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Modeling infinitely many agents

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This paper offers a resolution to an extensively studied question in theoretical economics: which measure spaces are suitable for modeling many economic agents? We propose the condition of "nowhere equivalence" to characterize those measure spaces that can be effectively used to model the space of many agents. In particular, this condition is shown to be more general than various approaches that have been proposed to handle the shortcoming of the Lebesgue unit interval as an agent space. We illustrate the minimality of the nowhere equivalence condition by showing its necessity in deriving the determinateness property, the existence of equilibria, and the closed graph property for equilibrium correspondences in general equilibrium theory and game theory.

Keywords. Agent space, nowhere equivalence, Nash equilibrium, Walrasian equilibrium, determinateness property, closed graph property, relative saturation, atomless independent supplement, conditional atomlessness.

JEL classification. C7, D0, D5.

1. Introduction

A typical economic model starts with an agent space. When a model considers a fixed finite number of agents, the most natural agent space is the set $\{1, 2, \ldots, n\}$ for some

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positive integer n. One may also need to model the interaction of many agents so as to discover mass phenomena that do not necessarily occur in the case of a fixed finite number of agents. A well known example is the Edgeworth conjecture that the set of core allocations shrinks to the set of competitive equilibria as the number of agents goes to infinity, though the former set is, in general, strictly larger than the latter set for an economy with a fixed finite number of agents.[1]

To avoid complicated combinatorial arguments that may involve multiple steps of approximations for a large but finite number of agents, it is natural to consider economic models with an infinite number of agents. The mathematical abstraction of an atomless (countably additive) measure space of agents provides a convenient idealization for a large but finite number of agents.[2] The archetype space in such a setting is the Lebesgue unit interval. However, a number of desirable properties fail to hold in various situations when the underlying agent space is the Lebesgue unit interval. We focus on the following three problems in this paper. The first is the determinateness problem in general equilibrium theory and game theory; that is, two economies (games) with identical distributions on the space of characteristics may not have the same set of distributions of equilibria, as shown in Examples 1 and 3 of Section 2. The second problem is the nonexistence of pure-strategy Nash equilibria in games with many agents. The third problem is the dissonance between an idealized economy (game) and its discretized versions. Namely, the equilibria in a sequence of economies (games) converging to a limit economy (game) may fail to converge to an equilibrium of the limit economy (game); see Examples 2 and 4 in Section 2.

A basic and natural question arises: which measure spaces are most suitable for modeling many economic agents? The key point is that an economic agent with a given characteristic often has multiple optimal choices. Various equilibrium properties may require different agents with the same characteristic to select different optimal choices (see Remark 4 below). To allow such heterogeneity, one needs to distinguish the agent space from the subspace generated by the mapping specifying the individual characteristics (i.e., the "characteristic type space" in Section 3).[3] For this purpose, we introduce the condition of "nowhere equivalence" to characterize those measure spaces that can be effectively used to model many agents. This condition requires that for any nontrivial collection of agents, when the agent space and the characteristic type space are restricted to such a collection, the former contains the latter strictly in terms of measure spaces.

We demonstrate that the nowhere equivalence condition can be used to handle the shortcoming of the Lebesgue unit interval, especially for the three problems as discussed above. To resolve the same or related problems with the Lebesgue unit interval, various approaches have been proposed, such as distributional equilibria, standard representations, hyperfinite agent spaces, saturated probability spaces, and agent spaces

[1] See, for example, Debreu and Scarf (1963), Hildenbrand (1974), Anderson (1978), and McLean and Postlewaite (2005).

[2] For some classical references, see, for example, Milnor and Shapley (1961), Aumann (1964), Hildenbrand (1974), and Hammond (1979).

[3] The shortcoming of the Lebesgue unit interval is due to the possible identity of the spaces of agents and their characteristic types.

with the condition of "many more agents than strategies." It is shown that our condition is more general than all these approaches. More importantly, we demonstrate the minimality of the nowhere equivalence condition by showing its necessity in resolving the three problems mentioned earlier, which is discussed below in more details.

We first consider the determinateness problem in general equilibrium theory. As pointed out in Kannai (1970, p. 811), Gérard Debreu remarked that there exists a serious difficulty with large economies in the sense that large economies with the same distribution on agents' characteristics need not have the same set of distributions for core allocations (i.e., Walrasian allocations). Kannai (1970, p. 811) presented a concrete example illustrating that point using the Lebesgue unit interval as the underlying agent space; see Example 1 below for details. It was then conjectured by Robert Aumann that the closure of the sets of distributions for the core allocations are the same for large economies with the same distribution on agents' characteristics; see Kannai (1970, p. 813). This conjecture was resolved in Hart et al. (1974). To show that the distribution of agents' characteristics is a concise and accurate description of a large economy, Hart et al. (1974) proposed the approach of "standard representation," which assumes the agent space to be the product of the space of characteristics and the Lebesgue unit interval.[4] Based on this approach, they obtained the determinateness property for large economies instead of the "same closure property" in Aumann's conjecture. It is easy to show that the nowhere equivalence condition is sufficient for the validity of such determinateness property. What is surprising is that our condition is also necessary for this property to hold. Both results are stated in Theorem 1 below.

Next, we move to games with many agents. Example 3 presents a simple game and its variation to illustrate a similar determinateness problem in the setting of large games. Khan and Sun (1999, p. 472) presented a rather difficult example of two large games (based on the Lebesgue agent space) with the same distribution on agents' characteristics, where one has a Nash equilibrium while the other does not. It implies that the respective sets of distributions for the Nash equilibria in these two large games do not have the same closure, in contrast to the "same closure" conjecture of Aumann for the case of large economies.[5] Khan and Sun (1999) resolved the existence and determinateness issues of large games by working with a hyperfinite agent space.[6] Motivated by the consideration of social identities as in Akerlof and Kranton (2000) and Brock and Durlauf (2001), Khan et al. (2013) introduced a more general class of large games in which agents have names and social types/biological traits.[7] Corresponding to Theorem 1, Theorem 2

[4]The construction of a standard representation leads to a continuum of agents with each characteristic, which allows a natural distinction between the spaces of agents and their characteristic types. Indeed, Lemma 1 shows that the nowhere equivalence condition is satisfied by the approach of standard representation.

[5]In the large economies considered here, one needs to work with the aggregate demand in a finite-dimensional commodity space. In contrast, the example in Khan and Sun (1999) involves the infinite-dimensional space of all the distributions on the action space $[-1, 1]$.

[6]Lemma 3 shows that the nowhere equivalence condition is always satisfied by a hyperfinite agent space. Thus, the shortcoming of the Lebesgue unit interval can be avoided.

[7]The large games considered in Khan and Sun (1999) can be viewed as large games with a single trait.

considers the sufficiency and necessity of the nowhere equivalence condition for the determinateness property and existence of Nash equilibria in large games (with or without traits).

Finally, we consider the fundamental issue of whether an economy with an atomless probability space of agents is, in general, a good proxy for large finite-agent economies. Example 2 presents a sequence of finite-agent economies converging to a limit large economy with the Lebesgue agent space. However, a converging sequence of equilibria corresponding to the given sequence of finite-agent economies fails to converge to an equilibrium of the limit economy. Theorem 1 demonstrates that the nowhere equivalence condition is not only sufficient but also necessary for modeling the agent space in a large limit economy. Similar issues arise in large games. Example 4 provides a simple game and its discretizations to show that the Nash equilibria in a sequence of finite-agent games, converging to a limit large game with the Lebesgue agent space, fail to converge to a Nash equilibrium of the limit game.[8] We show in Theorem 2 that the nowhere equivalence condition is also necessary and sufficient for the closed graph property to hold in large games.[9]

The rest of the paper is organized as follows. Section 2 presents several examples in general equilibrium theory and game theory to illustrate the shortcoming of the Lebesgue unit interval as an agent space. In Section 3, we introduce the nowhere equivalence condition for agent spaces, and then show that this condition is necessary and sufficient for obtaining the determinateness property, the existence of equilibria, and the closed graph property for equilibrium correspondences in general equilibrium theory and game theory. In Section 4, we show that the nowhere equivalence condition is more general than various earlier approaches proposed to handle the shortcoming of the Lebesgue unit interval. All the proofs are collected in the Appendix.

2. Examples

In this section, we present five examples in general equilibrium theory and game theory to illustrate the shortcoming of the Lebesgue unit interval as an agent space.[10] We first consider the determinateness problem and the closed graph property in large economies in Section 2.1. Similar issues in large games together with the relevant purification problem are then discussed in Section 2.2. The proofs of the claims in these examples are given in Appendix A.2.

[8]A rather involved example provided by Qiao and Yu (2014) shows the existence of a convergent sequence of exact Nash equilibria in a sequence of finite-agent games such that the idealized limit game of the sequence does not have any Nash equilibrium.

[9]Khan et al. (2013) and Qiao and Yu (2014) considered the existence issue of pure-strategy Nash equilibria and the closed graph property in large games with traits via a saturated agent space; for details about saturated agent spaces, see Section 4.5. More generally, the nowhere equivalence condition allows the possibility to work with both saturated and nonsaturated agent spaces and unifies various earlier approaches (including the distributional approach, the standard representation, and the saturated probability spaces).

[10]Some regularity properties (such as convexity, compactness, purification, and upper hemicontinuity) of the distribution of correspondences also fail to hold when the underlying measure space is the Lebesgue unit interval; see Examples 1–3 of Sun (1996).

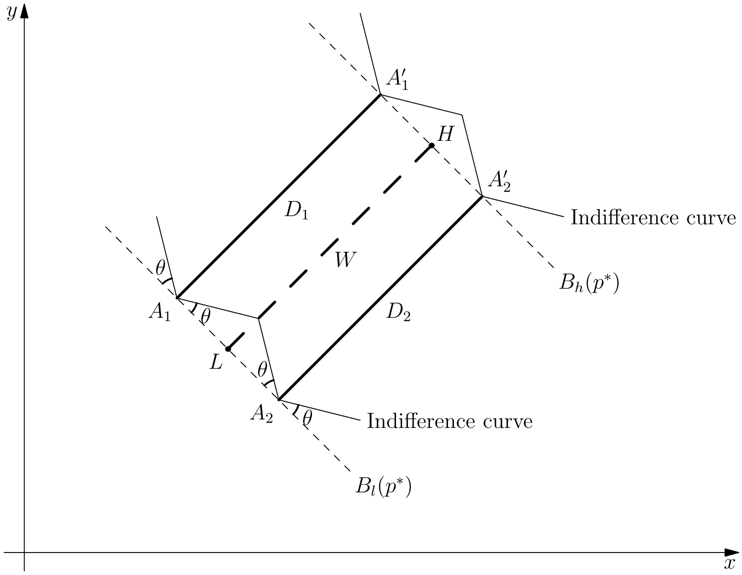

FIGURE 1. Indifference curves and budget lines.

2.1 *Examples on large economies*

The following example, taken from Kannai (1970), demonstrates the determinateness problem of large economies: the set of distributions of the Walrasian allocations is not completely determined by the distribution of agents' characteristics in terms of their preferences and endowments.

EXAMPLE 1. Consider two economies $\mathscr{E}_1$ and $\mathscr{E}_2$ with two goods. The agent space for each of the economies $\mathscr{E}_1$ and $\mathscr{E}_2$ is the Lebesgue unit interval $(I, \mathcal{B}, \eta)$, where $I = [0, 1]$, $\mathcal{B}$ is the Borel σ-algebra, and η is the Lebesgue measure. In both economies, all the agents have the same preference, and the corresponding indifference curves are parallel as shown in Figure 1. For $\ell = 1, 2$, the line segment D_ℓ is represented by $y = x + \frac{3}{2} - \ell$ for $x \in [(2\ell + 1)/4, (2\ell + 5)/4]$ with the endpoints $A_\ell = ((2\ell + 1)/4, (7 - 2\ell)/4)$ and $A'_\ell = ((2\ell + 5)/4, (11 - 2\ell)/4)$. The set of endowments is represented by the line segment W: $y = x$ for $x \in [1, 2]$ with the endpoints $L = (1, 1)$ and $H = (2, 2)$. Let $\mathbf{p}^* = (1, 1)$. The parallel dashed lines $B_l(\mathbf{p}^*)$ and $B_h(\mathbf{p}^*)$ are perpendicular to the parallel line segments D_1 and D_2. The angle θ is chosen to be sufficiently small so that the preference is monotonic.[11]

The endowments of the economies $\mathscr{E}_1$ and $\mathscr{E}_2$ are, respectively, given by

$$\mathbf{e}_1(i) = (1 + i, 1 + i), \quad \text{and} \quad \mathbf{e}_2(i) = \begin{cases} (1 + 2i, 1 + 2i) & \text{if } i \in \left[0, \frac{1}{2}\right), \\ (2i, 2i) & \text{if } i \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

[11]It suffices if θ is less than 45 degrees. For a formal definition of monotonic preferences, see Section 3.2.

Since all the agents in these two economies have the same preference and their endowments generate the same distribution (i.e., $\eta \circ \mathbf{e}_1^{-1} = \eta \circ \mathbf{e}_2^{-1}$), the two economies have the same distribution of agents' characteristics. For $\ell = 1, 2$, let $\mathcal{DW}(\mathscr{E}_\ell)$ be the set of distributions of all the Walrasian allocations in the economy $\mathscr{E}_\ell$.[12] Then a basic observation in Kannai (1970) is the following claim.

Claim 1. *We have $\mathcal{DW}(\mathscr{E}_1) \neq \mathcal{DW}(\mathscr{E}_2)$; in particular, the uniform distribution μ on $D_1 \cup D_2$ is in $\mathcal{DW}(\mathscr{E}_2)$, but not in $\mathcal{DW}(\mathscr{E}_1)$.*

The above claim means that the set of distributions of all the Walrasian allocations is not completely determined by the distribution of agents' characteristics.

In the second example, we construct a sequence of finite-agent economies such that (i) each economy has a Walrasian equilibrium, (ii) the sequence of distributions of agents' characteristics in those economies converges weakly to the distribution of agents' characteristics in the economy $\mathscr{E}_1$ as defined in Example 1, and (iii) the weak limit of the sequence of distributions of the Walrasian allocations in the finite-agent economies cannot be induced by any Walrasian allocation in the limit economy $\mathscr{E}_1$. This means that the closed graph property for the Walrasian equilibrium correspondence may fail for large economies.

Example 2. Let $\mathscr{E}_1$ be the economy as defined in Example 1, which will be discretized to generate a sequence of finite-agent economies $\{\mathscr{E}_1^k\}_{k\in\mathbb{Z}_+}$, where $\mathbb{Z}_+$ is the set of positive integers. For each $k \in \mathbb{Z}_+$, we take the probability space $(\Omega^k, \mathcal{F}^k, P^k)$ to be the agent space of the economy $\mathscr{E}_1^k$, where $\Omega^k = \{1, 2, \ldots, 2k\}$, $\mathcal{F}^k$ is the power set of Ω^k, and P^k is the counting probability measure over $\mathcal{F}^k$. For each economy $\mathscr{E}_1^k$ ($k \in \mathbb{Z}_+$), all the agents have the same preference as in Example 1 while the endowment for agent $j \in \Omega^k$ is given by $\mathbf{e}_1^k(j) = (1 + j/(2k), 1 + j/(2k))$. For each $k \in \mathbb{Z}_+$ and $j \in \Omega^k$, let

$$\mathbf{f}_1^k(j) = \begin{cases} \left(\frac{3}{4} + \frac{j}{2k}, \frac{5}{4} + \frac{j}{2k}\right) & \text{if } j \text{ is odd,} \\ \left(\frac{5}{4} + \frac{j}{2k}, \frac{3}{4} + \frac{j}{2k}\right) & \text{if } j \text{ is even.} \end{cases}$$

Claim 2. *For each $k \in \mathbb{Z}_+$, $(\mathbf{f}_1^k, \mathbf{p}^*)$ is a Walrasian equilibrium of the economy $\mathscr{E}_1^k$, where $\mathbf{p}^* = (1, 1)$.*

It is easy to see that the sequence of endowment distributions $\{P^k \circ (\mathbf{e}_1^k)^{-1}\}_{k\in\mathbb{Z}_+}$ on the line segment W converges weakly to the endowment distribution $\eta \circ (\mathbf{e}_1)^{-1}$ of the economy $\mathscr{E}_1$. Since all the agents in the relevant economies have the same preference, the distribution of agents' characteristics for the economy $\mathscr{E}_1^k$ converges weakly to the distribution of agents' characteristics for the economy $\mathscr{E}_1$ as k goes to infinity. Furthermore, it is clear that the sequence $\{P^k \circ (\mathbf{f}_1^k)^{-1}\}_{k\in\mathbb{N}_+}$ of distributions of the Walrasian allocations converges weakly to the uniform distribution μ on $D_1 \cup D_2$. However,

[12] See Section 3.2 for the formal definition of Walrasian allocations.

Claim 1 shows that the economy $\mathscr{E}_1$ does not have a Walrasian allocation with the distribution μ. This means that the limit distribution of the Walrasian allocations in a sequence of finite-agent economies may not be induced by any Walrasian allocation in the limit economy.

2.2 *Examples on large games*

As in the survey by Khan and Sun (2002), a large game with an atomless probability space of agents and a common compact metric action space A is a measurable mapping from the agent space to the space $C(A \times \mathcal{M}(A))$ of real-valued continuous functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology, where $\mathcal{M}(A)$ is the space of Borel probability measures on A with the topology of weak convergence of measures. Hence, the payoff function of an individual agent as an element of $C(A \times \mathcal{M}(A))$ depends on her action $a \in A$ and a societal action distribution $\nu \in \mathcal{M}(A)$. In this subsection, we consider the determinateness problem and the failure of the closed graph property together with the relevant purification problem in the setting of large games.

We first consider the determinateness problem of large games in the sense that the set of distributions of pure-strategy Nash equilibria may not be completely determined by the distribution of agents' characteristics in terms of their payoff functions.

Example 3. Consider two games G_1 and G_2. In each game, the agent space is the Lebesgue unit interval $(I, \mathcal{B}, \eta)$ as in Example 1, and the common action space is $A = [-1, 1]$. The payoffs for the games G_1 and G_2 are defined as follows. For agent $i \in [0, 1]$, action $a \in A$, and societal action distribution $\nu \in \mathcal{M}(A)$,

$$G_1(i, a, \nu) = -(a+i)^2 \cdot (a-i)^2 \quad \text{and} \quad G_2(i, a, \nu) = \begin{cases} G_1(2i, a, \nu) & \text{if } i \in \left[0, \frac{1}{2}\right), \\ G_1(2i-1, a, \nu) & \text{if } i \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It is clear that G_1 and G_2 are measurable from $(I, \mathcal{B}, \eta)$ to $C(A \times \mathcal{M}(A))$ and induce the same distribution. Hence the two games have the same distribution of agents' characteristics. For $\ell = 1, 2$, let $\mathcal{D}(G_\ell)$ be the set of distributions of pure-strategy Nash equilibria in the game G_ℓ.[13] Then we have the following claim.

Claim 3. *We have $\mathcal{D}(G_1) \neq \mathcal{D}(G_2)$; in particular, the uniform distribution μ on A is in $\mathcal{D}(G_2)$, but not in $\mathcal{D}(G_1)$.*

The above claim means that the set of distributions of pure-strategy Nash equilibria in a game is not completely determined by the distribution of agents' characteristics.

In the next example, we construct a sequence of finite-agent games such that (i) each game has a pure-strategy Nash equilibrium, (ii) the sequence of distributions of agents' characteristics in those games converges weakly to the distribution of agents' characteristics in the game G_1 as defined in Example 3, and (iii) the weak limit of the sequence

[13] See Section 3.2 for the formal definition of pure-strategy Nash equilibria in large games.

of distributions of the pure-strategy Nash equilibria in the finite-agent games cannot be induced by any pure-strategy Nash equilibrium in the limit game G_1. This means that the closed graph property for the Nash equilibrium correspondence may fail for large games.

