, Theorem 8). Therefore, the Filippov–Cesari existence theorem applies (see Cesari 1983).

Uniqueness. We can write the objective, up to constant terms, as

$$\int_0^T (1 + e^{-x_t})(w_t - v - c(u_t))\,dt,$$

or, using the likelihood ratio $l_t := p_t/(1 - p_t) > 0$,

$$J(l) := \int_0^T (1 + l_t)(w_t - v - c(u_t))\,dt.$$

Suppose that there are two distinct optimal trajectories l_1 and l_2, with associated controls u_1 and u_2. Assume without loss of generality that

$$l_{1,t} < l_{2,t} \quad \text{for all } t \in (0, T].$$

We analyze the modified objective function

$$\tilde{J}(l) := \int_0^T (1 + l_t)(w_t - v - \tilde{c}_t(u_t))\,dt,$$

in which we replace the cost function $c(u_t)$ with

$$\tilde{c}_t(u) := \begin{cases} \alpha_t u & \text{if } u \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}] \\ c(u) & \text{if } u \notin [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}], \end{cases}$$

where

$$\alpha_t := \frac{\max\{c(u_{1,t}), c(u_{2,t})\} - \min\{c(u_{1,t}), c(u_{2,t})\}}{\max\{u_{1,t}, u_{2,t}\} - \min\{u_{1,t}, u_{2,t}\}}.$$

(If $u_{1,t} = u_{2,t} =: u_t$ for some t, set α_t equal to $c'(u_t)$.) Because $\tilde{c}_t(u) \geq c(u)$ for all t, u, the two optimal trajectories l_1 and l_2, with associated controls u_1 and u_2, are optimal for the modified objective $\tilde{J}(l)$ as well. Furthermore, $\tilde{J}(l_1) = J(l_1)$ and $\tilde{J}(l_2) = J(l_2)$.

We construct a feasible path l_t and its associated control $u_t \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ that attains a higher payoff $\tilde{J}(l)$ and therefore a strictly higher payoff $J(l)$. Suppose $u_t \in [u_{1,t}, u_{2,t}]$ for all t. Letting $g_t := w_t - v + \alpha\lambda - \dot{\alpha}_t$, we rewrite the modified objective as

$$\int_0^T l_t g_t \, dt - \int_0^T \dot{\alpha}_t \ln l_t \, dt + \alpha_T l_T + \alpha_T \ln l_T + \text{constant}.$$

We now consider a continuous function $\varepsilon_t \geq 0$ and two associated variations on the paths l_1 and l_2:

$$l'_{1,t} := (1 - \varepsilon_t) l_{1,t} + \varepsilon_t l_{2,t}$$
$$l'_{2,t} := (1 - \varepsilon_t) l_{2,t} + \varepsilon_t l_{1,t}.$$

Because l_1 and l_2 are optimal, for any ε_t it must be the case that

$$\tilde{J}(l_1) - \tilde{J}(l'_1) \geq 0$$
$$\tilde{J}(l_2) - \tilde{J}(l'_2) \geq 0.$$

We can write these payoff differences as

$$\int_0^T \varepsilon_t(l_{1,t} - l_{2,t}) g_t \, dt + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{2,t} - l_{1,t}}{l_{1,t}} \, dt$$

$$+ \alpha_T \varepsilon_T (l_{1,T} - l_{2,T}) - \alpha_T \varepsilon_T \frac{l_{2,T} - l_{1,T}}{l_{1,T}} + o(\|\varepsilon\|) \geq 0$$

$$\int_0^T \varepsilon_t(l_{2,t} - l_{1,t}) g_t \, dt + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{1,t} - l_{2,t}}{l_{2,t}} \, dt$$

$$+ \alpha_T \varepsilon_T (l_{2,T} - l_{1,T}) - \alpha_T \varepsilon_T \frac{l_{1,T} - l_{2,T}}{l_{2,T}} + o(\|\varepsilon\|) \geq 0.$$

Letting

$$\rho_t := l_{1,t}/l_{2,t} < 1 \quad \text{for all } t > 0,$$

we can sum the previous two conditions (up to the second-order term). Finally, integrating by parts, we obtain the condition,

$$\int_0^T \left[\frac{\dot{\varepsilon}_t}{\varepsilon_t} \left(2 - \rho_t - \frac{1}{\rho_t} \right) + \dot{\rho}_t \frac{1 - \rho_t^2}{\rho_t^2} \right] \alpha_t \varepsilon_t \, dt \geq 0,$$

which must hold for all ε_t. Using the fact that $\dot{\rho} = \rho(u_2 - u_1)$ we have

$$\int_0^T \left[-\frac{\dot{\varepsilon}_t}{\varepsilon_t}(1 - \rho_t) + (u_{2,t} - u_{1,t})(1 + \rho_t) \right] \alpha_t \varepsilon_t \frac{1 - \rho_t}{\rho_t} \, dt \geq 0. \tag{13}$$

We now identify bounds on the function ε_t so that both variations l'_1 and l'_2 are feasible and their associated controls lie in $[\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ for all t. Consider the identities

$$\dot{l}'_1 = -l'_{1,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t(l_{2,t} - l_{1,t}) - \lambda l'_{1,t} - (1 - \varepsilon_t)u_{1,t}l_{1,t} - \varepsilon_t u_{2,t}l_{2,t}$$

$$\dot{l}'_2 = -l'_{2,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t(l_{1,t} - l_{2,t}) - \lambda l'_{2,t} - \varepsilon_t u_{1,t}l_{1,t} - (1 - \varepsilon_t)u_{2,t}l_{2,t}.$$

We therefore have the expressions for the function $\dot{\varepsilon}/\varepsilon$ associated with each variation:

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t} - u_t)\frac{1-\varepsilon_t}{\varepsilon_t}l_{1,t} + l_{2,t}(u_{2,t} - u_t)}{l_{2,t} - l_{1,t}} \tag{14}$$

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t} - u_t)l_{1,t} + \frac{1-\varepsilon_t}{\varepsilon_t}l_{2,t}(u_{2,t} - u_t)}{l_{1,t} - l_{2,t}}. \tag{15}$$

In particular, whenever $u_{2,t} > u_{1,t}$, the condition

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{1 - \varepsilon_t}{\varepsilon_t}\frac{l_{2,t}(u_{2,t} - u_{1,t})}{l_{2,t} - l_{1,t}}, \frac{l_{1,t}(u_{2,t} - u_{1,t})}{l_{2,t} - l_{1,t}}\right]$$

ensures the existence of two effort levels $u_t \in [u_{1,t}, u_{2,t}]$ that satisfy conditions (14) and (15) above. Similarly, whenever $u_{1,t} > u_{2,t}$, we have the bound

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{l_{1,t}(u_{1,t} - u_{2,t})}{l_{2,t} - l_{1,t}}, \frac{1 - \varepsilon_t}{\varepsilon_t}\frac{l_{2,t}(u_{2,t} - u_{1,t})}{l_{2,t} - l_{1,t}}\right].$$

Note that $\dot{\varepsilon}_t/\varepsilon_t = 0$ is always contained in both intervals.

