Efficient and strategy-proof allocation mechanisms in economies with many goods

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In this paper, we show that in pure exchange economies where the number of goods equals or exceeds the number of agents, any Pareto-efficient and strategy-proof allocation mechanism always allocates the total endowment to some single agent even if the receivers vary.

KEYWORDS. Social choice, strategy-proofness, Pareto efficiency, exchange economy.

JEL CLASSIFICATION. D71.

1. INTRODUCTION

Following the seminal work of Hurwicz (1972), the manipulability and efficiency of allocation mechanisms in pure exchange economies have been studied intensively. Zhou (1991) established that any Pareto-efficient and strategy-proof allocation mechanism is dictatorial in exchange economies with two agents having classical (i.e., continuous, strictly monotonic, and strictly convex) preferences. The dictatorship result in two-agent economies has been strengthened by being proven in the domain of restricted preferences.[1]

Compared with the result in two-agent economies, it is an open question whether Pareto-efficient and strategy-proof allocation mechanisms can be characterized in economies with many agents. This is the issue that we examine in this paper. In many-agent economies, there actually exist Pareto-efficient, strategy-proof, and nondictatorial allocation mechanisms. Satterthwaite and Sonnenschein (1981) constructed such a mechanism, relying on the reverse dictator's preference, to select one agent among the remaining agents, who is allocated the total endowment. Kato and Ohseto (2002) constructed a mechanism in economies with four or more agents, such that all agents have the opportunity to be allocated the total endowment. A specific feature shared by all known Pareto-efficient and strategy-proof allocation mechanisms is that some single agent receives the whole amount of goods even if the receivers vary. Such a mechanism is called alternately dictatorial. The natural question to be asked is whether there exists a Pareto-efficient, strategy-proof, and nonalternately dictatorial allocation mechanism.

[1]See Schummer (1997), Ju (2003), Hashimoto (2008), and Momi (2013a). Nicoló (2004), however, showed a Pareto-efficient, strategy-proof, and nondictatorial mechanism in the domain of Leontief preferences.

In this paper, we show that in exchange economies where the number of goods equals or exceeds the number of agents, any Pareto-efficient and strategy-proof allocation mechanism is alternately dictatorial. We believe that our method and result provide a first step toward solving the general question without conditions on numbers of goods and agents.

In the following subsections, we discuss our method in detail and compare our result with those of related papers to highlight the contributions in this paper.

1.1 *Approach*

In this paper, we study what we call the option set in detail. An agent's option set, when the other agents' preferences are fixed, is defined as the union of the agent's consumption bundles allocated by a mechanism while his preference changes. Then the agent's consumption bundle allocated by a strategy-proof mechanism should be the most preferred one in the option set with respect to his preference. Therefore, the option set completely describes how the agent's consumption changes under a strategy-proof mechanism in response to changes in his preference when the other agents' preferences are fixed.

Note that the option set must be either the zero consumption bundle or the total endowment, under an alternately dictatorial mechanism. In this paper, we assume that a positive consumption bundle different from the total endowment is allocated by a mechanism, and we then study the agent's option set in a neighborhood of the consumption bundle. Through the analysis of the option set, we investigate how consumption changes in response to changes in the agent's preference, and we obtain allocations that contradict the Pareto efficiency and strategy-proofness of the mechanism.

This is the approach that Hashimoto (2008) and Momi (2013a, 2013b) followed to analyze two-agent economies. In two-agent economies, we can determine an agent's option set as the inverted image of an upper contour set of the other agent's preference. This easily yields the dictatorship result in two-agent economies. In economies with many agents, we cannot obtain the exact shape of the option set. However, we can still derive the topological properties of the option set. Roughly speaking, we show that in a neighborhood of an allocated consumption bundle that is neither zero nor the total endowment, the option set is the smooth surface of a strictly convex set. This property is sufficient to yield allocations that contradict Pareto efficiency and strategy-proofness.

Throughout the paper, the condition that the number of goods equals or exceeds the number of agents plays an important role. In the core part of the paper, we deal with homothetic preferences and consider preferences that satisfy independence in the following sense. An efficient allocation given by a Pareto-efficient mechanism uniquely determines the supporting price, and the supporting price vector uniquely determines the direction of possible consumption of each agent with a homothetic preference. We refer to this direction vector as the consumption-direction vector of the agent. If these consumption-direction vectors are independent among agents, then there is a unique way to scale these vectors so that they sum up to the total endowment. That is, a supporting price induced by a Pareto-efficient mechanism determines the allocation itself

when the consumption-direction vectors are independent. In other words, a consumption bundle of an agent uniquely determines the other agents' consumption bundles under independence of the consumption-direction vectors. This works in many-agent economies as in two-agent economies: an agent's consumption immediately determines the other agent's consumption as the rest of the goods, which makes two-agent economies decisively tractable. It is clear that we need at least as many goods as the number of agents for the existence of a preference profile satisfying the independence of the consumption-direction vectors.

Through the option set, we know how an agent's consumption changes in response to changes in his preference, as mentioned above. If the consumption-direction vectors are independent, this change of an agent's consumption exactly determine how other agents' consumption changes, that is, how the allocation itself changes. Using this, the proof in this paper proceeds as follows. We suppose that an agent is allocated consumption that is neither zero nor the total endowment. First, we establish that the agent's option set is the smooth surface of a strictly smooth set in a neighborhood of the consumption bundle. Then we observe that such an option set induces allocations that contradict the Pareto efficiency and strategy-proofness. This implies that any agent's consumption is either zero or the total endowment; that is, the allocation is alternately dictatorial.

1.2 *Related literature*

As mentioned above, the general characterization of Pareto-efficient and strategy-proof mechanisms in many-agent economies is still an open problem.

Some studies show the incompatibility of Pareto efficiency and strategy-proofness with allocation restrictions. Serizawa (2002) shows the incompatibility with the individual rationality restriction, where agents originally possess their initial endowments and a mechanism is assumed to allocate consumption that benefits all agents. Serizawa and Weymark (2003) show the incompatibility with the minimum consumption guarantee restriction, where the consumption of each agent is assumed to be away from zero by some minimum distance. Momi (2013b) shows the incompatibility with a simple positivity restriction, where a mechanism is assumed to allocate positive consumption to all agents. Alternatively, Barberà and Jackson (1995) discard Pareto efficiency and characterize strategy-proof mechanisms satisfying the individual rationality restriction. These allocation restrictions are so strong that they exclude any mechanism wherein some agents receive zero consumption. In particular, alternately dictatorial allocations violate these restrictions. Therefore, as long as there are at least as many goods as agents, the result in this paper, where Pareto efficiency and strategy-proofness induce alternately dictatorial allocations, yields the above-mentioned results by Serizawa (2002), Serizawa and Weymark (2003), and Momi (2013b) as a corollary.

Some studies investigate the nonbossiness condition. A mechanism is called nonbossy if a change in the preference of an agent does not affect the allocation as long as it does not affect the agent's own consumption. Momi (2013b) shows that any Pareto-efficient, strategy-proof, and nonbossy mechanism is dictatorial. Goswami et al. (2014)

show that any Pareto-efficient, strategy-proof, nonbossy, and continuous mechanism is dictatorial even in the restricted domain of quasi-linear preferences. See Hatfield (2009) for a study of Pareto-efficient, strategy-proof, and nonbossy mechanisms in the context of allocating indivisible goods. The nonbossiness condition almost immediately implies that an alternately dictatorial mechanism is dictatorial; that is, it excludes a mechanism where receivers of the total endowment vary. Therefore, as long as there are at least as many goods as agents, the above-mentioned result by Momi (2013b) is obtained as a corollary of this paper's result.

The characterization of Pareto-efficient and strategy-proof mechanisms has been obtained not only for two-agent economies but also for three-agent economies. Momi (2013b) proves that any Pareto-efficient and strategy-proof allocation mechanism in three-agent economies is either dictatorial or of the Satterthwaite and Sonnenschein (1981) type. Unfortunately, this approach crucially relies on the assumption of three agents and it seems difficult to extend it to economies with more agents. Although Momi (2013b) studies the option set, he focuses on a preference profile where two out of three agents have the same preference. In such a case, these two agents can be identified, and the option set of the other agent is given by the turned-over image of an upper contour set of this preference, as in two-agent economies.

Alternatively, given the assumption on the numbers of agents and goods, the current paper's result does not cover the case of economies with three agents and two goods, which is covered by Momi (2013b). However, it might be possible to extend our proof to cover this case. As mentioned in the previous subsection, and as is seen in the proof, what is crucial is that the consumption bundle of an agent uniquely determines the other agents' consumption. Suppose that the number of agents exceeds the number of goods by exactly 1. If the consumption bundle of an agent is determined, then the other agents' consumption should be determined uniquely under independence of their consumption-direction vectors because their total consumption equals the total endowment minus the predetermined consumption. However, such a slight extension is of minor importance. The interesting and challenging question is, of course, whether we can have a Pareto-efficient, strategy-proof, and nonalternately dictatorial allocation mechanism without any restrictions on the numbers of agents and goods.

The rest of the paper is organized as follows. Section 2 describes the model and results. Section 3 explains some technical aspects of the paper and demonstrates a technique for constructing a preference. Section 4 reveals the properties of the option set. Sections 5 and 6 provide the proofs of the results in Section 2. Section 7 provides concluding remarks. The Appendix contains proofs of all lemmas and propositions in Sections 3 and 4.

2. Model and results

We consider an economy with N agents, indexed by $\mathbf{N} = \{1, \ldots, N\}$, where $N \geq 2$, and L goods, indexed by $\mathbf{L} = \{1, \ldots, L\}$, where $L \geq 2$. The consumption set for each agent is R_+^L. A consumption bundle for agent $i \in \mathbf{N}$ is a vector $x^i = (x_1^i, \ldots, x_L^i) \in R_+^L$. The total endowment of goods for the economy is $\Omega = (\Omega_1, \ldots, \Omega_L) \in R_{++}^L$. An allocation is a

vector $\mathbf{x} = (x^1, \ldots, x^N) \in R_+^{LN}$. Thus, the set of feasible allocations for the economy with N agents and L goods is

$$ X = \left\{ \mathbf{x} \in R_+^{LN} : \sum_{i \in \mathbf{N}} x^i \leq \Omega \right\}. $$

A preference R is a complete, reflexive, and transitive binary relation on R_+^L. The corresponding strict preference P_R and indifference I_R are defined in the usual way. For any x and x' in R_+^L, xP_Rx' implies that xRx' and not $x'Rx$, and xI_Rx' implies that xRx' and $x'Rx$. Given a preference R and a consumption bundle $x \in R_+^L$, the upper contour set of R at x is $\mathrm{UC}(x; R) = \{x' \in R_+^L : x'Rx\}$, and the lower contour set of R at x is $\mathrm{LC}(x; R) = \{x' \in R_+^L : xRx'\}$. We let $\mathrm{I}(x; R) = \{x' \in R_+^L : x'I_Rx\}$ denote the indifference set of R at x, and let $\mathrm{P}(x; R) = \{x' \in R_+^L : x'P_Rx\}$ denote the strictly preferred set of R at x.

A preference R is continuous if $\mathrm{UC}(x; R)$ and $\mathrm{LC}(x; R)$ are both closed for any $x \in R_+^L$. A preference R is strictly convex on R_{++}^L if $\mathrm{UC}(x; R)$ is a strictly convex set in R^L for any $x \in R_{++}^L$. A preference R is monotonic if, for any x and x' in R_+^L, $x > x'$ implies that xRx'.[2] A preference R is strictly monotonic on R_{++}^L if, for any x and x' in R_{++}^L, $x > x'$ implies that xP_Rx'.[3] A preference R is homothetic if, for any x and x' in R_+^L and any $t > 0$, xRx' implies that $(tx)R(tx')$. A preference R is smooth if for any $x \in R_{++}^L$, there exists a unique vector $p \in S_+^{L-1} \equiv \{x \in R_+^L : \|x\| = 1\}$ such that p is the normal of a supporting hyperplane to $\mathrm{UC}(x; R)$ at x. We call the vector p the gradient vector of R at x, and write $p = p(R, x)$. Note that if R is smooth, strictly convex on R_{++}^L, and strictly monotonic on R_{++}^L, then the gradient vector is positive in the positive orthant: $p(R, x) \in S_{++}^{L-1} \equiv \{x \in R_{++}^L : \|x\| = 1\}$ for any $x \in R_{++}^L$.

We call a preference classical when it is continuous, strictly convex on R_{++}^L, and strictly monotonic on R_{++}^L, and we let $\mathcal{R}_C$ denote the set of classical preferences. Furthermore, we let $\mathcal{R}$ denote the set of classical, smooth, and homothetic preferences. In this paper, we prove the results in the restricted domain $\mathcal{R}$ and then extend them to $\mathcal{R}_C$.

A preference profile is an N-tuple $\mathbf{R} = (R^1, \ldots, R^N) \in \mathcal{R}^N$. We write the subprofile obtained by removing R^i from $\mathbf{R}$ as $\mathbf{R}^{-i} = (R^1, \ldots, R^{i-1}, R^{i+1}, \ldots, R^N)$ and write the profile $(R^1, \ldots, R^{i-1}, \bar{R}^i, R^{i+1}, \ldots, R^N)$ as $(\bar{R}^i, \mathbf{R}^{-i})$. We also write $\mathbf{R}^{-\{i,j\}}$ to denote the subprofile obtained by removing R^i and R^j from $\mathbf{R}$.

A social choice function $f : \mathcal{R}^N \to X$ assigns a feasible allocation to each preference profile in $\mathcal{R}^N$. For a preference profile $\mathbf{R} \in \mathcal{R}^N$, the outcome chosen can be written as $f(\mathbf{R}) = (f^1(\mathbf{R}), \ldots, f^N(\mathbf{R}))$, where $f^i(\mathbf{R})$ is the consumption bundle allocated to agent i by f.

DEFINITION 1. A social choice function $f : \mathcal{R}^N \to X$ is strategy-proof if $f^i(\mathbf{R})R^if^i(\bar{R}^i, \mathbf{R}^{-i})$ for any $i \in \mathbf{N}$, any $\mathbf{R} \in \mathcal{R}^N$, and any $\bar{R}^i \in \mathcal{R}$.

A feasible allocation is Pareto efficient if there is no other feasible allocation that benefits someone without making anyone else worse off. That is, $\mathbf{x} \in X$ is Pareto efficient

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[2]For vectors x and x' in R^L, $x > x'$ denotes that $x_l \geq x'_l$ for any $l \in \mathbf{L}$ and $x \neq x'$.
[3]Therefore, if R is continuous, strictly convex on R_{++}^L, and strictly monotonic on R_{++}^L, then $\mathrm{UC}(x; R) \subset R_{++}^L$ for any $x \in R_{++}^L$ and the boundary ∂R_+^L is an indifference set.

for preference profile $\mathbf{R}$ if there exists no $\bar{\mathbf{x}} \in X$ such that $\bar{x}^i R^i x^i$ for any $i \in \mathbf{N}$ and $\bar{x}^j P_{R^j} x^j$ for some $j \in \mathbf{N}$. We say that a social choice function is Pareto efficient if it always assigns a Pareto-efficient allocation.

DEFINITION 2. A social choice function $f : \mathcal{R}^N \to X$ is Pareto efficient if $f(\mathbf{R})$ is Pareto efficient for any $\mathbf{R} \in \mathcal{R}^N$.

We say that a social choice function is dictatorial if there exists an agent who is always allocated the total endowment.

DEFINITION 3. A social choice function $f : \mathcal{R}^N \to X$ is dictatorial if there exists $i \in \mathbf{N}$ such that $f^i(\mathbf{R}) = \Omega$ for any $\mathbf{R} \in \mathcal{R}^N$.

We say that a social choice function is alternately dictatorial if it always allocates the total endowment to some single agent. Note that under an alternately dictatorial social choice function, the identity of the receiver of the total endowment may vary depending on preference profiles.

DEFINITION 4. A social choice function $f : \mathcal{R}^N \to X$ is alternately dictatorial if, for any $\mathbf{R} \in \mathcal{R}^N$, there exists $i_{\mathbf{R}} \in \mathbf{N}$ such that $f^{i_{\mathbf{R}}}(\mathbf{R}) = \Omega$.

This paper's main result is as follows.

THEOREM. *When $L \geq N$, a Pareto-efficient and strategy-proof social choice function $f : \mathcal{R}^N \to X$ is alternately dictatorial.*

This is proved in the preference domain $\mathcal{R}$. Let $\bar{\mathcal{R}}$ be a preference domain such that $\mathcal{R} \subset \bar{\mathcal{R}} \subset \mathcal{R}_C$, and let us extend Definitions 1–4 to $\bar{\mathcal{R}}$.

COROLLARY. *When $L \geq N$, a Pareto-efficient and strategy-proof social choice function $f : \bar{\mathcal{R}}^N \to X$ is alternately dictatorial.*

3. PRELIMINARY RESULTS

In this section, we explain some technical aspects of this paper. We introduce a metric in the space of preferences and show a technique of preference construction. Furthermore, we define the pseudo-efficiency of a social choice function and the feasible consumption set for an agent.

As in the previous works, including Serizawa (2002) and Momi (2013b), we introduce the Kannai metric into $\mathcal{R}$ following Kannai (1970), to discuss the continuity in $\mathcal{R}$. For $x \in R_+^L \setminus 0$, we let $[x]$ denote the ray starting from zero and passing through x: $[x] = \{y \in R_+^L : y = tx, t \geq 0\}$. We define $\mathbf{1} \equiv (1, \ldots, 1) \in R_+^L$ so that $[\mathbf{1}]$ denotes the principal diagonal of R_+^L. Using these definitions, the Kannai metric $d(R, R')$ for continuous and monotonic preferences R and R' is defined as

$$d(R, R') = \max_{x \in R_+^L} \frac{\left\| I(x; R) \cap [\mathbf{1}] - I(x; R') \cap [\mathbf{1}] \right\|}{1 + \|x\|^2},$$

where $\| \cdot \|$ denotes the Euclidean norm in R^L. With the Kannai metric, $\mathcal{R}$ is a metric space. See Kannai (1970) for details.

We often discuss the distance between upper contour sets of preferences. Note that any homothetic preference is identified by one indifference set or upper contour set in R_{++}^L because the other indifference sets of the homothetic preference are determined by similarity transformations. For any subsets A and B in R^L, we write $A + B = \{a + b \in R^L : a \in A, b \in B\}$ and $A - B = \{a - b \in R^L : a \in A, b \in B\}$. We call a subset $A \subset R_{++}^L$ monotonic when $x + R_+^L \subset A$ holds for any $x \in A$, and we define M as the set of closed and monotonic subsets of R_{++}^L. For any $A \in M$, the homothetic, continuous, and monotonic preference R_A that has A as its upper contour set is uniquely determined. We define the distance $d_M(A, B)$ between A and B in M as

$$d_M(A, B) = d(R_A, R_B).$$

Thus, the convergence of candidate upper contour sets with respect to this metric implies the convergence of the corresponding preferences with respect to the Kannai metric.[4]

In this paper, we use $B_\epsilon(\bar{R}) \subset \mathcal{R}$ to denote the open ball set of preferences in $\mathcal{R}$, with center $\bar{R}$ and radius $\epsilon > 0$: $B_\epsilon(\bar{R}) = \{R \in \mathcal{R} : d(R, \bar{R}) < \epsilon\}$. We use $B \subset \mathcal{R}$ to denote an open ball set of preferences without a specified center or radius,

For a preference $R \in \mathcal{R}$ and a consumption bundle $x \in R_+^L$, a preference $\bar{R}$ is called a Maskin monotonic transformation (MMT, hereafter) of R at x if $\bar{x} \in \mathrm{UC}(x; \bar{R})$ and $\bar{x} \neq x$ implies that $\bar{x} P_R x$. It is well known that if an agent receives x at a preference profile **R**, strategy-proofness implies that this agent receives the same consumption bundle x when his preference is subject to an MMT at x. As shown in Momi (2013b, Lemma 4), for a preference $R \in \mathcal{R}$ and a consumption bundle $x \in R_{++}^L$, there exists a preference that is an MMT of R at x in any neighborhood of R.

For a price vector $p \in S_{++}^{L-1}$ and a consumption vector $x \in R_{++}^L$, we let $p^\perp$ denote the hyperplane perpendicular to p, i.e., $p^\perp = \{y \in R^L : py = 0\}$, and let $H(x; p)$ denote the upper right-hand side half-space of the hyperplane, i.e., $H(x; p) = y + p^\perp + R_+^L$.

Now, we demonstrate a technique for constructing a preference that satisfies a given pair of a gradient vector and a consumption bundle in a neighborhood of a given preference. Let $\bar{R} \in \mathcal{R}$ be a preference that has a gradient vector $\bar{p} \in S_{++}^{L-1}$ at a consumption bundle $\bar{x} \in R_{++}^L$, as drawn in Figure 1. We let (x_n, p_n) be another pair consisting of a consumption bundle and a price vector, and construct a preference R_n that has p_n as the gradient vector at x_n. This n in x_n should not be confused with the subscripts labeling goods. It is not difficult to imagine such a preference R_n in a neighborhood of $\bar{R}$ if

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[4]Note the difference between the Kannai metric for preferences and the Hausdorff metric for the corresponding indifference or upper contour sets. Fix a consumption vector x. the convergence $R \to R^*$ with respect to the Kannai metric does not generally imply convergence $\mathrm{I}(x; R) \to \mathrm{I}(x; R^*)$ with respect to the Hausdorff metric because the indifference sets are not bounded. For example, let $\mathrm{P} \subset S_{++}^{L-1}$ be a compact set and let $K = \bigcup_{z \in \mathrm{P}}[z]$ denote the union of rays $[z]$ passing through $z \in \mathrm{P}$. Then $\mathrm{I}(x; R) \cap K$ is compact and the convergence $R \to R^*$ implies the convergence $\mathrm{I}(x; R) \cap K \to \mathrm{I}(x; R^*) \cap K$ with respect to the Hausdorff metric. Alternatively, the convergence of candidate indifference or upper contour sets with respect to the Hausdorff metric implies the convergence of the corresponding preferences with respect to the Kannai metric.
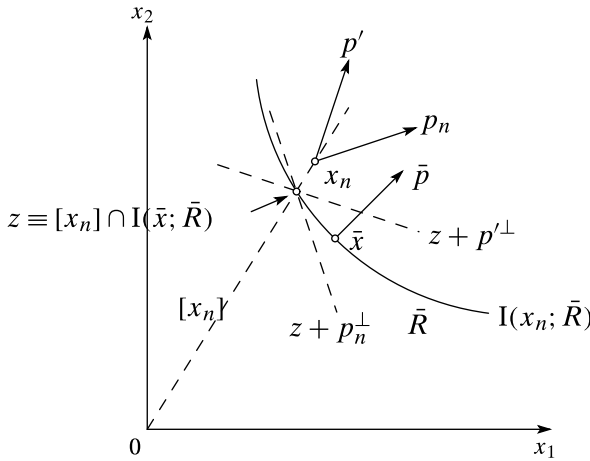
FIGURE 1. Preference construction.

p_n and x_n are sufficiently close to $\bar{p}$ and $\bar{x}$, respectively. Furthermore, we can have the preference R_n so that its gradient vector at $\bar{x}$ is $\bar{p}$ if $p_n \bar{x} > p_n([x_n] \cap I(\bar{x}; \bar{R}))$, as drawn in Figure 1. To understand this condition, consider another price vector p' such that $p' \bar{x} \leq p'([x_n] \cap I(\bar{x}; \bar{R}))$, as drawn in the figure. It is clear that any strictly convex preference with the gradient vector p' at x_n cannot have $\bar{p}$ as its gradient vector at $\bar{x}$. This is summarized as Lemma 1. Note that the condition $p_n([\bar{x}] \cap I(x_n; \bar{R})) > p_n x_n$ in the lemma is equivalent to the above-mentioned $p_n \bar{x} > p_n([x_n] \cap I(\bar{x}; \bar{R}))$, because the preference $\bar{R}$ is homothetic. See the Appendix for the proof of Lemma 1.

LEMMA 1. *Let a preference $\bar{R} \in \mathcal{R}$ have a gradient vector $\bar{p} \in S_{++}^{L-1}$ at a consumption bundle $\bar{x} \in R_{++}^L$: $\bar{p} = p(\bar{R}, \bar{x})$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be sequences of consumption bundles and price vectors that converge to $\bar{x}$ and $\bar{p}$, respectively: $x_n \to \bar{x}$ and $p_n \to \bar{p}$ as $n \to \infty$. For any $\epsilon > 0$, there exists $\bar{n}$ such that for any $n > \bar{n}$ there exists a preference $R_n \in B_{\epsilon}(\bar{R})$ such that $p(R_n, x_n) = p_n$. Furthermore, if $p_n([\bar{x}] \cap I(x_n; \bar{R})) > p_n x_n$ holds for $n > \bar{n}$, we can have $R_n \in B_{\epsilon}(\bar{R})$ satisfying $p(R_n, \bar{x}) = \bar{p}$ in addition to $p(R_n, x_n) = p_n$.*

We say that a social choice function is pseudo-efficient if it allocates a Pareto-efficient allocation or allocates zero consumption to all agents.

