Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition[1]

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Distributional constraints are common features in many real matching markets, such as medical residency matching, school admissions, and teacher assignment. We develop a general theory of matching mechanisms under distributional constraints. We identify the necessary and sufficient condition on the constraint structure for the existence of a mechanism that is stable and strategy-proof for the individuals. Our proof exploits a connection between a matching problem under distributional constraints and a matching problem with contracts.

Keywords. Matching with constraints, medical residency matching, school choice, stability, strategy-proofness, matching with contracts, hierarchy.

JEL classification. C70, D47, D61, D63.

1. Introduction

The theory of two-sided matching has been extensively studied since the seminal contribution by Gale and Shapley (1962), and has been applied to match individuals and institutions in various markets in practice (e.g., students and schools, doctors and hospitals, and workers and firms). However, many real matching markets are subject to distributional constraints, i.e., caps are imposed on the numbers of individuals who can

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be matched to some subsets of institutions. Traditional theory cannot be applied to such settings because it has assumed away those constraints.

The objective of this paper is to understand the implication of the structure of distributional constraints by investigating the extent to which a desirable mechanism can be designed under constraints. More specifically, we identify the necessary and sufficient condition on the constraint structure for the existence of a mechanism that is stable and strategy-proof for the individuals.[2] The necessary and sufficient condition is that constraints form a "hierarchy," that is, for any pair of subsets of institutions that are subject to constraints, the two are disjoint or one is a subset of the other.[3,4]

To understand the implications of our result, consider matching doctors with hospitals. To take an example from a real market, consider the medical matching market in Japan. As is the case in many countries, the Japanese government desires to keep the geographical imbalance of doctors in check. For that purpose, it imposes caps on the numbers of doctors in different prefectures that partition the country. Kamada and Kojima (2015a) propose a mechanism that produces a stable matching and is strategy-proof for doctors. Since partitions are a special case of hierarchy, their result can be derived as a corollary of our result.[5] The same positive conclusion holds even if the geographical constraints are imposed on a hierarchy of regions. For example, in Japan, each prefecture is divided into smaller geographic units called *Iryo-ken* (medical areas) in which various health care policies are implemented.[6] This is a case of a hierarchy, and hence our result implies that a desirable mechanism exists.

Next consider medical match with geographical constraints, but assume that the government also desires to impose caps on medical specialties.[7] Suppose for now that the government decides to impose nationwide caps on each specialty. In Japanese medical match, for example, a hospital program may be in the "obstetrics" region at the same time as it is in the "Tokyo" region, where neither of the two is a subset of the other. Our result implies that there exists no mechanism that is stable and strategy-proof for doctors in this environment. However, to the extent that doctors treat patients in person,

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[2]We formally define a stability concept under constraints in Section 2. Our stability concept reduces to the standard stability concept of Gale and Shapley (1962) if there are no binding constraints.

[3]Such a hierarchical structure is called a laminar family in the mathematics literature.

[4]Budish et al. (2013) consider a random object allocation problem with floor and ceiling constraints and show that "bi-hierarchical" constraints are necessary and sufficient for implementability of random assignments. Although the condition they reach is similar to ours, their implementability is unrelated to stability and strategy-proofness that we study here. See also Milgrom (2009), who considers hierarchical constraints in an auction setting. His analysis is unrelated to ours as, among other things, he considers a setting with continuous transfer and does not analyze stability.

[5]To see this connection, we also need to verify that stability in Kamada and Kojima (2015a) corresponds to stability in the present paper. See Kamada and Kojima (2015b) for details.

[6]Article 30-4 of the Medical Care Act of Japan.

[7]In the Japanese context, this concern is clearly exemplified by a proposal made to the governmental committee meeting that is titled "Measures to Address Regional Imbalance and Specialty Imbalance of Doctors" (Ministry of Health, Labour and Welfare, Japan 2008). In 2008, Japan took a measure that intended to address (only) regional imbalance, but even in 2007, the sense of crisis about specialty imbalance is shared by the private sector as well: In a sensationally titled article, "Obstetricians Are in Short Supply! Footsteps of Obstetrics Breakdown," NTT Com Research (2007) reports that many obstetrics hospitals have been closed even in urban areas such as Tokyo.

imposing specialty constraints in each region rather than nationally may be reasonable.[8] Constraints form a hierarchy in such a situation, so our result implies that there exists a mechanism with our desired properties.

Let us now consider field and financial constraints in college admission. In Hungarian college admission, there are upper-bound constraints on the number of students in terms of different fields of study as well as whether the study is subsidized by the government. In this problem, the constraints formed a hierarchy until 2007, but then the constraints were modified such that they do not form a hierarchy anymore (Biró et al. 2010). Thus, our result implies that there exists a mechanism that achieves stability and strategy-proofness for students in the old environment, but no mechanism can achieve these properties after the change.

Our result also has implications for the design of school choice mechanisms (Abdulkadiroğlu and Sönmez 2003). Maintaining diversity is a major concern in school choice, and many school districts have used policies to achieve this goal.[9] Suppose first that the school district wants to maintain a certain balance among the student body at each school in terms of socioeconomic class. Different socioeconomic classes form a partition (and thus a hierarchy), so a stable and strategy-proof mechanism exists. By contrast if, for instance, the school district desires to maintain balance in both socioeconomic class and gender, then the constraints do not form a hierarchy, so a desirable mechanism does not exist.[10]

In addition to its applied value, we believe that our analytical approach is of independent interest. The approach for showing the sufficiency of a hierarchy is to find a connection between our model and the "matching with contracts" model (Hatfield and Milgrom 2005).[11] More specifically, we define a hypothetical matching problem between doctors and *the hospital side* instead of doctors and hospitals. That is, we regard the hospital side as a hypothetical consortium of hospitals that acts as one agent.

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[8]For instance, the policy maker may decide not to give authority to claim caps and preferences to nationwide organizations such as the Japan Society of Obstetrics & Gynecology, but to give it to each of the regional organizations such as the Tokyo Association of Obstetricians and Gynecologists (see http://www.jsog.or.jp for the former organization and http://www.taog.gr.jp for the latter).

[9]For example, New York City (NY), Chicago (IL), Seattle (WA), Jefferson County (KY), Louisville (KY), Minneapolis (MN), and White Plains (NY) (Abdulkadiroğlu et al. 2005, Hafalir et al. 2013, Dur et al. 2016).

[10]Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005) show that a stable and strategy-proof mechanism exists under constraints on socioeconomic class and give an example that shows nonexistence when constraints are also imposed on gender. There are three main differences between their work and ours. First, we consider a different formulation than theirs. Second, our characterization of the necessary condition for existence is new. Third, we show existence under hierarchical constraints. See also Roth (1991) on gender balance in labor markets, Ergin and Sönmez (2006), Hafalir et al. (2013), Ehlers et al. (2014), and Echenique and Yenmez (2015) on diversity in schools, Westkamp (2013) on trait-specific college admission, Abraham et al. (2007) on project-specific quotas in projects–students matching, and Biró et al. (2010) on college admission with multiple types of tuitions.

[11]Fleiner (2003) considers a framework that generalizes various mathematical results. A special case of his model corresponds to the model of Hatfield and Milgrom (2005), although not all results of the latter (e.g., those concerning incentives) are obtained in the former. See also Crawford and Knoer (1981), who observe that wages can represent general job descriptions in their model, given their assumption that firm preferences satisfy separability.

By imagining that the hospital side (hospital consortium) makes a common employment decision, we can account for interrelated doctor assignments across hospitals, an inevitable feature in markets under distributional constraints. This association necessitates, however, that we distinguish a doctor's matching in different hospitals. We account for this complication by constructing the hospital side's choice defined as one over *contracts* rather than doctors, where a contract specifies a doctor–hospital pair to be matched. Once this connection is established, we can show that results in the matching-with-contract model can be applied to our matching model under distributional constraints.[12] This method shows that there exists a mechanism that is stable and strategy-proof for doctors. Note that our technique is different from those in school choice with diversity constraints, which take schools' aggregated preferences over different types of students *as primitives*. By contrast, aggregated preferences of the hospital side are *not* primitives of our model, and our contribution is to construct an appropriate hypothetical model with an aggregated choice function. We envision that analyzing a hypothetical model of matching with contracts may prove to be a useful approach for tackling complex matching problems one may encounter in the future.[13,14]

A recent paper by Hatfield et al. (2015) characterizes the class of choice functions over abstract contracts such that there exists a mechanism that is stable and strategy-proof for doctors (see a related contribution by Hirata and Kasuya (2017) as well). Our result and theirs are independent because our model and stability concept are different from theirs. Specifically, our model is that of matching with constraints and, even though we can associate our stability concept to theirs through our technique, these concepts are still not equivalent (Kamada and Kojima 2015a).