Finally, because $\rho_0 = 1$ and $\rho_t < 1$ for all $t > 0$, we must have $u_{1,t} > u_{2,t}$ for $t \in [0, t^*)$ with $t^* > 0$. Therefore, we can construct a path ε_t that satisfies

$$(u_{2,t} - u_{1,t})\frac{1 + \rho_t}{1 - \rho_t} < \frac{\dot{\varepsilon}_t}{\varepsilon_t} < 0 \quad \forall t \in [0, t^*),$$

with $\varepsilon_0 > 0$ and $\varepsilon_t \equiv 0$ for all $t \geq t^*$. Substituting into condition (13) immediately yields a contradiction. $\qquad\square$

PROOF OF PROPOSITION 1. Applying Pontryagin's theorem yields (7). It also follows that the effort and belief (x, u) trajectories satisfy

$$c''(u)(1 + e^x)\dot{u} = (\lambda + u)c'(u) - c(u) + w_t - v \tag{16}$$

$$\dot{x} = \lambda + u,$$

with boundary conditions

$$x_0 = x^0$$

$$u_T = 0. \tag{17}$$

Differentiating (16) further, we obtain

$$(c''(u)(1+e^x))^2 u_t''$$
$$= ((\lambda + u)c''(u)u_t' + w_t')c''(u)(1+e^x)$$
$$- ((\lambda + u)c'(u) + w_t - c(u) - v)(c'''(u)u_t'(1+e^x) + e^x(\lambda + u)c''(u)).$$

Thus, when $u_t' = 0$, we obtain

$$c''(u)(1+e^x)u_t'' = w_t'.$$

Combined with the transversality condition (17) and our assumption that the first-order condition holds at all times, this immediately implies the conclusion. □

PROOF OF PROPOSITION 2. From (12) we obtain the expression

$$u'(x) = \frac{(\lambda + u)c'(u) - c(u) - v}{c''(u)(1+e^x)(\lambda + u)},$$

which must hold for the optimal trajectory (in the (x, u) plane) for the social planner. Denote this trajectory x^{FB}. The corresponding law of motion for the agent's optimum trajectory x^* given w is

$$\frac{1}{x'(u)} = \frac{(\lambda + u)c'(u) - c(u) + w_t - v}{c''(u)(1+e^x)(\lambda + u)}.$$

(Note that, not surprisingly, time matters.) This implies that (in the (x, u) plane) the trajectories x^{FB} and x^* can only cross one way, if at all, with x^* being the flatter one. Yet the (decreasing) transversality curve of the social planner, implicitly given by

$$(1+e^{x_T})c'(u_T) = 1,$$

lies above the (decreasing) transversality curve of the agent, which is defined by $u_T = 0$.

Suppose now that the trajectory x^{FB} ends (on the transversality curve) at a lower belief x_T^{FB} than x^*: then it must be that effort u was higher throughout along that trajectory than along x^* (since the latter is flatter, x^{FB} must have remained above x^* throughout). But since the end value of the belief x is simply $x^0 + \int_0^T u_s \, ds$, this contradicts $x_T^{FB} < x_T^*$.

It follows that for a given x, the effort level u is higher for the social planner.

The same reasoning implies the second conclusion: if $x_T^{FB} = x_T^*$, so that total effort is the same, yet the trajectories can only cross one way (with x^* being flatter), it follows that x^* involves lower effort first, and then larger effort, i.e., the agent backloads effort. □

A.2 *Proofs for Section 3*

First, we describe and prove the omitted characterization in the case of linear cost. Proposition 4 describes the equilibrium in the linear case

PROPOSITION 4. *With linear cost, any equilibrium path consists of at most four phases, for some $0 \le t_1 \le t_2 \le t_3 \le T$:*

   (i) During $[0, t_1]$, no effort is exerted.

   (ii) During $(t_1, t_2]$, effort is interior, i.e., $u_t \in (0, \bar{u})$.

   (iii) During $(t_2, t_3]$, effort is maximal.

   (iv) During $(t_3, T]$, no effort is exerted.

   Any of these intervals might be empty.[36]

PROOF OF PROPOSITION 4. We prove the following statements:

1. If there exists $t \in (0, T)$ such that $\phi_t > 0$, then there exists $t' \in [t, T]$ such that $u_s = \bar{u}$ for $s \in [t, t']$ and $u_s = 0$ for $s \in (t', T]$.

2. If there exists $t \in (0, T)$ such that $\phi_t < 0$, then either $u_s = 0$ for all $s \in [t, T]$ or $u_s = 0$ for all $s \in [0, t]$.

This implies the desired decomposition. For the first part, note that either $u_s = \bar{u}$ for all $s > t$ or there exists t'' such that both $\phi_{t''} > 0$ (so in particular $u_{t''} = \bar{u}$) and $\dot{\phi}_{t''} < 0$. Because p_t decreases over time and $u_s \leq u_{t''}$ for all $s > t''$, it follows that $w_s < w_{t''}$, and so $\dot{\phi}_s < \dot{\phi}_{t''} < 0$. Hence ϕ can cross 0 only once for values above t, establishing the result. For the second part, note that either $u_s = 0$ for all $s \geq t$ or there exists $t'' \geq t$ such that $\phi_{t''} < 0$ (so in particular $u_{t''} = 0$) and $\dot{\phi}_{t''} > 0$. Because p_t decreases over time and $u_s \geq u_{t''}$ for all $s < t''$, it follows that $w_s \geq w_{t''}$, and so $\dot{\phi}_s > \dot{\phi}_{t''} > 0$. For all $s < t''$, $\phi_s < 0$ and $\dot{\phi}_s > 0$. Hence, $u_s = 0$ for all $s \in [0, t]$.   □

PROOF OF LEMMA 3. We can apply the continuous-time average cost optimality equation for multichain processes (see, e.g., Theorem 5.7 of Guo and Hernández-Lerma 2009). To verify their Assumption 5.4, we introduce a discount rate r (or a sequence thereof, tending to 0). Assume a public randomization device, which, given the negative result, is irrelevant. This is a game between a long-run player—the worker—and a sequence of short-run players—the market. Monitoring has a product structure, since wages are perfectly observed while the occurrence of breakthroughs relies on the worker's action only. It follows that attention can be restricted to strategies that depend on the public history only. Hence, there is a well defined highest and lowest equilibrium payoff for the agent, independent of the entire history. Let us denote by $\bar{v} \geq \underline{v}$ the corresponding average values. Given the randomization device, attention can be restricted to the strategies that achieve these values. Note that $\underline{v} = \lambda$: never exerting any effort is an option, and guarantees λ as a payoff; conversely, if the market never expects any effort, not exerting any is a best reply. To achieve $\bar{v}$, it is best to promise $\bar{v}$ as continuation utility if a breakthrough occurs, and threaten with reversion to $\underline{v}$ if no such breakthrough obtains with the minimum probability necessary to induce the desired level of effort. Let γ denote the rate at which reversion occurs in the absence of breakthroughs. It must hold

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[36]Here and elsewhere, the choices at the extremities of the intervals are irrelevant, and our specification is arbitrary in this respect.

that $\gamma = \mathcal{O}(dt)$, because the probability that a breakthrough occurs is of order dt. Hence, γ is at most $z\,dt$, for some $z > 0$, but then it holds that

$$\bar{v} = \max_{u \geq 0}\{r\,dt(\bar{w} - c(u)) + (1 - r\,dt)(\bar{v} - (1 - (\lambda + u)\,dt)z\,dt \cdot (\bar{v} - \underline{v}))\} + o(dt),$$

where $\bar{w}$ is the wage paid to the worker (equal to his expected output in the best equilibrium). Clearly the optimum effort is 0, as the impact of effort on the probability of punishment is of second order, while the cost is of first order. Verification of Theorem 5.7 of Guo and Hernández-Lerma (2009) is immediate. □

PROOF OF THEOREM 1. We start by establishing that equilibrium effort and wages are strictly positive (for $t < T$) and single-peaked. We then use these properties to prove uniqueness and existence.

