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On asymmetric reserve prices

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We investigate equilibrium bidding in first-price auctions with asymmetric reserve prices. For example, the auctioneer may set a low reserve price for one subset of bidders and a high reserve price for others. When used to pursue a distributional objective, lowering the reserve price for some bidders channels benefits toward marginal agents in the favored group and does not adversely impact nonfavored bidders whose reserve price is unchanged. Even in symmetric environments, when the valuation distribution is not regular, introducing asymmetric reserve prices can increase the auctioneer's revenue compared to an optimal common reserve price. Implications for auction design are considered.

Keywords. First-price auction, asymmetric auctions, reserve price, mechanism design, affirmative action, procurement.

JEL classification. D44.

The skewing of an economic contest to favor some participants is a frequent practical concern, particularly in the design of auctions or other procurement schemes. Bidder-specific subsidies, bidding credits, and set-asides have received much attention in the literature and in practice (Ayres and Cramton 1996, Rothkopf et al. 2003, Marion 2007, Krasnokutskaya and Seim 2011, Pai and Vohra 2014, Athey et al. 2013, Loertscher and Marx 2016). These policies allows an auctioneer to bias an auction in pursuit of an ancillary goal, such as a distributional or an affirmative-action objective. For example, political motives may impel a government to favor domestic bidders over foreigners in a procurement auction or a privatization sale (McAfee and McMillan 1989). Targeting benefits toward a specific interest group, such as small-businesses or women and minority-owned firms, is also a common goal.[1]

An underexplored policy lever that can tilt an auction in one direction or another is the reserve price or, rather, the departure from a common reserve price. The idea

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[1] Mummalaneni (2016) offers a recent empirical study of a procurement process incorporating this goal.

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is simple. Bidders are partitioned into groups and a bidder in group k must bid above r_k to be eligible to win the auction. Bidders facing a lower minimum admissible bid become advantaged. Whereas similar arrangements should usually appear in revenue-maximizing mechanisms when bidders are ex ante asymmetric (Myerson 1981), such reserve-price asymmetries can also boost revenue in some symmetric environments, distributional objectives notwithstanding. In fact, the procedure's simplicity may make it preferable in practice to otherwise complex optimal mechanisms.

To make our argument concrete, we focus on the first-price, sealed-bid auction for one item, a common selling procedure. Compared to a first-price auction with a revenue-maximizing common reserve price, lowering the reserve price faced by some bidders can

(i) increase the welfare of bidders enjoying the lower reserve price

(ii) increase the welfare of bidders whose reserve price was unchanged

(iii) increase the auctioneer's expected revenue.

In sum, a Pareto improvement is possible. In the analysis to follow, we precisely explain when and why these effects occur. However, they are apparent even in a simple example.

Example 1. Suppose an item is sold by a first-price, sealed-bid auction.[2] There are two risk-neutral bidders with independent and identically distributed valuations. With probability 0.6, a bidder's valuation is 1; with probability 0.4, it is 2. This distribution is common knowledge.

If the auctioneer sets a common reserve price, he has two reasonable options. When the reserve price is 1, a low-valuation bidder bids 1 while a high-valuation bidder adopts a mixed strategy.[3] The expected revenue is 1.16. If instead the auctioneer sets a reserve price of 2, a low-valuation bidder does not bid and a high-valuation bidder bids 2. The expected revenue increases to 1.28. Thus, $r^* = 2$ is the revenue-maximizing reserve price.

Now suppose the auctioneer adopts an asymmetric policy. He sets bidder 1's reserve price at $r_1 = 1$ and bidder 2's reserve price at $r_2 = 2$. In this case, bidder 1 always bids 1 in equilibrium, regardless of his valuation. Bidder 2 bids 2 only if his valuation is high; else, he does not bid. The expected revenue is now 1.4. Therefore, the asymmetric policy benefits the auctioneer.[4] Surprisingly, the bidders also gain from the asymmetry. Bidder 2 is no worse off than had the common reserve price $r^* = 2$ been maintained. His surplus is still zero. But bidder 1 now enjoys a positive surplus in expectations. Therefore, moving from $r^* = 2$ to the asymmetric policy (weakly) benefits everyone. $\Diamond$

[2] I thank Richard Zeckhauser for suggesting the inclusion of an example of this form.

[3] In equilibrium, a high-valuation bidder places a bid on the interval $(1, 1.4]$, randomizing according to the cumulative distribution function $(3b - 3)/(4 - 2b)$.

[4] The auctioneer cannot do any better: 1.4 is also the expected revenue generated by the revenue-maximizing direct mechanism, which does have a symmetric implementation. This mechanism allocates the good to the agent declaring the highest valuation. Ties are resolved with a fair coin flip. If both agents announce a valuation of 1, the good sells for that price. Else, if at least one agent declares a value of 2, the winner pays 1.625.

The preceding example is not a knife-edge case and its underlying logic applies more broadly, as illustrated by Example 2 below. In this paper we aim to generalize the preceding situation and its conclusions to a richer, yet familiar, environment. We focus on the classic independent private value model of a first-price auction for a single item (Vickrey 1961). In our baseline model bidders are ex ante identical with valuations drawn from a commonly known distribution. We partition bidders into groups and a bidder in group k needs to bid at least r_k to be eligible to win the item.[5] Agents are risk neutral, reserve prices are openly posted, and group membership is common knowledge. As usual, Bayesian–Nash equilibrium is the solution concept.

Two complementary points guide our investigation. First, as noted above, heterogenous reserve prices can bias an auction in favor of one party or another. Thus, they are a policy that can be deployed in pursuit of a distributional objective. They are distinct from other interventions, such as bid subsidies or set-asides. Reserve-price adjustments are able to channel benefits toward marginal or weaker bidders in favored groups without compromising too much the incentives of inframarginal or stronger bidders. Other policies, such as blanket subsidies given to all favored bidders, are less discerning. Moreover, lowering the reserve price faced by some bidders does not harm nonfavored bidders (relative to the uniform reserve-price status quo), which may enhance acceptance of the otherwise discriminatory procedure. Therefore, reserve-price adjustments complement more traditional policies and allow an auctioneer to pursue new and distinct objectives.

Second, as illustrated by the preceding example, reserve-price adjustments can boost auction revenues—even in an ex ante symmetric setting. However, generalizing this insight requires care. Riley and Samuelson (1981) and Myerson (1981) have characterized revenue-maximizing auctions, which often feature a common reserve price.[6] We never contradict their seminal results. Rather, when a common reserve price alone does not characterize a revenue-maximizing mechanism, asymmetric reserve prices allow an auctioneer to gain some of the revenue benefits of "ironing" while retaining the institutional simplicity and practicality of a standard auction with simple payment rules.[7] The revenue gains are (necessarily) not always maximal, but they can be substantial. An auctioneer troubled by the perceived favoritism may randomly assign "favored" and "unfavored" status to bidders beforehand without compromising any ex ante revenue implications.

While there is broad theoretical and practical interest in implementing biased auctions, the equilibrium consequences of group-specific minimal bids have received little attention. This is surprising given that many foundational studies have accommodated

[5]We formulate our model as a high-bid auction where the highest bidder wins. Our results have natural analogues in a procurement application, where the lowest bidder wins. In this case, reserve prices correspond to maximal admissible bids. The translation between high-bid and procurement models is outlined by de Castro and de Frutos (2010).

[6]When bidders are ex ante asymmetric, the optimal auction is a discriminatory procedure and may involve multiple reserve prices. Deb and Pai (2017) show how it can be implemented through a symmetric mechanism.

[7]See Skreta (2007), Monteiro and Svaiter (2010), and Toikka (2011) for further discussion on ironing.

this policy lever. Myerson (1981) shows that personalized reserve prices, among other bid adjustments, are a feature of revenue-maximizing auctions. This conclusion is easy to appreciate when bidders are ex ante asymmetric. We argue that reserve-price asymmetries can be beneficial in symmetric environments and are simpler to employ than many optimal procedures. Similarly, the literature on equilibrium existence in first-price auctions notes the admissibility of asymmetric reserve prices (Athey 2001, Reny and Zamir 2004). Though the existence question is settled, the economic implications of such asymmetries remain to be characterized. We hope that the analysis below can serve as a stepping stone toward use of this auction-design tool.[8] At times, a Pareto improvement over common practice may be possible.

To ensure a manageable analysis, we pragmatically shut down many of the embellishments that add subtlety to the reserve-price setting process, such as risk aversion (Hu et al. 2010), common values (Milgrom and Weber 1982, Levin and Smith 1996, Quint 2017), signalling (Cai et al. 2007), or loss aversion (Rosenkranz and Schmitz 2007). Likewise, our model's institutional features are straightforward. We preclude resale (cf. Haile 2000, Loertscher and Marx 2016) and we assume that reserve prices are not secret (cf. Ashenfelter 1989, Vincent 1995, Li and Perrigne 2003, Brisset and Naegelen 2006). We also bracket a reserve price's implications for entry, which may be significant (Samuelson 1985, Moreno and Wooders 2011).[9]

This paper is organized as follows. In Section 1, we introduce the baseline symmetric model of a first-price auction with group-specific reserve prices. Though the symmetric model's tractability is attractive, practical implementation of group-specific reserve prices often occurs when bidders are asymmetric. To facilitate such applications, in Section 2, we extend our analysis to the asymmetric case, where groups of bidders differ from one another. Despite the considerable added complexity, the intuitions from the symmetric case continue to apply. This extension draws on prior studies of asymmetric first-price auctions, particularly Lebrun (1997, 1999, 2006), Lizzeri and Persico (2000), and Maskin and Riley (2003). Fibich and Gavious (2003), Kirkegaard (2009), Kaplan and Zamir (2012), and Mares and Swinkels (2014) provide some recent results concerning asymmetric first-price auctions, which are generally very challenging to analyze.

[8] Albano et al. (2006) also propose reserve-price differences among bidders in a procurement application, particularly in asymmetric settings. Flambard and Perrigne (2006) consider an "optimal auction" counterfactual with bidder-specific reserve prices in their study of snow removal contracts in Montreal. Krasnokutskaya and Seim (2011, p. 2685) mention asymmetric reserve prices as a possible policy intervention in their study of highway procurement in California; however, they do not investigate its implications.

[9] Samuelson (1985) assumes that agents decide to enter the auction after knowing their valuations and the reserve price is the only screening tool available to the auctioneer. To maximize revenue, the seller should set a positive reserve price. (We have restated his conclusion in terms of a high-bid auction; it is originally a model of procurement.) Moreno and Wooders (2011) allow an auctioneer to screen separately along entry-cost and value dimensions. When both entry fees and reserve prices are available, they conclude that buyers should be screened by entry fees alone with the reserve price set to zero. When entry fees are not feasible, a positive reserve price is optimal. Levin and Smith (1994) consider an entry model with homogenous entry costs. In their setting, a reserve price of zero is optimal for the seller, even if he cannot change entry fees.

In Sections 3 and 4, we turn to applications of our equilibrium characterization. For simplicity, we return to the case where bidders are ex ante symmetric. We investigate implications for welfare and revenue. We also contrast asymmetric reserve prices with bid subsidies, discussed above, and we revisit Riley and Samuelson's (1981) comparison of reserve prices and entry fees. We conclude with brief remarks concerning other auction formats, such as the second-price and the all-pay auction, when agents face different reserve prices.

Proofs related to the symmetric model examined in Sections 1 and 3 are presented in the Appendix. The Supplemental Material, available in a supplementary file on the journal website, http://econtheory.org/supp/1824/supplement.pdf, collects proofs pertaining to Sections 2 and 4.

1. The symmetric model

There are N bidders and a single item is available for purchase. Each bidder has a private type s corresponding to his value for the item. The seller's value is zero. Though relaxed below, for the moment assume that all bidders are ex ante identical. Bidders' types are independently and identically distributed according to the cumulative distribution function (c.d.f.) $F(\cdot)$ on the interval $[0, \bar{s}]$. Unless noted otherwise, $F(\cdot)$ is absolutely continuous with a continuous, strictly positive, and bounded density, $f(\cdot)$.

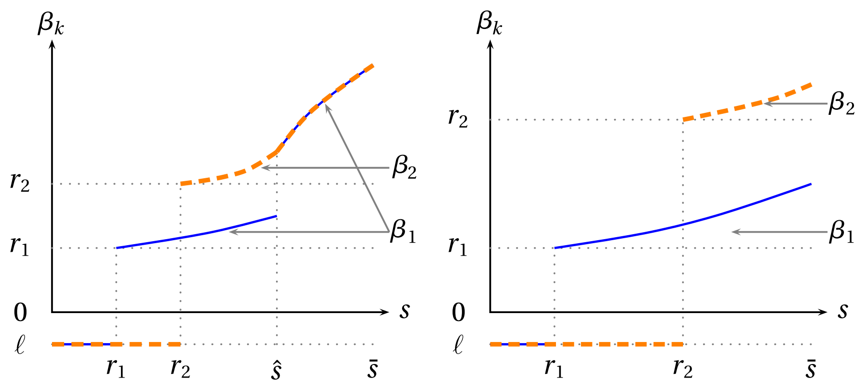
The rules of the first-price auction with a reserve price r are well known. Each bidder submits a bid $b \in \{\ell\} \cup [r, \infty)$, $r > 0$. All bids $[r, \infty)$ are *competitive bids*. The agent submitting the highest competitive bid wins the auction and makes a payment equal to his bid. As usual, ties are resolved with a uniform randomization. The bid $\ell < r$ is a *noncompetitive bid*, equivalent to not participating in the auction. An agent bidding ℓ receives a payoff of zero, irrespective of the others' bids. The payoff of a type-s bidder who wins the auction with bid b is $s - b$; a nonwinner's payoff is zero.

