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Ranking by rating

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Ranking by rating consists in evaluating the performances of items using exogenous rating functions and ranking these items according to their performance ratings. Any such method is separable: the ordering of two items does not depend on the performances of the remaining items. When performances belong to a finite set, ranking by rating is characterized by separability and a property of consistency; this characterization generalizes to the infinite case under a continuity axiom. Consistency follows from separability and symmetry or from monotonicity alone. When performances are vectors in $\mathbb{R}^m_+$, a separable, symmetric, monotonic, continuous, and invariant method must rank items according to a weighted geometric mean of their performances along the m dimensions.

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1. Introduction

Rankings are ubiquitous: we rank products and services such as cars, restaurants, scientific journals, web pages, songs; people such as athletes, students, chess players; institutions and groups such as schools, universities, academic departments, football teams, and even cities or countries.

Two types of ranking methods are widely used. Under the simplest and more traditional ones, an item's performance is rated using a set of exogenously specified and weighted criteria, and the items are then ranked according to their ratings. This is, for instance, how a ranking of students is usually computed from their performances at an exam. Under more sophisticated methods, the weights of the criteria used to evaluate an item's performance vary with the performances of all items. This is typically how academic journals are ranked according to the citations they receive from other journals, or web pages according to how they are linked to other pages: a reference from a highly ranked journal or web page carries an endogenously greater weight.

Methods of the first type are *separable*: the ranking of two items does not depend on the performances of the remaining items. This offers a guarantee of transparency, which probably accounts for the popularity of these methods. Methods of the second type are not separable.

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In the current note, we are exclusively concerned with separable methods. Our formal model has n items, each of which is characterized by a possibly different set of conceivable performances. No a priori structure is imposed on this set. A ranking method is modelled as a function that computes an ordering of the items for every performance profile. We call *ranking by rating* the class of methods where each item's performance is evaluated using an exogenous rating function defined over the set of its possible performances, and the items are ranked according to the resulting performance ratings. The main question we ask is whether all separable methods are of this type, and, if not, under which conditions that may be the case.

Our results are rather elementary and perhaps folk knowledge, but were, to the best of our knowledge, in need of a proof. We show in Section 3 that there exist separable methods other than ranking by rating. Those methods need not be degenerate and can be quite flexible; their range may include all the linear orderings of the items.

Theorem 1 in Section 4 shows that if the performance sets are finite, a ranking method is ranking by rating if and only if it is separable and consistent. Consistency here means that if a change in an item's performance improves its relative ranking against some other item at a given profile, the same change does not decrease its relative ranking against any item at any profile. Corollary 1 extends Theorem 1 to the infinite case under an added continuity requirement.

Section 5 identifies two natural conditions under which a separable method is necessarily consistent. Symmetry, which applies when the performance sets of all items coincide, requires that permuting the performances of two items results in permuting their positions in the ranking. Monotonicity applies when each performance set is ordered, and requires that a higher performance improves an item's position in the ranking. Theorem 2 shows that, in the finite case, a separable ranking method that is symmetric *or* monotonic is of the ranking-by-rating type, and Corollary 2 extends this conclusion to the infinite case under continuity.

Section 6 studies the particular case of our model where an items' possible performances are multidimensional and partially ordered: they are represented by vectors in $\mathbb{R}^m_+$. If the partial performances along the various dimensions are measured in noncomparable units, the ranking should not change when the partial performances of all items along a given dimension are multiplied by the same positive number. We show that a separable, symmetric, monotonic, and continuous method satisfying this invariance condition must rank items according to a weighted geometric mean of their performances along the m dimensions. Section 7 argues that, from an axiomatic viewpoint, the simple geometric mean method is a serious competitor of the nonseparable methods proposed in the literature.

2. Related literature

A sizable literature addresses the problem of characterizing separable orderings defined over a set of multidimensional alternatives such as a subset of $\mathbb{R}^m_+$. Separability, in that literature, means that the ordering of two alternatives whose coordinates coincide along one dimension does not change with the value of that coordinate. The seminal contribution is that of Gorman (1968), who shows that, under suitable (and important) topological assumptions, such an ordering can be represented by an additively separable function. Bradley et al. (2005) show that Gorman's result does not carry over to the finite case, and they study properties of discrete separable orderings. Despite a formal similarity, our work is essentially unrelated to that literature. Even when the sets of possible performances of the n items are infinite, we are interested in ordering only the finite sets containing precisely n performances, one for each item. We want to order all such sets, and our separability condition is precisely a restriction on how these different rankings should be related: the ordering of two performances should not depend on what the remaining performances are.

Our separability condition *is* closely related to Arrow's (1951) axiom of Independence of Irrelevant Alternatives and its weakening by Hansson (1973). Arrow's aggregation problem, however, cannot be rephrased as a ranking problem of the type we analyze. To be sure, candidates (or social alternatives) may be regarded as items, and each candidate's performance may be defined as the list of ranks he occupies in the preferences of the voters. But the set of possible performance profiles is *not* a Cartesian product: two candidates cannot both be ranked first by the same voter. Likewise, our separability condition is related to the independence condition used by Rubinstein (1980) to axiomatize the Copeland ranking method for tournaments, but the problem of ranking the participants in a tournament also lacks the Cartesian product structure imposed by our model.

A huge literature deals with the multidimensional submodel discussed in Section 6 and, more specifically, with the case where items and dimensions coincide: this is indeed a suitable framework to discuss the popular issue of how to rank academic journals or web pages. In contrast to the current paper, that literature is concerned with *nonseparable* methods and is, therefore, only tangentially related to our work.[1]

Two classes of methods have received considerable attention. The first class consists of variants of the eigenvector solution based on the Perron–Frobenius theorem popularized by Landau (1895), Wei (1952), Kendall (1955), Berge (1958), Keener (1993), and others; see Vigna (2009) for a survey. Under all such methods, the ranking of the items is determined by their coordinates of the Perron vector of some irreducible nonnegative matrix, but the methods differ in how they construct this matrix from the performance data. For the problem of ranking web pages, examples include the PageRank method (Brin and Page 1998), the HITS (Hyperlink-Induced Topic Search) method (Kleinberg 1999), and the SALSA (Stochastic Approach for Link- Structure Analysis) method (Lempel and Moran 2000); see Fercoq (2012) for a survey. For the problem of ranking journals, examples include the method of Leibowitz and Palmer (1984) and the intensity-invariant modification of Pinski and Narin (1976). Axiomatizations are offered by Palacios-Huerta and Volij (2004), Altman and Tennenholtz (2005), and Slutzki and Volij (2006).