Example 4. Let G_1 be the game as defined in Example 3, which will be discretized to generate a sequence of finite-agent games $\{G_1^k\}_{k\in\mathbb{Z}_+}$. For each $k\in\mathbb{Z}_+$, we take the probability space $(\Omega^k, \mathcal{F}^k, P^k)$ to be the agent space of the game G_1^k, where $\Omega^k = \{1, 2, \dots, 2k\}$, $\mathcal{F}^k$ is the power set of Ω^k, and P^k is the counting probability over $\mathcal{F}^k$. The payoff function for agent $j\in\Omega^k$ in the game G_1^k is

$$G_1^k(j, a, \nu) = -\left(a + \frac{j}{2k}\right)^2 \cdot \left(a - \frac{j}{2k}\right)^2$$

for her own action $a\in A$ and societal action distribution $\nu\in\mathcal{M}(A)$. For each $k\in\mathbb{Z}_+$ and $j\in\Omega^k$, let

$$f_1^k(j) = (-1)^j \frac{j}{2k}.$$

For each agent $j\in\Omega^k$ in the game G_1^k, $f_1^k(j)$ is a dominant action, which means that f_1^k is a pure-strategy Nash equilibrium of the game G_1^k for each $k\in\mathbb{Z}_+$. It is easy to verify that the sequence $\{P^k\circ(G_1^k)^{-1}\}_{k\in\mathbb{Z}_+}$ of distributions of agents' characteristics converges weakly to the distribution of agents' characteristics $\eta\circ G_1^{-1}$ in the game G_1. Furthermore, the sequence of equilibrium distributions $\{P^k\circ(f_1^k)^{-1}\}_{k\in\mathbb{Z}_+}$ converges weakly to the uniform distribution μ on A. However, by Claim 3, the uniform distribution μ cannot be induced by any pure-strategy Nash equilibrium of the limit game G_1. That is, the closed graph property for the Nash equilibrium correspondence fails.

The last example of this section demonstrates the purification problem of large games.[14]

Example 5. Let G_1 be the game as defined in Example 3. Define a measurable mapping g_1 from $(I, \mathcal{B}, \eta)$ to $\mathcal{M}(A)$ by letting

$$g_1(i) = \tfrac{1}{2}\delta_i + \tfrac{1}{2}\delta_{-i}$$

for $i\in I$, where δ_i and δ_{-i} are the Dirac measures on A at the points i and $-i$, respectively. Because i and $-i$ are the two dominant actions for each agent $i\in I$, g_1 is clearly a mixed-strategy Nash equilibrium of the game G_1. For notational simplicity, denote $C(A\times\mathcal{M}(A))$ by $\mathcal{U}$. Let τ be the joint distribution on $\mathcal{U}\times A$ induced by (G_1, g_1) in the sense that $\tau(C) = \int_I (\delta_{G_1(i)}\otimes g_1(i))(C)\,\mathrm{d}\eta(i)$ for any measurable subset C of $\mathcal{U}\times A$. A measurable mapping f_1 from $(I, \mathcal{B}, \eta)$ to A is said to be a purification of g_1 if the joint distribution $\eta\circ(G_1, f_1)^{-1}$ is τ; suppose that such a purification f_1 exists. It is then clear that for η-almost all $i\in I$, $f_1(i)$ is i or $-i$. As above, since i and $-i$ are the two dominant

[14]We thank an anonymous referee for the suggestion to add such an example.

actions for each agent $i \in I$, f_1 is a pure-strategy Nash equilibrium of the game G_1. It is easy to see that the marginal of τ on A is the uniform distribution μ, which is also the distribution induced by f_1 on A. It implies that $\mu \in \mathcal{D}(G_1)$, which is impossible by Claim 3.

3. Main results

3.1 *Nowhere equivalence*

In a typical economic model, each agent is described by some characteristics, such as the strategy/action set, payoff, preference, endowment/income, information, social types/biological traits. The mapping from the agent space to the space of characteristics generates a sub-σ-algebra on the agent space, which can be seen as the preimage of Borel measurable sets in the space of characteristics. Thus, it is natural to restrict our attention to a sub-σ-algebra that is to be used in modeling agents' characteristics. The corresponding restricted probability space on such a sub-σ-algebra is called the *characteristic type space*. In this subsection, we introduce the condition of nowhere equivalence to characterize the relationship between the agent space and the characteristic type space.

Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space with a complete countably additive probability measure P,[15] and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. The probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{G}, P)$ are used to model the agent space and the characteristic type space, respectively.[16]

For any nonnegligible subset $D \in \mathcal{F}$, i.e., $P(D) > 0$, the restricted probability space $(D, \mathcal{G}^D, P^D)$ is defined as follows: $\mathcal{G}^D$ is the σ-algebra $\{D \cap D' : D' \in \mathcal{G}\}$ and P^D is the probability measure rescaled from the restriction of P to $\mathcal{G}^D$; the restricted probability space $(D, \mathcal{F}^D, P^D)$ is defined similarly.

Let X and Y denote Polish (complete separable metrizable topological) spaces, let $\mathcal{B}(X)$ denote the Borel σ-algebra on X, and let $\mathcal{M}(X)$ denote the space of Borel probability measures on X with the topology of weak convergence of measures. Note that $\mathcal{M}(X)$ is again a Polish space; see Theorem 15.15 in Aliprantis and Border (2006). For any $\mu \in \mathcal{M}(X \times Y)$, let μ_X and μ_Y be the marginals of μ on X and Y, respectively.

Now we are ready to present the following definition.

Definition 1. A σ-algebra $\mathcal{F}$ is said to be *nowhere equivalent* to a sub-σ-algebra $\mathcal{G}$ if for every nonnegligible subset $D \in \mathcal{F}$, there exists an $\mathcal{F}$-measurable subset D_0 of D such that $P(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{G}^D$, where $D_0 \triangle D_1$ is the symmetric difference $(D_0 \setminus D_1) \cup (D_1 \setminus D_0)$.

[15]A probability space $(\Omega, \mathcal{F}, P)$ (or its σ-algebra) is atomless if for any nonnegligible subset $E \in \mathcal{F}$, there is an $\mathcal{F}$-measurable subset E' of E such that $0 < P(E') < P(E)$.

[16]Here $\mathcal{G}$ can be viewed as the σ-algebra induced by a mapping from the agent space to the space of characteristics. For example, consider a large economy $\mathscr{E}$ with the space of payoff functions $\mathcal{U}$ and the space of endowments E. Let $\mathcal{B}(U)$ and $\mathcal{B}(E)$ be the corresponding Borel σ-algebras on $\mathcal{U}$ and E, respectively. The mapping $\mathscr{E}$ from I to $\mathcal{U} \times E$ assigns for each agent a payoff function and an initial endowment. Then $\mathcal{G}$ is the σ-algebra generated by $\mathscr{E}$; that is, $\mathcal{G}$ is the preimage of $\mathcal{B}(\mathcal{U}) \otimes \mathcal{B}(E)$ of the function $\mathscr{E}$.

Let $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{G}, P)$ model the respective spaces of agents and characteristic types. For any given nonnegligible group D of agents, $\mathcal{F}^D$ is the collection of subgroups of agents in D, while $\mathcal{G}^D$ represents the set of all characteristic-generated subgroups of agents in D. The condition that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ means that for any nonnegligible group D of agents, $\mathcal{F}^D$ is always strictly richer than $\mathcal{G}^D$.

The following lemma provides a simple example to demonstrate how to restore the nowhere equivalence condition if it fails to hold. The idea is similar to that of the standard representation in Section 4.3.[17] Its proof is left to Appendix A.1.

Lemma 1. *Suppose that $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{G}, P)$ model the respective spaces of agents and characteristic types. Let $\Omega' = \Omega \times I$, $\mathcal{F}' = \mathcal{F} \otimes \mathcal{B}$, $\mathcal{G}' = \mathcal{G} \otimes \{\varnothing, I\}$, and $P' = P \otimes \eta$ on $\mathcal{F}'$, where $(I, \mathcal{B}, \eta)$ is the Lebesgue unit interval. Then $\mathcal{F}'$ is nowhere equivalent to $\mathcal{G}'$ under P'.*

The above lemma implies that when the Lebesgue unit interval models both the spaces of agents and characteristic types, the nowhere equivalence condition can always be achieved by using the Lebesgue unit square as an extended agent space. In particular, each agent name $\omega \in \Omega$ corresponds to a continuum of agents in the form (ω, t), $t \in [0, 1]$ with the same characteristic. This is a continuum version of the classical sequential replica model as in Debreu and Scarf (1963). However, such a formulation may not be appropriate when the characteristics are meant to capture individual idiosyncrasies.

3.2 *Applications*

In this subsection, we present several applications to illustrate the usefulness of the nowhere equivalence condition. Our first aim is to show that the nowhere equivalence condition can be used to handle the shortcoming of the Lebesgue unit interval. In particular, we focus on the following results from general equilibrium theory and game theory: (i) the determinateness in large economies, (ii) the closed graph property in large economies, (iii) the determinateness in large games with traits, (iv) the existence of pure-strategy Nash equilibria in large games with traits, and (v) the closed graph property in large games with traits. More importantly, we point out that the nowhere equivalence condition is minimal in the sense that it is necessary to derive these desirable economic results.

We follow the notation in the previous subsection, and use $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{G}, P)$ to model the respective spaces of agents and characteristic types. We assume that $\mathcal{G}$ is a countably generated sub-σ-algebra of $\mathcal{F}$ throughout this subsection.[18] The proofs of the results in this subsection are given in Appendixes A.3 and A.4.

Large economies Let $\mathbb{R}^\ell_+$ be the commodity space and let $\mathscr{P}_{\text{mo}}$ be the space of monotonic preference relations on $\mathbb{R}^\ell_+$. We endow the space $\mathscr{P}_{\text{mo}}$ with the metric of closed

[17]We thank the editor for suggesting this lemma.

[18]A probability space (or its σ-algebra) is said to be countably generated if its σ-algebra can be generated by countably many measurable subsets together with the null sets.

convergence.[19] Let $\mathcal{B}(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+})$ be the Borel σ-algebra on $\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}$. A *large economy* is a $\mathcal{G}$-measurable mapping $\mathscr{E}$ from the agent space $(\Omega, \mathcal{F}, P)$ to the space of characteristics $\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}$ such that for each $\omega \in \Omega$, $\mathscr{E}(\omega) = (\succsim_{\omega}, \mathbf{e}(\omega))$, and the mean endowment $\int_{\Omega} \mathbf{e} \, \mathrm{d}P$ is finite and strictly positive, where $(\Omega, \mathcal{G}, P)$ represents the characteristic type space.[20]

An integrable function $\mathbf{f}$ from the agent space $(\Omega, \mathcal{F}, P)$ to the commodity space $\mathbb{R}^{\ell}_{+}$ is called a *Walrasian allocation* for the economy $\mathscr{E}: (\Omega, \mathcal{G}, P) \to \mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}$ if there is a nonzero price vector $\mathbf{p} \in \mathbb{R}^{\ell}_{+}$ such that the following statements hold:

(i) For P-almost all $\omega \in \Omega$, $\mathbf{f}(\omega) \in D(\mathbf{p}, \succsim_{\omega}, \mathbf{e}(\omega))$, where $D(\mathbf{p}, \succsim_{\omega}, \mathbf{e}(\omega))$ is the set of all maximal elements for $\succsim_{\omega}$ in the budget set $\{\mathbf{x} \in \mathbb{R}^{\ell}_{+} : \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{e}(\omega)\}$.

(ii) We have $\int_{\Omega} \mathbf{f} \, \mathrm{d}P = \int_{\Omega} \mathbf{e} \, \mathrm{d}P$.

Here, $(\mathbf{f}, \mathbf{p})$ is called a *Walrasian equilibrium* and $\mathbf{p}$ is called a *Walrasian equilibrium price*. Let $W^{\mathcal{F}}(\mathscr{E})$ and $\mathcal{D}W^{\mathcal{F}}(\mathscr{E})$ denote the respective sets of all $\mathcal{F}$-measurable Walrasian allocations in the economy $\mathscr{E}$ and of their distributions.

The first issue we address is the determinateness problem in large economies. As shown in Example 1, the set of distributions of Walrasian allocations is not completely determined by the distribution of agents' characteristics. That is, there exist two atomless economies $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same distribution of agents' characteristics, and a Walrasian allocation f_2 of $\mathscr{E}_2$ such that the distribution of f_2 cannot be induced by any Walrasian allocation of $\mathscr{E}_1$.

We also consider the closed graph property for the Walrasian allocations in the setting of large economies. The basic idea of this property is quite clear: it simply asserts that any converging sequence of Walrasian allocations of a sequence of economies converging (in some sense) to a limit economy converges to a Walrasian allocation of the limit economy. Here is the formal definition. Let $\mathscr{E}$ be a $\mathcal{G}$-measurable economy from the agent space $(\Omega, \mathcal{F}, P)$ to the space of characteristics $\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}$. Let $\{(\Omega^k, \mathcal{G}^k, P^k)\}_{k \in \mathbb{Z}_+}$ be a sequence of probability spaces where Ω^k is finite, $\mathcal{G}^k$ is the power set of Ω^k, and $\sup_{\omega \in \Omega^k} P^k(\omega) \to 0$ as k goes to infinity. For each $k \in \mathbb{Z}_+$, let a finite-agent economy $\mathscr{E}^k$ be a mapping from $(\Omega^k, \mathcal{G}^k, P^k)$ to $\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}$. The Walrasian equilibrium correspondence $W^{\mathcal{F}}$ is said to have the *closed graph property* for the economy $\mathscr{E}$ if

(i) for any sequence of finite-agent economies $\{\mathscr{E}^k\}_{k \in \mathbb{Z}_+}$ which converges weakly to $\mathscr{E}$ in the sense that $\{P^k \circ (\mathscr{E}^k)^{-1}\}_{k \in \mathbb{Z}_+}$ converges weakly to $P \circ \mathscr{E}^{-1}$ and $\int_{\Omega^k} \mathbf{e}^k \, \mathrm{d}P^k$ converges to $\int_{\Omega} \mathbf{e} \, \mathrm{d}P$, and

(ii) for any sequence of Walrasian allocations $\{\mathbf{f}^k\}_{k \in \mathbb{Z}_+}$ ($\mathbf{f}^k$ is a Walrasian allocation in $\mathscr{E}^k$ for each $k \in \mathbb{Z}_+$) such that $\{P^k \circ (\mathbf{f}^k)^{-1}\}_{k \in \mathbb{Z}_+}$ converges weakly to a distribution μ on $\mathbb{R}^{\ell}_{+}$,

[19]The lemma on Hildenbrand (1974, p. 98) shows that the space $\mathscr{P}_{\text{mo}}$ with the metric of closed convergence is a G_{δ} set in a compact metric space. By the classical Alexandroff lemma (see Aliprantis and Border 2006, p. 88), $\mathscr{P}_{\text{mo}}$ is a Polish space.

[20]Another popular model for large economies is the model of replica economies; see, for example, Debreu and Scarf (1963), Hildenbrand (1974), and McLean and Postlewaite (2002).

there exists a Walrasian allocation $\mathbf{f}$: $(\Omega, \mathcal{F}, P) \to \mathbb{R}_+^\ell$ in the economy $\mathscr{E}$ such that $P \circ \mathbf{f}^{-1} = \mu$.[21] Example 2 shows the failure of the closed graph property for large economies in general.

The following theorem demonstrates that the nowhere equivalence condition is both sufficient and necessary for the determinateness and the closed graph property to hold in large economies.

Theorem 1 (Large economy). *The following statements are equivalent.*

(i) The σ-algebra $\mathcal{F}$ is nowhere equivalent to its sub-σ-algebra $\mathcal{G}$.

(ii) For any two $\mathcal{G}$-measurable large economies $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same distribution of agents' characteristics, $\mathcal{D}W^{\mathcal{F}}(\mathscr{E}_1)$ and $\mathcal{D}W^{\mathcal{F}}(\mathscr{E}_2)$ are the same, where $\mathcal{D}W^{\mathcal{F}}(\mathscr{E}_i)$ denotes the set of distributions of all the $\mathcal{F}$-measurable Walrasian allocations in the economy $\mathscr{E}_i$ for $i = 1, 2$.

(iii) The correspondence $W^{\mathcal{F}}$ of $\mathcal{F}$-measurable Walrasian allocations has the closed graph property for any $\mathcal{G}$-measurable economy $\mathscr{E}$.[22]

Remark 1. Recall that in Example 1, $\mathcal{F}$ is the Borel σ-algebra $\mathcal{B}$ on $[0, 1]$. However, the sub-σ-algebra $\mathcal{G}$, which is generated by $\mathscr{E}_1$, is also the Borel σ-algebra $\mathcal{B}$. Thus, statement (i) in Theorem 1 is not satisfied ($\mathcal{B}$ is not nowhere equivalent to itself) and the determinateness problem could occur.

Theorem 1(iii) states the closed graph property for large economies only in terms of a sequence of finite-agent spaces $\{(\Omega^k, \mathcal{G}^k, P^k)\}_{k\in\mathbb{Z}_+}$. In fact, the proof in the Appendix for the implication (i) $\Rightarrow$ (iii) in Theorem 1 does not rely on the finiteness of Ω^k. Thus, the closed graph property holds when one removes the restriction that Ω^k is finite for each $k \in \mathbb{Z}_+$, which also implies that $\mathcal{D}W^{\mathcal{F}}(\mathscr{E})$ is closed.

Large games Large games and their applications have been extensively studied.[23] Motivated by the consideration of social identities as in Akerlof and Kranton (2000) and Brock and Durlauf (2001), Khan et al. (2013) provided a treatment of large games in which individual agents have names as well as traits, and an agent's dependence on society is formulated as a joint probability measure on the space of actions and traits.

The agent space of a large game with traits is modeled by an atomless probability space $(\Omega, \mathcal{F}, P)$. Let A be a compact metric space that serves as the common action space for all the agents, and let T be a Polish space representing the traits of agents

[21] The main purpose of introducing a large economy is to model economies with large but finitely many agents. The closed graph property as defined here relates the limit economy to the relevant large finite economies.

[22] Kannai (1970) considered a similar convergence problem in which a sequence of continuous representations of finite-agent economies converges to a limit economy almost surely, while Hildenbrand (1974) adopted the distributional approach without explicit agent spaces. Thus, their approaches do not provide a direct relationship between the limit economy and the relevant large finite economies.

[23] See, for example, the survey by Khan and Sun (2002). For some recent applications of large games, see Angeletos et al. (2007), Guesnerie and Jara-Moroni (2011), Peters (2010), and Rauh (2007).

endowed with a Borel probability measure ρ. Let $\mathcal{M}(T \times A)$ be the space of Borel probability distributions on $T \times A$, and let $\mathcal{M}^{\rho}(T \times A)$ be the subspace of $\mathcal{M}(T \times A)$ such that for any $\nu \in \mathcal{M}^{\rho}(T \times A)$, its marginal probability ν_T (on T) is ρ. The set $\mathcal{M}^{\rho}(T \times A)$ is the space of societal responses. The space of agents' payoff functions $\mathcal{V}$, which can be regarded as the set of agents' characteristics, is the space of all continuous functions on the product space $A \times \mathcal{M}^{\rho}(T \times A)$ endowed with its sup-norm topology.

A *large game with traits* is a $\mathcal{G}$-measurable function $G = (\alpha, v)$ from the agent space $(\Omega, \mathcal{F}, P)$ to $T \times \mathcal{V}$ such that $P \circ \alpha^{-1} = \rho$, where $(\Omega, \mathcal{G}, P)$ is the characteristic type space.[24] A *pure-strategy Nash equilibrium* of the large game with traits G is an $\mathcal{F}$-measurable function $g\colon (\Omega, \mathcal{F}, P) \to A$ such that for P-almost all $\omega \in \Omega$,

$$v_\omega\big(g(\omega), P \circ (\alpha, g)^{-1}\big) \geq v_\omega\big(a, P \circ (\alpha, g)^{-1}\big) \quad \text{for all } a \in A,$$

where v_ω represents the function $v(\omega, \cdot)$.[25] Let $NE^{\mathcal{F}}(G)$ and $\mathcal{D}^{\mathcal{F}}(G)$ be the respective sets of $\mathcal{F}$-measurable pure-strategy Nash equilibria in the game G and of their distributions.

In Example 3, we have shown that the set of distributions of Nash equilibria is not completely determined by the distribution of agents' characteristics. Namely, for two large games G_1 and G_2 with $P \circ G_1^{-1} = P \circ G_2^{-1}$, the sets of distributions of Nash equilibria of games G_1 and G_2 could be different. Khan and Sun (1999) provided an example of two large games G_1 and G_2 with the same distribution of agents' characteristics such that G_1 has a Nash equilibrium, but G_2 does not. As a result, the closures of the sets of distributions for the Nash equilibria in their example are never equal, which indicates that the determinateness property may fail more severely for large games.