Positive effort. We know from transversality that $u_T = 0$. Furthermore, the arbitrage equation (7) implies that whenever $u_t = 0$ in equilibrium, the derivative of effort is given by

$$c''(u_t)\dot{u}_t = p_t\lambda(p_t - 1) < 0.$$

Because u_t is continuous on $[0, T]$, it then follows that $u_t > 0$ for all $t < T$.

Single-peakedness. Single-peakedness of effort is almost immediate. Substituting the equilibrium expression $w_t = (\lambda + u_t)/(1 + e^{x_t})$ in the boundary value problem (16). Differentiating u_t' further, we obtain

$$u_t' = 0 \quad \Rightarrow \quad c''(u)(1 + e^{-x})u_t'' = -(w_t)^2,$$

which implies that the function u is at most first increasing then decreasing.

We now argue that the wage is single-peaked. In terms of x, the wage is given by

$$w(x) = \frac{\lambda + u(x)}{1 + e^x}, \quad \text{and so}$$

$$w'(x) = \frac{u'(x)}{1 + e^x} - \frac{\lambda + u(x)}{(1 + e^x)^2}e^x,$$

so that $w'(x) = 0$ is equivalent to

$$u'(x) = w(x)e^x.$$

As in the proof of Lemma 1, when $w'(x) = 0$, we have

$$w''(x) = \frac{u''(x) - u'(x)}{1 + e^x}.$$

Furthermore, we know that

$$u'(x) = \frac{(\lambda + u)c'(u) - c(u) + \frac{\lambda + u}{1 + e^x} - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

Mimicking the proof of Lemma 1, we conclude that $w'(x) = 0$ implies

$$u''(x) - u'(x) = -\frac{u'(x)(3c'' + (\lambda + u)c''')e^x}{c''(1 + e^x)} < 0,$$

if, as we have assumed, $c'' + (\lambda + u)c''' > 0$. Therefore, we also have single-peaked (at most increasing, then decreasing) wages. (More generally, if $c''' < 0$ but $3c'' + (\lambda + u)c'''$ is increasing in u, then the wage can be increasing on at most one interval.)

Uniqueness. Assume an equilibrium exists, and note that, given a final belief x_T, the pair of differential equations for u and x (along with the transversality condition) admits a unique solution, pinning down, in particular, the effort exerted by and the wage received by the agent. Therefore, if two (or more) equilibria existed for some values (x_0, T), it would have to be the case that each of them is associated with a different terminal belief x_T. However, we shall show that, for any x_0, the time it takes to reach a terminal belief x_T is a continuous, strictly increasing function $T(x_T)$; therefore, no two different terminal beliefs can be reached in the same time T.

To prove this claim, fix the horizon T. The two differential equations obeyed by the (x, u) trajectory are

$$\dot{x} = \lambda + u$$
$$\dot{u} = \frac{(\lambda + u)c'(u), -c(u) + \frac{\lambda + u}{1 + e^x} - v}{c''(u)(1 + e^x)}.$$

Therefore, the equilibrium effort $u(x)$ satisfies

$$\frac{du}{dx} =: f(u, x) = \frac{(\lambda + u)c'(u) - c(u) - v + \frac{\lambda + u}{1 + e^x}}{(\lambda + u)c''(u)(1 + e^x)}, \tag{18}$$

with $u(x_T) = 0$. We now define

$$\phi(u) := v + c(u) - (\lambda + u)c'(u).$$

(To simplify notation, we do not yet impose the equilibrium restriction $v = \lambda$.) Note that $\phi(u)$ is strictly decreasing (because c is convex) and that $f(u, x) < 0 \Rightarrow \phi(u) > 0$. We can also write

$$(\lambda + u(x))c''(u(x))(1 + e^x)u'(x) = \frac{\lambda + u}{1 + e^x} - \phi(u(x)),$$

which means that the composite function $\phi(u(x))$ is strictly positive. By definition (and convexity of c) it is increasing in x whenever $u'(x) \leq 0$ and it is decreasing in x whenever $u'(x) > 0$. Because $u(x)$ is single-peaked, this implies $\phi(u(x))$ is positive everywhere.

The time required to reach a terminal belief x_T is given by

$$T(x_T) := \int_{x_0}^{x_T} \frac{1}{\lambda + u(x, x_T)} \, dx,$$

where $u(x, x_T)$ refers to the solution to the differential equation (18) with the boundary condition at x_T. Differentiating with respect to x_T, we obtain

$$T'(x_T) = \frac{1}{\lambda} - \int_{x_0}^{x_T} \frac{1}{(\lambda + u(x, x_T))^2} \frac{\partial u(x, x_T)}{\partial x_T} \, dx.$$

To obtain an expression for $\partial u(x, x_T)/\partial x_T$, rewrite $u(x, x_T)$ as

$$u(x, x_T) = -\int_x^{x_T} f(u(s, x_T), s) \, ds;$$

therefore

$$\frac{\partial u(x, x_T)}{\partial x_T} = -f(0, x_T) - \int_x^{x_T} \frac{\partial f(u(s, x_T), s)}{\partial u} \frac{\partial u(s, x_T)}{\partial x_T} \, ds.$$

Now, we let

$$g(x, x_T) := \frac{\partial u(x, x_T)}{\partial x_T},$$

and note that $g(x, x_T)$ satisfies the differential equation

$$\frac{\partial g(x, x_T)}{\partial x_T} = \frac{\partial f(u(x, x_T), x)}{\partial u} g(x, x_T),$$

with boundary condition $g(x_T, x_T) = -f(0, x_T)$. This implies

$$\frac{\partial u(x, x_T)}{\partial x_T} = -f(0, x_T) e^{-\int_x^{x_T} \partial f(u(s, x_T), s)/\partial u \, ds}.$$

We now rewrite the integrand as

$$\frac{\partial f(u, x)}{\partial u} = A(u, x) + \frac{1}{1 + e^x} - \frac{c'''(u) f(u, x)}{c''(u)},$$

$$A(u, x) := \frac{\phi(u)}{(\lambda + u)^2 c''(u)(1 + e^x)}.$$

Using the definition of $f(u, x)$ to compute $f(0, x_T)$, we have

$$\begin{aligned}
\frac{\partial u(x, x_T)}{\partial x_T} &= -\frac{-v + \frac{\lambda}{1 + e^{x_T}}}{\lambda c''(0)(1 + e^{x_T})} \frac{1 + e^{-x_T}}{1 + e^{-x}} \frac{c''(0)}{c''(u(x, x_T))} e^{-\int_x^{x_T} A(u(s, x_T), s) \, ds} \\
&= \frac{(v/\lambda - 1) e^{-x_T} + v/\lambda}{1 + e^{x_T}} \frac{1}{(1 + e^{-x}) c''(u(x, x_T))} e^{-\int_x^{x_T} A(u(s, x_T), s) \, ds} \\
&= \frac{(v/\lambda - 1) e^{-x_T} + v/\lambda}{1 + e^{x_T}} \frac{(\lambda + u(x, x_T))^2 e^x}{\phi(u(x, x_T))} A(u(x, x_T), x) e^{-\int_x^{x_T} A(u(s, x_T), s) \, ds}.
\end{aligned}$$