The second class contains methods based on Sinkhorn's (1967) algorithm for solving the so-called matrix scaling problem. For web page ranking, examples of such methods

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[1] Another difference is that, with a few exceptions such as Altman and Tennenholtz (2005), the literature focuses on cardinal methods. Such methods compute a score for each item, and the differences in scores are deemed meaningful.

include those of Smith (2005), Knight (2008), and Govan et al. (2009). For the case where items and dimensions need not coincide, Demange (2014) proposes and axiomatizes the so-called handicap-based method.

3. SEPARABILITY

Let $N = \{1, \ldots, n\}$ be a finite set of items, $n \geq 3$. Each item $i \in N$ is characterized by a nonempty set of possible performances A_i. A *performance profile* is a list $a = (a_1, \ldots, a_n) \in A_N := \bigtimes_{i \in N} A_i$. Let $\mathcal{R}_N$ denote the set of orderings[2] on N. A *(ranking) method* is a function $R : A_N \to \mathcal{R}_N$ that assigns to each performance profile a an ordering $R(a)$ of the items. The statement $(i, j) \in R(a)$, also written $iR(a)j$, means that the method R considers i at least as strong as j when the performance profile is a. Let $P(a)$ and $I(a)$ denote, respectively, the antisymmetric and symmetric components of $R(a)$. If $R(a)$ is a linear ordering, it will sometimes be convenient to express it by listing the items according to their rank; for instance, the linear ordering $iR(a)j \Leftrightarrow i \leq j$ will be written $R(a) = 1, 2, \ldots, n$.

A method R is a *ranking-by-rating method* if there exist real-valued functions $v_1, \ldots, v_n$ defined, respectively, on $A_1, \ldots, A_n$, such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$. We call $v_1, \ldots, v_n$ *rating functions*.

If R is a ranking-by-rating method, the relative ordering of two items depends only on the performances of these items. Formally, R satisfies the following property.

Separability. For all $i, j \in N$ and $a, a' \in A_N$, $[a_i = a'_i$ and $a_j = a'_j] \Rightarrow [iR(a)j \Leftrightarrow iR(a')j]$.

The following example shows that Separability does not characterize the ranking-by-rating methods.

EXAMPLE 1. Let $N = \{1, 2, 3\}$, $A_i = \{0, 1\}$ for all $i \in N$, and

$$R(a) = \begin{cases} 1\,2\,3 & \text{if } a_1 = a_2, \\ 2\,1\,3 & \text{if } a_1 \neq a_2. \end{cases}$$

Since 3 is always ranked last, the relative ordering of 1 and 3 and the relative ordering of 2 and 3 are constant. The relative ordering of 1 and 2 varies, but it does not depend on a_3. Thus, R is separable. If the rating functions v_1, v_2, and v_3 represent R, we should have

$$\begin{aligned} 1P(0,0,0)2 &\quad\Rightarrow\quad v_1(0) > v_2(0), \\ 2P(1,0,0)1 &\quad\Rightarrow\quad v_2(0) > v_1(1), \\ 1P(1,1,0)2 &\quad\Rightarrow\quad v_1(1) > v_2(1), \\ 2P(0,1,0)1 &\quad\Rightarrow\quad v_2(1) > v_1(0). \end{aligned}$$

Since these inequalities are incompatible, R is not a ranking-by-rating method. ◊

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[2] By an ordering we mean a complete, reflexive, and transitive binary relation. If this relation is also antisymmetric, we call it a linear ordering.

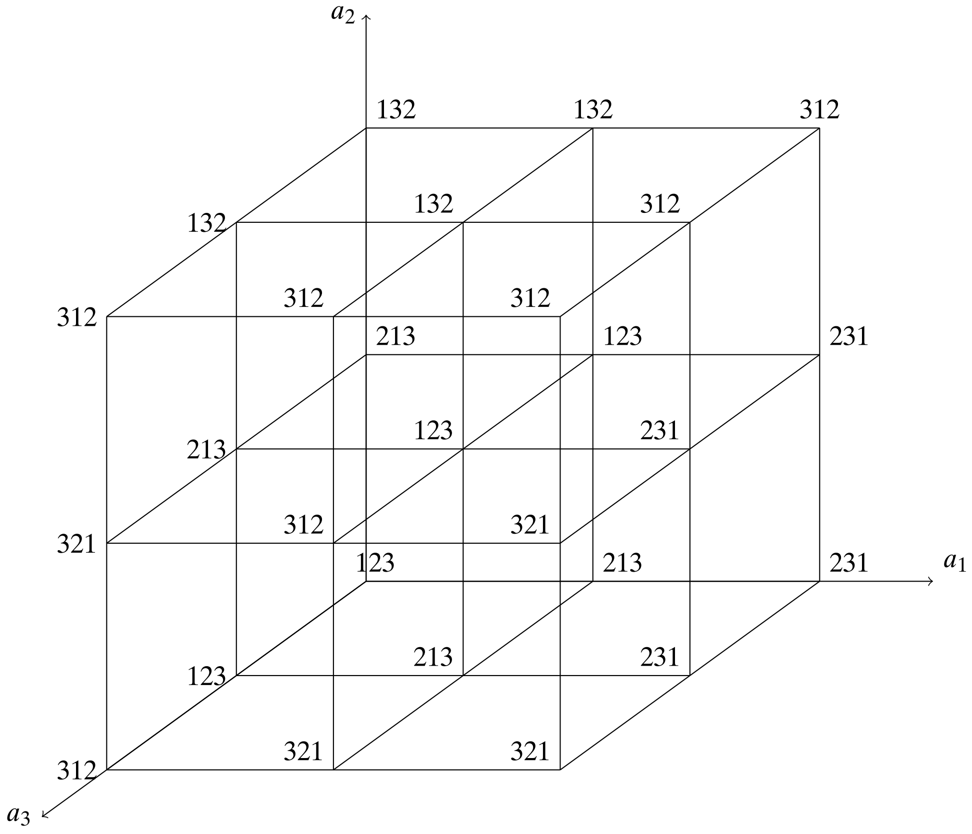

Figure 1. A full-range separable method not of the ranking-by-rating type.

In this example, the range of R is very small. But there exist separable methods whose range contains all strict orderings on N that are not ranking-by-rating methods. For instance, let $N = \{1, 2, 3\}$ and $A_i = \{0, 1, 2\}$ for all $i \in N$, and consider the method R depicted in Figure 1. It is tedious but straightforward to check that R is separable, and the same argument as above shows that it is not a ranking-by-rating method.