Rath et al. (1995) presented a large game (without traits) with an uncountable action space in which a pure-strategy Nash equilibrium does not exist. Khan et al. (2013) provided a large game with finite actions and an uncountable trait space, which does not have a pure-strategy Nash equilibrium either.

Let G be a $\mathcal{G}$-measurable large game with traits from $(\Omega, \mathcal{F}, P)$ to $T \times \mathcal{V}$. We follow Section 5 in Khan et al. (2013). Let T be compact, and let $\tilde{\mathcal{V}}$ be the space of all bounded continuous functions on the product space $A \times \mathcal{M}(T \times A)$. Let $\{(\Omega^k, \mathcal{G}^k, P^k)\}_{k \in \mathbb{Z}_+}$ be a sequence of probability spaces where Ω^k is finite, $\mathcal{G}^k$ is the power set of Ω^k, and $\sup_{\omega \in \Omega^k} P^k(\omega) \to 0$ as k goes to infinity. For each $k \in \mathbb{Z}_+$, let a finite-agent game $G^k = (\alpha^k, v^k)$ be a mapping from $(\Omega^k, \mathcal{G}^k, P^k)$ to $T \times \tilde{\mathcal{V}}$. The Nash equilibrium correspondence $NE^{\mathcal{F}}$ is said to have the *closed graph property* for the game G if

(i) for any sequence of finite-agent games $\{G^k\}_{k \in \mathbb{Z}_+}$ which converges weakly to G in the sense that $\{P^k \circ (G^k)^{-1}\}_{k \in \mathbb{Z}_+}$ converges weakly to $P \circ G^{-1}$, and

(ii) for any sequence of Nash equilibria $\{f^k\}_{k \in \mathbb{Z}_+}$ (f^k is a pure-strategy Nash equilibrium of G^k for each $k \in \mathbb{Z}_+$) such that $\{P^k \circ (f^k)^{-1}\}_{k \in \mathbb{Z}_+}$ converges weakly to a distribution μ on A,

[24] When the trait space is a singleton, we simply call such a large game with traits as a large game.

[25] Hereafter, the term "Nash equilibrium" refers to a pure-strategy Nash equilibrium when there is no confusion.

there exists a pure-strategy Nash equilibrium $f\colon (\Omega, \mathcal{F}, P) \to A$ in the game G such that $P \circ f^{-1} = \mu$. Example 4 shows the failure of the closed graph property for large games in general.

As an analog of Theorem 1, Theorem 2 below shows that if we distinguish the agent space $(\Omega, \mathcal{F}, P)$ from the characteristic type space $(\Omega, \mathcal{G}, P)$, then we are able to fully characterize the determinateness property, the equilibrium existence, and the closed graph property in the setting of large games with traits via the nowhere equivalence condition.[26]

Theorem 2 (Large game). *The following statements are equivalent.*

(i) The σ-algebra $\mathcal{F}$ is nowhere equivalent to its sub-σ-algebra $\mathcal{G}$.

(ii) For any two $\mathcal{G}$-measurable large games with traits G_1 and G_2 with the same distribution of agents' characteristics, $\mathcal{D}^{\mathcal{F}}(G_1)$ and $\mathcal{D}^{\mathcal{F}}(G_2)$ are the same, where $\mathcal{D}^{\mathcal{F}}(G_i)$ denotes the set of distributions of all the $\mathcal{F}$-measurable pure-strategy Nash equilibria in the game G_i for $i = 1, 2$.

(iii) Any $\mathcal{G}$-measurable large game with traits G has an $\mathcal{F}$-measurable pure-strategy Nash equilibrium g.

(iv) The correspondence $NE^{\mathcal{F}}$ of $\mathcal{F}$-measurable Nash equilibria has the closed graph property for any $\mathcal{G}$-measurable large game with traits G.[27]

In the setting of large games (with or without traits), Keisler and Sun (2009) and Khan et al. (2013) considered the necessity result for the existence of Nash equilibria, and Khan et al. (2013) and Qiao and Yu (2014) studied the necessity result for the closed graph property. All these papers worked with saturated agent spaces, which exclude the Lebesgue unit interval. Our Theorem 2 shows that the nowhere equivalence condition is also necessary for the existence of Nash equilibria and the closed graph property in large games with traits. Such results not only extend the earlier work, but also allow the possibility to work with nonsaturated agent spaces (by Lemma 5).[28]

Remark 2 (Necessity for the existence of Nash equilibria in large games). Since large games form a subclass of large games with traits, the existence of Nash equilibria in large games under the nowhere equivalence condition follows from Theorem 2. Furthermore, we also prove the necessity of nowhere equivalence for the existence of Nash equilibria in large games that goes beyond Theorem 2; see Remark 5 in Appendix A.4.

[26]The determinateness and existence problems in large games were resolved in Khan and Sun (1999) by working with a hyperfinite agent space; see Section 4.4 for more discussions.

[27]The closed graph property for large games with traits was shown in Khan et al. (2013) and Qiao and Yu (2014) by working with a saturated agent space, while Green (1984) considered a similar property based on the distributional approach without specifying the finite-agent spaces. Similar to Remark 1, our closed graph property for large games still holds when one removes the restriction that Ω^k is finite for each $k \in \mathbb{Z}_+$, which also implies that $\mathcal{D}^{\mathcal{F}}(G)$ is closed.

[28]To prove the necessity of the nowhere equivalence condition for the other statements of Theorems 1 and 2, we need to construct new large economies/games that are substantially more complicated than those in Examples 1–5; see Remark 6 for more details.

4. Unification

As discussed in Introduction, the condition of nowhere equivalence unifies various approaches that have been proposed to handle the shortcoming of the Lebesgue unit interval as an agent space. In this section, we discuss the unification in detail.

4.1 *Equivalent conditions*

In the following paragraphs, we introduce three equivalent conditions to nowhere equivalence, which are useful for discussing the unification of various previous approaches. We follow the notation in Section 3.

Definition 2. (i) The σ-algebra $\mathcal{F}$ is *conditional atomless* over $\mathcal{G}$ if for every $D \in \mathcal{F}$ with $P(D) > 0$, there exists an $\mathcal{F}$-measurable subset D_0 of D such that on some set of positive probability,

$$0 < P(D_0 \mid \mathcal{G}) < P(D \mid \mathcal{G}).$$

(ii) The σ-algebra $\mathcal{F}$ is said to be *relatively saturated* with respect to $\mathcal{G}$ if for any Polish spaces X and Y, any measure $\mu \in \mathcal{M}(X \times Y)$, and any $\mathcal{G}$-measurable mapping g from Ω to X with $P \circ g^{-1} = \mu_X$, there exists an $\mathcal{F}$-measurable mapping f from Ω to Y such that $\mu = P \circ (g, f)^{-1}$.

(iii) The sub-σ-algebra $\mathcal{G}$ admits an *atomless independent supplement* in $\mathcal{F}$ if there exists another sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$ such that $(\Omega, \mathcal{H}, P)$ is atomless, and for any $C_1 \in \mathcal{G}$ and $C_2 \in \mathcal{H}$, $P(C_1 \cap C_2) = P(C_1) \cdot P(C_2)$.

Condition (i), which is simply called "$\mathcal{F}$ is atomless over $\mathcal{G}$" in Definition 4.3 of Hoover and Keisler (1984), is a generalization of the usual notion of atomlessness. In particular, if $\mathcal{G}$ is the trivial σ-algebra, then $\mathcal{F}$ is atomless over $\mathcal{G}$ if and only if $\mathcal{F}$ is atomless. The concept of "relative saturation" refines the concept of "saturation" used in Corollary 4.5(i) of Hoover and Keisler (1984); see Section 4.5 for the formal definition of saturation. Condition (iii) simply indicates the abundance of events in $\mathcal{F}$ but independent of $\mathcal{G}$. The following lemma shows that all four of its conditions are equivalent if $\mathcal{G}$ is countably generated.

Lemma 2. *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. If $\mathcal{G}$ is countably generated,*[29] *then the following statements are equivalent:*[30]

(i) The σ-algebra $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

[29]The implication (iii) $\Rightarrow$ (iv) may not be true without the condition that $\mathcal{G}$ is countably generated. For example, if $\mathcal{G}$ is saturated and $\mathcal{F} = \mathcal{G}$, the statement (iii) holds while the statement (iv) is certainly false. Other implications are still true even though $\mathcal{G}$ is not countably generated.

[30]We thank an anonymous referee for pointing out an additional equivalent condition as follows. For every $D \in \mathcal{F}$ with $P(D) > 0$, there exists some $\epsilon > 0$ and an $\mathcal{F}$-measurable subset D_0 of D such that $P(D_0 \triangle D_1) \geq \epsilon$ for any $D_1 \in \mathcal{G}^D$.

(ii) *The σ-algebra $\mathcal{F}$ is conditional atomless over $\mathcal{G}$.*

(iii) *The σ-algebra $\mathcal{F}$ is relatively saturated with respect to $\mathcal{G}$.*

(iv) *The sub-σ-algebra $\mathcal{G}$ admits an atomless independent supplement in $\mathcal{F}$.*

4.2 *Distributional equilibria*

In the standard approach, a large economy/game is described by a measurable mapping from the agent space to the space of characteristics, and an equilibrium allocation/strategy profile is a measurable mapping from the agent space to the commodity/action space. In Hildenbrand (1974), the distributional approach was introduced in terms of the distribution of agents' characteristics without an explicit agent space, and the notion of Walrasian equilibrium distribution was proposed as a probability distribution on the product space of characteristics and commodities. The same idea was used in Mas-Colell (1984) for the notion of Nash equilibrium distribution in large games. Note that the joint distribution of a large economy/game and its equilibrium allocation/strategy profile automatically gives an equilibrium distribution. In the following discussion, we illustrate the point that given any large economy/game F as in Hildenbrand (1974) and Mas-Colell (1984), any equilibrium distribution τ associated with the corresponding distribution of F can be realized as the joint distribution of F and f, where f is an equilibrium allocation/strategy profile for the large economy/game F.[31] Throughout this subsection, we follow the definitions and notation in Section 3.2. All the proofs are given in Appendix A.5.

4.2.1 *Distributional approach in large economies* Let $D(\mathbf{p}, \succ, \mathbf{e})$ be the demand correspondence when the price vector, preference, and endowment are $\mathbf{p}$, $\succ$, and $\mathbf{e}$, respectively. Define a subset $E_{\mathbf{p}}$ of $\mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell \times \mathbb{R}_+^\ell$ as

$$E_{\mathbf{p}} = \{(\succ, \mathbf{e}, \mathbf{x}) \in \mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell \times \mathbb{R}_+^\ell : \mathbf{x} \in D(\mathbf{p}, \succ, \mathbf{e})\}.$$

Definition 3. A *distribution economy* with the preference space $\mathscr{P}_{\text{mo}}$ and the endowment space $\mathbb{R}_+^\ell$ is a Borel probability measure μ on the space of agents' characteristics $\mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell$.

A *Walrasian equilibrium distribution* of a distribution economy μ is a Borel probability measure τ on $(\mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell) \times \mathbb{R}_+^\ell$ with the following properties:

(i) The marginal distribution of τ on the space of characteristics $\mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell$ is μ.

(ii) There exists a nonzero price vector $\mathbf{p} \in \mathbb{R}_+^\ell$ such that $\tau(E_{\mathbf{p}}) = 1$.

(iii) We have $\int_{\mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell} \mathbf{e}\, \mathrm{d}\mu = \int_{\mathbb{R}_+^\ell} \mathbf{x}\, \mathrm{d}\nu$, where ν is the marginal distribution of τ on the space of consumption (the second $\mathbb{R}_+^\ell$ in $(\mathscr{P}_{\text{mo}} \times \mathbb{R}_+^\ell) \times \mathbb{R}_+^\ell$); i.e., mean supply equals mean demand.

[31] For some additional references on the distributional approach and its applications, see Daron and Wolitzky (2011), Eeckhout and Kircher (2010), Green (1984), and Noguchi and Zame (2006). The idea to obtain a measurable equilibrium allocation/strategy profile as described above also applies to the equilibrium distributions considered in all these papers.

The economy $\mathscr{E}_2$ and its Walrasian allocation f_2 in Example 1 can induce a joint distribution τ on $(\mathscr{P}_{\text{mo}} \times \mathbb{R}^\ell_+) \times \mathbb{R}^\ell_+$. As the economies $\mathscr{E}_1$ and $\mathscr{E}_2$ have the same distribution on agents' characteristics, they have the same set of Walrasian equilibrium distributions; see Hildenbrand (1974, p. 159). As a result, τ is also a Walrasian equilibrium distribution in the economy $\mathscr{E}_1$. Let μ be the marginal distribution of τ on the commodity space. We have shown that the economy $\mathscr{E}_1$ does not have a Walrasian allocation f_1 such that $\eta \circ f_1^{-1} = \mu$; this implies that τ cannot be realized as the joint distribution of $\mathscr{E}_1$ and any Walrasian allocation.

The "only if" part of the following corollary is a direct consequence of the relative saturation property.[32] Suppose that $\mathscr{E}$ is a $\mathcal{G}$-measurable economy from the agent space $(\Omega, \mathcal{F}, P)$ to the space of agents' characteristics $\mathscr{P}_{\text{mo}} \times \mathbb{R}^\ell_+$. Under the nowhere equivalence condition, every Walrasian equilibrium distribution τ of the induced distribution economy $P \circ \mathscr{E}^{-1}$ can be realized by an $\mathcal{F}$-measurable Walrasian allocation f in the sense that τ is the joint distribution of $(\mathscr{E}, f)$. The "if" part follows from the implication (ii) $\Rightarrow$ (i) in Theorem 1.

Corollary 1. *The σ-algebra $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ if and only if for any $\mathcal{G}$-measurable economy $\mathscr{E}\colon (\Omega, \mathcal{F}, P) \to \mathscr{P}_{\text{mo}} \times \mathbb{R}^\ell_+$ and any Walrasian equilibrium distribution τ of the induced distribution economy $P \circ \mathscr{E}^{-1}$, there exists an $\mathcal{F}$-measurable Walrasian allocation f such that $P \circ (\mathscr{E}, f)^{-1} = \tau$.*

4.2.2 *Distributional approach in large games* Let A be a compact metric space and let $\mathcal{U}$ be the space of all continuous functions on the product space $A \times \mathcal{M}(A)$ endowed with the sup-norm topology.

Definition 4. A *measure game* with the action space A is a Borel probability measure $\kappa \in \mathcal{M}(\mathcal{U})$.

A *Nash equilibrium distribution* of a measure game κ is a Borel probability measure $\tau \in \mathcal{M}(\mathcal{U} \times A)$ such that $\tau_{\mathcal{U}} = \kappa$ and

$$\tau\big(\{(u, x)\colon u(x, \tau_A) \geq u(a, \tau_A) \text{ for all } a \in A\}\big) = 1,$$

where $\tau_{\mathcal{U}}$ and τ_A are the marginal distributions of τ on $\mathcal{U}$ and A, respectively.

The following result, which is a corollary of Theorem 2, is a parallel result of Corollary 1 in the setting of large games.

Corollary 2. *The σ-algebra $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ if and only if for any $\mathcal{G}$-measurable game $G\colon (\Omega, \mathcal{F}, P) \to \mathcal{U}$, any Nash equilibrium distribution $\tau \in \mathcal{M}(\mathcal{U} \times A)$ of the induced measure game $P \circ G^{-1}$, there exists an $\mathcal{F}$-measurable Nash equilibrium $g\colon \Omega \to A$ such that $P \circ (G, g)^{-1} = \tau$.*

[32]Recall that the relative saturation property is equivalent to the nowhere equivalence condition, as shown in Lemma 2.

Remark 3. In Example 5, the failure of purification in a particular sense is discussed: given a large game G and a mixed-strategy profile g, one may not be able to find a pure-strategy profile f such that (G, f) and (G, g) have the same distribution. If $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$, Lemma 2 implies that for any $\mathcal{G}$-measurable large game G and any mixed-strategy profile g, there exists an $\mathcal{F}$-measurable pure-strategy profile f such that G and f induce the same joint distribution as G and g. The converse direction of this claim is also true. That is, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ if for any $\mathcal{G}$-measurable large game G and mixed-strategy profile g, there exists an $\mathcal{F}$-measurable pure-strategy profile f such that (G, f) and (G, g) induce the same distribution.[33]

4.2.3 *A principal–agent model with labor coercion* In this subsection, we provide a concrete example that shows how the nowhere equivalence condition is used to extract a pure-strategy Nash equilibrium from a given Nash equilibrium distribution.

Consider the principal–agent model of Daron and Wolitzky (2011), which contains coercive activities by employers. They proved in Proposition 12 the existence of a Nash equilibrium distribution, while pure-strategy Nash equilibria are used extensively in their context. Below, we show that a pure-strategy Nash equilibrium can be easily obtained via the nowhere equivalence condition.

Example 6. There is a population of mass 1 of producers, and a population of mass $L < 1$ of (identical) agents. At the initial stage, each producer is randomly matched with a worker with probability L. A successfully matched producer with productivity x will then choose a level of guns $g \geq 0$ at the cost $m(g)$ and offer a contract specifying an output-dependent wage-punishment pair $(w^y, p^y) \geq 0$ for $y \in \{h, l\}$, corresponding to high (x) and low (0) output, respectively.

- If the agent rejects the contract, then the producer receives a payoff 0 and the agent receives a payoff $\bar{u} - g$, where $\bar{u}$ is the outside option of the agent and $-g$ is the punishment enforced by the principal before the agent escapes.
- If the agent accepts the contract offer, then she chooses an effort level $a \in [0, 1]$ at private cost $c(a)$. Given output y, the producer's payoff is $P \cdot y - w^y - m(g)$ and the agent's payoff is $w^y - p^y - c(a)$, where P is the market price.

It is shown in Daron and Wolitzky (2011, Section 2.2) that the market price P and the outside option $\bar{u}$ are functions of the distribution of efforts a and the aggregate level of coercion g, respectively. Suppose that the functions P, $\bar{u}$, and m are all continuous, and that c is twice differentiable.

To establish the existence of an equilibrium contract, Daron and Wolitzky (2011) reformulated the model as a large game and adopted the distributional approach.[34] The

[33]Indeed, we prove a stronger result in the above Corollary 1: to derive the nowhere equivalence condition, one can focus on purification of mixed-strategy Nash equilibria instead of purification of all mixed-strategy profiles.

[34]See Appendix A of Daron and Wolitzky (2011). For more discussions, see Section 2.2 and Proposition 1 therein.

name space is $I = [\underline{x}, \bar{x}]$ ($\underline{x} > 0$) endowed with Borel σ-algebra $\mathcal{B}(I)$ and uniform distribution η, where $x \in I$ is the productivity of some producer. A producer with productivity x chooses (q, g) from the common action space $A = [0, \bar{x}] \times [0, \bar{g}]$ to maximize the payoff function

$$\pi(x)(q, g, Q, G) = q \cdot P(Q) - \frac{q}{x}\left[\left(1 - \frac{q}{x}\right) \cdot c'\left(\frac{q}{x}\right) + c\left(\frac{q}{x}\right) + \bar{u}(G) - g\right]_+ - \left(1 - \frac{q}{x}\right)\left[-\frac{q}{x} \cdot c'\left(\frac{q}{x}\right) + c\left(\frac{q}{x}\right) + \bar{u}(G) - g\right]_+ - m(g),$$

where (Q, G) is the aggregation of all producers' actions, and $[z]_+ = \max\{z, 0\}$.

In Proposition 12 of Daron and Wolitzky (2011), they showed that a Nash equilibrium distribution exists. We prove the existence of a pure-strategy Nash equilibrium under the nowhere equivalence condition. One can extend the name space $(I, \mathcal{B}(I), \eta)$ to $(I, \mathcal{F}, \eta')$ such that $\mathcal{B}(I)$ is a sub-σ-algebra of $\mathcal{F}$, η' coincides with η when restricted on $\mathcal{B}(I)$, and $\mathcal{F}$ is nowhere equivalent to $\mathcal{B}(I)$ under η'. Therefore, given a Nash equilibrium distribution, a corresponding $\mathcal{F}$-measurable pure-strategy Nash equilibrium f can be obtained via Corollary 2.