This paper is a part of our research agenda to study matching with constraints. Kamada and Kojima (2015a) consider the case in which constraints form a partition and show that a desirable mechanism exists. In practice, however, non-partitional constraints are prevalent. So as to understand what kinds of applications can be accommodated, the present paper does not presume a partition structure, and instead investigates a conceptual question. More specifically, we find a necessary and sufficient condition for the existence of a desirable mechanism in terms of the constraint structure. Kamada and Kojima (2017) investigate how to define the "right" stability concept for matching with constraints, defining what they call strong stability and weak stability. They show that strong stability suffers from a nonexistence problem whenever constraints are "nontrivial," while weakly stable matchings exist for a wide range of constraints and are efficient. Stability defined in the present paper is weaker than strong stability but stronger than weak stability. In contrast to strong and weak stability, the

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[12]Specifically, we invoke results by Hatfield and Milgrom (2005), Hatfield and Kojima (2009, 2010), and Hatfield and Kominers (2011, 2012).

[13]Indeed, after we circulated the first draft of the present paper, this technique was adopted by other studies such as Goto et al. (2014a, 2014b), and Kojima et al. (2017) in the context of matching with distributional constraints. See Sönnez and Switzer (2013) for a more direct application of the model of matching with contracts, where a cadet can be matched with a branch under one of two possible contracts. See also Sönmez (2013) and Kominers and Sönmez (2016).

[14]See also Hatfield et al. (2013), who connect stability in a trading network with stability in an associated model of many-to-one two-sided matching with transfer due to Kelso and Crawford (1982).

definition of stability in this paper uses information about policy goals of regions in terms of how to allocate their limited seats.[15] Thus our stability is particularly appropriate when such information is available. Instead of exploring many kinds of stability concepts, the present paper fixes one stability concept and examines the implication of constraint structures for the existence of desirable mechanisms.

The rest of this paper proceeds as follows. In Section 2, we present the model. Section 3 states the main results. Section 4 concludes. The Appendix provides the formal proof of the result as well as a number of discussions.

2. Model

This section introduces a model of matching under distributional constraints. We describe the model in terms of matching between doctors and hospitals with "regional caps," that is, upper bounds on the number of doctors that can be matched to hospitals in each region. However, the model is applicable to various other situations in and out of the residency matching context. For example, in medical residency applications, a region can represent a geographical region, medical specialty, or a combination of them.[16] Another example is school choice, where a region can represent a socioeconomic class of students.

We begin with preliminary definitions for two-sided matching in Section 2.1. Then Section 2.2 introduces our model of matching with constraints.

2.1 *Preliminary definitions*

Let there be a finite set of doctors D and a finite set of hospitals H. Suppose that $|D| \geq 2$. Each doctor d has a strict preference relation $\succ_d$ over the set of hospitals and being unmatched (being unmatched is denoted by $\varnothing$). For any $h, h' \in H \cup \{\varnothing\}$, we write $h \succeq_d h'$ if and only if $h \succ_d h'$ or $h = h'$. Each hospital h has a strict preference relation $\succ_h$ over the set of subsets of doctors. For any $D', D'' \subseteq D$, we write $D' \succeq_h D''$ if and only if $D' \succ_h D''$ or $D' = D''$. We denote by $\succ = (\succ_i)_{i \in D \cup H}$ the preference profile of all doctors and hospitals.

Doctor d is said to be *acceptable* to hospital h if $d \succ_h \varnothing$.[17] Similarly, h is acceptable to d if $h \succ_d \varnothing$. It will turn out that only rankings of acceptable partners matter for our analysis, so we often write only acceptable partners to denote preferences. For example,

$$\succ_d: h, h'$$

means that hospital h is the most preferred, h' is the second most preferred, and h and h' are the only acceptable hospitals under preferences $\succ_d$ of doctor d.

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[15]This usage of such information will be expressed in the definition of "illegitimate" doctor–hospital pairs.

[16]In real medical matching, a hospital may have multiple residency programs. These programs may differ from one another in terms of emphasis on specialties, for example. In such a case, the term "a hospital" should be understood to mean a residency program.

[17]We denote a singleton set $\{x\}$ by x when there is no confusion.

Each hospital $h \in H$ is endowed with a (physical) *capacity* q_h, which is a nonnegative integer. We say that preference relation $\succ_h$ is *responsive with capacity q_h* (Roth 1985) if the following conditions hold:

(i) For any $D' \subseteq D$ with $|D'| \leq q_h$, $d \in D \setminus D'$ and $d' \in D'$, $(D' \cup d) \setminus d' \succeq_h D'$ if and only if $d \succeq_h d'$.

(ii) For any $D' \subseteq D$ with $|D'| \leq q_h$ and $d' \in D'$, $D' \succeq_h D' \setminus d'$ if and only if $d' \succeq_h \varnothing$.

(iii) For any $D' \subseteq D$ with $|D'| > q_h$, $\varnothing \succ_h D'$.

In words, preference relation $\succ_h$ is responsive with a capacity if the ranking of a doctor (or keeping a position vacant) is independent of her colleagues, and any set of doctors exceeding its capacity is unacceptable. We assume that preferences of each hospital h are responsive with capacity q_h throughout the paper.

A *matching* μ is a mapping that satisfies (i) $\mu_d \in H \cup \{\varnothing\}$ for all $d \in D$, (ii) $\mu_h \subseteq D$ for all $h \in H$, and (iii) for any $d \in D$ and $h \in H$, $\mu_d = h$ if and only if $d \in \mu_h$. That is, a matching simply specifies which doctor is assigned to which hospital (if any).

A matching μ is *individually rational* if (i) for each $d \in D$, $\mu_d \succeq_d \varnothing$, and (ii) for each $h \in H$, $d \succeq_h \varnothing$ for all $d \in \mu_h$, and $|\mu_h| \leq q_h$. That is, no agent is matched with an unacceptable partner and each hospital's capacity is respected.

Given matching μ, a pair (d, h) of a doctor and a hospital is called a *blocking pair* if $h \succ_d \mu_d$ and either (i) $|\mu_h| < q_h$ and $d \succ_h \varnothing$, or (ii) $d \succ_h d'$ for some $d' \in \mu_h$. In words, a blocking pair is a pair of a doctor and a hospital who want to be matched with each other (possibly rejecting their partners in the prescribed matching) rather than following the proposed matching.

2.2 *Model with constraints*

Region structure   A collection $R \subseteq 2^H \setminus \{\varnothing\}$ is called a *set of regions*. Assume $\{h\} \in R$ for all $h \in H$ and $H \in R$.

A collection of regions $S \subset R$ is called a *partition of $r \in R$* if $S \neq \{r\}$, $\bigcup_{r' \in S} r' = r$, and $r_1 \cap r_2 = \varnothing$ for all $r_1, r_2 \in S$ with $r_1 \neq r_2$. A partition S of r is called a *largest partition of r* if there exists no partition $S' \neq S$ of r such that $r' \in S$ implies $r' \subseteq r''$ for some $r'' \in S'$. Note that, for a given region r, there can be more than one largest partition of r. We denote by $\mathcal{LP}(r)$ the collection of largest partitions of r. For $r \in R$ and $S \in \mathcal{LP}(r)$, we refer to each element of S as a *subregion of r with respect to S*.

A set of regions R is a *hierarchy* if $r, r' \in R$ implies $r \subseteq r'$ or $r' \subseteq r$ or $r \cap r' = \varnothing$.

Below is an example of a set of regions.