In the symmetric auction just described, equilibrium bidding is well understood. As shown by Riley and Samuelson (1981), there exists a unique Bayesian–Nash equilibrium where all bidders adopt the bidding strategy

$$\beta(s) = \begin{cases} \ell & \text{if } s < r, \\ s - \displaystyle\int_r^s \left[\frac{F(z)}{F(s)}\right]^{N-1} dz & \text{if } s \geq r. \end{cases}$$

To introduce different reserve prices, we maintain the above environment but we partition bidders into two groups $k \in \{1, 2\}$. (The analysis readily extends to more than two groups.) Each group has N_k members and group membership is common knowledge.[10] All bidders in group k face a posted reserve price of r_k and must bid at least this amount to be eligible to win the item. Hence, the set of valid bids for a group-k bidder is $\{\ell\} \cup [r_k, \infty)$. Without loss of generality, we assume that $0 < r_1 \leq r_2 < \bar{s}$. We focus on equilibria in *group symmetric strategies* where all members of group-k adopt a common bidding strategy, $\beta_k(s)$. As we show below, this restriction is without loss of generality.

[10]We assume that N_1 and N_2 are fixed. In practice, the seller may be able to control this partition.



(a) Equilibrium strategy when $|r_2 - r_1|$ is small (Theorem 1a).

(b) Equilibrium strategies when $|r_2 - r_1|$ is large (Theorem 1b).

FIGURE 1. Equilibrium strategies in the first-price auction with reserve prices $r_1 < r_2$.

Before stating the auction's equilibrium, we outline the associated intuition with reference to Figure 1. Slightly perturbing the symmetric model generates what we call a *semi-separating* equilibrium (Figure 1(a)). Such an equilibrium appears when $|r_1 - r_2|$ is positive, but small. In this equilibrium, bidders from different groups adopt distinct strategies, but with intersecting ranges of competitive bids. The intuition supporting this equilibrium is as follows. First, bidders with valuations $s < r_k$ always bid ℓ. Entering the auction in such a case leads to a negative expected payoff. Second, bidders in group 1 of type $s \in [r_1, r_2)$ find it worthwhile to bid at least r_1 but less than r_2. Third, a group-1 bidder of type $s \geq r_2$ faces a trade-off. A bid between r_1 and r_2 offers a modest chance of winning at a relatively low price. A bid above r_2 increases the probability of winning, but it comes at a cost. A high bid defeats all group-1 agents bidding less than r_2 and (perhaps) some additional group-2 bidders. Whether a low or a high bid is optimal depends on the bidder's valuation and on the strategy adopted by bidders in the other group. As s increases, the higher bid becomes more attractive until at a critical value $\hat{s}$, it becomes the better option. A bidder in group 1 with valuation $\hat{s}$ is indifferent between the low bid and matching the bid placed by a group-2 bidder of the same type. Thus, a group-1 agent's strategy jumps discontinuously at $\hat{s}$. Finally, bidders from groups 1 and 2 adopt the same strategy when $s > \hat{s}$.

The following theorem formalizes the preceding paragraph. First, it provides a sufficient (and necessary) condition ensuring that a semi-separating equilibrium exists. Second, it defines the critical value $\hat{s}$. Finally, equilibrium strategies are stated.

THEOREM 1a. *Let $r_1 < r_2$ and suppose*

$$\int_{r_1}^{\bar{s}} F(r_2)^{N_2} F(z)^{N_1 - 1}\, dz < \int_{r_2}^{\bar{s}} F(z)^{N_2 - 1}\, dz. \tag{1}$$

Define $\hat{s}$ as the unique value satisfying

$$\int_{r_1}^{\hat{s}} F(r_2)^{N_2} F(z)^{N_1 - 1}\, dz = \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2 - 1}\, dz. \tag{2}$$

There exists a Bayesian–Nash equilibrium where all bidders in group k adopt the strategy

$$\beta_k(s) = \begin{cases} \ell & \text{if } s < r_k, \\ s - \displaystyle\int_{r_k}^{s} \left[\frac{F(z)}{F(s)}\right]^{N_k-1} dz & \text{if } s \in [r_k, \hat{s}], \\ s - \displaystyle\int_{r_2}^{\hat{s}} \frac{F(\hat{s})^{N_1} F(z)^{N_2-1}}{F(s)^{N_1+N_2-1}}\, dz - \int_{\hat{s}}^{s} \left[\frac{F(z)}{F(s)}\right]^{N_1+N_2-1} dz & \text{if } s > \hat{s}. \end{cases}$$

When condition (1) is satisfied, there exists a unique value $\hat{s} < \bar{s}$ satisfying (2) (see Lemma 1 in the Appendix). When his valuation is $\hat{s}$, a group-1 bidder finds it worthwhile to discontinuously increase his bid above r_2, matching the bids of group-2 bidders as in Figure 1(a).

As the difference in reserve prices increases, group-1 agents find bids above r_2 less attractive. If $|r_2 - r_1|$ is sufficiently large, the ranges of bids submitted by members of each group separate fully. Figure 1(b) illustrates a *separating equilibrium* of this form. Members of each group bid *as if* the other group is absent; hence, everyone bids according to the symmetric equilibrium strategy associated with their reserve price.

Theorem 1b. *Let $r_1 < r_2$ and suppose*

$$\int_{r_1}^{\bar{s}} F(r_2)^{N_2} F(z)^{N_1-1}\, dz \ge \int_{r_2}^{\bar{s}} F(z)^{N_2-1}\, dz. \tag{3}$$

There exists a Bayesian–Nash equilibrium where all bidders in group k adopt the strategy

$$\beta_k(s) = \begin{cases} \ell & \text{if } s < r_k, \\ s - \displaystyle\int_{r_k}^{s} \left[\frac{F(z)}{F(s)}\right]^{N_k-1} dz & \text{if } s \ge r_k. \end{cases}$$

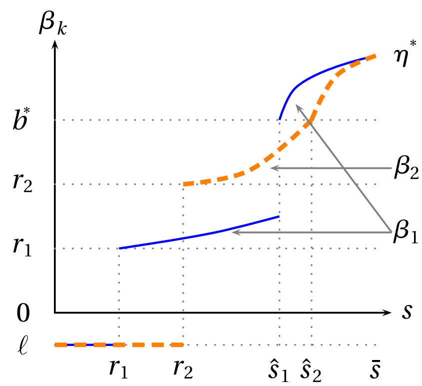
Theorems 1a and 1b characterize this auction's unique equilibrium. We discuss equilibrium uniqueness in the next section.

2. An asymmetric model

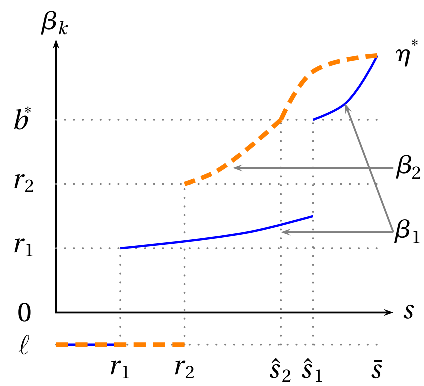
The preceding analysis assumes that all bidders are ex ante identical. As shown below, this symmetry is useful when investigating the economic implications of our model. In practice, however, an asymmetric policy is most likely to be implemented when there is a perceived ex ante difference between groups of bidders.[11] In this section, we extend our model to accommodate this case. Otherwise maintaining the setup from above, suppose that the c.d.f. of a typical group-k bidder's valuation is now $F_k(\cdot)$ with density $f_k(\cdot)$. We further assume that

$$\frac{d}{ds}\left(\frac{F_k(s)}{F_{k'}(s)}\right) < 0 \tag{4}$$

[11] For example, a government may wish to favor "small" companies in a procurement process. Due to economies of scale, "large" bidders may have systematically lower costs or higher valuations.



(a) Case 1: $\hat{s}_1 < \hat{s}_2$.



(b) Case 2: $\hat{s}_1 > \hat{s}_2$.

FIGURE 2. Group-symmetric equilibria in an asymmetric first-price auction.

for all $s \in (r_2, \bar{s}]$. Condition (4) has a straightforward economic interpretation. If condition (4) holds for all s, then $F_{k'}(s)$ dominates $F_k(s)$ in terms of the reverse hazard rate. Thus, $F_{k'}(s)$ first-order stochastically dominates $F_k(s)$. The condition lets us appeal to several results due to Lebrun (1997, 1999) from his analysis of asymmetric first-price auctions. These results considerably simplify our analysis, especially in relation to Theorem 2a below.

There are two cases, again depending on the relative difference between r_1 and r_2. Our discussion parallels the symmetric case. When $|r_2 - r_1|$ is small, we ought to anticipate an equilibrium featuring a discontinuous bidding strategy with bidders in group 1 bidding below and then above r_2. Figure 2 sketches two examples of how this may look when bidders are asymmetric. The cases differ in the relative location of the jump discontinuity in $\beta_1(s)$. The figures' resemblance to Figure 1(a) should be evident as the underlying intuition is the same. Prior to characterizing the equilibrium in this case, we explain the proposed construction in greater detail.

The strategies of both groups have two components. First, some ranges of bids are specific to a particular group. Only group-1 bidders of type $s \in [r_1, \hat{s}_1)$ bid in the range $[r_1, r_2)$. Similarly, only group-2 bidders of type $s \in [r_2, \hat{s}_2)$ bid in the range $[r_2, b^*)$. Standard reasoning suggests that the familiar function

$$b_k(s) = s - \int_{r_k}^{s} \left[\frac{F_k(z)}{F_k(s)} \right]^{N_k - 1} dz \tag{5}$$

must define the strategy of a group-k agent when bids are segregated in such a fashion.

Second, agents from both groups of type $s > \hat{s}_k$ place bids above b^*. Agents' strategies in this range must correspond to solutions of a system of differential equations derived from a family of first-order conditions, as shown by Lizzeri and Persico (2000), Maskin and Riley (2003), or Lebrun (1997, 1999, 2004, 2006) , among others. This is a well known property of the first-price auction. A key property of this system is that its solution must ensure that all type-$\bar{s}$ bidders place the same bid, denoted by η^* in Figure 2. This is the maximal bid placed in the auction.

Viewed independently, both of the above equilibrium features are docile. The key complication is the transition between them, which is defined by the values $\hat{s}_1$, $\hat{s}_2$, and b^*. These values are endogenously determined in equilibrium; we cannot confidently presume that $\hat{s}_1 = \hat{s}_2 = \hat{s}$, as in the symmetric case. To resolve such ambiguities, we follow preceding studies by considering a candidate maximal bid as the initial condition for the system of differential equations characterizing agents' equilibrium bids above (some) $b^* \geq r_2$. We then "shoot backward" toward the origin to construct the equilibrium strategy.[12] By varying the candidate maximal bid, we can pin-down appropriate values for $\hat{s}_1$, $\hat{s}_2$, and b^*. Among other considerations, these values are chosen to satisfy an indifference condition for bidders in group 1 whose valuation is $\hat{s}_1$ and a continuity-of-bidding-strategy requirement for bidders in group 2 whose valuation is $\hat{s}_2$. This argument is possible because the bidding strategies (i.e., the solutions to the appropriate differential equations) are monotone in the maximal bid (Lebrun 1997, 1999).

The following theorem formalizes the preceding discussion. Its proof builds on the equilibrium characterizations of Lebrun (1997, 1999). The proof, along with other results pertaining to this section, is presented in the Supplemental Material.

Theorem 2a. *Suppose*

$$\int_{r_1}^{\bar{s}} F_2(r_2)^{N_2} F_1(z)^{N_1-1}\,dz < \int_{r_2}^{\bar{s}} F_2(z)^{N_2-1}\,dz. \tag{6}$$

There exist constants $\hat{s}_1$ and $\hat{s}_2$ such that the auction has a Bayesian–Nash equilibrium where all bidders in group k adopt the strategy

$$\beta_k(s) = \begin{cases} \ell & \text{if } s < r_k, \\ b_k(s) & \text{if } s \in [r_k, \hat{s}_k], \\ \hat{b}_k(s) & \text{if } s > \hat{s}_k. \end{cases}$$

Furthermore, the following statements hold:

(a) The function $b_k\colon [r_k, \hat{s}_k] \to [r_k, \bar{s}]$ is given by

$$b_k(s) := s - \int_{r_k}^{s} \left[\frac{F_k(z)}{F_k(s)}\right]^{N_k-1} dz.$$

(b) The function $\hat{b}_k\colon [\hat{s}_k, \bar{s}] \to [r_2, \bar{s}]$ is defined as $\hat{b}_k := \phi_k^{-1}$, where (ϕ_1, ϕ_2) is the unique solution to the system of differential equations

$$\phi_k'(b) = \frac{1}{N_k + N_j - 1} \frac{F_k\big(\phi_k(b)\big)}{f_k\big(\phi_k(b)\big)} \left[\frac{1}{\phi_k(b) - b} + \frac{N_j\big(\phi_k(b) - \phi_j(b)\big)}{\big(\phi_k(b) - b\big)\big(\phi_j(b) - b\big)}\right], \tag{7}$$

$k, j \in \{1, 2\}$ and $k \neq j$, satisfying the boundary conditions

[12] Marshall et al. (1994) and Gayle and Richard (2008) adopt this technique to solve for equilibria numerically. See also Lebrun (1999).