4. CONSISTENCY

The separable method in Example 1 is inconsistent: a change in item 1's performance from $a_1 = 0$ to $a_1 = 1$ improves that item's position in the ranking when $(a_2, a_3) = (1, 1)$, but deteriorates it when $(a_2, a_3) = (0, 0)$. We show in this section that a separable method that does not exhibit this type of inconsistency is a ranking-by-rating method provided that (i) the set of performance profiles is finite or (ii) it is a connected topological space and the ranking method is continuous.

The following notation will be useful: if $i \in N$, $a \in A$, and $\alpha \in A_i$, then (α, a_{-i}) is the performance profile obtained from a by replacing a_i with α. We write A_{-i} for $\bigtimes_{j \in N \setminus \{i\}} A_j$. The formal property relevant to our analysis is the following.

Consistency. For all $i \in N$ and $\alpha, \beta \in A_i$, if there exists $a_{-i} \in A_{-i}$ and $j \in N \setminus \{i\}$ such that $iP(\alpha, a_{-i})jR(\beta, a_{-i})i$ or $iR(\alpha, a_{-i})jP(\beta, a_{-i})i$, then there do not exist $b_{-i} \in A_{-i}$ and $k \in N \setminus \{i\}$ such that $iR(\beta, b_{-i})kR(\alpha, b_{-i})i$.

If, starting from some profile, the ranking of item i relative to j improves when i's performance switches from β to α, this reveals that the method R deems α a stronger performance than β. In that case, a switch from β to α should never deteriorate i's ranking, and, at any profile where i is tied with some item k, it should push i strictly above k.

Lemma 1. *If a ranking method $R : A_N \to \mathcal{R}_N$ is separable and consistent, then there exists an ordering $\succsim$ on $X := \{(i, \alpha) : i \in N$ and $\alpha \in A_i\}$ such that $iR(a)j \Leftrightarrow (i, a_i) \succsim (j, a_j)$ for all $i, j \in N$ and all $a \in A_N$.*

The proof is given in the Appendix. As a direct corollary to Lemma 1, we obtain the central result of this section.

Theorem 1. *Suppose that $A_1, \ldots, A_n$ are finite. A ranking method $R : A_N \to \mathcal{R}_N$ is separable and consistent if and only if it is a ranking-by-rating method.*

Proof. If there exist functions $v_1 \in \mathbb{R}^{A_1}, \ldots, v_n \in \mathbb{R}^{A_n}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$, it is straightforward to check that the method R is separable and consistent.

Conversely, if R is separable and consistent, Lemma 1 guarantees that there exists an ordering $\succsim$ on $X := \{(i, \alpha) : i \in N$ and $\alpha \in A_i\}$ such that $iR(a)j \Leftrightarrow (i, a_i) \succsim (j, a_j)$ for all $i, j \in N$ and $a \in A_N$. Because X is finite, the ordering $\succsim$ admits a numerical representation: there exists a function $V : X \to \mathbb{R}$ such that $V(i, a_i) \geq V(j, a_j) \Leftrightarrow (i, a_i) \succsim (j, a_j)$. If, for each $i \in N$, we define the function $v_i : A_i \to \mathbb{R}$ by $v_i(a_i) = V(i, a_i)$, then $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$, proving that R is a ranking-by-rating method. □

The finiteness assumption in Theorem 1 is used to ensure the representability of the ordering $\succsim$, but the result is easily adapted to the infinite case. Assume that A_N is a perfectly separable topological space[3] and suppose that the ranking method is continuous in the sense that any strict ordering of two items is robust to small changes in the performance profile.

Continuity. For all $i, j \in N$, the set $\{a \in A_N \mid iP(a)j\}$ is relatively open in A_N.

A straightforward application of Theorem II in Debreu (1954), which we omit, then delivers the following corollary to Theorem 1.

Corollary 1. *If A_N is a perfectly separable topological space, then a ranking method $R : A_N \to \mathcal{R}_N$ is separable, consistent, and continuous if and only if there exist continuous functions $v_1 \in \mathbb{R}^{A_1}, \ldots, v_n \in \mathbb{R}^{A_n}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$.*

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[3] This means that there exists a countable class of open sets such that every open set in A_N is a union of sets in that class.

5. SYMMETRY AND MONOTONICITY

Consistency may seem complicated and somewhat contrived. The current section identifies two simpler conditions, each of which guarantees that a separable method is consistent. Both arise naturally in particular cases of our model. The first condition, Symmetry, applies when the performance sets of all items coincide, that is, when $A_1 = \cdots = A_n = A$. In that case, $A_N = A^N$, and we call a method $R : A^N \to \mathcal{R}_N$ symmetric if permuting the performances of two items results in permuting their positions in the ranking.

Symmetry. For all $i, j \in N$, all $a \in A^N$, and every bijection π from N to N, $iR(a)j \Leftrightarrow \pi(i)R(\pi a)\pi(j)$, where πa is the performance profile defined by $(\pi a)_{\pi(i)} = a_i$ for all $i \in N$.

The second condition, Monotonicity, applies when each performance set A_i is endowed with a linear ordering $\geq_i$. We call a ranking method monotonic if a higher performance improves an item's position in the ranking.

Monotonicity. For all distinct $i, j \in N$ and $a, a' \in A_N$, $[iR(a)j \text{ and } a'_i >_i a_i] \Rightarrow [iP(a'_i, a_{-i})j]$.

This is a strict form of monotonicity. Note that it implies $[iR(a)j \text{ and } a_j >_j a'_j] \Rightarrow [iP(a'_j, a_{-j})j]$ for all distinct $i, j \in N$ and $a, a' \in A_N$.

The crucial observation is recorded in the following lemma.

LEMMA 2. *Let $R : A_N \to \mathcal{R}_N$ be a separable ranking method. Suppose that (i) $A_1 = \cdots = A_n = A$ and R is symmetric or (ii) $A_1, \ldots, A_n$ are linearly ordered and R is monotonic. Then R is consistent.*

PROOF. Let $R : A_N \to R_N$ be a separable ranking method.