EXAMPLE 1. There are hospitals h_1, h_2, and h_3. The regions are

$$R = \{H, r_1, r_2, \{h_1\}, \{h_2\}, \{h_3\}\},$$

where $r_1 = \{h_1, h_2\}$ and $r_2 = \{h_2, h_3\}$. See Figure 1(a) for a graphical representation. In this example, the largest partitions of H are $S = \{r_1, \{h_3\}\}$ and $S' = \{\{h_1\}, r_2\}$. Regions r_1 and $\{h_3\}$ are subregions of H with respect to S, and $\{h_1\}$ and r_2 are subregions of H
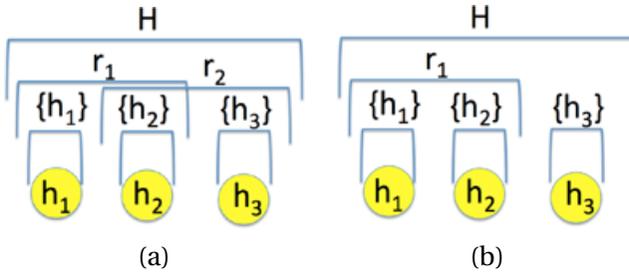
FIGURE 1. An example of sets of regions in Example 1. (a) A non-hierarchical set of regions R. (b) A hierarchical set of regions R'.

with respect to S'. The set of regions R is not a hierarchy because $r_1 \not\subseteq r_2$, $r_2 \not\subseteq r_1$, and $r_1 \cap r_2 = \{h_2\} \neq \varnothing$. By contrast, $R' := R \setminus \{r_2\}$ is a hierarchy (see Figure 1(b) for a graphical representation).   ◇

Regional preferences   When a given region is faced with applications by more doctors than the regional cap, the region has to allocate limited seats among its subregions. We consider the situation in which regions are endowed with policy goals in terms of doctor allocations, and formalize such policy goals using the concept of "regional preferences."[18] For example, a situation in which the government has a policy goal regarding doctor allocations across different geographic regions can be captured by the regional preferences of the grand region H. The regional preferences of a geographical region r whose subregions are all singleton sets may capture r's policy goal in terms of doctor allocations across different hospitals in r.

For each $r \in R$ that is not a singleton set and $S \in \mathcal{LP}(r)$, a regional preference for r, denoted $\rhd_{r,S}$, is a weak ordering over $W_{r,S} := \{w = (w_{r'})_{r' \in S} | w_{r'} \in \mathbb{Z}_+ \text{ for every } r' \in S\}$. That is, $\rhd_{r,S}$ is a binary relation that is complete and transitive (but not necessarily antisymmetric). We write $w \rhd_{r,S} w'$ if and only if $w \unrhd_{r,S} w'$ holds but $w' \unrhd_{r,S} w$ does not. Vectors such as w and w' are interpreted to be supplies of acceptable doctors to the subregions of region r, but they only specify how many acceptable doctors apply to hospitals in each subregion and no information is given as to who these doctors are. We denote by $\unrhd$ the profile $(\unrhd_{r,S})_{r \in R, S \in \mathcal{LP}(r)}$.

Given $\unrhd_{r,S}$, a function

$$\tilde{\text{Ch}}_{r,S} : W_{r,S} \times \mathbb{Z}_+ \to W_{r,S}$$

is an *associated quasi-choice rule* if $\tilde{\text{Ch}}_{r,S}(w; t) \in \arg\max_{\unrhd_{r,S}}\{w' | w' \leq w, \sum_{r' \in S} w'_{r'} \leq t\}$ for any nonnegative integer vector $w = (w_{r'})_{r' \in S}$ and nonnegative integer t.[19] Intuitively,

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[18]One interpretation is that each region has real preferences over doctor distributions. An alternative interpretation is that the regional preferences are not real preferences possessed by the region, but simply a rationing criterion imposed by the central government, for instance. The latter interpretation is analogous to that of "priority orders" in school choice (Abdulkadiroğlu and Sönmez 2003), which are not real preferences possessed by individual schools, but a criterion given by the school district.

[19]For any two vectors $w = (w_{r'})_{r' \in S}$ and $w' = (w'_{r'})_{r' \in S}$, we write $w \leq w'$ if and only if $w_{r'} \leq w'_{r'}$ for all $r' \in S$. We write $w \lneq w'$ if and only if $w \leq w'$ and $w_{r'} < w'_{r'}$ for at least one $r' \in S$. For any $W'_{r,S} \subseteq W_{r,S}$, $\arg\max_{\unrhd_{r,S}} W'_{r,S}$ is the set of vectors $w \in W'_{r,S}$ such that $w \unrhd_{r,S} w'$ for all $w' \in W'_{r,S}$.

$\tilde{Ch}_{r,S}(w, t)$ is a best vector of numbers of doctors allocated to subregions of r given a vector of numbers w under the constraint that the sum of the numbers of doctors cannot exceed the quota t. Note that there may be more than one quasi-choice rule associated with a given weak ordering $\unrhd_{r,S}$ because the set $\arg\max_{\unrhd_{r,S}}\{w'|w' \leq w, \sum_{r' \in S} w'_{r'} \leq t\}$ may not be a singleton for some w and t.

We assume that the regional preferences $\unrhd_{r,S}$ satisfy $w \rhd_{r,S} w'$ if $w' \lneq w$. This condition formalizes the idea that region r prefers to fill as many positions in its subregions as possible. This requirement implies that any associated quasi-choice rule is *acceptant* in the sense that, for each w and t, if there exists $r' \in S$ such that $[\tilde{Ch}_{r,S}(w; t)]_{r'} < w_{r'}$, then $\sum_{r'' \in S}[\tilde{Ch}_{r,S}(w; t)]_{r''} = t$.[20] This captures the idea that the social planner should not waste caps allocated to the region: If some doctor is rejected by a hospital even though she is acceptable to the hospital and the hospital's capacity is not binding, then the regional cap should be binding.

We say that $\unrhd_{r,S}$ is *substitutable* if there exists an associated quasi-choice rule $\tilde{Ch}_{r,S}$ that satisfies

$$w \leq w' \quad \text{and} \quad t \geq t' \quad \Rightarrow \quad \tilde{Ch}_{r,S}(w; t) \geq \tilde{Ch}_{r,S}(w'; t') \wedge w.$$

Throughout our analysis, we assume that $\unrhd_{r,S}$ is substitutable for any $r \in R$ and $S \in \mathcal{LP}(r)$. To understand this condition, notice that this condition can be decomposed into two parts, as follows:

$$w \leq w' \quad \Rightarrow \quad \tilde{Ch}_{r,S}(w; t) \geq \tilde{Ch}_{r,S}(w'; t) \wedge w, \quad \text{and} \tag{1}$$

$$t \geq t' \quad \Rightarrow \quad \tilde{Ch}_{r,S}(w; t) \geq \tilde{Ch}_{r,S}(w; t'). \tag{2}$$

Condition (1) imposes a condition on the quasi-choice rule for different vectors w and w' with a fixed parameter t, while condition (2) places restrictions for different parameters t and t' with a fixed vector w. The former condition requires that, given cap t, when the supply of doctors increases, the number of accepted doctors at a hospital can increase only when the hospital has accepted all acceptable doctors under the original supply profile. This condition is similar to the standard substitutability condition (Roth and Sotomayor 1990, Hatfield and Milgrom 2005) except that (i) it is defined over vectors that only specify how many doctors apply to hospitals in the region, so it does not distinguish different doctors, and (ii) it deals with multi-unit supplies (that is, coefficients in w can take integers different from 0 or 1).[21] Appendix B formally establishes an equivalence between condition (1) and familiar conditions from matching with contracts.

Condition (2) requires that the choice increases (in the standard vector sense) if the allocated quota is increased. One may interpret this condition as a "normal goods" assumption whereby an increase in the "budget" (the allocated quota) leads to a weak increase in demand for each "good" (the weight corresponding to each subregion).[22]

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[20]This condition is a variant of properties used by Alkan (2001) and Kojima and Manea (2010) in the context of choice functions over matchings.

[21]Condition (1) is also similar to *persistence* by Alkan and Gale (2003), who define the condition on a choice function in a slightly different context.

[22]We are grateful to an anonymous referee for suggesting this interpretation of condition (2).

Stability   We assume that each region $r \in R$ is endowed with a nonnegative integer κ_r called a *regional cap*.[23] We denote by $\kappa = (\kappa_r)_{r \in R}$ the profile of regional caps across all regions in R. A matching is *feasible* if $|\mu_r| \leq \kappa_r$ for all $r \in R$, where $\mu_r = \bigcup_{h \in r} \mu_h$. In other words, feasibility requires that the regional cap for every region is satisfied. For $R' \subseteq R$, we say that μ is *Pareto superior* to μ' for R' if $(|\mu_{r'}|)_{r' \in S} \trianglerighteq_{r,S} (|\mu'_{r'}|)_{r' \in S}$ for all (r, S), where $r \in R'$ and $S \in \mathcal{LP}(r)$, with at least one of the relations holding strictly. Given a matching μ, denote by $\mu^{d \to h}$ the matching such that $\mu^{d \to h}_{d'} = \mu_{d'}$ for all $d' \in D \setminus \{d\}$ and $\mu^{d \to h}_d = h$.