Therefore, plugging into the expression for $T'(x_T)$, we obtain

$$T'(x_T) = \frac{1}{\lambda} - \frac{(v/\lambda - 1) e^{-x_T} + v/\lambda}{1 + e^{x_T}} \int_{x_0}^{x_T} \frac{e^x A(u(x, x_T), x)}{\phi(u(x, x_T))} e^{-\int_x^{x_T} A(u(s, x_T), s) \, ds} \, dx.$$

Integrating by parts and substituting $v = \lambda$, we obtain

$$T'(x_T) = \frac{1}{\lambda(1+e^{x_T})} + \frac{1}{1+e^{x_T}} \frac{e^{x_0} A(u(x_0,x_T),x_0)}{\phi(u(x_0,x_T))} e^{-\int_{x_0}^{x_T} A(u(s,x_T),s)\,ds}$$

$$+ \frac{1}{1+e^{x_T}} \int_{x_0}^{x_T} \frac{\partial(e^x/\phi(u(x,x_T)))}{\partial x} e^{-\int_x^{x_T} A(u(s,x_T),s)\,ds}\, dx.$$

Note that the first two terms are positive, and that the derivative inside the last term simplifies to

$$\frac{\partial}{\partial x} \frac{e^x}{\phi(u(x,x_T))} = \frac{1}{(1+e^{-x})} \frac{1}{\phi(u(x,x_T))} \left(\frac{\lambda + u(x,x_T)}{(1+e^x)\phi(u(x,x_T))} + e^x \right),$$

which is also positive since $\phi(u(x,x_T)) > 0$ for all x. This establishes the result.

Existence. We have established that the time necessary to reach the terminal belief is a continuous and strictly increasing function. Therefore, the terminal belief reached in equilibrium is itself given by a strictly increasing function

$$x_T(T) : \mathbb{R}_+ \to [x_0, \infty).$$

Since there exists a unique path consistent with optimality for each terminal belief, given a deadline T, we can establish existence by constructing the associated equilibrium outcome and, in particular, the equilibrium wage path. Existence and uniqueness of an optimal strategy for the worker, after any (on- or off-path) history, follow then from Lemma 2.   □

A.3 *Proofs for Section 5*

PROOF OF THEOREM 2. The proof is divided into several steps. Consider the maximization program $\mathcal{P}$ in the text: we begin by conjecturing a full-zero (or FO) solution, i.e., a solution in which the agent first exerts maximum effort, then no effort; we show that this solution solves a relaxed program; and finally we verify that it also solves the original program.

Candidate solution. Consider the following compensation scheme: pay a wage $w_t = 0$ for $t \in [0, t_0] \cup [t_1, T]$, a constant wage $w_t = v - \alpha\lambda$ for $t \in [t_0, t_1]$, and a lump sum L at $t = T$. The agent exerts maximal effort for $t \le t_1$ and zero thereafter. Furthermore, the agent is indifferent among all effort levels for $t \in [t_0, t_1]$.

For short enough deadlines, there exists a payment scheme of this form that induces full effort throughout, i.e., $t_0 > 0$ and $t_1 = T$, and leaves the agent indifferent between effort levels at T. Whenever this is the case, we take this to be our candidate solution. The conditions that pin down this solution are given by indifference at T and by zero profits at $t = 0$. Recall the definition of ϕ_t from the proof of Proposition 1. The conditions are then given by

$$\phi_T = (k - \alpha - L)e^{-x_T} - \alpha = 0 \qquad (19)$$

$$\int_0^{t_0} (1+e^{-x_s}) \frac{\lambda + u}{1+e^{x_s}}\, ds + \int_{t_0}^T (1+e^{-x_s}) \left(\frac{\lambda + \bar{u}}{1+e^{x_s}} - v + \alpha\lambda \right) ds - (1+e^{-x_T})L = 0. \qquad (20)$$

As T increases, $t_0 \to 0$. Let T^* denote the longest deadline for which this solution induces full effort throughout. The threshold T^* is the unique solution to (19) and (20) with $x_T = x_0 + (\lambda + \bar{u})T$ and $t_0 = 0$.

LEMMA 5. *The candidate solution is the unique compensation scheme that induces full effort on $[0, T^*]$.*

PROOF. Delaying any payment from t to t' would induce the agent to shirk at t' because he is now indifferent for $t \leq t_1$. Anticipating payments while preserving zero profits ex ante would lead the agent to shirk at t. To see this, notice that if the firm wants to hold the ex ante profit level constant and shift wages across time periods, it can do so by setting

$$\Delta w_1 = -\frac{1 + e^{-x_2}}{1 + e^{-x_1}} \Delta w_2.$$

Then by construction,

$$\Delta w_1 + \Delta w_2 = -(e^{-x_1} \Delta w_1 + e^{-x_2} \Delta w_2).$$

Therefore, by delaying payments (in a profit-neutral way and without affecting effort), incentives at time t can be increased. Consider the function

$$\phi_t = \phi_T - \int_t^T e^{-x_s}(w_s - v + \alpha\lambda)\, ds$$

and two times t_1 and t_2. Indeed, if $\Delta w_2 > 0$, then $\Delta w_1 < 0$ and $\Delta w_1 + \Delta w_2 > 0$, which increases ϕ_1. Conversely, anticipating payments reduces incentives ϕ_1.   ◁

For $T > T^*$, we cannot obtain full effort throughout. Our candidate solution is then characterized by $t_0 = 0$, $t_1 < T$, indifference at $t = T$, and zero profits at $t = 0$. The final belief is given by $x_T = x_t + \lambda(T - t) + \bar{u}(t_1 - t)$. It is useful to rewrite our three conditions in beliefs space. We have

$$(k - \alpha - L)e^{-x_T} - \alpha + (v/\lambda - \alpha)(e^{-x_1} - e^{-x_T}) = 0 \tag{21}$$

$$e^{-x_0} - e^{-x_T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}(e^{-x_0} - e^{-x_1} + x_1 - x_0) - (1 + e^{-x_T})L = 0 \tag{22}$$

$$\frac{x_T - x_1}{\lambda} + \frac{x_1 - x_0}{\lambda + \bar{u}} - T = 0, \tag{23}$$

which determine the three variables (L, x_1, x_T) as a function of x_0 and T. To compute the solution, we can solve the second one for L and the third for x_T, and obtain one equation in one unknown for x_1.

We can now compute the agent's payoff under this compensation scheme. Namely,

$$\tilde{W}(x_0, T) = \int_0^{t_1} (1 + e^{-x_s})(v - \alpha\lambda - \alpha\bar{u} - v)\, ds - \int_{t_1}^T (1 + e^{-x_s})v\, ds + (1 + e^{-x_T})(L - k)$$

$$= -\int_{x_0}^{x_1} (1 + e^{-x})\alpha\, dx - \int_{x_1}^{x_T} (1 + e^{-x})\frac{v}{\lambda}\, dx + (1 + e^{-x_T})(L - k),$$

where (L, x_1, x_T) are the solution to (21)–(23) given (x_0, T). Plugging in the value of L from (20), we can rewrite payoffs as

$$\tilde{W}(x_0, T) = -\int_{x_0}^{x_1} \left(\frac{v - \alpha\lambda}{\lambda + \bar{u}} + e^{-x} \left(\frac{v + \bar{u}\alpha}{\bar{u} + \lambda} - 1 \right) \right) dx$$
$$- \int_{x_1}^{x_T} \left(\frac{v}{\lambda} + e^{-x} \frac{v - \lambda}{\lambda} \right) dx - (1 + e^{-x_T})k.$$

Now fix x_0 and T. We denote by $J(x)$ the payoff under an offer that follows our candidate solution to an agent who holds belief x. This requires solving the system (21)–(23) as a function of the current belief and the residual time. In particular, we have $J(x) = \tilde{W}(x, T - (x - x_0)/(\lambda + \bar{u}))$ when $x < x_1(x_0, T)$ and $J(x) = \tilde{W}(x, T - (x_1 - x_0)/(\lambda + \bar{u}) - (x - x_1)/\lambda)$ when $x \geq x_1(x_0, T)$.

