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Benchmarking

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We investigate a normative theory of incomplete preferences in the context of preliminary screening procedures. We introduce a theory of ranking in the presence of objectively incomparable marginal contributions (apples and oranges). Our theory recommends *benchmarking*, a method under which an individual is deemed more accomplished than another if and only if she has achieved more benchmarks, or important accomplishments. We show that benchmark rules are characterized by four axioms: transitivity, monotonicity, incomparability of marginal gains, and incomparability of marginal losses.

Keywords. Benchmarking, axioms, measurement, apples and oranges, incompleteness, closure operator, homomorphisms.

JEL classification. A10, C71, D63.

Introduction

This paper is devoted to a normative theory of incomplete preferences. Our aim is to understand and provide a recommendation for a class of preliminary screening procedures that a committee or group may use to simplify decision making. Whereas most work featuring incomplete preferences views the incompleteness as a mostly behavioral phenomenon, here the lack of completeness is intended to be a normatively compelling property that a group should require.

We illustrate our main ideas with an example. Employers making hiring decisions commonly follow a two-stage process. The pile of candidates is first winnowed according to objective criteria. Difficult cases that remain are then resolved by an executive decision.

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The objective criteria used to sort between candidates in the first stage may not be directly comparable. For example, an academic employer may care about both research and teaching, but will often not have a clear view as to how to trade off the two. In idiomatic English, this is referred to as the problem of comparing apples and oranges.

We provide a general theory of ranking in the presence of objects whose marginal values are incomparable. Because of this inherent incommensurability, we necessarily seek a ranking that is incomplete. One can view this ranking as an objective "meta-ranking," in which all interested parties agree. The meta-ranking is an object that can be "completed" in later stages by specific parties. A large university may complete this ranking differently than a small teaching college.

To understand our properties, consider our example of the two kinds of activities pursued by academic economists: (a) publishing papers and (b) teaching classes. Their marginal values depend on the other items on the résumé. Suppose an economist can improve her résumé by either publishing an extra paper or teaching an extra class. Which of the two is more desirable?

Our first axiom, *incomparability of marginal gains*, requires that we not make this choice: when both an additional publication and an additional teaching experience add value to a résumé, these marginal contributions must not themselves be comparable. We formalize this idea by requiring that the two résumés that result from the addition of either of the accomplishments (but not both) are themselves incomparable.

We generalize the statement to sets of accomplishments: if two disjoint sets of accomplishments are added to a given résumé, and each results in a strict marginal gain, then these marginal contributions should not be comparable.

Our second axiom is similar in nature and motivation to the first, except that it concerns deletions from (and not additions to) the candidate's résumé. Suppose that both the removal of a publication and the removal of a teaching experience would both weaken the candidate. Which deletion should weaken the candidate more? *Incomparability of marginal losses* requires that these two marginal losses not be comparable, again with a generalization to the removal of disjoint sets of accomplishments.

To illustrate the structure imposed by these new axioms, we provide an example in which incomparability of marginal losses is violated (under the assumption of transitivity). Consider three candidates, Alice, Bob, and Charles, who are candidates on the academic job market. Alice has taken all requisite courses, but has not yet defended her dissertation. Bob has defended his dissertation, but has not yet completed his course work. Charles has both finished all courses and successfully defended his dissertation. (We may think of Charles' résumé as being the union of Alice's and Bob's résumés, i.e., $A \cup B$.)

Charles qualifies for a PhD and, therefore, might be viewed as better than either Alice or Bob. (That is, $A \cup B \succ A$ and $A \cup B \succ B$.) But, from a pragmatic perspective, neither Alice nor Bob can be hired by an academic department, and, hence, they may be considered the same as a candidate with a blank résumé ($\varnothing$). (Thus, $A \sim \varnothing$ and $B \sim \varnothing$.) To the employer, the course work and the dissertation function as "extreme complements"; neither of them has any value on its own, but together they are valued highly.

This type of ranking is ruled out by the combination of the incomparability of marginal losses and transitivity axioms. Specifically, removing Charles' course work results in a strict loss ($A \cup B \succ B$). Removing his dissertation also results in a strict loss ($A \cup B \succ A$). Incomparability of marginal losses requires that Alice and Bob must be incomparable, but we know that $A \sim \varnothing \sim B$.

To these axioms we add two ancillary conditions. First, *monotonicity* requires that additional accomplishments never harm the candidate. Second, *transitivity* ensures that the method of comparison is a ranking.

We show that any method consistent with our axioms can be identified with a fixed list of relevant accomplishments, which we call *benchmarks*. One candidate is stronger than another if and only if the former has achieved all of the latter's benchmarks. Every set of benchmarks gives us a different ranking method; conversely, every benchmarking rule respects the incommensurability of apples and oranges.

In terms of interpretation, observe that our primary axioms, incomparability of marginal gains and losses, refer to what happens when accomplishments are either *added to* or *deleted from* a résumé. These properties are "local." For example, consider incomparability of marginal gains. The axiom requires that, for a given résumé, if each of two distinct accomplishments strictly adds value to that résumé, then the new résumés with either of the accomplishments added are incomparable. But the property only requires the marginal contributions to be incomparable *if* both add value to the given résumé. In principle, the accomplishments may add value to one résumé, but have no effect on another. This contrasts with our derived concept of a benchmark. The addition of a benchmark strictly adds value to any résumé, and the deletion of it strictly hurts a résumé. In this sense, the notion of a benchmark is a "global" concept.

Our model allows the flexibility of understanding some accomplishments as entailing others. There are at least two natural interpretations. The "cumulative accomplishments" interpretation is based on the idea that an accomplishment may be worth more when repeated. Being cited twice is better than being cited once. The alternative interpretation is that some accomplishments are normatively superior to others. One may think, for example, that one programming language is more advanced than another, or that one journal is superior to another. Neither of these examples can be easily represented in our framework without an entailment relation.

That these two interpretations are similar, however, becomes clear if we phrase them in the form of "at least" statements. If a paper has at least two citations, it follows that this paper has received at least one citation. If programming in Python is viewed to be objectively more advanced than programming in Matlab, then the accomplishment of programming in a language at least as advanced as Python entails the accomplishment of programming in a language at least as advanced as Matlab. Similarly, if there is an objective ranking of journals, so that all economists would agree that *Journal of Good Economics* is more prestigious than *Journal of Bad Economics*, then publishing at least one paper in a journal at least as good as *Journal of Good Economics* implies that one has published at least one paper in a journal at least as good as *Journal of Bad Economics*. The "at least" language allows us to naturally combine these interpretations: the accomplishment of publishing at least two papers in a journal at least as good as *Journal of*

Bad Economics would also entail the accomplishment of publishing at least one paper in a journal at least as good as *Journal of Bad Economics*, but not the accomplishment of publishing at least one paper in a journal at least as good as *Journal of Good Economics*.

We take this entailment relation as a primitive of the model. Admissible sets of accomplishments are restricted to respect the relation.

This entailment relation generates a particularly appealing structure for rankings that are required to be *complete*; that is, in which all pairs are comparable. The set of benchmarks for such a relation must be ordered according to it. The structure here allows an easy comparison to previous works that required completeness. It also allows us to easily construct new benchmarking rules that are complete.

Benchmarking is used in a variety of settings. Universities may use benchmarks in admissions. Investment firms use benchmarks when deciding between projects. Consumers use benchmarks when buying computers. Governments frequently use benchmark rules in assigning priorities in procurement contracts. For the criteria used in determining eligibility for federal contracts, see 48 C.F.R. Chapter 1. Schools may be compared among benchmarks, such as the scores that their students have received in math, science, history, and literature. While this practice is controversial, its study is important; large sums of federal money are allocated to schools according to these metrics. In particular, the "adequate yearly progress" requirements of the No Child Left Behind Act (see 20 U.S.C. 6311) can function like a benchmark rule under which schools are compared to themselves in prior years. Over $14 billion was allocated for these grants in 2014.

A common benchmark used in governmental hiring is the veterans' preference, according to which a military veteran is deemed superior to a non-veteran when they would otherwise be equivalent. Benchmarking may not provide a complete ranking: Alice may be a veteran without work experience, while Bob may be a non-veteran with work experience. But the benchmark rule nonetheless provides valuable comparisons that can be used by decision makers.