(i) Suppose first that $A_1 = \cdots = A_n = A$ and that $R : A_N = A^N \to \mathcal{R}_N$ is symmetric. If R is not consistent, we may assume without loss of generality that there exist $\alpha, \beta \in A$, $a, b \in A^N$, and $i \in N \setminus \{1\}$ such that

$$1P(\alpha, a_{-1})2R(\beta, a_{-1})1 \quad \text{or} \quad 1R(\alpha, a_{-1})2P(\beta, a_{-1})1 \tag{1}$$

and

$$1R(\beta, b_{-1})iR(\alpha, b_{-1})1. \tag{2}$$

Case 1. $i \neq 2$. From (1) and Separability,

$$1P(\alpha, a_2, b_{-12})2R(\beta, a_2, b_{-12})1 \quad \text{or} \quad 1R(\alpha, a_2, b_{-12})2P(\beta, a_2, b_{-12})1. \tag{3}$$

From (2) and Separability,

$$1R(\beta, a_2, b_{-12})iR(\alpha, a_2, b_{-12})1. \tag{4}$$

Because $R(\alpha, a_2, b_{-12})$ and $R(\beta, a_2, b_{-12})$ are transitive, (3) and (4) imply, after some manipulations,

$$iP(\alpha, a_2, b_{-12})2R(\beta, a_2, b_{-12})i \quad \text{or} \quad iR(\alpha, a_2, b_{-12})2P(\beta, a_2, b_{-12})i,$$

which contradicts Separability.

Case 2. $i=2$. From (1) and Separability,

$$\begin{aligned} &1P(\alpha, a_2, b_2, b_{-123})2R(\beta, a_2, b_2, b_{-123})1 \quad \text{or} \\ &1R(\alpha, a_2, b_2, b_{-123})2P(\beta, a_2, b_2, b_{-123})1. \end{aligned} \tag{5}$$

From (2) and Separability, and since $i=2$,

$$1R(\beta, b_2, a_2, b_{-123})2R(\alpha, b_2, a_2, b_{-123})1.$$

Exchanging the performances of items 2 and 3, Symmetry now implies

$$1R(\beta, a_2, b_2, b_{-123})3R(\alpha, a_2, b_2, b_{-123})1. \tag{6}$$

Because $R(\alpha, a_2, b_2, b_{-123})$ and $R(\beta, a_2, b_2, b_{-123})$ are transitive, (5) and (6) imply, after some manipulations,

$$3P(\alpha, a_2, b_2, b_{-123})2R(\beta, a_2, b_2, b_{-123})3 \quad \text{or} \quad 3R(\alpha, a_2, b_2, b_{-123})2P(\beta, a_2, b_2, b_{-123})3,$$

which contradicts Separability again.

(ii) Suppose next that each performance set A_i is endowed with a linear order $\geq_i$ and that $R: A_N \to \mathcal{R}_N$ is monotonic. This case is straightforward. If R is not consistent, we may assume without loss of generality that there exist $\alpha, \beta \in A_1$, $a, b \in A_N$, and $i \in N \setminus \{1\}$ such that (1) and (2) hold. Clearly, $\alpha \neq \beta$. Since $\geq_1$ is a linear ordering, either $\alpha >_1 \beta$ or $\beta >_1 \alpha$. If $\alpha >_1 \beta$, then (2) contradicts Monotonicity. If $\beta >_1 \alpha$, then (1) contradicts Monotonicity. ☐

Notice that Separability is not used in part (ii) of the proof: we have, in fact, proved that every monotonic method, separable or not, is consistent.

In the finite case, Theorem 1 and Lemma 2 imply that a separable method that is symmetric or monotonic is a ranking-by-rating method. Of course, Symmetry and Monotonicity translate into restrictions on the rating functions: under Symmetry these functions must all coincide; under Monotonicity they must be increasing. When the performance sets are infinite, Corollary 1 and Lemma 2 deliver a similar representation with continuous rating functions. These results are recorded below; we omit their obvious proofs.

THEOREM 2. *(i) If $A_1 = \cdots = A_n = A$ is a finite set, then a ranking method $R: A^N \to \mathcal{R}_N$ is separable and symmetric if and only if there exists a function $v: A \to \mathbb{R}$ such that $iR(a)j \Leftrightarrow v(a_i) \geq v(a_j)$ for all $i, j \in N$ and all $a \in A^N$.*

(ii) If $A_1, \ldots, A_n$ are linearly ordered finite sets, then a ranking method $R: A_N \to \mathcal{R}_N$ is separable and monotonic if and only if there exist increasing functions $v_1 \in \mathbb{R}^{A_1}, \ldots, v_n \in \mathbb{R}^{A_n}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$.

Corollary 2. *(i) If* $A_1 = \cdots = A_n = A$ *is a perfectly separable topological space, then a ranking method* $R : A^N \to \mathcal{R}_N$ *is separable, symmetric, and continuous if and only if there exists a continuous function* $v : A \to \mathbb{R}$ *such that* $iR(a)j \Leftrightarrow v(a_i) \geq v(a_j)$ *for all* $i, j \in N$ *and all* $a \in A^N$.

(ii) If A_N *is a perfectly separable topological space and each of* $A_1, \dots, A_n$ *is endowed with a linear order, then a ranking method* $R : A_N \to \mathcal{R}_N$ *is separable, monotonic, and continuous if and only if there exist increasing and continuous functions* $v_1 \in \mathbb{R}^{A_1}, \dots, v_n \in \mathbb{R}^{A_n}$ *such that* $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ *for all* $i, j \in N$ *and all* $a \in A_N$.

In statement (ii) of Corollary 2, each item's performance set is assumed to be *completely* ordered. The result does not extend to the case where these sets are only partially ordered and the method is assumed to be monotonic with respect to that partial order.

Example 2. Let $N = \{1, 2, 3\}$ and $A_i = A = [0, 1]^2$ for all $i \in N$. Endow A with the usual partial order $\geq$. A generic performance profile is a vector $a = (a_1, a_2, a_3) = ((a_1^1, a_1^2), (a_2^1, a_2^2), (a_3^1, a_3^2)) \in A^{\{1,2,3\}}$. Define the functions w_1, w_2, and w_3 from $A^{\{1,2,3\}}$ to $\mathbb{R}$ by

$$\begin{aligned} w_1(a) &= \left(1 - a_2^2\right)a_1^1 + \left(1 - a_2^1\right)a_1^2, \\ w_2(a) &= \frac{1}{2}a_2^1 + \frac{1}{2}a_2^2, \\ w_3(a) &= -1 \end{aligned}$$

for all $a \in A^{\{1,2,3\}}$. Note that $w_1(a)$ varies with a_2. Define the method R by

$$iR(a)j \Leftrightarrow w_i(a) \geq w_j(a)$$

for all $a \in A^{\{1,2,3\}}$ and all $i, j \in \{1, 2, 3\}$. Since $w_1(a), w_2(a) \geq 0$ for all a, item 3 is ranked last at every performance profile. Moreover, the ranking of items 1 and 2 does not change with 3's performance. So R is separable. Since w_1, w_2, and w_3 are continuous, R is also continuous. Furthermore, it is monotonic because w_1 is increasing in a_1 and w_2 is increasing in a_2.