DEFINITION 5. A social choice function $f : \mathcal{R}^N \to X$ is pseudo-efficient if, for any $\mathbf{R} \in \mathcal{R}^N$, $f(\mathbf{R})$ is Pareto efficient or $f^i(\mathbf{R}) = 0$ for any $i \in \mathbf{N}$.

It is clear that if a social choice function is Pareto efficient, then it is also pseudo-efficient. Until the last step in the proof of the theorem provided in Section 6, we prove all lemmas and propositions with a pseudo-efficient social choice function rather than with a Pareto-efficient social choice function. This is because, in the proof of the theorem, we apply the lemmas and propositions to a subeconomy $\mathbf{N}' \subset \mathbf{N}$ with N' ($< N$) agents. Note that pseudo-efficiency in such a subeconomy does not contradict Pareto efficiency in the whole economy. Let a social choice function f be Pareto efficient in

the whole economy $\mathbf{N}$. If f allocates the entire amount of goods to members of a subeconomy $\mathbf{N}' \subset \mathbf{N}$ or to agents outside the subeconomy, then the social choice function f restricted to the subeconomy $\mathbf{N}'$ is pseudo-efficient.

Furthermore, in most parts of the proofs in this paper, we deal with a social choice function that only locally satisfies efficiency and strategy-proofness. As mentioned above, $B \subset \mathcal{R}$ denotes an open ball set of preferences. We write B^i to clarify that it is a set of agent i's preferences and we write $\mathbf{B} = \prod_{i=1}^{N} B^i$ to denote the product of such open ball sets over agents. We say that a social choice function is strategy-proof on $\mathbf{B}$ if $f^i(\mathbf{R}) R^i f^i(\bar{R}^i, \mathbf{R}^{-i})$ for any $i \in \mathbf{N}$, any $\mathbf{R} \in \mathbf{B}$, and any $\bar{R}^i \in B^i$. We say that a social choice function f is Pareto efficient on $\mathbf{B}$ if $f(\mathbf{R})$ is Pareto efficient for any $\mathbf{R} \in \mathbf{B}$. We say that a social choice function f is pseudo-efficient on $\mathbf{B}$ if, for any $\mathbf{R} \in \mathbf{B}$, $f(\mathbf{R})$ is Pareto efficient or $f^i(\mathbf{R}) = 0$ for any $i \in \mathbf{N}$.

We let A denote the feasible consumption set for an agent:

$$A = \left\{ x \in R_+^L : 0 \le x_l \le \Omega_l, l \in \mathbf{L} \right\}.$$

It is not difficult to observe that if a social choice function is pseudo-efficient on $\mathbf{B}$, then for any $\mathbf{R} \in \mathbf{B}$, any agent's consumption $f^i(\mathbf{R})$ is not on the boundary of A except zero and Ω; that is, $f^i(\mathbf{R}) \in \text{int } A \cup \{0, \Omega\}$ for any i and any $\mathbf{R} \in \mathbf{B}$, where int A denotes the interior of A.

We first observe that $f^i(\mathbf{R})$ cannot be on the boundary of the consumption set, ∂R_+^L, except for the zero vector. Suppose $f^i(\mathbf{R}) \in \partial R_+^L \setminus 0$ for an agent i. If there exists another agent j who receives consumption in the interior of the consumption set, then reallocating $f^i(\mathbf{R})$ to agent j makes agent j better off without worsening agent i because the boundary of the consumption set is an indifferent set, and agent i is indifferent between zero consumption and any consumption on the boundary. This contradicts the pseudo-efficiency of f. If there exists no agent who receives consumption in the interior of the consumption set, then reallocating the total amount of goods to some agent j makes agent j better off without negatively impacting any other agent, for the same reason. This contradicts the pseudo-efficiency of f. Therefore, $f^i(\mathbf{R}) \notin \partial R_+^L \setminus 0$.

If $f_l^i(\mathbf{R}) = \Omega_l$ for some good l and $f^i(\mathbf{R}) \ne \Omega$ for an agent i, then there exists another agent j such that $f^j(\mathbf{R}) \in \partial R_+^L \setminus 0$, which is a contradiction, as shown above. Therefore, $f_l^i(\mathbf{R}) \ne \Omega_l$ for any l, except for the case where $f^i(\mathbf{R}) = \Omega$. Thus, we have $f^i(\mathbf{R}) \in \text{int } A \cup \{0, \Omega\}$ for any i and any $\mathbf{R} \in \mathbf{B}$.

4. Option set

In this section, we study the option set and show that it is the smooth surface of a strictly convex set in a neighborhood of a consumption bundle in the interior of the feasible consumption set.

We consider a social choice function f that is pseudo-efficient and strategy-proof on a product set $\mathbf{B} = \prod_{i=1}^{N} B^i$.

The option set is defined as follows. For agent i, when the other agents' preferences $\bar{\mathbf{R}}^{-i} \in \mathbf{B}^{-i} \equiv \prod_{j \ne i} B^j$ are fixed, we define the option set, $G^i(\bar{\mathbf{R}}^{-i}) \subset R_+^L$, as the union of the

agent's consumption bundles given by f over his preferences in B^i:

$$G^i(\bar{\mathbf{R}}^{-i}) = \bigcup_{R^i \in B^i} f^i(R^i, \bar{\mathbf{R}}^{-i}).$$

The fact that we omit the domain B^i in the notation of $G^i(\bar{\mathbf{R}}^{-i})$ should not cause any confusion. We are interested in the option set in a neighborhood of a specific consumption bundle $f^i(\mathbf{R})$, and the option set around the consumption bundle does not depend on the domain $\mathbf{B}$ as long as it includes the preference profile $\mathbf{R}$.

Because of the strategy-proofness on $\mathbf{B}$, $f^i(R^i, \bar{\mathbf{R}}^{-i})$ should be the most preferred consumption bundle in $G^i(\bar{\mathbf{R}}^{-i})$ with respect to $R^i \in B^i$. In two trivial cases, $G^i(\bar{\mathbf{R}}^{-i})$ is given by single-element sets because of strategy-proofness. If $f^i(R^i, \bar{\mathbf{R}}^{-i}) = 0$ for some $R^i \in B^i$, then $G^i(\bar{\mathbf{R}}^{-i}) = \{0\}$. If $f^i(R^i, \bar{\mathbf{R}}^{-i}) = \Omega$ for some $R^i \in B^i$, then $G^i(\bar{\mathbf{R}}^{-i}) = \Omega$. Note that these are the cases that occur under an alternately dictatorial social choice function. As mentioned in Section 3, $f^i(\mathbf{R}) \in \text{int } A \cup \{0, \Omega\}$ for any i and any $\mathbf{R} \in \mathbf{B}$. Going forward, we investigate the case where $f^i(R^i, \bar{\mathbf{R}}^{-i}) \in \text{int } A$ for an agent i and any $R^i \in B^i$. Note that as long as $f^i(R^i, \bar{\mathbf{R}}^{-i}) \in \text{int } A$, the allocation $f(R^i, \bar{\mathbf{R}}^{-i})$ is Pareto efficient under the pseudo-efficiency of f on $\mathbf{B}$.

To study the option set, independence of the preferences is required in the following sense. At a Pareto-efficient allocation $f(\mathbf{R})$, all agents share the same gradient vector at their consumption as long as consumption is positive and the gradient vector is well defined. We call this vector the price vector at allocation $f(\mathbf{R})$ and write $p(\mathbf{R}, f) \in S_{++}^{L-1}$.

Alternatively, for a preference $R \in \mathcal{R}$ and a price vector $p \in S_{++}^{L-1}$, we let $g(R, p) \in S_{++}^{L-1}$ denote the normalized consumption vector where the gradient vector of R is p. Note that the normalized consumption vector is a continuous function of p because preference R is smooth and strictly convex in R_{++}^L. Therefore, if normalized consumption vectors $g(R^i, p)$, $i = 1, \ldots, N$, are linearly independent at $p = \bar{p}$, the linear independence also holds in a small neighborhood of $\bar{p}$.

As mentioned above, the price vector depends on the social choice function: $p = p(\mathbf{R}, f)$. We call $g(R^i, p(\mathbf{R}, f))$ agent i's consumption-direction vector at the preference profile $\mathbf{R}$ under f because his consumption $f^i(\mathbf{R})$ should be on the ray $[g(R^i, p(\mathbf{R}, f))]$. We can write $f^i(\mathbf{R}) = \|f^i(\mathbf{R})\| g(R^i, p(\mathbf{R}, f))$.

We focus on a preference profile $\bar{\mathbf{R}} \in \mathbf{B}$ such that the consumption-direction vectors are independent. The role of this independence should be clear. As consumption vectors $f^i(\bar{\mathbf{R}})$, $i = 1, \ldots, N$, are on the rays $[g(\bar{R}^i, p(\bar{\mathbf{R}}, f))]$, $i = 1, \ldots, N$, respectively, and they sum to the total endowment Ω, the consumption vectors should be determined uniquely if the consumption-direction vectors are independent. Note that we need $L \geq N$ for the independence to hold.

The rest of this section discusses the topological properties of the option set when an agent i receives a consumption bundle in the interior of the feasible consumption set: $f^i(\bar{\mathbf{R}}) \in \text{int } A$. As a result, we see that in a neighborhood of $f^i(\bar{\mathbf{R}})$, the option set $G^i(\bar{\mathbf{R}}^{-i})$ is the $L - 1$-dimensional smooth surface of a strictly convex set, as drawn in Figure 2.

Keep in mind that such an $L - 1$-dimensional option set is obtained under the assumption that $f^i(\bar{\mathbf{R}}) \in \text{int } A$. In the next section, we observe that such an option set
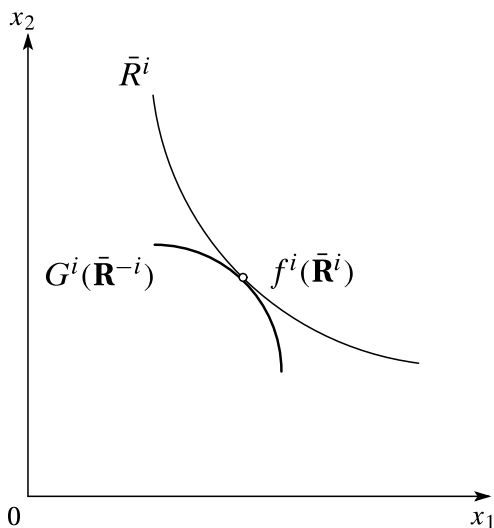
FIGURE 2. The option set.

induces allocations incompatible with the Pareto efficiency and strategy-proofness of the social choice function to prove the alternately dictatorial result.

The next lemma is an immediate consequence of the independence of the consumption-direction vectors. Recall that the proofs of all lemmas and propositions in this section are provided in the Appendix.

LEMMA 2. *Suppose that a social choice function f is pseudo-efficient on a product set of open balls $\mathbf{B} = \prod_{i=1}^{N} B^i$ and that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$. Let $\tilde{\mathbf{R}} = (\tilde{R}^1, \ldots, \tilde{R}^N) \in \mathbf{B}$ be a preference profile such that each $\tilde{R}^i$ has the same gradient vector as $\bar{R}^i$ at $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$. If $f^j(\tilde{\mathbf{R}}) = f^j(\bar{\mathbf{R}}) \neq 0$ for an agent j, then $f(\tilde{\mathbf{R}}) = f(\bar{\mathbf{R}})$.*

As mentioned, $f^i(\mathbf{R})$ is the most preferred consumption bundle in the option set $G^i(\mathbf{R}^{-i})$ with respect to $R^i \in B^i$. We want to establish the reverse: the most preferred consumption bundle in $G^i(\mathbf{R}^{-i})$ with respect to $R^i \in B^i$ is the consumption bundle allocated to agent i by the social choice function when his preference is R^i. For tractability, we initially consider the most preferred consumption bundle in the option set's closure, $\overline{G^i(\mathbf{R}^{-i})}$. By considering the closure, we ensure the existence of the most preferred consumption vector in $\overline{G^i(\mathbf{R}^{-i})}$ with respect to any preference R^i.

Lemma 3 shows that if a consumption bundle is the most preferred in the closure of the option set with respect to some preference, then the consumption bundle is actually an element of the option set.

LEMMA 3. *Suppose that a social choice function f is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^{N} B^i$. If x is the most preferred consumption bundle in $\overline{G^i(\mathbf{R}^{-i})}$ with respect to $R^i \in B^i$, then $x \in G^i(\mathbf{R}^{-i})$. In particular, if $\hat{R}^i \in B^i$ is an MMT of R^i at x, then $f^i(\hat{R}^i, \mathbf{R}^{-i}) = x$.*

An immediate consequence of Lemma 3 is that if x is the unique, most preferred consumption bundle in $\overline{G^i(\mathbf{R}^{-i})}$ with respect to $R^i \in B^i$, then $x = f^i(\mathbf{R})$. The next lemma proves that the most preferred consumption bundle is actually unique. Thus, for any preference $R^i \in B^i$, the most preferred consumption bundle in $\overline{G^i(\mathbf{R}^{-i})}$ with respect to the preference is unique, and it is exactly the consumption bundle $f^i(\mathbf{R})$ allocated to the agent for the preference by the social choice function f.

LEMMA 4. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^{N} B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and that $f^i(\bar{\mathbf{R}}) \in$ int A for an agent i. Then $f^i(\bar{\mathbf{R}})$ is the unique most preferred consumption bundle in $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ with respect $\bar{R}^i \in B^i$.*

As a consequence of Lemma 4, the next proposition shows that $f(\cdot, \bar{\mathbf{R}}^{-i})$ is a continuous function of R^i in a neighborhood of $\bar{R}^i$.

PROPOSITION 1. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^{N} B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and that $f^i(\bar{\mathbf{R}}) \in$ int A for an agent i. Then $f(\cdot, \bar{\mathbf{R}}^{-i})$ is a continuous function in a neighborhood of $\bar{R}^i$.*

Next, we show that $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ is the surface of a convex set in the sense that in a neighborhood of $f^i(\bar{\mathbf{R}})$, $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ is in the lower left-hand side of the hyperplane $f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f)^{\perp}$ for any R^i in a neighborhood of $\bar{R}^i$.

PROPOSITION 2. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^{N} B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and that $f^i(\bar{\mathbf{R}}) \in$ int A for an agent i. For any R^i in a neighborhood of $\bar{R}^i$, $\overline{G^i(\bar{\mathbf{R}}^{-i})} \subset f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f)^{\perp} - R_{+}^{L}$ holds in a neighborhood of $f^i(\bar{\mathbf{R}})$; that is, there exist positive scalars $\bar{\epsilon}$ and ϵ' such that*

$$D_{\bar{\epsilon}}\big(f^i(\bar{\mathbf{R}})\big) \cap \overline{G^i(\bar{\mathbf{R}}^{-i})} \subset f^i\big(R^i, \bar{\mathbf{R}}^{-i}\big) + p\big((R^i, \bar{\mathbf{R}}^{-i}), f\big)^{\perp} - R_{+}^{L} \tag{1}$$

for any $R^i \in B_{\epsilon'}(\bar{R}^i)$, where $D_{\bar{\epsilon}}(f^i(\bar{\mathbf{R}}))$ is the open ball in R_{+}^{L} with center $f^i(\bar{\mathbf{R}})$ and radius $\bar{\epsilon}$.

This proposition asserts the convexity of the option set in the following sense. For $x' = f^i(R^{i\prime}, \mathbf{R}^{-i})$ and $x'' = f^i(R^{i\prime\prime}, \mathbf{R}^{-i})$ in $\overline{G^i(\bar{\mathbf{R}}^{-i})}$, if a ray $[sx' + (1-s)x'']$ with $s \in (0, 1)$ has an intersection with $\overline{G^i(\bar{\mathbf{R}}^{-i})}$, then the intersection is written as $t(sx' + (1-s)x'')$ with a scalar $t \geq 1$ greater than or equal to 1. If the intersection is given as $t(sx' + (1-s)x'')$ with $t < 1$, then there exists R^i such that $f^i(R^i, \bar{\mathbf{R}}^{-i})$ is arbitrarily close to the intersection because of the definition of $\overline{G^i(\bar{\mathbf{R}}^{-i})}$. Then at least one of x' and x'' is in the upper right-hand side of the hyperplane $f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f))^{\perp}$ regardless of the value of the price vector, which contradicts Proposition 2.

Next, we show that $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ is a surface of a strictly convex set; that is, the t in the previous paragraph is strictly greater than 1. In other words, for any R^i in a neighborhood of $\bar{R}^i$, $f^i(R^i, \bar{\mathbf{R}}^{-i})$ is the unique intersection between $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ and the hyperplane $f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f)^\perp$.

PROPOSITION 3. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and that $f^i(\bar{\mathbf{R}}) \in$ int A for an agent i. Then, for any R^i in a neighborhood of $\bar{R}^i$, $f^i(R^i, \bar{\mathbf{R}}^{-i})$ is the unique intersection between $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ and $f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f)^\perp$ in a neighborhood of $f^i(R^i, \bar{\mathbf{R}})$.*

The next proposition shows that $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ is a smooth surface in the sense that at each point $f^i(R^i, \bar{\mathbf{R}}^{-i})$, the hyperplane tangent to $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ is unique.

PROPOSITION 4. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and that $f^i(\bar{\mathbf{R}}) \in$ int A for an agent i. For any R^i in a neighborhood of $\bar{R}^i$, $f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f)^\perp$ is the unique hyperplane tangent to $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ at $f^i(R^i, \bar{\mathbf{R}}^{-i})$.*

Finally, we show that $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ coincides with $G^i(\bar{\mathbf{R}}^{-i})$ and that it is an $L - 1$-dimensional manifold in a neighborhood of the consumption bundle $f^i(\bar{\mathbf{R}}) \in$ int A, where the consumption-direction vectors are independent.

PROPOSITION 5. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$ and that $f^i(\bar{\mathbf{R}}) \in$ int A for an agent i. In a neighborhood of $f^i(\bar{\mathbf{R}})$, $G^i(\bar{\mathbf{R}}^{-i})$ coincides with $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ and it is an $L - 1$-dimensional manifold.*

5. PROOF OF THEOREM

In the previous section, we showed that given the assumption that an agent is allocated positive consumption in the interior of the feasible consumption set, the agent's option set is the $L - 1$-dimensional smooth surface of a strictly convex set in a neighborhood of the consumption bundle. In this section, we observe that such an option set induces allocations that are incompatible with the Pareto efficiency and strategy-proofness of the social choice function. This implies that any agent should be allocated zero consumption or the total endowment.

In the next proposition, we show that if a social choice function satisfies pseudo-efficiency and strategy-proofness locally in a neighborhood of a preference profile where the consumption-direction vectors are independent, then any agent receives zero consumption or the total endowment at the preference profile.
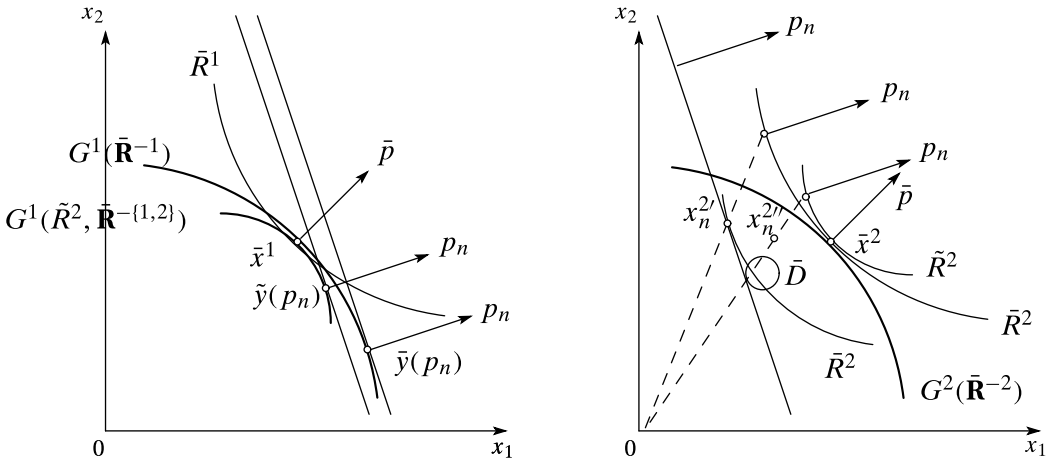
FIGURE 3. Proof of Proposition 6.

PROPOSITION 6. *We let f be a social choice function that is pseudo-efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^{N} B^i$. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$. Then $f^i(\bar{\mathbf{R}}) \in \{0, \Omega\}$ for any $i \in \mathbf{N}$.*

PROOF. We let f be a social choice function that is pseudo-efficient and strategy-proof on $\mathbf{B} = \prod_{i=1}^{N} B^i$. Suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \mathbf{B}$. As shown in Section 3, $f^i(\bar{\mathbf{R}}) \in \text{int } A \cup \{0, \Omega\}$ for any $i \in \mathbf{N}$. We assume that there exists an agent who receives consumption in int A at $\bar{\mathbf{R}}$, and show a contradiction.

As f is pseudo-efficient on $\mathbf{B}$, if an agent receives consumption in int A at $\bar{\mathbf{R}}$, there exists another agent who also receives consumption in int A. Without loss of generality, we assume that agents 1 and 2 are such agents: $f^i(\bar{\mathbf{R}}) \in \text{int } A$, $i = 1, 2$. We write $f(\bar{\mathbf{R}}) = \bar{\mathbf{x}} = (\bar{x}^1, \ldots, \bar{x}^N)$ and $p(\bar{\mathbf{R}}, f) = \bar{p}$. Because of the independence of the consumption-direction vectors, the results in the previous section hold for agents 1 and 2: in a neighborhood of their consumption bundles, each of their option sets is the $L-1$-dimensional smooth surface of a strictly convex set, as drawn in Figure 2.

We let $\tilde{R}^2 \in B^2$ be an MMT of $\bar{R}^2$ at $\bar{x}^2$ such that $\tilde{R}^2$ and $\bar{R}^2$ have the same gradient vector only at consumption on the ray $[\bar{x}^2]$. For example, see Momi (2013b, Lemma 4) for the construction of such an MMT in any neighborhood of $\bar{R}^2$. Then $f(\tilde{R}^2, \bar{\mathbf{R}}^{-2}) = \bar{\mathbf{x}}$ as in Lemma 2. Figure 3 describes agent 2's preferences $\bar{R}^2$ and $\tilde{R}^2$ and the corresponding agent 1's option sets. In the following proof, we observe that these induce allocations that contradict strategy-proofness with respect to agent 1.

Now, we focus on the option sets $G^1(\bar{\mathbf{R}}^{-1})$ and $G^1(\tilde{R}^2, \bar{\mathbf{R}}^{-\{1,2\}})$. As $f(\bar{\mathbf{R}}) = f(\tilde{R}^2, \bar{\mathbf{R}}^{-2}) = \bar{\mathbf{x}}$, both of these option sets are tangent to the hyperplane $\bar{x}^1 + \bar{p}^\perp$ at $\bar{x}^1$. We observe that $p((R^1, \bar{R}^2, \bar{\mathbf{R}}^{-\{1,2\}}), f) \neq p((R^1, \tilde{R}^2, \bar{\mathbf{R}}^{-\{1,2\}}), f)$ for any R^1 in a neighborhood of $\bar{R}^1$ unless the gradient vector of R^1 at x^1 is $\bar{p}$. If these price vectors coincide, then the hyperplanes tangent to the option set $G^2(R^1, \bar{\mathbf{R}}^{-\{1,2\}})$ at agent 2's two consumption

bundles, $f^2(R^1, \bar{R}^2, \bar{\mathbf{R}}^{-\{1,2\}})$ and $f^2(R^1, \tilde{R}^2, \bar{\mathbf{R}}^{-\{1,2\}})$, have the same normal vector. This contradicts the strict convexity of the option set shown in Proposition 3 because these two consumption bundles of agent 2 do not coincide by our choice of $\tilde{R}^2$.