Federal courts evaluating the decisions of administrative agencies often use benchmark rules in determining whether to uphold the agency decision. For example, the Federal Communications Commission was held to have acted unreasonably in awarding a television station to a broadcaster who was weaker than another on all relevant criteria used by the commission.[1] Administrative agencies themselves may choose whether particular criteria are important enough to qualify as benchmarks.[2] Benchmarking may be used when comparing scholars by their citation profiles, as in Chambers and Miller (2014b). Here, an accomplishment is a pair of two numbers (x, y), where the individual has at least x publications with at least y citations each. The step-based indices characterized by Chambers and Miller (2014b) are all benchmark rules, but the reverse is not true. For example, the h-index (Hirsch 2005) and the $i10$-index are two popular

[1] *Central Florida Enterprises, Inc. v. Federal Communications Commission*, 598 F.2d 37 (1979).

[2] For an example, see *Baltimore Gas & Electric Co. v. Natural Resources Defense Council*, 462 U.S. 87 (1983), which upheld the decision of the Nuclear Regulatory Commission to ignore potential harm from the accidental release of spent nuclear fuel from long term storage.

measures of scholarly accomplishment; each is a step-based index. However, many believe that multiple such indices should be used in practice. A method of comparison that determines Alice to be better than Bob if she is better according to both measures would not be a step-based index, but would be a benchmark rule. Benchmark rules are more versatile in that they can be applied to a wider array of problems than can be the step-based indices.

We are not the first to study two-stage procedures in a decision or choice-theoretic framework. For example, the pioneering paper of Manzini and Mariotti (2007) uncovers the choice-theoretic implications of an individual who lexicographically chooses from two asymmetric binary relations. Our work can be viewed as axiomatizing a particular form for the "first stage" relation described in this paper.

The relationship between the step-based indices and the benchmark rules is not a coincidence, but can be seen in the axioms as well. Our axioms imply two properties, meet and join dominance, which are weaker forms of the lattice-theoretic notions of meet and join homomorphisms used in Chambers and Miller (2014a, 2014b). These properties were first studied in economics by Kreps (1979) and have since been studied in a wide variety of settings. For example, Hougaard and Keiding (1998), Christensen et al. (1999), and Chambers and Miller (2014a, 2014b) study these axioms in the context of measurement, while Miller (2008), Chambers and Miller (2011), Dimitrov Dinko and Mishra (2012), Leclerc (2011), and Leclerc and Monjardet (2013) study them in the context of aggregation. In fact, the results of Chambers and Miller (2014b) can be derived from the much more general results we establish here. The axiomatic system studied in that work is equivalent to the combination of the four main axioms studied here together with completeness. In this sense, a main contribution of the present work is the absence of a completeness assumption. We leave a detailed connection to Kreps (1979) to a later section.

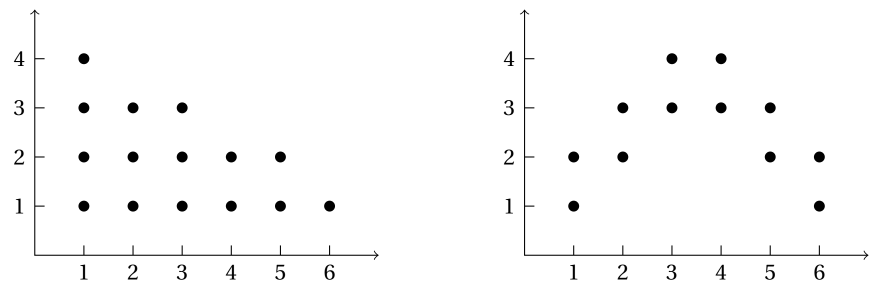
Our work is also related to prior literature on incomplete preferences. It is a relatively easy corollary of our main result that $\succeq$ satisfies our axioms if and only if there is a family $\mathcal{R}$ of *complete* relations satisfying our axioms such that for all x, y, $x \succeq y$ if and only if for all $\succeq^* \in \mathcal{R}$, $x \succeq^* y$. This explains our claim that $\succeq$ serves as a Pareto relation for interested parties. One can think of this representation as being a "vector-valued" utility, where each $\succeq^*$ represents a component of the vector. Results of this type were pioneered by Dubra et al. (2004) for the expected-utility case. Other such results include Bewley (2002) for the case of Savage acts, Duggan (1999) for the case of general binary relations, Donaldson and Weymark (1998), Dushnik and Miller (1941), and Szpilrajn (1930).[3]

The model

Let A be a nonempty set of accomplishments and let $\leq$ be a partial order on A for which the set $\{a \in A : a \leq x\}$ is finite for all $x \in A$.[4] The relation $\leq$ represents a "entailment"

[3]See also the work of Ok (2002) on incomplete preferences in economic environments, and a choice-theoretic foundation for incomplete preferences (Eliaz and Ok 2006).

[4]A partial order is a binary relation that is *reflexive*, *transitive*, and *antisymmetric*. The finiteness assumption is not substantive, and can be weakened by adding an appropriate continuity hypothesis. We assume it as it renders the proofs more transparent by enabling the use of standard mathematical induction.



(a) Comprehensive. (b) Not Comprehensive.

Figure 1. Sets in $\mathbb{N}^2$.

relation. For two accomplishments $a, b \in A$, if b has been achieved and $a \leq b$, then a has necessarily been achieved. Several interpretations of this relation were discussed in the Introduction; we present an interpretation involving the ranking of scholars in an example below. If one does not wish to consider a entailment relation, it is enough to consider the relation given by $\{(a, a) : a \in A\}$; that is, equality.

A subset $B \subseteq A$ is *comprehensive* if for all $b \in B$ and $a \in A$, $a \leq b$ implies that $a \in B$. For example, the subset of $\mathbb{N}^2$ (endowed with the usual order) depicted in Figure 1(a) is comprehensive; for any point (x, y) in the set, all points (x', y') with $x' \leq x$ and $y' \leq y$ are in the subset. By contrast, the subset depicted in Figure 1(b) is not comprehensive; the point $(2, 3)$ is in the subset, but the points $(2, 1)$ and $(1, 3)$ are not.

Let $\mathcal{X}$ be the set of all finite comprehensive subsets of A.[5] An element of $\mathcal{X}$ represents a logically consistent set of accomplishments.

We are interested in binary relations $\succeq$ on $\mathcal{X}$ used to compare logically consistent sets of accomplishments. For two sets $X, Y \in \mathcal{X}$, we write $X \parallel Y$ to denote that X and Y are not comparable with respect to $\succeq$; i.e., neither $X \succeq Y$ nor $Y \succeq X$.[6]

We define four axioms on binary relations $\succeq$ on $\mathcal{X}$. The first two, transitivity and monotonicity, are standard in the literature. Transitivity requires that the binary relation $\succeq$ be a ranking. Monotonicity requires that the binary relation $\succeq$ prefer more to less.

Transitivity. For all $X, Y, Z \in \mathcal{X}$, if $X \succeq Y$ and $Y \succeq Z$, then $X \succeq Z$.

Monotonicity. For all $X, Y \in \mathcal{X}$, if $X \supseteq Y$, then $X \succeq Y$.

Incomparability of marginal gains was described in the Introduction. Imagine that we start from a "baseline" set of accomplishments $(X \cap Y)$. Adding the marginal set of accomplishments $(X \backslash Y)$ results in X, which is deemed better than $(X \cap Y)$. Adding the marginal $(Y \backslash X)$ results in Y, which is also deemed better than $(X \cap Y)$. Since $(X \backslash Y) \cap (X \backslash Y) = \varnothing$, there are no common accomplishments in the marginals. Incomparability of marginal gains requires that these marginal contributions be unranked, which implies that X and Y must themselves be unranked.

Incomparability of Marginal Gains. For all $X, Y \in \mathcal{X}$, if $X \succ (X \cap Y)$ and $Y \succ (X \cap Y)$, then $X \parallel Y$.

[5] The set $\varnothing$ is obviously finite and comprehensive.

[6] To avoid confusion, we emphasize that there are three important binary relations in this paper: $\leq$, $\succeq$, and $\subseteq$. Incomparability of $\succeq$ is denoted $\parallel$. No notation is needed for incomparability of $\leq$ or $\subseteq$.