This method is not a ranking-by-rating method. By definition of w_1, w_2, w_3, and R,

$$\begin{aligned} &1P\big((1, 0), (1, 0), (0, 0)\big)2 \\ &2P\big((0, 1), (1, 0), (0, 0)\big)1 \\ &1P\big((0, 1), (0, 1), (0, 0)\big)2 \\ &2P\big((1, 0), (0, 1), (0, 0)\big)1. \end{aligned}$$

If v_1, v_2, and v_3 were rating functions from A to $\mathbb{R}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq w_j(a_j)$ for all $a \in A^{\{1,2,3\}}$ and all $i, j \in \{1, 2, 3\}$, then

$$v_1(1, 0) > v_2(1, 0)$$

$$
\begin{aligned}
v_2(1,0) &> v_1(0,1)\\
v_1(0,1) &> v_2(0,1)\\
v_2(0,1) &> v_1(1,0),
\end{aligned}
$$

which are incompatible inequalities. ◊

6. INVARIANCE

This section studies the case where the performance sets of the items coincide and are endowed with a partial order structure. More precisely, we assume that there is a finite set of criteria $M = \{1, \dots, m\}$ and $A_i = A = \mathbb{R}^M_{++}$ for each item $i \in N$. A generic performance for item i is a vector $a_i = (a_i^1, \dots, a_i^m) \in A$. A performance profile is a matrix $a = (a_i^h) \in A^N$: rows correspond to items, columns correspond to criteria, and the number a_i^h measures item i's performance according to criterion h. We write $b_i > a_i$ if $b_i^h \geq a_i^h$ for all $h \in M$ and $b_i \neq a_i$. With a slight abuse of our earlier terminology, we now call R monotonic if $[iR(a)j$ and $a'_i > a_i] \Rightarrow [iP(a'_i, a_{-i})j]$ for all distinct $i, j \in N$ and $a, a' \in A_N$.

An important concern in this multidimensional framework is expressed by the condition of Invariance. The condition says that the ordering of the items should remain unchanged when their performances according to a given criterion are all multiplied by the same positive number. This is compelling if performances are measured on noncomparable scales across criteria.

To express the condition formally, we use the following notation. For every $\lambda = (\lambda^1, \dots, \lambda^m) \in \mathbb{R}^M_{++}$, let $dg(\lambda)$ denote the $m \times m$ diagonal matrix whose hth diagonal entry is λ^h. With this notation, $a \cdot dg(\lambda)$ is the performance matrix obtained by multiplying each column h of a by λ^h.

Invariance. For all $a \in A^N$ and $\lambda \in \mathbb{R}^M_{++}$, $R(a \cdot dg(\lambda)) = R(a)$.

Let Δ^M_{++} denote the relative interior of the unit simplex of $\mathbb{R}^M$.

THEOREM 3. *Let $A_1 = \cdots = A_n = A = \mathbb{R}^M_{++}$. A ranking method $R : A^N \to \mathcal{R}_N$ is separable, symmetric, monotonic, continuous, and invariant if and only if there exists $\beta = (\beta^1, \dots, \beta^m) \in \Delta^M_{++}$ such that*

$$
iR(a)j \Leftrightarrow \prod_{h \in M} \left(a_i^h\right)^{\beta^h} \geq \prod_{h \in M} \left(a_j^h\right)^{\beta^h} \quad \textit{for all } i, j \in N \textit{ and all } a \in A^N. \tag{7}
$$

PROOF. The "if" statement is clear. The proof of the "only if" statement is a straightforward consequence of Corollary 2 and Osborne's (1976) characterization of the monotonic transformations of the weighted geometric means.

Fix a separable, symmetric, monotonic, continuous, and invariant method R. By statement (i) in Corollary 2, there exists a continuous function $w : \mathbb{R}^M_{++} \to \mathbb{R}$ such that

$$
iR(a)j \Leftrightarrow w(a_i) \geq w(a_j) \quad \text{for all } i, j \in N \text{ and all } a \in A^N. \tag{8}
$$

Since R is monotonic, w is increasing: $a_i < a_j \Rightarrow w(a_i) < w(a_j)$. Because R is invariant, w is ordinally invariant in the sense that

$$
w(a_i) \leq w(a_j) \quad \Leftrightarrow \quad w\left(\lambda^1 a_i^1, \dots, \lambda^m a_i^m\right) \leq w\left(\lambda^1 a_j^1, \dots, \lambda^m a_j^m\right)
$$

for all $\lambda \in \mathbb{R}^M_{++}$. By Osborne (1976), there exist $\beta = (\beta^1, \dots, \beta^m) \in \mathbb{R}^M_{++}$ and an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$w(a_i) = f\left(\prod_{h \in M} (a_i^h)^{\beta^h} \right) \tag{9}$$

for all $a_i \in A$. (In Osborne's theorem, w is nondecreasing and $\beta \in \mathbb{R}^M_+$. In our case, the fact that w is increasing guarantees that $\beta \in \mathbb{R}^M_{++}$. The normalization $\beta \in \Delta^M_{++}$ is innocuous.) Statement (7) now follows from (8) and (9). □

Of course, the weighted geometric mean numerical representation in Theorem 3 is unique only up to an increasing transformation.

7. SEPARABLE GRADING

We conclude with a defense of separability in the context of *cardinal* evaluation of multidimensional performances. A *grading method* is a function $G : A^N \to \Delta^N$, where Δ^N denotes the unit simplex of $\mathbb{R}^N$. The vector $G(a) = (G_1(a), \dots, G_n(a))$ is the grade distribution assigned by the method G to the performance matrix a. The grade of item i, $G_i(a)$, is interpreted as a cardinal measure of its multidimensional performance. A grading method G clearly induces a ranking method R_G defined on A^N by $iR_G(a)j \Leftrightarrow G_i(a) \geq G_j(a)$, but the information contained in the grade distribution $G(a)$ is richer than that in the induced ranking $R_G(a)$. As mentioned in footnote 1, grading methods are the traditional object of study in the literature on multidimensional performance evaluation. We call G *ordinally separable* if R_G is separable.

Assuming that performances are cardinally measurable on noncomparable scales, two properties of grading methods appear to be essential. The first is the cardinal version of the invariance axiom discussed earlier.