For a price vector $p \in S_{++}^{L-1}$ in a neighborhood of $\bar{p}$, we let $\bar{y}(p)$ denote the point on $G^1(\bar{\mathbf{R}}^{-1})$ such that $\bar{y}(p) + p^{\perp}$ is the hyperplane tangent to $G^1(\bar{\mathbf{R}}^{-1})$ at $\bar{y}(p)$, and we let $\tilde{y}(p)$ denote the point on $G^1(\tilde{R}^2, \bar{\mathbf{R}}^{-\{1,2\}})$ such that $\tilde{y}(p) + p^{\perp}$ is the hyperplane tangent to $G^1(\tilde{R}^2, \bar{\mathbf{R}}^{-\{1,2\}})$ at $\tilde{y}(p)$. Because of Propositions 1–5 in the previous section, $\tilde{y}(p)$ and $\tilde{y}(p)$ are determined uniquely for any p in a neighborhood of $\bar{p}$. Note that we can pick a price vector p in any neighborhood of $\bar{p}$ such that the hyperplanes $\bar{y}(p) + p^{\perp}$ and $\tilde{y}(p) + p^{\perp}$ are different. The existence of such a p should be clear because nonexistence of such a p in a neighborhood of $\bar{p}$ implies the coincidence of $G^1(\bar{\mathbf{R}}^{-1})$ and $G^1(\tilde{R}^2, \bar{\mathbf{R}}^{-\{1,2\}})$ in a neighborhood of $\bar{x}^1$, which contradicts the discussion in the previous paragraph. Thus, we let $\{p_n\}_{n=1}^{\infty}$ be a sequence of price vectors converging to $\bar{p}$: $p_n \to \bar{p}$ as $n \to \infty$, such that $\bar{y}(p_n) + (p_n)^{\perp}$ and $\tilde{y}(p_n) + (p_n)^{\perp}$ are different for any n. Then we have $\bar{y}(p_n) \to \bar{x}^1$ and $\tilde{y}(p_n) \to \bar{x}^1$ as $n \to \infty$.

For each sufficiently large n, we pick agent 1's preferences, $R_n^{1\prime}$ and $R_n^{1\prime\prime}$ in B^1, satisfying the following conditions: (i) the gradient vector of $R_n^{1\prime}$ at $\bar{y}(p_n)$ is p_n, (ii) the gradient vector of $R_n^{1\prime\prime}$ at $\tilde{y}(p_n)$ is p_n, and (iii) either $\bar{y}(p_n) \in P(\tilde{y}(p_n); R_n^{1\prime\prime})$ or $\tilde{y}(p_n) \in P(\bar{y}(p_n); R_n^{1\prime})$.

For example, such preferences can be obtained as follows. As the hyperplanes $\tilde{y}(p_n) + (p_n)^{\perp}$ and $\bar{y}(p_n) + (p_n)^{\perp}$ are different, $\bar{y}(p_n)$ is in the upper right-hand side of $\tilde{y}(p_n) + (p_n)^{\perp}$ or $\tilde{y}(p_n)$ is in the upper right-hand side of $\bar{y}(p_n) + (p_n)^{\perp}$. Here, we assume the former, as drawn in Figure 3, and construct $R_n^{1\prime}$ and $R_n^{1\prime\prime}$ satisfying $\bar{y}(p_n) \in P(\tilde{y}(p_n); R_n^{1\prime})$. A symmetric discussion can be applied for the other case.

To obtain $R_n^{1\prime} \in B^1$ satisfying (i), we directly apply the preference construction in Lemma 1 so that $\bar{R}$, $(\bar{x}, \bar{p})$, and (x_n, p_n) in Lemma 1 correspond to $\bar{R}^1$, $(\bar{x}^1, \bar{p})$, and $(\bar{y}(p_n), p_n)$, respectively, in the present setup. As $\bar{y}(p_n)$ and p_n converge to $\bar{x}^1$ and $\bar{p}$, respectively, as $n \to \infty$, $R_n^{1\prime}$ converges to $\bar{R}^1$, as shown in Lemma 1, and hence $R_n^{1\prime}$ is in B^1 for a sufficiently large n.

Alternatively, $R_n^{1\prime\prime} \in B^1$ satisfying (ii) and (iii) is obtained as follows. If n satisfies $\bar{y}(p_n) \in P(\tilde{y}(p_n); \bar{R}^1)$, we construct $R_n^{1\prime\prime}$ in a neighborhood of $\bar{R}^1$ satisfying (ii) by directly applying the preference construction in the first part of Lemma 1 so that $\bar{R}$, $(\bar{x}, \bar{p})$, and (x_n, p_n) in Lemma 1 correspond to $\bar{R}^1$, $(\bar{x}^1, \bar{p})$, and $(\tilde{y}(p_n), p_n)$, respectively, in the present setup. As $\tilde{y}(p_n)$ and p_n converge to $\bar{x}^1$ and $\bar{p}$, respectively, as $n \to \infty$, $R_n^{1\prime\prime}$ converges to $\bar{R}^1$, as shown in Lemma 1, and hence $R_n^{1\prime\prime}$ is in B^1 for a sufficiently large n. If $\bar{y}(p_n) \in P(\tilde{y}(p_n); \bar{R}^1)$, then (iii) is satisfied with $R_n^{1\prime\prime}$ when n is sufficiently large because $R_n^{1\prime\prime}$ is then sufficiently close to $\bar{R}^1$.

Even if n satisfies $\bar{y}(p_n) \notin P(\tilde{y}(p_n); \bar{R}^1)$, we construct $R_n^{1\prime\prime}$ as we did in the proof of Lemma 1. We let ϵ_n be a scalar smaller than the distance between the hyperplanes $\bar{y}(p_n) + p_n^{\perp}$ and $\tilde{y}(p_n) + p_n^{\perp}$, and define E_n as the upper contour set of $\bar{R}^1$ at $\bar{y}(p_n) - \epsilon_n p_n$ cut off by the hyperplane $\bar{y}(p_n) + (p_n)^{\perp}$: $E_n = \text{UC}(\bar{y}(p_n) - \epsilon_n p_n; \bar{R}^1) \cap H(\bar{y}(p_n); p_n)$. Then we consider a preference R^1 such that R^1 has the gradient vector p_n at $\tilde{y}(p_n)$ and the strictly preferred set at $\tilde{y}(p_n)$ includes E_n: $E_n \subset P(\tilde{y}(p_n); R^1)$. The existence of such a preference R^1 is clear because the set E_n is in the upper right-hand side of the hyperplane $\tilde{y}(p_n) + (p_n)^{\perp}$ and away from the hyperplane. Then we define F_n as

the intersection of the upper contour set of R^1 at $\tilde{y}(p_n)$ and that of $\bar{R}^1$ at $\bar{y}(p_n) - \epsilon_n p_n$:
$F_n = \text{UC}(\tilde{y}(p_n); R^1) \cap \text{UC}(\bar{y}(p_n) - \epsilon_n p_n; \bar{R}^1)$. Note that $\bar{y}(p_n)$ is in the interior of F_n. This
F_n cannot be an upper contour set of a smooth preference because it has the edge at the
intersection $\text{I}(\tilde{y}(p_n); R^1) \cap \text{I}(\bar{y}(p_n) - \epsilon_n p_n; \bar{R}^1)$.

We let ϵ'_n be a scalar smaller than the distance between $\bar{y}(p_n)$ and the boundary of
F_n. Using this ϵ'_n, we round the edge of F_n as in Lemma 6 and let $R_n^{1''}$ be the preference
whose upper contour set is the smoothed set. That is, we let $\bar{D}_{\epsilon'_n} \subset R_+^L$ denote a closed
ball with radius ϵ'_n and define a closed set C_n as the union of such closed balls with radius
ϵ'_n included in F_n. That is, $C_n = \bigcup_{\bar{D}_{\epsilon'_n} \subset F_n} \bar{D}_{\epsilon'_n}$. We define $\tilde{R}_n^{1''}$ as the preference such that it
has C_n as an upper contour set. It is clear from the construction that $\tilde{y}(p_n) + (p_n)^\perp$ is the
supporting hyperplane of C_n at $\tilde{y}(p_n)$ and that $\bar{y}(p_n)$ is in the interior of C_n, and hence,
$R_n^{1''}$ satisfies (ii) and (iii). To observe that $R_n^{1''}$ is in B^1 for a sufficiently large n, note that
the set F_n converges to $\text{UC}(\bar{x}^1; \bar{R}^1)$ as $n \to \infty$. As $n \to \infty$, the scalar ϵ'_n we used to round
the edge of F_n converges to 0. Thus $R_n^{1''}$ converges to $\bar{R}^1$ as $n \to \infty$.

We write $f(R_n^{1'}, \bar{R}^{-1}) = \mathbf{x}'_n = (x_n^{1'}, \ldots, x_n^{N'})$ and $f(R_n^{1''}, \bar{R}^2, \bar{R}^{-\{1,2\}}) = \mathbf{x}''_n = (x_n^{1''}, \ldots,$
$x_n^{N''})$. This n in $x_n^{i'}$ and $x_n^{i''}$ should not be confused with the subscripts labeling goods.
It is clear that $x_n^{1'} = \bar{y}(p_n)$ and $x_n^{1''} = \tilde{y}(p_n)$.

We now focus on agent 2's preferences. Note that $x_n^{2'}$ is the most preferred consumption bundle in $G^2(R_n^{1'}, \bar{R}^{-\{1,2\}})$ with respect to $\bar{R}^2$, and the gradient vector of $\bar{R}^2$ at $x_n^{2'}$
is p_n. Furthermore, $x_n^{2''}$ is the most preferred consumption bundle in $G^2(R_n^{1''}, \bar{R}^{-\{1,2\}})$
with respect to $\tilde{R}^2$, and the gradient vector of $\tilde{R}^2$ at $x_n^{2''}$ is p_n. A strictly convex preference cannot have the same gradient vector p_n at both consumption bundles $x_n^{2'}$ and $x_n^{2''}$.
We show that for a sufficiently large n, we have agent 2's preference in B^2 such that the
gradient vector at $x_n^{2'}$ is p_n and the gradient vector at $x_n^{2''}$ is arbitrarily close to p_n.

For example, such a preference can be obtained as follows. We let D' be a closed ball
with a small radius tangent to the hyperplane $x_n^{2'} + p_n^\perp$ at $x_n^{2'}$ in the upper right-hand
side of the hyperplane such that $D' \setminus x_n^{2'}$ is included in $\text{P}(x_n^{2'}; \bar{R}^2)$. Then we consider the
intersection of the ray $[x_n^{2''}]$ and the hyperplane $x_n^{2'} + p_n^\perp$, and we let $\bar{D}$ be the closed ball
with the same radius as D' tangent to the hyperplane $x_n^{2'} + p_n^\perp$ at the intersection in the
upper right-hand side of the hyperplane, as drawn in Figure 3.

We now define K_n as the convex hull of $\text{UC}(x_n^{2'}; \bar{R}^2) \cup \bar{D}$: $K_n = \text{co}(\text{UC}(x_n^{2'}; \bar{R}^2) \cup \bar{D})$.
The convex hull K_n cannot be an upper contour set of a preference because it is not
strictly convex. We construct a preference by Lemma 5. We fix any positive unit vector
$a \in S_{++}^{L-1}$ and consider the $L-1$-dimensional linear space $a^\perp$. For $y \in a^\perp$, we let $L(y)$
denote the half-line starting from y and extending in the direction of the vector a: $L(y) =$
$\{x \in R^L | x = y + ta, t \geq 0\}$. We let $0 < s < 1$ be a scalar and define $\hat{R}_{n,s}^2$ as the preference
that has the following as an indifference set:

$$\bigcup_{y \in a^\perp} \{s(L(y) \cap \partial K_n) + (1-s)(L(y) \cap \text{I}(x_n^{2'}; \bar{R}^2))\}.$$

From the construction, the gradient vector of $\hat{R}_{n,s}^2$ at $x_n^{2'}$ is p_n and the gradient vector at
$x_n^{2''}$ converges to p_n as s converges to 1. We observe that $\hat{R}_{n,s}^2 \in B^2$ for any $s \in (0, 1)$ when
n is sufficiently large. As $n \to \infty$, p_n converges to $\bar{p}$, and both $x_n^{1'}$ and $x_n^{1''}$ converge to

$\bar{x}^1$. Hence, both $x_n^{2\prime}$ and $x_n^{2\prime\prime}$ converge to $\bar{x}^2$. Therefore, as $n \to \infty$, $x_n^{2\prime\prime}$ becomes closer to $x_n^{2\prime}$ and the intersection of $[x_n^{2\prime\prime}]$ and the hyperplane $x_n^{2\prime} + p_n^{\perp}$ also becomes closer to $x_n^{2\prime}$. Then the set K_n in the above construction becomes closer to the upper contour set $\mathrm{UC}(x_n^{2\prime}; \bar{R}^2)$ of the preference $\bar{R}^2$. That is, the preference $\hat{R}_{n,s}^2$ becomes closer to $\bar{R}^2$.

Now, we set a sufficiently large n so that $R_n^{1\prime}, R_n^{1\prime\prime} \in B^1$ and $\hat{R}_{n,s}^2 \in B^2$ for any $s \in (0,1)$. We write $f(R_n^{1\prime}, \hat{R}_{n,s}^2, \bar{\mathbf{R}}^{-\{1,2\}}) = \hat{\mathbf{x}}_{n,s} = (\hat{x}_{n,s}^1, \ldots, \hat{x}_{n,s}^N)$ and $f(R_n^{1\prime\prime}, \hat{R}_{n,s}^2, \bar{\mathbf{R}}^{-\{1,2\}}) = \check{\mathbf{x}}_{n,s} = (\check{x}_{n,s}^1, \ldots, \check{x}_{n,s}^N)$, and we show that these allocations contradict the strategy-proofness of the social choice function when s is sufficiently close to 1.

Alternatively, as the preference $\hat{R}_{n,s}^2$ has the gradient vector p_n at $x_n^{2\prime}$, the most preferred consumption in $G^2(R_n^{1\prime}, \bar{\mathbf{R}}^{-\{1,2\}})$ with respect to $\hat{R}_{n,s}^2$ is $x_n^{2\prime}$. Therefore $\hat{\mathbf{x}}_{n,s} = \mathbf{x}_n'$, as shown in Lemma 2. In particular, $\hat{x}_{n,s}^1 = x_n^{1\prime} = \bar{y}(p_n)$

Alternatively, the gradient vector of $\hat{R}_{n,s}^2$ at $x_n^{2\prime\prime}$ converges to p_n as $s \to 1$ as shown above. Therefore, as $s \to 1$, the most preferred consumption bundle in $G^2(R_n^{1\prime\prime}, \bar{\mathbf{R}}^{-\{1,2\}})$ with respect to $\hat{R}_{n,s}^2$ converges to $x_n^{2\prime\prime}$. Then $\check{\mathbf{x}}_{n,s}$ converges to $\mathbf{x}_n''$. In particular, $\check{x}_{n,s}^1$ converges to $x_n^{1\prime\prime} = \tilde{y}(p_n)$ as $s \to 1$. As $R_n^{1\prime}$ and $R_n^{1\prime\prime}$ satisfy (iii), this implies that $\hat{x}_{n,s}^1 \in \mathrm{P}(\check{x}_{n,s}^1; R_n^{1\prime\prime})$ or $\check{x}_{n,s}^1 \in \mathrm{P}(\hat{x}_{n,s}^1; R_n^{1\prime})$ for a value of s sufficiently close to 1. This contradicts the strategy-proofness of f on $\mathbf{B}$ with respect to agent 1 because $\hat{x}_{n,s}^1$ and $\check{x}_{n,s}^1$ are consumption bundles allocated to agent 1 for his preferences $R_n^{1\prime}$ and $R_n^{1\prime\prime}$, respectively, when the other agents' preferences are $(\hat{R}_{n,s}^2, \bar{\mathbf{R}}^{-\{1,2\}})$. □

As a preparatory step in proving the theorem, we let $\mathbf{R}^* = (R^{1*}, \ldots, R^{N*})$ be a preference profile such that the consumption directions $g(R^{i*}, p)$, $i = 1, \ldots, N$, are independent for any price vector $p \in S_{++}^{L-1}$. For example, let R^{i*}, $i = 1, \ldots, N$, be the preferences represented by Cobb–Douglas utility functions $u^{i\bar{\alpha}^i}(x) = (x_1)^{\bar{\alpha}_1^i} \cdots (x_L)^{\bar{\alpha}_L^i}$, where the parameter vectors $\bar{\alpha}^i = (\bar{\alpha}_1^i, \ldots, \bar{\alpha}_L^i)$, $i = 1, \ldots, N$, are independent among agents. Then each $g(R^{i*}, p)$ is parallel to $(\bar{\alpha}_1^i/p_1, \ldots, \bar{\alpha}_L^i/p_L)$, and the consumption directions are independent with any price vector.

We show that for any preference profile in a neighborhood of $\mathbf{R}^*$, the consumption-direction vectors are independent at any Pareto-efficient allocation. For a scalar $\epsilon \geq 0$, we let $\mathrm{P}_\epsilon = \{p \in S_{++}^{L-1} : \Omega = \sum_{i=1}^N t^i g(R^i, p), \text{ where } t^i \geq 0, R^i \in B_\epsilon(R^{i*}), i \in \mathbf{N}\}$ denote the set of price vectors at the possible Pareto-efficient allocations, with preferences R^i in the ϵ-neighborhoods of R^{i*}, $i = 1, \ldots, N$. Note that the closure of P_ϵ, i.e., $\overline{\mathrm{P}_\epsilon}$, is away from the boundary ∂S_{++}^{L-1} when ϵ is sufficiently small because P_ϵ converges to P_0 as $\epsilon \to 0$ and because P_0, the set of price vectors at possible Pareto-efficient allocations with R^{i*}, $i \in \mathbf{N}$, is away from the boundary. We fix a scalar $\epsilon' > 0$ so that $\overline{\mathrm{P}_{\epsilon'}} \subset S_{++}^{L-1}$.

For any fixed price vector p, the consumption direction vector $g(R^i, p)$ converges to $g(R^{i*}, p)$ as R^i converges to R^{i*} with respect to the Kannia metric. Therefore, we let $\epsilon'' > 0$ be a sufficiently small scalar such that $g(R^i, p)$, $i = 1, \ldots, N$, are independent for any $R^i \in B_{\epsilon''}(R^{i*})$, $i = 1, \ldots, N$, and any $p \in \overline{\mathrm{P}_{\epsilon'}}$. We define $\hat{\epsilon} = \min\{\epsilon', \epsilon''\}$. Thus, for any preferences R^i in $B_{\hat{\epsilon}}(R^{i*})$, $i = 1, \ldots, N$, the consumption directions $g(R^i, p)$, $i = 1, \ldots, N$, are independent with any price vector at the possible Pareto-efficient allocations with such preferences.

We now prove the theorem. We suppose that a social choice function f is Pareto efficient and strategy-proof on the whole domain $\mathcal{R}^N$. As mentioned in Section 3,

$f^i(\mathbf{R}) \in \text{int } A \cup \{0, \Omega\}$ for any agent i and any preference profile $\mathbf{R}$. Therefore, all we have to show is that $f^i(\mathbf{R}) \notin \text{int } A$ for any $i = 1, \ldots, N$ and any $\mathbf{R} \in \mathcal{R}^N$. We assume that an agent receives consumption in int A at some preference profile $\bar{\mathbf{R}} \in \mathcal{R}^N$ (where the consumption-direction vectors are not independent). We show a contradiction.

We repeat the replacement of the preferences of agents who receive consumption in int A as follows. We define $\bar{\epsilon} = \hat{\epsilon}/N$.

Step 1. We first pick an agent i_1 who receives consumption in int A at $\bar{\mathbf{R}}$. We replace his preference with R^{i_1*}. Let $\mathbf{R}' = (R^{1\prime}, \ldots, R^{N\prime})$ denote the new preference profile after this replacement: $R^{i_1\prime} = R^{i_1*}$ and $R^{i\prime} = \bar{R}^i$ for $i \neq i_1$.

Agent i_1's consumption at $\mathbf{R}'$ is neither 0 nor Ω because of the strategy-proofness of f. Therefore, there exists another agent i_2 who also receives consumption in int A at $\mathbf{R}'$ because of the Pareto efficiency of f. We replace agent i_2's preference with R^{i_2*} and let $\mathbf{R}''$ denote the profile after this replacement. Note that agent i_2's consumption at $\mathbf{R}''$ is neither 0 nor Ω because of the strategy-proofness.

Then we consider the replacement of $\bar{R}^{i_1}$ and $\bar{R}^{i_2}$ with any $R^{i_1} \in B_{\bar{\epsilon}}(R^{i_1*})$ and any $R^{i_2} \in B_{\bar{\epsilon}}(R^{i_2*})$, respectively. As a result of this replacement, there should exist a preference profile $\tilde{\mathbf{R}}'' = (\tilde{R}^{1\prime\prime}, \ldots, \tilde{R}^{N\prime\prime})$ where $\tilde{R}^{i_1\prime\prime} \in B_{\bar{\epsilon}}(R^{i_1*})$ and $\tilde{R}^{i_2\prime\prime} \in B_{\bar{\epsilon}}(R^{i_1*})$, $\tilde{R}^{i\prime\prime} = \bar{R}^i$ for any agents other than i_1 and i_2, and there exists a third agent i_3 other than i_1 and i_2 who receives consumption in int A at $\tilde{\mathbf{R}}''$.

If such a preference profile does not exist, then let f'' be the social choice function f restricted to agents i_1 and i_2, with other agents' preferences fixed to $\bar{R}^i$, $i \notin \{i_1, i_2\}$. Then f'' is a social choice function in the two-agent economy of agents i_1 and i_2 that is Pareto efficient and strategy-proof on $B_{\bar{\epsilon}}(R^{i_1*}) \times B_{\bar{\epsilon}}(R^{i_2*})$ and that allocates consumption in int A to both agents at (R^{i_1*}, R^{i_2*}). This contradicts Proposition 6.

Step 2. Now we replace agent i_3's preference with R^{i_3*}. Let $\mathbf{R}''' = (R^{1\prime\prime\prime}, \ldots, R^{N\prime\prime\prime})$ denote the preference profile after this replacement: $R^{i_1\prime\prime\prime} = \tilde{R}^{i_1\prime\prime}$, $R^{i_2\prime\prime\prime} = \tilde{R}^{i_2\prime\prime}$, $R^{i_3\prime\prime\prime} = R^{i_3*}$, and $R^{i\prime\prime\prime} = \bar{R}^i$ for $i \notin \{i_1, i_2, i_3\}$. Note that agent i_3's consumption at $\mathbf{R}'''$ is neither 0 nor Ω because of the strategy-proofness.

Then we consider the replacement of $R^{i_1\prime\prime\prime}$, $R^{i_2\prime\prime\prime}$, and $R^{i_3\prime\prime\prime}$ with any $R^{i_1} \in B_{\bar{\epsilon}}(R^{i_1\prime\prime\prime})$, $R^{i_2} \in B_{\bar{\epsilon}}(R^{i_2\prime\prime\prime})$, and $R^{i_3} \in B_{\bar{\epsilon}}(R^{i_3\prime\prime\prime})$, respectively. As a result, there should exist a preference profile $\tilde{\mathbf{R}}''' = (\tilde{R}^{1\prime\prime\prime}, \ldots, \tilde{R}^{N\prime\prime\prime})$ where $\tilde{R}^{i\prime\prime\prime} \in B_{\bar{\epsilon}}^i(R^{i\prime\prime\prime})$ for $i \in \{i_1, i_2, i_3\}$, $\tilde{R}^{i\prime\prime\prime} = \bar{R}^i$ for $i \notin \{i_1, i_2, i_3\}$, and there exists a fourth agent $i_4 \notin \{i_1, i_2, i_3\}$ who receives consumption in int A at $\tilde{\mathbf{R}}'''$.

If such a preference profile does not exist, then let f''' be the social choice function f restricted to agents i_1, i_2, and i_3 with other agents' preferences fixed to $\bar{R}^i$, $i \notin \{i_1, i_2, i_3\}$. Then f''' is a social choice function in the three-agent economy of agents i_1, i_2, and i_3 that is pseudo-efficient and strategy-proof on $B_{\bar{\epsilon}}(R^{i_1\prime\prime\prime}) \times B_{\bar{\epsilon}}(R^{i_2\prime\prime\prime}) \times B_{\bar{\epsilon}}(R^{i_3\prime\prime\prime})$ and that allocates consumption in int A to at least two agents at the preference profile $(R^{i_1\prime\prime\prime}, R^{i_2\prime\prime\prime}, R^{i_3\prime\prime\prime})$.[5] Furthermore, $\mathbf{R}'''$ was obtained as a result of the replacement in the previous step and the replacement of agent i_3's preference: $R^{i\prime\prime\prime} = \tilde{R}^{i\prime\prime} \in$

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[5]For example, suppose that $f^{i_1}(\mathbf{R}''') \in \text{int } A$, $f^{i_2}(\mathbf{R}''') = 0$, $f^{i_3}(\mathbf{R}''') \in \text{int } A$, and $f^i(\mathbf{R}''') = 0$ for $i \notin \{i_1, i_2, i_3\}$. Then the agents i_1, i_2, and i_3 may receive zero consumption without violating the strategy-proofness and Pareto efficiency of the social choice function f when agent i_2's preference is replaced. To deal with this case, we proved the lemmas and propositions with a pseudo-efficient social choice function.

$B^i_{\bar{\epsilon}}(R^{i*})$ for $i \in \{i_1, i_2\}$ and $R^{i_3\prime\prime\prime} = R^{i_3*}$. Therefore, $R^{i\prime\prime\prime} \in B_{\bar{\epsilon}}(R^{i*}) \subset B_{\hat{\epsilon}}(B^{i*})$ for any $i \in \{i_1, i_2, i_3\}$. The consumption-direction vectors of these three agents are independent at $(R^{i_1\prime\prime\prime}, R^{i_2\prime\prime\prime}, R^{i_3\prime\prime\prime})$. This contradicts Proposition 6.