Finally, we denote by $Y(x)$ the agent's continuation payoff at x under the original scheme. Notice that the bound in (11) ensures that

$$\frac{\lambda + \bar{u}}{1 + e^{x_t}} \geq v - \alpha\lambda$$

for all $t \leq t_1$ and for all T. This means the firm is running a positive flow profit when paying $v - \alpha\lambda$ during a full effort phase, hence effort at t contributes positively to the lump sum L. In other words, the firm does not obtain positive profits when the agent's continuation value is $Y(x)$. The details on how to derive this bound can be found in the working paper.

Original and relaxed programs. Consider the original program $\mathcal{P}$, and rewrite it in terms of the log-likelihood ratios x_t, up to constant terms:

$$W(t, x_t) = \max_{w,u} \int_t^T (1 + e^{-x_s})(w_s - v - \alpha u_s)\, ds - k e^{-x_T}$$

$$\text{s.t.} \quad u = \arg\max_u \int_t^T (1 + e^{-x_s})(w_s - v - \alpha u_s)\, ds - k e^{-x_T}$$

$$\forall \tau \geq t: \quad \int_\tau^T (1 + e^{-x_s})(w_s - v - \alpha u_s)\, ds - k e^{-x_T} \geq W(\tau, x_\tau) \tag{24}$$

$$0 \leq \int_0^T (1 + e^{-x_t}) \left(\frac{\lambda + u_t}{1 + e^{x_t}} - w_t \right) dt. \tag{25}$$

We first argue that the nonnegative profit constraint (25) will be binding. This is immediate if we observe that constraint (24) implies the firm cannot make positive profits on any interval $[t, T]$, $t \geq 0$. If it did, the worker could be poached by a competitor who offers, for example, the same wage plus a signing bonus. We now consider a relaxed problem in which we substitute (24) and (25) with the nonpositive profit constraint:

$$W(t, x_t) = \max_{w,u} \int_0^T (1 + e^{-x_t})(w_t - v - \alpha u_t)\, dt - k e^{-x_T}$$

$$\text{s.t.} \quad u = \arg\max_u \int_0^T (1 + e^{-x_t})(w_t - v - \alpha u_t)\, dt - k e^{-x_T}$$

$$0 \geq \int_\tau^T (1 + e^{-x_t})\left(\frac{\lambda + u_t}{1 + e^{x_t}} - w_t\right) dt \quad \text{for all } \tau \leq T.$$

We then use the following result to further relax this program.

LEMMA 6. *Let $T > T^*$ and consider our candidate solution described in (21)–(23). If another contract generates a strictly higher surplus $W(0, x_0)$, then it must yield a strictly higher x_T.*

PROOF. We use the fact that our solution specifies maximal frontloading of effort, given x_T. Notice that we can rewrite the social surplus (which is equal to the agent's payoff at time 0) as

$$-(1 + k - \alpha)e^{-x_T} - \alpha x_T - \int_0^T (1 + e^{-x_t})(v - \alpha\lambda)\, dt + \text{constant.} \tag{26}$$

Therefore, for a given x_T, surplus is maximized by choosing the highest path for x_t, which is obtained by frontloading effort. Furthermore, (26) is strictly concave in x_T. Because $T > T^*$, we know from Proposition 2 that under any nonnegative payment function w, the agent works strictly less than the social planner. Since the agent receives the entire surplus, his ex ante payoff is then strictly increasing in x_T. ◁

We therefore consider the even more relaxed problem $\mathcal{P}'$, which is given by

$$\max_{w,u} x_T$$

$$\text{s.t.} \quad u = \arg\max_u \int_0^T (1 + e^{-x_t})(w_t - v - \alpha u_t)\, dt - k e^{-x_T}$$

$$0 \geq \int_\tau^T (1 + e^{-x_t})\left(\frac{\lambda + u_t}{1 + e^{x_t}} - w_t\right) dt \quad \text{for all } \tau \leq T.$$

In Lemma 7, whose proof is omitted, we prove that our candidate solves the relaxed program. We also show that the agent's continuation value under the original contract is higher than the value of the best contract offered at a later date, and hence that we have found a solution to the original program $\mathcal{P}$.

LEMMA 7. *Let $T > T^*$. The candidate solution described in (21)–(23) solves the relaxed program $\mathcal{P}'$. Furthermore, under the candidate solution, constraint (24) in the original program never binds (except at $t = 0$).* □

A.4 *Proofs for Section 6*

PROOF OF LEMMA 4. Suppose that the equilibrium effort is zero on some open set Ω. Consider the sets $\Omega_{t'} = \{(x, s) : s \in (t', T]\}$ such that the trajectory starting at (x, s) intersects Ω. Suppose that u is not identically zero on Ω_0 and let $\tau = \inf\{t' : u = 0 \text{ on } \Omega_{t'}\}$. That is, for all $t' < \tau$, there exists $(x, s) \in \Omega_{t'}$ such that $u(x, s) > 0$. Suppose first that we take

$(x, \tau) \in \Omega_\tau$. According to the definition of τ and Ω_τ, there exists $(x_k, k) \in \Omega$ such that the trajectory starting at (x, τ) intersects Ω at (x_k, k) and along the path the effort is zero. We can write the payoff

$$V(x, \tau) = \int_x^{x_k} \frac{1+e^{-s}}{1+e^{-x}} \left(\frac{\lambda}{1+e^s} - v \right) \frac{1}{\lambda} \, ds + \frac{1+e^{-x_k}}{1+e^{-x}} V(x_k, k)$$

or, rearranging,

$$(1+e^{-x})V(x, \tau) = -(e^{-x_k} - e^{-x})\left(1 - \frac{v}{\lambda} \right) - \frac{v}{\lambda}(x_k - x) + (1+e^{-x_k})V(x_k, k),$$

where $V(x_k, k)$ is differentiable. The Hamilton–Jacobi–Bellman (HJB) equation (a function of (x, τ)) can be derived from

$$V(x, \tau) = \frac{\lambda + \hat{u}}{1+e^x} \, dt - v \, dt + \max_u \left[-\alpha u \, dt + \left(1 - \frac{\lambda + u}{1+e^x} \, dt + o(dt) \right) \right.$$

$$\left. \times \left(V(x, \tau) + V_x(x, \tau)(\lambda + u) \, dt + V_t(x, \tau) \, dt + o(dt) \right) \right],$$

which gives, taking limits,

$$0 = \frac{\lambda + \hat{u}}{1+e^x} - v + \max_{u \in [0, \bar{u}]} \left[-\alpha u - \frac{\lambda + u}{1+e^x} V(x, \tau) + V_x(x, \tau)(\lambda + u) + V_t(x, \tau) \right].$$