Cardinal Invariance. For all $a \in A^N$ and $\lambda \in \mathbb{R}^M_{++}$, $G(a \cdot dg(\lambda)) = G(a)$.

The second condition is Homogeneity. It requires that if an item's performances with respect to all criteria are multiplied by the same positive number, the ratio of that item's grade to any other item's grade is multiplied by the same number. This is compelling if performances with respect to each criterion are cardinally measurable. For every $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^N_{++}$, let $dg(\mu)$ denote the $n \times n$ diagonal matrix whose ith diagonal entry is μ_i. With this notation, $dg(\mu) \cdot a$ is the performance matrix obtained by multiplying each row i of a by μ_i.

Homogeneity. For all $a \in A^N$ and $\mu \in \mathbb{R}^N_{++}$, $G(dg(\mu) \cdot a)$ is proportional to $dg(\mu) \cdot G(a)$.

Most popular grading methods fail at least one of these two axioms. In fact, Cardinal Invariance and Homogeneity together have far-reaching consequences. Call a performance matrix $a \in A^N$ *doubly balanced* if $\sum_{i \in N} a_i^h = 1$ for all $h \in M$ and $\sum_{h \in M} a_i^h = m/n$ for all $i \in N$. Let A^N_* denote the set of doubly balanced matrices. Sinkhorn (1967) proves that for every matrix $a \in A^N$, there exist a unique vector $\lambda(a) \in \mathbb{R}^M_{++}$ and a unique vector $\mu(a) \in \mathbb{R}^N_{++}$ such that $dg(\mu(a)) \cdot a \cdot dg(\lambda(a)) =: a^*$ is doubly balanced. This means that every positive matrix a can be reduced to a uniquely defined doubly balanced matrix a^* by rescaling its rows and columns. It follows that acardinally invariant and homogeneous grading method is completely determined by its behavior on the doubly balanced matrices.

Corollary to Sinkhorn's Theorem. A grading method $G : A^N \to \Delta^N$ *is cardinally invariant and homogeneous if and only if there exists a function* $G^* : A^N_* \to \Delta^N$ *such that* $G(a)$ *is proportional to* $(dg(\mu(a)))^{-1} \cdot G^*(a^*)$ *for all* $a \in A^N$.

Building on this corollary, Demange (2014) pins down the handicap-based grading method by adding to Cardinal Invariance and Homogeneity the Uniformity axiom, which requires that all items should be tied when the performance matrix is doubly balanced.

Uniformity. For all $a \in A^N_*$, $G(a) = (\frac{1}{n}, \ldots, \frac{1}{n})$.

The handicap-based method is *not* ordinally separable. Its computation requires an iterative procedure, and the grades it yields vary with the performance matrix in ways that are difficult to apprehend.

It may, therefore, be worth pointing out that the *geometric mean grading method*

$$G_i^{gm}(a) = \frac{\prod_{h\in M} (a_i^h)^{\frac{1}{m}}}{\sum_{j\in N} \prod_{h\in M} (a_j^h)^{\frac{1}{m}}}$$

is cardinally invariant, homogeneous, and ordinally separable. Thus, while Cardinal Invariance, Homogeneity, Ordinal Separability, and Uniformity are incompatible, the first three of these axioms are not.[4] If the first two are considered a must, we are left with a choice between the last two. Uniformity amounts to imposing the *arithmetic* mean criterion on the doubly balanced matrices. This creates an obvious tension with Cardinal Invariance and is probably not compelling. Ranking item 1 above 2 and 3 for the matrix

$$\begin{pmatrix} 3/9 & 3/9 & 3/9 \\ 2/9 & 3/9 & 4/9 \\ 4/9 & 3/9 & 2/9 \end{pmatrix},$$

as the geometric mean does, seems to be reasonable and is supported by an argument of variability aversion: the fact that the scores of item 1 coincide on all criteria gives them added value. Separability may well be worth sacrificing Uniformity.[5]

---

[4] In fact, Cardinal Invariance, Homogeneity, and the requirement that an items' grade should be the geometric mean of its performances when the performance matrix is doubly balanced, together pin down G^{gm}, thereby implying Ordinal Separability: this is again a consequence of the above corollary to Sinkhorn's theorem.

[5] The method G^{gm} is also ordinally monotonic in the sense that $R_{G^{gm}}$ is monotonic. The handicap-based method has not been shown to be ordinally monotonic.

8. CONCLUSION

We have shown that separability is not a characteristic property of ranking-by-rating methods. These methods are also consistent: if a change in an item's performance improves its relative ranking against some other item at a given profile, the same change never decreases its relative ranking against any item at any profile. Separability and Consistency do characterize ranking by rating in the finite case, and this characterization generalizes to the infinite case under a continuity axiom. Consistency follows from Separability and Symmetry or from Monotonicity alone. When performances are vectors in $\mathbb{R}^m_+$, a separable, symmetric, monotonic, continuous, and invariant method must rank items according to a weighted geometric mean of their performances along the m dimensions.

These results hold when the domain of performance profiles is a Cartesian product. Without that assumption, the implications of Separability are far from clear and deserve further study.

APPENDIX

PROOF OF LEMMA 1. If $i, j \in N$, $i \neq j$, $a \in A$, $\alpha \in A_i$, and $\beta \in A_j$, let (α, β, a_{-ij}) denote the performance profile obtained from a by replacing a_i with α and a_j with β. Let $A_{-ij} = \bigtimes_{k \in N\setminus\{i,j\}} A_k$ and define the binary relations $\succ$, $\sim$, and $\succsim$ on X as

$$
\begin{aligned}
(i,\alpha) \succ (j,\beta) \Leftrightarrow {} & \big[i \neq j \text{ and } \exists a_{-ij} \in A_{-ij} \text{ such that } iP(\alpha,\beta,a_{-ij})j\big] \quad \text{or} \\
& \begin{bmatrix} i = j \text{ and } \exists a_{-i} \in A_{-i} \text{ and } \exists k \in N \setminus \{i\} \text{ such that} \\ iP(\alpha, a_{-i})kR(\beta, a_{-i})i \text{ or } iR(\alpha, a_{-i})kP(\beta, a_{-i})i \end{bmatrix}, \\
(i,\alpha) \sim (j,\beta) \Leftrightarrow {} & \big[i \neq j \text{ and } \exists a_{-ij} \in A_{-ij} \text{ such that } iI(\alpha,\beta,a_{-ij})j\big] \quad \text{or} \\
& \begin{bmatrix} i = j \text{ and } \forall a_{-i} \in A_{-i} \text{ and } \forall k \in N \setminus \{i\}, \\ iR(\alpha, a_{-i})k \Leftrightarrow iR(\beta, a_{-i})k \end{bmatrix}
\end{aligned}
$$

and

$$(i,\alpha) \succsim (j,\beta) \quad \Leftrightarrow \quad (i,\alpha) \succ (j,\beta) \quad \text{or} \quad (i,\alpha) \sim (j,\beta).$$