(i) $\phi_1(\eta^*) = \phi_2(\eta^*) = \bar{s}$ *for some* $\eta^* \in (r_2, \bar{s})$

(ii) $\phi_1(b^*) = \hat{s}_1$ *and* $\phi_2(b^*) = \hat{s}_2$ *where* $b^* = b_2(\hat{s}_2)$.

Condition (6) is the direct analogue of condition (1) from Theorem 1a. We are unaware of any closed-form or compact expression characterizing the critical values $\hat{s}_1$ and $\hat{s}_2$ (cf. (2) above). Their identification is indirect and is outlined in the proof of Theorem 2a. As in the discussion above, η^* is the maximal bid placed in the auction and $\hat{s}_1$ is the point of discontinuity in the bidding strategy of a group-1 bidder. The strategy of a group-2 bidder is continuous at $\hat{s}_2$.

When the difference between r_1 and r_2 is large, bidders from group 1 always bid below r_2. Thus, bids are segregated as in the symmetric model.

Theorem 2b. *Suppose*

$$\int_{r_1}^{\bar{s}} F_2(r_2)^{N_2} F_1(z)^{N_1-1}\, dz \geq \int_{r_2}^{\bar{s}} F_2(z)^{N_2-1}\, dz.$$

There exists a Bayesian–Nash equilibrium where all bidders in group k adopt the strategy

$$\beta_k(s) = \begin{cases} \ell & \text{if } s < r_k, \\ s - \displaystyle\int_{r_k}^{s} \left[\frac{F_k(z)}{F_k(s)}\right]^{N_k-1} dz & \text{if } s \geq r_k. \end{cases} \tag{8}$$

The uniqueness of equilibrium in the first-price auction was studied by Lizzeri and Persico (2000), Maskin and Riley (2003), and Lebrun (1997, 1999, 2004, 2006), and their basic arguments carry over to our model.

Theorem 3. *Under the conditions of Theorems 2a and 2b, the auction's equilibrium is unique.*[13]

We prove Theorem 3 in the Supplemental Material, but the reasoning is familiar and we briefly outline it here. First, we consider the case of the equilibrium defined in Theorem 2a. We hypothesize that there exists an equilibrium where all group-1 bidders place a bid strictly above r_2 with positive probability. Given this hypothesis, we argue that such an equilibrium must be characterized by the strategy defined in Theorem 2a and explained in the preceding discussion. Confirming this claim requires several steps.

Standard arguments advanced by the authors cited above confirm that the equilibrium must be in nondecreasing, pure strategies that are differentiable almost everywhere. Since the valuation distributions share a common support, all bidders must share a common maximal bid, which we denote by η^*. Near this common maximal

[13]As usual, uniqueness is up to bids placed by a zero measure of each bidders' types. For example, a type-$\hat{s}_1$ group-1 bidder is indifferent between a bid below and a bid above r_2. He may randomize between them without upsetting the equilibrium.

bid, it is well known that the agents' bidding strategies are characterized by a system of differential equations. As shown by Lebrun (1997, Section 5) and Lebrun (1999), under the assumptions of Theorem 2a, this system has a unique solution. Given the symmetry among bidders in each group, this system of differential equations simplifies to (7).

In the special case where all bidders face a common reserve price, Lebrun (1999) shows that all equilibria must be in continuous strategies. As illustrated above, this observation does not carry over to our setting. In particular, bidder i in group 1 may wish to increase his bid discontinuously from a value below r_2 to a value above r_2. In fact, the equilibrium strategy of a group-1 bidder must exhibit a discontinuity conditional on the bidder placing a bid above r_2.[14] We can adapt the arguments presented by Lebrun (1999) to show that this type of discontinuity is the only kind that can occur in our setting. Moreover, it can be established that there exists a common value $\hat{s}_1$ at which each group-1 bidder's strategy jumps discontinuously from a bid below r_2 to a bid above r_2.

Given the common point of discontinuity in the strategies of group-1 agents, standard arguments let us conclude that the equilibrium bidding strategy of a group-1 bidder when placing bids below r_2 must coincide with the usual symmetric equilibrium bidding strategy, as defined in (5). An analogous conclusion applies to group-2 bidders whenever they place bids in a range that is avoided by bidders from group 1. Together these facts let us conclude that all equilibria where all group-1 bidders bid above r_2 with positive probability are characterized by Theorem 2a. Finally, we can show that two equilibria as described by Theorem 2a cannot coexist.

Next we consider the converse case and we hypothesize that there exists an equilibrium where all group-1 bidders bid only below r_2. Thus, the bids of both groups are segregated. Standard arguments readily show that if there exists an equilibrium with this feature, bidders in group k must bid according to (8).

We conclude with some housekeeping points. First, we rule out the possibility of an equilibrium where some group-1 bidders place a bid above r_2 with positive probability while others do not. And second, we show that an equilibrium where all group-1 bidders bid above r_2 with positive probability cannot coexist with an equilibrium where all group-1 bidders bid exclusively below r_2. The uniqueness of the equilibria reported in Theorems 2a and 2b follows from these final observations. A corollary establishes the equilibrium's uniqueness in the symmetric model.

3. Welfare and revenue

In this section we investigate the welfare and revenue implications of group-specific reserve prices. For tractability, we consider the symmetric model from Section 1. This is arguably the most difficult environment in which an asymmetric policy may offer an improvement upon its symmetric analogue, particularly concerning revenues. A continuity argument ensures that analogous implications apply to nearby asymmetric cases.

[14] The binding reserve price for bidders in group 2 implies that the probability that a group-1 bidder wins the auction increases discontinuously at r_2. Thus, any group-1 bidder of type $s \geq r_2$ placing a bid of $r_2 - \varepsilon$ can substantially improve his payoff by bidding infinitesimally above r_2.

3.1 *Welfare*

As noted in the introduction, reserve-price differences allow an auctioneer to channel benefits to a favored group. For instance, under the conditions of Theorem 1a, the equilibrium expected utility of a type-s group-1 agent as a function of r_1 and r_2 is

$$V_1(r_1, r_2|s) = \begin{cases} 0 & \text{if } s < r_1, \\ F(r_2)^{N_2} \int_{r_1}^{s} F(z)^{N_1-1}\, dz & \text{if } s \in [r_1, \hat{s}], \\ F(r_2)^{N_2} \int_{r_1}^{\hat{s}} F(z)^{N_1-1}\, dz + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\, dz & \text{if } s \in (\hat{s}, \bar{s}]. \end{cases}$$

Likewise, the equilibrium expected utility of a type-s group-2 agent is

$$V_2(r_1, r_2|s) = \begin{cases} 0 & \text{if } s < r_2, \\ F(\hat{s})^{N_1} \int_{r_2}^{s} F(z)^{N_2-1}\, dz & \text{if } s \in [r_2, \hat{s}], \\ F(\hat{s})^{N_1} \int_{r_2}^{\hat{s}} F(z)^{N_2-1}\, dz + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\, dz & \text{if } s \in (\hat{s}, \bar{s}]. \end{cases}$$

When $r_1 < r_2$, $V_1(r_1, r_2|s) \geq V_2(r_1, r_2|s)$ with a strict inequality for all $s \in (r_1, \hat{s})$. Thus, asymmetric reserve prices target benefits toward relatively disadvantaged, or marginal, group-1 members. Reductions in r_1 also lead to across-the-board welfare gains.

Theorem 4. *Consider a semi-separating equilibrium with r_2 fixed. A reduction of r_1 strictly increases the equilibrium expected payoff of all bidders who place a competitive bid in the auction.*

For agents enjoying a lower reserve price, Theorem 4 is almost trivial. Less clear is the impact on agents whose reserve price is unchanged because their opponents are now advantaged. When r_1 declines, each group-2 bidder gains from the equilibrium response of group-1 bidders. The ability to bid less encourages members of group 1 with relatively low valuations to reduce their bid, which increases the probability that a group-2 member wins the auction. Furthermore, it reduces the effective competition faced by members of group 2. Both effects contribute to the welfare gain.

Theorem 4 is relevant for an auctioneer with ancillary distributional or affirmative action objectives. If the status quo policy is an auction with a common reserve price, then lowering the reserve price of a favored group directly benefits its members. Moreover, members of the unfavored group gain as well, likely reducing opposition to the discriminatory change.[15]

[15] When $r_2 > r_1$, a reduction in r_2 (with r_1 fixed) benefits group-2 bidders. A low-valuation group-1 bidder is worse off. He now wins with a lower probability. A high-valuation group-1 bidder is better off. The magnitude of the equilibrium strategy's "jump" is smaller and he does not need to bid as much in equilibrium as beforehand.

3.2 *Revenue*

An auctioneer may be able to boost revenues by reducing the reserve price faced by some bidders in an auction. By generalizing Example 1, we see one reason this may be so.

Example 2. Consider a first-price auction with two bidders $\{1, 2\}$ and discrete valuations. With probability p, a bidder's valuation is $\underline{s} > 0$; with probability $1 - p$, it is $\bar{s} > \underline{s}$. When the reserve price is $\underline{s}$, the auction's expected revenue is $\bar{s} - (\bar{s} - \underline{s})p(2 - p)$. If the reserve price is $\bar{s}$, the expected revenue is $(1 - p^2)\bar{s}$. Finally, if bidder 1 faces a reserve price of $r_1 = \underline{s}$ and bidder 2 faces a reserve price of $r_2 = \bar{s}$, the expected revenue is $p\underline{s} + (1 - p)\bar{s}$. The asymmetric policy generates more revenue than the other options if and only if

$$p \geq \frac{\bar{s} - \underline{s}}{\bar{s}}. \tag{9}$$

Hence, compared to a reserve price of $\bar{s}$, the auctioneer gains by reducing bidder 1's reserve price if the likelihood of bidder 1 having a low valuation exceeds the (percentage) revenue loss associated with bidder 1's reduction in bid. The greater probability of a successful sale compensates for the (possibly) lower final price.[16] ♢

While Example 2's intuition is straightforward, its translation to a case with a continuum of valuations is less immediate. To make headway, we follow Myerson (1981) by first writing the auction's expected revenue as a function of the induced equilibrium allocation rule. To provide this formulation succinctly, we introduce some notation. Let $\tilde{s}_k$ be the highest valuation among group-k members. This value's c.d.f. is $G_k(s) := F(s)^{N_k}$. Let $g_k(s)$ be the associated probability density function (p.d.f.). As shown by Lemma 3 in the Appendix, the auction's expected revenue can be written as

$$\int_0^{\bar{s}} \int_0^{\bar{s}} \left[\sum_{k=1}^{2} \psi_k(\tilde{s}_1, \tilde{s}_2) J(\tilde{s}_k) \right] g_1(\tilde{s}_1) g_2(\tilde{s}_2)\, d\tilde{s}_1\, d\tilde{s}_2, \tag{10}$$

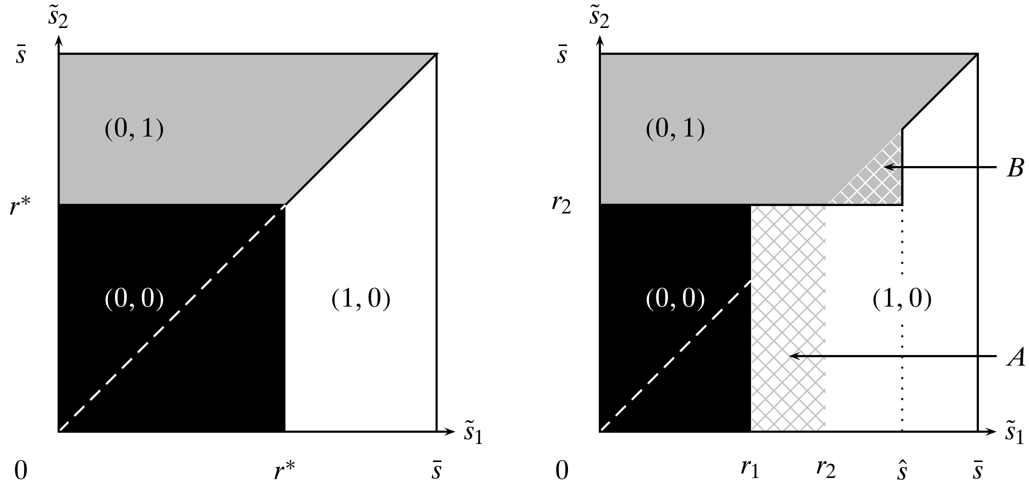
where

$$J(s) := s - \frac{1 - F(s)}{f(s)}$$

is the virtual valuation of a type-s bidder and the function $\psi_k(\tilde{s}_1, \tilde{s}_2)$ is the equilibrium (group) allocation rule. The function $\psi_k(\tilde{s}_1, \tilde{s}_2)$ specifies the probability that a group-k member wins the auction given the realized type profile. This function varies with the prevailing reserve prices. When there is a common reserve price, i.e., $r_1 = r_2 = r^*$, then

$$\psi_k(\tilde{s}_1, \tilde{s}_2) = \begin{cases} 0 & \text{if } \tilde{s}_k < r^*, \\ \dfrac{1}{2} & \text{if } \tilde{s}_k = \tilde{s}_j, \tilde{s}_k \geq r^*, \\ 1 & \text{if } \tilde{s}_k > \tilde{s}_j, \tilde{s}_k \geq r^*, \end{cases} \quad k \neq j. \tag{11}$$

[16]The asymmetric policy is revenue-equivalent to the optimal mechanism when (9) holds.