Step 3. We repeat the process. We replace agent i_4's preference with R^{i_4*}. Let $\mathbf{R}'''' = (R^{1\prime\prime\prime\prime}, \ldots, R^{N\prime\prime\prime\prime})$ denote the preference profile after this replacement: $R^{i\prime\prime\prime\prime} = \tilde{R}^{i\prime\prime\prime}$ for $i \in \{i_1, i_2, i_3\}$, $R^{i_4\prime\prime\prime\prime} = R^{i_4*}$, and $R^{i\prime\prime\prime\prime} = \bar{R}^i$ for $i \notin \{i_1, i_2, i_3, i_4\}$. Note that agent i_4's consumption at $\mathbf{R}''''$ is neither 0 nor Ω because of the strategy-proofness.

Then we consider the replacement of the preferences of these agents i_1, i_2, i_3, and i_4 with any preferences in the $\bar{\epsilon}$-neighborhoods $B_{\bar{\epsilon}}(R^{i\prime\prime\prime\prime})$ of $R^{i\prime\prime\prime\prime}$, $i \in \{i_1, i_2, i_3, i_4\}$, respectively. As a result, there should exist a preference profile $\tilde{\mathbf{R}}'''' = (\tilde{R}^{1\prime\prime\prime\prime}, \ldots, \tilde{R}^{N\prime\prime\prime\prime})$ where $\tilde{R}^{i\prime\prime\prime\prime} \in B^i_{\bar{\epsilon}}(R^{i\prime\prime\prime\prime})$ for $i \in \{i_1, i_2, i_3, i_4\}$, $\tilde{R}^{i\prime\prime\prime\prime} = \bar{R}^i$ for $i \notin \{i_1, i_2, i_3, i_4\}$, and there exists a fifth agent $i_5 \notin \{i_1, i_2, i_3, i_4\}$ who receives positive consumption in int A at $\tilde{\mathbf{R}}''''$.

If such a preference profile does not exist, then let f'''' be the social choice function f restricted to agents i_1, i_2, i_3, and i_4, with other agents' preferences fixed to $\bar{R}^i$, $i \notin \{i_1, i_2, i_3, i_4\}$. Then f'''' is a social choice function in the four-agent economy of agents i_1, i_2, i_3, and i_4 that is pseudo-efficient and strategy-proof on $B_{\bar{\epsilon}}(R^{i_1\prime\prime\prime\prime}) \times B_{\bar{\epsilon}}(R^{i_2\prime\prime\prime\prime}) \times B_{\bar{\epsilon}}(R^{i_3\prime\prime\prime\prime}) \times B_{\bar{\epsilon}}(R^{i_4\prime\prime\prime\prime})$ and that allocates consumption in int A to at least two agents at the preference profile $(R^{i_1\prime\prime\prime\prime}, R^{i_2\prime\prime\prime\prime}, R^{i_3\prime\prime\prime\prime}, R^{i_4\prime\prime\prime\prime})$. Furthermore, $\mathbf{R}''''$ was obtained as a result of the replacement in the previous step and the replacement of agent i_4's preference: $R^{i\prime\prime\prime\prime} = \tilde{R}^{i\prime\prime\prime} \in B^i_{\bar{\epsilon}}(R^{i\prime\prime\prime})$ for $i \in \{i_1, i_2, i_3\}$ and $R^{i_4\prime\prime\prime\prime} = R^{i_4*}$. Combined with the result $R^{i\prime\prime\prime} \in B_{\bar{\epsilon}}(R^{i*})$ for any $i \in \{i_1, i_2, i_3\}$ in the previous step, we have $R^{i\prime\prime\prime\prime} \in B_{2\bar{\epsilon}}(R^{i*}) \subset B_{\hat{\epsilon}}(R^{i*})$ for any $i \in \{i_1, i_2, i_3, i_4\}$. Therefore, the consumption-direction vectors of these four agents are independent at $(R^{i_1\prime\prime\prime\prime}, R^{i_2\prime\prime\prime\prime}, R^{i_3\prime\prime\prime\prime}, R^{i_4\prime\prime\prime\prime})$. This contradicts Proposition 6.

We repeat the process until Step $N - 2$. In Step k ($\leq N - 2$), we replace agent i_{k+1}'s preference with $R^{i_{k+1}*}$, where agent i_{k+1} receives consumption that is neither 0 nor Ω. We let $\mathbf{R}^{\langle k+1 \rangle} = (R^{1\langle k+1 \rangle}, \ldots, R^{N\langle k+1 \rangle})$ denote the preference profile after the replacement. We consider replacing these $k + 1$ agents' preferences with preferences in the $\bar{\epsilon}$ neighborhoods of $R^{i\langle k+1 \rangle}$, $i \in \{i_1, \ldots, i_{k+1}\}$, respectively, and obtain preference profile $\tilde{\mathbf{R}}^{\langle k+1 \rangle} = (\tilde{R}^{1\langle k+1 \rangle}, \ldots, \tilde{R}^{N\langle k+1 \rangle})$, where $\tilde{R}^{i\langle k+1 \rangle} \in B_{\bar{\epsilon}}(R^{i\langle k+1 \rangle})$ for $i \in \{i_1, \ldots, i_{k+1}\}$, $\tilde{R}^{i\langle k+1 \rangle} = \bar{R}^i$ for $i \notin \{i_1, \ldots, i_{k+1}\}$, and there exists a $(k + 2)$th agent $i_{k+2} \notin \{i_1, \ldots, i_{k+1}\}$ who receives consumption in int A at $\tilde{\mathbf{R}}^{\langle k+1 \rangle}$.

If such a preference profile does not exists, then $f^{\langle k+1 \rangle}$, which is the social choice function f restricted to agents $i_1, \ldots, i_{k+1}$ with other agents' preferences fixed to $\bar{R}^i$, is a social choice function in the $(k + 1)$-agent economy that is pseudo-efficient and strategy-proof on $B_{\bar{\epsilon}}(R^{i_1\langle k+1 \rangle}) \times \cdots \times B_{\bar{\epsilon}}(R^{i_{k+1}\langle k+1 \rangle})$ and it allocates consumption in int A to at least two agents at the preference profile $(R^{i_1\langle k+1 \rangle}, \ldots, R^{i_{k+1}\langle k+1 \rangle})$. Furthermore, $\mathbf{R}^{\langle k+1 \rangle}$ was obtained as a result of the replacement in Step $k - 1$ and the replacement of agent i_{k+1}'s preference: $R^{i\langle k+1 \rangle} = \tilde{R}^{i\langle k \rangle} \in B^i_{\bar{\epsilon}}(R^{i\langle k \rangle})$ for $i \in \{i_1, \ldots, i_k\}$ and $R^{i_{k+1}\langle k+1 \rangle} = R^{i_{k+1}*}$. Combined with the result $R^{i\langle k \rangle} \in B_{(k-2)\bar{\epsilon}}(R^{i*})$ for any $i \in \{i_1, \ldots, i_k\}$ in Step $k - 1$, we have $R^{i\langle k+1 \rangle} \in B_{(k-1)\bar{\epsilon}}(R^{i*}) \subset B_{\hat{\epsilon}}(R^{i*})$ for any $i \in \{i_1, \ldots, i_{k+1}\}$. Therefore, the consumption-direction vectors of these $k + 1$ agents are independent at $(R^{i_1\langle k+1 \rangle}, \ldots, R^{i_{k+1}\langle k+1 \rangle})$. This contradicts Proposition 6.

Finally, in Step $N - 2$, we replace agent i_{N-1}'s preference with $R^{i_{N-1}*}$, where agent i_N receives consumption that is neither 0 nor Ω. We let $\mathbf{R}^{\langle N-1 \rangle} = (R^{1\langle N-1 \rangle}, \ldots, R^{N\langle N-1 \rangle})$

denote the preference profile after the replacement. We consider replacing these $N - 1$ agents' preferences in the $\bar{\epsilon}$-neighborhoods of $R^{i\langle N-1\rangle}$, $i \in \{i_1, \ldots, i_{N-1}\}$, respectively, and obtain preference profile $\tilde{\mathbf{R}}^{\langle N-1\rangle} = (\tilde{R}^{1\langle N-1\rangle}, \ldots, \tilde{R}^{N\langle N-1\rangle})$, where $\tilde{R}^{i\langle N-1\rangle} \in B_{\bar{\epsilon}}(R^{i\langle N-1\rangle})$ for $i \in \{i_1, \ldots, i_{N-1}\}$, $\tilde{R}^{i\langle N-1\rangle} = \bar{R}^i$ for $i \notin \{i_1, \ldots, i_{N-1}\}$, that is, for $i = i_N$, and the agent i_N receives consumption in int A at $\tilde{\mathbf{R}}^{\langle N-1\rangle}$. Note that $R^{i\langle N-1\rangle} \in B_{(N-3)\bar{\epsilon}}(R^{i*}) \subset B_{\hat{\epsilon}}(R^{i*})$ for $i \in \{i_1, \ldots, i_{N-1}\}$.

We replace agent i_N's preference R^{i_N} with R^{i_N*} and let $\mathbf{R}^{\langle N\rangle}$ denote the preference profile after this replacement. Because of strategy-proofness, agent i_N receives consumption that is neither 0 nor Ω at $\mathbf{R}^{\langle N\rangle}$. Then the Pareto-efficient and strategy-proof social choice function f allocates consumption in int A to at least two agents at $\mathbf{R}^{\langle N\rangle} \in \prod_{i=1}^N B_{\hat{\epsilon}}(R^{i*})$, where the consumption-direction vectors are independent. This contradicts Proposition 6. This ends the proof of the theorem.

6. Proof of Corollary

The theorem implies that if a social choice function $f : \mathcal{R}^N \to X$ is pseudo-efficient and strategy-proof, then $f^i(\mathbf{R}) \in \{0, \Omega\}$ for any agent i and any $\mathbf{R} \in \mathcal{R}^N$. We prove the corollary by repeatedly applying this result, as we repeatedly applied Proposition 6 to prove the theorem.

We let $f : \bar{\mathcal{R}}^N \to X$ be a Pareto-efficient and strategy-proof social choice function defined on a domain $\bar{\mathcal{R}}^N$, where $\mathcal{R} \subset \bar{\mathcal{R}} \subset \mathcal{R}_C$. As in the case of $\mathbf{R} \in \mathcal{R}^N$, as mentioned in Section 3, $f^i(\mathbf{R}) \in \text{int } A \cup \{0, \Omega\}$ for any agent i and any preference profile $\mathbf{R} \in \bar{\mathcal{R}}^N$. Therefore, all we have to prove is that $f^i(\mathbf{R}) \notin \text{int } A$ for any agent i and any $\mathbf{R} \in \bar{\mathcal{R}}^N$. We suppose that $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N) \in \bar{\mathcal{R}}^N$ is a preference profile where an agent receives consumption in int A, and show a contradiction.

We let $\mathbf{R}^* = (R^{1*}, \ldots, R^{N*}) \in \mathcal{R}^N$ be a preference profile in $\mathcal{R}^N$. We repeat the replacement of the preferences of agents who receive positive consumption as follows. We first pick an agent i_1 who receives consumption in int A at $\bar{\mathbf{R}}$. We replace his preference with R^{i_1*}. Let $\mathbf{R}' = (R^{1\prime}, \ldots, R^{N\prime})$ denote the new preference profile after this replacement: $R^{i_1\prime} = R^{i_1*}$ and $R^{i\prime} = \bar{R}^i$ for $i \neq i_1$. As agent i_1's consumption at $\mathbf{R}'$ is neither 0 nor Ω, there exists another agent i_2 who receives positive consumption in int A at $\mathbf{R}'$. We replace this agent's preference with R^{i_2*} and let $\mathbf{R}''$ denote the preference profile after this replacement. Note that agent i_2's consumption at $\mathbf{R}''$ is neither 0 nor Ω.

Then we consider the replacement of agent i_1's and agent i_2's preferences with any preferences in $\mathcal{R}$. As a result of the replacement, there should exist a preference profile $\tilde{\mathbf{R}}'' = (\tilde{R}^{1\prime\prime}, \ldots, \tilde{R}^{N\prime\prime})$, where $\tilde{R}^{i_1\prime\prime} \in \mathcal{R}$, $\tilde{R}^{i_2\prime\prime} \in \mathcal{R}$, and $\tilde{R}^{i\prime\prime} = \bar{R}^i$ for $i \notin \{i_1, i_2\}$, and there exists another agent i_3, different from i_1 and i_2, receiving positive consumption in int A at $\tilde{\mathbf{R}}''$. If such a preference profile does not exist, then let f'' be the social choice function f restricted to agents i_1 and i_2, with other agents preferences fixed to $\bar{R}^i$, $i \notin \{i_1, i_2\}$. Then f'' is a social choice function in the two-agent economy of agents i_1 and i_2 that is Pareto efficient and strategy-proof on $\mathcal{R} \times \mathcal{R}$ and allocates consumption in int A to both agents at (R^{i_1*}, R^{i_2*}). This contradicts the theorem.

Now we replace agent i_3's preference with R^{i_3*} and let $\mathbf{R}'''$ denote the preference profile after the replacement: $R^{i_1\prime\prime\prime} = \tilde{R}^{i_1\prime\prime}$, $R^{i_2\prime\prime\prime} = \tilde{R}^{i_1\prime\prime}$, $R^{i_3\prime\prime\prime} = R^{i_3*}$, and $R^{i\prime\prime\prime} = \bar{R}^i$ for

$i \notin \{i_1, i_2, i_3\}$. Note that agent i_3's consumption at $\mathbf{R}'''$ is neither 0 nor Ω. Then we consider the replacement of the preferences of agents i_1, i_2, and i_3 with any preferences in $\mathcal{R}$. As a result of the replacement, there exists a preference profile $\tilde{\mathbf{R}}'''$, where $\tilde{R}^{i'''} \in \mathcal{R}$ for $i \in \{i_1, i_2, i_3\}$, $\tilde{R}^{i'''} = \bar{R}^i$ for $i \notin \{i_1, i_2, i_3\}$, and there exists a fourth agent $i_4 \notin \{i_1, i_2, i_3\}$ who receives consumption in int A at $\tilde{\mathbf{R}}'''$. If such a preference profile does not exist, then let f''' be the social choice function f restricted to agents i_1, i_2, and i_3, with other agents' preferences fixed to $\bar{R}^i$, $i \notin \{i_1, i_2, i_3\}$. Then f''' is a social choice function in the three-agent economy of agents i_1, i_2, and i_3 that is pseudo-efficient and strategy-proof on $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$ and allocates positive consumption in int A to at least two agents at the preference profile $(R^{i_1'''}, R^{i_2'''}, R^{i_3'''})$. This contradicts the theorem.

We replace the fourth agent's preference with R^{i_4*} and consider the replacement of these four agents' preferences with any preferences in $\mathcal{R}$. Repeating this process, we finally obtain a preference profile in $\mathcal{R}^N$ where at least two agents have consumption in int A. This contradicts the theorem and ends the proof of the corollary.

7. Concluding remarks

In this paper, we prove that as long as there are at least as many goods as agents, a Pareto-efficient and strategy-proof social choice function is alternately dictatorial. Our proof is based on the analysis of the option set. We show that the option set is the smooth surface of a strictly convex set if a consumption bundle is allocated in the interior of the feasible consumption set. Then we observe that such an option set induces allocations that contradict Pareto efficiency and strategy-proofness to prove that any agent is allocated zero consumption or the total endowment. As far as we are aware, this paper is the first to investigate the option set in many-agent economies. We believe that this approach will be useful for the study of the properties of a strategy-proof social choice function in more general setups.

The difficulty in dealing with economies with many agents is that the price vector does not uniquely determine the allocation. In other words, the consumption bundle of an agent does not determine the other agents' consumption. This is in sharp contrast to two-agent economies, where one agent's consumption determines the other's uniquely, because their consumption bundles sum to the total endowment under Pareto efficiency. In this paper, to overcome this difficulty, we made the key assumption that there are at least as many goods as agents. Under this assumption, it is possible to find a preference profile that ensures the independence of the consumption-direction vectors. If the consumption-direction vectors are independent, then an agent's consumption determines other agents' consumption uniquely. Using this property, we first prove the alternately dictatorial result at such a preference profile and then extend it to preference profiles that may not satisfy the independence of the consumption-direction vectors.

It is still an open question whether a Pareto-efficient and strategy-proof social choice function is alternately dictatorial in economies where the number of agents exceeds the number of goods. It is clear that consumption-direction vectors are dependent in such an economy and that our approach cannot be applied directly to this type of economy. In future research, we hope to find answers to this challenging question.
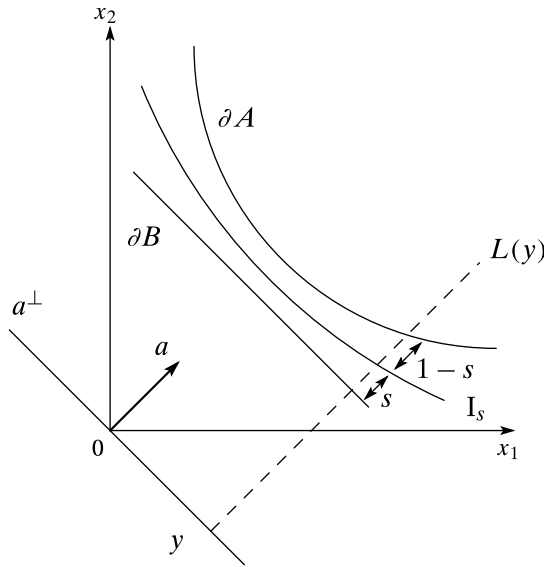
FIGURE 4. Construction of I_s.

APPENDIX

A.1 *Technical results*

In this section, we prove some technical results concerning preference construction that we repeatedly use in the proofs of the lemmas and propositions. We fix any positive unit vector $a \in S_{++}^{L-1}$ and consider the $L-1$-dimensional hyperplane $a^\perp$. For $y \in a^\perp$, we let $L(y)$ denote the half-line starting from y and extending in the direction of the vector a: $L(y) = \{x \in R^L | x = y + ta, t \geq 0\}$. For any consumption vector $x \in R_+^L$ and any preference $R \in \mathcal{R}$, the indifference set $I(x; R)$ intersects with $L(y)$ only once for any $y \in a^\perp$ because R is strictly monotonic in R_{++}^L.

If $A \subset R_{++}^L$ is a closed, strictly convex set with smooth boundary satisfying $x + R_+^L \subset A$ for any $x \in A$, then there exists a preference $R \in \mathcal{R}$ such that A equals an upper contour set of R. Let $B \subset R_{++}^L$ be a closed, convex set with a smooth boundary satisfying $x + R_+ \subset B$ for any $x \in B$. When B does not satisfy strict convexity, it cannot be an upper contour set of a preference in $\mathcal{R}$. We often make the set B into a strictly convex set by considering a convex combination of B with a strictly convex set A as follows. With a parameter $s \in [0, 1]$, we define I_s as

$$I_s = \bigcup_{y \in a^\perp} \{s(L(y) \cap \partial A) + (1 - s)(L(y) \cap \partial B)\}. \tag{2}$$

Figure 4 depicts the construction of I_s.

LEMMA 5. *For any $s \in (0, 1)$, the set $I_s + R_{++}^L$, where I_s is constructed by (2), is a closed, strictly convex set and its boundary I_s is smooth.*

PROOF. We choose any orthogonal unit vectors $e_1, \ldots, e_{L-1}$ that span $a^\perp$, and introduce a new orthogonal coordinate system $(z_1, \ldots, z_L)$ so that the set of vectors $e_1, \ldots, e_{L-1}$, and a is its basis. That is, $z = (z_1, \ldots, z_L)$ in this coordinate system corresponds to $z_1 e_1 + \cdots + z_{L-1} e_{L-1} + z_L a$ in the original coordinate system $(x_1, \ldots, x_L)$. We write z_{-L} to denote the first $L-1$ elements: $z_L = (z_1, \ldots, z_{L-1})$.

As A and B are closed sets satisfying $x + R_+^L \subset A$ for any $x \in A$ and $x + R_+^L \subset B$ for any $x \in B$, the half-line $L(y)$ intersects with the boundaries ∂A and ∂B only once for any $y \in a^\perp$. Thus, we let $\alpha : R^{L-1} \to R_+$ and $\beta : R^{L-1} \to R_+$ be functions in the coordinate system $(z_1, \ldots, z_l)$ so that the graphs equal the boundaries ∂A and ∂B, respectively: $\partial A = \{(z_{-L}, \alpha(z_{-L})) \in R^L | z_{-L} \in R^L\}$, $\partial B = \{(z_{-L}, \beta(z_{-L})) \in R^L | z_{-L} \in R^L\}$.

We let $\gamma_s : R^L \to R_+$ denote the function defined as $\gamma_s(z_{-L}) = s\alpha(z_{-L}) + (1-s)\beta(z_{-L})$. Then I_s defined by (2) equals the graph of γ_s and the set $I_s + R_+^L$ equals $\{(z_{-L}, z_L) \in R^L | z_L \geq \gamma_s(z_{-L})\}$ in the coordinate system $(z_1, \ldots, z_L)$. Therefore, $I_s + R_+^L$ is a closed set.

For the strict convexity of $I_s + R_+^L$, we need to prove the strict convexity of the function γ_s. Pick any $z'_{-L} \in R^{L-1}$ and $z''_{-L} \in R^{L-1}$, and let $z'''_{-L} = qz'_{-L} + (1-q)z''_{-L}$ with any scalar $q \in (0,1)$. All we have to show is

$$\gamma_s(z'''_{-L}) < q\gamma_s(z'_{-L}) + (1-q)\gamma_s(z''_{-L}). \tag{3}$$

The left-hand side of (3) is $\gamma_s(z'''_{-L}) = s\alpha(z'''_{-L}) + (1-s)\beta(z'''_{-L})$. The right-hand side of (3) is $q\gamma_s(z'_{-L}) + (1-q)\gamma_s(z''_{-L}) = q\{s\alpha(z'_{-L}) + (1-s)\beta(z'_{-L})\} + (1-q)\{s\alpha(z''_{-L}) + (1-s)\beta(z''_{-L})\}$. As A is strictly convex, the function α is strictly convex, and hence $\alpha(z'''_{-L}) < q\alpha(z'_{-L}) + (1-q)\alpha(z''_{-L})$. As B is convex, the function β is convex, and hence $\beta(z'''_{-L}) \leq q\beta(z'_{-L}) + (1-q)\beta(z''_{-L})$. Therefore, we have the inequality (3).

As both A and B have smooth boundaries, the functions α and β are differentiable.[6] Then γ_s is also differentiable, and I_s is a smooth boundary of the set $I_s + R_+^L$.[7]   □

We often turn a convex set into a convex set with a smooth boundary by rounding its edges. Let $A \subset R^L$ be a closed convex set such that its interior is not empty. We let $\bar{D}_\epsilon \subset R^L$ denote a closed ball with radius ϵ and define a closed set C as the union of such closed balls with radius ϵ included in A: $C = \bigcup_{\bar{D}_\epsilon \subset A} \bar{D}_\epsilon$. If ϵ is sufficiently small, then there exists a closed ball with radius ϵ that is a subset of A, and hence C is not an empty set.

LEMMA 6. *Let $A \subset R^L$ be a closed, convex set such that* int $A \neq \varnothing$. *When ϵ is sufficiently small, $C = \bigcup_{\bar{D}_\epsilon \subset A} \bar{D}_\epsilon$ is a closed, convex set with a smooth boundary. If A is a strictly convex set, then C is a strictly convex set.*

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[6]A function $g : R^{L-1} \to R$ is differentiable at z_{-L} if there exists $(T_1, \ldots, T_{L-1}) \in R^{L-1}$ such that

$$\lim_{h \to 0} \frac{\|g(z_{-L} + h) - g(z_{-L}) - T_1 h_1 - \cdots - T_{L-1} h_{L-1}\|}{\|h\|} = 0,$$

where $h = (h_1, \ldots, h_{L-1})$. Then $(-T_1, \ldots, -T_{L-1}, 1)$ is the normal vector of the supporting hyperplane to the graph of g at $(z_{-L}, g(z_{-L}))$ in the coordinate system $(z_1, \ldots, z_L)$.

[7]In the coordinate system $(z_1, \ldots, z_L)$, if $T^A = (T_1^A, \ldots, T_{L-1}^A, 1)$ and $T^B = (T_1^B, \ldots, T_{L-1}^B, 1)$ are the normal vectors of the supporting hyperplanes to A and B at $(z_{-L}, \alpha(z_{-L}))$ and $(z_{-L}, \beta(z_{-L}))$, respectively, then $sT^A + (1-s)T^B$ is the normal vector of the supporting hyperplane to I_s at $(z_{-L}, \gamma_s(z_{-L}))$.