4.3 *Standard representation*

Hart et al. (1974) modeled the agent space as the product of the space of characteristics and the Lebesgue unit interval to obtain the exact determinateness property for large economies. Let ϱ be a distribution of agents' characteristics. Define an atomless economy $\mathscr{E}^{\varrho}$ as the "standard representation" of ϱ as follows. The atomless probability space of agents is given by $(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}) \times I$ with the product measure $P = \varrho \otimes \eta$, where $(I, \mathcal{B}, \eta)$ is the Lebesgue unit interval. The mapping $\mathscr{E}^{\varrho}$ is the projection from the agent space to the space of characteristics such that

$$\mathscr{E}^{\varrho}(\succsim, \mathbf{e}, i) = (\succsim, \mathbf{e}) \quad \text{for every } (\succsim, \mathbf{e}, i) \in \mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+} \times I.$$

Hart et al. (1974) showed that (i) the closure of the sets of distributions for all the Walrasian allocations is the same, provided that the distributions of two large economies are the same, and (ii) in a standard representation, the set of distributions of all the Walrasian allocations is closed. Thus, the determinateness problem was resolved via standard representations.

In the construction of a standard representation, the σ-algebra induced by $\mathscr{E}^{\varrho}$ is $\mathcal{B}(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}) \otimes \{I, \varnothing\}$, which admits an atomless independent supplement $\{\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}, \varnothing\} \otimes \mathcal{B}$ in $\mathcal{B}(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}) \otimes \mathcal{B}$. Hence, $\mathcal{B}(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}) \otimes \mathcal{B}$ is nowhere equivalent to $\mathcal{B}(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}) \otimes \{I, \varnothing\}$. Let $\mathcal{F} = \mathcal{B}(\mathscr{P}_{\text{mo}} \times \mathbb{R}^{\ell}_{+}) \otimes \mathcal{B}$. By Remark 1, $\mathcal{DW}^{\mathcal{F}}(\mathscr{E}^{\varrho})$ is closed for the large economy $\mathscr{E}^{\varrho}$.

Next, we consider a related result in Noguchi (2009, Corollary 1) that used the idea of standard representation in the setting of large games. Suppose that A is a compact metric action space and that $\mathcal{U}$ is the space of all continuous functions on the product space $A \times \mathcal{M}(A)$ endowed with the sup-norm topology and the resulting Borel σ-algebra $\mathcal{B}(\mathcal{U})$.

Let $\kappa \in \mathcal{M}(\mathcal{U})$ be a measure game and let τ be a Nash equilibrium distribution of κ as in Definition 4. Let π be the projection from $\mathcal{U} \times I$ to $\mathcal{U}$, where $(I, \mathcal{B}, \eta)$ is the Lebesgue unit interval. It was shown in Corollary 1 of Noguchi (2009) that there exists a pure-strategy Nash equilibrium f from $\mathcal{U} \times I$ to A such that $(\kappa \otimes \eta)(\pi, f)^{-1} = \tau$. This result is a special case of our Corollary 2 by taking $\Omega = \mathcal{U} \times I$, $\mathcal{F} = \mathcal{B}(\mathcal{U}) \otimes \mathcal{B}$, and $\mathcal{G} = \mathcal{B}(\mathcal{U}) \otimes \{\varnothing, I\}$.

4.4 *Hyperfinite agent space*

Khan and Sun (1999) argued that hyperfinite Loeb counting probability spaces[35] are particularly valuable for modeling situations where individual players are strategically negligible. In particular, such agent spaces allow one to go back and forth between exact results for the ideal case and approximate results for the asymptotic large finite case.[36] Hyperfinite Loeb counting probability spaces also have a nice property called *homogeneity*.[37] In particular, a probability space $(\Omega, \mathcal{F}, P)$ is said to be homogeneous if for any two random variables x and y on Ω with the same distribution, there is a bijection h from Ω to Ω such that both h and its inverse are measure-preserving, and $x(\omega) = y(h(\omega))$ for P-almost all $\omega \in \Omega$. Based on the homogeneity property, we can prove the following simple lemma, which shows that the σ-algebra in a hyperfinite Loeb counting probability space is nowhere equivalent to any of its countably generated sub-σ-algebras.

Lemma 3. *Let $(\Omega, \mathcal{F}, P)$ be a hyperfinite Loeb counting probability space, and let $\mathcal{G}$ be a countably generated sub-σ-algebra of $\mathcal{F}$. Then $\mathcal{G}$ admits an atomless independent supplement $\mathcal{H}$ in $\mathcal{F}$ and, hence, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

Proof. Since $\mathcal{G}$ is countably generated, there exists a mapping $g\colon \Omega \to ([0,1], \mathcal{B})$ that generates $\mathcal{G}$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$. Let $\mu = P \circ g^{-1}$. By the atomlessness property of $(\Omega, \mathcal{F}, P)$, there exists a mapping $(g', f')\colon \Omega \to [0,1] \times [0,1]$ such that $P \circ (g', f')^{-1} = \mu \otimes \eta$, where η is the Lebesgue measure on $[0,1]$. Note that g and g' share the same distribution on $\mathcal{F}$. By the homogeneity property, there is an $\mathcal{F}$-measurable and measure-preserving bijection h on Ω such that $g = g' \circ h$ P-almost surely. Let $f = f' \circ h$. Then (g, f) induces the distribution $\mu \otimes \eta$ and, hence, the sub-σ-algebra $\mathcal{H}$ generated by f is independent of $\mathcal{G}$. □

Rauh (2007) developed an equilibrium sequential search model that can accommodate heterogeneity in buyers' search costs and demand functions, and firms' cost functions (with general demand and cost functions). The existence of pure-strategy Nash equilibria is proved by assuming the name space of firms to be a hyperfinite Loeb counting probability space. Below, we show that the existence of a pure-strategy Nash equilibrium can still be obtained when the nowhere equivalence condition is imposed on the agent space; such a result extends the theorem of Rauh (2007).

[35] Such probability spaces were introduced by Loeb (1975). For the construction, see Loeb and Wolff (2015).

[36] See Brown and Robinson (1975), Brown and Loeb (1976), and the references in Anderson (1991) for the use of hyperfinite agent spaces in different contexts.

[37] See Proposition 9.2 of Keisler (1984).

Example 7. Let the agent space be a hyperfinite Loeb counting probability space $(J, \mathcal{J}, \mu_J)$. The space of feasible prices is a compact set $P = [0, \bar{p}] \cup \{\tilde{p}\}$, where $\bar{p} > 0$ is the price above which buyers do not pay and $\tilde{p} < 0$ is the shutdown price chosen by firms that elect not to operate. Let $\mathcal{D}$ be the set of all cumulative distribution functions (abbreviated as cdf henceforth) with support contained in P. The set P is compact; so is the set $\mathcal{D}$.

Let $\mathcal{C}$ be the set of all continuous functions from $[0, \bar{y}]$ to $\mathbb{R}_+$ endowed with the sup-norm topology, where $\bar{y}$ is the upper bound of the demand. The supply side of the market is then characterized by a measurable function $S\colon J \to \mathcal{C}$, which assigns a cost function to each firm. If firm j charges the price p, then its expected demand is $D(p \mid F)$, where $F \in \mathcal{D}$ is the cdf induced by the prices of other firms. Let $\pi\colon P \times \mathcal{D} \times \mathcal{C} \to \mathbb{R}$ be

$$\pi(p, F, C) = \begin{cases} p \cdot D(p \mid F) - C\big(D(p \mid F)\big) & \text{if } p \in [0, \bar{p}], \\ 0, & \text{if } p = \tilde{p}. \end{cases}$$

Let $\mathscr{P}$ be the space of all continuous functions from $P \times \mathcal{D}$ to $\mathbb{R}$ endowed with the sup-norm topology and let $\Pi\colon J \to \mathscr{P}$ be defined by $\Pi(j) = \pi(\cdot, \cdot, S(j))$. It is shown in Rauh (2007, Proposition 4) that π is continuous and Π is $\mathcal{J}$-measurable.

A search market equilibrium is a $\mathcal{J}$-measurable price profile $f\colon J \to P$ such that for μ_J-almost all $j \in J$, $\Pi(j)(f(j), F) \geq \Pi(j)(p, F)$ for all $p \in P$, where $F \in \mathcal{D}$ is the cdf induced by f. Rauh (2007) proved that a search market equilibrium exists. To reflect the dependence of Π on the mapping S, we use Π_S to denote Π.

We replace the agent space $(J, \mathcal{J}, \mu_J)$ by an atomless probability space $(\Omega, \mathcal{F}, Q)$ with a sub-σ-algebra $\mathcal{G}$ such that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. Consider any new $\mathcal{G}$-measurable game $\Pi_{S'}\colon \Omega \to \mathscr{P}$ with a $\mathcal{G}$-measurable cost function profile $S'\colon \Omega \to \mathcal{C}$ such that $\Pi_{S'}(\omega) = \pi(\cdot, \cdot, S'(\omega))$. Since the hyperfinite Loeb counting agent space $(J, \mathcal{J}, \mu_J)$ is atomless, one can always find a $\mathcal{J}$-measurable cost function profile $S\colon J \to \mathcal{C}$ with the same distribution as that of S', which implies that $Q \circ \Pi_{S'}^{-1} = \mu_J \circ \Pi_S^{-1}$. Since Rauh (2007) showed the existence of a search market equilibrium $f\colon J \to P$, the relative saturation property implies the existence of an $\mathcal{F}$-measurable function $g\colon \Omega \to P$ such that $Q \circ (\Pi_{S'}, g)^{-1} = \mu_J \circ (\Pi_S, f)^{-1}$. By similar arguments as in the proof for (i) $\Rightarrow$ (ii) of Theorem 2, g is a pure-strategy Nash equilibrium of the game $\Pi_{S'}$.

4.5 *Saturated agent space*

The following concept of a saturated probability space was introduced in Hoover and Keisler (1984).

Definition 5. An atomless probability space $(S, \mathcal{S}, Q)$ is said to have the *saturation property* for a probability distribution μ on the product of Polish spaces X and Y if for every random variable $f\colon S \to X$, which induces the distribution as the marginal distribution of μ over X, there is a random variable $g\colon S \to Y$ such that the induced distribution of the pair (f, g) on $(S, \mathcal{S}, Q)$ is μ.

A probability space $(S, \mathcal{S}, Q)$ is said to be *saturated* if for any Polish spaces X and Y, $(S, \mathcal{S}, Q)$ has the saturation property for every probability distribution μ on $X \times Y$.

As noted in Hoover and Keisler (1984), any atomless hyperfinite Loeb counting space is saturated. It is pointed out in Keisler and Sun (2009) that one can usually transfer a result on hyperfinite Loeb counting spaces to a result on saturated probability spaces via the saturation property.

The following statement is an obvious corollary of Lemma 2.

Corollary 3. *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Then the following statements are equivalent:*[38]

(i) The set $(\Omega, \mathcal{F}, P)$ is saturated.

(ii) The σ-algebra $\mathcal{F}$ is nowhere equivalent to any countably generated sub-σ-algebra.

(iii) The σ-algebra $\mathcal{F}$ is conditional atomless over any countably generated sub-σ-algebra.

(iv) The σ-algebra $\mathcal{F}$ is relatively saturated with respect to any countably generated sub-σ-algebra.

(v) Any countably generated σ-algebra admits an atomless independent supplement in $\mathcal{F}$.

Theorem 2 together with Remark 2 shows that any $\mathcal{G}$-measurable large game with/without traits has an $\mathcal{F}$-measurable Nash equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. The following result clearly follows from this characterization and Corollary 3; see also Keisler and Sun (2009) and Khan et al. (2013) in the settings of large games and large games with traits, respectively.

Corollary 4. *Let $(\Omega, \mathcal{F}, P)$ be an atomless agent space. Any large game with/without traits G has a Nash equilibrium if and only if $(\Omega, \mathcal{F}, P)$ is saturated.*

It is well known that the Lebesgue unit interval is countably generated and, hence, not saturated (see Keisler and Sun (2009)). However, one can extend the Lebesgue unit interval $(I, \mathcal{B}, \eta)$ to a saturated probability space $(I, \mathcal{F}, \eta')$ as in Kakutani (1944). Since $\mathcal{B}$ is countably generated, $\mathcal{B}$ admits an atomless countably generated independent supplement $\mathcal{H}$ in $\mathcal{F}$. Thus, for any $\mathcal{B}$-measurable large game, Theorem 2 implies that there always exists a $\sigma(\mathcal{B} \cup \mathcal{H})$-measurable Nash equilibrium. Note that $\sigma(\mathcal{B} \cup \mathcal{H})$ is again countably generated. Example 3 of Rath et al. (1995) shows the nonexistence of Nash equilibrium for a large game with the Lebesgue unit interval as the agent space. Khan and Zhang (2012) present a countably generated Lebesgue extension as the agent space such that the large game in Rath et al. (1995, Example 3) has a Nash equilibrium. Their result is a special case of our Theorem 2 since their countably generated Lebesgue extension includes an atomless independent supplement of $\mathcal{B}$.

[38] Condition (iii) is called $\aleph_1$-atomless in Hoover and Keisler (1984), and the equivalence between (i) and (iii) is shown in Corollary 4.5(i) therein. For additional equivalent conditions, see Fact 2.5 in Keisler and Sun (2009).

4.6 *Many more players than strategies*

Rustichini and Yannelis (1991) proposed the following "many more players than strategies" condition, which aims to solve the convexity problem of the Bochner integral of a correspondence from a finite measure space to an infinite-dimensional Banach space.[39] Yannelis (2009) used this condition to prove the existence of equilibria in large economies with asymmetric information.

For any given atomless agent space $(\Omega, \mathcal{F}, P)$, let $L^\infty(P)$ be the Banach space of all essentially bounded functions endowed with the norm $\|\cdot\|_\infty$, and let $L_E^\infty(P)$ be the subspace of $L^\infty(P)$ with the elements that vanish off E. Let $\text{card}(K)$ be the cardinality of a set K and let $\dim(Y)$ be the cardinality of a Hamel basis in a vector space Y (i.e., the algebraic dimension of the vector space Y).

Assumption (Many more players than strategies). *Let Z be an infinite-dimensional Banach space. For each $E \in \mathcal{F}$ with $P(E) > 0$, we assume that* $\dim(L_E^\infty(P)) > \dim(Z)$.

The following lemma shows that the above assumption implies the nowhere equivalence condition.

Lemma 4. *Let $(\Omega, \mathcal{F}, P)$ be an atomless agent space, let $(\Omega, \mathcal{G}, P)$ be a space of characteristic types, and let Z be an infinite-dimensional Banach space. Suppose that the assumption "many more players than strategies" holds for $(\Omega, \mathcal{F}/\mathcal{G}, P)$ and Z. Then $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

Proof. As discussed in Remark 4 of Rustichini and Yannelis (1991), an infinite-dimensional Banach space cannot have a countable Hamel basis. The condition of many more players than strategies implies that for each $E \in \mathcal{F}$ with $P(E) > 0$, $\dim(L_E^\infty(P))$ is uncountable and, hence, the subspace $(E, \mathcal{F}^E, P^E)$ is not countably generated. Thus, $\mathcal{F}$ is nowhere equivalent to its countably generated sub-σ-algebra $\mathcal{G}$. □

Appendix

A.1 *Proofs of Lemmas 1 and 2*

Proof of Lemma 2. *(i) ⇒ (ii).* Suppose that $\mathcal{F}$ is not conditional atomless over $\mathcal{G}$. Then there exists a subset $D \in \mathcal{F}$ with $P(D) > 0$ such that for any $\mathcal{F}$-measurable subset D_0 of D, we have $P(D_0 \mid \mathcal{G}) = 0$ or $P(D_0 \mid \mathcal{G}) = P(D \mid \mathcal{G})$ for P-almost all $\omega \in \Omega$. For such an $\mathcal{F}$-measurable set D_0, let $E = \{\omega\colon P(D_0 \mid \mathcal{G}) = P(D \mid \mathcal{G})\}$. Then we have $E \in \mathcal{G}$ and $P(D_0 \mid \mathcal{G}) = P(D \mid \mathcal{G}) \cdot 1_E = P(D \cap E \mid \mathcal{G})$ for P-almost all $\omega \in \Omega$, where 1_E is the indicator function of E. Thus, we can obtain

$$\begin{aligned} P(D_0) &= \int_\Omega 1_{D_0}\, \mathrm{d}P = \int_\Omega P(D_0 \mid \mathcal{G})\, \mathrm{d}P = \int_\Omega P(D \cap E \mid \mathcal{G})\, \mathrm{d}P \\ &= \int_\Omega 1_{D\cap E}\, \mathrm{d}P = P(D \cap E). \end{aligned}$$

[39]For further discussions on the relevance of this condition in general equilibrium theory, see Tourky and Yannelis (2001).

Next, we have

$$P(D_0 \cap E^c) = \int_\Omega P(D_0 \cap E^c \mid \mathcal{G}) \, \mathrm{d}P = \int_\Omega 1_{E^c} P(D_0 \mid \mathcal{G}) \, \mathrm{d}P$$
$$= \int_\Omega 1_{E^c} \cdot P(D \mid \mathcal{G}) \cdot 1_E \, \mathrm{d}P = 0.$$

Let $D_0' = D_0 \cap E^c$. Then D_0' is a P-null set and $(D_0 \setminus D_0') \subseteq D \cap E$. Thus, $P(D_0 \setminus D_0') = P(D \cap E)$ and $P(D_0 \triangle (D \cap E)) = 0$. This means that for any $\mathcal{F}$-measurable subset D_0 of D, there exits a set $D_1 = D \cap E$ in $\mathcal{G}^D$ such that $P(D_0 \triangle D_1) = 0$, which contradicts the assumption that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

(ii) $\Rightarrow$ *(i)*. Suppose that $\mathcal{F}$ is not nowhere equivalent to $\mathcal{G}$. Then there exists a subset $D \in \mathcal{F}$ with $P(D) > 0$ such that for any $\mathcal{F}$-measurable subset D_0 of D, there exists a subset $E \in \mathcal{G}$ with $P(D_0 \triangle (E \cap D)) = 0$. Thus, we have $P(D_0 \mid \mathcal{G}) = P(E \cap D \mid \mathcal{G}) = 1_E \cdot P(D \mid \mathcal{G})$ for P-almost all $\omega \in \Omega$, which contradicts the assumption that $\mathcal{F}$ is conditional atomless over $\mathcal{G}$.

(i) $\Rightarrow$ *(iii)*. Suppose that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. Since g is $\mathcal{G}$-measurable, $\mathcal{F}$ is also nowhere equivalent to $\sigma(g)$, where $\sigma(g)$ denotes the σ-algebra generated by g. By (i) $\Rightarrow$ (ii), $\mathcal{F}$ is conditional atomless over $\sigma(g)$. The claim then holds by referring to the proof of Corollary 4.5(i) in Hoover and Keisler (1984) (g and f here are x_1 and x_2 therein).

(iii) $\Rightarrow$ *(iv)*. Since $\mathcal{G}$ is countably generated, there exists a mapping g from Ω to $[0, 1]$ such that the σ-algebra $\mathcal{G}$ is generated by g. Let η be the Lebesgue measure on $[0, 1]$ and let $\mu = (P \circ g^{-1}) \otimes \eta$. Since $\mathcal{F}$ is relatively saturated with respect to $\mathcal{G}$, there exists an $\mathcal{F}$-measurable mapping f from Ω to $[0, 1]$ such that $P \circ (g, f)^{-1} = \mu$. It is clear that f is independent of g and generates an atomless σ-algebra. Therefore, the σ-algebra generated by f is atomless and independent of $\mathcal{G}$.

(iv) $\Rightarrow$ *(ii)*. This is exactly Lemma 4.4(iv) of Hoover and Keisler (1984). □

Proof of Lemma 1. The σ-algebra $\mathcal{G}' = \mathcal{G} \otimes \{\varnothing, I\}$, which admits an atomless independent supplement $\{\Omega, \varnothing\} \otimes \mathcal{B}$ in $\mathcal{F}' = \mathcal{F} \otimes \mathcal{B}$ under P'. By Lemma 2, $\mathcal{F}'$ is nowhere equivalent to $\mathcal{G}'$ under P'. □

The following lemma shows that for any atomless probability space, we can always find an atomless sub-σ-algebra to which the original σ-algebra is nowhere equivalent. This means that the condition of nowhere equivalence can be generally satisfied.