(a) (ψ_1, ψ_2) with a common reserve price r^*.

(b) (ψ_1, ψ_2) with reserve prices $r_1 < r_2$.

FIGURE 3. Allocation rules in the first-price auction.

If instead $r_1 < r_2$, up to a set of zero measure,

$$\begin{aligned}
\psi_1(\tilde{s}_1, \tilde{s}_2) &= \begin{cases} 0 & \text{if } \tilde{s}_1 < r_1, \\ 1 & \text{if } r_1 < \tilde{s}_1 < \hat{s}, \tilde{s}_2 < r_2, \\ 1 & \text{if } \tilde{s}_1 > \hat{s}, \tilde{s}_1 > \tilde{s}_2, \end{cases} \quad \text{and} \\
\psi_2(\tilde{s}_1, \tilde{s}_2) &= \begin{cases} 0 & \text{if } \tilde{s}_2 < r_2, \\ 1 & \text{if } r_2 < \tilde{s}_2 < \hat{s}, \tilde{s}_1 < \hat{s}, \\ 1 & \text{if } \tilde{s}_2 > \hat{s}, \tilde{s}_2 > \tilde{s}_1. \end{cases}
\end{aligned} \tag{12}$$

Figure 3 illustrates (ψ_1, ψ_2) in $(\tilde{s}_1, \tilde{s}_2)$ space in each of the preceding cases.

When does allocation rule (12) lead to greater revenue than (11)? First, Riley and Samuelson (1981) show that a revenue-maximizing common reserve price solves $J(r^*) = 0$. Furthermore, if $J(s)$ is increasing, there exists a unique r^* such that $J(r^*) = 0$ and a first-price auction with reserve price r^* is an optimal selling *mechanism*—any other selling procedure necessarily generates (weakly) lower expected revenues (Myerson 1981). When $J(s)$ is increasing, the distribution of valuations is commonly called *regular*. Therefore, an irregular distribution of valuations is a prerequisite for different reserve prices to boost revenues above the common reserve-price benchmark when valuations assume a continuum of valuations.[17]

[17]The connotation surrounding "regular" and "irregular" distributions is unfortunate. The latter are empirically relevant and easily arise in practice. For instance, multimodal distributions are often irregular. To illustrate, consider a mixture of two normal distributions with different means:

$$F(s) = \frac{1}{2}\Phi\left(\frac{s-1}{0.25}\right) + \frac{1}{2}\Phi\left(\frac{s-1.9}{0.25}\right).$$

Though irregularity is a necessary condition, it is of course not sufficient. To further discipline our analysis, we henceforth assume that the revenue-maximizing common reserve price r^* has been identified and the auctioneer considers *lowering* the reserve price faced by agents in group 1 holding fixed the reserve price faced by bidders in group 2, i.e., $r_1 < r_2 = r^*$.[18] In this case, (11) and (12) differ only in two contingencies. We illustrate these cases as the cross-hatched regions A and B in Figure 3(b). When $(\tilde{s}_1, \tilde{s}_2) \in A$, the good is allocated to a group-1 bidder who bids between r_1 and r_2, and all group-2 agents have a valuation below r_2. When $(\tilde{s}_1, \tilde{s}_2) \in B$, the highest-bidding group-2 agent wins the auction, but his valuation is less than the highest valuation among agents in group 1. If the asymmetric policy is to generate greater expected revenue, the net revenue contribution of the above changes must be positive. More formally,

$$\begin{aligned} &\Gamma(r_1, r_2) \\ &\quad := \overbrace{G_2(r_2)\int_{r_1}^{r_2} J(\tilde{s}_1)g_1(\tilde{s}_1)\,d\tilde{s}_1}^{\text{(A)}} \\ &\qquad + \underbrace{\int_{r_2}^{\hat{s}} \big(G_1(\hat{s}) - G_1(\tilde{s}_2)\big)J(\tilde{s}_2)g_2(\tilde{s}_2)\,d\tilde{s}_2 - \int_{r_2}^{\hat{s}} \big(G_2(\tilde{s}_1) - G_2(r_2)\big)J(\tilde{s}_1)g_1(\tilde{s}_1)\,d\tilde{s}_1}_{\text{(B)}} > 0. \end{aligned} \tag{13}$$

Term (A) is the change in revenue associated with allocating the item to an agent when $(\tilde{s}_1, \tilde{s}_2) \in A$. Term (B) is the change in revenue associated with the reallocation of the item from a group-1 agent to a group-2 agent when $(\tilde{s}_1, \tilde{s}_2) \in B$.

If $\Gamma(r_1, r_2) > 0$, term (A), term (B), or both must be strictly positive. In principle, each of these cases is possible. First, consider term (A). It is strictly positive only if $J(s) > 0$ for some $s \in (r_1, r_2)$. When $r_2 = r^*$, as assumed above, then $J(r_2) = 0$. Thus, if (A) is positive, $J(s)$ must be nonmonotone when $s < r_2$, crossing zero multiple times. Figure 4 illustrates a simple case.

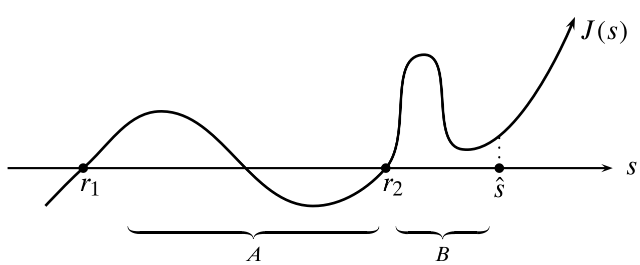


FIGURE 4. Gains and losses from reserve prices $r_1 < r_2$.

While the normal distribution is regular, $F(s)$ is not. The valuation $J(s)$ crosses zero at 0.972, 1.310, and 1.492.

[18] A parallel analysis is possible for the $r^* = r_1 < r_2$ case. As the reasoning is similar, we omit it for brevity.

Suppose there exists an $r_1 < r_2$ that ensures that term (A) is positive. It is instructive to see why r_1 was *not* the initial revenue-maximizing common reserve price. When the common reserve price is r_1, we can write the auction's expected revenue as

$$\int_{r_1}^{\bar{s}} J(s)g(s)\,ds = \int_{r_1}^{r_2} J(s)g(s)\,ds + \int_{r_2}^{\bar{s}} J(s)g(s)\,ds,$$

where $g(s)$ is the p.d.f. associated with the c.d.f. $G(s) := F(s)^N$; $N = N_1 + N_2$ is the total number of bidders in the auction. If r_2 is the revenue-maximizing uniform reserve price, it follows that $\int_{r_1}^{r_2} J(s)g(s)\,ds \leq 0$. Nevertheless, it is possible that $\int_{r_1}^{r_2} J(s)g_1(s)\,ds > 0$, which appears in term (A) of (13).[19] The distribution $G(s)$ likelihood ratio dominates $G_1(s)$. Thus, $G_1(s)$ places less probability weight on valuations near r_2, where $J(s) < 0$, than $G(s)$. By allowing bids from only a few agents, the auctioneer skews the distribution of the highest valuation among participating bidders toward lower values where $J(s)$ is positive.[20]

Similar nonmonotonicities are necessary for term (B) in (13) to be positive. When $(\tilde{s}_1, \tilde{s}_2) \in B$, the item is allocated to the highest-valuation group-2 agent, but there exists a group-1 agent with a higher valuation, i.e., $\tilde{s}_1 > \tilde{s}_2$. If $J(s)$ is not monotone on $(r_2, \hat{s})$, as in Figure 4, the winning group-2 agent may have a higher virtual valuation, i.e., $J(\tilde{s}_1) < J(\tilde{s}_2)$. If on average such inefficient allocations occur sufficiently frequently, expected revenue will rise. The intensity of this effect depends on the distribution of valuations, the severity of $J(s)$'s nonmonotonicity, and the relative sizes of groups 1 and 2.

The underlying mechanics behind the revenue gains described above can be interpreted as an approximation of Myerson's (1981) ironing procedure, which is a common step when deriving the optimal selling mechanism when $J(s)$ is not monotone. The lottery derived with the ironing procedure pools agents with high signals and low virtual valuations with agents with low signals and high virtual valuations. Relative to an (ex post) efficient allocation, the lottery skews the expected allocation toward agents with lower valuations, but higher virtual valuations. As explained above, a similar effect operates in our setting.

An implication of the above analysis is that a nontrivial gap between r_1 and r_2 is usually required to increase revenues. Furthermore, the change in revenue is sensitive to the form of $J(s)$ and $F(s)$ over an entire range of values, and it depends on $\hat{s}$, which itself is endogenously defined in equilibrium. Identifying assumptions on $F(s)$ ensuring that $J(s)$ exhibits the right peaks and valleys for inequality (13) to hold is far from trivial.

Some insight can be gleamed by considering a local argument. For instance, when does an arbitrarily small reduction of r_1 relative to the revenue-maximizing common reserve price r^* generate a revenue gain? Clearly, if $J(s)$ is continuous and increasing

[19] The distribution noted in footnote 17 has this property when, for instance, $N_1 = 2$, $N_2 = 5$, $r_1 = 0.9724$, and $r_2 = 1.4924$. The revenue-maximizing common reserve price is $r^* = r_2$.

[20] Intuitively, the reserve price $r^* = r_2$ is too high when there are fewer bidders. Generally, optimal reserve price depends on the number of bidders when $F(\cdot)$ is not regular. It is independent of the number of bidders when $F(\cdot)$ is regular (Riley and Samuelson 1981).

at r^*, a sufficiently small gap between r_1 and $r_2 = r^*$ will decrease revenues. Both terms (A) and (B) in (13) will be negative.[21] If, however, we relax our maintained assumption that $f(s)$ is continuous, then even a small difference between r_1 and r_2 can be revenue enhancing.[22] To simplify notation, for any function $h(\cdot)$, let $h(r^-) := \lim_{s\to r^-} h(s)$ and $h(r^+) := \lim_{s\to r^+} h(s)$.

Theorem 5. *Let r^* be the revenue-maximizing (common) reserve price. Suppose $J(s)$ is twice continuously differentiable, except at r^* where (i) $f(r^{*-}) < f(r^{*+}) < \infty$, (ii) $J(r^{*-}) = 0$, and (iii) the left and right derivatives of $J(\cdot)$ exist and $|J'(r^{*-})| < \infty$ and $J'(r^{*+}) < 0$. For all $\varepsilon > 0$ sufficiently small, a first-price auction with reserve prices $r_1 = r^* - \varepsilon$ and $r_2 = r^*$ generates strictly greater revenue than a first-price auction with common reserve price r^*.*

Relating to prior discussion, under the conditions of Theorem 5, term (B) in (13) is positive while term (A) is negative, but much smaller in magnitude. The net effect is a revenue gain. It is important to note, however, that Theorem 5 seeks to squeeze out a revenue gain from an arbitrarily small reduction in r_1, which necessitates its particular technical restrictions around r^*. A nontrivial gap between r_1 and r_2 is often more effective.

Example 3. Suppose there are two bidders, one in each group. The c.d.f. of each bidder's valuation is

$$F(s) = \begin{cases} s & \text{if } s \in [0, 1/2), \\ \dfrac{1}{2} + \dfrac{4s-1}{6}\sqrt{2s-1} & \text{if } s \in [1/2, 1]. \end{cases}$$

Though unfamiliar, the c.d.f. $F(\cdot)$ is very close to the uniform distribution. In fact, $|s - F(s)| < 0.016$ for all $s \in [0, 1]$. Figure 5(a) presents the associated $J(s)$ function. The revenue-maximizing common reserve price is $r^* = 1/2$. To illustrate the gain from an asymmetric reserve-price policy, we plot a normalization of $\Gamma(r_1, 1/2)$, defined in (13), in Figure 5(b). The realized gain $\Gamma(r_1, 1/2)$ is scaled by the additional revenue one can gain by implementing Myerson's (1981) optimal auction instead of the uniform reserve-price auction.[23] Hence, we compare the realized revenue gain with the theoretical maximum revenue gain. When $r_1 = 0.487$, the asymmetric policy attains 72 percent of this theoretical maximal additional revenue. ◊

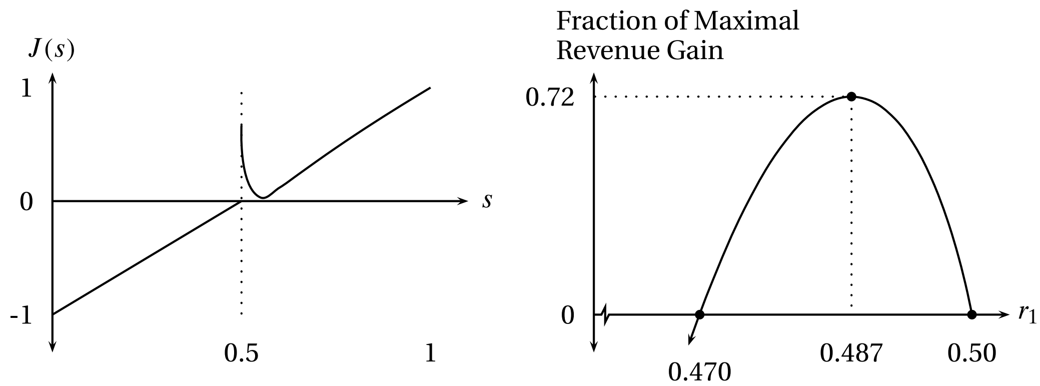
4. Comparisons and extensions

In this section, we provide additional context for our analysis by presenting extensions and variants of our model.