Given these concepts, let us now introduce two key notions to define stability. First, we say that a pair (d, h) is *infeasible* at μ if $\mu^{d \to h}$ is not feasible. Second, we say that a pair (d, h) is *illegitimate* at μ if there exists $r \in R$ with $\mu_d, h \in r$ and $|\mu_r| = \kappa_r$ such that $\mu^{d \to h}$ is not Pareto superior to μ for $\{r' \in R | \mu_d, h \in r'$ and $r' \subseteq r\}$.

DEFINITION 1. A matching μ is *stable* if it is feasible, individually rational, and if (d, h) is a blocking pair then $d' \succ_h d$ for all doctors $d' \in \mu_h$ and (d, h) is either infeasible or illegitimate at μ.

The standard definition of stability without regional caps requires individual rationality and the absence of blocking pairs. With regional caps, however, there are cases in which every feasible and individually rational matching admits a blocking pair. For this reason, we allow for the presence of some blocking pairs. To keep the spirit of stability, however, we require that only certain kinds of blocking pairs remain. Specifically, we demand that all remaining blocking pairs be either infeasible or illegitimate. Below we provide justification for the choice of these restrictions.

A pair of a doctor d and a hospital h is infeasible if moving d to h while keeping other parts of the matching unchanged leads to a violation of a regional cap. To the extent that regional caps encode what matchings are allowed in the given situation, a demand by a blocking pair that would cause a violation of a regional cap does not have the same normative support as in the case without regional caps. For this reason, our stability concept allows for infeasible blocking pairs to remain.

A doctor–hospital pair is illegitimate if the movement of doctor d to h does not lead to a Pareto superior distribution of doctors for a certain set of regions. We require any region r' in this set to satisfy two conditions. First, we require that r' contains both hospitals μ_d (the original hospital for d) and h, as this corresponds to the case in which r' is in charge of controlling distributions of doctors involving these hospitals. Second, the region r' should be currently "constrained." That is, it is a subset of some region r whose regional cap is full in the present matching: In such a case, the region r should ration the distribution of doctors among its subregions, each of which needs to ration the distribution among its subregions, and so forth, which indirectly constrains the number of doctors that can be matched in r'. The requirement that r' is (indirectly) constrained limits the case in which a blocking pair is declared illegitimate. This is in line with our

motivation to keep the spirit of stability, i.e., to allow for the presence of blocking pairs only in a conservative manner.[24]

The implicit idea behind the definition is that the government or some authority can interfere and prohibit a blocking pair from being executed if regional caps are an issue. Thus, our preferred interpretation is that stability captures a normative notion that it is desirable to implement a matching that respects participants' preferences to the extent possible. Justification of the normative appeal of stability is established in the recent matching literature, and Kamada and Kojima (2015a) offer further discussion on this point, so we refer interested readers to that paper for details.

REMARK 1. Kamada and Kojima (2017) define two other stability concepts, which they call strong stability and weak stability. Strong stability is more demanding than stability, requiring that any blocking pairs lead to infeasibility. Weak stability is less demanding than stability, allowing for some blocking pairs that are neither infeasible nor illegitimate. Kamada and Kojima (2017) establish that a matching is strongly stable if and only if it is stable for all possible regional preference profiles, while a matching is weakly stable if and only if there exists a regional preference profile under which it is stable.

REMARK 2. Kamada and Kojima (2017) show that any weakly stable matching is (constrained) efficient, i.e., there is no feasible matching μ' such that $\mu'_i \succeq_i \mu_i$ for all $i \in D \cup H$ and $\mu'_i \succ_i \mu_i$ for some $i \in D \cup H$.[25] Because stability implies weak stability, a stable matching is efficient for any regional preferences, which provides one normative appeal of our stability concept.

REMARK 3. Kamada and Kojima (2017) demonstrate that (i) there does not necessarily exist a strongly stable matching, and (ii) if a mechanism produces a strongly stable matching whenever one exists, then it is not strategy-proof for doctors. Given these negative findings, the present paper focuses on stability as defined in Definition 1.

REMARK 4. Practically, moving a doctor from one hospital to another involves administrative tasks on the part of relevant regions (we give examples of possible organizations dealing with such administration in the Introduction); hence disallowing only those blocking pairs that Pareto-improve the relevant regions is, in our view, the most plausible notion in our environment. An alternative notion of illegitimacy may be that a doctor–hospital pair is said to be illegitimate if moving doctor d to h leads to a Pareto inferior distribution of doctors for the set of regions that we consider here. This notion not only is inconsistent with our view just described, but also leads to nonexistence.[26]

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[24]For the case of hierarchies, we provide an alternative interpretation of stability in Remark 6 in the Appendix.

[25]Since regional caps are a primitive of the environment, we consider a *constrained* efficiency concept.

[26]To see this, consider a two-doctor two-hospital example with one region with cap 1, where doctor $i \in \{1, 2\}$ likes hospital i best and hospital $j \neq i$ second, and hospital i likes doctor $j \neq i$ best and doctor i second. If the region containing hospitals 1 and 2 is indifferent between allocations $(1, 0)$ and $(0, 1)$, then there exists no matching that satisfies the strengthened version of stability. A similar logic is used to show that there exists no strongly stable matching in Kamada and Kojima (2017).

Mechanism   Recall that κ denotes the profile of regional caps and that $\rhd$ denotes the profile of regional preferences. A *mechanism* φ is a function that maps preference profiles to matchings for a given profile $(\kappa, \rhd)$. The matching under φ at preference profile $\succ$ is denoted $\varphi^{\kappa, \rhd}(\succ)$, and agent i's match is denoted by $\varphi_i^{\kappa, \rhd}(\succ)$ for each $i \in D \cup H$.

A mechanism φ is said to be *stable* if, for each $(\kappa, \rhd)$ and a preference profile $\succ$, the matching $\varphi^{\kappa, \rhd}(\succ)$ is stable.

A mechanism φ is said to be *strategy-proof for doctors* if there do not exist $(\kappa, \rhd)$, a preference profile $\succ$, a doctor $d \in D$, and preferences $\succ'_d$ of doctor d such that

$$\varphi_d^{\kappa, \rhd}(\succ'_d, \succ_{-d}) \succ_d \varphi_d^{\kappa, \rhd}(\succ).$$

That is, no doctor has an incentive to misreport her preferences under the mechanism.[27]

3. Results

This section presents our analysis of the model. We start by analyzing the case of hierarchy (Section 3.1), and then proceed to the analysis of the case of non-hierarchy (Section 3.2). We conclude the section by stating a necessary and sufficient condition that characterizes the set of constraint structures admitting the existence of a desirable mechanism (Section 3.3).

3.1 *Hierarchy*

THEOREM 1. *Fix D, H, and a set of regions R. If R is a hierarchy, then there exists a mechanism that is stable and strategy-proof for doctors.*

PROOF SKETCH. Our proof is constructive. In particular, we introduce a new algorithm, called the flexible deferred acceptance algorithm. That algorithm is a generalization of the deferred acceptance algorithm, in which the acceptance by hospitals in each step is made in a coordinated way that is consistent with the regional constraints.[28] Our proof strategy is to show that this mechanism has the desired properties.

The main idea for establishing those properties is to connect our matching model with constraints to the "matching with contracts" model (Hatfield and Milgrom 2005). More specifically, given the original matching model under constraints, we define an "associated model," which is a hypothetical matching model between doctors and the "hospital side" instead of doctors and hospitals. In the associated model, we regard the hospital side as a hypothetical consortium of all hospitals that acts as one agent, and imagine that the hospital side (hospital consortium) makes a coordinated employment decision. This association enables us to account for the fact that acceptance of a doctor by a hospital in one region may depend on doctor applications to other hospitals in the

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[27]Roth (1982) shows that there is no mechanism that produces a stable matching for all possible preference profiles and is strategy-proof for both doctors and hospitals even in a market without regional caps.

[28]The flexible deferred acceptance is also a generalization of the algorithm under the same name in Kamada and Kojima (2015a), where it is defined only for partitional constraints and a specific type of regional preferences.

same region, an inevitable feature in markets under distributional constraints. It necessitates, however, that we distinguish a doctor's placements in different hospitals. We account for this complication by defining the hospital side's choice function over *contracts* rather than doctors, where a contract specifies a doctor–hospital pair to be matched. We construct such a choice function by using two pieces of information: the preferences of all the hospitals and regional preferences. The idea is that each hospital's preferences are used for choosing doctors *given the number of allocated slots*, while regional preferences are used to regulate slots allocated to different hospitals. In other words, regional preferences trade off multiple hospitals' desires to accept more doctors, when accepting more is in conflict with the regional cap. With the help of this association, we demonstrate that any stable allocation in the associated model with contracts induces a stable matching in the original model with distributional constraints (Proposition 2).