Lemma 5. *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Then there exists an atomless and countably generated sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

Proof. Consider the product space $(I \times I, \mathcal{B} \otimes \mathcal{B}, \eta \otimes \eta)$ of two Lebesgue unit intervals, where $I = [0, 1]$, $\mathcal{B}$ is the Borel σ-algebra, and η is the Lebesgue measure. Since $(\Omega, \mathcal{F}, P)$ is atomless, there exists a measurable mapping $f = (h, g)$ from Ω to $[0, 1] \times [0, 1]$ that induces $\eta \otimes \eta$. Then h and g are independent and generate atomless sub-σ-algebras $\mathcal{H}$ and $\mathcal{G}$ of $\mathcal{F}$, respectively. It is clear that $\mathcal{G}$ is countably generated and admits an atomless independent supplement $\mathcal{H}$ in $\mathcal{F}$. By Lemma 2, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. □

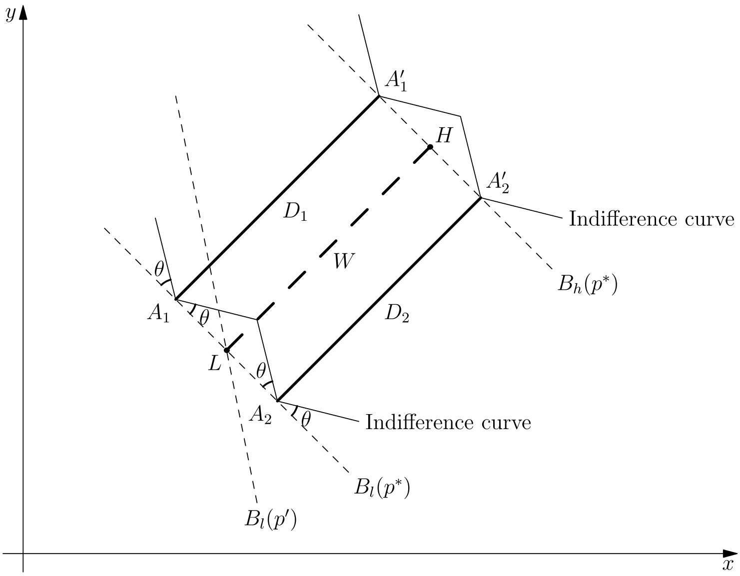

FIGURE 2. Indifference curves and budget lines.

A.2 *Proofs of Claims 1–3 in Section 2*

PROOF OF CLAIM 1. For the convenience of the reader, we provide the detailed proof of the claim, which is not included in Kannai (1970).

Let $\mathbf{p}^* = (1, 1)$ and

$$\mathbf{f}_2(i) = \begin{cases} \left(2i + \frac{3}{4}, 2i + \frac{5}{4}\right) & \text{if } i \in \left[0, \frac{1}{2}\right), \\ \left(2i + \frac{1}{4}, 2i - \frac{1}{4}\right) & \text{if } i \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We verify that $(\mathbf{f}_2, \mathbf{p}^*)$ is a Walrasian equilibrium in the economy $\mathscr{E}_2$. For the price $\mathbf{p}^* = (1, 1)$, the best response for agent i must be the intersection of her budget line and the set $D_1 \cup D_2$. If $i \in [0, \frac{1}{2})$, then $\mathbf{f}_2(i) \cdot (1, 1) = 4i + 2 = \mathbf{e}_2(i) \cdot (1, 1)$; if $i \in [\frac{1}{2}, 1]$, then $\mathbf{f}_2(i) \cdot (1, 1) = 4i = \mathbf{e}_2(i) \cdot (1, 1)$. Thus, $\mathbf{f}_2(i)$ is on the budget line of agent i. Furthermore, if $i \in [0, \frac{1}{2})$, then $(2i + \frac{5}{4}) - (2i + \frac{3}{4}) = \frac{1}{2}$; that is, $\mathbf{f}_2(i) \in D_1$. If $i \in [\frac{1}{2}, 1]$, then $(2i - \frac{1}{4}) - (2i + \frac{1}{4}) = -\frac{1}{2}$; that is, $\mathbf{f}_2(i) \in D_2$. Thus, $\mathbf{f}_2(i) \in D_1 \cup D_2$ for any $i \in [0, 1]$.

The distribution of $\mathbf{f}_2$ is the uniform distribution μ on the two darkened line segments $D_1 \cup D_2$ in Figure 2. Thus, the uniform distribution on $D_1 \cup D_2$ is in the set $\mathcal{DW}(\mathscr{E}_2)$.

Next, fix any Walrasian allocation $\mathbf{f}_1$ in the economy $\mathscr{E}_1$. Assume that $\mathbf{f}_1$ induces the uniform distribution μ on the set $D_1 \cup D_2$ with the corresponding Walrasian equilibrium price $\mathbf{p}'$. We show that $\mathbf{p}' = \mathbf{p}^* = (1, 1)$.

Suppose that $\mathbf{p}' = (a, b)$ with $a > b > 0$ (since the preference is monotonic, the equilibrium price for each good must be positive). For the agent $i \in I$ with initial endowment $L = (1, 1)$, denote her budget line by $B_l(\mathbf{p}')$ as shown in Figure 2.

It is clear from Figure 2 that there is a subsegment of D_1 with positive length such that every point on this subsegment is in the interior of the budget set. Thus, the probability measure induced by any Walrasian allocation under the price $\mathbf{p}'$ assigns zero probability on this subsegment of D_1, which implies that $\eta \circ \mathbf{f}_1^{-1}$ cannot be the uniform distribution on the set $D_1 \cup D_2$. Similar arguments also apply to the case $\mathbf{p}' = (a, b)$ with $b > a > 0$. Therefore, we must have $\mathbf{p}' = \mathbf{p}^* = (1, 1)$.

As discussed in the first paragraph of the proof of this claim, the best response of agent $i \in I$ is the intersection of the budget line and the set $D_1 \cup D_2$. That is, the best response correspondence is

$$\mathbf{F}(i) = \left\{\left(i + \tfrac{3}{4}, i + \tfrac{5}{4}\right), \left(i + \tfrac{5}{4}, i + \tfrac{3}{4}\right)\right\} \quad \text{for each } i \in I.$$

Since $\mathbf{f}_1$ is a Walrasian allocation in the economy $\mathscr{E}_1$, we have $\mathbf{f}_1(i) \in \mathbf{F}(i)$ for η-almost all $i \in I$. For $j = 1, 2$, let $C_j = \mathbf{f}_1^{-1}(D_j)$; we have $\mathbf{f}_1(C_j) \subseteq D_j$. Since $\mathbf{f}_1$ induces the μ on the set $D_1 \cup D_2$, we know that $\mu(\mathbf{f}_1(C_j)) = \eta(C_j)$. Now consider the case $j = 1$. Since μ is the uniform distribution on $D_1 \cup D_2$ and $\mathbf{f}_1(i) = (i + \frac{3}{4}, i + \frac{5}{4})$ for any $i \in C_1$, we can obtain $\mu(\mathbf{f}_1(C_1)) = \eta(C_1)/2$. Similarly, we have $\mu(\mathbf{f}_1(C_2)) = \eta(C_2)/2$. Therefore, for each $j = 1, 2$, $\eta(C_j) = \eta(C_j)/2$, which implies that $\eta(C_j) = 0$. This is a contradiction. Therefore, the uniform distribution on $D_1 \cup D_2$ is not in the set $\mathcal{DW}(\mathscr{E}_1)$ and, hence, $\mathcal{DW}(\mathscr{E}_1) \neq \mathcal{DW}(\mathscr{E}_2)$. □

Remark 4. For any $i \in [0, \frac{1}{2})$, agents i and $\frac{1}{2} + i$ have the same characteristics (preference and endowment) in $\mathscr{E}_2$. However, their equilibrium consumptions as given by $\mathbf{f}_2$ are $(2i + \frac{3}{4}, 2i + \frac{5}{4})$ and $(2i + \frac{5}{4}, 2i + \frac{3}{4})$, respectively, which are different. In fact, it is easy to see that a stronger result holds as follows.

Let $\mathbf{g}_2$ be any Walrasian allocation in the economy $\mathscr{E}_2$ that induces the uniform distribution μ on $D_1 \cup D_2$, and let J be the set of all agents $j \in [0, \frac{1}{2})$ such that agent $j \in J$ and agent $\frac{1}{2} + j$ choose the same consumption. Suppose that $\eta(J) > 0$. As in Proof of Claim 1 above, the equilibrium price must be $\mathbf{p}^* = (1, 1)$, and for any $j \in [0, \frac{1}{2})$, agents j and $\frac{1}{2} + j$ have the same set of optimal consumptions $\mathbf{G}(j) = \{(2j + \frac{3}{4}, 2j + \frac{5}{4}), (2j + \frac{5}{4}, 2j + \frac{3}{4})\}$. Since $\mathbf{g}_2$ is a Walrasian allocation, there is a subset $\bar{J}$ of J with $\eta(J \setminus \bar{J}) = 0$ such that for all $j \in \bar{J}$, $\mathbf{g}_2(j)$ is $(2j + \frac{3}{4}, 2j + \frac{5}{4})$ or $(2j + \frac{5}{4}, 2j + \frac{3}{4})$. Let $\bar{J}_1 = \{j \in \bar{J} : \mathbf{g}_2(j) = (2j + \frac{3}{4}, 2j + \frac{5}{4})\}$ and $\bar{J}_2 = \{j \in \bar{J} : \mathbf{g}_2(j) = (2j + \frac{5}{4}, 2j + \frac{3}{4})\}$. Without loss of generality, assume that $\eta(\bar{J}_1) > 0$. Since for any $j \in \bar{J}_1$, agents j and $\frac{1}{2} + j$ choose the same consumption, the set $\{(2j + \frac{5}{4}, 2j + \frac{3}{4}) : j \in \bar{J}_1\}$ has measure zero under the distribution induced by $\mathbf{g}_2$ but positive measure under the uniform distribution μ on $D_1 \cup D_2$, which is a contradiction. Therefore, $\eta(J) = 0$. That is, except for a null set of agents, different agents with the same characteristics have to select different optimal choices. A similar result also holds for the game G_2 in Claim 3.

Proof of Claim 2. For any fixed $k \in \mathbb{Z}_+$, we consider the economy $\mathscr{E}_1^k$ with finite agent space Ω^k. For the price $\mathbf{p}^* = (1, 1)$, the best response for agent $j \in \Omega^k$ must be the intersection of her budget line and the set $D_1 \cup D_2$. For any agent $j \in \Omega^k$, since

$$\left(\frac{3}{4} + \frac{j}{2k}\right) + \left(\frac{5}{4} + \frac{j}{2k}\right) = 2 + \frac{j}{k},$$

$\mathbf{f}_1^k(j)$ is on her budget line. Moreover, when j is an odd number, we have

$$\left(\frac{5}{4}+\frac{j}{2k}\right)-\left(\frac{3}{4}+\frac{j}{2k}\right)=\frac{1}{2},$$

which implies that $\mathbf{f}_1^k(j)\in D_1$; when j is an even number, we have

$$\left(\frac{3}{4}+\frac{j}{2k}\right)-\left(\frac{5}{4}+\frac{j}{2k}\right)=-\frac{1}{2},$$

which implies that $\mathbf{f}_1^k(j)\in D_2$.

The total supply of the first good in the market is $2k+\sum_{j=1}^{2k} j/(2k)$. The total demand of the first good in the market is

$$k\cdot\left(\frac{3}{4}+\frac{5}{4}\right)+\sum_{j=1}^{2k}\frac{j}{2k}=2k+\sum_{j=1}^{2k}\frac{j}{2k}.$$

Thus, the total demand and the total supply of the first good are the same. Similarly, one can check that the market clearing condition is also satisfied for the second good. Therefore, $(\mathbf{f}_1^k,\mathbf{p}^*)$ is a Walrasian equilibrium in the economy $\mathscr{E}_1^k$. □

Proof of Claim 3. Let

$$f_2(i)=\begin{cases}2i & \text{if } i\in\left[0,\frac{1}{2}\right),\\ 1-2i & \text{if } i\in\left[\frac{1}{2},1\right].\end{cases}$$

It is clear that f_2 is a pure-strategy Nash equilibrium of the game G_2 and it induces the uniform distribution μ on $A=[-1,1]$.

For any pure-strategy Nash equilibrium f_1 of the game G_1, we must have $f_1(i)\in\{i,-i\}$ for η-almost all $i\in[0,1]$, which means that $f_1(i)=i$ on a Borel measurable set $C\subseteq[0,1]$, and $f_1(i)=-i$ on $[0,1]\setminus C$. Let $D=\{-i\mid i\in[0,1]\setminus C\}$. Then we have

$$\begin{aligned}\left(\eta\circ f_1^{-1}(C),\eta\circ f_1^{-1}(D)\right)&=\left(\eta(C),1-\eta(C)\right)\\&\neq\left(\frac{\eta(C)}{2},\frac{1-\eta(C)}{2}\right)\\&=\left(\mu(C),\mu(D)\right).\end{aligned}$$

That is, there does not exist a pure-strategy Nash equilibrium f_1 of G_1 such that f_1 and f_2 have the same distribution. □

A.3 *Proof of Theorem 1*

We divide the proof of Theorem 1 into five parts. In the first two parts, we prove the directions (i) ⇒ (ii) and (i) ⇒ (iii). In Part 3, we present two lemmas and one example that are used for the proof of (ii) ⇒ (i) and (iii) ⇒ (i) in Parts 4 and 5, respectively.

Part 1: Proof of (i) $\Rightarrow$ (ii) Let $\mathscr{E}_1 = (\succ_1, \mathbf{e}_1)$ and $\mathscr{E}_2 = (\succ_2, \mathbf{e}_2)$ be two $\mathcal{G}$-measurable economies with the same distribution. Then we have $P \circ \mathbf{e}_1^{-1} = P \circ \mathbf{e}_2^{-1}$, which implies that the two economies $\mathscr{E}_1$ and $\mathscr{E}_2$ have the same mean endowment. That is, $\int_\Omega \mathbf{e}_1 \, \mathrm{d}P = \int_\Omega \mathbf{e}_2 \, \mathrm{d}P$.

Next, let $\mathbf{f}_1$ be a Walrasian allocation of $\mathscr{E}_1$ with the corresponding equilibrium price $\mathbf{p}$. Then we have $\int_\Omega \mathbf{f}_1 \, \mathrm{d}P = \int_\Omega \mathbf{e}_1 \, \mathrm{d}P$. Because $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$, $\mathcal{F}$ is relatively saturated with respect to $\mathcal{G}$ by Lemma 2. Since $P \circ \mathscr{E}_1^{-1} = P \circ \mathscr{E}_2^{-1}$, there exists an $\mathcal{F}$-measurable mapping $\mathbf{f}_2 \colon \Omega \to \mathbb{R}_+^\ell$ such that $P \circ (\mathscr{E}_1, \mathbf{f}_1)^{-1} = P \circ (\mathscr{E}_2, \mathbf{f}_2)^{-1}$. Hence, $P \circ \mathbf{f}_1^{-1} = P \circ \mathbf{f}_2^{-1}$, which implies that $\int_\Omega \mathbf{f}_1 \, \mathrm{d}P = \int_\Omega \mathbf{f}_2 \, \mathrm{d}P$. Therefore, $\int_\Omega \mathbf{f}_2 \, \mathrm{d}P = \int_\Omega \mathbf{e}_2 \, \mathrm{d}P$.

As in the distributional approach for large economies in Section 4.2, define a set $E_\mathbf{p}$ such that

$$E_\mathbf{p} = \{(\succ, \mathbf{e}, \mathbf{x}) \in \mathscr{P}_{\mathrm{mo}} \times \mathbb{R}_+^\ell \times \mathbb{R}_+^\ell : \mathbf{x} \in D(\mathbf{p}, \succ, \mathbf{e})\},$$

where $D(\mathbf{p}, \succ, \mathbf{e})$ is the demand correspondence for the price vector $\mathbf{p}$, preference $\succ$, and endowment $\mathbf{e}$. Since $P \circ (\mathscr{E}_1, \mathbf{f}_1)^{-1}(E_\mathbf{p}) = 1$, we have $P \circ (\mathscr{E}_2, \mathbf{f}_2)^{-1}(E_\mathbf{p}) = 1$. Thus, the set $H = \{\omega \in \Omega : (\mathscr{E}_2(\omega), \mathbf{f}_2(\omega)) \in E_\mathbf{p}\}$ has probability 1 under P. This means that for any $\omega \in H$, $\mathbf{f}_2(\omega) \in D(\mathbf{p}, \mathscr{E}_2(\omega))$. Hence, $\mathbf{f}_2$ is a Walrasian allocation in $\mathscr{E}_2$ with the equilibrium price $\mathbf{p}$.

Since $P \circ \mathbf{f}_2^{-1} = P \circ \mathbf{f}_1^{-1}$, the arbitrary choice of $\mathbf{f}_1^{-1}$ implies that $\mathcal{DW}^\mathcal{F}(\mathscr{E}_1) \subseteq \mathcal{DW}^\mathcal{F}(\mathscr{E}_2)$. By symmetry, we also have $\mathcal{DW}^\mathcal{F}(\mathscr{E}_2) \subseteq \mathcal{DW}^\mathcal{F}(\mathscr{E}_1)$, which implies that $\mathcal{DW}^\mathcal{F}(\mathscr{E}_1) = \mathcal{DW}^\mathcal{F}(\mathscr{E}_2)$.

Part 2: Proof of (i) $\Rightarrow$ (iii) For any $k \in \mathbb{Z}_+$, let $\tau^k = P^k \circ (\mathscr{E}^k, \mathbf{f}^k)^{-1}$ and $\mu^k = P^k \circ (\mathbf{f}^k)^{-1}$. Assume that μ^k converges weakly to some $\mu \in \mathcal{M}(\mathbb{R}_+^\ell)$ as k goes to infinity. Since $P^k \circ (\mathscr{E}^k)^{-1}$ converges weakly to $P \circ \mathscr{E}^{-1}$, Lemma 2.1 in Keisler and Sun (2009) implies that there exists a subsequence of $\{\tau^k\}_{k \in \mathbb{Z}_+}$ (say itself), which converges weakly to some $\tau \in \mathcal{M}(\mathscr{P}_{\mathrm{mo}} \times \mathbb{R}_+^\ell \times \mathbb{R}_+^\ell)$. It is clear that the respective marginals of τ on $\mathscr{P}_{\mathrm{mo}} \times \mathbb{R}_+^\ell$ and $\mathbb{R}_+^\ell$ are $P \circ \mathscr{E}^{-1}$ and μ. Since $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$, there exists an $\mathcal{F}$-measurable function $\mathbf{f} \colon \Omega \to \mathbb{R}_+^\ell$ such that $P \circ (\mathscr{E}, \mathbf{f})^{-1} = \tau$. It is obvious that $P \circ \mathbf{f}^{-1} = \mu$. Theorem 3 in Hildenbrand (1974, p. 159) shows that τ is a Walrasian equilibrium distribution (as defined on p. 158 of Hildenbrand (1974)) corresponding to $P \circ \mathscr{E}^{-1}$. As shown in the proof of (i) $\Rightarrow$ (ii), $\mathbf{f}$ is a Walrasian allocation in economy $\mathscr{E}$.

Part 3: The preparation for proving the necessity of nowhere equivalence The result in the following lemma is well known.[40] Here we give a simple and direct proof.

Lemma 6. *If (Ξ, Σ, Λ) is an atomless probability space and Σ is countably generated, then there exists a measure-preserving mapping ψ from (Ξ, Σ, Λ) to the Lebesgue unit interval $(I, \mathcal{B}, \eta)$ such that for any $E \in \Sigma$, there exists a set $E' \in \mathcal{B}$ such that $\Lambda(E \triangle \psi^{-1}(E')) = 0$.*

Proof. Since Σ is countably generated, Theorem 6.5.5 in Bogachev (2007) implies that there is a measurable mapping ψ_1 from Ξ to I such that ψ_1 could generate the σ-algebra Σ. The induced measure $\Lambda \circ \psi_1^{-1}$ on I is atomless since (Ξ, Σ, Λ) is also atomless.