[21] If $|r_1 - r_2| \to 0$, then $\hat{s} \to r_2^+$. Thus, $J(s)$ is increasing for all $s \in (r_1, \hat{s})$.

[22] Discontinuities in $f(s)$ may occur when bidders are a pooled sample of agents whose types are originally distributed with different supports. Additionally, while the proof of Theorem 1a assumes $F(s)$ is differentiable, this requirement can be relaxed to differentiability almost everywhere.

[23] The optimal auction involves a lottery. If $s_1 < 0.5$ and $s_2 < 0.5$, the item is left unallocated. When s_1 and s_2 are both between 0.5 and 0.6, the winner is chosen by a coin flip. Otherwise, the agent with the highest type gets the item.



(a) The function $J(s)$ in Example 3.

(b) Normalized change in revenue with reserve prices r_1 and $r_2 = 1/2$.

FIGURE 5. Virtual valuation and revenue gain in Example 3.

4.1 *Bid subsidies*

Asymmetric reserve prices can bias an auction in favor of some bidders. A traditional instrument used to tilt an auction in this manner is the bid subsidy. A bidder who bids b and benefits from a subsidy will have his bid evaluated *as if* it is $(1+\alpha)b$, with $\alpha > 0$ being the subsidy rate. As we illustrate in the following example, subsidies and asymmetric reserve prices are not equivalent instruments. While the example departs from our baseline setting (bidders are asymmetric and one agent's valuations assume discrete values), there exist nearby continuous cases where similar conclusions apply.

EXAMPLE 4. Consider a first-price auction with two bidders. Bidder 1's valuation s_1 is distributed uniformly on the unit interval. Bidder 2's valuation is $3/4$ with probability $1-\gamma$ and 0 with probability γ. To minimize the number of cases, assume $0 < \gamma < 1/3$. To avoid technical complications due to the discrete budget distribution, suppose ties are broken in favor of bidder 1. The revenue-maximizing common reserve price is $r^* = 3/4$. Both bidders bid $3/4$ when their valuation exceeds $3/4$ and the resulting expected revenue is

$$R^*(\gamma) = \frac{3}{4} - \frac{9\gamma}{16}.$$

Now suppose bidder 1's reserve price is lowered to $r_1 = 1/2$. For bidder 2, $r_2 = 3/4$ as before. Now bidder 1 will bid $1/2$ when $s_1 < \hat{s}_1 = \frac{3-2\gamma}{4-4\gamma}$; when $s_1 > \hat{s}_1$, he bids $3/4$. The resulting revenue is

$$R^{\text{Asym}}(\gamma) = \frac{\gamma(4\gamma - 9) + 6}{8 - 8\gamma}.$$

It is simple to verify that the reserve-price asymmetry boosts revenue, i.e., $R^{\text{Asym}}(\gamma) > R^*(\gamma)$.

Suppose instead the auctioneer maintains the common reserve price $r^* = 3/4$ but he subsidizes bidder 1. Specifically, and in line with common practice, he adopts a linear subsidy. If bidder 1 bids b_1, he pays b_1 when he is the winner. He wins the auction when $(1+\alpha)b_1 \geq \max\{b_2, r^*\}$, where b_2 is bidder 2's bid. The parameter $\alpha \geq 0$ is the subsidy rate. Note that this policy allows bidder 1 to bid (and pay) less than the reserve price $r^* = 3/4$ and still win the auction. Thus, it is seemingly similar to introducing a reserve-price asymmetry.

In the resulting equilibrium, bidder 2 continues to bid 3/4 when his valuation is high. Bidder 1 bids $3/(4+4\alpha)$ when $s_1 \geq \check{s}_1 = 3/(4+4\alpha)$. Due to his subsidy, bidder 1 wins the auction with this bid. When $s_1 < \check{s}_1$, bidder 1 does not enter the auction. The resulting revenue is

$$R^{\alpha}(\gamma) = \frac{12 + 21\alpha - 9\gamma - 9\alpha\gamma}{16(\alpha+1)^2}.$$

Since $\gamma < 1/3$, $R^{\alpha}(\gamma)$ is decreasing in α. Therefore, the revenue-maximizing subsidy is zero and thus $R^{\alpha}(\gamma) \leq R^*(\gamma) < R^{\text{Asym}}(\gamma)$. ◊

In the preceding example, the subsidy is costly in terms of revenue as it allows bidder 1 to pay a reduced price for the item even when his valuation is very high. In contrast, the reduction in r_1 allows bidder 1 to pay a reduced price mainly when his valuation is relatively low and he is more disadvantaged. The reserve-price asymmetry "gives a leg up" to marginal types in the favored group. Agents with high valuations are still incentivized to bid aggressively.

It is important to emphasize that reserve-price adjustments and subsidies are complementary policies. Whether one is preferable depends on the auctioneer's objective, the default status quo (i.e., what will happen in the absence of any intervention), and the particulars of the problem. More elaborate objectives may require both to be used in tandem.

4.2 *Entry fees*

The revenue equivalence theorem identifies reserve prices and entry fees as equivalent instruments in a symmetric setting. Riley and Samuelson (1981) show that an auctioneer can set a reserve price of r or charge an entry fee of $c = rF(r)^{N-1}$ to earn the same expected revenue. Group-specific entry fees and group-specific reserve prices are not equivalent policies. To see why, consider an auction with entry fees and define $\check{s}_k$ as the lowest valuation for which a group-k bidder finds it worthwhile to pay the entry fee c_k and to place a bid in the auction. The proof of the following theorem and corollary are provided in the Supplemental Material.

Theorem 6. *Let* $0 \leq \check{s}_1 < \check{s}_2 \leq \bar{s}$ *and define*

$$c_1 := F(\check{s}_1)^{N_1-1} F(\check{s}_2)^{N_2} \check{s}_1,$$

$$c_2 := F(\check{s}_2)^{N_2} \int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\, dz + F(\check{s}_1)^{N_1-1} F(\check{s}_2)^{N_2} \check{s}_1$$

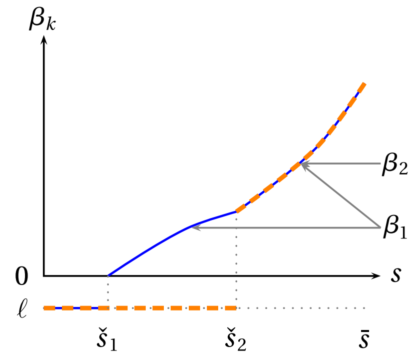


FIGURE 6. Equilibrium bidding in the first-price auction with entry fees $c_1 < c_2$.

as the entry fees charged to bidders of group 1 *and group* 2*, respectively.*[24] *There exists a group-symmetric equilibrium with entry fees where*

$$\beta_1(s) = \begin{cases} \ell & \text{if } s < \check{s}_1, \\ s - \displaystyle\int_{\check{s}_1}^{s} \left[\frac{F(z)}{F(s)}\right]^{N_1-1} dz - \frac{c_1}{F(s)^{N_1-1}F(\check{s}_2)^{N_2}} & \text{if } s \in [\check{s}_1, \check{s}_2), \\ s - \displaystyle\int_{\check{s}_2}^{s} \left[\frac{F(z)}{F(s)}\right]^{N_1+N_2-1} dz - \frac{c_2}{F(s)^{N_1+N_2-1}} & \text{if } s \geq \check{s}_2, \end{cases}$$

$$\beta_2(s) = \begin{cases} \ell & \text{if } s < \check{s}_2, \\ s - \displaystyle\int_{\check{s}_2}^{s} \left[\frac{F(z)}{F(s)}\right]^{N_1+N_2-1} dz - \frac{c_2}{F(s)^{N_1+N_2-1}} & \text{if } s \geq \check{s}_2. \end{cases}$$

Figure 6 illustrates the defined equilibrium strategy. As suggested by the notation, $\check{s}_k$ is the cutoff type of a group-k bidder placing a competitive bid in the auction. Welfare and revenue implications are now easy to derive.

COROLLARY 1. *Consider a first-price auction with entry fees $c_1 \leq c_2$. Suppose that the entry fee of group-*1 *bidders is reduced to $c_1' < c_1$ while the entry fee of group-*2 *bidders remains constant, i.e., $c_2 = c_2'$. The expected payoff of a group-*1 *bidder improves while the expected payoff of a group-*2 *bidder declines.*

The lower fee aids group-1 agents directly and encourages greater entry. Consequently, group-2 members need to bid more aggressively, which reduces their expected payoff.

From Theorem 6, we see that a group-1 bidder is never outbid by a group-2 bidder with a lower valuation. Hence, entry fees and reserve prices are not revenue equivalent as the induced equilibrium allocation rules differ. A revenue ranking is possible when the valuation distribution is regular.

[24] Formally, $\check{s}_k$ is a function of the posted entry fees. Defining the equilibrium strategy with reference to the participants' cutoff types instead of the entry fees directly is equivalent and analytically simpler.

Corollary 2. *Consider a first-price auction with reserve prices and a first-price auction with entry fees. Suppose reserve prices and entry fees are set so that the same types of bidders place competitive bids in each auction, i.e., $r_k = \check{s}_k$ for all k. If $J(s)$ is nondecreasing, the expected revenue of the auction with entry fees exceeds the expected revenue of the auction with reserve prices.*

4.3 *Other auction formats*

Though our study focuses on the first-price auction, we briefly remark on reserve-price asymmetries in other auction formats. In a second-price auction, the bidder submitting the highest competitive bid wins the item. If a bidder in group k wins the auction, he pays a price equal to the second-highest bid or to r_k, whichever is greater. Reserve prices do not alter the usual dominant-strategy equilibrium of the second-price auction. It is weakly dominant for a group-k bidder to bid his value if it exceeds r_k and bid ℓ otherwise.

First- and second-price auctions differ along both welfare and revenue dimensions. First, lowering r_1 in a second-price auction, without changing r_2, improves the welfare of group-1 agents but fails to strictly benefit bidders in group 2. Second, the two mechanisms generate different revenue. In fact, the second-price auction's equilibrium allocation rule has the same form as the first-price auction's allocation rule with entry fees. Hence, Corollary 2 applies. Also assuming a regular type distribution, Mares and Swinkels (2014) identify a similar superiority of second-price auctions, though accommodating a different policy parameter. They consider handicap auctions where the favored party's bid is subsidized. To derive their ranking, Mares and Swinkels (2014) note that the equilibrium allocation rule in the (optimal) second-price handicap auction is a smaller departure from the efficient allocation rule. A similar geometric comparison underlies Corollary 2.

Another important auction format is the all-pay auction. In this mechanism the highest bidder wins, but every bidder makes a payment equal to his bid. Though unusual as a selling procedure, all-pay auctions have been employed to study political lobbying and other contest-like activities (Baye et al. 1993). Biasing all-pay contests has attracted considerable research, with policies such as bid caps (Che and Gale 1998), handicaps, and head starts (Kirkegaard 2012, 2013) being proposed and evaluated. With group-specific reserve prices, the qualitative nature of equilibria in the all-pay auction mirrors the first-price case. Group-symmetric equilibria may be either semi-separating or separating. We explore the details in a working paper (Kotowski 2014).

5. Concluding remarks

We have developed a simple model that accommodates asymmetric reserve prices in standard auctions. While treating possibly symmetric bidders asymmetrically may at first seem arbitrary, nonintuitive, or unfair, it may constitute a simple and worthwhile policy change for all parties involved. Furthermore, such asymmetric policies offer an alternative route when pursuing ancillary distributional objectives. They complement traditional approaches, such as subsidies, and may at times yield preferable outcomes.

Appendix A: Proofs for Section 1

Lemma 1. *Suppose* $0 \le r_1 < r_2 < \bar{s}$ *and let* $N_k \ge 1$. *Define the functions*

$$p(s) := F(r_2)^{N_2} \int_{r_1}^{s} F(z)^{N_1 - 1}\, dz,$$

$$q(s) := \int_{r_2}^{s} F(s)^{N_1} F(z)^{N_2 - 1}\, dz.$$

If $p(\bar{s}) \le q(\bar{s})$, *then there exists a unique* $\hat{s} \in [r_2, \bar{s}]$ *such that* $p(\hat{s}) = q(\hat{s})$. *Otherwise, if* $p(\bar{s}) > q(\bar{s})$, *then* $p(s) > q(s)$ *for all* $s \in [r_2, \bar{s}]$.

Proof. Suppose $p(\bar{s}) \le q(\bar{s})$; $p(s)$ and $q(s)$ are continuous and strictly increasing. Moreover, $p(r_2) > 0 = q(r_2)$. By the intermediate value theorem, there exists $\hat{s} \in (r_2, \bar{s}]$ such that $p(\hat{s}) = q(\hat{s})$. To verify uniqueness it is sufficient to observe that $p'(s) = F(r_2)^{N_2} F(s)^{N_1 - 1}$,

$$q'(s) = F(s)^{N_1 + N_2 - 1} + \int_{r_2}^{s} \frac{d}{ds} F(s)^{N_1} F(z)^{N_2 - 1}\, dz,$$

and $q'(s) > p'(s)$ for all $s > \hat{s} > r_2$. Hence, $q(s) > p(s)$ for all $s > \hat{s}$. Suppose instead that $p(\bar{s}) > q(\bar{s})$. Since $q'(s) > p'(s)$ for all $s \in (r_2, \bar{s}]$, $p(s) > q(s)$ for all $s \in [r_2, \bar{s}]$. □

Proof of Theorem 1a. The following argument is standard, but depends on checking many cases.[25] We rule out deviations by players to alternative bids in the ranges of β_1 and β_2. (Other bids are easily seen to be dominated.) For notation, let $U_k(b|s)$ be the expected utility of a bidder in group k of type s when he bids b given that all other bidders are following the strategy prescribed in the theorem.

