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One-dimensional mechanism design

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We prove a general possibility result for collective decision problems where individual allocations are one-dimensional, preferences are single-peaked (strictly convex), and feasible allocation profiles cover a closed convex set. Special cases include the celebrated median voter theorem (Black 1948, Dummett and Farquharson 1961) and the division of a nondisposable commodity by the uniform rationing rule (Sprumont 1991).

We construct a canonical peak-only rule that equalizes, in the leximin sense, individual gains from an arbitrary benchmark allocation: it is efficient, group-strategyproof, fair, and (for most problems) continuous. These properties leave room for many other rules, except for symmetric nondisposable division problems.

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JEL classification. D63, D71, D82.

1. Introduction and the punchline

Single-peaked preferences played an important role in the birth of social choice theory and mechanism design. Black observed in 1948 that the majority relation is transitive when candidates are aligned and preferences are single-peaked (Black 1948): this result inspired Arrow to develop the social choice approach with arbitrary preferences. Dummett and Farquharson noted in 1961 that the median peak (the majority winner) defines an incentive compatible voting rule (Dummett and Farquharson 1961); they also conjectured that no voting rule is incentive compatible under general preferences, which was proven true 12 years later by Gibbard and by Satterthwaite (Gibbard 1973, Satterthwaite 1975).

Two decades and many more impossibility theorems later, single-peaked preferences reappeared in the problem of allocating a single *nondisposable* commodity (e.g., a workload) when the aggregate demand may be above or below the amount to be divided. Inspired by Benassy's earlier observation (Benassy 1982) that uniform rationing of a single commodity prevents the strategic inflation of individual demands, Sprumont

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(1991) characterized the uniform rationing rule by combining the three perennial goals of prior-free mechanism design: efficiency, strategyproofness, and fairness.

This striking "if and only if" result is almost alone of its kind in the literature on private goods allocation mechanisms (briefly reviewed in Section 3). By contrast, in the voting model with single-peaked preferences over a line, there are *many* efficient, strategyproof, and fair voting rules, known as the generalized median rules (Moulin 1980).

We define a family of collective decision problems encompassing voting, nondisposable division, its many variants and extensions (see Section 3), and much more. Each participant is interested in a one-dimensional "personal" allocation, his/her preferences are single-peaked (strictly convex) over this allocation, and some abstract constraints limit the set X of feasible allocation profiles. The feasible set X is a line in the voting model, and a simplex in the nondisposable division model; in general it is any closed convex set. Our main result is that we can always design allocations rules that are efficient, group-strategyproof (hence prior-free incentive compatible), and fair. Loosely speaking, in convex economies where each agent consumes a single commodity, the mechanism designer hits no impossibility wall.

The proof constructs such a rule (in fact, a family of them) with the help of the leximin ordering, an important concept in post-Rawls welfare economics. Recall that the welfare profile w beats profile w' for this ordering if the smallest coordinate is larger in w than in w', or when these are equal, if the second smallest coordinate is larger in w than in w', and so on. In our model we fix a benchmark allocation ω that is fair in the sense that it respects the symmetries of the set of feasible allocation profiles. Then we equalize, as much as permitted by feasibility, individual benefits away from ω in the direction of individual peaks: that is, the profile of actual benefits maximizes the leximin ordering. In addition to meeting the three basic goals, the corresponding rule is continuous in the profile of peaks for almost all shapes of X (more on this in the next section). We call it the uniform gains rule to stress its similarity with the uniform rationing rule. Indeed in the nondisposable division problem, the two rules coincide.

A natural question is to describe the entire set of rules efficient, group-strategyproof, and fair. We do not answer this question for general feasible sets, but we have some precise answers when the set X is fully symmetric, i.e., invariant by any permutation of the agents. In that case the dimension of X can only be 1, $n-1$, or n (where n is the number of agents). Loosely put, our results are as follows.

If X is of dimension 1, we have a voting problem, where the rules in question are the generalized median rules, described by $n-1$ free parameters.

If X is of dimension $n-1$, the sum of individual allocations is constant, which generalizes nondisposable division. The only symmetric allocation ω is equal split: the corresponding uniform gains rule generalizes Sprumont's uniform rationing rule, and remains the only rule efficient, group-strategyproof, and fair.

If X is of dimension n, the set of such rules is of infinite dimension (except for $n=2$), so the mechanism designer faces an embarrassment of riches.

2. Overview of the results

After reviewing the literature in Section 3, we define the model in Section 4. A one-dimensional problem consists of the set N of agents and the set X of feasible allocation profiles: it is a closed convex subset of $\mathbb{R}^N$. Agent i has single-peaked preferences over the projection X_i of X onto his coordinate.

Two standard notions of prior-free incentive compatibility are defined in Section 5: strategyproofness (SP) prevents individual strategic misreport, while strong group-strategyproofness (SGSP) rules out coordinated moves by a group of agents and guarantees nonbossiness to boot. Under single-peaked preferences we expect a group-strategyproof revelation rule to be also peak-only: it only elicits individual peak allocations and ignores preferences across the peak. This is true in our general model provided the rule is continuous in the reports (Lemma 1).

The well known *fixed priority* mechanisms are, as usual, both efficient and SGSP. Therefore, the point of our main result is to provide a *fair* mechanism that achieves these properties. We define three fairness requirements in Section 6. *Symmetry* (horizontal equity) says that the rule must respect the symmetries between agents: if a permutation σ of the agents leaves X invariant, then relabeling agents according to σ will simply permute their allocations. Next, *no envy* says that if X is invariant by permuting i and j, then i weakly prefers her own allocation x_i to j's allocation x_j. Finally, given any benchmark allocation ω in X, the *ω-guarantee property* requires each agent i to weakly prefer her allocation x_i to ω_i. As long as ω respects the symmetries of X, all three requirements are compatible.

We state the main result in Section 7. Given any symmetric allocation ω in X, we define the uniform-gains rule f^ω that selects the allocation in X where the profile of gains from ω_i toward the peak p_i maximizes the leximin ordering. This peak-only direct revelation mechanism is efficient, SGSP, symmetric, envy-free, and guarantees ω; it is also continuous if X is either a polytope (a finite intersection of half-spaces) or strictly convex and of dimension n.

Sections 8 and 9 provide some insights into the structure of the set of efficient symmetric rules meeting SGSP: we call such rules *focal*. In particular, we ask whether the set of rules f^ω uncovered in the Theorem 1 exhausts all focal rules. With the exception of two-person problems and of a family of problems generalizing Sprumont's model, the answer is "no."

In Section 8 we consider *fully symmetric* problems, i.e., such that X is invariant by all permutations of the agents. This implies X is of dimension 1, $n-1$, or n.

Voting problems are those where X is of dimension 1. The uniform gains rule f^ω is but one of many more generalized median rules,[1] i.e., the most strongly biased in favor of the status quo outcome ω: so as to elect another outcome, *all* individual peaks must be to the right of ω (or all to its left), and then the rule selects the peak closest to ω (Proposition 1).

[1]Such a rule is defined by $n-1$ arbitrary fixed ballots: it selects the median of the n "live" plus the $n-1$ fixed ballots. In the rule f^ω the fixed ballots are $n-1$ copies of ω.

When X is symmetric and of dimension $n-1$ the sum $\sum_N x_i$ must be constant so we can interpret X as a generalized division problem: the original nondisposable division model is the instance where the only additional constraint is the nonnegativity of shares. There is only one symmetric allocation ω, and the uniform gains rule f^ω is the *unique* continuous focal rule (Proposition 2). This result applies to a much larger class of problems than Sprumont's characterization (Sprumont 1991, Ching 1994). However, it requires more properties: SGSP in lieu of SP, and continuity.

If X is of dimension n, the set of focal rules is of infinite dimension if $n \geq 3$, but if $n = 2$, it coincides with the one-dimensional family f^ω parametrized by ω (Proposition 3).

Finally Section 9 explains why, when the set X of feasible allocations is not fully symmetric, we expect the set of focal rules (they must still respect the partial symmetries of X) to be extremely large even if X is of dimension 2. We use a very simple three-person workload division problem to make this point. Workers $i = 1, 2$ each bring some amount x_i of input, and worker 3 must process the total output; the feasibility constraint is $x_3 = x_1 + x_2$. Symmetry rules out discrimination between workers 1 and 2, but it imposes no restriction to the relative treatment of 3 versus 1 and 2. We describe three quite different subfamilies of focal rules, corresponding to sharply different power distributions between the players. Even in such a simple problem, the full menu of focal rules is worthy of further research.

After some concluding comments in Section 10, Section 11 collects the proofs of the Theorem 1 and Propositions 2 and 3.

3. Related literature

There is a folk impossibility result about the design of (prior-free) strategyproof mechanisms: in economies where agents consume two or more commodities, a strategyproof mechanism must be either inefficient, grossly unfair, or both. To mention only a few salient contributions to this theme, Hurwicz conjectured (Hurwicz 1972) and then Zhou proved (Zhou 1991) that the strategyproof and efficient allocation of private goods cannot guarantee "voluntary trade" (everyone weakly improves upon his initial endowment ω_i: see the ω-guarantee axiom in Section 6); it cannot treat agents symmetrically either (Serizawa 2002). In abstract quasi-linear economies, no strategyproof mechanism can be efficient (Green and Laffont 1977); ditto in public good economies (Barberà and Jackson 1994). And the related, more general, concept of ex post implementation hits similar impossibility walls when individual allocations are of dimension 2 or more (Jehiel et al. 2006).

By contrast we show a broad possibility result in economies where each agent consumes a unique divisible commodity, possibly a different commodity for different agents.

After the Gibbard–Satterthwaite theorem, a substantial literature on voting rules looked for restrictions to the domain of preferences eschewing the impossibility. The single-peaked domain was extended in a variety of ways. If outcomes are arranged on a tree, the Condorcet winner still defines a focal voting rule (Demange 1982). If outcomes are a product of lines, there is a natural extension of single-peakedness in which coordinate-wise majority still yields a strategyproof and symmetric rule, though efficiency is replaced by the much weaker *unanimity* property[2] (Barberà et al. 1991, 1993, 1997b), another instance of the "no rule is perfect in dimension 2 or more" result. Trees and products of lines are special cases of abstract convex sets, where we have a general characterization of strategyproof rules (Nehring and Puppe 2007a, 2007b).

Still in the voting context, recent results provide an endogenous characterization of (a generalization of) single-peaked domains by the fact that we can find strategyproof peak-only voting rules that are symmetric and unanimous (Bogomolnaia 1998, Chatterji et al. 2013, Chatterji and Massó 2015).

Following Sprumont's result, the nondisposable division problem received much attention as well. On the one hand, if viewed as a fair division method, it can be axiomatized in a variety of ways without invoking its incentive compatibility properties; see, for instance, Schummer and Thomson (1997), Thomson (1994), and Thomson (1997). On the other hand, it can be adapted and generalized to a variety of alternative models; for instance, to the random distribution of indivisible units (Hatsumi and Serizawa 2009) or the balancing of supply and demand in one-dimensional economies (Klaus et al. 1998). A good survey of the literature on strategyproof voting and nondisposable division rules up to 2001 is Barberà (2001).

More recently the rationing model has been extended to allow multiple resources and bipartite constraints (Bochet et al. 2013), and so has the supply and demand balancing model (Bochet et al. 2012, Chandramouli and Sethuraman 2011).

If we drop the fairness requirement in Sprumont's nondisposable division problem, there is an infinite dimensional set of efficient and strategyproof division rules: Barberà et al. (1997a), Moulin (1999), Ehlers (2002). See also the discussion of asymmetric rules in the bipartite rationing (Flores-Szwagrzak 2017) and supply–demand (Flores-Szwagrzak 2012) models. The same is true in our general model. However the strength of our Proposition 3 is that we find an infinite dimensional set of fair (symmetric) rules even when the feasible set is fully symmetric.

In modern welfare economics the leximin ordering was introduced by Sen (Sen 1970) as a tool to implement Rawls' egalitarian program. Maximizing this ordering is sometimes called *practical egalitarianism*, as it guarantees efficiency while deviating as little as possible from the ideal of full equality of welfares. This ordering was axiomatized first as a social welfare ordering (Hammond 1976, d'Aspremont and Gevers 1977) and then as an axiomatic bargaining solution (Imai 1983, Thomson and Lensberg 1989, Chun and Peters 1989). It also plays a key role in the recent design of good mechanisms for two problems: the assignment of objects when preferences are dichotomous[3] (Bogomolnaia and Moulin 2004), and the fair division of multiple divisible commodities when all agents have Leontief preferences (Ghodsi et al. 2011, Li and Xue 2013). See also the generalization of these two results in Kurokawa et al. (2015).

[2] Outcome x is elected if it is the peak of all voters.

[3] Each agent wants at most one indivisible object and partitions objects into two indifference classes; allocations are random.

4. The model and some examples

The finite set of agents is N and $n = |N|$. An allocation profile is $x = (x_i)_{i \in N} \in \mathbb{R}^N$. The set of feasible allocations is a *closed* subset X of $\mathbb{R}^N$. The projection X_i of X on the ith coordinate is the set of agent i's feasible allocations; the cartesian product of these closed sets is $X_N = \prod_{i \in N} X_i$.