[40]This result plays a key role in obtaining the necessity of saturation in Keisler and Sun (2009). See Fremlin (1989) for a general result.

Moreover, by Theorem 16 (p. 409) in Royden (1988), $(I, \mathcal{B}, \Lambda \circ \psi_1^{-1})$ is isomorphic to the Lebesgue unit interval $(I, \mathcal{B}, \eta)$; denote this isomorphism by ψ_2. Let $\psi = \psi_2 \circ \psi_1$. Then ψ satisfies the requirement. □

As in Section 3.1, for a given $D \in \mathcal{F}$ with $P(D) > 0$, $\mathcal{G}^D$ denotes the σ-algebra $\{D \cap D' \colon D' \in \mathcal{G}\}$ while $P|_D$ and P^D represent, respectively, the restriction of P to D and the probability measure rescaled from the restriction of P to D. Recall that the probability space $(\Omega, \mathcal{F}, P)$ is assumed to be atomless, but $(\Omega, \mathcal{G}, P)$ may not be atomless. There exist two disjoint $\mathcal{G}$-measurable subsets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \Omega$, and $(\Omega_1, \mathcal{G}^{\Omega_1}, P|_{\Omega_1})$ and $(\Omega_2, \mathcal{G}^{\Omega_2}, P|_{\Omega_2})$ are, respectively, atomless and purely atomic.[41] Let $P(\Omega_1) = \gamma$. If $\gamma = 0$, then $(\Omega, \mathcal{G}, P)$ is purely atomic and the nowhere equivalence condition is automatically satisfied. Thus, we only need to consider the case $0 < \gamma \leq 1$.

Lemma 6 shows that there exists a measure-preserving mapping

$$\phi\colon \big(\Omega_1, \mathcal{G}^{\Omega_1}, P|_{\Omega_1}\big) \to \big([0, \gamma), \mathcal{B}^{[0,\gamma)}, \eta_1\big) \tag{1}$$

such that for any $E \in \mathcal{G}^{\Omega_1}$, there exists a subset $E' \in \mathcal{B}^{[0,\gamma)}$ with $P(E \triangle \phi^{-1}(E')) = 0$, where η_1 is the Lebesgue measure on $\mathcal{B}^{[0,\gamma)}$.

The following lemma provides another characterization of the nowhere equivalence condition.

Lemma 7. *Let $\bar{\mathbb{Z}}_+$ be an infinite subset of the set $\mathbb{Z}_+$ of positive integers. If for each $n \in \bar{\mathbb{Z}}_+$, there exists an $\mathcal{F}$-measurable partition $\{E_1, E_2, \ldots, E_n\}$ of Ω_1 such that $P^{\Omega_1}(E_j) = 1/n$ and E_j is independent of $\mathcal{G}^{\Omega_1}$ under P^{Ω_1} for $j = 1, 2, \ldots, n$, then $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ under P.*

Proof. We prove this result by contradiction. Suppose that $\mathcal{F}$ is not nowhere equivalent to $\mathcal{G}$ under P. Then there exists a nonnegligible subset $D \in \mathcal{F}$ such that for any $L_1 \in \mathcal{F}^D$, there exists a subset $L_2 \in \mathcal{G}^D$ with $P(L_1 \triangle L_2) = 0$. If $P(\Omega_2 \cap D) > 0$, then $\mathcal{G}^{\Omega_2 \cap D}$ is purely atomic while $\mathcal{F}^{\Omega_2 \cap D}$ is atomless, which is a contradiction. Thus, we can assume that D is a subset of Ω_1. Choose a sufficiently large integer $n \in \bar{\mathbb{Z}}_+$ so that $1/n < \frac{1}{2}P^{\Omega_1}(D)$. Let $E_1, E_2, \ldots, E_n$ be n subsets satisfying the assumption of this lemma.

For P^{Ω_1}-almost all $\omega \in \Omega_1$, we have

$$P^{\Omega_1}\big(E_j \cap D \mid \mathcal{G}^{\Omega_1}\big) \leq P^{\Omega_1}\big(E_j \mid \mathcal{G}^{\Omega_1}\big) = P^{\Omega_1}(E_j) = \frac{1}{n} < \frac{1}{2}P^{\Omega_1}(D)$$

for $j = 1, 2, \ldots, n$. Denote $E = \{\omega \in \Omega_1 \colon P^{\Omega_1}(D \mid \mathcal{G}^{\Omega_1}) > \frac{1}{2}P^{\Omega_1}(D)\}$. Then it is clear that $E \in \mathcal{G}^{\Omega_1}$ and $P^{\Omega_1}(E) > 0$. For each j, there exists a set $C_j \in \mathcal{G}^{\Omega_1}$ such that $P((E_j \cap D) \triangle (C_j \cap D)) = 0$. Thus, for P^{Ω_1}-almost all $\omega \in \Omega_1$, we have

$$1_{C_j} \cdot P^{\Omega_1}\big(D \mid \mathcal{G}^{\Omega_1}\big) = P^{\Omega_1}\big(C_j \cap D \mid \mathcal{G}^{\Omega_1}\big) = P^{\Omega_1}\big(E_j \cap D \mid \mathcal{G}^{\Omega_1}\big) < \frac{1}{2}P^{\Omega_1}(D),$$

which implies that $P^{\Omega_1}(C_j \cap E) = 0$. As usual, 1_{C_j} denotes the indicator function of the set C_j.

[41] A measure space is purely atomic if it has no atomless part.

Next, we have

$$\begin{aligned} P^{\Omega_1}\left(D\cap\left(\bigcup_{j=1}^n C_j\right)\right) &= \int_{\Omega_1} P^{\Omega_1}\left(D\cap\left(\bigcup_{j=1}^n C_j\right)\,\Big|\,\mathcal{G}^{\Omega_1}\right)\mathrm{d}P^{\Omega_1} \\ &= \int_{\Omega_1} P^{\Omega_1}\left(D\cap\left(\bigcup_{j=1}^n E_j\right)\,\Big|\,\mathcal{G}^{\Omega_1}\right)\mathrm{d}P^{\Omega_1} \\ &= \int_{\Omega_1} P^{\Omega_1}\left(D\mid\mathcal{G}^{\Omega_1}\right)\mathrm{d}P^{\Omega_1} \\ &= P^{\Omega_1}(D) \end{aligned}$$

and, hence, $P^{\Omega_1}(D\setminus(\bigcup_{j=1}^n C_j))=0$. Moreover,

$$\begin{aligned} P^{\Omega_1}(D\cap E) &= \int_{\Omega_1} P^{\Omega_1}\left(D\cap E\mid\mathcal{G}^{\Omega_1}\right)\mathrm{d}P^{\Omega_1} = \int_{\Omega_1} 1_E\cdot P^{\Omega_1}\left(D\mid\mathcal{G}^{\Omega_1}\right)\mathrm{d}P^{\Omega_1} \\ &> \frac{1}{2}P^{\Omega_1}(D)\cdot P^{\Omega_1}(E)>0. \end{aligned}$$

Thus, we have $P^{\Omega_1}((\bigcup_{j=1}^n C_j)\cap E)>0$, which contradicts the fact that $P^{\Omega_1}(C_j\cap E)=0$ for $j=1,2,\dots,n$. Therefore, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ under P. □

In the following illustration, we construct an example for proving the necessity of nowhere equivalence in the next two parts.

Example 8. Fix an integer $n\geq 1$. Consider the following two economies $\mathcal{E}_1$ and $\mathcal{E}_2$ with two goods. The agent space for each of the economies $\mathcal{E}_1$ and $\mathcal{E}_2$ is the probability space $(\Omega,\mathcal{F},P)$. In both economies, all the agents have the same preference, and the corresponding indifference curves are parallel as shown in Figure 3. For $i=1,2,\dots,2n$, the ith line segment D_i is represented by $y=x+1-(2i-1)/(2n)$ for $x\in[(2n+2i-1)/(4n),(6n+2i-1)/(4n)]$.

The set of endowments is represented by the line segment W: $y=x$ for $x\in[1,2]$ with endpoints $L=(1,1)$ and $H=(2,2)$. Let $\mathbf{p}^*=(1,1)$. The parallel dashed lines $B_l(\mathbf{p}^*)$ and $B_h(\mathbf{p}^*)$ are perpendicular to the line segment D_i for each i. The angle θ is chosen to be sufficiently small so that the preference is monotonic.

The endowment in the economy $\mathcal{E}_1$ is given by

$$\mathbf{e}_1(\omega)=\begin{cases}\left(1+\dfrac{\phi(\omega)}{\gamma},1+\dfrac{\phi(\omega)}{\gamma}\right) & \text{if } \omega\in\Omega_1,\\ (0,0) & \text{if } \omega\in\Omega_2,\end{cases}$$

and the endowment in the economy $\mathcal{E}_2$ is given by

$$\mathbf{e}_2(\omega)=\begin{cases}\left(1+2n\left(\dfrac{\phi(\omega)}{\gamma}-\dfrac{i-1}{2n}\right),1+2n\left(\dfrac{\phi(\omega)}{\gamma}-\dfrac{i-1}{2n}\right)\right) & \text{if } \omega\in\Omega_1^i,\\ (0,0) & \text{if } \omega\in\Omega_2,\end{cases}$$

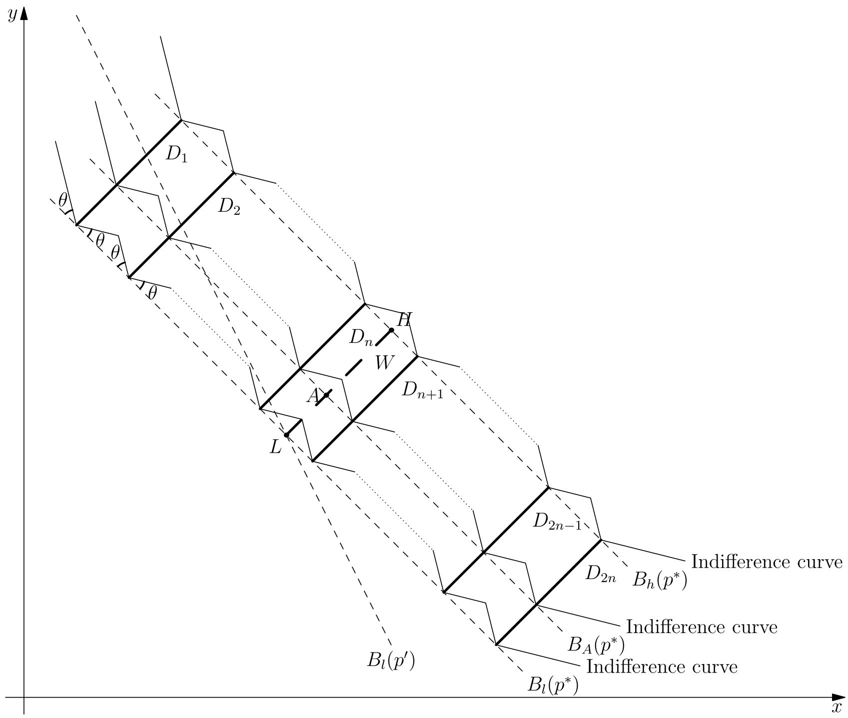

FIGURE 3. Indifference curves and budget lines.

where $\Omega_1^i = \{\omega \in \Omega_1 : \phi(\omega) \in [(i-1)/(2n)\gamma, i/(2n)\gamma)\}$ for $i = 1, 2, \ldots, 2n$ and ϕ has been defined in (1). Since the mapping ϕ is measure-preserving, the distributions of economies $\mathscr{E}_1$ and $\mathscr{E}_2$ are the same in the sense that $P \circ \mathbf{e}_1^{-1} = P \circ \mathbf{e}_2^{-1}$.[42]

Part 4: Proof of (ii) ⇒ (i) We prove this direction based on Lemma 7 and Example 8.

It is clear that both economies in Example 8 are $\mathcal{G}$-measurable. Since the distributions of economies $\mathscr{E}_1$ and $\mathscr{E}_2$ are the same, statement (ii) in Theorem 1 says that $\mathcal{DW}^{\mathcal{F}}(\mathscr{E}_1) = \mathcal{DW}^{\mathcal{F}}(\mathscr{E}_2)$.

We first show that $(\mathbf{f}_2, \mathbf{p}^*)$ is a Walrasian equilibrium of economy $\mathscr{E}_2$, where $\mathbf{p}^* = (1, 1)$ and

$$\mathbf{f}_2(\omega) = \begin{cases} \left(\dfrac{2n+2i-1}{4n} + 2n\left(\dfrac{\phi(\omega)}{\gamma} - \dfrac{i-1}{2n}\right), \right. \\ \left. \quad \dfrac{6n-2i+1}{4n} + 2n\left(\dfrac{\phi(\omega)}{\gamma} - \dfrac{i-1}{2n}\right)\right) & \text{if } \omega \in \Omega_1^i, \\ (0, 0) & \text{if } \omega \in \Omega_2 \end{cases}$$

for $i = 1, 2, \ldots, 2n$.

For the price vector $\mathbf{p}^*$, the best response for agent $\omega \in \Omega_1$ must be the intersection of her budget line and the set $\bigcup_{i=1}^{2n} D_i$.

[42] Example 1 is a special case of Example 8 when $n = 1$ and $(\Omega, \mathcal{F}, P)$ is the Lebesgue unit interval.

For agent $\omega \in \Omega_1^i$, since $\mathbf{f}_2(\omega) \cdot \mathbf{p}^* = \mathbf{e}_2(\omega) \cdot \mathbf{p}^*$, $\mathbf{f}_2(\omega)$ is on her budget line. Moreover, we have

$$\left(\frac{6n-2i+1}{4n} + 2n\left(\frac{\phi(\omega)}{\gamma} - \frac{i-1}{2n}\right)\right) - \left(\frac{2n+2i-1}{4n} + 2n\left(\frac{\phi(\omega)}{\gamma} - \frac{i-1}{2n}\right)\right) = 1 - \frac{2i-1}{2n}$$

and, hence, $\mathbf{f}_2(\omega) \in D_i$. Thus, the measure induced by $\mathbf{f}_2$ on D_i is uniform with total measure $\gamma/(2n)$ for each $i = 1, 2, \ldots, 2n$.

The total supply of the first good in the economy $\mathscr{E}_2$ is

$$\sum_{i=1}^{2n} \int_{\Omega_1^i} \left(1 + 2n\left(\frac{\phi(\omega)}{\gamma} - \frac{i-1}{2n}\right)\right) \mathrm{d}P(\omega) = \gamma + \alpha,$$

where $\alpha = \sum_{i=1}^{2n} \int_{\Omega_1^i} 2n(\phi(\omega)/\gamma - (i-1)/(2n)) \,\mathrm{d}P(\omega)$. The total demand of the first good as given by $\mathbf{f}_2$ in $\mathscr{E}_2$ is

$$\sum_{i=1}^{2n} \int_{\Omega_1^i} \left(\frac{2n+2i-1}{4n} + 2n\left(\frac{\phi(\omega)}{\gamma} - \frac{i-1}{2n}\right)\right) \mathrm{d}P(\omega) = \frac{\gamma}{2n} \sum_{i=1}^{2n} \frac{2n+2i-1}{4n} + \alpha = \gamma + \alpha.$$

Thus, the total demand and the total supply of the first good are the same. Similarly, one can check that the market clearing condition is also satisfied for the second good. Hence, $(\mathbf{f}_2, \mathbf{p}^*)$ is a Walrasian equilibrium in the economy $\mathscr{E}_2$. Note that $P \circ \mathbf{f}_2^{-1} = \gamma\mu + (1-\gamma)\delta_{(0,0)}$, where μ is the uniform distribution on the set $\bigcup_{i=1}^{2n} D_i$ and $\delta_{(0,0)}$ is the Dirac measure at the point $(0, 0)$.

Since $\mathcal{DW}^{\mathcal{F}}(\mathscr{E}_1) = \mathcal{DW}^{\mathcal{F}}(\mathscr{E}_2)$, economy $\mathscr{E}_1$ has an $\mathcal{F}$-measurable Walrasian allocation $\mathbf{f}_1$ with the same distribution as $\mathbf{f}_2$. Thus, there exist $2n$ disjoint $\mathcal{F}$-measurable subsets $E_1, E_2, \ldots, E_{2n}$ of Ω_1 such that $\mathbf{f}_1(\omega) \in D_i$ for $\omega \in E_i$, $i = 1, 2, \ldots, 2n$. Let h be a mapping from the set $\bigcup_{i=1}^{2n} D_i$ to $[0, \gamma) \times \{1, 2, \ldots, 2n\}$ such that

$$h(x, y) = \left(\gamma\left(x - \frac{2n+2i-1}{4n}\right), i\right)$$

if $(x, y) \in D_i$ for some $i = 1, 2, \ldots, 2n$. Then $1/\gamma P \circ \mathbf{f}_1^{-1} \circ h^{-1}$ is the uniform distribution on the set $[0, \gamma) \times \{1, 2, \ldots, 2n\}$.

Next, let $\mathbf{p}' = (a, b)$ be the corresponding equilibrium price of the Walrasian allocation $\mathbf{f}_1$ in economy $\mathscr{E}_1$. We show that $\mathbf{p}' = \mathbf{p}^* = (1, 1)$. Suppose that $a > b > 0$ (since the preference is monotonic, the equilibrium price for each good must be positive). For the agent $\omega_L \in \Omega_1$ with the initial endowment $L = (1, 1)$, denote her budget line by $B_l(\mathbf{p}')$ as shown in Figure 3. It is clear from Figure 3 that there is a subsegment of D_1 with positive length such that every point on this subsegment is in the interior of ω_L's budget set. Since the preference is monotonic, every point on this subsegment cannot be a best response for the agent ω_L and, hence, cannot be a best response for any agent $\omega \in \Omega_1$. This means that $\mathbf{f}_1(\omega)$ does not belong to this subsegment of D_1 for P-almost all $\omega \in \Omega_1$, which contradicts the fact that $P \circ \mathbf{f}_1^{-1} = P \circ \mathbf{f}_2^{-1} = \gamma\mu + (1-\gamma)\delta_{(0,0)}$, where μ is uniform on the set $\bigcup_{i=1}^{2n} D_i$. Similar arguments also apply to the case $b > a > 0$.

We can now focus on the Walrasian equilibrium price $\mathbf{p}^* = (1,1)$ in $\mathscr{E}_1$. As discussed above, for agent $\omega \in \Omega_1$, the set of optimal consumptions within her budget is the intersection of her budget line and the set $\bigcup_{i=1}^{2n} D_i$:

$$\left\{\left(\frac{2n+2i-1}{4n} + \frac{\phi(\omega)}{\gamma}, \frac{6n-2i+1}{4n} + \frac{\phi(\omega)}{\gamma}\right)\right\}_{i=1}^{2n}.$$

Then we have $h \circ \mathbf{f}_1(\omega) \in \{(\phi(\omega), i)\}_{i=1}^{2n}$.

For any $C \in \mathcal{G}^{\Omega_1}$, Lemma 6 implies that there exists a subset $C_1 \in \mathcal{B}^{[0,\gamma)}$ such that $P(C \triangle \phi^{-1}(C_1)) = 0$. Then we have

$$\begin{aligned} P(E_j \cap C) &= P\big(E_j \cap \phi^{-1}(C_1)\big) = P\big(h \circ \mathbf{f}_1 \in \big(C_1 \times \{j\}\big)\big) \\ &= \frac{1}{2n}\eta(C_1) = \frac{1}{2n}P(\phi \in C_1) = \frac{1}{2n}P(C) \end{aligned}$$

and

$$P(E_j) = P(E_j \cap \Omega_1) = \frac{1}{2n}P(\Omega_1),$$

and, hence, $P^{\Omega_1}(E_j) = P(E_j)/P(\Omega_1) = 1/(2n)$. Thus, E_j is independent of $\mathcal{G}^{\Omega_1}$ under P^{Ω_1} for $j = 1, 2, \dots, 2n$. By Lemma 7, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ under P.