So as to use this association, we show that the key conditions in the associated model—the substitutes condition and the law of aggregate demand—are satisfied (Proposition 1).[29] This enables us to invoke existing results for matching with contracts, namely that an existing algorithm called the cumulative offer process finds a stable allocation, and it is (group) strategy-proof for doctors in the associated model (Hatfield and Milgrom 2005, Hatfield and Kojima 2009, Hatfield and Kominers 2012).

The full proof is given in the Appendix, and it formalizes this idea by introducing the flexible deferred acceptance algorithm. We establish that, with the hierarchical region structure, the outcome of the cumulative offer process in the associated model corresponds to the matching produced by the flexible deferred acceptance algorithm in the original model (Remark 7). This correspondence establishes that the flexible deferred acceptance algorithm finds a stable matching in the original problem and this algorithm is (group) strategy-proof for doctors, proving Theorem 1. For illustration, our proof approach is represented as a chart in Figure 2.   □

3.2 *Non-Hierarchy*

THEOREM 2. *Fix D, H, and a set of regions R. If R is not a hierarchy, then there exists no mechanism that is stable and strategy-proof for doctors.*

PROOF SKETCH. Proving the theorem in the general environment is somewhat involved and is given in the Appendix. Here we illustrate the main idea by an example. Consider the problem described in Example 1. In that example, there are three hospitals, h_1, h_2, and h_3. The regions are $R = \{H, r_1, r_2, \{h_1\}, \{h_2\}, \{h_3\}\}$, where $r_1 = \{h_1, h_2\}$ and $r_2 = \{h_2, h_3\}$. Recall Figure 1(a) for a graphical representation. Note that $h_1 \in r_1 \setminus r_2$, $h_2 \in r_1 \cap r_2$, and $h_3 \in r_2 \setminus r_1$, so R is not a hierarchy. We will show that there does not exist a mechanism that is stable and strategy-proof for doctors.

Clearly, $S_1 := \{\{h_1\}, \{h_2\}\}$ is the unique largest partition of r_1. Similarly, $S_2 := \{\{h_2\}, \{h_3\}\}$ is the unique largest partition of r_2. Suppose that, under $\triangleright_{r_1, S_1}$, region r_1

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[29]Substitutability of regional preferences plays a crucial role in the proof of Proposition 1, and we regard it as reasonable in our applications. Substitutability is commonly assumed in various domains in the matching literature.
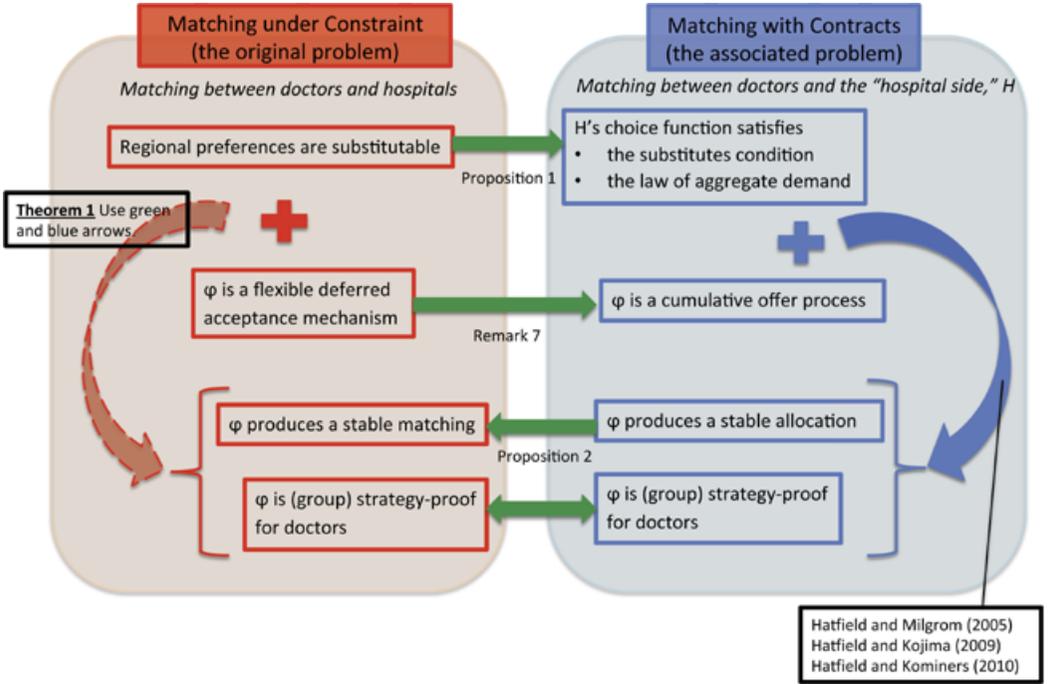
FIGURE 2. Proof sketch for Theorem 1.

prefers a vector such that the coordinate corresponding to $\{h_1\}$ is 1 and the other coordinate is 0 to a vector such that the coordinate corresponding to $\{h_2\}$ is 1 and the other coordinate is 0. Also suppose that, under $\rhd_{r_2, S_2}$, region r_2 prefers a vector such that the coordinate corresponding to $\{h_2\}$ is 1 and the other coordinate is 0 to a vector such that the coordinate corresponding to $\{h_3\}$ is 1 and the other coordinate is 0. Finally, let $\kappa_{r_1} = \kappa_{r_2} = 1$ and, for each $\tilde{r} \in R \setminus \{r_1, r_2\}$, $\kappa_{\tilde{r}}$ is sufficiently large so that it never binds.[30]

Suppose that there are two doctors, d_1 and d_2. Finally, assume that preferences of doctors and hospitals are

$$\succ_{d_1}: h_3, \qquad \succ_{d_2}: h_2, h_1,$$

$$\succ_{h_1}: d_2, d_1, \qquad \succ_{h_2}: d_1, d_2, \qquad \succ_{h_3}: d_2, d_1,$$

and the capacity of each hospital is sufficiently large so that it never binds.[31]

By inspection, it is straightforward to see that μ and μ' defined by

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 & \varnothing \\ \varnothing & d_2 & \varnothing & d_1 \end{pmatrix} \quad \text{and} \quad \mu' = \begin{pmatrix} h_1 & h_2 & h_3 \\ d_2 & \varnothing & d_1 \end{pmatrix}.$$

are the only stable matchings given the above preferences.

We consider two cases.

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[30] For example, let $\kappa_{\tilde{r}} = 3$ for each $\tilde{r} \in R \setminus \{r_1, r_2\}$.
[31] For instance, let $q_{h_1} = q_{h_2} = q_{h_3} = 3$.

Case 1. Suppose that a mechanism produces μ given the above preference profile. Consider d_1's preferences

$$\succ'_{d_1}: h_1, h_2, h_3.$$

Under the preference profile $(\succ'_{d_1}, \succ_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, it is straightforward to check that μ' is a unique stable matching. Note that d_1 is matched to h_3 under this new preference profile, which is strictly better under $\succ_{d_1}$ than d_1's match $\varnothing$ under the original preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1} \succ_{h_2}, \succ_{h_3})$. This implies that if a mechanism is stable and produces μ given preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, then it is not strategy-proof for doctors.

Case 2. Suppose that a mechanism produces μ' given the above preference profile. Consider d_2's preferences

$$\succ'_{d_2}: h_2, h_3, h_1.$$

Under the preference profile $(\succ_{d_1}, \succ'_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, it is straightforward to check that μ is a unique stable matching. Note that d_2 is matched to h_2 under this new preference profile, which is strictly better under $\succ_{d_2}$ than d_2's match h_1 under the original preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1} \succ_{h_2}, \succ_{h_3})$. This implies that if a mechanism is stable and produces μ' given preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, then it is not strategy-proof for doctors.   □

3.3 *The necessary and sufficient condition*

Theorems 1 and 2 imply the following corollary, which characterizes the set of constraint structures that admit the existence of a desirable mechanism.