Agent i's preferences $\succeq_i$ are single-peaked over X_i if (i) there is some $p_i \in X_i$—the peak—that $\succeq_i$ ranks strictly above any other, (ii) $\succeq_i$ increases strictly with x_i on $X_i \cap]-\infty, p_i]$ and decreases strictly on $X_i \cap [p_i, +\infty[$, and (iii) $\succeq_i$ is continuous. Note that in all our results the set X_i is convex, and in that case single-peakedness simply means that $\succeq_i$ is strictly convex and continuous.

We write $\mathcal{SP}(X_i)$ for the set of such preferences, and write the domain of preferences profiles as $\mathcal{SP}(X_N) = \prod_{i \in N} \mathcal{SP}(X_i)$. A preference profile is $\succeq = (\succeq_i)_{i \in N} \in \mathcal{SP}(X_N)$, and $p = (p_i)_{i \in N} \in X_N$ is a profile of individual peaks.

Definition 1. A one-dimensional allocation problem is a triple $(N, X, \succeq)$ where X is closed in $\mathbb{R}^N$ and $\succeq \in \mathcal{SP}(X_N)$.

Definition 2. Fixing the pair (N, X), a rule (aka a revelation mechanism) is a (single-valued) mapping F choosing a feasible allocation for each allocation problem

$$F : \mathcal{SP}(X_N) \to X \quad \text{written as } F(\succeq) = x.$$

A rule F is peak-only if it is described by a (single-valued) mapping

$$f : X_N \to X \quad \text{written as } f(p) = x$$

such that for all $\succeq \in \mathcal{SP}(X_N)$ with profile of peaks $p \in X_N$ we have $F(\succeq) = f(p)$.

A peak-only rule is a particularly simple direct revelation mechanism because participants only need to report their peak, so an agent does not even need to figure out how she compares allocations across her peak.

Example 1 (Voting). Here the feasible set is a closed interval of the diagonal $\Delta = \{x \in \mathbb{R}^N | x_i = x_j \text{ for all } i, j \in N\}$. $\lozenge$

Example 2 (Nondisposable division (Sprumont 1991)). The feasible set is the simplex $X = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } \sum_{i \in N} x_i = 1\}$. $\lozenge$

Example 2* (Bipartite rationing (Bochet et al. 2013, Flores-Szwagrzak 2017)). Here we have a set A of partially heterogenous resources and we must distribute the amount r_a of resource a among agents in N. Compatibility constraints prevent some agents from consuming certain resources: for instance, a is a type of job requiring certain skills and agent i's skills allow him to do only some of the jobs (see Bochet et al. 2013 for more examples). Formally agent i can only consume a subset $\theta(i)$ of the resources (and each resource can be consumed by at least one agent). If y_{ia} is how much i consumes of resource a, the feasibility constraints are

$$y_{ia} > 0 \implies a \in \theta(i) \quad \text{and} \quad \sum_i y_{ia} = r_a \quad \text{for all } a. \tag{1}$$

All resources that agent i can consume are perfect substitutes for her: she cares only about her total share $x_i = \sum_a y_{ia}$, over which her preferences are single-peaked. The constraints (1) generate a convex compact set of matrices $[y_{ia}]$, so the corresponding set of vectors $(x_i)_{i\in N}$ covers a convex compact $X \subset \mathbb{R}^N$. ◊

Example 3 (Balancing demand and supply). This is the problem, closely related to Example 2, where each agent i can be a supplier or a demander of the nondisposable commodity. Normalizing initial endowments at zero and ignoring bankruptcy constraints, we get the feasible set $X = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = 0\}$. If $p_i < 0$ (resp. $p_i > 0$), agent i wishes to be a net supplier (resp. demander) of the commodity. Here the familiar voluntary trade requirement corresponds to the ω-guarantee axiom below where $\omega = 0$ is the no-trade outcome. ◊

Example 3* (Bipartite demand–supply (Bochet et al. 2012, Flores-Szwagrzak 2012)). This is a variant of Example 3 where transfers between two given agents may or may not be feasible, and such constraints are described by an arbitrary graph with agents on the vertices. We omit the formal description for brevity. ◊

Example 4 (Bilateral workload). We have a fixed partition of N as $L \cup R$, and we set $X = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } \sum_{i\in L} x_i = \sum_{j\in R} x_j\}$. We think of two teams L and R who choose individual workloads x_k and must coordinate the total workload across the two teams (as in a production chain where R is upstream of L). If R consists of a single "manager," this is a moneyless version of the principal–agent problem, where the principal wishes to adjust total output to his own target level, while the workers' individual targets should also be taken into account (the manager is no dictator). This modifies Example 3 because the role of agents as suppliers or demanders is fixed exogenously; moreover, voluntariness of trade is not assumed. As a result, we show in Section 9 that the set of focal rules becomes much larger. ◊

Our last example is one where the feasible set is of dimension n.

Example 5 (Location). Initially the agents live at 0; they wish to locate somewhere on the real line. The stand alone cost of moving agent i to location x_i is x_i^2, and in addition there are externalities—positive or negative—to locate x_i near x_j. The agents share a total relocation budget of 1. Formally,

$$x \in X \quad \overset{\text{def}}{\iff} \quad \sum_{i\in N} x_i^2 - \pi \sum_{i,j\in N} x_i x_j \leq 1,$$

where $-2/(n-1) < \pi < 2$ ensures that X is convex and compact. ◊

The externality factor π is positive if there are economies of scale in building two nearby homes; it is negative if two nearby homes must be isolated from one another, e.g., for privacy.

If $\pi = 0$, we interpret 0 as the default level of the parameter x_i, which is costly to adjust up or down, and there is a cap on total expense: think of temperature in a row of offices, or the carbon dioxide (CO_2) emissions of the different plants of the firm.

5. Efficiency and incentives

Definition 3. The rule F at (N, X) is *efficient* (EFF) if for any $\succeq \in \mathcal{SP}(X_N)$, the allocation $x = F(\succeq)$ is Pareto optimal at $\succeq$.[4]

The rule F at (N, X) is *continuous* (CONT) if F is continuous for the topology of the closed convergence[5] in $\mathcal{SP}(X_N)$.

We let the reader check that if F is peak-only and represented by f, it is continuous if and only if f is continuous in $\mathbb{R}^N$.

Next we define three increasingly more demanding versions of incentive compatibility. Fixing (N, X), a profile of preferences $\succeq \in \mathcal{SP}(X_N)$, and a coalition $M \subseteq N$, we say that M *can misreport* at $\succeq$ if there is some $\succeq'_{[M]} \stackrel{\text{def}}{=} (\succeq'_i)_{i \in M} \in \mathcal{SP}(X_M)$ such that $x'_i \succ_i x_i$ for all $i \in M$, where $x = F(\succeq)$ and $x' = F(\succeq'_{[M]}, \succeq_{[N \setminus M]})$. We say that M *can weakly misreport* at $\succeq$ if under the same premises we have $x'_i \succeq_i x_i$ for all $i \in M$ and at least one is a strict preference.

Definition 4. The rule F is *strategyproof* (SP) if no single agent can misreport at any profile in $\mathcal{SP}(X_N)$.

The rule F is *group-strategyproof* (GSP) if no coalition can misreport at any profile in $\mathcal{SP}(X_N)$.

The rule F is *strongly group-strategyproof* (SGSP) if no coalition can weakly misreport at any profile in $\mathcal{SP}(X_N)$.

In general GSP (or SGSP) is considerably stronger than SP, the voting problem being an exception.[6] We recall two well known facts that are useful below.

Lemma 1. *Fix (N, X) and a rule F at (N, X) strongly group-strategyproof and continuous. Then F is peak-only; moreover, the mapping $p \to f(p)$ representing F is weakly increasing and "uncompromising": for all $p \in X_N$ and all $i \in N$,*

$$f_i(p) = x_i < p_i \quad (\textit{resp. } x_i > p_i) \quad \Longrightarrow$$
$$f(p'_i, p_{-i}) = f(p) \quad \textit{for all } p'_i \geq x_i \quad (\textit{resp. } p'_i \leq x_i).$$

[4] There is no $y \in X$ such that $y_i \succeq_i x_i$ for all i, with at least one strict preference.

[5] See Hildenbrand (1974) Section 1.2, p. 96.

[6] See Barberà et al. (2010) for a detailed discussion of the connections between the two concepts in domains more general than single-peaked.

Proof. For peak-onlyness we fix $i \in N$ and $\succeq_{[N \setminus i]} \in \mathcal{SP}(X_{N \setminus i})$. We assume $\succeq_i^1, \succeq_i^2 \in \mathcal{SP}(X_i)$ have the same peak p_i but $x_i^1 = F_i(\succeq_i^1, \succeq_{N \setminus i}) \neq x_i^2 = F_i(\succeq_i^2, \succeq_{N \setminus i})$ and derive a contradiction. By SP the peak p_i must be strictly between x_i^1 and x_i^2, else agent i can misreport at one of $(\succeq_i^1, \succeq_{[N \setminus i]})$ or $(\succeq_i^2, \succeq_{[N \setminus i]})$. Now CONT implies that the range of $\succeq_i \longrightarrow x_i = F_i(\succeq_i, \succeq_{[N \setminus i]})$ is connected so it contains p_i and this yields a profitable misreport at both $(\succeq_i^1, \succeq_{[N \setminus i]})$ and $(\succeq_i^2, \succeq_{[N \setminus i]})$. We have shown $F_i(\succeq_i^1, \succeq_{N \setminus i}) = F_i(\succeq_i^2, \succeq_{N \setminus i})$, i.e., an agent's allocation depends only upon her own reported peak.

Now assume $F_j(\succeq_i^1, \succeq_{N \setminus i}) = x_j^1 \neq x_j^2 = F_j(\succeq_i^2, \succeq_{N \setminus i})$ for some $j \neq i$: by the previous argument and SGSP, agent j is indifferent between these two allocations; therefore, the peak p_j is in $]x_j^1, x_j^2[$. Now we can move continuously from $\succeq_i^1$ to $\succeq_i^2$ while keeping the same peak p_i; the range of x_j contains p_j so that coalition $\{i, j\}$ can weakly misreport at $(\succeq_i^1, \succeq_{N \setminus i})$ (and at $(\succeq_i^2, \succeq_{N \setminus i})$). This is a contradiction of SGSP so we conclude $F(\succeq_i^1, \succeq_{N \setminus i}) = F(\succeq_i^2, \succeq_{N \setminus i})$. Peak-onlyness is now clear.

Next to uncompromisingness. The standard proof that $f_i(p_i', p_{-i}) = f_i(p)$ under the premises of the implication is omitted for brevity. Just as above, we go from there to $f(p_i', p_{-i}) = f(p)$ by SGSP. □

It is a folk result that a *fixed priority* rule (also called *serial dictatorship*) is both efficient and group-strategyproof. In our model define the *slice* of X at $\widetilde{x}_{[M]}$ as $X[\widetilde{x}_{[M]}] = \{x_{[N \setminus M]} \in \mathbb{R}^{N \setminus M} | (\widetilde{x}_{[M]}, x_{[N \setminus M]}) \in X\}$: it is closed and possibly empty. Given the priority ordering $1, 2, \ldots,$ the rule gives peak p_1 to agent 1 (this is feasible by definition of X_1), then gives to agent 2 his best allocation x_2 in (the projection on the second coordinate of) $X[p_1]$, then next gives to agent 3 her best allocation x_3 in (the projection on the third coordinate of) $X[(p_1, x_2)]$, and so on. If X is convex each step is well defined as we maximize a single-peaked preference in a closed real interval. The rule is peak-only, efficient, and strongly group-strategyproof (instead of just GSP). It is continuous as well if X is either a polytope or strictly convex and of full dimension (the proof is similar to that in Steps 8, 9, and 10 in Section 11.1).[7]

The strength of our Theorem 1 is to achieve all the properties in Definitions 3 and 4 by a rule treating the participants fairly.

6. Fairness

In our model the feasible set X may not treat all agents symmetrically, so the familiar horizontal equity property needs to be adjusted with the help of a few definitions.

Let $S(N)$ be the set of all permutations σ of N. Permuting coordinates according to σ changes x to $x^\sigma = (x_{\sigma(i)})_{i \in N}$ and $\succeq$ to $\succeq^\sigma = (\succeq_{\sigma(i)})_{i \in N}$. We call $\sigma \in S(N)$ a *symmetry of* X if $X^\sigma = X$, and write their set $S(N, X)$. We call ω a *symmetric element* of X if $\omega \in X$ and $\omega^\sigma = \omega$ for all $\sigma \in S(N, X)$. We say that X is *fully symmetric* if $S(N, X) = S(N)$.

In Examples 1, 2, 3, and 5, the set X is fully symmetric; in Example 4, $S(N, Z)$ contains the permutations leaving both L and R unchanged, but not those swapping agents

[7] We can of course define the mechanism when X is not convex: it retains the properties EFF and GSP, but is not necessarily SGSP, peak-only, or continuous.

between the two groups. Similarly in Examples 2* and 3*, the set $S(N, X)$ corresponds to the symmetries of the bipartite graph of compatibilities.