Therefore, if $u(x, \tau) > 0$,

$$-\frac{V(x, \tau)}{1+e^x} - \alpha + V_x(x, \tau) \geq 0, \quad \text{or} \quad (1+e^{-x})V_x(x, \tau) - e^{-x}V(x, \tau) \geq \alpha(1+e^{-x})$$

or, finally,

$$\frac{\partial}{\partial x}[(1+e^{-x})V(x, \tau)] - \alpha(1+e^{-x}) \geq 0.$$

Notice, however, by direct computation, that because low effort is exerted from (x, τ) to (x_k, k), for all points (x_s, s) on this trajectory, $s \in (\tau, k)$,

$$\frac{\partial}{\partial x}[(1+e^{-x_s})V(x_s, s)] - \alpha(1+e^{-x_s}) = -e^{-x_s}\left(1 + \alpha - \frac{v}{\lambda} \right) + \frac{v}{\lambda} - \alpha \leq 0,$$

so that because $x < x_s$ and $1 + \alpha - v/\lambda > 0$,

$$\frac{\partial}{\partial x}[(1+e^{-x})V(x, \tau)] - \alpha(1+e^{-x}) < 0,$$

a contradiction to $u(x, \tau) > 0$.

If instead $u(x, \tau) = 0$ for all $(x, \tau) \in \Omega_\tau$, then there exists $(x', t') \to (x, \tau) \in \Omega_\tau$, $u(x', t') > 0$. Because u is upper semicontinuous, for every $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}$ of (x, τ) such that $u < \varepsilon$ on $\mathcal{N}$. Hence

$$\lim_{(x', t') \to (x, \tau)} \frac{\partial}{\partial x}[(1+e^{x'})V(x', t')] - \alpha(1+e^{x'}) = \frac{\partial}{\partial x}[(1+e^{-x})V(x, \tau)] - \alpha(1+e^{-x}) < 0,$$

a contradiction.   □

PROOF OF THEOREM 3. We start with (i). That is, we show that $u(x, t) = \bar{u}$ for $x < \underline{x}_t$ in all equilibria. We first define $\underline{x}$ as the solution to the differential equation

$$\left(\lambda(1+\alpha) - v + (\lambda+\bar{u})\alpha e^{\underline{x}(t)} + \bar{u} - ((1+k)(\lambda+\bar{u}) - (v+\alpha\bar{u}))e^{-(\lambda+\bar{u})(T-t)}\right)$$
$$\times \left(\frac{\underline{x}'(t)}{\lambda+\bar{u}} - 1\right) = -\bar{u} \tag{27}$$

subject to $\underline{x}(T) = x^*$. This defines a strictly increasing function of slope larger than $\lambda+\bar{u}$, for all $t \in (T - t^*, T]$, with $\lim_{t \uparrow t^*} \underline{x}(T-t) = -\infty$.[37] Given some equilibrium, and an initial value (x_t, t), let $u(\tau; x_\tau)$ denote the value at time $\tau \geq t$ along the equilibrium trajectory. For all t, let

$$\tilde{x}(t) := \sup\{x_t : \forall \tau \geq t : u(\tau; x_t) = \bar{u} \text{ in all equilibria}\},$$

with $\tilde{x}(t) = -\infty$ if no such x_t exists. By definition the function $\tilde{x}$ is increasing (in fact, for all $\tau \geq t$, $\tilde{x}(\tau) \geq \tilde{x}(t) + (\lambda + \bar{u})(\tau - t)$), and so it is a.e. differentiable (set $\tilde{x}'(t) = +\infty$ if x jumps at t). Whenever finite, let $s(t) = \tilde{x}'(t)/(\tilde{x}'(t) - \lambda) > 0$. Note that, from the transversality condition, $\tilde{x}(T) = x^*$. In an abuse of notation, we also write $\tilde{x}$ for the set function $t \to [\lim_{t' \uparrow t} \tilde{x}(t'), \lim_{t' \downarrow t} \tilde{x}(t')]$.

We first argue that the incentives to exert high effort decrease in x (when varying the value x of an initial condition (x, t) for a trajectory along which effort is exerted throughout). Indeed, recall that high effort is exerted if and only if

$$\frac{\partial}{\partial x}(V(x, t)(1 + e^{-x})) \geq \alpha(1 + e^{-x}). \tag{28}$$

The value $V^H(x, t)$ obtained from always exerting (and being paid for) high effort is given by

$$(1 + e^{-x})V^H(x, t) = \int_t^T (1 + e^{-x_s})\left[\frac{\lambda+\bar{u}}{1 + e^{x_s}} - v - \alpha\bar{u}\right]ds - k(1 + e^{-x_T})$$
$$= (e^{-x} - e^{-x_T})\left(1 - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right) - (T - t)(v + \alpha\bar{u}) - k(1 + e^{-x_T}), \tag{29}$$

where $x_T = x + (\lambda + \bar{u})(T - t)$. Therefore, using (28), high effort is exerted if and only if

$$k - \left(1 + k - \frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)(1 - e^{-(\lambda+\bar{u})(T-t)}) \geq \alpha(1 + e^x).$$

---

[37]The differential equation for $\underline{x}$ can be implicitly solved, which yields

$$\ln \frac{k-\alpha}{\alpha} = (\underline{x}_t + (\lambda+\bar{u})(T-t)) + \frac{\bar{u}}{\lambda(1+\alpha) + \bar{u} - v} \ln(k-\alpha)\bar{u}(\lambda+\bar{u})$$

$$- \frac{\bar{u}}{\lambda(1+\alpha) + \bar{u} - v} \ln\left(e^{(\lambda+\bar{u})(T-t)}(\lambda(1+\alpha) + \bar{u} - v)(\lambda(1+\alpha) - v + \alpha(\lambda+\bar{u})e^{\underline{x}_t})\right.$$

$$- (\lambda(1+\alpha) - v)(\lambda(1+\alpha) + \bar{u} - v + (k-\alpha)(\lambda+\bar{u}))\Big).$$

Note that the left-hand side is independent of x, while the right-hand side is increasing in x. Therefore, if high effort is exerted throughout from (x, t) onward, then it is also from (x', t) for all $x' < x$.