Incomparability of marginal losses is similar. Imagine starting from a baseline set of accomplishments $(X \cup Y)$. Removing the marginal $(Y \backslash X)$ results in X, which is deemed worse than $(X \cup Y)$. Similarly, removing the marginal $(X \backslash Y)$ results in Y, which is deemed worse than $(X \cup Y)$. As $(X \backslash Y)$ and $(Y \backslash X)$ have no common elements, incomparability of marginal losses requires that X and Y be unranked.

Incomparability of Marginal Losses. For all $X, Y \in \mathcal{X}$, if $(X \cup Y) \succ X$ and $(X \cup Y) \succ Y$, then $X \parallel Y$.

A *benchmarking rule* is a binary relation $\succeq$ on $\mathcal{X}$ for which there exists $\mathcal{B} \subseteq A$ such that, for all $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if $\mathcal{B} \cap X \supseteq \mathcal{B} \cap Y$. We refer to elements of $\mathcal{B}$ as *benchmarks*. Our main result is a characterization of benchmarking rules.[7]

Theorem 1. *A binary relation $\succeq$ on $\mathcal{X}$ satisfies transitivity, monotonicity, incomparability of marginal gains, and incomparability of marginal losses if and only if it is a benchmarking rule. Furthermore, the four axioms are independent.*

As an example, we can consider the environment in which $\leq$ is the trivial order; that is, $\leq = \{(a, a) : a \in A\}$. In this case, Theorem 1 states that a binary relation $\succeq$ on 2^A satisfies the four axioms if and only if there is a set $B \subseteq A$ for which, for all $X, Y \in 2^A$, $X \succeq Y$ if and only if $(X \cap B) \supseteq (Y \cap B)$.

Proof of Theorem 1. Only if: Let $\succeq$ satisfy the four axioms. For $x \in A$, we define $\mathcal{K}(x) \equiv \{a \in A : a \leq x\}$. Note that for each $x \in A$, $\mathcal{K}(x) \in \mathcal{X}$. For $x, y \in A$, we write $y < x$ if $y \leq x$ and $y \neq x$. For $x \in A$ and $Z \in \mathcal{X}$, we say that x *covers* Z if $x \notin Z$ and $y < x$ implies that $y \in Z$.

Let

$$\mathcal{B} = \left\{x \in A : \mathcal{K}(x) \succ \mathcal{K}(x) \backslash \{x\}\right\}.$$

It is sufficient to show that for all $C, Z \in \mathcal{X}$, $(Z \cap \mathcal{B}) \subseteq (C \cap \mathcal{B})$ if and only if $C \succeq Z$.

The proof now proceeds in four steps.

Step 1. *By transitivity, monotonicity, and incomparability of marginal gains, for all $x \in \mathcal{B}$ and $Z \in \mathcal{X}$ such that x covers Z, $Z \cup \{x\} \succ Z$.*

By monotonicity, $(Z \cup \{x\}) \succeq Z$. Suppose, to the contrary, that $(Z \cup \{x\}) \sim Z$. By monotonicity, $\mathcal{K}(x) \subseteq (Z \cup \{x\})$ implies that $(Z \cup \{x\}) \succeq \mathcal{K}(x)$. By the definition of $\mathcal{B}$, $\mathcal{K}(x) \succ \mathcal{K}(x) \backslash \{x\}$. By transitivity, $Z \succ \mathcal{K}(x) \backslash \{x\}$. Because $Z \cap \mathcal{K}(x) = \mathcal{K}(x) \backslash \{x\}$, it follows from incomparability of marginal gains that $Z \parallel \mathcal{K}(x)$. Consequently, as $(Z \cup \{x\}) \sim Z$, we infer that $(Z \cup \{x\}) \parallel \mathcal{K}(x)$, a contradiction, which proves the claim.

Step 2. *By transitivity, monotonicity, and incomparability of marginal losses, for all $x \notin \mathcal{B}$ and $Z \in \mathcal{X}$ such that x covers Z, $Z \cup \{x\} \sim Z$.*

[7] The finiteness of the sets in $\mathcal{X}$ is necessary for our result. In the infinite case, an example of a non-benchmarking rule that satisfies our axioms can be constructed using free ultrafilters.

By monotonicity, $(Z \cup \{x\}) \succeq Z$. Suppose, to the contrary, that $(Z \cup \{x\}) \succ Z$. By monotonicity, $\mathcal{K}(x)\backslash\{x\} \subseteq Z$ implies that $Z \succeq \mathcal{K}(x)\backslash\{x\}$. By transitivity and the fact that $x \notin \mathcal{B}$, $(Z \cup \{x\}) \succ \mathcal{K}(x)$. Because $Z \cup \mathcal{K}(x) = (Z \cup \{x\})$, it follows from incomparability of marginal losses that $Z \parallel \mathcal{K}(x)$. Consequently, as $x \notin \mathcal{B}$, $\mathcal{K}(x)\backslash\{x\} \sim \mathcal{K}(x)$, so that $Z \parallel (\mathcal{K}(x)\backslash\{x\})$, a contradiction, which proves the claim.

Step 3. *For all $C, Z \in \mathcal{X}$, $Z \succ (Z \cap C)$ if and only if $\mathcal{B} \cap (Z\backslash C) \neq \varnothing$,*

Let $C, Z \in \mathcal{X}$. Let $\{z_1, \dots, z_k\} \subseteq Z$ be a sequence such that (i) $Z\backslash\{z_1, \dots, z_k\} = Z \cap C$ and (ii) $x < z_i$ implies that $x \in Z\backslash\{z_i, \dots, z_k\}$.[8] By Step 1, if $z_i \in \mathcal{B}$, then $Z\backslash\{z_{i+1}, \dots, z_k\} \succ Z\backslash\{z_i, \dots, z_k\}$. By Step 2, if $z_i \notin \mathcal{B}$, then $Z\backslash\{z_{i+1}, \dots, z_k\} \sim Z\backslash\{z_i, \dots, z_k\}$. By transitivity, it follows that $\mathcal{B} \cap (Z\backslash C) \neq \varnothing$ if and only if $Z \succ (Z \cap C)$.

Step 4. *Completion of the argument.*

First, we prove that if $C \succeq Z$, then $(Z \cap \mathcal{B}) \subseteq (C \cap \mathcal{B})$. Let $C, Z \in \mathcal{X}$ such that $C \succeq Z$. We need to show that $(Z\backslash C) \cap \mathcal{B} = \varnothing$. To the contrary, suppose that $(Z\backslash C) \cap \mathcal{B} \neq \varnothing$. By Step 3, $Z \succ (Z \cap C)$. By transitivity, $C \succ (Z \cap C)$. By incomparability of marginal gains, $Z \parallel C$, a contradiction, which proves the claim.

Next we prove that if $(Z \cap \mathcal{B}) \subseteq (C \cap \mathcal{B})$, then $C \succeq Z$. Let $C, Z \in \mathcal{X}$ such that $(Z \cap \mathcal{B}) \subseteq (C \cap \mathcal{B})$. Because $(Z \cap \mathcal{B}) \subseteq (C \cap \mathcal{B})$, it follows that $\mathcal{B} \cap (Z\backslash C) = \varnothing$. Therefore, by Step 3, $Z \sim (C \cap Z)$. By monotonicity, $C \succeq (C \cap Z)$ and, thus, by transitivity, $C \succeq Z$.

If: Let $\succeq$ be a benchmarking rule with benchmarks $\mathcal{B}$. The transitivity of $\succeq$ follows from the transitivity of $\supseteq$. To see this, let $X \succeq Y$ and $Y \succeq Z$. It follows that $\mathcal{B} \cap X \supseteq \mathcal{B} \cap Y$ and $\mathcal{B} \cap Y \supseteq \mathcal{B} \cap Z$. Hence, $\mathcal{B} \cap X \supseteq \mathcal{B} \cap Z$ and, therefore, $X \succeq Z$. To see that $\succeq$ is monotonic, note that $X \supseteq Y$ implies that $\mathcal{B} \cap X \supseteq \mathcal{B} \cap Y$ and, therefore, that $X \succeq Y$.