Of special interest are the simple permutations τ_{ij} exchanging i and j while leaving all other coordinates constant. If τ_{ij} is a symmetry of X, we think of agents i and j as having identical opportunities in X, and the no envy test where i compare his allocation to j's allocation is meaningful.

Definition 5. Given (N, X), the rule F meets *symmetry* (SYM) if we have $F(\succeq) = x \Longrightarrow F(\succeq^{\sigma}) = x^{\sigma}$ for every $\sigma \in S(N, X)$.

Given (N, X), the rule F meets *no envy* (NE) if whenever $\tau_{ij} \in S(N, X)$ and $F(\succeq) = x$ we have $x_i \succeq_i x_j$.

Given an allocation $\omega \in X$, the rule F meets ω-*guaranteed* (ω-G) if $F(\succeq) = x$ implies $x_i \succeq_i \omega_i$ for all i.

As in axiomatic bargaining, the ω-G property views ω as a default option (e.g., status quo ante) that each agent can revert to.

The three fairness axioms are not logically connected to one another. They have most bite when the problem (N, X) is fully symmetric: all agents have the same feasible set X_i and no envy applies to every pair of agents.

The affine subspace $H[X]$ spanned by X is the set of vectors $\lambda x + (1 - \lambda)y$, where $x, y \in X$ and $\lambda \in \mathbb{R}$. If X is fully symmetric, so is $H[X]$, and if X is not a singleton, $H[X]$ is of dimension at least 1. It is easy to check that there are only three types of fully symmetric affine subspaces:

- The (one-dimensional) diagonal D of $\mathbb{R}^N$ (Example 1).
- A $(n - 1)$-dimensional subspace orthogonal to D (Examples 2 and 3).
- The full space $\mathbb{R}^N$ (Example 5).

As usual the dimension of X is defined as that of $H[X]$.

7. Main result: The uniform gains rules

We pick an arbitrary ω in X, not necessarily symmetric. So as to define the uniform gains rule f^{ω}, we need a couple of definitions and some notation.

Recall that the *leximin* ordering $\succeq_{\text{lxmin}}$ of $\mathbb{R}^N$ is a symmetric version of the lexicographic ordering $\succeq_{\text{lexic}}$ of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^N$ we set

$$x \succeq_{\text{lxmin}} y \quad \overset{\text{def}}{\Longleftrightarrow} \quad x^* \succeq_{\text{lexic}} y^*, \tag{2}$$

where $x^* \in \mathbb{R}^n$ has the same set of coordinates as x (including possible repetitions) rearranged increasingly: $\min_N x_i = x^{*1} \leq x^{*2} \leq \cdots \leq x^{*n} = \max_N x_i$. It is well known that $\succeq_{\text{lxmin}}$ is an ordering (complete, transitive) of $\mathbb{R}^N$ with convex upper contours, but is discontinuous and cannot be represented by a utility function. Over a compact set its maximum always exists but may not be unique; however, its maximum over a *convex* compact set is unique.[8]

[8]See Lemma 1.1 in Moulin (1988). The simple argument is in Step 1 in Section 11.1.

In $\mathbb{R}^N$ we use the notation $[a, b] \stackrel{\text{def}}{=} \{x | \min\{a_i, b_i\} \leq x_i \leq \max\{a_i, b_i\} \text{ for all } i\}$ and $|a| = (|a_i|)_{i \in N}$. Given a profile of peaks p, the rule f^ω chooses an allocation x in $[\omega, p]$. The vector $|x - \omega|$ is the profile of gains from the benchmark ω, when we measure each ordinal welfare gain as $|x_i - \omega_i|$. The rule equalizes gains across agents as much as permitted by feasibility,

$$f^\omega(p) = x \stackrel{\text{def}}{\Longleftrightarrow} \Big\{x \in X \cap [\omega, p] \text{ and } |x - \omega| = \arg \max_{\Delta(\omega, p)} \succeq_{\text{lxmin}}\Big\}, \tag{3}$$

where

$$z \in \Delta(\omega, p) \stackrel{\text{def}}{\Longleftrightarrow} \big\{z = |x - \omega| \text{ for some } x \in X \cap [\omega, p]\big\}.$$

The allocation $f^\omega(p)$ is well defined because $\Delta(\omega, p)$ is convex and compact, so the maximum of $\succeq_{\text{lxmin}}$ exists and is unique. We write $\mathcal{F} = \{f^\omega\}$ for the set of rules thus constructed.

Theorem 1. *Fix (N, X) and a symmetric allocation ω in X. If X is closed and convex in $\mathbb{R}^N$ the (peak-only) uniform gains rule $f^\omega \in \mathcal{F}$ is efficient, symmetric, envy-free, strongly group-strategyproof, and ω-guaranteed.*

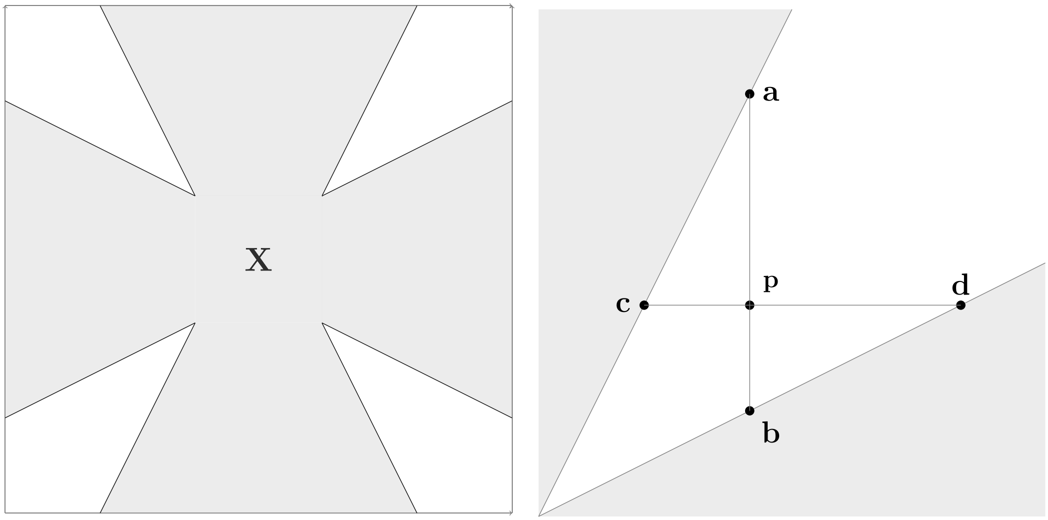
This rule is also continuous if $n = 2$ or if $n \geq 3$ and either X is a polytope, or X is strictly convex and of dimension n.

The proof is given in Section 11.1. There we show first that for any choice of ω, symmetric or not, f^ω meets EFF and SGSP, and obviously ω-G. It is then easy to check SYM when ω is symmetric in X, and NE when τ_{ij} is a symmetry of X. The proof of continuity when X is a polytope or is strictly convex and full dimensional takes more work: see Steps 8 and 9 of the proof. In Step 10 we also provide an example where X is convex and f^ω is discontinuous. Note that Theorem 1 implies that f^ω is continuous for all examples in Section 4.

The convexity of X is a sufficient condition for the existence of a focal rule (nonenvious as well), but it is by no means a necessary condition. Remark 3 in Section 8.3 gives a two-person example of a focal rule where X is not convex in $\mathbb{R}^2$.

Remark 1. Alternatively, for some nonconvex sets X even efficiency, strategyproofness, and continuity are incompatible. Figure 1 explains this in a two-person example. Assume such a rule F exists and fix a profile $\succeq = (\succeq_1, \succeq_2)$ with profile of peaks p. If agent 1 reports $\succeq_1'$ with peak c_1 instead of p_1, while agent 2 reports $\succeq_2$, then EFF implies $F(\succeq_1', \succeq_2) = c$. Thus agent 1 can achieve c_1 as well as d_1 by a similar argument. Set $F_1(\succeq) = x_1$ and assume $x_1 > p_1$: then there is a preference $\succeq_1^*$ with peak p_1 ranking c_1 above x_1. But by SP and CONT an agent's allocation depends only on her own reported peak;[9] therefore $F_1(\succeq_1^*, \succeq_2) = x_1$ while $F_1(\succeq_1', \succeq_2) = c_1$ and agent 1 can misreport. Inequality $x_1 < p_1$ is similarly impossible, so we conclude $F_1(\succeq) = p_1$. The same argument for agent 2 gives $F_2(\succeq) = p_2$ and we reach a contradiction.

[9]See the first part in the proof of Lemma 1 that only requires SP and CONT.



(a) (b)

Figure 1. Proof of Remark 1.

Remark 2. The rule f^ω measures all individual welfare gains as $|x_i - \omega_i|$. If we are not imposing either SYM or NE, we can use a different cardinalization for each i, for instance, $\lambda_i|x_i - \omega_i|$ (where $\lambda_i > 0$) and get a rule meeting EFF, SGSP, and CONT (the latter with the same qualifications as in the theorem). In the proof of Proposition 3 (Section 11.3) we use a more subtle alternative cardinalization respecting SYM and NE, where "gains" ($x_i - \omega_i > 0$) and "losses" ($x_i - \omega_i < 0$) are treated differently. This is how we show that $\mathcal{F}$ is typically much smaller than the set $\mathcal{G}$ of focal rules.

In the next section, we focus on the comparison of $\mathcal{F}$ and $\mathcal{G}$ when X is fully symmetric, hence of dimension 1, $n-1$, or n. If $\dim(X) = 1$, $\mathcal{F}$ is a one-dimensional subset of the $(n-1)$-dimensional $\mathcal{G}$ (Proposition 1). If $\dim(X) = n-1$, $\mathcal{F}$ and $\mathcal{G}$ coincide and contain a single rule (Proposition 2). If $\dim(X) = n$, $\mathcal{G}$ is of infinite dimension while $\mathcal{F}$ remains one-dimensional (Proposition 3).

8. Application to fully symmetric problems

8.1 *Voting:* $\dim(X) = 1$

This is Example 1. Let X_0 be the set of individual allocations common to all agents: a peak-only rule f is simply a mapping from X_0^N into X_0. Any allocation $\omega \in X \subseteq D$ is symmetric: $\omega_i = \omega_0 \in X_0$ for all i. To read definition (3), fix a profile of peaks $p \in X_0^N$ and some $x \in X \cap [\omega, p]$ such that $x_i = x_0$ for all i. If there are agents i, j such that $p_i \leq \omega_0 \leq p_j$, then $x = \omega$ because $x \in [\omega, p]$ implies $p_i \leq x_i \leq \omega_0 \leq x_j \leq p_j$. If $\omega_0 \leq p_i$ for all i, then $\omega_0 \leq x_0 \leq p^{*1}$ and $x_0 - \omega_0$ is maximal at $f^\omega(p) = p^{*1}$; similarly if $p_i \leq \omega_0$ for all i, we have $f^\omega(p) = p^{*n}$. We just proved the following proposition.

Proposition 1. *Given* (N, X_0) *and* $\omega_0 \in X_0$, *the uniform gains rule* f^ω *defined by (3) is*

$$f^\omega(p) = \text{median}\{p^{*1}, p^{*n}, \omega_0\}.$$

We have known for decades that a voting rule in (N, X_0) is efficient, symmetric, and strategyproof if and only if it is a *generalized median* rule (Moulin 1980, Sprumont 1995). Such a rule is defined by the choice of $(n-1)$ arbitrary parameters q_k in X_0, $1 \le k \le n-1$, interpreted as *fixed ballots*.[10] It picks the median of the fixed and the live ballots:

$$f(p) = \text{median}\{p_i, i \in N; q_k, 1 \le k \le n-1\}.$$

(It also meets SGSP and CONT.) The rule f^ω is the instance where all $n-1$ fixed ballots q_k are the status quo ω_0.

Note that if $n = 2$, every generalized median rule is also a uniform gains rule; therefore, the family of uniform gains rules $\{f^\omega | \omega_0 \in X_0\}$ is characterized by the combination of efficiency, symmetry, and strategyproofness.[11] This is no longer true for $n \ge 3$, where the set $\mathcal{G}$ of generalized median rules is of dimension $(n-1)$ while $\mathcal{F}$ is one dimensional.

8.2 *Dividing:* $\dim(X) = n-1$

Here $H[X]$ is orthogonal to the diagonal D of $\mathbb{R}^N$ and X takes the form $X = \{\sum_N x_i = \beta\} \cap C$, where β is a real number and C is convex, closed, fully symmetric, and of dimension n. Equal split is the only symmetric point in X: $\omega_i = (1/n)\beta$ for each i.