It is optimal for group-k bidders of type $s < r_k$ to bid ℓ. Thus, we focus on ruling out deviations to competitive bids greater than r_k by bidders of type $s \ge r_k$. We divide our analysis into cases depending on the group membership and the type of the bidder who considers deviating.

Case 1: A group-1 bidder of type $s \in [r_1, \hat{s}]$. When this bidder bids $\beta_1(s)$, his expected payoff is

$$U_1\big(\beta_1(s)|s\big) = F(r_2)^{N_2} \int_{r_1}^{s} F(z)^{N_1 - 1}\, dz.$$

(a) Suppose this bidder bids $\beta_1(t)$, $t \in [r_1, \hat{s}]$. Then $U_1(\beta_1(t)|s) = F(r_2)^{N_2} \times F(t)^{N_1 - 1}(s - t) + F(r_2)^{N_2} \int_{r_1}^{t} F(z)^{N_1 - 1}\, dz$. Therefore,

$$U_1\big(\beta_1(t)|s\big) - U_1\big(\beta_1(s)|s\big) = F(r_2)^{N_2} \int_{s}^{t} \big(F(z)^{N_1 - 1} - F(t)^{N_1 - 1}\big)\, dz \le 0.$$

Hence, this is not a profitable deviation.

[25] Krishna (2002, pp. 17–18) presents essentially the same reasoning for the case without reserve prices.

(b) Suppose this bidder bids $\beta_2(t)$, $t \in [r_2, \hat{s}]$. Then

$$U_1\big(\beta_2(t)|s\big) = F(\hat{s})^{N_1-1}F(t)^{N_2}(s-t) + F(\hat{s})^{N_1-1}F(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz.$$

Suppose first that $t \in [s, \hat{s}]$. Let $\Delta(t,s) := U_1(\beta_2(t)|s) - U_1(\beta_1(s)|s)$. Then

$$\frac{d}{ds}\Delta(t,s) = F(\hat{s})^{N_1-1}F(t)^{N_2} - F(r_2)^{N_2}F(s)^{N_1-1} \geq 0.$$

Hence, $\Delta(t,s) \leq \Delta(t,t) = F(\hat{s})^{N_1-1}F(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz - F(r_2)^{N_2} \times \int_{r_1}^{t} F(z)^{N_1-1}\,dz$. Since $r_2 \leq t \leq \hat{s}$,

$$\begin{aligned}
&\frac{d}{dt}\Delta(t,t) \\
&\quad = F(\hat{s})^{N_1-1}f(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz \\
&\qquad + F(\hat{s})^{N_1-1}F(t)^{N_2} - F(r_2)^{N_2}F(t)^{N_1-1} \\
&\quad \geq 0.
\end{aligned}$$

Thus, $\Delta(t,s) \leq \Delta(t,t) \leq \Delta(\hat{s},\hat{s}) = 0$. The final equality follows from the definition of $\hat{s}$. Thus, $U_1(\beta_2(t)|s) \leq U_1(\beta_1(t)|s)$ for all $t \in [s, \hat{s}]$.

If instead $t \in [r_2, s)$, then

$$\begin{aligned}
\frac{d}{dt}U_1\big(\beta_2(t)|s\big) &= F(\hat{s})^{N_1-1}\bigg(N_2F(t)^{N_2-1}f(t)(s-t) + f(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz\bigg) \\
&\geq 0.
\end{aligned}$$

Therefore, $U_1(\beta_2(t)|s) \leq U_1(\beta_2(s)|s) \leq U_1(\beta_1(s)|s)$. The final inequality follows from the preceding case. Thus, a deviation to the bid $\beta_2(t)$ is not profitable.

(c) Suppose this bidder bids $\beta_1(t)$, $t \in (\hat{s}, \bar{s}]$. The expected payoff from this bid is

$$\begin{aligned}
&U_1\big(\beta_1(t)|s\big) \\
&\quad = F(t)^{N_1+N_2-1}\left(s - t + \frac{\displaystyle\int_{r_2}^{\hat{s}} F(\hat{s})^{N_1}F(z)^{N_2-1}\,dz}{F(t)^{N_1+N_2-1}} + \int_{\hat{s}}^{t}\left[\frac{F(z)}{F(t)}\right]^{N_1+N_2-1} dz\right) \\
&\quad = F(t)^{N_1+N_2-1}(s-t) + \left[\int_{r_2}^{\hat{s}} F(\hat{s})^{N_1}F(z)^{N_2-1}\,dz\right] + \int_{\hat{s}}^{t} F(z)^{N_1+N_2-1}\,dz \\
&\quad = \left[\int_{r_1}^{\hat{s}} F(r_2)^{N_2}F(z)^{N_1-1}\,dz\right] + \underbrace{\int_{\hat{s}}^{t}\big[F(z)^{N_1+N_2-1} - F(t)^{N_1+N_2-1}\big]\,dz}_{(-)}
\end{aligned}$$

$$\begin{aligned}
&- \int_{s}^{\hat{s}} F(t)^{N_1+N_2-1}\,dz \\
&\le \int_{r_1}^{\hat{s}} F(r_2)^{N_2} F(z)^{N_1-1}\,dz - \int_{s}^{\hat{s}} F(r_2)^{N_2} F(z)^{N_1-1}\,dz \\
&= \int_{r_1}^{s} F(r_2)^{N_2} F(z)^{N_1-1}\,dz = U_1\big(\beta_1(s)|s\big).
\end{aligned}$$

We have used the definition of $\hat{s}$ in moving from line 2 to line 3 (the terms in the square brackets). Thus, the bid $\beta_1(t)$ is not a profitable deviation.

The preceding cases confirm that no bidder in group 1 of type $s \in [r_1, \hat{s}]$ has a profitable deviation from $\beta_1(s)$.

Case 2: A group-1 bidder of type $s \in (\hat{s}, \bar{s}]$. Some algebra shows that $U_1(\beta_1(s)|s) = U_1(\beta_1(\hat{s})|\hat{s}) + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\,dz$.

(a) Suppose this bidder bids $\beta_1(t)$, $t \in [r_1, \hat{s}]$. The expected utility from placing this bid is

$$U_1\big(\beta_1(t)|s\big) = F(r_2)^{N_2} F(t)^{N_1-1}(s-t) + F(r_2)^{N_2} \int_{r_1}^{t} F(z)^{N_1-1}\,dz.$$

Since $\frac{d}{dt}U_1(\beta_1(t)|s) = F_2(r_2)^{N_2}(N_1 - 1)F(t)^{N_1-2}f(t)(s - t) \ge 0$, $U_1(\beta_1(t)|s) \le U_1(\beta_1(\hat{s})|s) \le U_1(\beta_1(\hat{s})|\hat{s}) \le U_1(\beta_1(s)|s)$. Thus, the bid $\beta_1(t)$, is not a profitable deviation.

(b) Suppose this bidder bids $\beta_2(t)$, $t \in [r_2, \hat{s}]$. The expected payoff from this bid is

$$U_1\big(\beta_2(t)|s\big) = F(\hat{s})^{N_1-1} F(t)^{N_2}(s-t) + F(\hat{s})^{N_1-1} F(t) \int_{r_2}^{t} F(z)^{N_2-1}\,dz.$$

Noting that $\frac{d}{dt}U_1(\beta_2(t)|s) \ge 0$, $U_1(\beta_2(t)|s) \le U_1(\beta_2(\hat{s})|s) \le U_1(\beta_2(\hat{s})|\hat{s}) = U_1(\beta_1(\hat{s})|\hat{s}) \le U_1(\beta_1(s)|s)$. Thus, the bid $\beta_2(t)$ is not a profitable deviation.

(c) Suppose this bidder bids $\beta_1(t)$, $t \in (\hat{s}, \bar{s}]$. By an argument like Case 1(a) above, we can confirm that $U_1(\beta_1(t)|s) \le U_1(\beta_1(s)|s)$ for all $t \in (\hat{s}, \bar{s}]$.

The preceding cases confirm that no bidder in group 1 of type $s \in (\hat{s}, \bar{s}]$ has a profitable deviation from $\beta_1(s)$.

Case 3: A group-2 bidder of type $s \in [r_k, \hat{s}]$. When this bidder bids $\beta_2(s)$, his expected payoff is $U_2(\beta_2(s)|s) = \int_{r_2}^{s} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz$.

(a) Suppose this bidder bids $\beta_2(t)$, $t \in [r_2, \hat{s}]$. By an argument like Case 1(a) above, we can confirm that $U_2(\beta_2(t)|s) \le U_2(\beta_2(s)|s)$ for all $t \in [r_2, \hat{s}]$.

(b) Suppose this bidder bids $\beta_2(t)$, $t \in (\hat{s}, \bar{s}]$. The expected utility from doing so is

$$\begin{aligned} U_2\big(\beta_2(t)|s\big) &= F(t)^{N_1+N_2-1}(s-t) + \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz \\ &\quad + \int_{\hat{s}}^{t} F(z)^{N_1+N_2-1}\,dz \\ &= \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz + \int_{\hat{s}}^{t} \big[F(z)^{N_1+N_2-1} - F(t)^{N_1+N_2-1}\big]\,dz \\ &\quad - \int_{s}^{\hat{s}} F(t)^{N_1+N_2-1}\,dz \\ &\le \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz - \int_{s}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz \\ &= \int_{r_2}^{s} F(r_2)^{N_2} F(z)^{N_1-1}\,dz = U_2\big(\beta_2(s)|s\big). \end{aligned}$$

Thus, the bid $\beta_2(t)$ is not a profitable deviation.

The preceding two cases confirm that no bidder in group 2 of type $s \in [r_2, \hat{s}]$ has a profitable deviation from $\beta_2(s)$.

Case 4: A group-2 bidder of type $s \in (\hat{s}, \bar{s}]$. As in the case of a group-1 bidder with a type in this range, we can write $U_2(\beta_1(s)|s) = U_2(\beta_1(\hat{s})|\hat{s}) + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\,dz$.

(a) Suppose this bidder bids $\beta_2(t)$, $t \in [r_2, \hat{s}]$. The expected payoff from doing so is $U_2(\beta_2(t)|s) = F(\hat{s})^{N_1} F(t)^{N_2-1}(s-t) + F(\hat{s})^{N_1} \int_{r_2}^{t} F(z)^{N_2-1}\,dz$. Since

$$\frac{d}{dt} U_2\big(\beta_2(t)|s\big) = F(\hat{s})^{N_1}(N_2-1)F_2(t)^{N_2-2} f(t)(s-t) \ge 0,$$

we see that $U_2(\beta_2(t)|s) \le U_2(\beta_2(\hat{s})|s) \le U_2(\beta_2(\hat{s})|\hat{s}) \le U_2(\beta_2(s)|s)$. Therefore, the bid $\beta_2(t)$ is not a profitable deviation.

(b) Suppose this bidder bids $\beta_2(t)$, $t \in (\hat{s}, \bar{s}]$. This case is the same as Case 2(c) above.

The preceding cases confirm that no bidder in group 2 of type $s \in (\hat{s}, \bar{s}]$ has a profitable deviation from $\beta_2(s)$.

Our argument has exhausted all cases. Thus, β is an equilibrium. □

Proof of Theorem 1b. A group-k bidder has no profitable deviation to any bid $\beta_k(t)$ for $t \in [r_k, \bar{s}]$. The argument from the proof of Theorem 1a, Cases 1(a) and 3(a), applies by replacing $\hat{s}$ with $\bar{s}$. It remains to show that no bidder in group 1 can benefit from the bid $\beta_2(t)$, $t \in [r_2, \bar{s}]$. Recalling that $F(\bar{s}) = 1$, the associated payoff is $U_1(\beta_2(t)|s) = F(t)^{N_2}(s-t) + F(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz$.

Suppose that $t \in [s, \bar{s}]$. Let $\Delta(t, s) := U_1(\beta_2(t)|s) - U_1(\beta_1(s)|s)$. Then

$$\frac{d}{ds}\Delta(t, s) = F(t)^{N_2} - F(r_2)^{N_2} F(s)^{N_1 - 1} \geq 0.$$

Hence, $\Delta(t, s) \leq \Delta(t, t) = F(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz - F(r_2)^{N_2}\int_{r_1}^{t} F(z)^{N_1-1}\,dz$. However, since $r_2 \leq t \leq \bar{s}$,

$$\frac{d}{dt}\Delta(t, t) = f(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz + F(t)^{N_2} - F(r_2)^{N_2} F(t)^{N_1-1} \geq 0.$$

Thus, $\Delta(t, s) \leq \Delta(t, t) \leq \Delta(\bar{s}, \bar{s}) \leq 0$, where the final inequality follows from (3). Thus, $U_1(\beta_2(t)|s) \leq U_1(\beta_1(s)|s)$ for all $t \in [s, \bar{s}]$.