Part 5: Proof of (iii) $\Rightarrow$ (i) Fix an integer $n \geq 1$. Let $\mathscr{E}_1$ be the economy as defined in Example 8,[43] which is discretized to generate a sequence of finite-agent economies $\{\mathscr{E}_1^k\}_{k\in\mathbb{Z}_+}$. For each $k \in \mathbb{Z}_+$, we take the probability space $(\Omega^k, \mathcal{G}^k, P^k)$ to be the agent space of the economy $\mathscr{E}_1^k$, where $\Omega^k = \{1, 2, \dots, 2nk-1, 2nk\}$, $\mathcal{G}^k$ is the power set of Ω^k, and P^k is the counting probability measure. For each economy $\mathscr{E}_1^k$, all the agents have the same preference; the indifference curves are shown in Figure 3. The endowment in the economy $\mathscr{E}_1^k$ is given by $\mathbf{e}_1^k(j) = (1 + j/(2nk), 1 + j/(2nk))$ for $j = 1, 2, \dots, 2nk$.

Let $\mathbf{p}^* = (1,1)$ and

$$\mathbf{f}_1^k(j) = \left(\frac{1}{2} + \frac{2i-1}{4n} + \frac{j}{2nk}, \frac{3}{2} - \frac{2i-1}{4n} + \frac{j}{2nk}\right)$$

for $j = 2nm + i$, where $m = 0, 1, \dots, k-1$ and $i = 1, 2, \dots, 2n$. We claim that for each $k \in \mathbb{Z}_+$, $(\mathbf{f}_1^k, \mathbf{p}^*)$ is a Walrasian equilibrium of the economy $\mathscr{E}_1^k$.

For the price $\mathbf{p}^*$, the best response for the agent $j \in \Omega^k$ must be the intersection of her budget line and the set $\bigcup_{i=1}^{2n} D_i$. Consider an agent $j = 2nm + i$, $m \in \{0, 1, \dots, k-1\}$ and $i \in \{1, 2, \dots, 2n\}$. We have

$$\left(\frac{1}{2} + \frac{2i-1}{4n} + \frac{j}{2nk}\right) + \left(\frac{3}{2} - \frac{2i-1}{4n} + \frac{j}{2nk}\right) = 2 + \frac{j}{nk}.$$

Thus, $\mathbf{f}_1^k(j)$ is on her budget line. Moreover, we have

$$\left(\frac{3}{2} - \frac{2i-1}{4n} + \frac{j}{2nk}\right) - \left(\frac{1}{2} + \frac{2i-1}{4n} + \frac{j}{2nk}\right) = 1 - \frac{2i-1}{2n},$$

[43] For the sake of simplicity, we assume that $\gamma = 1$.

which implies that $\mathbf{f}_1^k(j) \in D_i$. The total supply of the first good in the market is $2nk + \sum_{j=1}^{2nk} j/(2nk)$. The total demand of the first good in the market is

$$k \cdot \sum_{i=1}^{2n}\left(\frac{1}{2} + \frac{2i-1}{4n}\right) + \sum_{j=1}^{2nk} \frac{j}{2nk} = 2nk + \sum_{j=1}^{2nk} \frac{j}{2nk}.$$

Thus, the total demand and the total supply of the first good are the same. Similarly, one can check that the market clearing condition is also satisfied for the second good. Therefore, $(\mathbf{f}_1^k, \mathbf{p}^*)$ is a Walrasian equilibrium in the economy $\mathscr{E}_1^k$.

It is obvious that $P^k \circ (\mathbf{e}_1^k)^{-1}$ converges weakly to the uniform distribution on the line segment W; that is, $\mathscr{E}_1^k$ converges weakly to $\mathscr{E}_1$. Clearly, we have that $\int_{\Omega^k} \mathbf{e}^k \, \mathrm{d}P^k = 1/(2nk)\sum_{j=1}^{2nk}(1 + j/(2nk)) = 1 + (2nk - 1)/(4nk)$ approaches $\frac{3}{2} = \int_\Omega \mathbf{e} \, \mathrm{d}P$ when k goes to infinity. In addition, $P^k \circ (\mathbf{f}_1^k)^{-1}$ converges weakly to μ, where μ is the uniform distribution on the set $\bigcup_{i=1}^{2n} D_i$. Since the Walrasian equilibrium correspondence of the $\mathcal{G}$-measurable large economy $\mathscr{E}_1$ has the closed graph property, there exists a Walrasian allocation in the economy $\mathscr{E}_1$ that induces the distribution μ. As shown in the proof of (ii) $\Rightarrow$ (i), there exist $2n$ disjoint subsets $\{E_j\}_{j=1}^{2n}$ such that $P(E_j) = 1/(2n)$ and E_j is independent of $\mathcal{G}$ under P for each $j = 1, 2, \dots, 2n$. By Lemma 7, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

A.4 *Proof of Theorem 2*

We divide the proof of Theorem 2 into seven parts. In the first three parts, we prove the directions (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv). In Part 4, we present one lemma and one example that are used for the proof of (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i) in Parts 5 and 6, respectively. The proof of (iv) $\Rightarrow$ (i) is given in the last part.

Part 1: Proof of (i) $\Rightarrow$ (ii) Let $G_1 = (\alpha_1, v_1)$ and $G_2 = (\alpha_2, v_2)$ be two $\mathcal{G}$-measurable games with the same distribution, and let f_1 be an $\mathcal{F}$-measurable mapping from Ω to A such that f_1 is a Nash equilibrium of G_1. Denote $\lambda = P \circ f_1^{-1}$. Since $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$, Lemma 2 implies that $\mathcal{F}$ is relatively saturated with respect to $\mathcal{G}$. As $P \circ G_1^{-1} = P \circ G_2^{-1}$, there is an $\mathcal{F}$-measurable mapping f_2 from Ω to A such that $P \circ (G_2, f_2)^{-1} = P \circ (G_1, f_1)^{-1}$.

Denote $\nu = P \circ (\alpha_1, f_1)^{-1}$. Then $P \circ (\alpha_2, f_2)^{-1} = P \circ (\alpha_1, f_1)^{-1} = \nu$. Define a set B_ν such that

$$B_\nu = \{(t, v, x) \in T \times \mathcal{V} \times A\colon v(x, \nu) \geq v(a, \nu) \text{ for all } a \in A\}.$$

Since $P \circ (G_1, f_1)^{-1}(B_\nu) = 1$, we have $P \circ (G_2, f_2)^{-1}(B_\nu) = 1$. Thus, the set $H = \{\omega \in \Omega : (G_2(\omega), f_2(\omega)) \in B_\nu\}$ has probability 1 under P. This means that for any $\omega \in H$, $v_2(\omega, f_2(\omega), \nu) \geq v_2(\omega, a, \nu)$ for any $a \in A$. Hence, f_2 is a Nash equilibrium of G_2, which means that $\lambda = P \circ f_2^{-1} \in \mathcal{D}^{\mathcal{F}}(G_2)$. Thus, $\mathcal{D}^{\mathcal{F}}(G_1) \subseteq \mathcal{D}^{\mathcal{F}}(G_2)$. By symmetry, we also have $\mathcal{D}^{\mathcal{F}}(G_2) \subseteq \mathcal{D}^{\mathcal{F}}(G_1)$, which implies $\mathcal{D}^{\mathcal{F}}(G_1) = \mathcal{D}^{\mathcal{F}}(G_2)$.

Part 2: Proof of (i) ⇒ (iii) Assume that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ and that $G = (\alpha, v)\colon \Omega \to T \times \mathcal{V}$ is a $\mathcal{G}$-measurable game with traits. Pick a saturated probability space $(S, \mathcal{S}, Q)$.[44] Then there exists an $\mathcal{S}$-measurable game with traits $F = (\beta, u)$ from S to $T \times \mathcal{V}$ such that $Q \circ F^{-1} = P \circ G^{-1}$. Since $(S, \mathcal{S}, Q)$ is saturated, Theorem 1 of Khan et al. (2013) implies that F has a Nash equilibrium f from S to A. Define $\nu = Q \circ (F, f)^{-1} \in \mathcal{M}(T \times \mathcal{V} \times A)$. Since $\mathcal{F}$ is relatively saturated with respect to $\mathcal{G}$ by Lemma 2, there exists an $\mathcal{F}$-measurable mapping g from Ω to A such that $P \circ (G, g)^{-1} = \nu$. As shown in the proof of (i) ⇒ (ii), g is a Nash equilibrium of the game G.

Part 3: Proof of (i) ⇒ (iv) For all $k \in \mathbb{Z}_+$, let $\tau^k = P^k \circ (G^k, f^k)^{-1}$ and $\mu^k = P^k \circ (f^k)^{-1}$. Since $P^k \circ (G^k)^{-1}$ converges weakly to $P \circ G^{-1}$ and μ^k converges weakly to μ, Lemma 2.1 in Keisler and Sun (2009) implies that there exists a subsequence of $\{\tau^k\}_{k\in\mathbb{Z}_+}$ (say itself) that converges weakly to some $\tau \in \mathcal{M}(T \times \tilde{\mathcal{V}} \times A)$. It is obvious that the marginals of τ on $T \times \tilde{\mathcal{V}}$ and A are $P \circ G^{-1}$ and μ, respectively. Since $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$, there exists an $\mathcal{F}$-measurable function $f\colon \Omega \to A$ such that $P \circ (G, f)^{-1} = \tau$. Following the same argument in the last three paragraphs of the proof of Theorem 3 in Khan et al. (2013), f is a Nash equilibrium of G.

Part 4: The preparation for proving the necessity of nowhere equivalence As in the previous subsection, we assume that $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are disjoint and $\mathcal{G}$-measurable. Suppose that $(\Omega_1, \mathcal{G}^{\Omega_1}, P|_{\Omega_1})$ and $(\Omega_2, \mathcal{G}^{\Omega_2}, P|_{\Omega_2})$ are, respectively, atomless and purely atomic. Let $P(\Omega_1) = \gamma$. If $\gamma = 0$, then $(\Omega, \mathcal{G}, P)$ is purely atomic and the nowhere equivalence condition is automatically satisfied. Thus, we only need to consider the case that $0 < \gamma \le 1$. Lemma 6 shows that there exists a measure-preserving mapping $\phi\colon (\Omega_1, \mathcal{G}^{\Omega_1}, P) \to ([0, \gamma), \mathcal{B}^{[0,\gamma)}, \eta_1)$ such that for any $E \in \mathcal{G}^{\Omega_1}$, there exists a subset $E' \in \mathcal{B}^{[0,\gamma)}$ with $P(E \triangle \phi^{-1}(E')) = 0$, where η_1 is the Lebesgue measure on $\mathcal{B}^{[0,\gamma)}$.

Fix an integer $n \ge 2$. Let $A_1 = [0, 1]$, $A_2 = \{0, 1, \dots, n-1\}$, and $A_0 = A_1 \times A_2$ with the standard metric in $\mathbb{R}^2$. Let $\mathrm{d}(\cdot, \cdot)$ be the Prohorov metric on $\mathcal{M}(A_0)$. Define a probability measure η^2 on A_0 as follows. For any Borel measurable subset $E \subseteq A_1$ and any $j \in A_2$, $\eta^2(E \times \{j\}) = 1/(n\gamma)\eta(E \cap [0, \gamma))$. Let η^1 be a convex combination of η^2 and the Dirac measure concentrated at the point $(1, n-1)$: $\eta^1 = \gamma\eta^2 + (1-\gamma)\delta_{(1,n-1)}$.

Let $f\colon A_0 \times [0, 1] \to \mathbb{R}$ be defined as follows. For any $a_1 \in A_1$, $a_2 \in A_2$ and $b \in [0, 1]$,

$$
\begin{aligned}
& f\big((a_1, a_2), b\big) \\
&= \begin{cases} 0 & \text{if } b = 0 \text{ or } a_1 = kb, \\ \Big(\delta_j\big(\{a_2\}\big) - \frac{1}{2}\Big) & \\ \quad \cdot \min\big\{a_1 - (nk+j)b, (nk+j+1)b - a_1\big\} & \text{if } a_1 \in \big((nk+j)b, (nk+j+1)b\big) \end{cases}
\end{aligned}
$$

for some $k \in \mathbb{N}$ and $j \in A_2$, where $\mathbb{N}$ is the set of nonnegative integers. It is easy to show that the function f is continuous on $A_0 \times [0, 1]$. Figures 4–7 illustrate the function f with $b = \frac{1}{8}$ and $n = 4$.

[44] For the definition of a saturated probability space, see Section 4.5.

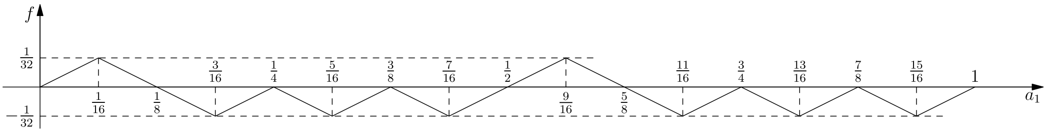

FIGURE 4. The graph of $f((a_1, 0), b)$.

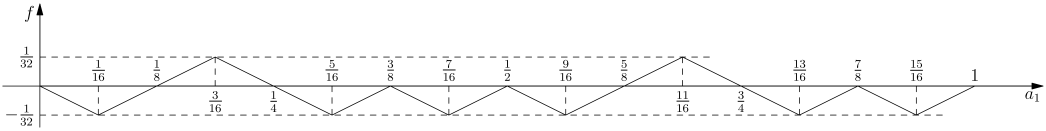

FIGURE 5. The graph of $f((a_1, 1), b)$.

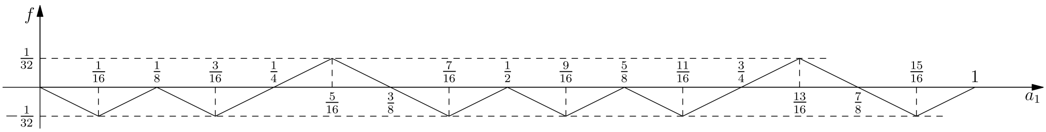

FIGURE 6. The graph of $f((a_1, 2), b)$.

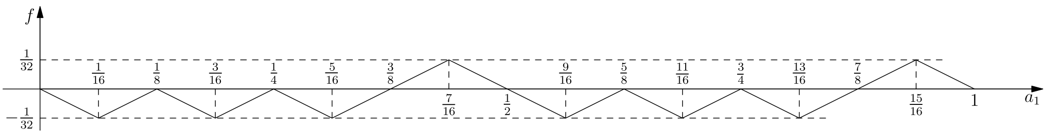

FIGURE 7. The graph of $f((a_1, 3), b)$.

Define a mapping u from $[0, \gamma)$ to the space of continuous functions on $A_0 \times \mathcal{M}(A_0)$ as follows. For any $i \in [0, \gamma)$, $a_1 \in A_1$, $a_2 \in A_2$, and $\nu \in \mathcal{M}(A_0)$,

$$u(i)\big((a_1, a_2), \nu\big) = f\big((a_1, a_2), f_0(\nu)\big) - |i - a_1|,$$

where $f_0(\nu) = (1/n)\, \mathrm{d}(\eta^1, \nu)$.

In the following paragraphs, we construct an example for proving the necessity of nowhere equivalence in the next part.

EXAMPLE 9. Fix an integer $n \geq 2$. Let $(\Omega, \mathcal{F}, P)$ be the player space and let $A = A_0$ be the action space. Define a mapping $G\colon \Omega \to \mathcal{U}$ as

$$G(\omega)\big((a_1, a_2), \nu\big) = \begin{cases} u\big(\phi(\omega)\big)\big((a_1, a_2), \nu\big) & \text{if } \omega \in \Omega_1, \\ a_1 + a_2 & \text{if } \omega \in \Omega_2 \end{cases}$$

for any $(a_1, a_2) \in A$ and $\nu \in \mathcal{M}(A)$.[45]

[45] The payoff function in our example for the case $n = 2$ is a variation of the payoff function used in Example 3 of Rath et al. (1995).

It is clear that u is a continuous mapping from $[0,\gamma)$ to $\mathcal{U}$ under the sup-norm and G is a constant function on Ω_2. Thus, G is a $\mathcal{G}$-measurable large game.

Lemma 8. *Let g be an $\mathcal{F}$-measurable Nash equilibrium for the above large game G. Then there exists an $\mathcal{F}$-measurable partition $\{\tilde{D}_0, \tilde{D}_1, \dots, \tilde{D}_{n-1}\}$ of Ω_1 such that for each $j = 0, 1, \dots, n-1$, (i) $P^{\Omega_1}(\tilde{D}_j) = 1/n$ and (ii) $\tilde{D}_j$ is independent of $\mathcal{G}^{\Omega_1}$ under the probability measure P^{Ω_1}.*

Proof. Let $g = (g_1, g_2)$ be an $\mathcal{F}$-measurable Nash equilibrium of the game G and let $\vartheta = P \circ g^{-1} \in \mathcal{M}(A)$.[46] We prove that $\vartheta = \eta^1$.

Suppose that $\vartheta \neq \eta^1$. Denote $b_0 = (1/n)\,\mathrm{d}(\vartheta, \eta^1)$. Then $0 < b_0 \leq 1/n$. For each agent $\omega \in \Omega_2$, since the payoff function is $a_1 + a_2$, it is obvious that $g(\omega) = (1, n-1)$. Next, fix an agent $\omega \in \Omega_1$ with $\phi(\omega) \neq kb_0$ for any $k \in \mathbb{N}$. Let (a_1^*, a_2^*) be a best response of this agent. For any $(x, j) \in A$ with $x \neq \phi(\omega)$,

$$\begin{aligned} &G(\omega)\big((x,j),\vartheta\big) - G(\omega)\big((\phi(\omega),j),\vartheta\big) \\ &\quad = f\big((x,j), f_0(\vartheta)\big) - \big|\phi(\omega) - x\big| - f\big((\phi(\omega),j), f_0(\vartheta)\big). \end{aligned}$$

When a_2 and b are fixed, $f((a_1, a_2), b)$ is a Lipschitz function in terms of a_1 with the Lipschitz constant $\frac{1}{2}$ and, hence, the expression above is negative. Thus, $g_1(\omega) = a_1^* = \phi(\omega)$ regardless of the value of a_2^*. There is a unique pair (k, j') with $k \in \mathbb{N}$ and $j' \in A_2$ such that $\phi(\omega) \in ((nk + j')b_0, (nk + j' + 1)b_0)$. For any $j \in A_2$, $G(\omega)((\phi(\omega), j), \vartheta) = f((\phi(\omega), j), f_0(\vartheta))$, which is positive only if $j = j'$, and, hence, $g_2(\omega) = a_2^* = j'$. Therefore,

$$g(\omega) = \begin{cases} \big(\phi(\omega), j'\big) & \text{if } \omega \in \Omega_1, \\ (1, n-1) & \text{if } \omega \in \Omega_2, \end{cases}$$

where j' is such that $\phi(\omega) \in ((nk + j')b_0, (nk + j' + 1)b_0)$ for some $k \in \mathbb{N}$.

We now show that $\mathrm{d}(\vartheta, \eta^1)$ is at most $\epsilon = (n-1)b_0$. Since ϑ coincides with η^1 on $[\gamma, 1] \times A_2$, we only need to calculate $\mathrm{d}(\vartheta, \eta^1)$ on $[0, \gamma) \times A_2$.

Fix $j = 0, 1, \dots, n-2$; let $W \times \{j\}$ be the support of ϑ on $[0,\gamma) \times \{j\}$. The set W should be a union of finite disjoint intervals; denote them by $W_1, W_2, \dots, W_m$ in increasing order. In particular, for each $\ell = 1, 2, \dots, m$, W_ℓ is in the form of $((nk+j)b_0, (nk+j+1)b_0)$ with or without its endpoints for some $k \in \mathbb{N}$. The distance between W_ℓ and $W_{\ell+1}$ is $(n-1)b_0$ for $\ell = 1, 2, \dots, m-1$. The length of W_ℓ is b_0 for $\ell = 1, 2, \dots, m-1$ and the length of W_m is at most b_0.