Example 6 (Nondisposable division, $X = \{x \ge 0, \sum_N x_i = 1\}$). Here f^ω is precisely Sprumont's uniform rationing rule φ, a fact that requires some explanation because the original definition in Sprumont (1991) of the rule φ is different. ◊

The key fact is that an efficient allocation must be "one-sided." Assume excess demand at p, i.e., $\sum_N p_i > 1$: then the allocation $x \in X$ is efficient *if and only if* $x_i \le p_i$ for all i. And $\varphi(p)$ is the most egalitarian efficient allocation; it is the only one in X that can be written as $\varphi_i(p) = \min\{\lambda, p_i\}$ for all i, for some parameter $\lambda \in [0, 1]$. To check $\varphi(p) = f^\omega(p)$ (where $\omega_i = 1/n$ for all i), we partition N as $N_- \cup N_+$, where $p_i \le 1/n$ in N_- and $p_i \ge 1/n$ in N_+ (assigning agents such that $p_i = 1/n$ arbitrarily). Then excess demand implies

$$\delta = \sum_{N_-}\left|p_i - \frac{1}{n}\right| \le \sum_{N_+}\left|p_i - \frac{1}{n}\right|.$$

Therefore, the maximum z of $\succeq_{\text{lxmin}}$ in $\Delta(\omega, p)$ has $z_i = |p_i - 1/n|$ in N_- and $z_j = \min\{\mu, |p_j - 1/n|\}$ in N_+ for some $\mu \ge 0$. The corresponding feasible allocation $x = f^\omega(p)$ is $x_i = \min\{\mu + 1/n, p_i\}$ for all i, and $\varphi(p) = f^\omega(p)$ follows.

[10]Note that q_k could be $\pm\infty$ if X_0 is unbounded.

[11]If we drop symmetry, the combination of efficiency, continuity, and strategyproofness characterizes a two-dimensional family described in the concluding remarks, Section 10.

Next assume excess supply: $\sum_N p_i < 1$. The allocation $x \in X$ is efficient *if and only if* $x_i \geq p_i$ for all i and the argument for $\varphi(p) = f^\omega(p)$ is similar.

This new interpretation of the uniform rule stresses the fact that an agent requesting her fair share of the resources ($p_i = 1/n$) is guaranteed to receive exactly that much.

Example 2* (Bipartite rationing). Recall that allocation x is feasible if and only if $x_i = \sum_A y_{ia}$ for some matrix of transfers $[y_{ia}]$ such that $y_{ia} > 0 \Longrightarrow a \in \theta(i)$ and $\sum_i y_{ia} = r_a$ for all a. A fully egalitarian allocation ($x_i = x_j$ for all i, j) is typically not feasible, but there is a canonical "most egalitarian" allocation ω that Lorenz dominates any other feasible allocation x: $\omega^{*1} \geq x^{*1}$, $\omega^{*1} + \omega^{*2} \geq x^{*1} + x^{*2}$, and so on.[12] Clearly ω is symmetric and f^ω is the most natural choice of a uniform gains rule. ◊

Mimicking the original definition of uniform rationing, we can also choose for each profile of peaks p the allocation $\varphi(p)$ that Lorenz dominates every other *efficient* allocation x: this rule is defined and axiomatized in Bochet et al. (2013). It turns out that φ guarantees ω as well however, unlike in the simple model of Example 6, the rules φ and f^ω are in general different.

Here is a three-person two-resource example: $N = \{A, B, C\}$, $Q = \{a, b\}$; $f(A) = f(B) = \{a\}$, $f(C) = \{a, b\}$, $r_a = 6$, $r_b = 5$. The egalitarian allocation is $\omega = (3, 3, 5)$ and it is chosen by both φ and f^ω whenever it is efficient. Now for $p = (1, 6, 11)$ the allocation x is efficient if and only if $x_A = 1$, $x_B + x_C = 10$, and $x_C \geq 5$. Then $\varphi(p) = (1, 5, 5)$ while $f^\omega(p) = (1, 4, 6)$.

Example 7 (Balancing demand and supply, $X = \{x \in \mathbb{R}^N \mid \sum_N x_i = 0\}$). Here the symmetric default allocation is $\omega = 0$ and f^0 is the well known rule that serves the short side while rationing uniformly the long side. That is, given p we let $N_+ = \{i \in N \mid p_i > 0\}$ be the set of agents with positive demand, and let $N_- = \{i \in N \mid p_i < 0\}$ be the set of those with positive supply. If $\sum_{N_+} p_i > \sum_{N_-} |p_i|$, we have excess demand, and each $i \in N_-$ (as well as any with $p_i = 0$) gets $x_i = p_i$ while agents in N_+ use the uniform rationing rule to divide $\sum_{N_-} |p_i|$. And a similar definition applies in case of excess supply. ◊

In the bipartite demand–supply model of Example 3*, the compatibility constraints ruling out transfers between certain agents complicate the description of feasible and efficient allocations: in particular, the agents who must be rationed at a given profile of peaks may contain both demanders and suppliers. But because trade must be voluntary, the default allocation is still $\omega = 0$ and the rule axiomatized in Bochet et al. (2012) equalizes the net gains of agents who must be rationed. Therefore it is precisely the rule f^0.

Our next result characterizes the uniform gains rule in all symmetric division problems.

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[12]The recursive definition of ω is as follows. Let N_1 be the largest solution of $\lambda_1 = \min_{S \subseteq N} \frac{\sum_{a \in \theta(S)} r_a}{|S|}$: then $x_i = \lambda_1$ for all $i \in N_1$; next N_2 is the largest solution of $\lambda_2 = \min_{S \subseteq N \setminus N_1} \frac{\sum_{a \in \theta(S) \setminus \theta(N_1)} r_a}{|S|}$ and $x_i = \lambda_2$ for all $i \in N_2$; and so on. See Bochet et al. (2013) for details.

Proposition 2. *Fix a fully symmetric division problem (N, X), where $X = \{\sum_N x_i = \beta\} \cap C$ and C is either a polytope or strictly convex and of dimension n. Then the uniform gains rule f^ω where $\omega_i = \beta/n$ for all i is the unique continuous focal rule (i.e., EFF, SYM, CONT, and SGSP).*

The proof is given in Section 11.2. This result is closely related—but not logically comparable—to the characterization of the uniform rationing rule in Example 2 by the combination of EFF, SYM, and SP (Sprumont 1991, Ching 1994). The proof of that result uses critically the fact that efficient allocations must be one-sided as explained in Example 2 above. One-sidedness no longer holds in a general symmetric division problem, which explains why Proposition 2 uses the stronger requirement SGSP and adds CONT.[13]

Here is an example where four partners divide 100 shares in a joint venture under the constraint that no two partners own more than 2/3 of the shares:

$$X = \left\{ x \in \mathbb{R}^4_+ \mid \sum_1^4 x_i = 100 \text{ and } x_i + x_j \leq 66 \text{ for all } i \neq j \right\}.$$

At the profile of peaks $p = (10, 15, 35, 40)$, the allocation $x = (17, 17, 30, 36)$ is efficient.

Another related result in Klaus et al. (1998) is about Example 7 discussed just before Proposition 2. The uniform gains rule f^0 is characterized by EFF, voluntary trade (i.e., 0-G), and SP: efficient allocations are one-sided so that the proof in Ching (1994) can be adapted. Proposition 2 is an alternative characterization of f^0 where voluntary trade is replaced by symmetry plus continuity, and SP is replaced by SGSP.

8.3 *Full dimension:* $\dim(X) = n$

Proposition 3. *(i) Assume $n = 2$ and the closed, convex subset X of $\mathbb{R}^N$ is symmetric and of dimension 2. The uniform gains rule f^ω with ω symmetric in X is the only continuous focal rule (i.e., EFF, SYM, CONT, and SGSP).*

(ii) Assume $n \geq 3$ and the closed, convex subset X of $\mathbb{R}^N$ is symmetric and of dimension n. The set of envy-free focal rules is of infinite dimension (while the set of symmetric uniform gains rules f^ω is of dimension 1).

We prove here statement (i) in one instance of Example 5 with two agents, and we explain in Section 11.3 how the argument applies to any full dimensional two-person problem. For statement (ii) we take again a simple three-person instance of Example 5 and construct a new one-dimensional family of continuous focal rules that are simple variants of, and different from, the uniform gains rules. The proof that we can similarly generate an infinite dimensional set of rules meeting all the required axioms is in Section 11.3.

[13] Yet a plausible conjecture is that Proposition 2 holds when SGSP is replaced by SP.

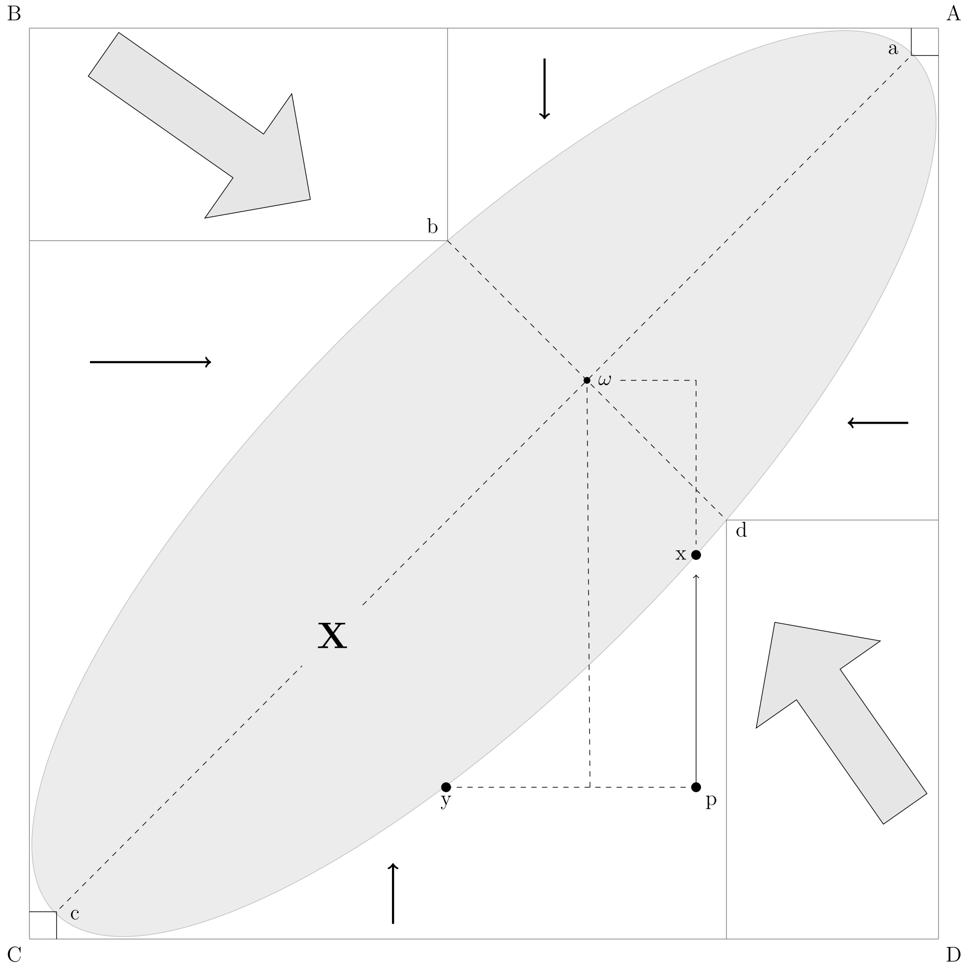

FIGURE 2. A uniform gains rule with two agents.

Statement (i) Consider the two-person problem

$$X = \left\{ x_1^2 + x_2^2 - \frac{8}{5} x_1 x_2 \leq 1 \right\}. \tag{4}$$

Figure 2 represents the feasible set X, where $X_i = [-5/3, 5/3]$ for $i = 1, 2$. Also represented are the symmetric point $\omega = (1/3, 1/3)$ and the four boundary points a, b, c, d of X critical to the construction of f^ω. By EFF we only need to describe $f^\omega(p)$ when p is outside X. Suppose p is to the Northeast (NE) of a. Outcome a is efficient at p and inside $[\omega, p]$; it also equalizes the benefits $|a_i - \omega_i|$; therefore $f^\omega(p) = a$. Similar arguments show that $f^\omega(p) = b$ for p in the Northwest (NW) of b, $f^\omega(p) = c$ if p is Southwest (SW) of c, and $f^\omega(p) = d$ if it is Southeast (SE) of d. Now take p SE of ω but SW of d shown in Figure 2: at outcome x the vector $(|x_1 - \omega_1|, |x_2 - \omega_2|) = (|p_1 - \omega_1|, |x_2 - \omega_2|)$ is leximin optimal for $x \in [\omega, p]$; thus $f^\omega(p) = x$. Thus we see that for any p outside X that is West of d, East of c, and South of ω, agent 1 gets her peak allocation and, conditional on this, x_2 is best for agent 2. Similar arguments in the three other remaining regions complete the description of f^ω.

Clearly f^ω is continuous: in fact for any two-agent problem (N, X) with X convex and closed, all rules f^ω are continuous. We omit the easy proof for brevity.

We show now that, conversely, any continuous focal rule F is precisely f^ω for some ω in the diagonal of X. The proof works by focusing on the choice of F at the four corners of X_{12} namely $A = (5/3, 5/3)$ in the NE corner, $B = (-5/3, 5/3)$ in the NW, and so on. By Lemma 1, F is peak-only so we write it as f. By EFF and SYM, we have $f(A) = a$, $f(C) = c$. Now by EFF, $f(B)$ is some point b on the NW frontier of X, and by SYM, $f(D) = d$ obtains from b by exchanging its coordinates. Call ω the intersection of the line bd and the diagonal: we show that $f = f^\omega$.