PROOF. We prove that C is a closed set. We let $\{x_n\}_{n=0}^{\infty}$ be a sequence of points in C converging to $\bar{x}$ and prove $\bar{x} \in C$. From the definition of C, there exists a closed ball $\bar{D}_{\epsilon}^n$ with radius ϵ such that $x_n \in \bar{D}_{\epsilon}^n \subset A$ for each n. Let c_n denote the center of the closed ball $\bar{D}_{\epsilon}^n$. As x_n is convergent, the union $\bigcup_n \bar{D}_{\epsilon}^n$ of the closed balls is bounded, and hence the sequence $\{c_n\}_{n=0}^{\infty}$ has a convergent subsequence $\{c_{n_k}\}_{k=0}^{\infty}$. We let $c_{n_k} \to \bar{c}$ as $k \to \infty$. We let $\bar{D}_{\epsilon}(c)$ denote the closed ball with center c and radius ϵ. Given $x_{n_k} \to \bar{x}$ as $k \to \infty$ and $x_{n_k} \in \bar{D}_{\epsilon}(c_k) \subset A$, we have $\bar{x} \in \bar{D}_{\epsilon}(\bar{c}) \subset A$. Therefore, $\bar{x} \in C$ from the definition of C.

We prove that C is a convex set. We let $x' \in C$, $x'' \in C$, and $s \in (0,1)$, and prove that $sx' + (1-s)x'' \in C$. From the definition of C, there exist closed balls $\bar{D}_{\epsilon}'$ and $\bar{D}_{\epsilon}''$ such that $x' \in \bar{D}_{\epsilon}' \subset A$ and $x'' \in \bar{D}_{\epsilon}'' \subset A$. Let K be the convex hull of $\bar{D}_{\epsilon}' \cup \bar{D}_{\epsilon}''$: $K = \{z \in R_+^L | z = rz' + (1-r)z'', z' \in \bar{D}_{\epsilon}', z'' \in \bar{D}_{\epsilon}'', r \in [0,1]\}$. It is clear that $sx' + (1-s)x'' \in K$. As K equals the union of closed balls with radius ϵ that have their centers between the centers of $\bar{D}_{\epsilon}'$ and $\bar{D}_{\epsilon}''$, there exists a closed ball $\bar{D}_{\epsilon}'''$ with radius ϵ such that $sx' + (1-s)x'' \in \bar{D}_{\epsilon}''' \subset K$. Convexity of A implies $K \subset A$. Therefore, $sx' + (1-s)x'' \in \bar{D}_{\epsilon}''' \subset K \subset A$ and $sx' + (1-s)x'' \in C$ from the definition of C.

We show that the boundary of C is smooth. If it is not smooth at a point x on the boundary of C, then there are two different hyperplanes tangent to C at x. However, as C is a union of closed balls, there should exist a closed ball with radius ϵ that is tangent to x and included in C. This is a contradiction.

Finally, we prove that C is a strictly convex set if A is also a strictly convex set. We suppose that C is not strictly convex. Given C is convex as shown above, this implies that there is a segment $[x', x'']$ in R^L such that the segment is on the boundary of C and C is tangent to an $L-1$-dimensional hyperplane H along the segment $[x', x'']$. Then there exist closed balls $\bar{D}_{\epsilon}'$ and $\bar{D}_{\epsilon}''$ in C with radius ϵ that are tangent to the hyperplane H at x' and x''. Let K be the convex hull of $\bar{D}_{\epsilon}' \cup \bar{D}_{\epsilon}''$ and let $s \in (0,1)$ be any scalar. The convex hull K is tangent to H along the segment $[x', x'']$, and there exists a closed ball $\bar{D}_{\epsilon}''' \subset K$ with radius ϵ that is tangent to the hyperplane H at $sx' + (1-s)x''$. Note that $K \subset C \subset A$ because C is a convex set. From the definition of C, if a closed ball $\bar{D}_{\epsilon}$ in C with radius ϵ touches the boundary of C, then the closed ball also touches the boundary of A. That is, if there exists a point $x \in \partial \bar{D}_{\epsilon} \cap \partial C$, then there exists a point y, which might be different from x, such that $y \in \partial \bar{D}_{\epsilon} \cap \partial A$. As $sx' + (1-s)x''$ is in $\partial \bar{D}_{\epsilon}''' \cap \partial C$, the closed ball $\bar{D}_{\epsilon}'''$ touches the boundary of A, that is, there exists y such that $y \in \partial \bar{D}_{\epsilon}''' \cap \partial A$. Let H' denotes the supporting hyperplane of $\bar{D}_{\epsilon}'''$ at y. Given that $\bar{D}_{\epsilon}'''$ is located between $\bar{D}_{\epsilon}'$ and $\bar{D}_{\epsilon}''$ in K, then, H', which is a supporting hyperplane of D_{ϵ}''', has an intersection with K other than y. Alternatively, strict convexity of A implies that A does not have an intersection with the hyperplane H' other than $y \in \partial A$. This is a contradiction. $\qquad \square$

LEMMA 7. *If $A \subset R^L$ and $B \subset R^L$ are closed convex sets with smooth boundaries, then the convex hull of their union, $\mathrm{co}(A \cup B)$, also has a smooth boundary.*

PROOF. Note that any point in the boundary of $\mathrm{co}(A \cup B)$ is either in the boundary of A, in the boundary of B, or a convex combination of points in the boundaries of A and B. That is, if $y \in \partial \, \mathrm{co}(A \cup B)$, then either $y \in \partial A$, $y \in \partial B$, or $y = sx' + (1-s)x''$, where $x' \in \partial A$, $x'' \in \partial B$, and $s \in (0,1)$.

We suppose that the boundary of $\text{co}(A \cup B)$ is not smooth at a point x on its boundary. That is, there are two different hyperplanes H' and H'' tangent to $\text{co}(A \cup B)$ at x. As mentioned above, $x \in \partial A$, $x \in \partial B$, or $x = sx' + (1 - s)x''$, where $x' \in \partial A$, $x'' \in \partial B$, and $s \in (0, 1)$. If $x \in \partial A$, then H' and H'' are tangent to A at x, and this contradicts the smoothness of the boundary of A. Similarly, $x \in \partial B$ contradicts the smoothness of ∂B.

We consider the case of $x = sx' + (1 - s)x''$, where $x' \in \partial A$, $x'' \in \partial B$, and $s \in (0, 1)$. We show that H' is tangent to $\text{co}(A \cup B)$ along the segment $[x', x'']$. Note that the segment $[x', x'']$ is in $\text{co}(A \cup B)$ because of the convexity of $\text{co}(A \cup B)$. Given $x = sx' + (1 - s)x'' \in H'$ with $s \in (0, 1)$, if the segment $[x', x'']$ is not in H', then x' and x'' are located in the two different sides of R^L separated by H'. This contradicts that H' is the hyperplane tangent to $\text{co}(A \cup B)$. Similarly, H'' is tangent to $\text{co}(A \cup B)$ along the segment $[x', x'']$. Then both H' and H'' are tangent to A at $x' \in \partial A$. This contradicts the smoothness of the boundary of A. □

A.2 *Proof of Lemma 1*

We show an example of preference construction. For any $x_n \in R^L_{++}$ and $p_n \in S^{L-1}_{++}$, we construct a preference R_n satisfying $p_n = p(R_n, x_n)$ as follows. We use the technical results in the previous section.

We let $\epsilon' > 0$ be a small scalar and define E as the intersection of the upper contour set $\text{UC}(x_n - \epsilon' p_n; \bar{R})$ of $\bar{R}$ at $x_n - \epsilon' p_n$ and the half-space $H(x_n + \epsilon' p_n; p_n)$: $E = \text{UC}(x_n - \epsilon' p_n; \bar{R}) \cap H(x_n + \epsilon' p_n; p_n)$.

We then pick a preference R' satisfying $E \subset \text{UC}(x_n; R')$ and $p_n = p(R', x_n)$. The existence of such a preference R' is clear from the fact that E is in the right-hand side of the hyperplane $x_n + p_n^\perp$ and is away from the hyperplane.

We provide an example of the construction of such a preference R'. We apply the methods in the previous section. We let $\text{UC}(x'; \bar{R})$ be an upper contour set of the preference $\bar{R}$ at some consumption vector x' such that $E \subset \text{UC}(x'; \bar{R})$. Applying Lemma 6, we make the convex set $H(x_n; p_n) \cap \text{UC}(x'; \bar{R})$ into a convex set with smooth boundary. We let C' denote the union of close balls with radius ϵ included in $H(x_n; p_n) \cap \text{UC}(x'; \bar{R})$: $C' = \bigcup_{\bar{D}_\epsilon \subset H(x_n; p_n) \cap \text{UC}(x'; \bar{R})} \bar{D}_\epsilon$, where $\bar{D}_\epsilon$ denotes a closed ball with radius ϵ. We chose ϵ sufficiently small so that $E \subset C'$. Since C' is not strictly convex, it cannot be an upper contour set of a preference. By applying Lemma 5, we make C' into a strictly convex set. We let $\tilde{R} \in \mathcal{R}$ be a preference such that $p_n = p(x_n; \tilde{R})$. We fix any positive unit vector $a \in S^{L-1}_{++}$ and consider the $L - 1$-dimensional hyperplane $a^\perp$. For $y \in a^\perp$, we let $L(y)$ denote the half-line starting from y and extending in the direction of the vector a: $L(y) = \{x \in R^L | x = y + ta, t \geq 0\}$. With parameter $s \in (0, 1)$, we define I_s as

$$I_s = \bigcup_{y \in a^\perp} \{s(L(y) \cap I(x_n; \tilde{R})) + (1 - s)(L(y) \cap \partial C')\}.$$

We let $R_s \in \mathcal{R}$ be the preference such that I_s is its indifference set. We set R' as a preference R_s with s sufficiently small so that $E \subset \text{UC}(x_n; R')$. We have $p_n = p(R', x_n)$ since we have chosen $\tilde{R}$ so that $p_n = p(x_n, \tilde{R})$.

We continue the construction of R_n. We define $F \subset R_{++}^L$ as the intersection of the upper contour set of R' at x_n and that of $\bar{R}$ at $x_n - \epsilon' p_n$: $F = \text{UC}(x_n; R') \cap \text{UC}(x_n - \epsilon' p_n; \bar{R})$. This F cannot be an upper contour set of a smooth preference because it has its edge at the intersection $\text{I}(x_n; R') \cap \text{I}(x_n - \epsilon' p_n; \bar{R})$. We round the edge by Lemma 6. We let $\epsilon'' < \epsilon'$ be a scalar smaller than ϵ'. We let $\bar{D}_{\epsilon''} \subset R_+^L$ denote a closed ball with radius ϵ'' and define a closed set C as the union of such closed balls with radius ϵ'' included in F: $C = \bigcup_{\bar{D}_{\epsilon''} \subset F} \bar{D}_{\epsilon''}$. We define R_n as the preference such that it has C as an upper contour set.

From the construction, R_n satisfies $p_n = p(R_n, x_n)$. We observe that R_n converges to $\bar{R}$ as $n \to \infty$ and ϵ' in the above construction converges to 0. As $\epsilon' \to 0$, the upper contour set $U(x_n; R_n)$ converges to $U(x_n; \bar{R}) \cap H(x_n; p_n)$. Furthermore, as $n \to \infty$, we have $p_n \to \bar{p}$ and $x_n \to \bar{x}$, and hence the set $U(x_n; \bar{R}) \cap H(x_n; p_n)$ converges to $U(\bar{x}; \bar{R}) \cap H(\bar{x}; \bar{p}) = U(\bar{x}; \bar{R})$. Therefore, as $n \to \infty$ and $\epsilon' \to 0$, the upper contour set of R_n converges to that of $\bar{R}$, that is, the preference R_n converges to $\bar{R}$.

We now prove the second part of the lemma. Remember that the preference R_n constructed above has C as its upper contour set and the set C is $F = \text{UC}(x_n; R') \cap \text{UC}(x_n - \epsilon' p_n; \bar{R})$, the edge of which is truncated. Therefore, any consumption bundle on the boundary of C, ∂C, is on either $\text{I}(x_n; R')$, $\text{I}(x_n - \epsilon' p_n; \bar{R})$ or on a set that arises from the edge truncation. If a consumption bundle on ∂C is on $\text{I}(x_n - \epsilon' p_n; \bar{R})$, then the gradient vector of R_n is the same as that of $\bar{R}$ at the consumption bundle. Therefore all we have to do is to construct C such that $[\bar{x}] \cap \partial C \in \text{I}(x_n - \epsilon' p_n; \bar{R})$ or, equivalently, $[\bar{x}] \cap \text{I}(x_n - \epsilon' p_n; \bar{R}) \in \partial C$. We write $y = [\bar{x}] \cap \text{I}(x_n - \epsilon' p_n; \bar{R})$.

When $p_n([\bar{x}] \cap \text{I}(x_n; \bar{R})) > p_n x_n$ holds, we can construct such a C by taking R', ϵ', and ϵ'' suitably in the above construction. The assumption that $p_n([\bar{x}] \cap \text{I}(x_n; \bar{R})) > p_n x_n$ implies that $[\bar{x}] \cap \text{I}(x_n; \bar{R})$ is in the interior of $H(x_n; p_n)$. Thus, we let ϵ' be sufficiently small so that y is in the interior of $H(x_n; p_n)$. We take the preference R' in the above construction so that it additionally satisfies $y \in \text{P}(x_n; R')$. The existence of such an R' is clear because y is in the right-hand side of the hyperplane $x_n + p_n^\perp$ and is away from the hyperplane. Then observe that the intersection of $[\bar{x}]$ and the boundary of F is on $\text{I}(x_n - \epsilon' p_n; \bar{R})$. Now we take a sufficiently small ϵ'' in the above construction so that y is not truncated in the process of rounding the edge. More rigorously, we take $\epsilon'' < \epsilon'$ such that the closed ball $\bar{D}_{\epsilon''} \subset \text{UC}(x_n - \epsilon' p_n; \bar{R})$ tangent to $\text{I}(x_n - \epsilon' p_n; \bar{R})$ at y is in $\text{P}(x_n; R')$. Then $\bar{D}_{\epsilon''} \subset \text{P}(x_n; R') \cap \text{UC}(x_n - \epsilon' p_n; \bar{R}) \subset F$, and y is not truncated; that is, y is on ∂C, as desired.

A.3 *Proof of Lemma 2*

As $f^j(\tilde{\mathbf{R}}) = f^j(\bar{\mathbf{R}}) \neq 0$ and $\tilde{R}^j$ and $\bar{R}^j$ have the same gradient vector at $f^j(\bar{\mathbf{R}})$, we have $p(\bar{\mathbf{R}}, f) = p(\tilde{\mathbf{R}}, f)$ and $g(\bar{R}^i, p(\bar{\mathbf{R}}, f)) = g(\tilde{R}^i, p(\tilde{\mathbf{R}}, f))$ for any i.

As consumption at preference profiles $\bar{\mathbf{R}}$ and $\tilde{\mathbf{R}}$, respectively, sum up to the total endowment Ω, we have $\sum_{i=1}^N f^i(\bar{\mathbf{R}}) = \sum_{i=1}^N \|f^i(\bar{\mathbf{R}})\| g(\bar{R}^i, p(\bar{\mathbf{R}}, f)) = \Omega$ and $\sum_{i=1}^N f^i(\tilde{\mathbf{R}}) = \sum_{i=1}^N \|f^i(\tilde{\mathbf{R}})\| g(\bar{R}^i, p(\tilde{\mathbf{R}}, f)) = \Omega$. Considering the difference between these equations, we have $\sum_{i \neq j}(\|f^i(\bar{\mathbf{R}})\| - \|f^i(\tilde{\mathbf{R}})\|) g(\bar{R}^i, p(\bar{\mathbf{R}}, f)) = 0$ because $f^j(\bar{\mathbf{R}}) = f^j(\tilde{\mathbf{R}})$ and $g(\bar{R}^i, p(\bar{\mathbf{R}}, f)) = g(\bar{R}^i, p(\tilde{\mathbf{R}}, f))$. As the consumption-direction vectors are independent, this

equation implies that $\|f^i(\bar{\mathbf{R}})\| = \|f^i(\tilde{\mathbf{R}})\|$ for all $i \neq j$. Thus, we obtain the equality of the two allocations: $f(\bar{\mathbf{R}}) = f(\tilde{\mathbf{R}})$.

A.4 *Proof of Lemma 3*

We let $\hat{R}^i \in B^i$ be an MMT of R^i at x and show that $x = f^i(\hat{R}^i, \mathbf{R}^{-i})$. As $\hat{R}^i$ is an MMT of R^i at x, $\mathrm{UC}(x; \hat{R}^i) \setminus x \subset \mathrm{P}(x; R^i)$. Therefore, x is the unique, most preferred consumption bundle in $\overline{G^i(\mathbf{R}^{-i})}$ with respect to $\hat{R}^i$.

 As $x \in \overline{G^i(\mathbf{R}^{-i})}$, there exists a $\hat{x} \in G^i(\mathbf{R}^{-i})$ arbitrarily close to x. Thus, if x is strictly preferred to $f^i(\hat{R}^i, \mathbf{R}^{-i})$ with respect to $\hat{R}^i$, then $\hat{x}$ is strictly preferred to $f^i(\hat{R}^i, \mathbf{R}^{-i})$. This contradicts the strategy-proofness of f. Therefore, $f^i(\hat{R}^i, \mathbf{R}^{-i}) \in \mathrm{UC}(x; \hat{R}^i)$. Then $f^i(\hat{R}^i, \mathbf{R}^{-i}) \neq x$ contradicts that x is the unique most preferred consumption bundle in $\overline{G^i(\mathbf{R}^{-i})}$ with respect to $\hat{R}^i$. Therefore, $x = f^i(\hat{R}^i, \mathbf{R}^{-i}) \in G^i(\mathbf{R}^{-i})$.

A.5 *Proof of Lemma 4*

Without loss of generality, we prove the statement for agent 1. We write $f(\bar{\mathbf{R}}) = \bar{\mathbf{x}} = (\bar{x}^1, \ldots, \bar{x}^N)$ and $\bar{p} = p(\bar{\mathbf{R}}, f)$. From the definitions, $\bar{x}^1$ is (one of) the most preferred consumption bundles in $G^1(\bar{\mathbf{R}}^{-1})$ with respect to $\bar{R}^1$, and no consumption bundle in $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ is strictly preferred to $\bar{x}^1$ with respect to $\bar{R}^1$.[8] We suppose that there exists another consumption bundle $\tilde{x}^1$ in $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ that is indifferent to $\bar{x}^1$ with respect to $\bar{R}^1$, and show a contradiction.

 We let $\tilde{R}^1$ be an MMT of $\bar{R}^1$ at $\tilde{x}^1$. As in Lemma 3, $\tilde{x}^1 = f^1(\tilde{R}^1, \bar{\mathbf{R}}^{-1})$. We write $f(\tilde{R}^1, \bar{\mathbf{R}}^{-1}) = \tilde{\mathbf{x}} = (\tilde{x}^1, \ldots, \tilde{x}^N)$ and let $\tilde{p} = p((\tilde{R}, \bar{\mathbf{R}}^{-1}), f)$ denote the price vector at this allocation. Figure 5 describes the situation where the closure of agent 1's option set has two most preferred consumption bundles $\bar{x}^1$ and $\tilde{x}^1$ with respect to the preference $\bar{R}^1$, contrary to the statement of the lemma. In the following proof, we observe that such an option set contradicts the pseudo-efficiency and strategy-proofness of the social choice function.

 We construct agent 1's new preferences R_t^1 as follows. In addition to $\tilde{R}^1$, which is an MMT of $\bar{R}^1$ at $\tilde{x}^1$, we pick $R^{1\prime}$, which is an MMT of $\bar{R}^1$ at $\bar{x}^1$. Let $\epsilon > 0$ be a sufficiently small scalar. For a parameter $t \in (1 - \epsilon, 1)$ sufficiently close to 1, we define K_t as the convex hull of the set $\mathrm{UC}(t\tilde{x}^1; \tilde{R}^1) \cup \mathrm{UC}(\bar{x}^1; R^{1\prime})$:

$$K_t = \mathrm{co}\big(\mathrm{UC}(t\tilde{x}^1; \tilde{R}^1) \cup \mathrm{UC}(\bar{x}^1; R^{1\prime})\big).$$

We set ϵ sufficiently small so that for any $t \in (1 - \epsilon, 1)$, $\tilde{p}$ and $\bar{p}$ are normal vectors of the supporting hyperplanes to K_t at $t\tilde{x}^1$ and $\bar{x}^1$, respectively.

 Applying Lemma 5 with K_t, we slightly modify the indifference set $\mathrm{I}(\bar{x}^1; \bar{R}^1)$. We consider the hyperplane $(\tilde{x}^1)^{\perp}$ passing through the origin and perpendicular to the vector $\tilde{x}^1$. For each $y \in (\tilde{x}^1)^{\perp}$, we let $L(y)$ denote the half-line starting from y and extending in

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 [8]As in the proof of Lemma 3, if there exists an $x \in \overline{G^1(\bar{\mathbf{R}}^{-1})}$ that is strictly preferred to $\bar{x}$ with respect to $\bar{R}^1$, then there exists a $\hat{x} \in G^1(\bar{\mathbf{R}}^{-1})$ that is strictly preferred to $\bar{x}$ with respect to $\bar{R}^1$.
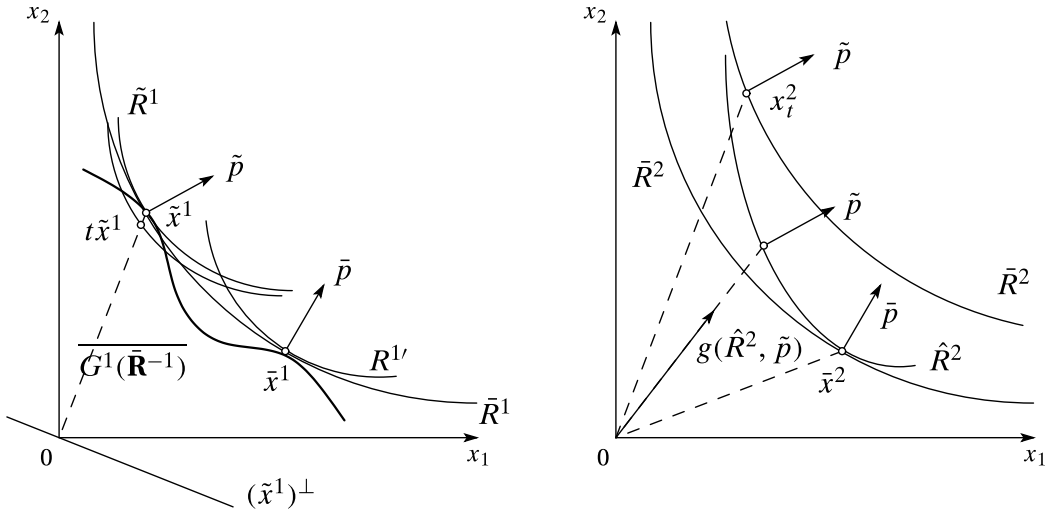
FIGURE 5. Proof of Lemma 4.

the direction of $\tilde{x}^1$: $L(y) = \{x \in R^L | x = y + r\tilde{x}^1, r \geq 0\}$. We fix a scalar $\bar{s} < 1$ close to 1 and define

$$I_t = \bigcup_{y \in (\tilde{x}^1)^\perp} \{\bar{s}(L(y) \cap I(\bar{x}^1; \bar{R}^1)) + (1 - \bar{s})(L(y) \cap \partial K_t)\}.$$

We let $R_t^1 \in \mathcal{R}$ be agent 1's preference that has I_t as its indifference set. Note that R_t^1 can be arbitrarily close to $\bar{R}^1$ by setting $\bar{s}$ close to 1. Therefore, we set $\bar{s}$ sufficiently close to 1 so that R_t^1 is in B^1 for any $t \in (1 - \epsilon, 1)$.

Note that both R_t^1 and $\bar{R}^1$ have the same gradient vectors $\tilde{p}$ and $\bar{p}$ at $\tilde{x}^1$ and $\bar{x}^1$, respectively. Observe that R_t^1 is an MMT of $\bar{R}^1$ at $\tilde{x}^1$. Furthermore, observe that $\mathrm{UC}(\bar{x}^1; R_t^1) \cap \mathrm{LC}(\bar{x}^1; \bar{R}^1)$ consists of the point $\bar{x}^1$ and a set in a neighborhood of $\tilde{x}^1$ that converges to $\tilde{x}^1$ as t converges to 1.

We write $f(R_t^1, \bar{\mathbf{R}}^{-1}) = \mathbf{x}_t = (x_t^1, \ldots, x_t^N)$. These t in x_t^i should not be confused with the subscripts labeling goods. Since R_t^1 is an MMT of $\bar{R}^1$ at $\tilde{x}^1$, $x_t^1 = \tilde{x}^1$.