Suppose instead that $t \in [r_2, s)$. Then

$$\frac{d}{dt}U_1\big(\beta_2(t)|s\big) = N_2 F(t)^{N_2-1} f(t)(s - t) + f(t)\int_{r_2}^{t} F(z)^{N_2-1}\,dz \geq 0.$$

Therefore, $U_1(\beta_2(t)|s) \leq U_1(\beta_2(s)|s) \leq U_1(\beta_1(s)|s)$, where the final inequality follows from the preceding case. Thus, a deviation to the bid $\beta_2(t)$ is not profitable. □

Appendix B: Proofs for Section 3

Lemma 2. *Suppose $\int_{r_1}^{\bar{s}} F(r_2)^{N_2} F(z)^{N_1-1}\,dz < \int_{r_2}^{\bar{s}} F(z)^{N_2-1}\,dz$. Let $\{r_1, r_2\}$ and $\{r_1', r_2'\}$ be two sets of reserve prices. Let $\hat{s}$ ($\hat{s}'$) be the point of discontinuity of the strategy of a typical group-1 bidder when the reserve prices are $\{r_1, r_2\}$ ($\{r_1', r_2'\}$). If $r_1' < r_1$ and $r_2' = r_2$, then $\hat{s}' > \hat{s}$.*

Proof. The values $\hat{s}$, r_1, and r_2 must satisfy

$$\varphi(\hat{s}, r_1, r_2) := \int_{r_1}^{\hat{s}} F(r_2)^{N_2} F(z)^{N_1-1}\,dz - \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz = 0.$$

Likewise, $\varphi(\hat{s}', r_1', r_2') = 0$. In each case, $\hat{s} > r_2$ and $\hat{s}' > r_2$. The function $\varphi(s, r_1, r_2)$ is differentiable in s and

$$\frac{\partial}{\partial s}\varphi(s, r_1, r_2) = F(r_2)^{N_2} F(s)^{N_1-1} - F(s)^{N_1+N_2-1} - \int_{r_2}^{s} \frac{d}{ds} F(s)^{N_1} F(z)^{N_2-1}\,dz.$$

Note that $\frac{\partial}{\partial s}\varphi(s, r_1, r_2) < 0$ when $s > r_2$. Since $\varphi(s, r_1, r_2)$ is decreasing in r_1 when r_2 is fixed, $\varphi(\hat{s}, r_1, r_2) < \varphi(\hat{s}, r_1', r_2) = \varphi(\hat{s}, r_1', r_2')$. Hence, if $\varphi(\hat{s}', r_1', r_2') = 0$, it follows that $\hat{s}' > \hat{s}$. □

Proof of Theorem 4. Let $r_1' < r_1$ and $r_2' = r_2$. From Lemma 2, we know that the reduction in r_1 shifts the point of discontinuity in the strategy of a group-1 bidder to the right, i.e., $\hat{s} < \hat{s}'$.

First, consider a bidder in group 1. If $s < \hat{s}$, then, by inspection, $V_1(r_1', r_2|s) > V_1(r_1, r_2|s)$. If $s \in (\hat{s}, \hat{s}']$,

$$\begin{aligned}
V_1(r_1, r_2|s) &= \int_{r_1}^{\hat{s}} F(r_2)^{N_2} F(z)^{N_1-1}\,dz + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\,dz \\
&= \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz + \int_{\hat{s}}^{\hat{s}'} F(z)^{N_1+N_2-1}\,dz - \int_{s}^{\hat{s}'} F(z)^{N_1+N_2-1}\,dz \\
&< \int_{r_2}^{\hat{s}} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz + \int_{\hat{s}}^{\hat{s}'} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz \\
&\quad - \int_{s}^{\hat{s}'} F(r_2)^{N_2} F(z)^{N_1-1}\,dz \\
&= \left[\int_{r_2}^{\hat{s}'} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz\right] - \int_{s}^{\hat{s}'} F(r_2)^{N_2} F(z)^{N_1-1}\,dz \\
&= \left[\int_{r_1'}^{\hat{s}'} F(r_2)^{N_2} F(z)^{N_1-1}\,dz\right] - \int_{s}^{\hat{s}'} F(r_2)^{N_2} F(z)^{N_1-1}\,dz = V_1(r_1', r_2|s).
\end{aligned}$$

The definition of $\hat{s}$ implies that the terms in square brackets are equal. Finally, the expected payoff of a group-1 bidder of type $s > \hat{s}'$ also increases:

$$\begin{aligned}
V_1(r_1, r_2|s) &= \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_1-1}\,dz + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\,dz \\
&= \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_1-1}\,dz + \int_{\hat{s}}^{\hat{s}'} F(z)^{N_1+N_2-1}\,dz + \int_{\hat{s}'}^{s} F(z)^{N_1+N_2-1}\,dz \\
&< \int_{r_2}^{\hat{s}} F(\hat{s}')^{N_1} F(z)^{N_1-1}\,dz + \int_{\hat{s}}^{\hat{s}'} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz + \int_{\hat{s}'}^{s} F(z)^{N_1+N_2-1}\,dz \\
&= \int_{r_2}^{\hat{s}'} F(\hat{s}')^{N_1} F(z)^{N_1-1}\,dz + \int_{\hat{s}'}^{s} F(z)^{N_1+N_2-1}\,dz = V_1(r_1', r_2|s).
\end{aligned}$$

Therefore, a group-1 bidder benefits from the reduction in r_1.

Next consider a bidder in group 2. Since $F(\hat{s})^{N_1} \int_{r_2}^{s} F(z)^{N_2-1}\,dz < F(\hat{s}')^{N_1} \times \int_{r_2}^{s} F(z)^{N_2-1}\,dz$, all group-2 bidders of type $s \in [r_2, \hat{s}]$ benefit from the reduction in r_1. As in the case above, a group-2 bidder of type $s \in (\hat{s}, \hat{s}']$ also sees his expected payoff rise:

$$\begin{aligned}
V_2(r_1, r_2|s) &= \int_{r_2}^{\hat{s}} F(\hat{s})^{N_1} F(z)^{N_2-1}\,dz + \int_{\hat{s}}^{s} F(z)^{N_1+N_2-1}\,dz \\
&< \int_{r_2}^{\hat{s}} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz + \int_{\hat{s}}^{s} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz \\
&= \int_{r_2}^{s} F(\hat{s}')^{N_1} F(z)^{N_2-1}\,dz = V_2(r_1', r_2|s).
\end{aligned}$$

For $s > \hat{s}'$, the argument presented above for a group-1 bidder applies. □

Lemma 3. *The expected revenue of the first-price auction with reserve prices $r_1 \le r_2$ can be written as in (10).*

Proof. Let $\xi_k^i(\mathbf{s})$ be the probability that bidder i in group k wins the auction as a function of the realized type profile $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2) = (s_1^1, s_1^2, \dots, s_1^{N_1}, s_2^1, s_2^2, \dots, s_2^{N_2})$. Up to a set of zero measure,

$$\xi_1^i(\mathbf{s}) = \begin{cases} 1 & \text{if } \max\{\mathbf{s}_1, r_1\} \le s_1^i \text{ and } \max\{\mathbf{s}_2\} < r_2, \\ 1 & \text{if } \max\{\mathbf{s}_1, \hat{s}\} \le s_1^i \text{ and } \max\{\mathbf{s}_2\} < s_1^i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\xi_2^i(\mathbf{s}) = \begin{cases} 1 & \text{if } \max\{\mathbf{s}_2, r_2\} \le s_2^i \text{ and } \max\{\mathbf{s}_2\} < \hat{s}, \\ 1 & \text{if } \max\{\mathbf{s}_2, r_2\} \le s_2^i \text{ and } \max\{\mathbf{s}_1\} < s_2^i, \\ 0 & \text{otherwise.} \end{cases}$$

From Myerson (1981), the auction's expected revenue can be written as

$$\int \cdots \int \left[\sum_{k=1}^{2} \sum_{i=1}^{N_k} \xi_k^i(\mathbf{s}) J\big(s_k^i\big) \right] f\big(s_1^1\big) \cdots f\big(s_2^{N_2}\big)\, ds_1^1 \cdots ds_2^{N_2}. \tag{14}$$

For $k \in \{1, 2\}$, define $\tilde{s}_k := \max\{s_k^1, \dots, s_k^{N_k}\}$ as the maximum signal among group-k bidders. Consider the allocation rule (ψ_1, ψ_2) defined in (12). We show that, up to a set of zero measure,

$$\sum_{i=1}^{N_k} \xi_k^i(\mathbf{s}) J\big(s_k^i\big) = \psi_k(\tilde{s}_1, \tilde{s}_2) J(\tilde{s}_k). \tag{15}$$

Case 1. Suppose $\tilde{s}_k < r_k$. In this case $\psi_k(\tilde{s}_1, \tilde{s}_2) = 0$. Similarly, $\tilde{s}_k < r_k \implies s_k^i < r_k \implies \xi_k^i(\mathbf{s}) = 0$ for all i in group k. Thus, both sides of (15) are zero.

Case 2. Suppose $\tilde{s}_2 < r_2$ and $\tilde{s}_1 > r_1$. For all i in group 2, $\xi_2^i(\mathbf{s}) = \psi_2(\tilde{s}_1, \tilde{s}_2) = 0$. Thus, (15) holds for $k = 2$. For $k = 1$, $\psi_1(\tilde{s}_1, \tilde{s}_2) = 1$ and $\xi_1^i(\mathbf{s}) = 0$ for all agents i except for the agent with the highest valuation in group 1, say $\hat{i}$. Thus, $\sum_{i=1}^{N_1} \xi_1^i(\mathbf{s}) J(s_1^i) = \xi_1^{\hat{i}}(\mathbf{s}) J(s_1^{\hat{i}}) = J(\tilde{s}_1)$.

Case 3. Suppose $\tilde{s}_1 < \hat{s}$ and $\tilde{s}_2 > r_2$. For all i in group 1, $\xi_1^i(\mathbf{s}) = \psi_1(\tilde{s}_1, \tilde{s}_2) = 0$. Thus, (15) holds for $k = 1$. For $k = 2$, $\psi_2(\tilde{s}_1, \tilde{s}_2) = 1$ and $\xi_2^i(\mathbf{s}) = 0$ for all agents i except for the agent with the highest valuation in group 2, say $\hat{i}$. Thus, $\sum_{i=1}^{N_2} \xi_2^i(\mathbf{s}) J(s_2^i) = \xi_2^{\hat{i}}(\mathbf{s}) J(s_2^{\hat{i}}) = J(\tilde{s}_2)$.

Case 4. Suppose $\tilde{s}_k > \max\{\hat{s}, \tilde{s}_{k'}\}$. In this case, $\xi_{k'}^i(\mathbf{s}) = \psi_{k'}(\tilde{s}_1, \tilde{s}_2) = 0$ for all i in group k'. Thus, (15) holds for $k' \ne k$. For group k, $\psi_k(\tilde{s}_1, \tilde{s}_2) = 1$ and $\xi_k^i(\mathbf{s}) = 0$ for all agents i except for the agent with the highest valuation in group k, say $\hat{i}$. Thus, $\sum_{i=1}^{N_k} \xi_k^i(\mathbf{s})J(s_k^i) = \xi_k^{\hat{i}}(\mathbf{s})J(s_k^{\hat{i}}) = J(\tilde{s}_k)$.

Thus, we can rewrite (14) as $\int \cdots \int [\sum_{k=1}^{2} \psi_k(\tilde{s}_1, \tilde{s}_2)J(\tilde{s}_k)]f(s_1^1)\cdots f(s_2^{N_2})\, ds_1^1 \cdots ds_2^{N_2}$. Replacing $f(s_1^1)\cdots f(s_2^{N_2})$ with the joint density of $\tilde{s}_1$ and $\tilde{s}_2$ completes the proof. □

The following lemma defines the function $r_1(s)$, which is used in the proof of Theorem 5 to follow. If r_2 is fixed and $\hat{s}$ is the point of discontinuity is the equilibrium strategy of a group-1 bidder, then $r_1(\hat{s})$ is the corresponding value of r_1.

Lemma 4. *Fix $r_2 > 0$ and define*

$$\varphi(s, r_1) := \int_{r_1}^{s} F(r_2)^{N_2} F(z)^{N_1-1}\, dz - \int_{r_2}^{s} F(s)^{N_1} F(z)^{N_2-1}\, dz$$

in an open neighborhood of $(s, r_1) = (r_2, r_2)$. There exists $\varepsilon > 0$ sufficiently small and a function $r_1(s)$ such that $\varphi(s, r_1(s)) = 0$ for all $s \in (r_2 - \varepsilon, r_2 + \varepsilon)$. Furthermore, $r_1(s)$ is continuously differentiable in a neighborhood of $s = r_2$, $r_1(r_2) = r_2$, $r_1'(r_2) = 0$, and $r_1'(s) < 0$ for $s \in (r_2, r_2 + \varepsilon)$.