Consider first the rectangle $[B, b]$: by uncompromisingness (Lemma 1), $f(p_1, B_2) = b$ for any $p_1 \in [B_1, b_1]$: $f(p) = b$ along the top edge of $[B, b]$. Repeating this argument we see that $f(p) = b$ holds along its left edge and then inside $[B, b]$ as well. Similarly $f = f^\omega$ in the three rectangles $[A, a]$, $[C, c]$, and $[D, d]$. Now consider the point p in Figure 2 that is neither in X nor in any of these four rectangles. By EFF, $f(p) = z$ is on the frontier of X between y and x. We assume $z_1 < x_1 = p_1$ and derive a contradiction. By uncompromisingness, we get $f(5/3, p_2) = f(p) = z$; but $(5/3, p_2) \in [D, d]$ so $f(5/3, p_2) = d$, a contradiction. We conclude that f and f^ω coincide in the triangular region bordered by $[D, d]$ and the SE frontier of X. Finally we repeat this argument in the seven other triangular regions.

Statement (ii) Consider the three-person problem

$$X = \{x_1^2 + x_2^2 + x_3^2 \leq 1\}.$$

We define a variant of the uniform gains rule f^0 where we discount losses (from zero) relative to gains (from zero). For any positive λ and real number y, we set $|y|_\lambda = y$ if $y \geq 0$, $|y|_\lambda = -y/\lambda$ if $y \leq 0$, and, for $z \in \mathbb{R}^3$, $|z|_\lambda$ is the profile of $|z_i|_\lambda$, $i = 1, 2, 3$. Then we define

$$f_\lambda(p) = x \overset{\text{def}}{\iff} \Big\{x \in X \cap [0, p] \text{ and } |x|_\lambda = \arg\max_{\Delta_\lambda(p)} \succeq_{\text{lxmin}}\Big\},$$

where $\Delta_\lambda(p) = \{|x|_\lambda, \text{for } x \in X \cap [0, p]\}$. Note that f_λ is symmetric in the sense of Definition 5. Also f_1 is simply the uniform gains rule f^0. But for $\lambda \neq 1$, the rule f_λ is clearly different than f_1, and $f_\lambda(0) = 0$ implies that it is different than any rule f^ω with $\omega \neq 0$.

Remark 3. Figure 3 shows a nonconvex feasible set $X$where the same construction as in the proof of Statement (i) above delivers the rule f^ω (still defined by (3)). It goes to show that convexity is not a necessary condition for the existence of a rule meeting all properties in Theorem 1. Note that, unlike in Figure 1, all horizontal and vertical slices (cross sections) of X are convex. A challenging open question is how to characterize the geometric properties of X for which focal rules exist. The difficulty is to ensure that in (3) the leximin ordering has a unique maximum in $\Delta(\omega, p)$. For instance, take $n = 2$ and $X = ([0, 1] \times [0, 2]) \cup ([0, 2] \times [0, 1])$ (the union of two rectangles). Vertical and horizontal slices of X are all convex, yet the leximin ordering has two maxima in $\Delta(\omega, p)$ for any symmetric feasible ω and any $p_i > 1$, $i = 1, 2$.

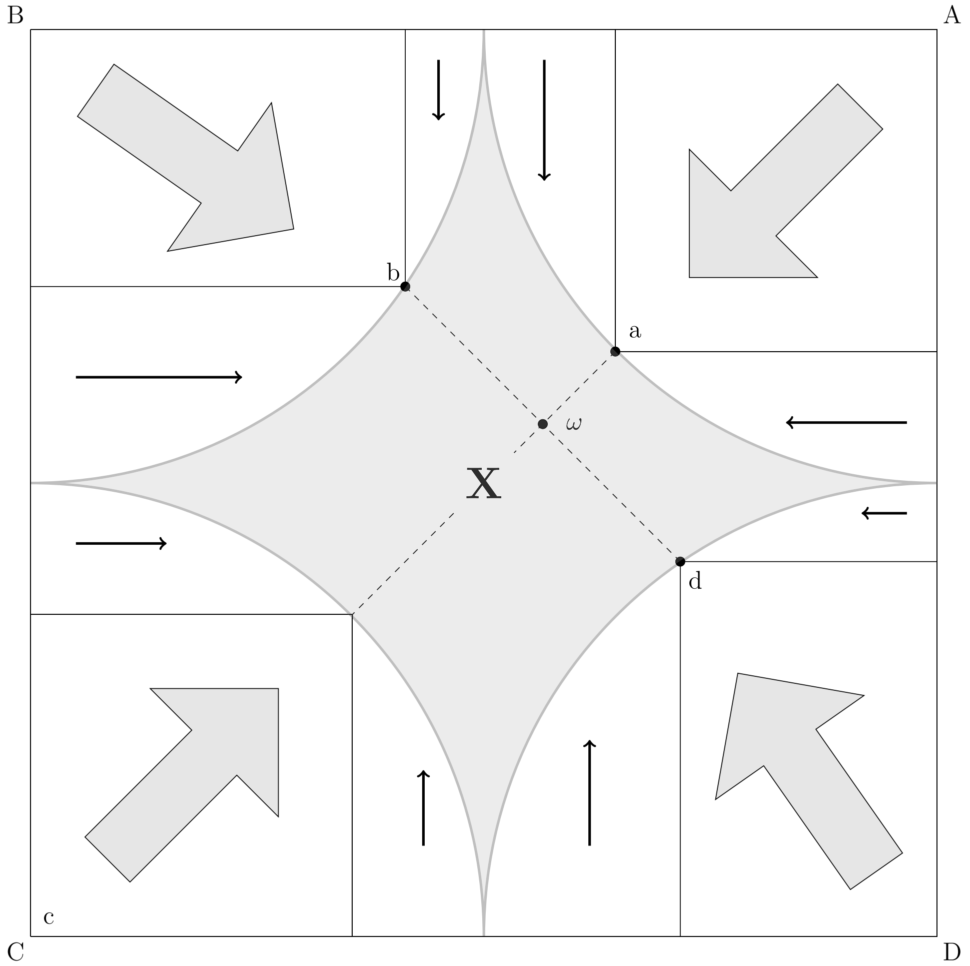

FIGURE 3. A uniform gains rule with non convex set X.

9. General problems: An embarrassment of riches

For general feasible sets X where SYM may have no bite at all, there are very few cases where can we characterize the entire set of continuous focal rules.

One well known case is when X is of dimension 1: barring trivial cases where some agents are dummies, all sets X_i are isomorphic to X and we can interpret the model as a voting problem, to which the general characterization in Moulin (1980) applies (it requires only SP in lieu of SGSP); the set of rules in question can then be of dimension as large as $2^n - 2$.

Another simple case is two-person problems, $n = 2$. Assuming for simplicity that X is compact and not symmetric in the agents, the three properties EFF, CONT, and SGSP characterize a four-dimensional family of rules, constructed by adapting the proof of statement (i) in Proposition 3. The four parameters are the values of the rule at the four corners of the rectangle X_{12}. In the typical instance (4) of Example 5, we can choose $f(A) = a'$ anywhere on the Northeast frontier of X in Figure 2, $f(B) = b'$ anywhere on its NW frontier, and so on. As in Figure 2, the rule f maps the entire rectangle $[a', A]$ to a', the rectangle $[b', B]$ to b', etc.; then the pattern of horizontal and vertical arrows is exactly as in Figure 2. Generalization to any shape of X and to unbounded sets is easy.

Now we show by an example that already for $n = 3$ and $\dim(X) = 2$, we can expect a complex and interesting set of focal rules. This makes a different point than statement (ii) in Proposition 3, where we construct a large set of focal rules that are mere variants of the canonical uniform gains rule. Here we find instead a menu of genuinely different power-sharing scenarios between the three participants.

Consider the the bilateral workload problem in Example 4, with two agents on one side and one on the other: $L = \{1, 2\}$ and $R = \{3\}$. Thus $X = \{x \in \mathbb{R}^3_+ | x_1 + x_2 = x_3\}$ is a two-dimensional polytope and the problem (N, X) treats agents 1 and 2 symmetrically. Let f be a continuous focal rule. We derive the general structure of f before describing appealing subfamilies.

Fixing p_3 for a while, consider the two-person allocation rule $(p_1, p_2) \to f_{-3}(p) = (x_1, x_2)$. It meets SGSP, SYM, and CONT. Suppose $x_1 < p_1 \le p_2$: by Lemma 1, we have $f_{-3}(p_2, p_2) = (x_1, x_2)$ so by SYM $x_1 = x_2$. Similarly if one of x_1, x_2 is outside $[p_1, p_2]$, we get $x_1 = x_2$. Next $x_1, x_2 \in]p_1, p_2[$ is ruled out by EFF because we can push each x_i toward p_i by the same small amount and keep their sum x_3 constant. Thus the only possible configurations are

$$x_1 = x_2 \notin [p_1, p_2] \quad \text{or} \quad p_i = x_i \ge x_j > p_j \quad \text{or} \quad p_i \ge x_i \ge x_j = p_j. \tag{5}$$

Let g^t be the *two-person* uniform gains rule in Proposition 2 applied to $Z(t) = \{(x_1, x_2) | x_i \ge 0 \text{ and } x_1 + x_2 = t\}$. It is extended to any profile (p_1, p_2) in $\mathbb{R}^2_+$ as $g^t(\min\{p_1, 1\}, \min\{p_2, 1\})$, which we simply write as $g^t(p_1, p_2)$. We let the reader check that for any (p_1, p_2), the allocation $(x_1, x_2) = g^t(p_1, p_2)$ is the only one in $Z(t)$ meeting (5). Therefore we can write the three-person rule f as

$$f(p) = \big(g^{x_3}(p_1, p_2), x_3\big), \quad \text{where } x_3 = f_3(p) \text{ for all } p \in \mathbb{R}^3_+.$$

In this way the real-valued function $p \to f_3(p)$ determines f entirely.

Symmetry of f means that f_3 is symmetric in p_1, p_2. Efficiency amounts to $f_3(p) \in [p_1 + p_2, p_3]$: indeed if x_3 is outside this interval and for instance $x_3 < t < p_1 + p_2, p_3$, the allocation $(g^t(p_1, p_2), t)$ Pareto dominates $(g^{x_3}(p_1, p_2), x_3)$; for fixed p_1, p_2 the mapping $p_3 \to f_3(p)$ must ensure agent 3's truthfulness, which means that it is the projection of p_3 on an interval independent of p_3. Putting these facts together we get the general form

$$f_3(p) = \text{median}\big\{p_3, J_-(p_1, p_2), J_+(p_1, p_2)\big\}, \tag{6}$$

where $J_\pm$ are symmetric, continuous functions such that

$$0 \le J_-(p_1, p_2) \le p_1 + p_2 \le J_+(p_1, p_2). \tag{7}$$

Of course SGSP imposes some further constraints on $J_\pm$. We describe three families of rules where SGSP holds, keeping in mind that they do not exhaust all focal rules f in this very simple allocation problem.

First family of focal rules

Say we want to guarantee the benchmark allocation $\omega = (\alpha, \alpha, 2\alpha) \in X$. This is a a supply–demand model similar to Example 3 between demanders 1 and 2, and supplier 3 where ω is the profile of initial endowments. Then

$$f_3(p) = \text{median}\{p_1 + p_2, p_3, 2\alpha\} \tag{8}$$

is the rule giving its peak to the short side and rationing the long side (here $J_-(p_1, p_2) = \min\{p_1 + p_2, 2\alpha\}$ and $J_+(p_1, p_2) = \max\{p_1 + p_2, 2\alpha\}$). We let the reader check the ω-G property.

The canonical rule f^ω also guarantees ω, but proves to be more complicated than the rule (8). Straightforward computations from definition (3) give the following J_- and J_+ in (6):

$$\begin{aligned} J_-(p_1, p_2) &= p_1 + p_2 \quad \text{if } 2p_1 + p_2, p_1 + 2p_2 \leq 3\alpha \\ &= \alpha + \frac{1}{2}\min\{p_1, \alpha\} + \frac{1}{2}\min\{p_2, \alpha\} \quad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} J_+(p_1, p_2) &= p_1 + p_2 \quad \text{if } 2p_1 + p_2, p_1 + 2p_2 \geq 3\alpha \\ &= \alpha + \frac{1}{2}\max\{p_1, \alpha\} + \frac{1}{2}\max\{p_2, \alpha\} \quad \text{otherwise.} \end{aligned}$$

Thus f^ω coincides with (8) if $p_1, p_2 \leq \alpha$ and if $\alpha \leq p_1, p_2$. But for instance if $p_3 < 2\alpha < p_1 + p_2$ and $p_1 < \alpha < p_2$, then $f_3(p)$ is smaller with f^ω than under rule (8), which may or may not favor agent 3 or agent 1.