To see that incomparability of marginal gains is satisfied, suppose that $X \succ (X \cap Y)$ and $Y \succ (X \cap Y)$. Since $X \succ (X \cap Y)$, $\mathcal{B} \cap (X\backslash Y) \neq \varnothing$, and since $Y \succ (X \cap Y)$, $\mathcal{B} \cap (Y\backslash X) \neq \varnothing$. Hence, $X \parallel Y$.

To see that incomparability of marginal losses is satisfied, suppose that $(X \cup Y) \succ X$ and $(X \cup Y) \succ Y$. Since $(X \cup Y) \succ X$, $\mathcal{B} \cap (Y\backslash X) \neq \varnothing$ and since $(X \cup Y) \succ Y$, $\mathcal{B} \cap (X\backslash Y) \neq \varnothing$. Thus, $X \parallel Y$. □

To prove the independence of the axioms, we provide four examples of rules that are not benchmarking rules. Each of these rules satisfies three of the axioms while violating the fourth.

Our first example is a class that we call the *benchmark-majority rules*; rankings that are formed through the majoritarian aggregation of a set of benchmark rules.

Benchmark-majority rules. The relation $\succeq$ is a *benchmark-majority rule* if there exists a finite set of benchmark rules $\{\succeq_i\}_{i=1}^n$ such that, for all $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if $|\{i : X \succeq_i Y\}| > \frac{n}{2}$.

[8] To construct such a sequence, let $k = |Z\backslash C|$, and inductively define z_k to be $\leq$ maximal in $Z\backslash C$ and z_i to be $\leq$ maximal in $(Z\backslash C)\backslash\{z_{i+1}, \dots, z_k\}$.

All members of the class satisfy monotonicity, incomparability of marginal gains, and incomparability of marginal losses. This class includes the benchmark rules, but also many non-benchmark methods that do not satisfy transitivity. For our purpose, it is sufficient to show that at least one member of this class is intransitive.

Claim 1. *Benchmark-majority rules satisfy monotonicity, incomparability of marginal gains, and incomparability of marginal losses, but are not necessarily transitive.*

Our second example is the trivial order, according to which distinct elements of $\mathcal{X}$ are not comparable. This rule satisfies all axioms except for monotonicity.

Trivial order. For all $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if $X = Y$.

Claim 2. *The trivial order satisfies transitivity, incomparability of marginal gains, and incomparability of marginal losses, but not monotonicity.*

A weak order is a binary relation that is complete and transitive. Our third example is the indirect utility ranking (see Kreps 1979).

Indirect utility rankings. The relation $\succeq$ is an *indirect utility ranking* if there exists a weak order $\overset{\bullet}{\succsim}$ over A such that, for $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if for each $c \in Y$, there is $a \in X$ for which $a \overset{\bullet}{\succsim} c$.

Claim 3. *Indirect utility rankings satisfy transitivity, monotonicity, and incomparability of marginal losses, but may fail incomparability of marginal gains.*

Our fourth example is the dual of the indirect utility ranking. There are several ways to understand it, but perhaps the easiest is that a set is as good as the worst element it does *not* contain.

Dual indirect utility rankings. The relation $\succeq$ is a *dual indirect utility ranking* if there exists a weak order $\overset{\bullet}{\succsim}$ over A such that, for $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if for each $a \notin X$, there is $c \notin Y$ for which $a \overset{\bullet}{\succsim} c$.

Claim 4. *Dual indirect rankings satisfy transitivity, monotonicity, and incomparability of marginal gains, but may fail incomparability of marginal losses.*

The relation $\succeq_{\mathcal{B}}$ is the benchmarking rule associated with $\mathcal{B}$ as a set of benchmarks. We provide a basic comparative static result that relates nested sets of benchmarks to benchmarking rules. When the set of benchmarks expands, the set of weak rankings between pairs decreases and conversely. An important implication of this result is that we can identify each benchmarking rule with a unique set of benchmarks.

Write $\succeq \subseteq \succeq'$ if and only if $X \succeq Y$ implies that $X \succeq' Y$.

Theorem 2. *For two sets of benchmarks $\mathcal{B}$ and $\mathcal{B}'$ with associated benchmarking rules $\succeq_{\mathcal{B}}$ and $\succeq_{\mathcal{B}'}$, $\mathcal{B} \subseteq \mathcal{B}'$ if and only if $\succeq_{\mathcal{B}'} \subseteq \succeq_{\mathcal{B}}$.*

Complete benchmarking rules enjoy additional structure. Namely, all benchmarks must be comparable according to the entailment relation $\leq$.

Completeness. For all $X, Y \in \mathcal{X}$, either $X \succeq Y$ or $Y \succeq X$.

The following corollary explains our particular interest in the relation $\leq$.

Corollary 1. *A benchmarking rule $\succeq$ is complete if and only if its associated set of benchmarks $\mathcal{B}$ has the property that for all $a, b \in \mathcal{B}$, either $a \leq b$ or $b \leq a$.*

Let us return to the case in which $\leq = \{(a, a) : a \in A\}$; that is the trivial order. Complete benchmarking rules in this environment are rather uninteresting. Here, the only sets B satisfying the property listed in Corollary 1 are the singleton sets and the empty set. This means that a complete benchmarking rule can take one of two forms. In the first case, it could be complete indifference (when $B = \varnothing$). If, instead, B is a singleton, say $B = \{b\}$, then the rule has exactly two indifference classes. Every X for which $b \in X$ is in the higher indifference class, and every X for which $b \notin X$ is in the lower. Hence, such a rule is a kind of "pass–fail" rule, whereby a pass is recorded if accomplishment b is achieved, and fail is recorded if not. So, if $|A| < +\infty$, there are exactly $|A| + 1$ complete benchmarking rules when $\leq$ is trivial.

In contrast, when $\leq$ is more interesting, say, as the standard order on $\mathbb{Z}^2_+$, a much richer structure of complete benchmarking rules emerges; in fact, Corollary 1 generalizes the earlier results of Chambers and Miller (2014b), where $\leq$ is the standard order on $\mathbb{Z}^2_+$. It is also conceptually related to earlier literature on efficiency measurement (e.g., Hougaard and Keiding 1998, Christensen et al. 1999, Chambers and Miller 2014a). This earlier literature works on a space in which there are no nontrivial finite comprehensive sets. Generalizing our results to such environments requires the imposition of continuity properties; we leave this to future research.

We present the corollary that affords the interpretation of benchmarking rules as Pareto relations of interested parties.

Corollary 2. *For any benchmarking rule $\succeq$, there is a family $\mathcal{R}$ of complete benchmarking rules for which for all $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if for all $\succeq^* \in \mathcal{R}$, $X \succeq^* Y$.*

Ranking scholars

Academic institutions often use influence measures to compare scholars in terms of citations to their scientific publications. Popular influence measures include the h-index (Hirsch 2005), which is the largest number h such that the scholar has at least h publications with at least h citations each, the i10-index, which is the number of publications with at least 10 citations each, and the citation count, which is the combined number of citations to all of the author's publications.[9] Chambers and Miller (2014b) study a model of influence measures and characterize the family of step-based indices. What were termed *steps* in this previous paper are the benchmarks here.

Influence measures can be studied in our framework. Let $A \equiv \mathbb{N} \times \mathbb{Z}_+$, the set of pairs of integers (m, n) where m is positive and n is nonnegative; each pair is the accomplishment that the scholar has published at least m papers with at least n citations

[9]These particular measures are widely used, in part, due to their inclusion in the internet service Google Scholar Profiles, available at http://scholar.google.com/.

each. Let $\leq$ be the natural order, where $(m,n) \leq (m',n')$ if and only if $m \leq m'$ and $n \leq n'$. For example, the accomplishment of publishing at least three papers with at least two citations each entails the accomplishment of publishing at least two papers with at least two citations each. A scholar is a comprehensive collection of these pairs; the set of scholars is equivalent to the set $\mathcal{X}$ of all finite comprehensive subsets of A. A step path (see Chambers and Miller 2014b) is a function $P : \mathbb{N} \to A$, such that $n > n'$ implies that $P(n) \geq P(n')$, where each $P(n)$ is a step. A step-based index is a ranking $\succeq$ for which, for $S, S' \in \mathcal{X}$, $S \succeq S'$ if and only if $P(n) \in S'$ implies that $P(n) \in S$ for all $n \in \mathbb{N}$.