Take a Borel subset $E \subseteq [0, \gamma)$. Without loss of generality, we may assume that E does not contain any endpoint of the subintervals $\{W_\ell\}_{\ell=1}^m$. For $\ell = 1, 2, \dots, m-1$, let $E_\ell = W_\ell \cap E$. Then the sets $E_\ell, E_\ell + b_0, \dots, E_\ell + (n-1)b_0$ are all disjoint subsets of $[0,\gamma)$, where $E_\ell + tb_0$ is the set $\{x + tb_0 \colon x \in E_\ell\}$ for any integer t. Furthermore, $(E_\ell + tb_0) \times \{j\}$ is included in $(E \times \{j\})^\epsilon$ for $t = 0, 1, \dots, n-1$, where $(E \times \{j\})^\epsilon$ is the ϵ-neighborhood of

[46] Without loss of generality, we assume that for any $\omega \in \Omega$, $G_\omega(g(\omega), \vartheta) \geq G_\omega(a, \vartheta)$ for each $a \in A$.

$E\times\{j\}$. Since $\eta(E_m)\le b_0$ and $\eta^1(D\times\{j\})=(1/n)\eta(D)$ for any Borel subset $D\subseteq[0,\gamma)$, we have

$$\begin{aligned}\vartheta\big(E\times\{j\}\big)&=\sum_{\ell=1}^{m-1}\vartheta\big(E_\ell\times\{j\}\big)+\vartheta\big(E_m\times\{j\}\big)\\&=\sum_{\ell=1}^{m-1}\eta(E_\ell)+\eta(E_m)\le n\sum_{\ell=1}^{m-1}\eta^1\big(E_\ell\times\{j\}\big)+b_0\\&=\sum_{\ell=1}^{m-1}\big(\eta^1\big(E_\ell\times\{j\}\big)+\eta^1\big((E_\ell+b_0)\times\{j\}\big)+\cdots\\&\quad+\eta^1\big(\big(E_\ell+(n-1)b_0\big)\times\{j\}\big)\big)+b_0\\&\le\eta^1\big(\big(E\times\{j\}\big)^\epsilon\big)+b_0.\end{aligned}$$

For $j=n-1$, let $W'\times\{n-1\}$ be the support of ϑ on $[0,\gamma)\times\{n-1\}$. The set W' should be a union of finite disjoint intervals; denote them by $W'_1, W'_2,\dots,W'_m$ in increasing order. The distance between W'_ℓ and $W'_{\ell+1}$ is $(n-1)b_0$ for $\ell=1,2,\dots,m-1$. In addition, the distance between $\{0\}$ and W'_1 is also $(n-1)b_0$. The length of W'_ℓ is b_0 for $\ell=1,2,\dots,m-1$ and the length of W'_m is at most b_0.

Take a Borel subset $E'\subseteq[0,\gamma)$. Without loss of generality, we may assume that E' does not contain any endpoint of the subintervals $\{W'_\ell\}_{\ell=1}^m$. For $\ell=1,2,\dots,m$, let $E'_\ell=W'_\ell\cap E'$. Then $E'_\ell, E'_\ell-b_0,\dots,E'_\ell-(n-1)b_0$ are all disjoint, and $(E'_\ell-tb_0)\times\{n-1\}$ is included in $(E'\times\{n-1\})^\epsilon$ for $t=0,1,\dots,n-1$. We have

$$\begin{aligned}\vartheta\big(E'\times\{n-1\}\big)&=\sum_{\ell=1}^{m}\vartheta\big(E'_\ell\times\{n-1\}\big)\big(E'_\ell\big)\\&=\sum_{\ell=1}^{m}\eta=n\sum_{\ell=1}^{m}\eta^1\big(E'_\ell\times\{n-1\}\big)\\&=\sum_{\ell=1}^{m}\big(\eta^1\big(E'_\ell\times\{n-1\}\big)+\eta^1\big(\big(E'_\ell-b_0\big)\times\{n-1\}\big)+\cdots\\&\quad+\eta^1\big(\big(E'_\ell-(n-1)b_0\big)\times\{n-1\}\big)\big)\\&\le\eta^1\big(\big(E'\times\{n-1\}\big)^\epsilon\setminus\big\{(1,n-1)\big\}\big).\end{aligned}$$

Fix any Borel subset $C\subseteq A$. Then $C=\bigcup_{k=0}^{n-1}(C_k\times\{k\})$, where $C_0, C_1,\dots,C_{n-1}\subseteq A_1$. Let $C_k^1=C_k\cap[0,\gamma)$ and $C_k^2=C_k\cap[\gamma,1]$ for $k=0,1,\dots,n-1$. We have

$$\begin{aligned}\vartheta(C)&=\sum_{k=0}^{n-1}\vartheta\big(C_k\times\{k\}\big)=\sum_{k=0}^{n-1}\vartheta\big(C_k^1\times\{k\}\big)+\sum_{k=0}^{n-1}\vartheta\big(C_k^2\times\{k\}\big)\\&\le\sum_{k=0}^{n-2}\big[\eta^1\big(\big(C_k^1\times\{k\}\big)^\epsilon\big)+b_0\big]+\eta^1\big(\big(C_{n-1}^1\times\{n-1\}\big)^\epsilon\setminus\big\{(1,n-1)\big\}\big)\end{aligned}$$

$$+\sum_{k=0}^{n-1}\eta^1\big(C_k^2\times\{k\}\big)$$
$$\leq \eta^1\big(C^\epsilon\big)+(n-1)b_0=\eta^1\big(C^\epsilon\big)+\epsilon.$$

Hence, $\mathrm{d}(\vartheta,\eta^1)\leq\epsilon=(n-1)b_0=(n-1)/n\,\mathrm{d}(\vartheta,\eta^1)$, which is a contradiction. Therefore, we have proved $\vartheta=\eta^1$ as claimed in the beginning of the proof of this lemma.

From now on, we work with the case that $\vartheta=\eta^1$. Then $f_0(\vartheta)=0$ and, hence, $f(a,f_0(\vartheta))=0$ for any $a\in A$, which implies that $G(\omega)((a_1,a_2),\vartheta)=-|\phi(\omega)-a_1|$ for any $\omega\in\Omega_1$. The best response correspondence is

$$H(\omega)=\begin{cases}\big\{\big(\phi(\omega),0\big),\dots,\big(\phi(\omega),n-1\big)\big\} & \text{if } \omega\in\Omega_1,\\ \big\{(1,n-1)\big\} & \text{if } \omega\in\Omega_2.\end{cases}$$

By the definition of Nash equilibria, $g(\omega)\in H(\omega)$ for P-almost all $\omega\in\Omega$.

For any $\tilde{C}\in\mathcal{G}^{\Omega_1}$, Lemma 6 implies that there exists an $\tilde{C}_1\in\mathcal{B}^{[0,\gamma)}$ such that $P(\tilde{C}\,\triangle\,\phi^{-1}(\tilde{C}_1))=0$. Define $\tilde{D}_j=\{\omega\in\Omega_1: g(\omega)=(\phi(\omega),j)\}$ for $j=0,1,\dots,n-1$. Thus, we have

$$P(\tilde{D}_j\cap\tilde{C})=P\big(\tilde{D}_j\cap\phi^{-1}(\tilde{C}_1)\big)=P\big(g\in\big(\tilde{C}_1\times\{j\}\big)\big)$$
$$=\eta^1\big(\tilde{C}_1\times\{j\}\big)=\frac{1}{n}\eta(\tilde{C}_1)=\frac{1}{n}P(\phi\in\tilde{C}_1)=\frac{1}{n}P(\tilde{C})$$

and, hence, $P^{\Omega_1}(\tilde{D}_j)=1/n$. Therefore, $\{\tilde{D}_0,\tilde{D}_1,\dots,\tilde{D}_{n-1}\}$ is an $\mathcal{F}$-measurable partition of Ω_1 such that for each $j=0,1,\dots,n-1$, (i) $P^{\Omega_1}(\tilde{D}_j)=1/n$ and (ii) $\tilde{D}_j$ is independent of $\mathcal{G}^{\Omega_1}$ under the probability measure P^{Ω_1}. □

Part 5: Proof of (iii) ⇒ (i) Consider the game G in Example 9. Define a game with traits $G'=(\alpha,v)$ from Ω to $T\times\mathcal{V}$, where T is a singleton, α is a constant mapping, and $v(\omega)(a,\nu)=G(\omega)(a,\nu_A)$ for $\omega\in\Omega$. Then G' is $\mathcal{G}$-measurable. For the new game G', there exists an $\mathcal{F}$-measurable Nash equilibrium g' of the game G' by statement (iii) of Theorem 2. It is easy to see that g' is also a Nash equilibrium of the game G because of the construction of the payoff function v. Therefore, Lemma 8 implies the existence of an $\mathcal{F}$-measurable partition $\{E_0,E_1,\dots,E_{n-1}\}$ of Ω_1 such that for each $j=0,1,\dots,n-1$, (i) $P^{\Omega_1}(E_j)=1/n$ and (ii) E_j is independent of $\mathcal{G}^{\Omega_1}$ under the probability measure P^{Ω_1}. Then $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ due to Lemma 7.

Remark 5. In Remark 2 we claimed that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ provided that any $\mathcal{G}$-measurable large game has an $\mathcal{F}$-measurable Nash equilibrium. This result follows from Lemmas 7 and 8.

Next we present another large game with traits F such that the action space is finite and the trait space is uncountable. We show that (iii) ⇒ (i) can be also proved via this game.

EXAMPLE 10. Fix an integer $n \geq 2$. Let $(\Omega, \mathcal{F}, P)$ be the agent space, let $T = A_1 = [0, 1]$ be the trait space with the measure $\rho = \gamma\eta_1 + (1-\gamma)\delta_1$ (δ_1 is the Dirac measure at 1), and let $A = A_2 = \{0, 1, \dots, n-1\}$ be the action space. The large game with traits $F = (\alpha, v)$ is defined as

$$\alpha(\omega) = \begin{cases} \phi(\omega) & \text{if } \omega \in \Omega_1, \\ 1 & \text{if } \omega \in \Omega_2, \end{cases}$$

and

$$v(\omega)(a, \nu) = \begin{cases} f\big(\phi(\omega), a, f_0(\nu)\big) & \text{if } \omega \in \Omega_1, \\ a & \text{if } \omega \in \Omega_2 \end{cases}$$

for any $a \in A$ and $\nu \in \mathcal{M}^\rho(T \times A)$.

LEMMA 9. *Let $\tilde{f}$ be an $\mathcal{F}$-measurable Nash equilibrium in the game F. Then there exists an $\mathcal{F}$-measurable partition $\{E_0, E_1, \dots, E_{n-1}\}$ of Ω_1 such that for each $j = 0, 1, \dots, n-1$, (i) $P^{\Omega_1}(E_j) = 1/n$ and (ii) E_j is independent of $\mathcal{G}^{\Omega_1}$ under the probability measure P^{Ω_1}.*

PROOF. Let $\tilde{f}$ be an $\mathcal{F}$-measurable Nash equilibrium of F, $\vartheta = P \circ (\alpha, \tilde{f})^{-1}$ and let $b_0 = (1/n)\,\mathrm{d}(\vartheta, \eta^1)$.

Suppose that $b_0 > 0$. Note that for any agent $\omega \in \Omega_1$ with $\phi(\omega) \neq kb_0$ for any $k \in \mathbb{N}$, the best response must be j' such that $\phi(\omega) \in ((nk + j')b_0, (nk + j' + 1)b_0)$ for some $k \in \mathbb{N}$. Thus, $(\alpha, \tilde{f}) = g$ for P-almost all ω, where g is the best response in the proof of the case $\vartheta \neq \eta^1$ in Lemma 8. As shown in the proof of Lemma 8, we must have $b_0 = 0$, which is a contradiction. Thus, $\vartheta = \eta^1$, which means that $f_0(\vartheta) = 0$. Then for any agent $\omega \in \Omega_1$, any $a \in A$ is a best response. Furthermore, for any agent $\omega \in \Omega_2$, $n-1$ is the best response. We can regard $(\alpha, \tilde{f})$ as the function g in the proof of Lemma 8 for the case $\vartheta = \eta^1$. Then the rest is clear. □

Part 6: Proof of (ii) ⇒ (i) By Lemma 5, there is an atomless σ-algebra $\mathcal{H}_1$ on Ω_1 such that $\mathcal{H}_1 \subseteq \mathcal{G}^{\Omega_1}$ and $\mathcal{G}^{\Omega_1}$ is nowhere equivalent to $\mathcal{H}_1$ under the probability measure P^{Ω_1}. Let $\mathcal{H} = \sigma(\mathcal{H}_1 \cup \mathcal{G}^{\Omega_2})$, which is the σ-algebra generated by $\mathcal{H}_1$ and $\mathcal{G}^{\Omega_2}$. Then $\mathcal{F}$ is nowhere equivalent to $\mathcal{H}$ under P.

For any $\mathcal{G}$-measurable large game G, since $\mathcal{H}^{\Omega_1}$ is atomless and $\mathcal{H}^{\Omega_2}$ coincides with $\mathcal{G}^{\Omega_2}$, there is an $\mathcal{H}$-measurable game G_1 such that $P \circ G_1^{-1} = P \circ G^{-1}$. Since $\mathcal{F}$ is nowhere equivalent to $\mathcal{H}$, G_1 has an $\mathcal{F}$-measurable Nash equilibrium g_1. By the condition that $\mathcal{D}^{\mathcal{F}}(G_1) = \mathcal{D}^{\mathcal{F}}(G)$, $\mathcal{D}^{\mathcal{F}}(G)$ is nonempty. Thus, the $\mathcal{G}$-measurable large game G has an $\mathcal{F}$-measurable Nash equilibrium. By Remark 5, we know that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

Part 7: Proof of (iv) ⇒ (i) For the sake of simplicity, we restrict our attention to the case that $(\Omega, \mathcal{G}, P)$ is atomless. By Lemma 6, there exists a measure-preserving mapping $\phi\colon (\Omega, \mathcal{G}, P) \to ([0, 1], \mathcal{B}, \eta)$ such that $\mathcal{G} = \phi^{-1}(\mathcal{B})$.

Fix an integer $n \geq 2$. Let $A_1 = [0,1]$ and $A_2 = \{1, 2, \ldots, n\}$. Let μ be the uniform distribution on $A_1 \times A_2$. Define a large game G with the player space Ω and the action space $A_1 \times A_2$ as follows. For any $\omega \in \Omega$, $a_1 \in A_1$, $a_2 \in A_2$, and $\nu \in \mathcal{M}(A_1 \times A_2)$,

$$G(\omega)(a_1, a_2, \nu) = -\big(\phi(\omega) - a_1\big)^2.$$

For each $k \in \mathbb{Z}_+$, let $\Omega^k = \{1, 2, \ldots, nk\}$ and let P^k be the counting probability measure on Ω^k. We define a game G^k with the player space Ω^k and the action space $A_1 \times A_2$ as follows. For each $m = 0, 1, \ldots, k-1$ and $\ell = 1, 2, \ldots, n$, given the action (a_1, a_2) and the action distribution ν of all other players, player $(mn + \ell)$'s payoff is

$$G^k(mn + \ell)(a_1, a_2, \nu) = -\left(\frac{m}{k} - a_1\right)^2.$$

For player $(mn + \ell)$, define

$$f^k(mn + \ell) = \left(\frac{m}{k}, \ell\right).$$

Since ϕ is measure-preserving, G^k converges weakly to G. In addition, it is clear that f^k is a pure-strategy Nash equilibrium of G^k for each $k \in \mathbb{Z}_+$ and that $P^k \circ (f^k)^{-1}$ converges weakly to the uniform distribution μ on the set $A_1 \times A_2$. Since the $\mathcal{G}$-measurable large game G has the closed graph property, there exists an $\mathcal{F}$-measurable Nash equilibrium f of G such that $P \circ f^{-1} = \mu$.

Let $f = (f_1, f_2)$ with f_1 and f_2 taking values in A_1 and A_2, respectively. Then $f_1 = \phi$. For $\ell = 1, 2, \ldots, n$, let $E_\ell = f_2^{-1}(\ell)$. It is easy to show that $\{E_1, E_2, \ldots, E_n\}$ is an $\mathcal{F}$-measurable partition of Ω such that for each $\ell = 1, 2, \ldots, n$, (i) $P(E_\ell) = 1/n$ and (ii) E_ℓ is independent of $\mathcal{G}$ under the probability measure P.[47] By Lemma 7, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

Remark 6. We have shown the necessity of the nowhere equivalence condition for statements (ii)–(iv) in Theorem 2. If $\mathcal{F}$ is not nowhere equivalent to $\mathcal{G}$, then there exists an $\mathcal{F}$-measurable nonnegligible set D such that $\mathcal{F}^D$ and $\mathcal{G}^D$ are essentially the same. If D is $\mathcal{G}$-measurable, then Lemma 6 says that there is a measure-preserving mapping ψ from $(\Omega, \mathcal{G}, P)$ to the Lebesgue unit interval such that the restriction of ψ to D induces an isomorphism between the restricted measure algebra on D and the measure algebra on a subinterval with the Lebesgue measure. Such a mapping ψ allows one to construct counterexamples by transferring Examples 1–5. Keisler and Sun (2009), Khan et al. (2013), and Qiao and Yu (2014) proved their necessity results for saturated agent spaces following such an approach. However, the set D is not necessarily $\mathcal{G}$-measurable in our setting. Hence, that approach is not applicable for proving (ii)/(iii) $\Rightarrow$ (i) in Theorem 1 and (ii)/(iii)/(iv) $\Rightarrow$ (i) in Theorem 2. To obtain the necessity of nowhere equivalence via Lemma 7, we need to construct new large economies/games that are substantially more complicated than those in Examples 1–5.

[47] Since the product measure μ is induced by (f_1, f_2), $\mathcal{G} = \sigma(f_1)$ is independent of $\sigma(f_2)$.

A.5 *Proofs of Corollaries 1 and 2*

Proof of Corollary 1. The "only if" part follows immediately from the relative saturation. We only prove the "if" part.

Assume that $\mathscr{E}_1$ and $\mathscr{E}_2$ are two arbitrary $\mathcal{G}$-measurable economies from $(\Omega, \mathcal{F}, P)$ to $\mathscr{P}_{\mathrm{mo}} \times \mathbb{R}^{\ell}_{+}$ with the same distribution. Suppose that $\mathscr{E}_1$ has a Walrasian allocation $\mathbf{f}_1$. Since $\mathscr{E}_1$ and $\mathscr{E}_2$ have the same distribution, $P \circ (\mathscr{E}_1, \mathbf{f}_1)^{-1}$ is a Walrasian equilibrium distribution of the distribution economy $P \circ \mathscr{E}_2^{-1}$; see Hildenbrand (1974, p. 159).

By the assumption of the corollary, there exists an $\mathcal{F}$-measurable Walrasian allocation $\mathbf{f}_2$ of the economy $\mathscr{E}_2$ such that $P \circ (\mathscr{E}_2, \mathbf{f}_2)^{-1} = P \circ (\mathscr{E}_1, \mathbf{f}_1)^{-1}$ and, hence, $P \circ \mathbf{f}_2^{-1} = P \circ \mathbf{f}_1^{-1}$. That is, the set of distributions of Walrasian allocations in $\mathscr{E}_1$ is a subset of the set of distributions of Walrasian allocations in $\mathscr{E}_2$. Analogously, the latter is also a subset of the former. Therefore, $\mathcal{DW}^{\mathcal{F}}(\mathscr{E}_1) = \mathcal{DW}^{\mathcal{F}}(\mathscr{E}_2)$. By Theorem 1, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. □

Proof of Corollary 2. The "only if" part follows immediately from the relative saturation. We only prove the "if" part.

Assume that G_1 and G_2 are two arbitrary $\mathcal{G}$-measurable games from $(\Omega, \mathcal{F}, P)$ to $\mathcal{U}$ with the same distribution. Suppose that G_1 has an $\mathcal{F}$-measurable Nash equilibrium g_1. Since G_1 and G_2 have the same distribution, $P \circ (G_1, g_1)^{-1}$ is a Nash equilibrium distribution of G_2.

By the assumption of the corollary, there exists an $\mathcal{F}$-measurable Nash equilibrium g_2 of the game G_2 such that $P \circ (G_2, g_2)^{-1} = P \circ (G_1, g_1)^{-1}$. That is, the set of distributions of Nash equilibria in G_1 is a subset of the set of distributions of Nash equilibria in G_2. Analogously, the latter is also a subset of the former. Therefore, $\mathcal{D}^{\mathcal{F}}(G_1) = \mathcal{D}^{\mathcal{F}}(G_2)$. By Theorem 2, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. □

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