Fix an equilibrium and a state (x_0, t_0) such that $x_0 + (\lambda + \bar{u})(T - t_0) < x^*$. Note that the equilibrium trajectory must eventually intersect some state $(\tilde{x}_t, t)$. We start again from the formula for the payoff:

$$(1 + e^{-x_0})V(x_0, t_0) = \int_{t_0}^{t} \left[e^{-x_s}(\lambda + u(x_s, s)) - (1 + e^{-x_s})(v + \alpha u(x_s, s)) \right] ds$$

$$+ (1 + e^{-\tilde{x}_t})V^H(\tilde{x}_t, t).$$

Let $W(\tilde{x}_t) = V^H(\tilde{x}_t, t)$ (since $\tilde{x}$ is strictly increasing, it is well defined). Differentiating with respect to x_0 and taking limits as $(x_0, t_0) \to (\tilde{x}_t, t)$, we obtain

$$\lim_{(x_0, t_0) \to (\tilde{x}_t, t)} \frac{\partial (1 + e^{-x_0})V(x_0, t_0)}{\partial x_0}$$

$$= [e^{-\tilde{x}_t}\lambda - (1 + e^{-\tilde{x}_t})v]\frac{s(\tilde{x}_t) - 1}{\lambda} + s(\tilde{x}_t)[W'(\tilde{x}_t)(1 + e^{-\tilde{x}_t}) - W(\tilde{x}_t)e^{-\tilde{x}_t}].$$

If less than maximal effort can be sustained arbitrarily close to, but before the state $(\tilde{x}_t, t)$ is reached, it must be that this expression is no more than $\alpha(1 + e^{-\tilde{x}_t})$ in some equilibrium, by (28). Rearranging, this means that

$$\left(1 - W(x) + (1 + e^x)\left(W'(x) - \frac{v}{\lambda} \right) \right)s(x) + \left(\frac{v}{\lambda} - \alpha \right)e^x \leq 1 + \alpha - \frac{v}{\lambda}$$

for $x = \tilde{x}_t$. Given the explicit formula for W (see (29)) and since $s(\tilde{x}_t) = \tilde{x}_t'/(\tilde{x}_t' - \lambda)$, we can rearrange this to obtain an inequality for $\tilde{x}_t$. The derivative $\tilde{x}_t'$ is smallest, and thus the solution $\tilde{x}_t$ is highest, when this inequality binds for all t. The resulting ordinary differential equation is precisely (27).

Next, we turn to (ii). That is, we show that $u(x, t) = 0$ for $x > \bar{x}_t$ in all equilibria. We define $\bar{x}$ by

$$\bar{x}_t = \ln\left[k - \alpha + \left(\frac{v + \bar{u}\alpha}{\lambda + \bar{u}} - (1 + k) \right)(1 - e^{-(\lambda + \bar{u})(T - t)}) \right] - \ln \alpha, \tag{30}$$

which is well defined as long as $k - \alpha + ((v + \bar{u}\alpha)/(\lambda + \bar{u}) - (1 + k))(1 - e^{-(\lambda + \bar{u})(T - t)}) > 0$. This defines a minimum time $T - t^*$ mentioned above, which coincides with the asymptote of $\underline{x}$ (as can be seen from (27)). It is immediate to check that $\bar{x}$ is continuous and strictly increasing on $[T - t^*, T]$, with $\lim_{t \uparrow t^*} \bar{x}_{T-t} = -\infty$, $x_T = x^*$, and, for all $t \in (T - t^*, T)$, $\bar{x}_t' > \lambda + \bar{u}$.

Let us define $W(x, t) = (1 + e^{-x})V(x, t)$, and re-derive the HJB equation. The payoff can be written as

$$W(x, t) = \left[(\lambda + u(x, t))e^{-x} - (1 + e^{-x})(v + \alpha u) \right] dt + W(x + dx, t + dt),$$

which gives

$$0 = (\lambda + u(x,t))e^{-x} - v(1 + e^{-x}) + W_t(x,t) + \lambda W_x(x,t) + \max_{u \in [0,\bar{u}]} (W_x(x,t) - \alpha(1 + e^{-x}))u.$$

As we already know (see (28)), effort is maximum or minimum depending on $W_x(x,t) \lessgtr \alpha(1 + e^{-x})$. Let us rewrite the previous equation as

$$v - \alpha\lambda - W_t(x,t) = ((1 + \alpha)\lambda - v + u(x,t))e^{-x} + \lambda(W_x(x,t) - \alpha(1 + e^{-x}))$$
$$+ (W_x(x,t) - \alpha(1 + e^{-x}))^+ \bar{u}.$$

Given W_x, W_t is maximized when effort $u(x,t)$ is minimized: the lower is $u(x,t)$, the higher is $W_t(x,t)$, and hence the lower is $W(x,t - dt) = W(x,t) - W_t(x,t) dt$. Note also that along any equilibrium trajectory, no effort is ever strictly optimal (by (iv)). Hence, $W_x(x,t) \geq \alpha(1 + e^{-x})$, and, therefore, again $u(x,t)$ (or $W(x,t - dt)$) is minimized when $W_x(x,t) = \alpha(1 + e^{-x})$: to minimize $u(x,t)$ while preserving incentives to exert effort, it is best to be indifferent whenever possible.

Hence, integrating over the equilibrium trajectory starting at (x,t),

$$(v - \alpha\lambda)(T - t) + k(1 + e^{-x_T}) + W(x,t)$$

$$= \int_t^T u(x_s, s)e^{-x_s} ds$$

$$+ \int_t^T \left[((1 + \alpha)\lambda - v)e^{-x_s} + (\lambda + \bar{u})(W_x(x_s, s) - \alpha(1 + e^{-x_s}))^+\right] ds.$$

We shall construct an equilibrium in which $W_x(x_s, s) = \alpha(1 + e^{-x_s})$ for all $x > \underline{x}_t$. Hence, this equilibrium minimizes

$$\int_t^T u(x_s, s)e^{-x_s} ds$$

along the trajectory, and since this is true from any point of the trajectory onward, it also minimizes $u(x_s, s)$, $s \in [t, T]$; the resulting $u(x,t)$ will be shown to be increasing in x and equal to $\bar{u}$ at $\bar{x}_t$.

Let us construct this interior effort equilibrium. Integrating (28) over any domain with nonempty interior, we obtain that

$$(1 + e^x)V(x,t) = e^x(\alpha x + c(t)) - \alpha \tag{31}$$

for some function $c(t)$. Because the trajectories starting at (x,t) must cross $\underline{x}$ (whose slope is larger than $\lambda + \bar{u}$), value matching must hold at the boundary, which means that

$$(1 + e^{\underline{x}_t})V^H(\underline{x}_t, t) = e^{\underline{x}_t}(\alpha\underline{x}_t + c(t)) - \alpha,$$

which gives $c(t)$ (for $t \geq T - t^*$). From (31), we then back out $V(x,t)$. The HJB equation then reduces to

$$v - \alpha\lambda = \frac{\lambda + u(x,t)}{1 + e^x} + V_t(x,t),$$

which can now be solved for $u(x, t)$. That is, the effort is given by

$$\lambda + u(x, t) = (1 + e^x)(v - \alpha\lambda) - \frac{\partial}{\partial t}[(1 + e^x)V(x, t)]$$

$$= (1 + e^x)(v - \alpha\lambda) - e^x c'(t).$$

It follows from simple algebra (c' is detailed below) that $u(x, t)$ is increasing in x. Therefore, the upper end $\bar{x}_t$ cannot exceed the solution to

$$\lambda + \bar{u} = (1 + e^{\bar{x}})(v - \alpha\lambda) - e^{\bar{x}}c'(t),$$

and so we can solve for

$$e^{\bar{x}} = \frac{\lambda(1 + \alpha) - v + \bar{u}}{v - \alpha\lambda - c'(t)}.$$

Note that from totally differentiating the equation that defines $c(t)$,

$$c'(t) = \underline{x}'(t)e^{-\underline{x}(t)}\left[\left(W'(\underline{x}(t)) - \alpha\right)(e^{\underline{x}(t)} + 1) - W(\underline{x}(t))\right]$$

$$= v - \alpha\lambda + e^{-\underline{x}(t)}(v - (1 + \alpha)\lambda),$$

where we recall that $\underline{x}$ is the lower boundary below which effort must be maximal, and $W(\underline{x}) = V^H(\underline{x}_t, t)$. This gives

$$e^{\bar{x}} = e^{\underline{x}}\frac{\lambda(1 + \alpha) - v + \bar{u}}{\lambda(1 + \alpha) - v}, \quad \text{or} \quad e^{\underline{x}} = \frac{\lambda(1 + \alpha) - v}{\lambda(1 + \alpha) - v + \bar{u}}e^{\bar{x}}.$$