COROLLARY 1. *Fix D, H, and a set of regions R. The following statements are equivalent.*

  (i) *R is a hierarchy.*

 (ii) *There exists a mechanism that is stable and strategy-proof for doctors.*

This corollary identifies the conditions on the markets for which we can find a mechanism that is stable and strategy-proof for doctors. Since our proof for the assertion that statement (i) implies statement (ii) is constructive (Theorem 1), for markets in which constraints are a hierarchy, we can directly use the mechanism we construct. Also, for markets in which constraints do not form a hierarchy, the corollary shows that there is no hope for adopting a mechanism that is stable and strategy-proof for doctors. These points are illustrated by the practical applications in the Introduction.

REMARK 5. In some applications, it may be desirable to require that there be no blocking pair (d, h) such that moving d to h while displacing some d' with $\mu_{d'} \neq h$ improves the doctor distribution. In this remark, let us consider a stronger notion than stability that additionally requires such a property.

It turns out that the same characterization result holds when we replace stability with the above stronger notion. More precisely, the set of regions is a hierarchy if

and only if there exists a mechanism that satisfies the stronger notion of stability and strategy-proofness for doctors.[32]

To see this, note first that necessity of a hierarchy is obvious from Corollary 1 because the stability concept is stronger. To obtain the intuition for sufficiency, recall that our flexible deferred acceptance algorithm corresponds to the cumulative offer process in an associated model with contracts. The latter algorithm finds a stable allocation in the associated model. A stable allocation in the associated model implies a stable matching in the original model under constraints, but the converse does not hold. In particular, a stable allocation eliminates a blocking pair (d, h) such that the doctor distribution is improved by moving d to h while displacing some d' with $\mu_{d'} \neq h$, implying the stronger property.

4. Conclusion

This paper presented a model of matching under distributional constraints. We identified the necessary and sufficient condition on the constraint structure for the existence of a mechanism that is stable and strategy-proof for the individuals. The necessary and sufficient condition is that the constraints form a hierarchy.

The fact that our condition is both sufficient and necessary gives us a clear guide to future research. First, our sufficiency result implies that, in applications with hierarchical constraints, one can utilize our theory to design a desirable mechanism. We hope that this result will stimulate future works on specific applications. Second, our necessity result suggests that if the constraints do not form a hierarchy, one needs to weaken either strategy-proofness for doctors or the stability concept as desiderata for a mechanism to be designed.[33] Then the critical questions are how such weakening should be done, and what mechanism satisfies the weakened criteria.

Finally, it is worth noting the connection between matching with constraints and matching with contracts that we used in proving the sufficiency result. This technique was subsequently adopted by other studies such as Goto et al. (2014a, 2014b), and Kojima et al. (2017). We envision that this approach may prove useful for tackling complex matching problems one may encounter in the future.

Appendix

Appendix A defines the flexible deferred acceptance mechanism under the assumption that the set of regions forms a hierarchy. Appendix B establishes several properties of

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[32]The formal definition of the stronger stability concept and the statement of the characterization result are presented in Appendix E.1. We are grateful to an anonymous referee for encouraging us to discuss this concept.

[33]In addition to the applications discussed in the Introduction, the case with floor constraints is worth mentioning. If we translate those constraints to ceiling constraints like ours, the resulting problem does not generally admit a hierarchy. Contributions in the literature studying floor constraints seek solutions different from ours. Fragiadakis and Troyan (2017), for instance, find a mechanism that is strategy-proof for students and has desirable efficiency properties. We plan to further study floor constraints in future work.

substitutability that prove useful in subsequent analysis. The proofs of our main results are provided in Appendix C (Theorem 1) and Appendix D (Theorem 2). Appendix E provides additional discussions.

Appendix A: Flexible deferred acceptance mechanism

Throughout this section, suppose that R is a hierarchy. It is straightforward to see that, for any non-singleton region $r \in R$, $\mathcal{LP}(r)$ is a singleton set. Given this fact, denote by $S(r)$ the unique element of $\mathcal{LP}(r)$, and call each element of $S(r)$ a *subregion* of r. We use simplified notation $\unrhd_r$ for $\unrhd_{r,S(r)}$ and $\tilde{\mathrm{Ch}}_r$ for $\tilde{\mathrm{Ch}}_{r,S(r)}$. We say that $r \in R$ is a *smallest common region* of hospitals h and h' if $h, h' \in r$ and there is no $r' \in R$ with $r' \subsetneq r$ such that $h, h' \in r'$. For any h and h', it is straightforward to see that a smallest common region of h and h' exists and is unique. Given this fact, denote the smallest common region of h and h' by $SC(h, h')$.

We say that region r is of *depth* k if $|\{r' \in R | r \subseteq r'\}| = k$. Note that the depth of a "smaller" region is larger. The standard model without regional caps can be interpreted as a model with regions of depths less than or equal to 2 (H and singleton sets), and the model of Kamada and Kojima (2015a) has regions of depths less than or equal to 3 (H, "regions," and singleton sets), both with κ_H sufficiently large.

We proceed to define a quasi-choice rule for the "hospital side," denoted $\tilde{\mathrm{Ch}}$: Let $\tilde{\kappa}_H = \kappa_H$. Given $w = (w_h)_{h \in H}$, we define $v^w_{\{h\}} = \min\{w_h, q_h, \kappa_{\{h\}}\}$ and, for each non-singleton region r, inductively define $v^w_r = \min\{\sum_{r' \in S(r)} v^w_{r'}, \kappa_r\}$. Intuitively, v^w_r is the maximum number that the input w can allocate to its subregions given the feasibility constraints that w and regional caps of subregions of r impose. Note that v^w_r is weakly increasing in w, that is, $w \geq w'$ implies $v^w_r \geq v^{w'}_r$.

We inductively define $\tilde{\mathrm{Ch}}(w)$ following a procedure starting from Step 1, where Step k for general k is as follows.

Step k. If all the regions of depth k are singletons, then let $\tilde{\mathrm{Ch}}(w) = (\tilde{\kappa}^w_{\{h\}})_{h \in H}$ and stop the procedure. For each non-singleton region r of depth k, set $\tilde{\kappa}^w_{r'} = [\tilde{\mathrm{Ch}}_r((v^w_{r''})_{r'' \in S(r)}; \tilde{\kappa}^w_r)]_{r'}$ for each subregion r' of r. Go to Step $k+1$.

That is, under $\tilde{\mathrm{Ch}}(w)$, doctors are allocated to subregions of H, and then the doctors allocated to region r are further allocated to subregions of r, and so forth until the bottom of the hierarchy is reached. In doing so, the capacity constraint of each hospital and the feasibility constraint are taken into account. For example, if the capacity is 5 at hospital h, then no more than five doctors at h are allocated to the regions containing h.

Assume that $\unrhd_r$ is substitutable for every region r. Now we are ready to define the *flexible deferred acceptance algorithm*:

For each region r, fix an associated quasi-choice rule $\tilde{\mathrm{Ch}}_r$ for which the conditions for substitutability are satisfied (note that the assumption that $\unrhd_r$ is substitutable assures the existence of such a quasi-choice rule).

Step 1. Begin with an empty matching, that is, a matching μ such that $\mu_d = \varnothing$ for all $d \in D$.

Step 2. Arbitrarily choose a doctor d, who is currently not tentatively matched to any hospital and who has not yet applied to all acceptable hospitals. If such a doctor does not exist, then terminate the algorithm.

Step 3. Let d apply to the most preferred hospital $\bar{h}$ at $\succ_d$ among the hospitals that have not rejected d so far. If d is unacceptable to $\bar{h}$, then reject this doctor and go back to Step 2. Otherwise, define vector $w = (w_h)_{h \in H}$ by

   (a) $w_{\bar{h}}$ is the number of doctors currently held at $\bar{h}$ plus 1

   (b) w_h is the number of doctors currently held at h if $h \neq \bar{h}$.

Step 4. Each hospital $h \in H$ considers the new applicant d (if $h = \bar{h}$) *and* doctors who are temporarily held from the previous step together. It holds its $[\tilde{\text{Ch}}(w)]_h$ most preferred applicants among them temporarily and rejects the rest (so doctors held at this step may be rejected in later steps). Go back to Step 2.

We define the *flexible deferred acceptance mechanism* to be a mechanism that produces, for each input, the matching given at the termination of the above algorithm.[34]
This algorithm is a generalization of the deferred acceptance algorithm of Gale and Shapley (1962) to the model with regional caps. The main differences are found in Steps 3 and 4. Unlike the deferred acceptance algorithm, this algorithm limits the number of doctors (tentatively) matched in each region r at κ_r. This results in rationing of doctors across hospitals in the region, and the rationing rule is governed by regional preferences $\rhd_r$. Clearly, this mechanism coincides with the standard deferred acceptance algorithm if all the regional caps are large enough and hence non-binding.