These are revealed performance comparison relations. The statement $(i,\alpha) \succ (j,\beta)$ means that performance α by item i is considered stronger than performance β by item j. This occurs when (i) i and j are distinct and R ranks i above j at some profile where their performances are α and β, or (ii) when i and j are the same item and the ranking of this item drops when its performance changes from α to β. Likewise, $(i,\alpha) \sim (j,\beta)$ means that performance α by item i is considered equivalent to performance β by item j. This occurs when (a) i and j are distinct and R ties them at some profile where their performances are α and β, or (b) i and j are the same item and the ranking of this item never changes when its performance switches from α to β.

Step 1. The binary relation $\sim$ is reflexive and symmetric, and $\succsim$ is complete.

These facts follow directly from the definition of $\sim$ and $\succsim$; they do not rely on the assumptions that R is separable and consistent.

Step 2. For all $(i, \alpha), (j, \beta) \in X$, exactly one of the following statements holds: (i) $(i, \alpha) \succ (j, \beta)$; (ii) $(j, \beta) \succ (i, \alpha)$; (iii) $(i, \alpha) \sim (j, \beta)$.

Let $(i, \alpha), (j, \beta) \in X$. By Step 1, at least one of the above statements (i), (ii), and (iii) holds. If both (i) and (ii) hold, we obtain a contradiction to Separability when $i \neq j$ and a contradiction to Consistency when $i = j$. If both (i) and (iii) hold or if both (ii) and (iii) hold, we obtain a contradiction to Separability when $i \neq j$ and an immediate contradiction when $i = j$.

Because of Steps 1 and 2, to prove that $\succsim$ is an ordering, it is enough to show that $\succ$ and $\sim$ are transitive. These are the next two steps.

Step 3. The binary relation $\succ$ is transitive.

Let (i_1, α), (i_2, β), $(i_3, \gamma) \in X$ be such that $(i_1, \alpha) \succ (i_2, \beta) \succ (i_3, \gamma)$.

Case 1. There exist $i_1 = i_2 = i_3$, say, $(1, \alpha) \succ (1, \beta) \succ (1, \gamma)$.

Then there exist $a_{-1}, b_{-1} \in A_{-1}$, and $i, j \in N \setminus \{1\}$ such that

$$1P(\alpha, a_{-1})iR(\beta, a_{-1})1 \quad \text{or} \quad 1R(\alpha, a_{-1})iP(\beta, a_{-1})1 \tag{10}$$

and

$$1P(\beta, b_{-1})jR(\gamma, b_{-1})1 \quad \text{or} \quad 1R(\beta, b_{-1})jP(\gamma, b_{-1})1. \tag{11}$$

To prove $(1, \alpha) \succ (1, \gamma)$, we show that neither $(1, \gamma) \succ (1, \alpha)$ nor $(1, \alpha) \sim (1, \gamma)$.

If $(1, \gamma) \succ (1, \alpha)$, then there exists $c_{-1} \in A_{-1}$ and $k \in N \setminus \{1\}$ such that

$$1P(\gamma, c_{-1})kR(\alpha, c_{-1})1 \quad \text{or} \quad 1R(\gamma, c_{-1})kP(\alpha, c_{-1})1. \tag{12}$$

If $1R(\beta, c_{-1})k$, (12) implies $1R(\beta, c_{-1})kR(\alpha, c_{-1})1$, which, combined with (10), contradicts Consistency. If $kR(\beta, c_{-1})1$, (12) implies $1R(\gamma, c_{-1})kR(\beta, c_{-1})1$, which, combined with (11), contradicts Consistency again.

If $(1, \alpha) \sim (1, \gamma)$, then for all $x_{-1} \in A_{-1}$ and all $k \in N \setminus \{1\}$, we have $1R(\alpha, x_{-1})k \Leftrightarrow 1R(\gamma, x_{-1})k$. Specializing this equivalence to $k = i$, $x_{-1} = a_{-1}$, and combining it with statement (10) yields $1P(\gamma, a_{-1})iR(\beta, a_{-1})1$ or $1R(\gamma, a_{-1})iP(\beta, a_{-1})1$. This statement, in conjunction with (11), contradicts Consistency.

Case 2. There exist $i_1 = i_2 \neq i_3$, say, $(1, \alpha) \succ (1, \beta) \succ (2, \gamma)$.

Then there exist $a_{-1} \in A_{-1}$ and $i \in N \setminus \{1\}$ such that (10) holds and there exists $b_{-12} \in A_{-12}$ such that

$$1P(\beta, \gamma, b_{-12})2. \tag{13}$$

By Consistency, (10) implies that there is no $c_{-1} \in A_{-1}$ such that $1R(\beta, c_{-1})2R(\alpha, c_{-1})1$. Therefore, (13) implies $1P(\alpha, \gamma, b_{-12})2$; hence, $(1, \alpha) \succ (2, \gamma)$.

Case 3. There exist $i_1 = i_3 \neq i_2$, say, $(1, \alpha) \succ (2, \beta) \succ (1, \gamma)$.

Then there exist $a_{-12}, b_{-12} \in A_{-12}$ such that $1P(\alpha, \beta, a_{-12})2P(\gamma, \beta, b_{-12})1$; hence, by Separability, $1P(\alpha, \beta, a_{-12})2P(\gamma, \beta, a_{-12})1$, which implies $(1, \alpha) \succ (1, \gamma)$.

Case 4. There exist $i_1 \neq i_2 = i_3$, say, $(1, \alpha) \succ (2, \beta) \succ (2, \gamma)$.

Then there exists $a_{-12} \in A_{-12}$ such that

$$1P(\alpha, \beta, a_{-12})2 \tag{14}$$

and there exist $b_{-2} \in A_{-2}$ and $i \in N \setminus \{2\}$ such that

$$2P(\beta, b_{-2})iR(\gamma, b_{-2})2 \quad \text{or} \quad 2R(\beta, b_{-2})iP(\gamma, b_{-2})2. \tag{15}$$

By Consistency, (15) implies that there is no $c_{-2} \in A_{-2}$ such that $2R(\gamma, c_{-2})1R(\beta, c_{-2})2$. Therefore, (14) implies $1P(\alpha, \gamma, a_{-12})2$; hence, $(1, \alpha) \succ (2, \gamma)$.