We can select a small positive vector $\alpha \in R_{++}^L$ such that for any $t \in (1 - \epsilon, 1)$, there exists an agent $i_t \in \{2, \ldots, N\}$ whose consumption $x_t^{i_t}$ is strictly preferred to $\bar{x}^{i_t} + \alpha$ with respect to $\bar{R}^{i_t}$: $x_t^{i_t} \in \mathrm{P}(\bar{x}^{i_t} + \alpha; \bar{R}^{i_t})$. Contrary to this, suppose that for any vector $\alpha \in R_{++}^L$, there exists some $t_\alpha \in (1 - \epsilon, 1)$ such that $\bar{x}^i + \alpha \in \mathrm{UC}(x_{t_\alpha}^i; \bar{R}^i)$ holds for any $i = 2, \ldots, N$. Then, with a sufficiently small α, $(\bar{x}^i + x_{t_\alpha}^i)/2$ is strictly preferred to $x_{t_\alpha}^i$ with respect to $\bar{R}^i$ for any $i = 2, \ldots, N$ by the strict convexity of the preferences. Similarly, $x_{t_\alpha}^1 = \tilde{x}^1$ and $(\bar{x}^1 + \tilde{x}_{t_\alpha}^1)/2$ is preferred to $x_{t_\alpha}^1$ with respect to $R_{t_\alpha}^1$ when t_α is sufficiently close to 1. These contradict the Pareto efficiency of $\mathbf{x}_{t_\alpha}$.

For each $i = 2, \ldots, N$, we let T_i denote the set of t such that x_t^i is strictly preferred to $\bar{x}^i + \alpha$ with respect to $\bar{R}^i$: $T_i = \{t \in (1 - \epsilon, 1) : x_t^i \in \mathrm{P}(\bar{x}^i + \alpha; \bar{R}^i)\}$. Note that there exists some i such that $T_i \cap [t, 1) \neq \varnothing$ for any $t \in (1 - \epsilon, 1)$. If there does not exist any such i, then for each $i = 2, \ldots, N$, there exists a t_i such that $T_i \cap [t_i, 1) = \varnothing$, and there exists no i

satisfying $x_t^i \in P(\bar{x}^i + \alpha, \bar{R}^i)$ for $t \in [\max\{t_2, \ldots, t_N\}, 1)$, which is a contradiction. Without loss of generality, we assume that agent 2 is such an agent: $T_2 \cap [t, 1) \neq \varnothing$ for any $t \in (1 - \epsilon, 1)$. From now on, we only consider t in T_2. In particular, we select a sequence of t's in T_2 converging to 1.

We pick $\hat{R}^2 \in B^2$ such that (i) $\hat{R}^2$ is an MMT of $\bar{R}^2$ at $\bar{x}^2$, (ii) $x_t^2 \in P(\bar{x}^2; \hat{R}^2)$ for $t \in T_2$, and (iii) $g(\hat{R}^2, \tilde{p}) \notin S(g(\bar{R}^2, \tilde{p}), \ldots, g(\bar{R}^N, \tilde{p}))$, where $S(g(\bar{R}^2, \tilde{p}), \ldots, g(\bar{R}^N, \tilde{p}))$ denotes the $N - 1$-dimensional linear space spanned by $g(\bar{R}^i, \tilde{p})$s, $i = 2, \ldots, N$. The condition (iii) requires that on the $L - 1$-dimensional indifference surface $I(\bar{x}^2; \hat{R}^2)$, the point where the gradient vector $\tilde{p}$ is not in the $N - 2$-dimensional set $I(\bar{x}^2; \hat{R}^2) \cap S(g(\bar{R}^2, \tilde{p}), \ldots, g(\bar{R}^N, \tilde{p}))$. As for (ii), we have $x_t^2 \in P(\bar{x}^2 + \alpha; \hat{R}^2) \subset P(\bar{x}^2; \hat{R}^2)$ for $t \in T_2$ as shown above. Thus, we have $\hat{R}^2 \in B^2$ satisfying (i)–(iii).

Observe that $f^2(\bar{R}^1, \hat{R}^2, \bar{R}^{-\{1,2\}}) = \bar{x}^2$ because $\hat{R}^2 \in B^2$ is an MMT of $\bar{R}^2$ at $\bar{x}^2$. Therefore, $f^1(\bar{R}^1, \hat{R}^2, \bar{R}^{-\{1,2\}}) = \bar{x}^1$ as shown in Lemma 2.

We write $f(R_t^1, \hat{R}^2, \bar{R}^{-\{1,2\}}) = \hat{\mathbf{x}}_t = (\hat{x}_t^1, \ldots, \hat{x}_t^N)$ and focus on $\hat{x}_t^1$ for $t \in T_2$. Facing the other agents' preferences $(\hat{R}^2, \bar{R}^{-\{1,2\}})$, agent 1 can achieve $\bar{x}^1$ by reporting $\bar{R}^1$, as mentioned above. Therefore, $\hat{x}_t^1$ should be preferred to $\bar{x}^1$ with respect to R_t^1. Alternatively, $\hat{x}_t^1$ is not strictly preferred to $\bar{x}^1$ with respect to $\bar{R}^1$. Therefore, $\hat{x}_t^1 \in UC(\bar{x}^1; R_t^1) \cap LC(\bar{x}^1; \bar{R}^1)$. Remember that this set consists of the point $\bar{x}^1$ and a set in a neighborhood of $\tilde{x}^1$ that converges to $\tilde{x}^1$ as t converges to 1. We investigate these two cases.

We consider the case where $\hat{x}_t^1 = \bar{x}^1$. As R_t^1 and $\bar{R}^1$ have the same gradient vector at $\bar{x}^1$, and $\hat{R}^2$ and $\bar{R}^2$ have the same gradient vector at $\bar{x}^2$, this case implies that $\hat{\mathbf{x}}_t = \bar{\mathbf{x}}$ as shown in Lemma 2. In particular, $\hat{x}_t^2 = \bar{x}^2$. Remember that $f^2(R_t^1, \bar{R}^2, \bar{R}^{-\{1,2\}}) = x_t^2$ and we have chosen $\hat{R}^2$ so that $x_t^2 \in P(\bar{x}^2; \hat{R}^2)$ for any $t \in T_2$. If agent 2 has preference $\hat{R}^2$ and faces the other agents' preferences $(R_t^1, \bar{R}^{-\{1,2\}})$, he can become better off by reporting $\bar{R}^2$ and achieving x_t^2 than reporting his true preference $\hat{R}^2$ and achieving $\hat{x}_t^2 = \bar{x}^2$. This contradicts the strategy-proofness of f on $\mathbf{B}$.

We now consider the case where $\hat{x}_t^1$ is in a set in a neighborhood of $\tilde{x}^1$ that converges to $\tilde{x}^1$ as t converges to 1. In this case, as t converges to 1, the gradient vector of R_t^1 at $\hat{x}_t^1$ converges to $\tilde{p}$. Therefore, the price vector at $\hat{\mathbf{x}}_t$ converges to $\tilde{p}$. In particular, the gradient of $\hat{R}^2$ at $\hat{x}_t^2$ converges to $\tilde{p}$ as $t \to 1$.

However, note the equation $\hat{x}_t^2 = \tilde{x}^2 + \sum_{i \neq 2}(\tilde{x}^i - \hat{x}_t^i)$ obtained from $\sum_{i=1}^N \tilde{x}^i = \Omega$ and $\sum_{i=1}^N \hat{x}_t^i = \Omega$. In the right-hand side of this equation, $\tilde{x}^1 - \hat{x}_t^1 \to 0$ as $t \to 1$, $\tilde{x}^i \in [g(\bar{R}^i, \tilde{p})]$ for $i \geq 2$, and $\hat{x}_t^i$ converges to some point on $[g(\bar{R}^i, \tilde{p})]$ for $i \geq 3$. Therefore, as $t \to 1$, $\hat{x}_t^2$ converges to a point in $S(g(\bar{R}^2, \tilde{p}), \ldots, g(\bar{R}^N, \tilde{p}))$, and the gradient vector of $\hat{R}^2$ at $\hat{x}_t^2$ does not converge to $\tilde{p}$ because of (iii). This contradicts the discussion in the previous paragraph.

A.6 *Proof of Proposition 1*

We only have to prove that if the consumption-direction vectors $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at $\bar{\mathbf{R}}$ and if $f^i(\bar{\mathbf{R}}) \in \operatorname{int} A$ for an agent i, then $f(\cdot, \bar{\mathbf{R}}^{-i})$ is a continuous function at $\bar{R}^i$. Suppose that the claim is true and let R^i be a preference close to $\bar{R}^i$. Then $f(R^i, \bar{\mathbf{R}}^{-i})$ is close to $f(\bar{\mathbf{R}})$ because of the supposed continuity, and hence the consumption-direction vectors of agents at the preference profile $(R^i, \bar{R}^i)$ are still

independent and $f^i(R^i, \bar{\mathbf{R}}^{-i}) \in \text{int } A$. Then $f(\cdot, \bar{\mathbf{R}}^{-i})$ is continuous at R^i because of the supposed claim.

We first prove that $f^i(\cdot, \bar{\mathbf{R}}^{-i})$ is a continuous function at $\bar{R}^i$. We let $\{R_n^i\}_{n=1}^{\infty}$ be a sequence of preferences in B^i converging to $\bar{R}^i$ as $n \to \infty$. There exists a convergent subsequence $\{f_{n_k}^i(R^i, \bar{\mathbf{R}}^{-i})\}_{k=1}^{\infty}$ because of the compactness of the feasible allocation set. We write $f^i(R_{n_k}^i, \bar{\mathbf{R}}^{-i}) \to x^{i*}$ as $k \to \infty$. All we have to show is that $x^{i*} = f^i(\bar{\mathbf{R}})$.

We observe that x^{i*} is indifferent to $f^i(\bar{\mathbf{R}})$ with respect to $\bar{R}^i$. If $x^{i*} \in \text{P}(f^i(\bar{\mathbf{R}}); \bar{R}^i)$, then $f^i(R_{n_k}^i, \bar{\mathbf{R}}^{-i}) \in \text{P}(f^i(\bar{\mathbf{R}}); \bar{R}^i)$ for a sufficiently large k. This contradicts the strategy-proofness on $\mathbf{B}$. If $f^i(\bar{\mathbf{R}}) \in \text{P}(x^{i*}; \bar{R}^i)$, then $f^i(\bar{\mathbf{R}}) \in \text{P}(f^i(R_{n_k}^i, \bar{\mathbf{R}}^{-i}); R_{n_k}^i)$ for a sufficiently large k because $f^i(R_{n_k}^i, \bar{\mathbf{R}}^{-i})$ converges to x^{i*} as $k \to \infty$ and $\text{P}(x; R_{n_k}^i)$ converges to $\text{P}(x; \bar{R}^i)$ at any consumption x as $k \to \infty$. Again, this contradicts the strategy-proofness on $\mathbf{B}$. Thus, x^{i*} is indifferent to $f^i(\bar{\mathbf{R}})$ with respect to $\bar{R}^i$. Alternatively, $x^{i*} \in \overline{G^i(\bar{\mathbf{R}}^{-i})}$ because $f^i(R_{n_k}^i, \bar{\mathbf{R}}^{-i}) \in G^i(\bar{\mathbf{R}}^{-i})$ for any k. Then $x^{i*} = f^i(\bar{\mathbf{R}})$ by Lemma 4.

We now prove the continuity of $f^j(\cdot, \bar{\mathbf{R}}^{-i})$ at $\bar{R}^i$, $j \neq i$. We proved the continuity of $f^i(\cdot, \bar{\mathbf{R}})$ at $\bar{R}^i$. That is, if $\{R_n^i\}_{n=1}^{\infty}$ is a sequence of preferences in B^i converging to $\bar{R}^i$ as $n \to \infty$, then $f^i(R_n^i, \bar{\mathbf{R}}^{-i})$ converges to $f^i(\bar{\mathbf{R}})$. According to this convergence, the gradient vector of R_n^i at $f^i(R_n^i, \bar{\mathbf{R}}^{-i})$ converges to the gradient vector of $\bar{R}^i$ at $f^i(\bar{\mathbf{R}})$; that is, $p((R_n^i, \bar{\mathbf{R}}^{-i}), f)$ converges to $p(\bar{\mathbf{R}}, f)$ and, hence, $g(\bar{R}^j, p((R_n^i, \bar{\mathbf{R}}^{-i}), f))$ converges to $g(\bar{R}^j, p(\bar{\mathbf{R}}, f))$ for any $j \neq i$.

As in the proof of Lemma 2, $f^j(\mathbf{R}) = \|f^j(\mathbf{R})\| g(R^j, p(\mathbf{R}, f))$ for any preference profile $\mathbf{R} \in \mathbf{B}$, and the sum of the agents' consumption equals the total endowment. Thus, we have two equalities: $f^i(\bar{\mathbf{R}}) + \sum_{j \neq i} \|f^j(\bar{\mathbf{R}})\| g(R^j, p(\bar{\mathbf{R}}, f)) = \Omega$ and $f^i(R_n^i, \bar{\mathbf{R}}^{-i}) + \sum_{j \neq i} \|f^j(R_n^i, \bar{\mathbf{R}}^{-i})\| g(R^j, p((R_n^i, \bar{\mathbf{R}}^{-i}), f)) = \Omega$ for any n. Considering the difference between these equalities, we have

$$
\begin{aligned}
0 = {} & f^i(\bar{\mathbf{R}}) - f^i(R_n^i, \bar{\mathbf{R}}^{-i}) \\
& + \sum_{j \neq i} \|f^j(\bar{\mathbf{R}})\| g(R^j, p(\bar{\mathbf{R}}, f)) - \sum_{j \neq i} \|f^j(R_n^i, \bar{\mathbf{R}}^{-i})\| g(R^j, p((R_n^i, \bar{\mathbf{R}}^{-i}), f)) \\
= {} & f^i(\bar{\mathbf{R}}) - f^i(R_n^i, \bar{\mathbf{R}}^{-i}) \\
& + \sum_{j \neq i} (\|f^j(\bar{\mathbf{R}})\| - \|f^j(R_n^i, \bar{\mathbf{R}}^{-i})\|) g(R^j, p(\bar{\mathbf{R}}, f)) \\
& + \sum_{j \neq i} \|f^j(R^{i(n)}, \bar{\mathbf{R}}^{-i})\| (g(R^j, p(\bar{\mathbf{R}}, f)) - g(R^j, p((R_n^i, \bar{\mathbf{R}}^{-i}), f))).
\end{aligned}
$$

As $n \to \infty$, the first and third elements in the last equation converge to 0. As $g(\bar{R}^j, p(\bar{\mathbf{R}}, f))$, $j \neq i$, are independent vectors, $\|f^j(R_n^i, \bar{\mathbf{R}}^{-i})\|$ converges to $\|f^j(\bar{\mathbf{R}})\|$ for any $j \neq i$. This implies that $f^j(R_n^i, \bar{\mathbf{R}}^{-i})$ converges to $f^j(\bar{\mathbf{R}})$ for any j as $n \to \infty$. That is, $f^j(\cdot, \bar{\mathbf{R}}^{-i})$ is continuous at $\bar{R}^i$ for any $j \neq i$.

A.7 *Proof of Proposition 2*

We first show that for any R^i in a neighborhood of $\bar{R}^i$, $\overline{G^i(\bar{\mathbf{R}}^{-i})} \subset f^i(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i}), f))^{\perp} - R_+^L$ holds in a neighborhood of $f^i(R^i, \bar{\mathbf{R}}^{-i})$. That is, there exists a positive

scalar ϵ_{R^i}, depending on R^i, such that

$$D_{\epsilon_{R^i}}\big(f^i(R^i, \bar{\mathbf{R}}^{-i})\big) \cap \overline{G^i(\bar{\mathbf{R}}^{-i})} \subset f^i(R^i, \bar{\mathbf{R}}^{-i}) + p\big((R^i, \bar{\mathbf{R}}^{-i}), f\big)^\perp - R^L_+. \tag{4}$$

Note that all we have to show is the existence of an $\epsilon_{\bar{R}^i}$ satisfying (4) at $\bar{\mathbf{R}}$, where consumption-direction vectors are independent and $f^i(\bar{\mathbf{R}}) \in \mathrm{int}\, A$. Suppose this claim to be true. If R^i is in a neighborhood of $\bar{R}^i$, then $f(R^i, \bar{\mathbf{R}}^{-i})$ is in a neighborhood of $f(\bar{\mathbf{R}})$ by Proposition 1, and, hence, the consumption-direction vectors of agents remain independent at the preference profile $(R^i, \bar{\mathbf{R}}^{-i})$ and $f^i(R^i, \bar{\mathbf{R}}^{-i}) \in \mathrm{int}\, A$. Then there exists an ϵ_{R^i} satisfying (4) by the supposed claim.

Contrary to the existence of an $\epsilon_{\bar{R}^i}$ satisfying (4), we suppose that $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ has an intersection with $f^i(\bar{\mathbf{R}}) + p(\bar{\mathbf{R}}, f)^\perp + R^L_{++}$ in any neighborhood of $f^i(\bar{\mathbf{R}})$. Then we have a preference in a neighborhood of $\bar{R}^i$ that has two most preferred consumption bundles in $\overline{G^i(\bar{\mathbf{R}}^{-i})}$, which contradicts Lemma 4. The rigorous proof proceeds as follows. We write $f(\bar{\mathbf{R}}) = (\bar{x}^1, \ldots, \bar{x}^N)$ and $p(\bar{\mathbf{R}}, f) = \bar{p}$.

We construct agent i's new preference as follows. For a parameter $\epsilon > 0$, we consider the set $H(\bar{x}^i; \bar{p}) \cap \mathrm{UC}(\bar{x}^i - \epsilon\bar{p}; \bar{R}^i)$. Applying Lemma 6, we round the edge of this set. We let $\epsilon'' < \epsilon$ be a scalar smaller than ϵ and consider a closed ball $\bar{D}_{\epsilon''}$ with radius ϵ''. We let C_ϵ be the union of such closed balls with radius ϵ'' included in the set $H(\bar{x}^i; \bar{p}) \cap \mathrm{UC}(\bar{x}^i - \epsilon\bar{p}; \bar{R}^i)$: $C_\epsilon = \bigcup_{\bar{D}_{\epsilon''} \subset H(\bar{x}^i; \bar{p}) \cap \mathrm{UC}(\bar{x}^i - \epsilon\bar{p}; \bar{R}^i)} \bar{D}_{\epsilon''}$. Note that the surface of the set C_ϵ is flat in a neighborhood of $\bar{x}^i$.

We apply Lemma 5 to make C_ϵ into an upper contour set of a preference. We fix any positive unit vector $a \in S^{L-1}_{++}$ and consider the $L-1$-dimensional linear space $a^\perp$. For $y \in a^\perp$, we let $L(y)$ denote the half-line starting from y and extending in the direction of the vector a: $L(y) = \{x \in R^L \mid x = y + ta, t \geq 0\}$. For a parameter $t \in (0, 1]$, we let $R^i_{\epsilon, t}$ be agent i's preference that has as its indifference set

$$\mathrm{I}_{\epsilon, t} = \bigcup_{y \in a^\perp} \big\{ t\big(L(y) \cap \mathrm{I}(\bar{x}^i; \bar{R}^i)\big) + (1 - t)\big(L(y) \cap \partial C_\epsilon\big) \big\}.$$

Observe that $R^i_{\epsilon, 1} = \bar{R}^i$ and that $R^i_{\epsilon, t}$ is an MMT of $R^i_{\epsilon, t'}$ at $\bar{x}^i$ for $t > t'$, and the indifference set $\mathrm{I}(\bar{x}^i; R^i_{\epsilon, t})$ becomes flatter in a neighborhood of $\bar{x}^i$ as $t \to 0$. Furthermore, observe that if ϵ is sufficiently small, then ∂C_ϵ is close to the indifference set $\mathrm{I}(\bar{x}^i; \bar{R}^i)$, and hence $R^i_{\epsilon, t}$ is close to $\bar{R}^i$ for any t. We let ϵ be sufficiently small such that $R^i_{\epsilon, t} \in B^i$ for any $t \in (0, 1]$.

For any t, $\bar{x}^i$ is an intersection between $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ and $\mathrm{UC}(\bar{x}^i; R^i_{\epsilon, t})$, and it is the unique intersection for $t = 1$. By the assumption that $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ has an intersection with $\bar{x}^i + \bar{p}^\perp + R^L_{++}$ in any neighborhood of $\bar{x}^i$, $\mathrm{UC}(\bar{x}^i; R^i_{\epsilon, t})$ has intersections with $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ other than $\bar{x}^i$ when t is small. We let $t(\epsilon)$ be the largest t such that $\mathrm{UC}(\bar{x}^i; R^i_{\epsilon, t})$ has such an intersection with $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ in $f^i(\bar{\mathbf{R}}) + p(\bar{\mathbf{R}}, f))^\perp + R^L_{++}$. Then, with respect to $R^i_{\epsilon, t(\epsilon)}$, there exist two most preferred consumption bundles in $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ and this contradicts Lemma 4. This ends the proof of the existence of ϵ_{R^i} satisfying (4).

We now prove the statement of the proposition. For example, we choose the scalars $\bar{\epsilon}$ and ϵ' satisfying (1) as follows. We consider ϵ_{R^i} satisfying (4) for each R^i in a neighborhood of $\bar{R}^i$. For $\epsilon > 0$, we consider the ϵ-neighborhood of $\bar{R}^i$, i.e., $B_{\epsilon(\bar{R}^i)}$, and

define $\alpha(\epsilon)$ as the infimum of ϵ_{R^i} for $R^i \in B_\epsilon(\bar{R}^i)$: $\alpha(\epsilon) = \inf_{R^i \in B_\epsilon(\bar{R}^i)} \epsilon_{R^i}$. Note that $\epsilon \mapsto \alpha(\epsilon)$ is a positive and decreasing function by the definition. Alternatively, we define $\beta(\epsilon)$ as the supremum of the distance between $f(R^i, \bar{\mathbf{R}}^{-i})$ and $f^i(\bar{\mathbf{R}})$ for $R^i \in B_\epsilon(\bar{R}^i)$: $\beta(\epsilon) = \sup_{R^i \in B_\epsilon(\bar{R}^i)} \|f(R^i, \bar{\mathbf{R}}^{-i}) - f^i(\bar{\mathbf{R}})\|$. Note that $\epsilon \mapsto \beta(\epsilon)$ is a positive and increasing function and $\beta(\epsilon) \to 0$ as $\epsilon \to 0$ because of the continuity of f. We pick a scalar ϵ' satisfying $\beta(\epsilon') < \frac{1}{2}\alpha(\epsilon')$ and define $\bar{\epsilon} = \beta(\epsilon')$. The existence of such a scalar ϵ' is ensured by the properties of the functions $\epsilon \mapsto \alpha(\epsilon)$ and $\epsilon \mapsto \beta(\epsilon)$ mentioned above.

It can be easily seen that these are the desired scalars. If R^i is in the neighborhood $B_{\epsilon'}(\bar{R}^i)$ of $\bar{R}^i$, then in the neighborhood $D_{\alpha(\epsilon')}(f^i(R^i, \bar{\mathbf{R}}^{-i}))$, the option set $\overline{G^i(\bar{\mathbf{R}}^{-i})}$ is in the lower left-hand side of the hyperplane $f(R^i, \bar{\mathbf{R}}^{-i}) + p((R^i, \bar{\mathbf{R}}^{-i})), f)^\perp$:

$$D_{\alpha(\epsilon')}\big(f^i\big(R^i, \bar{\mathbf{R}}^{-i}\big)\big) \cap \overline{G^i(\bar{\mathbf{R}}^{-i})} \subset f^i\big(R^i, \bar{\mathbf{R}}^{-i}\big) + p\big((R^i, \bar{\mathbf{R}}^{-i}), f\big)^\perp - R_+^L.$$

As the distance between $f^i(R^i, \bar{\mathbf{R}}^{-i})$ and $f(\bar{\mathbf{R}})$ is at most $\bar{\epsilon}$ and $\alpha(\epsilon') > 2\bar{\epsilon}$, we have $D_{\bar{\epsilon}}(f(\bar{\mathbf{R}})) \subset D_{\alpha(\epsilon')}(f^i(R^i, \bar{\mathbf{R}}^{-i}))$. Hence, we have (1).

A.8 *Proof of Proposition 3*

As in the proof of Proposition 2, we only have to prove that if the consumption-direction vectors are independent at $\bar{\mathbf{R}}$ and if $f^i(\bar{\mathbf{R}}) \in \text{int}\, A$, then the statement of the proposition holds at the preference profile $\bar{\mathbf{R}}$.

Without loss of generality, we prove the proposition for agent 1. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \dots, N$, are independent at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N) \in \mathbf{B}$ and $f^1(\bar{\mathbf{R}}) \in \text{int}\, A$. We write $f(\bar{\mathbf{R}}) = \bar{\mathbf{x}} = (\bar{x}^1, \dots, \bar{x}^N)$ and $p(\bar{\mathbf{R}}, f) = \bar{p}$. We have to prove that $\bar{x}^1$ is the unique intersection between $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ and $\bar{x}^1 + \bar{p}^\perp$ in a neighborhood of $\bar{x}^1$.