Appendix B: Remarks on substitutability

The substitutability condition plays an important role in our proofs. This section presents three remarks on substitutability.
First, conditions (1) and (2) are independent of each other. One might suspect that these conditions are related to responsiveness of preferences, but these conditions do not imply responsiveness. In Appendix E.3, we provide examples to distinguish these conditions.
Second, condition (1) is equivalent to

$$w \leq w' \quad \Rightarrow \quad \left[\tilde{\text{Ch}}_{r,S}(w; t)\right]_{r'} \geq \min\left\{\left[\tilde{\text{Ch}}_{r,S}(w'; t)\right]_{r'}, w_{r'}\right\}$$
$$\text{for every } r' \in S.$$

(B.1)

This condition says that when the supply of doctors is increased, the number of accepted doctors at a hospital can increase only when the hospital has accepted all acceptable

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[34]Note that this algorithm terminates in a finite number of steps because each doctor makes an application to a particular hospital at most once. In Appendix C we show that the outcome of the algorithm is independent of the order in which doctors make their applications during the algorithm.

doctors under the original supply profile. Formally, condition (B.1) is equivalent to

$$w \le w' \quad \text{and} \quad \left[\tilde{\text{Ch}}_{r,S}(w; t)\right]_{r'} < \left[\tilde{\text{Ch}}_{r,S}(w'; t)\right]_{r'} \Rightarrow \left[\tilde{\text{Ch}}_{r,S}(w, t)\right]_{r'} = w_{r'}. \tag{B.2}$$

To see that condition (B.1) implies condition (B.2), suppose that $w \le w'$ and $[\tilde{\text{Ch}}_{r,S}(w;$ $t)]_{r'} < [\tilde{\text{Ch}}_{r,S}(w'; t)]_{r'}$. These assumptions and condition (B.1) imply $[\tilde{\text{Ch}}_{r,S}(w; t)]_{r'} \ge w_{r'}$. Since $[\tilde{\text{Ch}}_{r,S}(w; t)]_{r'} \le w_{r'}$ holds by the definition of $\tilde{\text{Ch}}_{r,S}$, this implies $[\tilde{\text{Ch}}_{r,S}(w; t)]_{r'} = w_{r'}$. To see that condition (B.2) implies condition (B.1), suppose that $w \le w'$. If $[\tilde{\text{Ch}}_{r,S}(w; t)]_{r'} \ge [\tilde{\text{Ch}}_{r,S}(w'; t)]_{r'}$, the conclusion of (B.1) is trivially satisfied. If $[\tilde{\text{Ch}}_{r,S}(w;$ $t)]_{r'} < [\tilde{\text{Ch}}_{r,S}(w'; t)]_{r'}$, then condition (B.2) implies $[\tilde{\text{Ch}}_{r,S}(w; t,)]_{r'} = w_{r'}$; thus the conclusion of (B.1) is satisfied.

Finally, we establish a relation between substitutability and conditions familiar from matching with contracts. First, a quasi-choice rule $\tilde{\text{Ch}}_{r,S}$ is said to be *consistent* if, for any t, $\tilde{\text{Ch}}_{r,S}(w; t) \le w' \le w \Rightarrow \tilde{\text{Ch}}_{r,S}(w'; t) = \tilde{\text{Ch}}_{r,S}(w; t)$. Consistency requires that if $\tilde{\text{Ch}}_{r,S}(w; t)$ is chosen at w and the supply decreases to $w' \le w$ but $\tilde{\text{Ch}}_{r,S}(w; t)$ is still available under w', then the same choice $\tilde{\text{Ch}}_{r,S}(w; t)$ should be made under w' as well. Note that there may be more than one consistent quasi-choice rule associated with a given weak ordering $\unrhd_{r,S}$ because the set $\arg\max_{\unrhd_{r,S}}\{w'|w' \le w, \sum_{r' \in S} w'_{r'} \le t\}$ may not be a singleton for some $\unrhd_{r,S}$, w, and t. Note also that there always exists a consistent quasi-choice rule associated with a given weak ordering $\unrhd_{r,S}$.[35]

Next, given a quasi-choice rule $\tilde{\text{Ch}}_{r,S}$, define the associated *quasi-rejection rule* $\tilde{\text{R}}_{r,S}$ by $\tilde{\text{R}}_{r,S}(w; t) := w - \tilde{\text{Ch}}_{r,S}(w; t)$ for every $w \in W_{r,S}$ and $t \in \mathbb{Z}_+$. We say $\unrhd_{r,S}$ is *Hatfield–Milgrom-substitutable* (*HM-substitutable*) if there exists an associated quasi-choice rule $\tilde{\text{Ch}}_{r,S}$ such that its associated quasi-rejection rule $\tilde{\text{R}}_{r,S}$ satisfies $\tilde{\text{R}}_{r,S}(w; t) \le \tilde{\text{R}}_{r,S}(w'; t)$ for every $w, w' \in W_{r,S}$ with $w \le w'$ and $t \in \mathbb{Z}_+$. HM-substitutability is analogous to the standard substitutes condition in matching with contracts due to Hatfield and Milgrom (2005) except that (i) it is defined over vectors that only specify how many doctors apply to hospitals in the region, so it does not necessarily distinguish identity of contracts, and (ii) it deals with multi-unit supplies (that is, coefficients in w can take integers different from 0 or 1).

The following claim establishes an exact sense in which our substitutability concept is related to consistency and HM-substitutability.

CLAIM 1. *The following statements are equivalent.*[36]

   (i) $\tilde{\text{Ch}}$ *satisfies condition (1).*

   (ii) $\tilde{\text{Ch}}$ *is consistent and HM-substitutable.*

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[35]To see this point, consider preferences $\unrhd'_{r,S}$ such that $w \rhd'_{r,S} w'$ if $w \rhd_{r,S} w'$ and $w = w'$ if $w \unrhd'_{r,S} w'$ and $w' \unrhd'_{r,S} w$. The quasi-choice rule that chooses (the unique element of) $\arg\max_{\unrhd'_{r,S}}\{w'|w' \le w, \sum_{r' \in S} w'_{r'} \le t\}$ for each w is clearly consistent.

[36]Fleiner (2003) and Aygün and Sönmez (2012) prove results analogous to our claim that condition (1) implies consistency although they do not work on substitutability defined over the space of integer vectors. Conditions that are analogous to our consistency concept are used by Blair (1988), Alkan (2002), and Alkan and Gale (2003) in different contexts.

PROOF. We first establish that condition (1) implies consistency. To do so, fix $\unrhd_{r,S}$ and its associated quasi-choice rule $\tilde{\text{Ch}}_{r,S}$, and suppose that for some t, $\tilde{\text{Ch}}_{r,S}(w';t) \le w \le w'$. Suppose also that condition (1) holds. We will prove $\tilde{\text{Ch}}_{r,S}(w;t) = \tilde{\text{Ch}}_{r,S}(w';t)$. Condition (1) implies $w \le w' \Rightarrow \tilde{\text{Ch}}_{r,S}(w;t) \ge \tilde{\text{Ch}}_{r,S}(w';t) \wedge w$. Since $\tilde{\text{Ch}}_{r,S}(w';t) \le w$ implies $\tilde{\text{Ch}}_{r,S}(w';t) \wedge w = \tilde{\text{Ch}}_{r,S}(w';t)$, this means that $\tilde{\text{Ch}}_{r,S}(w';t) \le \tilde{\text{Ch}}_{r,S}(w;t) \le w'$. If $\tilde{\text{Ch}}_{r,S}(w;t) \ne \tilde{\text{Ch}}_{r,S}(w';t)$, then by the assumption that $\tilde{\text{Ch}}_{r,S}$ is acceptant, we must have $\tilde{\text{Ch}}_{r,S}(w;t) \rhd_{r,S} \tilde{\text{Ch}}_{r,S}(w';t)$. But then $\tilde{\text{Ch}}_{r,S}(w';t)$ cannot be an element of $\arg\max_{\unrhd_{r,S}}\{w''|w'' \le w', \sum_{r'\in S} w''_{r'} \le t\}$ because $\tilde{\text{Ch}}_{r,S}(w;t) \in \{w''|w'' \le w', \sum_{r'\in S} w''_{r'} \le t\}$. Hence we have $\tilde{\text{Ch}}_{r,S}(w';t) = \tilde{\text{Ch}}_{r,S}(w;t)$.