One can readily see that the set of step-based indices is identical to the set of complete benchmark rules in this setting. Corollary 2 explains the broader relation between benchmark rules and step-based indices. For every benchmark rule there is a collection $\{\succeq_i\}$ of step-based indices for which $x \succeq y$ if and only if $x \succeq_i y$ for all i; similarly every collection of step-based indices induces a benchmark rule.

For example, the h-index and the $i6$-index (the number of publications with at least six citations each) are step-based indices and benchmark rules. The benchmarks for these rules are depicted in Figure 2(a) and (b), respectively. One can see that they are complete; all benchmarks are comparable according to $\leq$. According to the h-index, scholars B and C are equivalent and each dominates scholar A. According to the $i6$-index, alternatively, scholar C dominates scholar A, and each of them dominates scholar B.

From these two indices we can construct a composite benchmark rule under which a scholar is at least as good as another if the former scholar is at least as good according to both the h-index and the $i6$-index. This is shown in Figure 2. Under this composite rule, scholars A and B are incomparable, but both are dominated by scholar C. In the context of ranking scholars, every collection of step-based indices can generate a benchmark rule in this manner, and every benchmark rule can be generated by some collection of step-based indices.

On the connection with Kreps

This paper is inspired by, and related to, the work of Kreps (1979). Kreps' work does not include an entailment relation $\leq$, and so can be thought of as the case in which $\leq$ is the trivial relation. Further, Kreps is interested in the set of *nonempty* subsets of some finite set A, whereas sets in our domain can be empty. Kreps' focus is decision theoretic. He interprets these sets as menus of objects from which a decision maker has the opportunity to choose from in an implicit second stage. Kreps' results do not hold exactly as stated here for the domain of all sets, so we are explicit here when the domain includes only nonempty sets.[10]

[10]For example, if the empty set were permitted in Proposition 2, then the representation would require it to be ranked strictly below all nonempty sets, but complete indifference of $\succeq$ obviously satisfies all of the axioms and does not have this representation. With the empty set, there are two possibilities: either it is strictly worse than all nonempty sets or it is weakly worse and indifferent to the lowest ranked nonempty set. Kreps avoids this distinction by restricting the domain to nonempty sets.

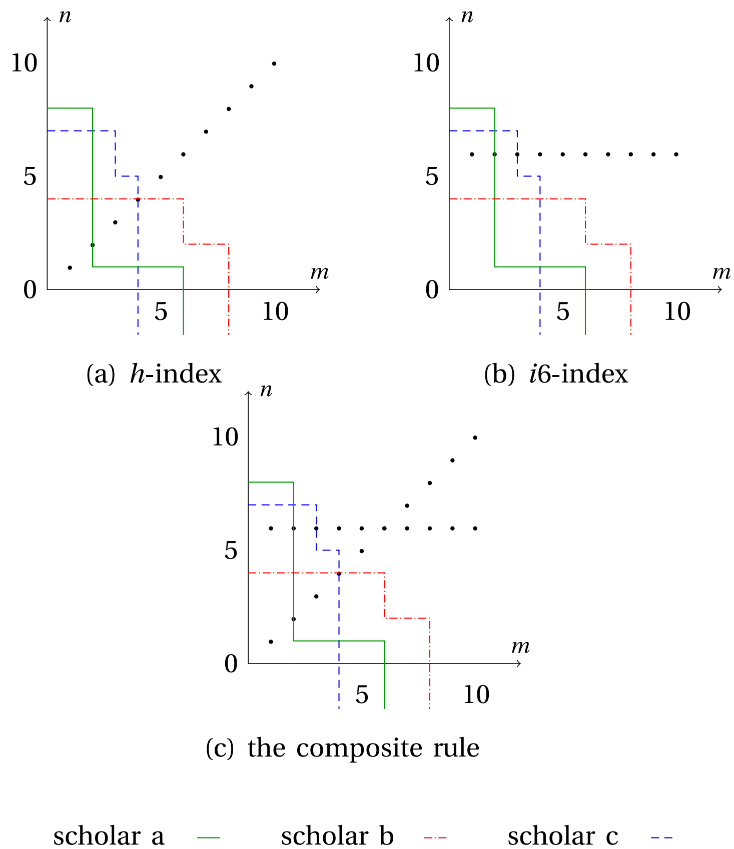


Figure 2. Ranking scholars.

Kreps does not use our properties per se, but he does discuss a variant of the following property, which we termed *join dominance* in other work: For all $X, Y \in \mathcal{X}$, $X \succeq Y$ implies $X \succeq (X \cup Y)$.[11] The following proposition is true.

Proposition 1. *Assume that $\succeq$ on $\mathcal{X}$ satisfies monotonicity and transitivity. Then $\succeq$ satisfies join dominance if and only if it satisfies incomparability of marginal losses.*

Proof. Suppose that incomparability of marginal losses, monotonicity, and transitivity are satisfied, and let $X \succeq Y$. By monotonicity, $(X \cup Y) \succeq X$; suppose, to the contrary, that $(X \cup Y) \succ X$. Then by transitivity, $(X \cup Y) \succ Y$ as well. By incomparability of marginal losses, we conclude $X \parallel Y$, a contradiction. Conversely, suppose that join dominance is satisfied and that $(X \cup Y) \succ X$ and $(X \cup Y) \succ Y$. We claim that $X \parallel Y$; if not, we can suppose, without loss of generality, that $X \succeq Y$. Then by join dominance, we have $X \succeq (X \cup Y)$, again a contradiction. $\square$

[11] Kreps' version of this axiom is a join homomorphism property (see Kreps (1979) property (1.2), p. 565). Join dominance is implied by the combination of Kreps' axiom and monotonicity, and the latter combination is equivalent to join dominance under the assumption of completeness.

We claim below that Kreps implicitly provided a characterization of relations $\succeq$ satisfying monotonicity, transitivity, and incomparability of marginal losses. Thus, aside from the technical details mentioned at the beginning of this section, our contribution is to uncover what happens when adding incomparability of marginal gains.

The complete case

Kreps uses join dominance on complete and transitive rankings $\succeq$ of $\mathcal{X}$ to derive a notion of indirect utility. The following result is not given the formal status of a theorem by Kreps, but is discussed in the paragraph following his introduction of the property.

Proposition 2. *A relation $\succeq$ on $\mathcal{X} \setminus \{\varnothing\}$ satisfies completeness, transitivity, monotonicity, and join dominance if and only if there is a complete and transitive relation $\succeq^*$ on A for which for all $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if for all $y \in Y$, there is $x \in X$ for which $x \succeq^* y$.*

We call the representation in Proposition 2 an *indirect utility* representation because the preference over menus (or budgets) is determined uniquely by a ranking over the singleton alternatives, with each menu essentially being indifferent to its best element.

To this, we have added another axiom. This axiom is dual to join dominance, and we call it *meet dominance*: for all $X, Y \in \mathcal{X}$, if $X \succeq Y$, then $(X \cap Y) \succeq Y$. As in Proposition 1, it is equivalent to incomparability of marginal gains in the presence of transitivity and monotonicity.[12]

Adding the meet dominance property to Kreps' properties and working on the domain of all sets delivers Corollary 1. This tells us that for each relation $\succeq$ over $\mathcal{X}$ satisfying the axioms join dominance, meet dominance, monotonicity, completeness, and transitivity, there is a linearly ordered subset of A, which are to be taken as benchmarks.[13] In particular, in the case where $\leq$ is trivial, there are exactly $|A| + 1$ linearly ordered sets: each $a \in A$ forms a singleton linearly ordered set, and the empty set is a linearly ordered set.

To see how this fits with a special case of Proposition 2, observe that each $b \in A$ induces a ranking $\succeq_b$ over A, where for all $a \in A \setminus \{b\}$, $\{b\} \succ_b \{a\}$, and for all $a, c \in A \setminus \{b\}$, $\{a\} \sim \{c\}$. Hence, $\succeq_b$ has exactly two indifference classes. At the top is b, and everything else is ranked strictly below b (and indifferent). For a benchmarking relation $\succeq$ with a singleton benchmark $\mathcal{B} = \{b\}$, by taking $\succeq_b$ to be $\succeq^*$ in Proposition 2, we get $\succeq$. Alternatively, if $\succeq$ is a benchmarking relation with the empty set as benchmarks, the ranking $\succeq^*$ derived in Proposition 2 corresponds to complete indifference over A.