Contrary to the statement of the proposition, we suppose that for any scalar $\epsilon > 0$, there exists $\tilde{x}_\epsilon^1$ in the ϵ-neighborhood $D_\epsilon(\bar{x}^1)$ of $\bar{x}^1$ such that $\tilde{x}_\epsilon^1$ is different from $\bar{x}^1$, and that $\tilde{x}_\epsilon^1$ is the intersection between $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ and $\bar{x}^1 + \bar{p}^\perp$. We show a contradiction.

We first observe that when ϵ is sufficiently small, the hyperplane $\bar{x}^1 + \bar{p}^\perp$ is tangent to $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ along the segment $[\bar{x}^1, \tilde{x}_\epsilon^1] \equiv \{t\bar{x}^1 + (1-t)\tilde{x}_\epsilon^1 \in R_+^L : 0 \le t \le 1\}$. We pick an arbitrary consumption bundle $x^1 \in (\bar{x}^1, \tilde{x}_\epsilon^1) \equiv \{t\bar{x}^1 + (1-t)\tilde{x}_\epsilon^1 \in R_+^L : 0 < t < 1\}$ and consider a preference R^1 in a neighborhood of $\bar{R}^1$ so that the gradient vector of R^1 at x^1 is $\bar{p}$. When ϵ is sufficiently small, x^1 is sufficiently close to $\bar{x}^1$, and we can have such a preference R^1 in a neighborhood of $\bar{R}^1$. We let $x^{1\prime}$ denote the most preferred consumption in $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ with respect to R^1 and let p' denote the gradient vector of R^1 at $x^{1\prime}$. As shown in Proposition 2, $x^{1\prime}$ is in the lower left-hand side of the hyperplane $\bar{x}^1 + \bar{p}^\perp$. Therefore, $\bar{p}x^{1\prime} \le \bar{p}\bar{x}^1 = \bar{p}\tilde{x}_\epsilon^1$, and hence $\bar{p}x^{1\prime} \le \bar{p}x^1$. By the same reasoning, $\bar{x}^1$ and $\tilde{x}_\epsilon^1$ are in the lower left-hand side of the hyperplane $x^{1\prime} + (p')^\perp$. Therefore, $p'\bar{x}^1 \le p'x^{1\prime}$ and $p'\tilde{x}_\epsilon^1 \le p'x^{1\prime}$, and hence $p'x^1 \le p'x^{1\prime}$. These two inequalities are satisfied for the two combinations $(\bar{p}, x^1)$ and $(p', x^{1\prime})$ of a gradient vector and a consumption bundle with the preference R^1 if and only if $\bar{p} = p'$ and $x^{1\prime} = x^1$. As our choice of x^1 is arbitrary on $(\bar{x}^1, \tilde{x}_\epsilon^1)$, this implies that $\bar{x}^1 + \bar{p}^\perp$ is tangent to $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ along the segment $[\bar{x}^1, \tilde{x}_\epsilon^1]$. From now on, we assume that ϵ is sufficiently small so that $\bar{x}^1 + \bar{p}^\perp$ is tangent to $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ along the segment $[\bar{x}^1, \tilde{x}_\epsilon^1]$.
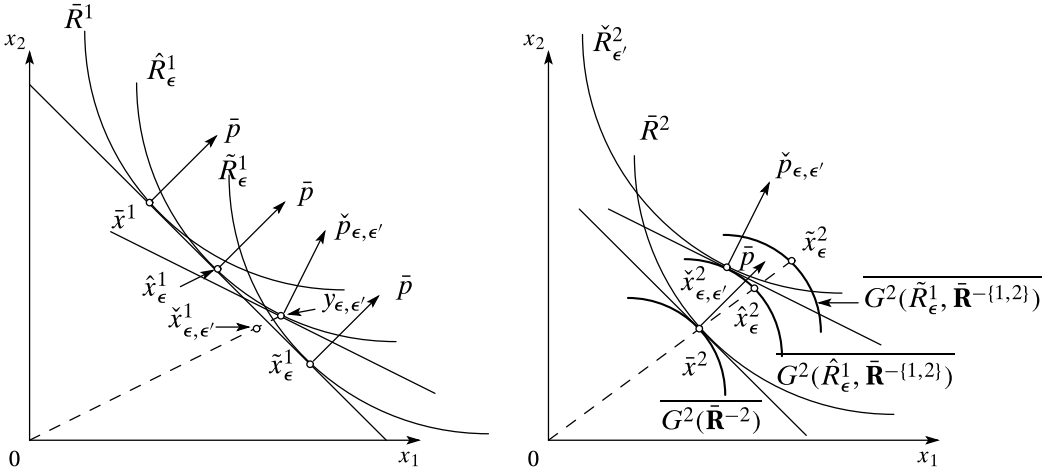
FIGURE 6. Proof of Proposition 3.

For each ϵ, we pick a consumption bundle $\hat{x}^1_\epsilon \in (\bar{x}^1, \tilde{x}^1_\epsilon)$ and a preference $\hat{R}^1_\epsilon$ such that the gradient vector of $\hat{R}^1_\epsilon$ at $\hat{x}^1_\epsilon$ is $\bar{p}$ and $\hat{R}^1_\epsilon$ converges to $\bar{R}^1$ as $\epsilon \to 0$. We can have such a preference because $\hat{x}^1_\epsilon \to \bar{x}$ as $\epsilon \to 0$. Similarly, we let $\tilde{R}^1_\epsilon$ be a preference such that the gradient vector of $\tilde{R}^1_\epsilon$ at $\tilde{x}^1_\epsilon$ is $\bar{p}$ and $\tilde{R}^1_\epsilon$ converges to R^1 as $\epsilon \to 0$. It is clear that $f^1(\tilde{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}) = \tilde{x}^1_\epsilon$, $f^1(\hat{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}) = \hat{x}^1_\epsilon$, and $p((\tilde{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}), f) = p((\hat{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}), f) = \bar{p}$. We write $f(\tilde{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}) = \tilde{\mathbf{x}}_\epsilon = (\tilde{x}^1_\epsilon, \ldots, \tilde{x}^N_\epsilon)$ and $f(\tilde{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}) = \hat{\mathbf{x}}_\epsilon = (\hat{x}^1_\epsilon, \ldots, \hat{x}^N_\epsilon)$.

For agent $i \neq 1$, the consumption bundles, $\bar{x}^i$, $\hat{x}^i_\epsilon$, and $\tilde{x}^i_\epsilon$ are all on the same ray $[g(\bar{R}^i, \bar{p})]$ because of the same price vector $\bar{p}$. Under the independence of the consumption-direction vectors, which holds for preferences in a neighborhood of $\bar{\mathbf{R}}$, $\tilde{x}^1_\epsilon \neq \bar{x}^1$ implies that there exists an agent $i \neq 1$ such that $\tilde{x}^i_\epsilon \neq \bar{x}^i$. Without loss of generality, we assume agent 2 is such an agent. Then $\hat{x}^2_\epsilon$ is between $\bar{x}^2$ and $\tilde{x}^2_\epsilon$ on the same ray, and either $\bar{x}^2 < \hat{x}^2_\epsilon < \tilde{x}^2_\epsilon$ or $\tilde{x}^2_\epsilon < \hat{x}^2_\epsilon < \bar{x}^2$ holds.

Figure 6 describes the situation where the closure of agent 1's option set is tangent to the hyperplane $\bar{x}^1 + \bar{p}^\perp$ along the segment $[\bar{x}^1, \tilde{x}^1_\epsilon]$, contrary to the statement of the proposition. In the following proof, we show that the other agents' option sets are also flat and observe that such flat option sets contradict the pseudo-efficiency and strategy-proofness of the social choice function.

We now show that when ϵ is sufficiently small, $\overline{G^2(\hat{R}^1_\epsilon, \bar{\mathbf{R}}^{-\{1,2\}})}$ is flat in a neighborhood of $\hat{x}^2_\epsilon$. We suppose that it is not flat in any neighborhood of $\hat{x}^2_\epsilon$, as drawn in Figure 5. We let $\epsilon' > 0$ be a scalar and let $\check{R}^2_{\epsilon'}$ be agent 2's preference in the ϵ'-neighborhood $B_{\epsilon'}(\bar{R}^2)$ of $\bar{R}^2$ such that the gradient vector $\check{p}_{\epsilon,\epsilon'}$ of $\check{R}^2_{\epsilon'}$ at the most preferred consumption bundle in $\overline{G^2(\hat{R}^1_\epsilon, \bar{\mathbf{R}}^{-\{1,2\}})}$ with respect to $\check{R}^2_{\epsilon'}$ is different from $\bar{p}$, as drawn in the figure. As the closure of the option set is not flat in any neighborhood of $\hat{x}^2_\epsilon$, we can have such a preference $\check{R}^2_{\epsilon'}$ in any ϵ'-neighborhood of $\bar{R}^2$. We write $f(\hat{R}^1_\epsilon, \check{R}^2_{\epsilon'}, \bar{\mathbf{R}}^{-\{1,2\}}) = \check{\mathbf{x}}_{\epsilon,\epsilon'} = (\check{x}^1_{\epsilon,\epsilon'}, \ldots, \check{x}^N_{\epsilon,\epsilon'})$.[9] We have $\check{p}_{\epsilon,\epsilon'} \to \bar{p}$ and $\check{\mathbf{x}}_{\epsilon,\epsilon'} \to \hat{\mathbf{x}}_\epsilon$ as $\epsilon' \to 0$.

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[9]The set $\overline{G^2(\hat{R}^1_\epsilon, \bar{\mathbf{R}}^{-\{1,2\}})}$ may have an edge at $\hat{x}^2_\epsilon$, and $\check{x}_{\epsilon,\epsilon'}$ may be $\hat{x}^2_\epsilon$.

As $\hat{x}^1_\epsilon$ is between $\bar{x}^1$ and $\tilde{x}^1_\epsilon$, either $\check{p}_{\epsilon,\epsilon'}\bar{x}^1 > \check{p}_{\epsilon,\epsilon'}\hat{x}^1_\epsilon > \check{p}_{\epsilon,\epsilon'}\tilde{x}^1_\epsilon$, $\check{p}_{\epsilon,\epsilon'}\bar{x}^1 < \check{p}_{\epsilon,\epsilon'}\hat{x}^1_\epsilon <$ $\check{p}_{\epsilon,\epsilon'}\tilde{x}^1_\epsilon$, or $\check{p}_{\epsilon,\epsilon'}\bar{x}^1 = \check{p}_{\epsilon,\epsilon'}\hat{x}^1_\epsilon = \check{p}_{\epsilon,\epsilon'}\tilde{x}^1_\epsilon$ holds. When $\check{p}_{\epsilon,\epsilon'}\bar{x}^1 > \check{p}_{\epsilon,\epsilon'}\hat{x}^1_\epsilon > \check{p}_{\epsilon,\epsilon'}\tilde{x}^1_\epsilon$, as drawn in Figure 2, or when $\check{p}_{\epsilon,\epsilon'}\bar{x}^1 = \check{p}_{\epsilon,\epsilon'}\hat{x}^1_\epsilon = \check{p}_{\epsilon,\epsilon'}\tilde{x}^1_\epsilon$, we focus on the two combinations $(\bar{x}^1, \bar{p})$ and $(\check{x}^1_{\epsilon,\epsilon'}, \check{p}_{\epsilon,\epsilon'})$ of a consumption bundle and a price vector. We consider a preference $R^1_{\epsilon,\epsilon'}$ in a neighborhood of $\bar{R}^1$ such that the gradient vector of $R^1_{\epsilon,\epsilon'}$ at $\bar{x}^1$ is $\bar{p}$ and that at $\check{x}^1_{\epsilon,\epsilon'}$ is $\check{p}_{\epsilon,\epsilon'}$: $\bar{p} = p(R^1_{\epsilon,\epsilon'}, \bar{x}^1)$ and $\check{p}_{\epsilon,\epsilon'} = p(R^1_{\epsilon,\epsilon'}, \check{x}^1_{\epsilon,\epsilon'})$.

For example, such a preference $R^1_{\epsilon,\epsilon'}$ can be obtained as follows. We write $y_{\epsilon,\epsilon'} = [\check{x}^1_{\epsilon,\epsilon'}] \cap I(\hat{x}^1_\epsilon; \hat{R}^1_\epsilon)$. We let $R^*_{\epsilon,\epsilon'}$ be an MMT of $\bar{R}^1$ at $\bar{x}^1$ and let $K_{\epsilon,\epsilon'}$ denote the convex hull of $\text{UC}(\hat{x}^1_\epsilon; \hat{R}^1_\epsilon) \cup \text{UC}(\bar{x}^1, R^*_{\epsilon,\epsilon'})$: $K_{\epsilon,\epsilon'} = \text{co}(\text{UC}(\hat{x}^1_\epsilon; \hat{R}^1_\epsilon) \cup \text{UC}(\bar{x}^1, R^*_{\epsilon,\epsilon'}))$. Observe that the convex hull $K_{\epsilon,\epsilon'}$ is tangent to the hyperplane $y_{\epsilon,\epsilon'} + \check{p}^\perp_{\epsilon,\epsilon'}$ at $y_{\epsilon,\epsilon'}$. This $K_{\epsilon,\epsilon'}$ cannot be an upper contour set of a preference because it is not a strictly convex set. We let $R_{\epsilon,\epsilon'}$ be a preference such that its gradient vector at $y_{\epsilon,\epsilon'}$ is $\check{p}_{\epsilon,\epsilon'}$ and its upper contour set at $y_{\epsilon,\epsilon'}$ includes $K_{\epsilon,\epsilon'}$: $K_{\epsilon,\epsilon'} \subset \text{UC}(y_{\epsilon,\epsilon'}; R_{\epsilon,\epsilon'})$. We use $R_{\epsilon,\epsilon'}$ and make $K_{\epsilon,\epsilon'}$ into an upper contour set of a preference by applying Lemma 5. We fix any positive unit vector $a \in S^{L-1}_{++}$ and consider the $L-1$-dimensional linear space $a^\perp$. For $y \in a^\perp$, we let $L(y)$ denote the half-line starting from y and extending in the direction of the vector a: $L(y) = \{x \in R^L | x = y + ta, t \geq 0\}$. We let $R'_{\epsilon,\epsilon'}$ be a preference, for which the indifference set at $y_{\epsilon,\epsilon'}$ is

$$\bigcup_{y \in a^\perp} \{s(L(y) \cap I(y_{\epsilon,\epsilon'}; R_{\epsilon,\epsilon'})) + (1-s)(L(y) \cap \partial K_{\epsilon,\epsilon'})\} \tag{5}$$

with a sufficiently small $s > 0$. Finally, we construct a preference $R^1_{\epsilon,\epsilon'}$ in a neighborhood of $R'_{\epsilon,\epsilon'}$ such that $\check{p}_{\epsilon,\epsilon'} = p(R^1_{\epsilon,\epsilon'}, \check{x}^1_{\epsilon,\epsilon'})$ and $\bar{p} = p(R^1_{\epsilon,\epsilon'}, \bar{x}^1)$ by directly applying the preference construction in Lemma 1 so that $\bar{R}$, $(\bar{x}, \bar{p})$, and (x_n, p_n) in Lemma 1 correspond to $R'_{\epsilon,\epsilon'}$, $(\check{x}_{\epsilon,\epsilon'}, \check{p}_{\epsilon,\epsilon'})$, and $(\bar{x}, \bar{p})$ in the present setup, respectively.

We observe that Lemma 1 can be applied to $R'_{\epsilon,\epsilon'}$ when ϵ, ϵ', and s in (5) are sufficiently small. As $\epsilon \to 0$, we have $\hat{x}_\epsilon \to \bar{x}$. As $\epsilon' \to 0$, we have $\check{x}_{\epsilon,\epsilon'} \to \hat{x}_\epsilon$ and $\check{p}_{\epsilon,\epsilon'} \to \bar{p}$. Thus, when ϵ and ϵ' are sufficiently small, we have $\check{x}^1_{\epsilon,\epsilon'}$ and $\check{p}_{\epsilon,\epsilon'}$ sufficiently close to $\bar{x}^1$ and $\bar{p}$, respectively. Therefore, we can apply the preference construction method in Lemma 1. The condition in Lemma 1 is $\bar{p}([\check{x}_{\epsilon,\epsilon'}] \cap I(\bar{x}; R'_{\epsilon,\epsilon'})) > \bar{p}\bar{x}$ in the present setup. We set ϵ and ϵ'. As $s \to 0$ in (5), $\text{UC}(\bar{x}^1; R'_{\epsilon,\epsilon'})$ converges to $K_{\epsilon,\epsilon'}$ and $[\check{x}_{\epsilon,\epsilon'}] \cap I(\bar{x}; R'_{\epsilon,\epsilon'})$ converges to $y_{\epsilon,\epsilon'}$, which satisfies $\bar{p}y_{\epsilon,\epsilon'} > \bar{p}\bar{x}$. Therefore, the condition is satisfied when s in (5) is sufficiently small.

Next, we observe that $R^1_{\epsilon,\epsilon'}$ is in a neighborhood of $\bar{R}^1$ when ϵ, ϵ', and s in (5) are sufficiently small. As $\epsilon \to 0$, we have $\hat{x}^1_\epsilon \to \bar{x}^1$ and $\hat{R}_\epsilon \to \bar{R}^1$, and, hence, $\text{UC}(\hat{x}^1_\epsilon; \hat{R}_\epsilon)$ converges to $\text{UC}(\bar{x}; \bar{R}^1)$. Since $\text{UC}(\bar{x}^1; R^*_{\epsilon,\epsilon'}) \subset \text{UC}(\bar{x}^1; \bar{R}^1)$, $K_{\epsilon,\epsilon'}$ converges to $\text{UC}(\bar{x}; \bar{R}^1)$. As $s \to 0$ in (5), $\text{UC}(y_{\epsilon,\epsilon'}; R'_{\epsilon,\epsilon'})$ converges to $K_{\epsilon,\epsilon'}$. Therefore, as $\epsilon \to 0$ and $s \to 0$, $\text{UC}(y_{\epsilon,\epsilon'}; R'_{\epsilon,\epsilon'})$ converges to $\text{UC}(\bar{x}^1; \bar{R}^1)$ and $R'_{\epsilon,\epsilon'}$ converges to $\bar{R}^1$. As $R^1_{\epsilon,\epsilon'}$ converges to $R'_{\epsilon,\epsilon'}$ as $\epsilon \to 0$ and $\epsilon' \to 0$, we have $R^1_{\epsilon,\epsilon'}$ in a neighborhood of $\bar{R}^1$ as desired when ϵ, ϵ', and s are sufficiently small.

With the preference $R^1_{\epsilon,\epsilon'}$, we have $f(R^1_{\epsilon,\epsilon'}, \check{R}^2_{\epsilon,\epsilon'}, \bar{R}^{-\{1,2\}}) = \check{x}_{\epsilon,\epsilon'}$ and $f(R^1_{\epsilon,\epsilon'}, \bar{R}^{-2}) = \bar{x}$. This contradicts the strategy-proofness of f on $\mathbf{B}$ with respect to agent 2 be-

cause $\check{x}_{\epsilon,\epsilon'} \to \hat{x}_\epsilon$ as $\epsilon' \to 0$, and $\hat{x}_\epsilon^2$, which is on the same ray as $\bar{x}^2$, satisfies $\hat{x}_\epsilon^2 < \bar{x}^2$ or $\bar{x}^2 < \hat{x}_\epsilon^2$ as mentioned above.

The discussion is symmetric when $\check{p}_{\epsilon,\epsilon'}\bar{x}^1 < \check{p}_{\epsilon,\epsilon'}\hat{x}_{\epsilon,\epsilon'}^1 < \check{p}_{\epsilon,\epsilon'}\tilde{x}_\epsilon^1$. We focus on the two pairs $(\check{x}_{\epsilon,\epsilon'}^1, \check{p}_{\epsilon,\epsilon'})$ and $(\tilde{x}_\epsilon^1, \bar{p})$. Similar to the discussion above, we can construct a preference $R_{\epsilon,\epsilon'}^1$ in a neighborhood of $\tilde{R}_\epsilon^1$, and thus in a neighborhood of $\bar{R}^1$, such that the gradient vectors of $R_{\epsilon,\epsilon'}^1$ at $\tilde{x}_\epsilon^1$ and $\check{x}_{\epsilon,\epsilon'}^1$ are $\bar{p}$ and $\check{p}_{\epsilon,\epsilon'}$, respectively. Then we have $f(R_{\epsilon,\epsilon'}^1, \check{R}_{\epsilon,\epsilon'}^2, \mathbf{R}^{-\{1,2\}}) = \check{x}_{\epsilon,\epsilon'}$, which converges to $\hat{x}_\epsilon$ as $\epsilon' \to 0$, and $f(R_{\epsilon,\epsilon'}^1, \bar{\mathbf{R}}^{-2}) = \tilde{x}_\epsilon$. This again contradicts the strategy-proofness of f on $\mathbf{B}$ with respect to agent 2. This ends the proof that $\overline{G^2(\hat{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}})}$ is flat in a neighborhood of $\hat{x}_\epsilon^2$ when ϵ is sufficiently small.

In addition to $\hat{x}_\epsilon^1$ and $\hat{R}_\epsilon^1$, we pick another consumption bundle $\acute{x}_\epsilon^1 (\neq \hat{x}_\epsilon^1)$ on $(\bar{x}^1, \tilde{x}_\epsilon^1)$ and another preference $\acute{R}_\epsilon^1$ for each ϵ such that the gradient vector of $\acute{R}_\epsilon^1$ at $\acute{x}_\epsilon^1$ is $\bar{p}$ and $\acute{R}_\epsilon^1$ converges to $\bar{R}^1$ as $\epsilon \to 0$. It is clear that $f^1(\acute{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}}) = \acute{x}_\epsilon^1$ and $p((\acute{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}}), f) = \bar{p}$. We write $f(\acute{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}}) = \acute{\mathbf{x}}_\epsilon = (\acute{x}_\epsilon^1, \ldots, \acute{x}_\epsilon^N)$. Similar to the discussion above, $\overline{G^2(\acute{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}})}$ is flat in a neighborhood of $\acute{x}_\epsilon^2$ when ϵ is sufficiently small. Under the independence of the consumption-direction vectors, $\acute{x}_\epsilon^2$ is on the same ray as $\hat{x}_\epsilon^2$ and $\acute{x}_\epsilon^2 \neq \hat{x}_\epsilon^2$ because of $\acute{x}_\epsilon^1 \neq \hat{x}_\epsilon^1$.

Now we exchange the roles of agents 1 and 2. Facing the closure of the option set $\overline{G^2(\hat{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}})}$, which is flat in a neighborhood of $\hat{x}_\epsilon^2$, we let ϵ'' be a scalar such that with respect to any preference R^2 in the ϵ''-neighborhood $B_{\epsilon''}(\bar{R}^2)$ of R^2, the most preferred consumption bundle in $\overline{G^2(\hat{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}})}$, which should be $f^2(\hat{R}_\epsilon^1, R^2, \mathbf{R}^{-\{1,2\}})$, is in the flat part. Then $p(\hat{R}_\epsilon^1, R^2, \bar{\mathbf{R}}^{\{1,2\}}) = \bar{p}$. In particular, for each sufficiently small scalar ϵ'', we let $\grave{R}_{\epsilon''}^2$ be agent 2's preference in $B_{\epsilon''}^2(\bar{R}^2)$ such that $f^2(\hat{R}_\epsilon^1, \grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}}) - \hat{x}_\epsilon^2 \notin S(g(\bar{R}^3; \bar{p}), \ldots, g(\bar{R}^N; \bar{p}))$, where $S(g(\bar{R}^3; \bar{p}), \ldots, g(\bar{R}^N; \bar{p}))$ denotes the $N-2$-dimensional linear space spanned by the consumption-direction vectors of agents $i = 3, \ldots, N$. We can have such a preference $\grave{R}_{\epsilon''}^2$ because we can have $f^2(\hat{R}_\epsilon^1, \grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}}) - \hat{x}_\epsilon^2$ in any $L-1$-dimensional directions on the flat part of $\overline{G^2(\hat{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}})}$. Note that the condition $f^2(\hat{R}_\epsilon^1, \grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}}) - \hat{x}_\epsilon^2 \notin S(g(\bar{R}^3; \bar{p}), \ldots, g(\bar{R}^N; \bar{p}))$ implies that $f^1(\hat{R}_\epsilon^1, \grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}}) \neq \hat{x}_\epsilon^1$ under the independence of the consumption-direction vectors.

We write $f(\hat{R}_\epsilon^1, \grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}}) = \mathbf{x}_{\epsilon,\epsilon''} = (\grave{x}_{\epsilon,\epsilon''}^1, \ldots, \grave{x}_{\epsilon,\epsilon''}^N)$. Similar to the discussion above, we now have that $\grave{x}_{\epsilon,\epsilon''}^1 (\neq \hat{x}_\epsilon^1)$ is on the same ray as $\hat{x}_\epsilon^1$ and $\overline{G^1(\grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}})}$ is flat in a neighborhood of $\grave{x}_{\epsilon,\epsilon''}^1$ when ϵ'' is sufficiently small.