Proof. We note that $\varphi(s, r_1)$ is continuously differentiable in both arguments. Moreover, $\partial\varphi(s, r_1)/\partial r_1 = -F(r_2)^{N_2}F(r_1)^{N_1-1} \neq 0$. Thus, by the implicit function theorem, there exists a continuously differentiable function $r_1(s)$ and $\varepsilon > 0$, sufficiently small, such that $\varphi(s, r_1(s)) = 0$ for all $s \in (r_2 - \varepsilon, r_2 + \varepsilon)$. Clearly, $r_1(r_2) = r_2$. Moreover,

$$\left.\frac{dr_1(s)}{ds}\right|_{s=r_2} = \left.\frac{F(r_2)^{N_2}F(s)^{N_1-1} - F(s)^{N_1+N_2-1} - \displaystyle\int_{r_2}^{s} \frac{d}{ds} F(s)^{N_1-1} F(z)^{N_2-1}\, dz}{F(r_2)^{N_2} F\big(r_1(s)\big)^{N_1-1}}\right|_{s=r_2} = 0.$$

Finally, noting Lemma 2, $r_1'(s) < 0$ when $s \in (r_2, r_2 + \varepsilon)$. □

Proof of Theorem 5. Define $\Gamma(r_1, r_2)$ as in (13). Define $r_1(s)$ as in Lemma 4. Thus, $\Gamma(r_1(\hat{s}), r_2)$ is the revenue difference between an auction with reserve prices $r_1(\hat{s})$ and r_2 and an auction with a common reserve price of r_2. In this case, $\hat{s}$ is the critical type in the bidding strategy of a group-1 agent.

To prove the theorem, it is sufficient to verify that for $\hat{s} \in (r_2, r_2 + \varepsilon)$,

$$\frac{d}{d\hat{s}} \Gamma\big(r_1(\hat{s}), r_2\big) > 0. \tag{16}$$

Computing this derivative, we see that (16) is satisfied if and only if

$$\begin{aligned} &g_1(\hat{s}) \cdot \left(\int_{r_2}^{\hat{s}} J(s) g_2(s)\, ds - \big(G_2(\hat{s}) - G_2(r_2)\big) J(\hat{s})\right) \\ &\quad > g_1\big(r_1(\hat{s})\big) \cdot \big(-J\big(r_1(\hat{s})\big)\big) \cdot \big(-G_2(r_2) r_1'(\hat{s})\big). \end{aligned} \tag{17}$$

From the theorem's conditions, we see that $J(s)$ is strictly positive and decreasing for $s \in (r_2, r_2 + \varepsilon)$. Moreover, $J(s) \leq 0$ when $s < r_2$ and $r_1'(s) \leq 0$ when $s > r_2$. Thus, both sides of (17) are nonnegative. We verify three facts that together are sufficient to confirm (17).

Fact 1: $\lim_{\hat{s}\to r_2^+} g_1(\hat{s}) > \lim_{\hat{s}\to r_2^+} g_1(r_1(\hat{s}))$. From Lemma 4, $\lim_{\hat{s}\to r_2^+} g_1(r_1(\hat{s})) = \lim_{s\to r_2^-} g_1(s)$ and

$$\lim_{\hat{s}\to r_2^+} g_1(\hat{s}) > \lim_{s\to r_2^-} g_1(s) \quad\iff\quad \lim_{\hat{s}\to r_2^+} f(\hat{s}) > \lim_{s\to r_2^-} f(s).$$

As $r_2 = r^*$, the claim follows from the theorem's assumptions.

Fact 2: $\lim_{\hat{s}\to r_2^+}(-G_2(r_2)r_1'(\hat{s})) = 0^+$. From Lemma 4, we see that $\lim_{\hat{s}\to r_2^+} r_1'(\hat{s}) = 0^-$. Hence the conclusion follows.

Fact 3: If $\gamma(\hat{s}) := \int_{r_2}^{\hat{s}} J(s)g_2(s)\,ds - (G_2(\hat{s}) - G_2(r_2))J(\hat{s})$ and $\zeta(\hat{s}) := -J(r_1(\hat{s}))$, then for all $\hat{s} \in (r_2, r_2 + \varepsilon)$, $\gamma(\hat{s}) > \zeta(\hat{s})$. Consider a Taylor approximation as $\hat{s} \to r_2^+$:

$$\begin{aligned}\gamma(\hat{s}) &\simeq \gamma(r_2^+) + \big(-G_2(r_2)J'(r_2^+)\big)\big(\hat{s} - r_2^+\big) + O\big(\hat{s}^2\big)\\ &= -G_2(r_2)J'\big(r_2^+\big)(\hat{s} - r_2) + O\big(\hat{s}^2\big).\end{aligned}$$

Similarly, we can approximate $\zeta(\hat{s})$,

$$\begin{aligned}\zeta(\hat{s}) &\simeq -J\big(r_2^-\big) + \big(-J'\big(r_2^-\big)r_1'\big(r_2^+\big)\big)\big(\hat{s} - r_2^+\big) + O\big(\hat{s}^2\big)\\ &= 0 + O\big(\hat{s}^2\big),\end{aligned}$$

since $\lim_{s\to r_2^+} r_1'(s) = 0$ and $\lim_{s\to r_2^-} |J'(s)| < \infty$. Noting that $J'(r_2^+) < 0$, we conclude that $\gamma(\hat{s}) > \zeta(\hat{s})$ for all $\hat{s}$ sufficiently close to r_2.

Therefore, there exists $\varepsilon > 0$ such that (17) is satisfied. □

References

Albano, Gian Luigi, Nicola Dimitri, Isabelle Perrigne, and Gustavo Piga (2006), "Fostering participation." In *Handbook of Procurement* (Nicola Dimitri, Gustavo Piga, and Giancarlo Spagnolo, eds.), 267–292, Cambridge University Press, New York, NY. [208]

Ashenfelter, Orley (1989), "How auctions work for wine and art." *Journal of Economic Perspectives*, 3, 23–36. [208]

Athey, Susan (2001), "Single crossing properties and the existence of pure strategy equilibria in games of incomplete information." *Econometrica*, 69, 861–889. [208]

Athey, Susan, Dominic Coey, and Jonathan Levin (2013), "Set-asides and subsidies in auctions." *American Economic Journal: Microeconomics*, 5, 1–27. [205]

Ayres, Ian and Peter Cramton (1996), "Deficit reduction through diversity: How affirmative action at the FCC increased auction competition." *Stanford Law Review*, 48, 761–815. [205]

Baye, Michael R., Dan Kovenock, and Casper G. de Vries (1993), "Rigging the lobbying process: An application of the all-pay auction." *American Economic Review*, 83, 289–294. [225]

Brisset, Karine and Florence Naegelen (2006), "Why the reserve price should not be kept secret." *Topics in Theoretical Economics*, 6, 1–17. [208]

Cai, Hongbin, John Riley, and Lixin Ye (2007), "Reserve price signaling." *Journal of Economic Theory*, 135, 253–268. [208]

Che, Yeon-Koo and Ian L. Gale (1998), "Caps on political lobbying." *American Economic Review*, 88, 643–651. [225]

de Castro, Luciano I. and María-Angeles de Frutos (2010), "How to translate results from auctions to procurements." *Economics Letters*, 106, 115–118. [207]

Deb, Rahul and Mallesh M. Pai (2017), "Discrimination via symmetric auctions." *American Economic Journal: Microeconomics*, 9, 275–314. [207]

Fibich, Gadi and Arieh Gavious (2003), "Asymmetic first-price auctions: A peturbation approach." *Mathematics of Operations Research*, 28, 836–852. [208]

Flambard, Véronique and Isabelle Perrigne (2006), "Asymmetry in procurement auctions: Evidence from snow removal contracts." *The Economic Journal*, 116, 1014–1036. [208]

Gayle, Wayne-Roy and Jean Francois Richard (2008), "Numerical solutions of asymmetric, first-price, independent private values auctions." *Computational Economics*, 32, 245–278. [213]

Haile, Philip A. (2000), "Partial pooling at the reserve price in auctions with resale opportunities." *Games and Economic Behavior*, 33, 231–248. [208]

Hu, Audrey, Steven A. Matthews, and Liang Zou (2010), "Risk aversion and optimal reserve prices in first- and second-price auctions." *Journal of Economic Theory*, 145, 1188–1202. [208]

Kaplan, Todd R. and Shmuel Zamir (2012), "Asymmetric first-price auctions with uniform distributions: Analytic solutions to the general case." *Economic Theory*, 50, 269–302. [208]

Kirkegaard, René (2009), "Asymmetric first price auctions." *Journal of Economic Theory*, 144, 1617–1635. [208]

Kirkegaard, René (2012), "Favoritism in asymmetric contests: Head starts and handicaps." *Games and Economic Behavior*, 76, 226–248. [225]

Kirkegaard, René (2013), "Handicaps in incomplete information all-pay auctions with a diverse set of bidders." *European Economic Review*, 64, 98–110. [225]

Kotowski, Maciej H. (2014), "On asymmetric reserve prices." Unpublished paper, Harvard University. [225]

Krasnokutskaya, Elena and Katja Seim (2011), "Bid preference programs and participation in highway procurement auctions." *American Economic Review*, 101, 2653–2686. [205, 208]

Krishna, Vijay (2002), *Auction Theory*. Academic Press, San Diego. [226]

Lebrun, Bernard (1997), "First price auctions in the asymmetric N bidder case." Unpublished paper, Université Laval, Cahier 9715. [208, 212, 213, 214, 215]

Lebrun, Bernard (1999), "First price auctions in the asymmetric N bidder case." *International Economic Review*, 40, 125–142. [208, 212, 213, 214, 215]

Lebrun, Bernard (2004), "Uniqueness of the equilibrium in first-price auctions." Unpublished paper, Discussion Paper Department of Economics, York University. [212, 214]

Lebrun, Bernard (2006), "Uniqueness of the equilibrium in first-price auctions." *Games and Economic Behavior*, 55, 131–151. [208, 212, 214]

Levin, Dan and James L. Smith (1994), "Equilibrium in auctions with entry." *American Economic Review*, 84, 585–599. [208]

Levin, Dan and James L. Smith (1996), "Optimal reservation prices in auctions." *The Economic Journal*, 106, 1271–1283. [208]

Li, Tong and Isabelle Perrigne (2003), "Timber sale auctions with random reserve prices." *Review of Economics and Statistics*, 85, 189–200. [208]

Lizzeri, Alessandro and Nicola Persico (2000), "Uniqueness and existence of equilibrium in auctions with a reserve price." *Games and Economic Behavior*, 30, 83–114. [208, 212, 214]

Loertscher, Simon and Leslie M. Marx (2016), "Auctions with bid credits and resale." Unpublished paper, Duke University. [205, 208]

Mares, Vlad and Jeroen M. Swinkels (2014), "On the analysis of asymmetric first price auctions." *Journal of Economic Theory*, 152, 1–40. [208, 225]

Marion, Justin (2007), "Are bid preferences benign? The effect of small business subsidies in highway procurment auctions." *Journal of Public Economics*, 91, 1591–1624. [205]

Marshall, Robert C., Michael J. Meurer, Jean-Francois Richard, and Walter Stromquist (1994), "Numerical analysis of asymmetric first price auctions." *Games and Economic Behavior*, 7, 193–220. [213]

Maskin, Eric and John Riley (2003), "Uniqueness of equilibrium in sealed high-bid auctions." *Games and Economic Behavior*, 45, 395–409. [208, 212, 214]

McAfee, R. Preston and John McMillan (1989), "Government procurement and international trade." *Journal of International Economics*, 26, 291–308. [205]

Milgrom, Paul R. and Robert J. Weber (1982), "A theory of auctions and competitive bidding." *Econometrica*, 50, 1089–1122. [208]

Monteiro, Paulo Klinger and Benar Fux Svaiter (2010), "Optimal auction with a general distribution: Virtual valuation without densities." *Journal of Mathematical Economics*, 46, 21–31. [207]

Moreno, Diego and John Wooders (2011), "Auctions with heterogeneous entry costs." *Rand Journal of Economics*, 42, 313–336. [208]

Mummalaneni, Simha (2016), "Affirmative action as a cost cutting tool in procurement markets." Unpublished paper, University of Washington. [205]

Myerson, Roger B. (1981), "Optimal auction design." *Mathematics of Operations Research*, 6, 58–73. [206, 207, 208, 217, 218, 220, 221, 232]

Pai, Mallesh M. and Rakesh Vohra (2014), "The design of affirmative action policy: A mechanism design approach." Unpublished paper, Rice University. [205]

Quint, Daniel (2017), "Common values and low reserve prices." *Journal of Industrial Economics*, 65, 363–396. [208]

Reny, Philip J. and Shmuel Zamir (2004), "On the existence of pure strategy monotone equilibria in asymmetric first-price auctions." *Econometrica*, 72, 1105–1125. [208]

Riley, John G. and William F. Samuelson (1981), "Optimal auctions." *American Economic Review*, 71, 381–392. [207, 209, 218, 220, 223]

Rosenkranz, Stephanie and Patrick W. Schmitz (2007), "Reserve prices in auctions as reference points." *The Economic Journal*, 117, 637–653. [208]

Rothkopf, Michael H., Ronald M. Harstad, and Yuhong Fu (2003), "Is subsidizing inefficient bidders actually costly?" *Management Science*, 49, 71–84. [205]

Samuelson, William F. (1985), "Competitive bidding with entry costs." *Economics Letters*, 17, 53–57. [208]

Skreta, Vasiliki (2007), "Optimal auctions with general distributions." Unpublished paper, NYU Working Paper No. 2451/26024. [207]

Toikka, Jusso (2011), "Ironing without control." *Journal of Economic Theory*, 146, 2510–2526. [207]

Vickrey, William (1961), "Counterspeculation, auctions and competitive sealed tenders." *Journal of Finance*, 16, 8–37. [207]

Vincent, Daniel R. (1995), "Bidding off the wall: Why reserve prices may be kept secret." *Journal of Economic Theory*, 65, 575–584. [208]

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