Because (27) is a differential equation characterizing $\underline{x}$, we may substitute for $\bar{x}$ from the last equation to obtain a differential equation characterizing $\bar{x}$, namely

$$\frac{\bar{u}}{1 - \frac{\bar{x}'(t)}{\lambda + \bar{u}}} + ((1 + k)(\lambda + \bar{u}) - (v + \alpha\bar{u}))e^{-(\lambda + \bar{u})(T - t)}$$

$$= \lambda(1 + \alpha) + \bar{u} - v + \frac{\alpha(\lambda + \bar{u})(\lambda(1 + \alpha) - v)}{\lambda(1 + \alpha) - v + \bar{u}}e^{\bar{x}},$$

with boundary condition $\bar{x}(T) = x^*$. It is simplest to plug in the formula given by (30) and verify that it is indeed the solution of this differential equation.

   Finally, we prove (iii). The same procedure applies to both, so let us consider $\bar{\sigma}$, the strategy that exerts high effort as long as $x < \bar{x}_t$ (and no effort above). We shall do so by "verification." Given our closed-form expression for $V^H(x, t)$ (see (29)), we immediately verify that it satisfies the constraint (35) for all $x \le \bar{x}_t$ (remarkably, $\bar{x}_t$ is *precisely* the boundary at which the constraint binds; it is strictly satisfied at $\underline{x}_t$ when considering $\underline{\sigma}$). Because this function $V^H(x, t)$ is differentiable in the set $\{(x, t) : x < \bar{x}_t\}$, and satisfies the HJB equation as well as the boundary condition $V^H(x, T) = 0$, it is a solution to the optimal control problem in this region (remember that the set $\{(x, t) : x < \bar{x}_t\}$ cannot be left under any feasible strategy, so that no further boundary condition needs to be verified). We can now consider the optimal control problem with exit region $\Omega := \{(x, t) : x = \bar{x}_t\} \cup \{(x, t) : t = T\}$ and salvage value $V^H(\bar{x}_t, t)$ or 0, depending on the exit point.

Again, the strategy of exerting no effort satisfies the HJB equation, gives a differentiable value on $\mathbb{R} \times [0, T] \setminus \Omega$, and satisfies the boundary conditions. Therefore, it is a solution to the optimal control problem.   □

Comparative statics for the boundary. We focus on the equilibrium that involves the largest effort region.

PROPOSITION 5. *The boundary of the maximal effort equilibrium $\underline{p}(t)$ is nonincreasing in k and v, and nondecreasing in α and λ.*

This proposition follows directly by differentiating expression (30) for the frontier $\bar{x}(t)$.

The effect of the maximum effort level $\bar{u}$ is ambiguous. Also, one might wonder whether increasing the penalty k increases welfare for some parameters, as it helps resolve the commitment problem. Unlike in the non-observable case, this turns out never to occur, at least in the maximum-effort equilibrium: increasing the penalty decreases welfare, though it increases total effort. The proof is available from the authors. Similarly, increasing v increases effort (in the maximum-effort equilibrium), though it decreases the worker's payoff.   □

PROOF OF PROPOSITION 3. (i) The equation defining the full effort frontier in the unobservable case $x_2(t)$ is given by

$$(k - \alpha)e^{-x_2-(\lambda+u)(T-t)} - \alpha - \int_{x_2}^{x_2+(\lambda+u)(T-t)} e^{-x}\left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda + \bar{u}}\right)dx. \qquad (32)$$

Plug the expression for $\bar{x}(t)$ given by (30) into (32) and notice that (32) cannot be equal to zero unless $\bar{x}(t) = x^*$ and $t = T$, or $\bar{x}(t) \to -\infty$. Therefore, the two frontiers cannot cross before the deadline T, but they have the same vertical asymptote.

Now suppose that $\phi'(x^*|\bar{u}) > 0$ so that the frontier $x_2(t)$ goes through (T, x^*). Consider the slopes of $x_2(t)$ and $\bar{x}(t)$ evaluated at (T, x^*). We obtain

$$[\bar{x}'(t) - x_2'(t)]_{t=T} \propto (\bar{u} + \lambda)(k - \alpha) > 0,$$

so the unobservable frontier lies above the observable one for all t.

Next suppose $\phi'(x^*|\bar{u}) < 0$, so there is no mixing at x^* and the frontier $x_2(t)$ does not go through (T, x^*). In this case, we still know the two cannot cross, and we also know that a point on $x_2(t)$ is the pre-image of (T, x^*) under full effort. Since we also know the slope $\bar{x}'(t) > \lambda + \bar{u}$, we again conclude that the unobservable frontier $x_2(t)$ lies above $\bar{x}(t)$.

Finally, consider the equation defining the no effort frontier $x_3(t)$,

$$(k - \alpha)e^{-x_3-\lambda(T-t)} - \alpha - \int_{x_3}^{x_3+\lambda(T-t)} e^{-x}\left(\frac{1}{1 + e^x} - \frac{v - \alpha\lambda}{\lambda}\right)dx = 0. \qquad (33)$$

Totally differentiating with respect to t shows that $x_3'(t) < \lambda$ (might be negative). Therefore, the no effort region does not intersect the full effort region defined by $\bar{x}(t)$ in the observable case.

(ii) To compare the effort regions in the unobservable case and the full effort region in the social optimum, consider the planner's frontier $x_P(t)$, which is given by

$$x_P(t) = \ln((1 + k - v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)) - \ln \alpha.$$

The slope of the planner's frontier is given by

$$x'_P(t) = \lambda \frac{(1 + k - v/\lambda)e^{-\lambda(T-t)}}{(1 + k - v/\lambda)e^{-\lambda(T-t)} + v/\lambda - \alpha} \in [0, \lambda].$$

In the equilibrium with unobservable effort, all effort ceases above the frontier $x_3(t)$ defined in (33) above, which has slope

$$x'_3(t) = \lambda \frac{((1 + e^{x_3 + \lambda(T-t)})^{-1} + k - v/\lambda)e^{-\lambda(T-t)}}{((1 + e^{x_3 + \lambda(T-t)})^{-1} + k - v/\lambda)e^{-\lambda(T-t)} + v/\lambda - \alpha - (1 + e^{x_3})^{-1}}.$$

We also know $x_3(T) = x^*$ and $x_P(T) = \ln((1 + k - \alpha)/\alpha) > x^*$. Now suppose toward a contradiction that the two frontiers crossed at a point (t, x). Plugging in the expression for $x_P(t)$ in both slopes, we obtain

$$x'_3(t) = \left(1 + \frac{v/\lambda - \alpha - s(t)}{(1 + k - v/\lambda + (1 - s(t)))e^{-\lambda(T-t)}}\right)^{-1}$$

$$> \left(1 + \frac{v/\lambda - \alpha}{(1 + k - v/\lambda)e^{-\lambda(T-t)}}\right)^{-1} = x'_P(t),$$

with

$$s(t) = 1/(1 + e^{x_P(t)}) \in [0, 1],$$

meaning the unobservable frontier would have to cross from below, a contradiction. □

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