Second family of focal rules

We now run a vote between the three agents to determine $f_3(p)$: thus agent $i = 1, 2$ reports $2p_i$, because if $f_3(p) = 2p_i$ the uniform rationing rule guarantees $x_i = p_i$. The simplest rule is majority voting:

$$f_3(p) = \text{median}\{2p_1, 2p_2, p_3\} = \text{median}\{2p^{*1}, 2p^{*2}, p_3\}. \tag{9}$$

More generally $p \to f_3(p)$ can be any three-person strategyproof voting rule respecting the symmetry between 1 and 2 and ensuring efficiency (7). Such rules take the form

$$f_3(p) = \text{median}\{\min\{2p^{*1}, \alpha\}, \max\{2p^{*2}, \beta\}, p_3\}$$

for some constants α, β such that $\alpha \leq \beta$. Note that agent 3 can enforce any x_3 in $[\alpha, \beta]$ while agents 1 and 2 together can only force $f_3(p)$ below β or above α.

A variant closer to the spirit of the first family is the rule

$$f_3(p) = \text{median}\{\min\{p_1 + p_2, 2\alpha\}, \max\{p_1 + p_2, 2\beta\}, p_3\}$$

with $\alpha \leq \beta$. Here agent 3 can also force x_3 anywhere in $[2\alpha, 2\beta]$, while if agent $i = 1, 2$ reports $p_i \in [\alpha, \beta]$, she guarantees only that x_i is somewhere in $[\alpha, \beta]$.

Conversely, if $\beta \leq \alpha$, then $f_3(p) = p_1 + p_2$ if $p_1 + p_2 \in [2\alpha, 2\beta]$, while the report $p_3 \in [2\alpha, 2\beta]$ only guarantees $x_3 \in [2\alpha, 2\beta]$.

Third family of focal rules

We fix $\gamma, \delta \geq 0$ and apply the general formula (6) with the functions

$$J_-(p_1, p_2) = \min\{p_1, (p_2 + \gamma)\} + \min\{(p_1 + \gamma), p_2\},$$
$$J_+(p_1, p_2) = \max\{p_1, (p_2 - \delta)\} + \max\{(p_1 - \delta), p_2\}.$$

For $\gamma = \delta = 0$, this is the simple majority rule (9). For general parameters γ, δ, the rule gives full power to agents 1 and 2 if their peaks are not too different: $f_3(p) = p_1 + p_2$ if $|p_1 - p_2| \leq \min\{\gamma, \delta\}$; but if $p_1 \geq p_2 + \max\{\gamma, \delta\}$, then $f_3(p) = \text{median}\{2p_1 - \delta, 2p_2 + \gamma, p_3\}$.

10. Conclusion

Allocation problems with one-dimensional individual allocations and single-peaked preferences allow much flexibility to the mechanism designer, even under the simultaneous constraints of efficiency, prior-free incentive compatibility, and fairness. Our results make two rather different points about the flexibility in question.

First, Theorem 1 says that Sprumont's uniform rationing rule is a template for constructing a focal rule in any one-dimensional problem provided the feasible set is convex and closed (continuity holds for many sets as well).

Second, the example developed in Section 9 shows that, as soon as X is not fully symmetric, even focal rules respecting its symmetries form a much richer set than the uniform gains rules and their variants (in statement (ii) of Proposition 3).

11. Proofs

11.1 *Main theorem*

Step 1: The leximin ordering. The *leximin* ordering $\succeq_{\text{lxmin}}$ of $\mathbb{R}^N$ is defined by (2) in Section 7. It is a *separable* ordering, which means that for any $x, y \in \mathbb{R}^N$ and any $i \in N$,

$$\{x \succeq_{\text{lxmin}} y \text{ and } x_i = y_i\} \Longrightarrow x_{-i} \succeq_{\text{lxmin}} y_{-i}$$

(where the second inequality is in $\mathbb{R}^{N \setminus i}$). Check now that $\succeq_{\text{lxmin}}$ has a unique maximum over any *convex* and compact set C of $\mathbb{R}^N$. Suppose instead that x and y are two such maximizers so that $x^{*1} = y^{*1} = a$ (recall that x^* rearranges the coordinates of x increasingly). Compare $S = \{i \in N | x_i = a\}$ with $T = \{j \in N | y_j = a\}$: if they are disjoint for all $k \in N$, we have $a \leq \min\{x_k, y_k\}$ and $a < \max\{x_k, y_k\}$ for all k; this implies $\min_{k \in N}(x_k + y_k)/2 > a$ and contradicts the optimality of x. Thus there is an agent labeled 1 in $S \cap T$ such that $x_1 = y_1 = a$. By separability, x_{-1} and y_{-1} maximize $\succeq_{\text{lxmin}}$ in the slice $C[a_{[1]}]$ and we can proceed by induction on $|N|$.

The upper contour sets of $\succeq_{\text{lxmin}}$ are convex (proof omitted) and, in particular, for all $u, v \in \mathbb{R}^N$,

$$u \succeq_{\text{lxmin}} v \quad \Longrightarrow \quad \big(\lambda u + (1-\lambda)v\big) \succeq_{\text{lxmin}} v \quad \text{for all } \lambda, 0 \le \lambda \le 1. \tag{10}$$

Throughout the rest of the proof we fix (N, X) with X convex and closed.

Step 2: Efficient allocations. Let $\mathcal{T}$ be the set of ordered partitions $\tau = (S_0, S_+, S_-)$ of N, where up to two components of τ can be empty (if all three are nonempty, τ is a partition of N). The signature $\tau = s(y)$ of $y \in \mathbb{R}^N$ is given by $S_0 = \{i \in N | y_i = 0\}$, $S_+ = \{i \in N | y_i > 0\}$, and $S_- = \{i \in N | y_i < 0\}$. We define a transitive but incomplete ordering $\trianglerighteq$ on $\mathcal{T}$ by

$$\tau^2 \trianglerighteq \tau^1 \overset{\text{def}}{\Longleftrightarrow} \{S_0^2 \supseteq S_0^1, S_+^2 \subseteq S_+^1, S_-^2 \subseteq S_-^1\}$$

and $\triangleright$ is the strict component of $\trianglerighteq$.

Fixing $\tau \in \mathcal{T}$, we define the τ-boundary of X as

$$\partial^\tau(X) = \big\{x \in X | \text{ for all } y\big\{y \neq x \text{ and } s(y-x) \trianglerighteq \tau\big\} \Longrightarrow y \notin X\big\}.$$

Lemma 2. *Fix a problem $(N, X, \succeq)$ with corresponding profile of peaks $p \in X_N$. If $p \in X$, then $x = p$ is the only Pareto optimal allocation. If $p \notin X$ then $x \in X$ is Pareto optimal for every profile $\succeq \in \prod_{i \in N} \mathcal{SP}(X_i)$ with peaks p if and only if $x \in \partial^{s(p-x)}(X)$.*

Proof. The first statement is clear. Next we fix $p \notin X$ and pick $x \in X$ such that $x \notin \partial^{s(p-x)}(X)$. There exists $y \in X \diagdown x$ such that $s(y-x) \trianglerighteq s(p-x)$. This implies $y_i = x_i$ for each i such that $x_i = p_i$, and for all j,

$$y_j > x_j \quad \Longrightarrow \quad p_j > x_j \quad \text{and} \quad y_j < x_j \quad \Longrightarrow \quad p_j < x_j.$$

From $y \neq x$ we see that not both S_+ and S_- are empty at $y - x$; therefore for $\varepsilon > 0$ small enough, $\varepsilon y + (1-\varepsilon)x$ stays in X and is a Pareto improvement of x.

Conversely, with $p \notin X$ still fixed, we pick $x \in X$ that is Pareto inferior to $y \in X$ at a profile $\succeq$ with those peaks p. Then $x_i = p_i \Longrightarrow y_i = x_i$ and $y_j > x_j \Longrightarrow p_j > x_j$; similarly $y_j < x_j \Longrightarrow p_j < x_j$, so we conclude $x \notin \partial^{s(p-x)}(X)$. □

Step 3: Defining f^ω. Recall the notation $|a| = (|a_i|)_{i \in N}$ and $[a, b] = \{x | \min\{a_i, b_i\} \le x_i \le \max\{a_i, b_i\}\}$. We fix $\omega \in X$ and define, for all $p \in \mathbb{R}^N$,

$$\Delta(\omega, p) = \big\{y \in \mathbb{R}^N | y = |z - \omega| \text{ for some } z \in X \cap [\omega, p]\big\},$$

$$f^\omega(p) = x \overset{\text{def}}{\Longleftrightarrow} \Big\{x \in X \cap [\omega, p] \text{ and } |x - \omega| = \arg \max_{\Delta(\omega, p)} \succeq_{\text{lxmin}}\Big\}.$$

This allocation is well defined: for any $x \in [\omega, p]$ we have $s(x - \omega) \trianglerighteq s(p - \omega)$ so in $[\omega, p]$ each $|x_i - \omega_i|$ is either $x_i - \omega_i$ or $\omega_i - x_i$ and the mapping $x \to |x - \omega|$ is linear and invertible in $X \cap [\omega, p]$; thus its image $\Delta(\omega, p)$ is convex and compact. By Step 1, $\succeq_{\text{lxmin}}$ has a unique maximum y in $\Delta(\omega, p)$, which comes from a unique x in $X \cap [\omega, p]$.

Step 4: f^ω is efficient. Fix p and set $x = f^\omega(p)$. If $p \in X$, then the maximum of $\succeq_{\text{lxmin}}$ on $\Delta(\omega, p)$ is clearly $|p-\omega|$; therefore $x = p$ as desired. Assume next $p \notin X$: by Lemma 2 we must check $x \in \partial^{s(p-x)}(X)$. Assume to the contrary there exists $y \in X \setminus x$ such that $s(y-x) \trianglerighteq s(p-x)$. Then $y_i = p_i$ whenever $x_i = p_i$; moreover $x_i < p_i \Longrightarrow x_i \leq y_i$ and $x_i > p_i \Longrightarrow x_i \geq y_i$. Therefore, for ε small enough, $y' = (1-\varepsilon)x + \varepsilon y$ stays in $X \cap [\omega, p]$. For all i we have $|y'_i - \omega_i| = |y'_i - x_i| + |x_i - \omega_i| \geq |x_i - \omega_i|$, with a strict inequality if $y_i \neq x_i$ (which does happen). We conclude that $|y' - \omega| \succ_{\text{lxmin}} |x - \omega|$, a contradiction.

Step 5: f^ω is SGSP. We fix ω and show first that f^ω meets a coalitional form of uncompromisingness (Lemma 1). For any $p, p' \in X_N$ with $x = f^\omega(p)$ we have

$$p' \in [x, p] \quad \Longrightarrow \quad f^\omega(p') = x. \tag{11}$$

Together $x \in [\omega, p]$ and $p' \in [x, p]$ imply $x \in [\omega, p']$. Now $|x-\omega|$ maximizes (uniquely) $\succeq_{\text{lxmin}}$ over $\Delta(\omega, p)$, and is in $\Delta(\omega, p') \subseteq \Delta(\omega, p)$: hence $|x - \omega|$ maximizes $\succeq_{\text{lxmin}}$ over $\Delta(\omega, p')$ as well.

Next we fix $p \in X_N$ with $x = f^\omega(p)$, and consider a coalition $M \subseteq N$ changing *all* its reports to $p'_{[M]}$ (so $p'_i \neq p_i$ for all $i \in M$), and such that everyone in M weakly prefers $x' = f^\omega(p'_{[M]}, p_{[N \setminus M]})$ to x. We claim that this implies $x' = x$. Hence M, as well as any coalition larger than M, cannot weakly misreport at p and we are done.

To prove the claim, consider first an agent i such that $p_i = \omega_i$. By definition of f^ω we have $x_i = p_i$, hence $x'_i = x_i$ as well, because agent i's welfare does not decrease at x'. So at profile $(p'_{[M]}, p_{[N\setminus M]})$ agent i's allocation is $x_i \neq p'_i$ and uncompromisingness (11) implies that everyone's allocation is unchanged if i reports instead $x_i = p_i$: $f^\omega(p'_{[M]}, p_{[N\setminus M]}) = f^\omega(p'_{[M\setminus i]}, p_{[(N\setminus M)\cup i]})$. Therefore we need only to prove the claim when $p_i \neq \omega_i$ for all $i \in M$.

For easier reading we assume, without loss of generality, $p_i > \omega_i$ for all i, so that $\omega_i \leq x_i \leq p_i$ for all i. Next we consider i such that $p'_i \leq \omega_i$: as i weakly prefers x'_i to x_i (at p_i), this implies $x'_i = x_i = \omega_i$ and we can again ignore those coordinates and prove the claim when $p'_i > \omega_i$ for all i.

We must have $p'_i \geq x_i$ for all $i \in M$; otherwise $x'_i \leq p'_i < x_i$ implies that i is strictly worse off at x'. Thus we can partition M as $M_+ \cup M_-$ where $p'_i > p_i \geq x_i$ in M_+, while $p_i > p'_i \geq x_i$ in M_- (one set $M_{+,-}$ could be empty).