The incomplete case

Less well known is the fact that Kreps generalizes Proposition 2 in the proof of his main theorem with an extremely clever argument. The following result can be easily derived

[12] Meet dominance is formally dual to join dominance in the following sense: given a relation $\succeq$ over $\mathcal{X}$ satisfying join dominance, the relation $\succeq^*$ defined by $X \succeq^* Y$ if and only if $(A \setminus Y) \succeq (A \setminus X)$ satisfies meet dominance, and conversely. This motivates our notion of dual indirect utility above. Observe that the properties of monotonicity and transitivity are self-dual in the same sense.

[13] Monotonicity is implied by the combination of completeness and join dominance.

by appropriately adapting the construction he develops there. We will not replicate the argument here as it can be understood through a careful reading of his proof or as a special case of our more general lattice-theoretic argument, which appears in Chambers et al. (2015).[14]

THEOREM 3. *Suppose that $\succeq$ over $\mathcal{X} \setminus \{\varnothing\}$ is monotonic, transitive, and satisfies join dominance. Then there is a family of complete and transitive relations $\{\succeq_\lambda\}_{\lambda\in\Lambda}$ over A, such that for all $X, Y \in \mathcal{X}$, $X \succeq Y$ if and only if for all $\lambda \in \Lambda$ and all $y \in Y$, there is $x \in X$ such that $x \succeq_\lambda y$.*

This representation can be viewed as a vector-valued indirect utility representation in the sense in the Introduction. Importantly, there is no uniqueness or identification result here. There are many representations of the type described in Theorem 3.

Even though our result stems from adding an axiom to Theorem 3, the nonuniqueness inherent in Theorem 3 introduces a difficulty. To prove Theorem 1, we need to establish that there is one vector-valued representation (out of potentially many) that takes the form we characterize. So we do not simply take a uniquely identified representation and find what happens when adding an axiom.

Theorem 1 can be related to Theorem 3 in the same way we related Corollary 1 and Proposition 2. Theorem 3 claims there is a set of relations, $\{\succeq_\lambda\}$. In the benchmarking case, suppose that $\succeq$ is associated with a set of benchmarks $\mathcal{B}$, which we will suppose is nonempty. Each $b \in \mathcal{B}$ is associated with a relation $\succeq_b$, as described after Proposition 2. We take the set of relations in Theorem 3 to be $\{\succeq_b\}_{b\in\mathcal{B}}$. Then, for any sets X, Y, observe that $X \succeq Y$ if and only if $b \in Y$ implies $b \in X$, which is the same as saying that for every $b \in \mathcal{B}$ and every $y \in Y$, there exists $x \in X$ such that $x \succeq_b y$.

Embedding our result into Kreps' main theorem

There is more to discuss. The primary focus of Kreps' work is to consider a complete relation $\succeq'$ over nonempty sets, from which an auxiliary relation $\succeq$ satisfying the axioms of Theorem 3 is described. The auxiliary relation $\succeq$ and the relation $\succeq'$ are related by extension: in Kreps' axiomatization, it follows that $X \succeq Y$ implies $X \succeq' Y$ and $X \succ Y$ implies $X \succ' Y$. Kreps does this by introducing another axiom, termed *ordinal submodularity* in Chambers and Echenique (2016):

Ordinal Submodularity. For all $X, Y, Z \in \mathcal{X}$, $X \sim' (X \cup Y)$ implies $(X \cup Z) \sim' (X \cup Y \cup Z)$.

Kreps uses ordinal submodularity to prove the following result.

[14] Kreps actually uses a slightly distinct collection of axioms to derive this result. He uses the axioms

(i) transitivity

(ii) monotonicity

(iii) $Z \subseteq Y \preceq X$ implies $Z \preceq X$

(iv) $X \succeq Y$ and $Z \succeq W$ implies $(X \cup Z) \succeq (Y \cup W)$

(v) for every X, there exists Y such that $X \succeq Z$ if and only if $Z \subseteq Y$.

THEOREM 4. *The following statements are equivalent for a binary relation $\succeq'$ on $X \setminus \{\varnothing\}$.*

(i) The relation $\succeq'$ satisfies completeness, transitivity, monotonicity, and ordinal submodularity.

(ii) There is a collection of utility functions $\{u_\lambda\}_{\lambda\in\Lambda}$, where $u_\lambda : A \to \mathbb{R}$, and a strictly monotone function $v : \mathbb{R}^\Lambda \to \mathbb{R}$ such that $U(X) = v(\max_{x\in X} u_\lambda(x))$ represents $\succeq'$.

(iii) There is a collection of utility functions $\{u_\lambda\}_{\lambda\in\Lambda}$, where $u_\lambda : A \to \mathbb{R}$, such that

$$U(X) = \sum_{\lambda\in\Lambda} \max_{x\in X} u_\lambda(x)$$

represents $\succeq'$.

We sketch the relationship between parts (i) and (ii) of the theorem. Define the auxiliary relation $\succeq$ on $\mathcal{X} \setminus \{\varnothing\}$ by $X \succeq Y$ if and only if $X \sim' (X \cup Y)$. Note that the axioms of part (i) imply that the axioms of Theorem 3 are satisfied for $\succeq$; in particular, ordinal submodularity is used to establish the transitivity of $\succeq$. From the definition of $\succeq$ it is simple to see that $X \succeq Y$ implies $X \succeq' Y$ and $X \succ Y$ implies $X \succ' Y$. Consequently, from Theorem 3 it follows that $\succeq$ has a vector utility representation; this, in turn, implies that there is a strictly monotone function v such that $U(X)$ represents $\succeq'$. For further details, see Kreps.

This suggests that adding an additional axiom to $\succeq'$ would be enough to guarantee that the endogenously derived $\succeq$ would also satisfy meet dominance.

Meet Join Consistency. For all $X, Y \in \mathcal{X}$, $X \succeq' (X \cup Y)$ implies $(X \cap Y) \succeq' Y$.[15]

COROLLARY 3. *Suppose that $\succeq'$ on $\mathcal{X}$ satisfies completeness, transitivity, monotonicity, ordinal submodularity, and meet join consistency. Then the relation $\succeq$ defined by $X \succeq Y$ if and only if $X \sim' (X \cup Y)$ satisfies the axioms of Theorem 1.*

PROOF. Most of the axioms were already established by Kreps, but we here establish meet dominance. The satisfaction of these axioms holds true even when including the empty set, as is easily verified.

So suppose that $X \succeq Y$. Then $X \succeq' (X \cup Y)$. Consequently, by meet join consistency, $(X \cap Y) \succeq' Y$. Hence, $(X \cap Y) \succeq Y$. □

Let us rule out the case of complete indifference by the following postulate.

Nondegeneracy. There exists $X, Y \in \mathcal{X}$ for which $X \succ' Y$.

We therefore obtain the following corollary.

COROLLARY 4. *Suppose that $|A| < +\infty$. A relation $\succeq'$ on $\mathcal{X}$ satisfies completeness, transitivity, monotonicity, nondegeneracy, ordinal submodularity, and meet join consistency if and only if there is a nonempty set $\mathcal{B} \subseteq A$ and a strictly monotonic capacity $v : 2^{\mathcal{B}} \to \mathbb{R}$ such that $U(X) = v(\mathcal{B} \cap X)$ represents $\succeq'$. Furthermore, the set $\mathcal{B}$ is unique.*

[15] An anonymous referee also suggests a weakened version: $X \succeq' Y \succ' (X \cap Y)$ implies $(X \cup Y) \succ' X$.

Proof. It is easy to verify that the axioms are satisfied if the representation holds.