We write $f(\acute{R}_\epsilon^1, \grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}}) = \dot{\mathbf{x}}_{\epsilon,\epsilon''} = (\dot{x}_{\epsilon,\epsilon''}^1, \ldots, \dot{x}_{\epsilon,\epsilon''}^N)$. As $\epsilon'' \to 0$, we have $\grave{R}_{\epsilon''}^2 \to \bar{R}^2$, and hence $\dot{\mathbf{x}}_{\epsilon,\epsilon''} \to \acute{\mathbf{x}}_{\epsilon,\epsilon''}$. Therefore, when ϵ'' is sufficiently small, $\dot{x}_{\epsilon,\epsilon''}^2$ is on the flat part of $\overline{G^2(\acute{R}_\epsilon^1, \mathbf{R}^{-\{1,2\}})}$ in a neighborhood of $\acute{x}_\epsilon^2$. As $\epsilon \to 0$, both $\hat{R}_\epsilon^1$ and $\acute{R}_\epsilon^1$ converges to $\bar{R}^1$ and, hence, both preferences become closer. Therefore, when ϵ is sufficiently small, $\dot{x}_{\epsilon,\epsilon''}^1$ is sufficiently close to $\grave{x}_{\epsilon,\epsilon''}^1$ and it is in the flat part of $\overline{G^1(\grave{R}_{\epsilon''}^2, \mathbf{R}^{-\{1,2\}})}$ in a neighborhood of $\grave{x}_{\epsilon,\epsilon''}^1$.

Now we consider four allocations $\hat{\mathbf{x}}_\epsilon$, $\acute{\mathbf{x}}_\epsilon$, $\mathbf{x}_{\epsilon,\epsilon''}$, and $\dot{\mathbf{x}}_{\epsilon,\epsilon''}$ with sufficiently small ϵ and ϵ''. Clearly, these satisfy four equations: $\sum_{i=1}^N \hat{x}_\epsilon^i = \Omega$, $\sum_{i=1}^N \acute{x}_\epsilon^i = \Omega$, $\sum_{i=1}^N \grave{x}_{\epsilon,\epsilon''}^i = \Omega$, and $\sum_{i=1}^N \dot{x}_{\epsilon,\epsilon''}^i = \Omega$. As the price vector $\bar{p}$ is the same in all allocations, and the preferences of all agents, except agents 1 and 2, are unchanged, we have, for $i = 3, \ldots, N$,

$\dot{x}^i_\epsilon = a_i \hat{x}^i_\epsilon$, $\dot{x}^i_{\epsilon,\epsilon''} = b_i \hat{x}^i_\epsilon$, and $\dot{x}^i_{\epsilon,\epsilon''} = c_i \hat{x}^i_\epsilon$ with some scalars a_i, b_i, and c_i. As for agent 1, we have $\dot{x}^1_{\epsilon,\epsilon''} = t\hat{x}^1_\epsilon$ and $\dot{x}^1_{\epsilon,\epsilon''} = t\hat{x}^1_\epsilon$ with a scalar t, because $\dot{x}^1_{\epsilon,\epsilon''}$ and $\hat{x}^1_\epsilon$ are on the same ray $[g(\hat{R}^1_\epsilon, \bar{p})]$, $\dot{x}^1_{\epsilon,\epsilon''}$ and $\hat{x}^1_\epsilon$ are on the same ray $[g(\hat{R}^1_\epsilon, \bar{p})]$, and the segments $[\dot{x}^1_{\epsilon,\epsilon''}, \dot{x}^1_{\epsilon,\epsilon''}]$ and $[\hat{x}^1_\epsilon, \dot{x}^1_\epsilon]$ are both perpendicular to $\bar{p}$. Similarly, we have $\dot{x}^2_\epsilon = s\hat{x}^2_\epsilon$ and $\dot{x}^2_{\epsilon,\epsilon''} = s\dot{x}^2_{\epsilon,\epsilon''}$ with a scalar s for agent 2. Thus, we have

$$\hat{x}^1_\epsilon + \hat{x}^2_\epsilon + \hat{x}^3_\epsilon + \cdots + \hat{x}^N_\epsilon = \Omega,$$

$$\dot{x}^1_\epsilon + s\hat{x}^2_\epsilon + a_3\hat{x}^3_\epsilon + \cdots + a_N\hat{x}^N_\epsilon = \Omega,$$

$$t\hat{x}^1_\epsilon + \dot{x}^2_{\epsilon,\epsilon'} + b_3\hat{x}^3_\epsilon + \cdots + b_N\hat{x}^N_\epsilon = \Omega,$$

$$t\dot{x}^1_\epsilon + s\dot{x}^2_{\epsilon,\epsilon'} + c_3\hat{x}^3_\epsilon + \cdots + c_N\hat{x}^N_\epsilon = \Omega.$$

From the first and second equations, we have

$$s\hat{x}^1_\epsilon - \dot{x}^1_\epsilon + (s - a_3)\hat{x}^3_\epsilon + \cdots + (s - a_N)\hat{x}^N_\epsilon = (s - 1)\Omega.$$

From the third and fourth equations, we have

$$t\left(s\hat{x}^1_\epsilon - \dot{x}^1_\epsilon\right) + (sb_3 - c_3)\hat{x}^3_\epsilon + \cdots + (sb_N - c_N)\hat{x}^N_\epsilon = (s - 1)\Omega.$$

Because of the independence of the consumption-direction vectors and $\hat{x}^1_\epsilon \neq 0$, Ω is not in the linear space spanned by $\hat{x}^3_\epsilon, \ldots, \hat{x}^N_\epsilon$. Thus, these two equations hold only if $s = 1$ or $t = 1$. However, this contradicts that $\dot{x}^1_{\epsilon,\epsilon''} \neq \hat{x}^1_\epsilon$ and $\dot{x}^2_\epsilon \neq \hat{x}^2_\epsilon$.

A.9 *Proof of Proposition 4*

As in the proof of Proposition 2, we only have to prove that if the consumption-direction vectors are independent at $\bar{\mathbf{R}}$ and if $f^i(\bar{\mathbf{R}}) \in \text{int } A$, then the statement of the proposition holds at the preference profile $\bar{\mathbf{R}}$.

   Without loss of generality, we prove the statement for agent 1. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \ldots, N$, are independent at $\bar{\mathbf{R}} = (\bar{R}^1, \ldots, \bar{R}^N)$ and that $f^1(\bar{\mathbf{R}}) \in \text{int } A$. We write $f(\bar{\mathbf{R}}) = \bar{\mathbf{x}} = (\bar{x}^1, \ldots, \bar{x}^N)$ and $\bar{p} = p(\bar{\mathbf{R}}, f)$.

   As shown in Proposition 2, $\bar{x}^1 + \bar{p}^\perp$ is a hyperplane tangent to $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ at x^1, and $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ is in the lower left-hand side of this hyperplane. We suppose that $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ has an edge at $\bar{x}^1$ and that there exists another hyperplane tangent to $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ at $\bar{x}^1$ with a normal vector $\tilde{p}$ different from $\bar{p}$. We show a contradiction.

   As $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ is in the lower left-hand side of $\bar{x}^1 + \bar{p}^\perp$, and in the lower left-hand side of $\bar{x}^1 + \tilde{p}^\perp$, any hyperplane $\bar{x}^1 + (t\bar{p}, +(1 - t)\tilde{p})^\perp$, $t \in [0, 1]$, is tangent to $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ at $\bar{x}^1$

   We let $\epsilon > 0$ be a small scalar. For each ϵ, we pick a preference $\tilde{R}^1_\epsilon$ in the ϵ-neighborhood $B_\epsilon(\bar{R}^1)$ of $\bar{R}^1$ such that the gradient vector of $\tilde{R}^1_\epsilon$ at $\bar{x}^1$ is different from $\bar{p}$ and $\bar{x}^1$ is the most preferred consumption bundle in $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ with respect to $\tilde{R}^1_\epsilon$. The existence of such a preference should be clear from the above discussion. We let $\tilde{p}_\epsilon$ denote the gradient vector of $\tilde{R}^1_\epsilon$ at $\bar{x}^1$: $\tilde{p}_\epsilon = p(\tilde{R}^1_\epsilon, \bar{x}^1)$. We write $f(\tilde{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}) = \tilde{\mathbf{x}}_\epsilon = (\tilde{x}^1_\epsilon, \ldots, \tilde{x}^N_\epsilon)$. It is clear that $\tilde{x}^1_\epsilon = \bar{x}^1$ and $\tilde{p}_\epsilon = p((\tilde{R}^1_\epsilon, \bar{\mathbf{R}}^{-1}), f)$. As $\epsilon \to 0$, $\tilde{R}_\epsilon \to \bar{R}^1$, $\tilde{p}_\epsilon \to \bar{p}$, and $\tilde{\mathbf{x}}_\epsilon \to \bar{\mathbf{x}}$.
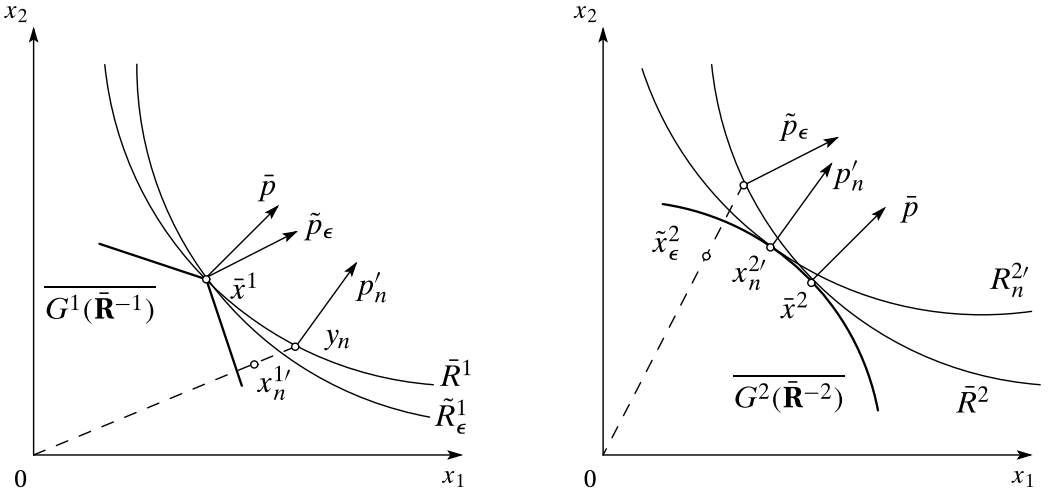
FIGURE 7. Proof of Proposition 4.

Figure 7 describes the situation where the closure of agent 1's option set has an edge at $\bar{x}^1$, contrary to the statement of the proposition. In the following proof we observe that this induces allocations that contradict the pseudo-efficiency and strategy-proofness of the social choice function f.

As $f^1(\bar{\mathbf{R}}) \in \text{int } A$ and $f^i(\bar{\mathbf{R}}) \in \text{int } A \cup \{0, \Omega\}$ for any i as observed at the end of Section 3, there exists another agent receiving positive consumption in int A. Without loss of generality, we assume that agent 2 is such an agent: $f^2(\bar{\mathbf{R}}) \in \text{int } A$.

As $\bar{R}^1$ and $\tilde{R}_\epsilon^1$ have different gradient vectors at $\bar{x}^1$, $\text{P}(\bar{x}^1; \tilde{R}_\epsilon^1) \setminus \text{UC}(\bar{x}^1; \bar{R}^1) \neq \varnothing$. Then there exists a consumption bundle y in any neighborhood of $\bar{x}^1$ such that y is indifferent to $\bar{x}^1$ with respect to $\bar{R}^1$ and y is strictly preferred to $\bar{x}^1$ with respect to $\tilde{R}_\epsilon^1$. We let $\{y_n\}_{n=1}^\infty$ be a sequence of such consumption bundles converging to $\bar{x}^1$ as $n \to \infty$: $y_n \in \text{I}(\bar{x}^1; \bar{R}^1)$, $y_n \in \text{P}(\bar{x}^1; \tilde{R}_\epsilon^1)$, and $y_n \to \bar{x}^1$ as $n \to \infty$. We let p_n' denote the gradient vector of $\bar{R}^1$ at y_n: $p_n' = p(\bar{R}^1, y_n)$.

We focus on agent 2. With each sufficiently large n, we let $x_n^{2'}$ be a consumption vector on $\overline{G^2(\bar{\mathbf{R}}^{-2})}$ such that the hyperplane $x_n^{2'} + p_n'^\perp$ is tangent to $\overline{G^2(\bar{\mathbf{R}}^{-2})}$ at $x_n^{2'}$. Such a consumption vector $x_n^{2'}$ is obtained uniquely because $\bar{x}^2$ is the unique intersection between $\overline{G^2(\bar{\mathbf{R}}^{-2})}$ and $\bar{x}^1 + \bar{p}^\perp$, as shown in Proposition 3, and $p_n' \to \bar{p}$ as $n \to \infty$. We have $x_n^{2'} \to \bar{x}^2$ as $n \to \infty$.[10]

We let $R_n^{2'}$ be agent 2's preference that has gradient vector p_n' at $x_n^{2'}$ and converges to $\bar{R}^2$ as $n \to \infty$. Such a preference can be obtained by directly applying the preference construction in the first part of Lemma 1 so that $\bar{R}$, $(\bar{x}, \bar{p})$, and (x_n, p_n) in Lemma 1 correspond to $\bar{R}^2$, $(\bar{x}^2, \bar{p})$, and $(x_n^{2'}, p_n')$, respectively, in the present setup. It is clear that with respect to $R_n^{2'}$, $x_n^{2'}$ is the most preferred consumption bundle in $\overline{G^2(\bar{\mathbf{R}}^{-2})}$, and, hence, $f^2(R_n^{2'}, \bar{\mathbf{R}}^{-2}) = x_n^{2'}$ and $p((R_n^{2'}, \bar{\mathbf{R}}^{-2}), f) = p_n^{2'}$. We write $f(R_n^{2'}, \bar{\mathbf{R}}^{-2}) = \mathbf{x}_n' = (x_n^{1'}, \dots, x_n^{N'})$. Observe that $x_n^{1'}$ is on the ray $[y_n]$.

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[10]The term $\overline{G^2(\bar{\mathbf{R}}^{-2})}$ may also have an edge at $\bar{x}^2$ and $x_n^{2'}$ may be $\bar{x}^2$.

For each sufficiently small ϵ, there exists a sufficiently large n, and we can have agent 2's preference $\hat{R}^2_{\epsilon,n}$ in B^2 such that (I) the gradient vector of $\hat{R}^2_{\epsilon,n}$ at $\tilde{x}^2_\epsilon$ is $\tilde{p}_\epsilon$, and (II) the gradient vector of $\hat{R}^2_{\epsilon,n}$ at $x^{2\prime}_n$ is p'_n. To obtain such a preference, we directly apply the preference construction in Lemma 1 so that $\bar{R}$, $(\bar{x}, \bar{p})$, and (x_n, p_n) in Lemma 1 correspond to $\bar{R}^2$, $(\tilde{x}^2_\epsilon, \tilde{p}_\epsilon)$, and $(x^{2\prime}_n, p'_n)$, respectively, in the present setup.

We observe that we can apply the preference construction in Lemma 1. As $\epsilon \to 0$, $\tilde{x}^2_\epsilon \to \bar{x}^2$ and $\tilde{p}_\epsilon \to \bar{p}$. As $n \to \infty$, $x^{2\prime}_n \to \bar{x}^2$ and $p^{2\prime}_n \to \bar{p}$. Therefore, when ϵ is sufficiently small and n is sufficiently large, $x^{2\prime}_n$ and p'_n are sufficiently close to $\tilde{x}^2_\epsilon$ and $\tilde{p}_\epsilon$, respectively, and, hence, we can apply Lemma 1. We set a sufficiently small ϵ. The strict convexity of $\bar{R}^2$ ensures that $\bar{p}([\tilde{x}^2_\epsilon] \cap \mathrm{I}(\bar{x}^2; \bar{R}^2)) > \bar{p}\bar{x}^2$. As $n \to \infty$, $x^{2\prime}_n \to \bar{x}^2$ and $p^{2\prime}_n \to \bar{p}$. Therefore, we have the condition in Lemma 1, $p'_n([\tilde{x}^2_\epsilon] \cap \mathrm{I}(x^{2\prime}_n; \bar{R}^2)) > p'_n x^{2\prime}_n$ in the present setup when n is sufficiently large.

We write $f(\hat{R}^2_{\epsilon,n}, \bar{\mathbf{R}}^{-2}) = \hat{\mathbf{x}}_{\epsilon,n} = (\hat{x}^1_{\epsilon,n}, \dots, \hat{x}^N_{\epsilon,n})$ and $f(\tilde{R}^1_\epsilon, \hat{R}^2_{\epsilon,n}, \bar{\mathbf{R}}^{-\{1,2\}}) = \check{\mathbf{x}}_{\epsilon,n} = (\check{x}^1_{\epsilon,n}, \dots, \check{x}^N_{\epsilon,n})$. We have $\check{x}^2_{\epsilon,n} = \tilde{x}^2_\epsilon$ because of (I) and, hence, $\check{\mathbf{x}}_{\epsilon,n} = \tilde{\mathbf{x}}_\epsilon$ by Lemma 2. Alternatively, we have $\hat{x}^2_{\epsilon,n} = x^{2\prime}_n$ because of (II) and, hence, $\hat{\mathbf{x}}_{\epsilon,n} = \mathbf{x}'_n$. In particular, note that $\hat{x}^1_{\epsilon,n} = x^{1\prime}_n$ and, hence, $\hat{x}^1_{\epsilon,n}$ is on the ray $[y_n]$.

We first consider the case where $\hat{x}^1_{\epsilon,n} \in \mathrm{P}(\bar{x}^1; \tilde{R}^1_\epsilon)$. If agent 1 has the preference $\tilde{R}^1_\epsilon$ and faces other agents' preferences $(\hat{R}^2_{\epsilon,n}, \bar{\mathbf{R}}^{-\{1,2\}})$, he is better off by reporting $\bar{R}^1$ and achieving $\hat{x}^1_{\epsilon,n}$ than reporting the true preference $\tilde{R}^1_\epsilon$ and achieving $\check{x}^1_{\epsilon,n} = \tilde{x}^1_\epsilon = \bar{x}^1$. This contradicts the strategy-proofness of f on $\mathbf{B}$.

Next, we consider the case $\hat{x}^1_{\epsilon,n} \notin \mathrm{UC}(\bar{x}^1; \bar{R}^1)$. If agent 1 has preference $\bar{R}^1$ and faces other agents' preferences $(\hat{R}^2_{\epsilon,n}, \bar{\mathbf{R}}^{-\{1,2\}})$, he is better off by reporting $\tilde{R}^1_\epsilon$ and achieving $\check{x}^1_{\epsilon,n} = \tilde{x}^1_\epsilon = \bar{x}^1$ than reporting his true preference $\bar{R}^1$ and achieving $\hat{x}^1_{\epsilon,n}$. Again, this contradicts the strategy-proofness of f on $\mathbf{B}$.

As $\hat{x}^1_{\epsilon,n} \in [y'_\epsilon]$, where $y_n \in \mathrm{I}(\bar{x}^1; \bar{R}^1)$ and $y_n \in \mathrm{P}(\bar{x}^1; \tilde{R}^1_\epsilon)$, only these two cases need to be considered.

A.10 *Proof of Proposition 5*

Without loss of generality, we prove the proposition for agent 1. We suppose that $g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \dots, N$, are independent at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N)$ and that $f^1(\bar{\mathbf{R}}) \in \mathrm{int}\, A$. We write $f(\bar{\mathbf{R}}) = \mathbf{x} = (\bar{x}^1, \dots, \bar{x}^N)$ and $p(\bar{\mathbf{R}}, f) = \bar{p}$.

We consider the Cobb–Douglas utility functions $u^\alpha(x) = x_1^{\alpha_1} \cdots x_L^{\alpha_L}$ with parameter $\alpha = (\alpha_1, \dots, \alpha_L) \in S^{L-1}_{++}$ and the preferences represented by these utility functions.

Observe that the gradient vector of the preferences represented by the utility function $u^\alpha(x)$ at a consumption bundle $x = (x_1, \dots, x_L)$ is given by the normalization of $(\frac{\alpha_1}{x_1}, \dots, \frac{\alpha_L}{x_L})$. Alternatively, if a preference represented by a Cobb–Douglas utility function $u^\alpha(x)$ has gradient vector $p = (p_1, \dots, p_L)$ at $x = (x_1, \dots, x_L)$, then the parameter α is the normalization of $(p_1 x_1, \dots, p_L x_L)$.

We let $u^{\alpha^*}(x)$ be the Cobb–Douglas utility function such that agent 1's preference $R^{1\alpha^*}$ represented by $u^{\alpha^*}(x)$ has gradient vector $\bar{p}$ at $\bar{x}^1$.

We consider a preference $R^{1\alpha}$ represented by a Cobb–Douglas utility function u^α, where α is in a neighborhood of α^*. Then, of course, $R^{1\alpha}$ is in a neighborhood of $R^{1\alpha^*}$.

As f is supposed to satisfy pseudo-efficiency and strategy-proofness only on $\mathbf{B}$, we let $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$ denote the most preferred consumption in $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ with respect to the preference $R^{1\alpha}$. If $R^{1\alpha} \in B^1$, then, of course, $f(R^{1\alpha}, \bar{\mathbf{R}}^{-1}) = \tilde{f}(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$.

We have $\tilde{f}(R^{1\alpha^*}, \bar{\mathbf{R}}^{-i}) = \bar{\mathbf{x}}$ and $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-i})$ is in a neighborhood of $\bar{x}^1$ when α is in a neighborhood of α^* because of the properties of $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ shown in Propositions 2–4. Observe that $\alpha \neq \alpha'$ implies that $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1}) \neq \tilde{f}^1(R^{1\alpha'}, \bar{\mathbf{R}}^{-1})$ because $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1}) = \tilde{f}^1(R^{1\alpha'}, \bar{\mathbf{R}}^{-1})$ and $\alpha \neq \alpha'$ imply that $R^{1\alpha}$ and $R^{1\alpha'}$ have different gradient vectors at the same consumption bundle, which contradicts Proposition 4. Thus, each $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$ in a neighborhood of $\bar{x}^1$ is identified with the corresponding parameter $\alpha \in S_{++}^{L-1}$ in a neighborhood of α^* and, hence, in a neighborhood of $\bar{x}^1$, $\bigcup_\alpha \tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$ is an $L-1$-dimensional manifold.

To end the proof, we prove that in a neighborhood of $\bar{x}^1$, $\bigcup_\alpha \tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$, $G^1(\bar{\mathbf{R}}^{-1})$, and $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ coincide. We let $\tilde{p}_\alpha$ denote the gradient vector of $R^{1\alpha}$ at $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$: $\tilde{p}_\alpha = p(R^{1\alpha}, \tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1}))$. When α is in a neighborhood of α^* and $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$ is in a neighborhood of $\bar{x}^1$, we construct a new preference $\tilde{R}^1 \in B^1$ such that the gradient vector of $\tilde{R}^1$ at $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$ is $\tilde{p}_\alpha$. For example such a preference $\tilde{R}^1$ can be obtained by applying the preference construction in the first part of Lemma 1 so that $\bar{R}$, $(\bar{x}, \bar{p})$, and (x_n, p_n) in Lemma 1 correspond to $\tilde{R}^1$, $(\bar{x}^1, \bar{p})$, and $(\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1}), \tilde{p}_\alpha)$, respectively, in the present setup. As α converges to α^*, $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$ converges to $\bar{x}^1$ and $\tilde{p}_\alpha$ converges to $\bar{p}$, and, hence, Lemma 1 is applicable and $\tilde{R}^1$ converges to $\bar{R}^1$, as shown in the lemma.

Then $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$, $\tilde{p}_\alpha$) is the most preferred consumption in $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ with respect to $\tilde{R}^1 \in B^1$, and $\tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1}) = f(\tilde{R}^1, \bar{\mathbf{R}}^{-1}) \in G^1(\bar{\mathbf{R}}^{-1})$, as in Lemma 3. Therefore, we have $\bigcup_\alpha \tilde{f}(R^{1\alpha}, \bar{\mathbf{R}}^{-1}) \subset G^1(\bar{\mathbf{R}}^{-1}) \subset \overline{G^1(\bar{\mathbf{R}}^{-1})}$ in a neighborhood of $\bar{x}^1$. To observe that any element of $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ in a neighborhood of $\bar{x}^1$ is included in $\bigcup_\alpha \tilde{f}^1(R^{1\alpha}, \bar{\mathbf{R}}^{-1})$, note that any ray $[y]$ in a neighborhood of $[\bar{x}^1]$ intersects with $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ once at the most because of strategy-proofness and, hence, $\overline{G^1(\bar{\mathbf{R}}^{-1})}$ is at most $L-1$ dimensional.

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