The coordinate-wise minimum of p and $(p'_{[M]}, p_{[N\setminus M]})$ is $q = (p_{[M_+]}, p'_{[M_-]}, p_{[N\setminus M]})$. From $q \in [x, p]$ and (11) we have $x = f^\omega(q)$. From $\Delta(\omega, q) \subseteq \Delta(\omega, (p'_{[M]}, p_{[N\setminus M]}))$ and the definition of f^ω we get $(x' - \omega) \succeq_{\text{lxmin}} (x - \omega)$. Applying property (10) to $u = x' - \omega$, $v = x - \omega$, and $\lambda = \varepsilon$, we deduce $((\varepsilon x' + (1-\varepsilon)x) - \omega) \succeq_{\text{lxmin}} (x - \omega)$. Now we claim that for ε positive and small enough, the profile $\varepsilon x' + (1-\varepsilon)x$ is in $\Delta(\omega, q)$. As $x = f^\omega(q)$, this gives $\varepsilon x' + (1-\varepsilon)x = x$ and the desired conclusion $x' = x$.

To prove the claim, observe that for all $i \notin M_+$ we have $\omega_i \leq x_i, x'_i \leq q_i$ by definition of q. Next for $i \in M_+$ such that $x_i < p_i = q_i$, we have $x_i \leq x'_i$ (because i weakly prefers x' to x) so the inequalities $\omega_i \leq \varepsilon x'_i + (1-\varepsilon)x_i \leq q_i$ hold for ε strictly positive and small enough; and for $i \in M_+$ such that $x_i = p_i = q_i$, we have $x'_i = p_i$ (again because i weakly improves from x to x') so that $\varepsilon x'_i + (1-\varepsilon)x_i = x_i$.

Step 6: f^ω is symmetric if ω is symmetric in X. Check that a symmetric point always exists. The set $S(N;X)$ of all symmetries of X is a group for the composition of permutations. For an arbitrary element x of X, we set $\omega = (1/|S(N;X)|)(\sum_{\sigma \in S(N;X)} x^\sigma)$: it is in X because it is convex, and it is symmetric in X by the group properties.

We check that f^ω is symmetric if (and only if) ω is symmetric. For any profile $p \in X_N$, we must show $f^\omega(p^\sigma) = f^\omega(p)^\sigma$ whenever $\sigma \in S(N;X)$. As $\succeq_{\text{lxmin}}$ is a symmetric ordering, we have $\arg\max_{B^\sigma} \succeq_{\text{lxmin}} = (\arg\max_B \succeq_{\text{lxmin}})^\sigma$ for any set B where the maximum is unique; moreover if $x^\sigma = x$, then $\Delta(\omega, p^\sigma) = \Delta(\omega, p)^\sigma$.

Step 7: f^ω is envy-free. Assume $\tau_{ij} \in S(N,X)$. The desired property $x_i \succeq_i x_j$ is clear if p_i and p_j are on both sides of $\omega_i = \omega_j$ because for agent i, allocation x_i is on the "good" side of ω_i while x_j is on the "bad" side. Now assume p_i and p_j are on the same side of ω_i, say $p_i, p_j \geq \omega_i$, and that agent i envies x_j: then $p_i > x_i \geq \omega_i$ and $x_j > x_i$ (x_j may be larger or smaller than p_i). We consider several allocations where coordinates other than i, j stay as in x, and for brevity we only mention these two coordinates: e.g., x is simply (x_i, x_j). By the symmetry assumption, $x' = (x_j, x_i)$ is in X, and by convexity, so is $x'' = ((1-\lambda)x_i + \lambda x_j, \lambda x_i + (1-\lambda)x_j)$. For λ small enough (in particular below 1/2), the allocation $(|x''_i - \omega_i|, |x''_j - \omega_j|)$ is in $\Delta(\omega, p)$ (recall $x_i < p_i$) and the shift from $(|x_i - \omega_i|, |x_j - \omega_j|)$ to $(|x''_i - \omega_i|, |x''_j - \omega_j|)$ is a Pigou–Dalton transfer; hence it improves the leximin ordering.

Step 8: f^ω is continuous if $n = 2$ or $n \geq 3$ and X is a polytope or is strictly convex and of dimension n. We apply repeatedly a simple version of Berge's maximum theorem. Let a, b vary in two metric spaces A, B; fix a real-valued function $a \to g(a)$ and a compact-valued function $b \to \Gamma(b)$ from B into A. If g is continuous and Γ is hemicontinuous (both upper and lower hemicontinuous), then the real-valued function $\gamma(b) = \max\{g(a) | a \in \Gamma(b)\}$ is continuous as well.

For any $(q, p) \in (\mathbb{R}^N_+)^2$ we set $\Phi(q, p) = X \cap [q, p]$ and we postpone to Step 9 the proof of the following fact: *if $n = 2$ or $n \geq 3$ and X is a polytope or is strictly convex and of dimension n, the convex-compact-valued function $(q, p) \to \Phi(q, p)$ is hemicontinuous on the closed convex subset of $(\mathbb{R}^N_+)^2$ where it is nonempty.* Then we show in Step 10 that f^ω may not be continuous when $n \geq 3$ and X is of dimension n but neither a polytope nor strictly convex.

Define an orthant Θ of $\mathbb{R}^N$ by fixing the sign of each coordinate: Θ is described by n inequalities $x_i \leq 0$ or $x_i \geq 0$, one for each coordinate i. It is enough to show that f^ω is continuous when $p - \omega$ varies in such an orthant, because the orthants are 2^n closed sets covering $\mathbb{R}^N$. Without loss of generality we focus on the orthant $\Theta = \mathbb{R}^N_+$, i.e., we prove continuity for the set of profiles p such that $p \geq \omega$. Here $f^\omega(p) - \omega$ maximizes $\succeq_{\text{lxmin}}$ over $(X - \omega) \cap [0, p - \omega]$. Using the normalization $\omega = 0$, this is simply written as

$$f^\omega(p) = \arg\max_{X \cap [0,p]} \succeq_{\text{lxmin}}.$$

We prove first that the mapping $p \to f^*(p)$ is continuous. Observe that $x \to x^*$ is continuous. Then check that the first coordinate of f^*,

$$f^{*1}(p) = \max\{x^{*1} | x \in \Phi(0, p)\},$$

is continuous in p: Berge's theorem applies because $x \to x^{*1}$ is continuous and $p \to \Phi(0, p)$ is hemicontinuous. With the notation e^S for the vector $(e^S)_i = 1$ if $i \in S$ and 0 otherwise, we write f^{*2} as

$$f^{*2}(p) = \max\{x^{*2} | x \in \Phi(f^{*1}(p)e^N, p)\}.$$

It is continuous by Berge's theorem because $x \to x^{*2}$ is continuous and $\Phi(f^{*1}(p)e^N, p)$ is hemicontinuous. Next we write

$$f^{*3}(p) = \max\left\{x^{*3} \middle| x \in \bigcup_{i \in N} \Phi(f^{*1}(p)e^i + f^{*2}(p)e^{N \setminus i}, p)\right\}.$$

Here again $\Phi(f^{*1}(p)e^i + f^{*2}(p)e^{N \setminus i}, p)$ is hemicontinuous and hemicontinuity is preserved by union, so the same argument applies. We define similarly $f^{*4}(p)$ in terms of the sets $\Phi(f^{*1}(p)e^i + f^{*2}(p)e^j + e^{N \setminus \{i,j\}}, p)$ and so on. We omit the details.

Thus f^* is continuous and we show now that f is too. Fix $p \in \mathbb{R}^N_+$ and let p^t, $t = 1, 2, \ldots$, be a sequence converging to p: if w is a limit point of the sequence $f(p^t)$ (i.e., the limit of one of its subsequences), then $w \in \Phi(0, p)$ because the graph of Φ is closed. Moreover $f^*(p^t)$ converges to w^* and to $f^*(p)$, by continuity of $x \to x^*$ and of f^*, respectively. Thus $w^* = f^*(p)$; hence w maximizes $\succeq_{\text{lxmin}}$ in $\Phi(0, p)$ and by Step 1 this unique maximum is $f(p)$.

Step 9: $\Phi(q, p) = [q, p] \cap X$ is hemicontinuous if $n = 2$ or $n \geq 3$ and X is a polytope or is strictly convex and of dimension n. Upper hemicontinuity is clear because the graph of Φ is closed. We let the reader check lower hemicontinuity when $n = 2$. Next we assume that X is a polytope and invoke an auxiliary result from the linear programming literature. Consider a polytope-valued function $b \to H(b) = \{x \in \mathbb{R}^{m_2} | Ax \leq b\}$, where $b \in \mathbb{R}^{m_1}$ and A is a fixed $m_1 \times m_2$ matrix. This function is hemicontinuous where it is nonempty (Theorem 14 in Wets 1985). If X is an intersection of half-spaces, the mapping $(q, p) \to [q, p] \cap X$ takes the form $b \to H(b)$ for $b = (-q, p, b_0)$, where b_0 is a constant vector. Therefore Φ is hemicontinuous as desired.

Finally we assume that X is strictly convex and of dimension n. We fix (q, p), an allocation $x \in [q, p] \cap X$, and a sequence (q^t, p^t) converging to (q, p) and such that $\Phi(q^t, p^t) \neq \varnothing$ for all t, and we must construct a sequence $x^t \in [q^t, p^t] \cap X$ converging to x.

Each limit point of an arbitrary sequence in $[q^t, p^t] \cap X$ is in $[q, p] \cap X$; therefore the desired conclusion holds if $[q, p] \cap X$ reduces to $\{x\}$. Assume from now on that $[q, p] \cap X$ contains z, $z \neq x$, and set $D = \|z - x\|_\infty$ (the supremum norm). Fix $\varepsilon > 0$ and consider $y = (\varepsilon/2D)z + (1 - (\varepsilon/2D))x$, also in $[q, p] \cap X$: we have $\|y - x\|_\infty \leq \varepsilon/2$, and by the full dimensionality of X there is some η, $0 < \eta \leq \varepsilon/2$, such that $\|y' - y\|_\infty \leq \eta \Longrightarrow y' \in X$. For each i the interval $[q_i^t, p_i^t]$ converges to $[q_i, p_i] \ni y_i$; hence for t large enough, $[q_i^t, p_i^t] \cap [y_i - \eta, y_i + \eta] \neq \varnothing$: we choose y'_i in this intersection for all i and we see that

$$y' \in [q^t, p^t] \cap X \quad \text{and} \quad \|y' - x\|_\infty \leq \varepsilon \tag{12}$$

holds for all t large enough. The construction of the desired sequence is now clear.

Step 10: An example where f^ω is discontinuous. We have $N = \{1, 2, 3\}$ and X is the cone with origin $a = (1, 1, 2)$ and for base, we have the two-dimensional disk

$$B = \left\{x \in \mathbb{R}^3 | x_1^2 + (x_2 - 1)^2 \leq 1 \text{ and } x_3 = 1\right\}.$$

That is, $x \in X$ if and only if $x = a + \theta(b - a)$ for some $b \in B$ and $\theta \geq 0$. It is easy to check that X is also represented by the inequalities

$$x_3 \leq 2 \quad \text{and} \quad (x_1 + 1)^2 + (x_2 - 1)^2 + 2(1 - x_1)x_3 \leq 4.$$

We choose $\omega = 0$ and check that f^ω is discontinuous at $p = a$.[14] By EFF, $f^\omega(a) = a$. Consider next $p^\delta = ((1 - \delta^2)^{1/2}, 1 - \delta, 2)$ converging to p for small positive δ. As the segment from ω to $(1, 1, 1)$ stays in X and $1 - \delta \leq (1 - \delta^2)^{1/2} \leq 2$, we get $f_2^\omega(p^\delta) = 1 - \delta$. Next the segment from $(1 - \delta, 1 - \delta, 1 - \delta)$ to $((1 - \delta^2)^{1/2}, 1 - \delta, (1 - \delta^2)^{1/2})$ stays in X, implying $f_1^\omega(p^\delta) = (1 - \delta^2)^{1/2}$. Finally $((1 - \delta^2)^{1/2}, 1 - \delta, 1)$ is on the boundary of X and raising x_3 any more takes us outside X; hence $f_3^\omega(p^\delta) = 1$. We conclude $\lim_{\delta \to 0} f^\omega(p^\delta) = (1, 1, 1) \neq a$.

11.2 *Proposition 2*

Fix $X = \{\sum_N x_i = \beta\} \cap C$, where C is fully symmetric, closed, and either a polytope or strictly convex and of dimension n. Recall the only symmetric ω divides β equally.

Step 1. We know from Theorem 1 that f^ω meets EFF, SYM, and SGSP. It is also continuous if C is a polytope because X is one too. If C is strictly convex and of dimension n, the set X is strictly convex but certainly not of dimension n so the continuity of f^ω requires checking. We can use the argument in Step 8 of the proof of Theorem 1 provided $(q, p) \to X \cap [q, p]$ is hemicontinuous.

To check the latter we adapt the argument in the second half of Step 9: we fix (q, p), $x \in [q, p] \cap X$, $(q^t, p^t) \to_t (q, p)$, and we construct $x^t \in [q^t, p^t] \cap X$ converging to x. As above we can assume there is some $z \neq x$ in $[q, p] \cap X$, and we construct $y = (\varepsilon/2D)z + (1 - (\varepsilon/2D))x$ that is in $[q, p] \cap X$ at distance $\varepsilon/2$ or less from x. By the full dimensionality of C there is some η, $0 < \eta \leq \varepsilon/2$, such that for all y',

$$\left\{\sum_N y'_i = \beta \text{ and } \|y' - y\|_\infty \leq \eta\right\} \Longrightarrow y' \in C.$$

We claim now that for t large enough we can find y' in $[q^t, p^t]$ such that $\sum_N y'_i = \beta$ and $\|y' - y\|_\infty \leq \eta$. Then (12) holds and we are done as in Step 9 above.