Case 5. Items i_1, i_2, and i_3 are all distinct, say, $(1, \alpha) \succ (2, \beta) \succ (3, \gamma)$.

Then there exist $a_{-12} \in A_{-12}$ and $b_{-23} \in A_{-23}$ such that

$$1P(\alpha, \beta, a_{-12})2P(\beta, \gamma, b_{-23})3. \tag{16}$$

Let $(\alpha, \beta, \gamma, a_{-123})$ be the profile obtained by replacing item 3's performance in (α, β, a_{-12}) with γ. By Separability, (16) implies $1P(\alpha, \beta, \gamma, a_{-123})2P(\alpha, \beta, \gamma, a_{-123})3$ and, hence, $1P(\alpha, \beta, \gamma, a_{-123})3$ (because $P(\alpha, \beta, \gamma, a_{-123})$ is transitive), and therefore $(1, \alpha) \succ (3, \gamma)$.

Step 4. The binary relation $\sim$ is transitive.

Let $(i_1, \alpha), (i_2, \beta), (i_3, \gamma) \in X$ be such that $(i_1, \alpha) \sim (i_2, \beta) \sim (i_3, \gamma)$.

Case 1. There exist $i_1 = i_2 = i_3$, say, $(1, \alpha) \sim (1, \beta) \sim (1, \gamma)$.

Then for all $a_{-1} \in A_{-1}$ and all $i \in N \setminus \{1\}$, $1R(\alpha, a_{-1})i \Leftrightarrow 1R(\beta, a_{-1})i \Leftrightarrow 1R(\gamma, a_{-1})i$. Therefore, $1R(\alpha, a_{-1})i \Leftrightarrow 1R(\gamma, a_{-1})i$; hence, $(1, \alpha) \sim (1, \gamma)$.

Case 2. There exist $i_1 = i_2 \neq i_3$, say, $(1, \alpha) \sim (1, \beta) \sim (2, \gamma)$.

Then for all $a_{-1} \in A_{-1}$ and all $i \in N \setminus \{1\}$,

$$1R(\alpha, a_{-1})i \quad \Leftrightarrow \quad 1R(\beta, a_{-1})i, \tag{17}$$

and there exists $b_{-12} \in A_{-12}$ such that

$$1I(\beta, \gamma, b_{-12})2. \tag{18}$$

Applying (17) with $a_{-1} = (\gamma, b_{-12})$ and $i = 2$ yields $1R(\alpha, \gamma, b_{-12})2 \Leftrightarrow 1R(\beta, \gamma, b_{-12})2$. Combining this with (18) implies $1I(\alpha, \gamma, b_{-12})2$; hence, $(1, \alpha) \sim (2, \gamma)$.

Case 3. There exist $i_1 = i_3 \neq i_2$, say, $(1, \alpha) \sim (2, \beta) \sim (1, \gamma)$.

Then there exist $a_{-12}, b_{-12} \in A_{-12}$ such that $1I(\alpha, \beta, a_{-12})2I(\gamma, \beta, b_{-12})1$; hence, by Separability, $1I(\alpha, \beta, a_{-12})2I(\gamma, \beta, a_{-12})1$, which implies

$$1R(\gamma, \beta, a_{-12})2R(\alpha, \beta, a_{-12})1. \tag{19}$$

To prove $(1, \alpha) \sim (1, \gamma)$, we must show that $1R(\alpha, c_{-1})i \Leftrightarrow 1R(\gamma, c_{-1})i$ for all $c_{-1} \in A_{-1}$ and all $i \in N \setminus \{1\}$. Suppose, on the contrary, that there exist $c_{-1} \in A_{-1}$ and $i \in N \setminus \{1\}$ such that $1P(\alpha, c_{-1})iR(\gamma, c_{-1})1$ or $1R(\alpha, c_{-1})iP(\gamma, c_{-1})1$. Then, by Consistency, there is no $d_{-1} \in A_{-1}$ such that $1R(\gamma, d_{-1})2R(\alpha, d_{-1})1$, contradicting (19).

Case 4. There exist $i_1 \neq i_2 = i_3$, say, $(1, \alpha) \sim (2, \beta) \sim (2, \gamma)$.

Then there exists $a_{-12} \in A_{-12}$ such that

$$1I(\alpha, \beta, a_{-12})2 \tag{20}$$

and, for all $b_{-2} \in A_{-2}$ and all $i \in N \setminus \{2\}$, we have $2R(\beta, b_{-2})i \Leftrightarrow 2R(\gamma, b_{-2})i$. Applying this equivalence with $b_{-2} = (\alpha, a_{-12})$ and $i = 1$, we get $2R(\alpha, \beta, a_{-12})1 \Leftrightarrow 2R(\alpha, \gamma, a_{-12})1$. Combining this with (20) yields $1I(\alpha, \gamma, a_{-12})2$; hence, $(1, \alpha) \sim (2, \gamma)$.

Case 5. Items i_1, i_2, and i_3 are all distinct, say, $(1, \alpha) \sim (2, \beta) \sim (3, \gamma)$.

Then there exist $a_{-12} \in A_{-12}$ and $b_{-23} \in A_{-23}$ such that $1I(\alpha, \beta, a_{-12})2I(\beta, \gamma, b_{-23})3$. By Separability, this implies $1I(\alpha, \beta, \gamma, a_{-123})2I(\alpha, \beta, \gamma, a_{-123})3$; hence, $1P(\alpha, \beta, \gamma, a_{-123})3$ (because $I(\alpha, \beta, \gamma, a_{-123})$ is transitive) and, therefore, $(1, \alpha) \succ (3, \gamma)$.

This completes the proof that $\succsim$ is an ordering.

Step 5. It remains to check that $iR(a)j \Leftrightarrow (i, a_i) \succsim (j, a_j)$ for all $i, j \in N$ and all $a \in A^N$.

Fix $i, j \in N$ and $a \in A^N$. The case $i = j$ being trivial, assume $i \neq j$. It follows directly from the definition of $\succ$ and $\sim$ that

$$iP(a)j \quad \Rightarrow \quad (i, a_i) \succ (j, a_j) \quad \text{and} \quad iI(a)j \quad \Rightarrow \quad (i, a_i) \sim (j, a_j),$$

hence, $iR(a)j \Rightarrow (i, a_i) \succsim (j, a_j)$. The converse implication follows from Step 2 and the completeness of $R(a)$. □

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