Conversely, observe that the derived representation $\succeq$ from the preceding corollary satisfies the axioms of Theorem 1, so that there is $\mathcal{B} \neq \varnothing$ for which $X \succeq Y$ if and only if $(Y \cap \mathcal{B}) \subseteq (X \cap \mathcal{B})$. Now observe that $X \succeq Y$ implies $X \succeq' Y$ by construction and, further, $X \succ Y$ implies $X \succ' Y$. This completes the proof, as we may now take v to be any function strictly monotonic in $\mathcal{B} \cap X$ that gives the desired representation $\succeq'$. Uniqueness of B is straightforward. □

Alternatively, it is not without loss of generality to assume a representation as in part (iii) of Theorem 4, where additivity across the elements $b \in \mathcal{B}$ prevails. Observe that additivity in this case means representation via a probability measure on $\mathcal{B}$. It is well known that a strictly monotonic capacity need not be ordinally equivalent to a probability measure.[16] In fact, Kraft et al. (1959) demonstrate that this is true even for a strictly monotonic capacity that satisfies a well known additivity axiom of decision theory. Kraft et al. (1959, Theorem 2) also provide conditions for a relation on a finite algebra to be represented by a probability measure. Scott (1964) provides a simplified proof and exposition based on the theorem of the alternative.

We do not describe the axiomatization here, but the conditions can be found in Scott (1964). We refer to these axioms as the axioms of Kraft et al. (1959).

Theorem 5. *Suppose $|A| < +\infty$. Then there is $\mathcal{B} \neq \varnothing$ and a probability measure π on $2^{\mathcal{B}}$ such that $X \succeq' Y \Leftrightarrow \pi(\mathcal{B} \cap X) \geq \pi(\mathcal{B} \cap Y)$ if and only if $\succeq'$ satisfies the axioms of Kraft et al. (1959).*

The question remains as to whether similar variants of these results hold with a nontrivial $\leq$. The answer is yes, and this type of question is pursued in more detail in Chambers et al. (2015).

Conclusion

We have described a method of comparison that we term benchmarking. Benchmark rules are characterized by four axioms: transitivity, monotonicity, incomparability of marginal gains, and incomparability of marginal losses.

Benchmark rules are not necessarily complete, and can be used in cases where completeness is considered undesirable or is otherwise not required. One can see that a benchmark rule will satisfy completeness if and only if the benchmarks are totally ordered. In the case of ranking scholars, this implies that they must form a step-based index.

In some cases it may seem as if, in practice, the benchmark rule is simply a comparison according to set inclusion. This does not necessarily mean that all potential accomplishments are benchmarks. Alternatively, it may be that only benchmarks are included on résumés. This would be expected if the rule were to be known with certainty.

[16] For example, take any strictly monotonic capacity on three elements $\{a, b, c\}$, where $v(\{a\}) = v(\{b\}) = v(\{c\})$, but $v(\{a, b\}) \neq v(\{a, c\})$.

In other cases, however, résumés often include accomplishments that are not benchmarks. One might ask why this would occur. As we noted, this might come across as a result of uncertainty about the rule. However, there are other possibilities. One is that the applicant finds it efficient to use the same résumé for multiple employers or in multiple markets, where different rules may be applied. Alternatively, as we explained in the Introduction, the benchmark rule might be used only as a first step in sorting applicants; the otherwise extraneous information might still be relevant when making an executive decision.

Appendix

Proof of Claim 1. Let $\succeq$ be a benchmark-majority rule. To see that $\succeq$ is monotone, let $X \supseteq Y$. Because the benchmark rules are monotone, it follows that $X \succeq_i Y$ for all i and, thus, $X \succeq Y$.

To see that $\succeq$ satisfies incomparability of marginal gains, let $X \succ (X \cap Y)$ and $Y \succ (X \cap Y)$, and suppose, to the contrary, that X and Y are comparable. Without loss of generality assume that $X \succeq Y$. By monotonicity of $\succeq_i$, $X \succeq_i (X \cap Y)$ and $Y \succeq_i (X \cap Y)$ for all i. It follows that for all i, $X \succeq_i Y$ only if $Y \sim_i (X \cap Y)$. Thus $|\{i : Y \sim_i (X \cap Y)\}| > \frac{n}{2}$. However, because $Y \succ (X \cap Y)$, it follows that $|\{i : Y \sim_i (X \cap Y)\}| \leq \frac{n}{2}$, a contradiction.

Incomparability of marginal losses follows from a similar argument.

To see that some benchmark-majority rules violate transitivity, consider the case where $A = \{1, \dots, n\}$ with $\leq$ being the trivial order, and suppose that $n \geq 3$. Let $\succeq_i$ be the benchmark rule with the set of benchmarks $\{i\}$. Then it is easy to see that, for all $k < n$, $\bigcup_{i=1}^{k}\{i\} \succeq \bigcup_{i=1}^{k+1}\{i\}$, but $A \succ \{1\}$. □

Proof of Claim 2. Transitivity of $\succeq$ follows from transitivity of $=$. The relation $\succeq$ is symmetric, so there are no strict rankings. Thus, this trivially satisfies incomparability of marginal gains and incomparability of marginal losses. However, this rule is not monotone. To see this, let $X \subsetneq Y$. Then $Y \not\succ X$, a violation of monotonicity. □

Proof of Claim 3. Clearly, $\succeq$ is transitive and monotonic.

The rule satisfies incomparability of marginal losses, as there is no $X, Y \in \mathcal{X}$ for which $(X \cup Y) \succ X$ and $(X \cup Y) \succ Y$. To see this, let $X, Y \in \mathcal{X}$ and suppose that $(X \cup Y) \succ X$ and $(X \cup Y) \succ Y$. Since $(X \cup Y) \succ X$, there is $a \in (Y \backslash X)$ for which $a \mathrel{\dot{\succ}} c$ for all $c \in X$, and since $(X \cup Y) \succ Y$, there is $d \in (X \backslash Y)$ for which $d \mathrel{\dot{\succ}} c$ for all $c \in Y$. Then we have $a \mathrel{\dot{\succ}} d$ and $d \mathrel{\dot{\succ}} a$, a contradiction.

However, this rule violates incomparability of marginal gains. To see this, let $A = \{1, 2\}$, where $\leq$ is defined so that no two distinct items are comparable. Observe that $\{1\} \succ \varnothing$ and that $\{2\} \succ \varnothing$, yet $\succeq$ is complete, and, hence, it cannot be the case that $\{1\} \parallel \{2\}$. □

Proof of Claim 4. Let $\succeq$ be a dual indirect utility ranking. To see that $\succeq$ is transitive, suppose that $X \succeq Y$ and $Y \succeq Z$. Then for all $x \notin X$, there is $y_x \notin Y$ for which $x \mathrel{\dot{\succcurlyeq}} y_x$, and for all $y \notin Y$, there is $z_y \notin Z$ for which $y \succsim z_y$. By transitivity of $\succsim$, for all $x \notin X$, $z_{y_x} \notin Z$ and $x \succsim z_{y_x}$. Hence, $X \succeq Z$.

To see monotonicity, suppose that $Y \subseteq X$. Then, whenever $x \notin X$, we know that $x \notin Y$; hence, since $x \succsim x$, we have $X \succeq Y$.

The rule satisfies incomparability of marginal gains, as there is no $X, Y \in \mathcal{X}$ for which $X \succ (X \cap Y)$ and $Y \succ (X \cap Y)$. To see this, let $X, Y \in \mathcal{X}$ and suppose, to the contrary, that $X \succ (X \cap Y)$ and $Y \succ (X \cap Y)$.

Since $X \succ (X \cap Y)$, it follows that $(X \cap Y) \succeq X$ is false. Hence, there is some $a \notin (X \cap Y)$ such that for all $z \notin X$, $z \succ a$. In particular, it follows easily that $a \in X \setminus Y$, as otherwise we would get $a \succ a$, since $a \notin X$.

Symmetrically, since $Y \succ (X \cap Y)$, it follows that there is some $b \in Y \setminus X$ such that for all $z \notin Y$, $z \succ b$.

Observe then that $a \succ b \succ a$, a contradiction.