To prove the claim we partition N as $N = N_+ \cup N_- \cup N_0$, where

$$i \in N_+ \quad \Longrightarrow \quad p_i^t \leq y_i; \qquad i \in N_- \quad \Longrightarrow \quad y_i \leq q_i^t; \qquad i \in N_0 \quad \Longrightarrow \quad q_i^t \leq y_i \leq p_i^t.$$

Note that up to two of these sets can be empty, and if, say, $y_i = q_i^t$, agent i can be placed in N_- or N_0.

[14] Given our choices of ω and p, the fact that X is unbounded from below is irrelevant: for instance, $X \cap \mathbb{R}_+^3$ works just as well.

From $p_i^t \to_t p_i$ and $y_i \leq p_i$ we see that for t large enough we have $y_i - p_i^t \leq \eta/n$, and similarly $q_i^t - y_i \leq \eta/n$. Now consider $\overline{y}$: $\overline{y}_i = p_i^t$ for $i \in N_+$; $\overline{y}_i = q_i^t$ for $i \in N_-$; $\overline{y}_i = y_i$ for $i \in N_0$.

We assume first $\sum_N \overline{y}_i < \beta$ and construct y' with the help of the fact

$$\sum_{N_+} \overline{y}_i + \sum_{N_- \cup N_0} \min\{\overline{y}_i + \eta, p_i^t\} \geq \beta. \tag{13}$$

Indeed if $p_i^t \leq \overline{y}_i + \eta$ for all $i \in N_- \cup N_0$ this follows because $[q^t, p^t] \cap X \neq \varnothing$ implies $\sum_N p_i^t \geq \beta$. And if $\overline{y}_i + \eta \leq p_i^t$ for some i, then

$$\sum_{N_- \cup N_0} \min\{\overline{y}_i + \eta, p_i^t\} \geq \eta + \sum_{N_- \cup N_0} \overline{y}_i \geq \eta + \sum_{N_- \cup N_0} y_i$$

while

$$\sum_{N_+} \overline{y}_i \geq \sum_{N_+} y_i - n_+ \frac{\eta}{n} \geq \sum_{N_+} y_i - \eta.$$

By inequality (13) we can raise (e.g., uniformly) $\overline{y}_i$ for each $i \in N_- \cup N_0$ up to $y_i' \leq \min\{\overline{y}_i + \eta, p_i^t\}$ such that $\sum_{N_+} \overline{y}_i + \sum_{N_- \cup N_0} y_i' = \beta$. Setting $y_i' = \overline{y}_i$ in N_+ completes the definition of y' in $[q^t, p^t] \cap X$ and at distance at most η from y, as announced in the claim. The case $\sum_N \overline{y}_i > \beta$ is treated similarly by lowering $\overline{y}_i$ in $N_+ \cup N_0$ so as to meet the equality constraint.

In the rest of the proof we fix a continuous focal rule F, i.e., meeting EFF, SYM, CONT, and SGSP. By Lemma 1, F is peak-only so we write it as f.

Step 2. For any $p \in X_N$ such that $x = f(p)$, and any two agents labeled 1, 2 such that $p_1 \geq p_2$, we claim that there is exactly three possible configurations of their allocations x_1, x_2:

$$p_1 \geq p_2 > x_1 = x_2 \quad \text{or} \quad x_1 = x_2 > p_1 \geq p_2 \quad \text{or} \quad p_1 \geq x_1 \geq x_2 \geq p_2.$$

Assume $p_1 \geq p_2 > x_1$. By uncompromisingness (Lemma 1) $f(p_2, p_2, p_{-1,2}) = x$; hence by SYM, $x_1 = x_2$. Then if $p_1 \geq p_2 > x_2$, we use $f(p_1, p_1, p_{-1,2}) = x$ and the cases $x_i > p_1 \geq p_2$ are similar.

The remaining case is $x_1, x_2 \in [p_1, p_2]$: we assume the configuration $p_1 \geq x_2 > x_1 \geq p_2$ and derive a contradiction. By SYM the allocation $(x_2, x_1, x_{-1,2})$ is in X and by convexity of X so is $((x_1 + x_2)/2, (x_1 + x_2)/2, x_{-1,2})$: the latter is Pareto superior to $f(p)$, a contradiction.

Step 3. We fix an arbitrary profile p and define $N_- = \{i \in N | p_i < x_i\}$, $N_0 = \{i \in N | p_i = x_i\}$, and $N_+ = \{i \in N | p_i > x_i\}$. By Step 2 and SYM, all i in N_- (resp. N_+) have the same allocation α_- (resp. α_+). Again by Step 2 and SYM for $j \in N_0$ and $i \in N_-$, inequality $p_j < \alpha_-$ is impossible: so $\alpha_- \leq p_j$ for all $j \in N_0$. A similar argument gives $p_j \leq \alpha_+$ for $j \in N_0$.

We claim that $x \in X \cap [\omega, p]$. From $\alpha_- \leq x_j \leq \alpha_+$ for all $j \in N_0$ and $\sum_N x_i = \beta$, we see that $\alpha_- \leq \omega_i = \beta/n \leq \alpha_+$; therefore $p_i < \alpha_- = x_i \leq \omega_i$ in N_-, and similarly $\omega_i \leq x_i = \alpha_+ < p_i$ in N_+. Finally $x_i = p_i$ in N_0.

Thus the allocation x is entirely described by the two numbers α_+, α_-, where $-\infty \leq \alpha_- \leq \beta/n \leq \alpha_+ \leq +\infty$. That is, if $p_i > \alpha_+$, agent i gets α_+, she gets α_- if $p_i < \alpha_-$, and she gets p_i if $\alpha_- \leq p_i \leq \alpha_+$. Note that $\alpha_+ = +\infty$ (resp. $\alpha_- = -\infty$) only if $N_+ = \varnothing$ (resp. $N_- = \varnothing$).

Now the equality $\sum_N x_i = \beta$ reduces to

$$\begin{aligned}\psi(\alpha_+) &= \big|\{i : \alpha_+ < p_i\}\big| \times \left(\alpha_+ - \frac{\beta}{n}\right) + \sum_{i:\frac{\beta}{n} \leq p_i \leq \alpha_+} \left(p_i - \frac{\beta}{n}\right) \\ &= \big|\{i : p_i < \alpha_-\}\big| \times \left(\frac{\beta}{n} - \alpha_-\right) + \sum_{i:\alpha_- \leq p_i \leq \frac{\beta}{n}} \left(\frac{\beta}{n} - p_i\right) = \chi(\alpha_-)\end{aligned} \tag{14}$$

and $(\alpha_+, \alpha_-) \in [\beta/n, +\infty[\times]-\infty, \beta/n]$ is a solution of this equation.

If $p^{*n} \leq \beta/n$, we have $\psi \equiv 0$ on $[\beta/n, +\infty[$ so the solutions are $\alpha_- = \beta/n$ and any α_+ in $[\beta/n, +\infty[$; the corresponding allocation is $x = \omega$. Similarly if $\beta/n \leq p^{*1}$, the solutions are $\alpha_+ = \beta/n$, any α_- in $]-\infty, \beta/n]$, and the allocation $x = \omega$ again.

If $p^{*1} < \beta/n < p^{*n}$, then ψ increases strictly from 0 on $[\beta/n, p^{*n}]$ after which it is constant; and χ is constant up to p^{*1}, then decreases strictly to 0 on $[p^{*1}, \beta/n]$.

Step 4. We fix p and compare $x = f(p)$ and $z = f^{\omega}(p)$. By Step 1, f^{ω} is a continuous focal rule just like f. Therefore by Steps 2 and 3 above, the allocation z is described just like x by two numbers γ_+, γ_- solving equation (14). If $p^{*n} \leq \beta/n$ or $\beta/n \leq p^{*1}$, the solution is unique and we are done. If $p^{*1} < \beta/n < p^{*n}$ and $z \neq x$, the monotonicity properties of ψ and χ imply $\{\gamma_+ > \alpha_+$ and $\gamma_- < \alpha_-\}$ or $\{\gamma_+ < \alpha_+$ and $\gamma_- > \alpha_-\}$. In the former case, z is Pareto superior to x and vice versa in the latter case. This is impossible because both rules are efficient.

11.3 *Proposition 3*

Statement (i). We let the reader check that the argument detailed for example (4) applies as well to any convex, compact X symmetric and of dimension 2; the shape of X inside X_{12} is the same, except when some of the four corners are actually feasible, but those cases are easy. A similar proof applies to the case where X is unbounded.

Statement (ii). Here we choose a function θ_0 from $\mathbb{R}$ into $\mathbb{R}_+ = [0, +\infty[$ such that its restriction θ_- to $\mathbb{R}_-$ is a decreasing bijection to $\mathbb{R}_+$, and its restriction θ_+ to $\mathbb{R}_+$ is an increasing bijection to $\mathbb{R}_+$. The canonical example used in the construction of f^{ω} is $\theta_0(x) = |x|$; in the example after Proposition 3 we used $\theta_0(x) = x$ if $x > 0$ and $\theta_0(x) = -x/\lambda$ if $x < 0$.

We construct a family of focal allocation rules for any choice of θ_0. We write $\theta(z) = (\theta_0(z_i))_{i \in N}$ for $z \in \mathbb{R}^N$. Fixing (N, X), a symmetric allocation $\omega \in X$, and θ_0, we define a new rule $f^{\omega,\theta}$ as

$$f^{\omega,\theta}(p) = x \overset{\text{def}}{\iff} \left\{x \in X \cap [\omega, p] \text{ and } \theta(x - \omega) = \arg \max_{\Delta^{\theta}(\omega,p)} \succeq_{\text{lxmin}}\right\},$$

$$y \in \Delta^{\theta}(\omega, p) \overset{\text{def}}{\iff} \left\{y = \theta(x - \omega) \text{ for some } x \in X \cap [\omega, p]\right\}.$$

When $\theta_-(z) = \theta_+(-z)$ this definition is exactly the same as (3). Not so otherwise, because θ treats differently a move above the default ω_i and one below it.

Then we follow step by step the proof of Theorem 1 to show that $f^{\omega,\theta}$ meets precisely the same properties as f^ω. The desired conclusion follows because the set of functions θ such that θ_- is not the mirror image of θ_+ is of infinite dimension.

As the range of $X \cap [\omega, p]$ by $x \to \theta(x - \omega)$ is a compact set, $\succeq_{\text{lxmin}}$ reaches its maximum in $\Delta^\theta(\omega, p)$. To prove uniqueness (despite the fact that this range may not be convex) we mimic the argument in Step 1 of the Theorem 1 proof, and use the same notation. Assume that x, y are two maximizers and that S, T are disjoints, and set $a = \theta(x)^{*1} = \theta(y)^{*1}$: then for all $k \in N$, we have $a \leq \min\{\theta_0(x_k), \theta_0(y_k)\}$ and $a < \max\{\theta_0(x_k), \theta_0(y_k)\}$, implying $\min_{k\in N} \theta_0((x + y)/2)_k > a$ and contradicting the optimality of x, y. Therefore S and T must intersect and by the separability of $\succeq_{\text{lxmin}}$ we can drop this coordinate; the induction proceeds as before.

The proofs of EFF, SGSP, SYM, and NE are exactly as in the proof of Theorem 1, so we do not repeat them.

Continuity is not much harder. We restrict attention first to an arbitrary orthant Θ and to the vectors p such that $p - \omega \in \Theta$. Because θ treats differently positive and negative deviations from ω, we keep Θ an arbitrary orthant; alternatively, normalizing ω to zero is without loss of generality. We set $h(p) = \theta(f^{0,\theta}(p))$ and prove first that h^* is continuous. As $\theta(x)^{*1}$ is continuous, Berge's theorem tells us that $h(p)^{*1} = \max\{\theta(x)^{*1} | x \in \Phi(0, p)\}$ is continuous as well. For the next coordinate we can write

$$\begin{aligned} h(p)^{*2} &= \max\{\theta(x)^{*2} | x \in \Phi(0, p) \text{ and } \theta(x) \geq h(p)^{*1} e^N\} \\ &= \max\{\theta(x)^{*2} | x \in \Phi(\theta_0^{-1}(h(p)^{*1}), p)\}, \end{aligned}$$

where Berge theorem applies again, so h^{*2} is continuous. And so on as in Step 8 of the proof of Theorem 1.

Once we know that h^* is continuous, we take a converging sequence $p^t \to p$ as before and w a limit point of $f(p^t)$, i.e., $w = \lim_{t'} f(p^{t'})$ for some subsequence t' of t (omitting the superscripts in f). Then $\theta(f(p^{t'}))^* \to \theta(w)^*$ because θ and $x \to x^*$ are continuous, and $\theta(f(p^t))^* \to \theta(f(p))^*$ by the continuity of h^*. Thus $\theta(f(p))^* = \theta(w)^*$ and $w \in \Phi(0, p)$ by the hemicontinuity of Φ. We conclude that $w = f(p)$ as was to be proved.

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