In general, this rule violates incomparability of marginal losses. To see this, let $A = \{1, 2\}$, where $\leq$ is defined so that no two distinct items are comparable, and let $\{1\} \sim \{2\}$. Observe that $\{1, 2\} \succ \{1\}$ and that $\{1, 2\} \succ \{2\}$, but $\{1\} \sim \{2\}$. □

Proof of Theorem 2. Let $\mathcal{B}$ and $\mathcal{B}'$ be sets of benchmarks with associated benchmarking rules $\succeq_{\mathcal{B}}$ and $\succeq_{\mathcal{B}'}$. We first prove that $\mathcal{B} \subseteq \mathcal{B}'$ if $\succeq_{\mathcal{B}'} \subseteq \succeq_{\mathcal{B}}$. To see this, let $\succeq_{\mathcal{B}'} \subseteq \succeq_{\mathcal{B}}$ and let $x \in \mathcal{B}$. Then $\mathcal{K}(x) \succ_{\mathcal{B}} \varnothing$. Because $\succeq_{\mathcal{B}'} \subseteq \succeq_{\mathcal{B}}$, it follows that $\varnothing \not\succeq_{\mathcal{B}'} \mathcal{K}(x)$. Because $\succeq_{\mathcal{B}'}$ is monotonic, it follows that $\mathcal{K}(x) \succ_{\mathcal{B}'} \varnothing$. Assume, to the contrary, that $x \notin \mathcal{B}'$; hence, $\mathcal{K}(x) \sim_{\mathcal{B}'} \mathcal{K}(x) \setminus \{x\}$. Because $x \in \mathcal{B}$, it follows that $\mathcal{K}(x) \succ_{\mathcal{B}} \mathcal{K}(x) \setminus \{x\}$, which contradicts the assumption that $\succeq_{\mathcal{B}'} \subseteq \succeq_{\mathcal{B}}$.

To prove the converse, let $\mathcal{B} \subseteq \mathcal{B}'$ and let $X \succeq_{\mathcal{B}'} Y$. Then $(Y \cap \mathcal{B}') \subseteq (X \cap \mathcal{B}')$. Because $\mathcal{B} \subseteq \mathcal{B}'$, it follows that $(Y \cap \mathcal{B}) \subseteq (X \cap \mathcal{B})$ and, hence, $X \succeq_{\mathcal{B}} Y$. □

Proof of Corollary 1. Suppose that $\succeq$ is a benchmarking rule with associated benchmarks $\mathcal{B}$.

First, suppose that $\mathcal{B}$ has the property that for all $a, b \in \mathcal{B}$, either $a \leq b$ or $b \leq a$. We will show that $\succeq$ is complete. Let $X, Y \in \mathcal{X}$. If $X \sim Y$, we are done, so suppose that $(\mathcal{B} \cap X) \neq (\mathcal{B} \cap Y)$. Without loss of generality, let $b \in (\mathcal{B} \cap X)$ and suppose that $b \notin (\mathcal{B} \cap Y)$. If $(\mathcal{B} \cap Y) = \varnothing$, we have that $X \succeq Y$, so assume that $(\mathcal{B} \cap Y) \neq \varnothing$. Let $c \in (\mathcal{B} \cap Y)$. Now either $b \leq c$ or $c \leq b$. If the former, then since Y is comprehensive, $b \in (\mathcal{B} \cap Y)$, which is false. Hence, $c \leq b$. As X is comprehensive, it follows that $c \in X$. We conclude that $(\mathcal{B} \cap Y) \subseteq (\mathcal{B} \cap X)$, so that $X \succeq Y$.

Conversely, suppose that $\succeq$ is complete and let $b, c \in \mathcal{B}$. Either $\mathcal{K}(b) \succeq \mathcal{K}(c)$ or conversely. Suppose without loss of generality that $\mathcal{K}(b) \succeq \mathcal{K}(c)$. We conclude that $c \in (\mathcal{B} \cap \mathcal{K}(c)) \subseteq (\mathcal{B} \cap \mathcal{K}(b))$. By definition of $\mathcal{K}(b)$, $c \in \mathcal{K}(b)$ implies that $c \leq b$. □

Proof of Corollary 2. Suppose that $\succeq$ is a benchmarking rule and let $\mathcal{B}$ be the associated set of benchmarks. For each $b \in \mathcal{B}$, let $\succeq_{\{b\}}$ be the associated benchmarking rule. Since $\{b\}$ is (vacuously) a chain, $\succeq_{\{b\}}$ is complete. Let $\mathcal{R} = \bigcup\{\succeq_{\{b\}} : b \in \mathcal{B}\}$.

Now let $X, Y \in \mathcal{X}$. Suppose that $X \succeq Y$. Then $(\mathcal{B} \cap Y) \subseteq (\mathcal{B} \cap X)$. In particular, for all $b \in \mathcal{B}$, we have $b \in Y$ implies $b \in X$; we conclude that $X \succeq_{\{b\}} Y$.

Conversely, suppose that for all $b \in \mathcal{B}$, we have that $X \succeq_{\{b\}} Y$. Then, if $b \in Y$, it follows that $b \in X$. We conclude that $X \succeq Y$. □

References

Bewley, Truman F. (2002), "Knightian decision theory. Part I." *Decisions in Economics and Finance*, 25, 79–110. [489]

Chambers, Christopher P. and Federico Echenique (2016), *Revealed Preference Theory*, volume 56 of Econometric Society Monographs. Cambridge University Press. [498]

Chambers, Christopher P. and Alan D. Miller (2011), "Rules for aggregating information." *Social Choice and Welfare*, 36, 75–82. [489]

Chambers, Christopher P. and Alan D. Miller (2014a), "Inefficiency measurement." *American Economic Journal: Microeconomics*, 6, 79–92. [489, 494]

Chambers, Christopher P. and Alan D. Miller (2014b), "Scholarly influence." *Journal of Economic Theory*, 151, 571–583. [488, 489, 494, 495]

Chambers, Christopher P., Alan D. Miller, and M. Bumin Yenmez (2015), "Closure and preferences." Unpublished paper, SSRN 2671963. [498, 500]

Christensen, Flemming, Jens Leth Hougaard, and Hans Keiding (1999), "An axiomatic characterization of efficiency indices." *Economics Letters*, 63, 33–37. [489, 494]

Dimitrov Dinko, Thierry Marchant and Debasis Mishra (2012), "Separability and aggregation of equivalence relations." *Economic Theory*, 51, 191–212. [489]

Donaldson, David and John A. Weymark (1998), "A quasiordering is the intersection of orderings." *Journal of Economic Theory*, 78, 382–387. [489]

Dubra, Juan, Fabio Maccheroni, and Efe A. Ok (2004), "Expected utility theory without the completeness axiom." *Journal of Economic Theory*, 115, 118–133. [489]

Duggan, John (1999), "A general extension theorem for binary relations." *Journal of Economic Theory*, 86, 1–16. [489]

Dushnik, Ben and E. W. Miller (1941), "Partially ordered sets." *American Journal of Mathematics*, 63, 600–610. [489]

Eliaz, Kfir and Efe A. Ok (2006), "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences." *Games and Economic Behavior*, 56, 61–86. [489]

Hirsch, Jorge E. (2005), "An index to quantify an individual's scientific research output." *Proceedings of the National Academy of Sciences of the United States of America*, 102, 16569–16572. [488, 494]

Hougaard, Jens Leth and Hans Keiding (1998), "On the functional form of an efficiency index." *Journal of Productivity Analysis*, 9, 103–111. [489, 494]

Kraft, Charles H., John W. Pratt, and Abraham Seidenberg (1959), "Intuitive probability on finite sets." *Annals of Mathematical Statistics*, 30, 408–419. [500]

Kreps, David M. (1979), "A representation theorem for "Preference for flexibility"." *Econometrica*, 47, 565–577. [489, 493, 495, 496]

Leclerc, Bruno (2011), "Galois connections in axiomatic aggregation." In *Formal Concept Analysis* (Petko Valtchev and Robert Jäschke, eds.), volume 6628 of Lecture Notes in Computer Science, 24–25, Springer, Berlin. [489]

Leclerc, Bruno and Bernard Monjardet (2013), "Aggregation and residuation." *Order*, 30, 261–268. [489]

Manzini, Paola and Marco Mariotti (2007), "Sequentially rationalizable choice." *American Economic Review*, 97, 1824–1839. [489]

Miller, Alan D. (2008), "Group identification." *Games and Economic Behavior*, 63, 188–202. [489]

Ok, Efe A. (2002), "Utility representation of an incomplete preference relation." *Journal of Economic Theory*, 104, 429–449. [489]

Scott, Dana (1964), "Measurement structures and linear inequalities." *Journal of Mathematical Psychology*, 1, 233–247. [500]

Szpilrajn, Edward (1930), "Sur l'extension de l'ordre partiel." *Fundamenta Mathematicae*, 16, 386–389. [489]

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