Next, we establish that condition (1) implies HM-substitutability. For that purpose, suppose condition (1) holds and let $w \le w'$. If HM-substitutability is violated, then there exists r' such that $[\tilde{\text{R}}_{r,S}(w;t)]_{r'} > [\tilde{\text{R}}_{r,S}(w';t)]_{r'}$. Then it follows that $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < [\tilde{\text{Ch}}_{r,S}(w';t)]_{r'}$. Then condition (B.2) implies that $[\tilde{\text{Ch}}_{r,S}(w,t)]_{r'} = w_{r'}$ holds, so $0 = [\tilde{\text{R}}_{r,S}(w;t)]_{r'} \le [\tilde{\text{R}}_{r,S}(w';t)]_{r'}$, contradicting the earlier inequality $[\tilde{\text{R}}_{r,S}(w;t)]_{r'} > [\tilde{\text{R}}_{r,S}(w';t)]_{r'}$.

Last, we establish that consistency and HM-substitutability imply condition (1). To do so, suppose, to the contrary, that HM-substitutability holds and $w \le w'$ but $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < \min\{[\tilde{\text{Ch}}_{r,S}(w';t)]_{r'}, w_{r'}\}$ for some r'. Now define w'' by

$$w''_{r''} = \begin{cases} w'_{r'} & \text{if } r'' = r', \\ w_{r''} & \text{otherwise.} \end{cases}$$

LEMMA 1. *We have* $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} = [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'}$.

PROOF. First note that consistency implies $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} \le [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'}$. Now suppose, to the contrary, that $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'}$. Then define w''' by

$$w'''_{r''} = \begin{cases} [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'} & \text{if } r'' = r', \\ w''_{r''} & \text{otherwise.} \end{cases}$$

By consistency, $[\tilde{\text{Ch}}_{r,S}(w''';t)]_{r'} = [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'}$. In particular, $[\tilde{\text{Ch}}_{r,S}(w''';t)]_{r'} = w'''_{r'}$, so $[\tilde{\text{R}}_{r,S}(w''';t)]_{r'} = 0 < [\tilde{\text{R}}_{r,S}(w;t)]_{r'}$.[37] But this is a contradiction to $w \le w'''$ and HM.[38]   □

To finish the proof, recall $w' \ge w''$, so by HM-substitutability, $[\tilde{\text{R}}_{r,S}(w';t)]_{r'} \ge [\tilde{\text{R}}_{r,S}(w'';t)]_{r'}$. Recalling $w'_{r'} = w''_{r'}$ by definition, this implies $[\tilde{\text{Ch}}_{r,S}(w';t)]_{r'} \le [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'}$. This and Lemma 1 imply $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} \ge [\tilde{\text{Ch}}_{r,S}(w';t)]_{r'}$, which contradicts the assumption that $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < \min\{[\tilde{\text{Ch}}_{r,S}(w';t)]_{r'}, w_{r'}\}$.   □

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[37]The assumption $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < \min\{[\tilde{\text{Ch}}_{r,S}(w';t)]_{r'}, w_{r'}\}$ implies $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < w_{r'}$ and, thus, $[\tilde{\text{R}}_{r,S}(w;t)]_{r'} > 0$.

[38]Note that $w'''_{r'} \ge w_{r'}$. This is because otherwise $w'''_{r'} = [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'} < w_{r'}$, so by consistency, $\tilde{\text{Ch}}_{r,S}(w;t) = \tilde{\text{Ch}}_{r,S}(w'';t)$, contradicting the assumption $[\tilde{\text{Ch}}_{r,S}(w;t)]_{r'} < [\tilde{\text{Ch}}_{r,S}(w'';t)]_{r'}$.

APPENDIX C: PROOF OF THEOREM 1

With the definition of the flexible deferred acceptance mechanism, we are now ready to present the following statement.

> Suppose that R is a hierarchy and $\unrhd_r$ is substitutable for every $r \in R$. Then the flexible deferred acceptance mechanism produces a stable matching for any input and is group strategy-proof for doctors.[39]

This statement suffices to show Theorem 1. Therefore, the remainder of this section establishes the above statement.

It is useful to relate our model to a (many-to-many) matching model with contracts (Hatfield and Milgrom 2005). Let there be two types of agents: doctors in D and the hospital side (thus there are $|D|+1$ agents in total). Note that we regard the hospital side, instead of each hospital, as an agent in this model. There is a set of contracts $X = D \times H$.

We assume that, for each doctor d, any set of contracts with cardinality 2 or more is unacceptable; that is, a doctor can sign at most one contract. For each doctor d, her preferences $\succ_d$ over $(\{d\} \times H) \cup \{\varnothing\}$ are given as follows.[40] We assume that $(d, h) \succ_d (d, h')$ in this model if and only if $h \succ_d h'$ in the original model, and that $(d, h) \succ_d \varnothing$ in this model if and only if $h \succ_d \varnothing$ in the original model.

For the hospital side, we assume that it has preferences and its associated choice rule $\mathrm{Ch}(\cdot)$ over all subsets of $D \times H$. For any $X' \subset D \times H$, let $w(X') := (w_h(X'))_{h \in H}$ be the vector such that $w_h(X') = |\{(d, h) \in X' | d \succ_h \varnothing\}|$. For each X', the chosen set of contracts $\mathrm{Ch}(X')$ is defined by

$$\mathrm{Ch}(X') = \bigcup_{h \in H} \{(d, h) \in X' | |\{d' \in D | (d', h) \in X', d' \succeq_h d\}| \leq [\tilde{\mathrm{Ch}}(w(X'))]_h\}.$$

That is, each hospital $h \in H$ chooses its $[\tilde{\mathrm{Ch}}(w(X'))]_h$ most preferred contracts from acceptable contracts in X'.

DEFINITION 2 (Hatfield and Milgrom 2005). Choice rule $\mathrm{Ch}(\cdot)$ satisfies the *substitutes condition* if there do not exist contracts $x, x' \in X$ and a set of contracts $X' \subseteq X$ such that $x' \notin \mathrm{Ch}(X' \cup \{x'\})$ and $x' \in \mathrm{Ch}(X' \cup \{x, x'\})$.

In other words, contracts are substitutes if adding a contract to the choice set never induces a region to choose a contract it previously rejected. Hatfield and Milgrom (2005) show that there exists a stable allocation (defined in Definition 4) when contracts are substitutes for the hospital side.

DEFINITION 3 (Hatfield and Milgrom 2005). Choice rule $\mathrm{Ch}(\cdot)$ satisfies the *law of aggregate demand* if for all $X' \subseteq X'' \subseteq X$, $|\mathrm{Ch}(X')| \leq |\mathrm{Ch}(X'')|$.[41]

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[39]The definition of strategy-proofness for doctors is in Section 2. The definition of group strategy-proofness for doctors can be found in Appendix E.2.

[40]We abuse notation and use the same notation $\succ_d$ for preferences of doctor d both in the original model and in the associated model with contracts.

[41]Analogous conditions called cardinal monotonicity and size monotonicity are introduced by Alkan (2002) and Alkan and Gale (2003) for matching models without contracts.

PROPOSITION 1. *Suppose that $\unrhd_r$ is substitutable for all $r \in R$.*

(i) *Choice rule $\mathrm{Ch}(\cdot)$ defined above satisfies the substitutes condition.*[42]

(ii) *Choice rule $\mathrm{Ch}(\cdot)$ defined above satisfies the law of aggregate demand.*

PROOF. (i) Fix $X' \subset X$. Suppose to the contrary, i.e., that there exist X', (d, h) and (d', h') such that $(d', h') \notin \mathrm{Ch}(X' \cup \{(d', h')\})$ and $(d', h') \in \mathrm{Ch}(X' \cup \{(d, h), (d', h')\})$. This will lead to a contradiction.

Let $w' = w(X' \cup \{(d', h')\})$ and $w'' = w(X' \cup \{(d, h), (d', h')\})$. The proof consists of three steps.

Step 1. In this step we observe that $\tilde{\kappa}_{\{h'\}}^{w'} < \tilde{\kappa}_{\{h'\}}^{w''}$. To see this, note that otherwise we would have $\tilde{\kappa}_{\{h'\}}^{w'} \geq \tilde{\kappa}_{\{h'\}}^{w''}$; hence, by the definition of Ch we must have $[\mathrm{Ch}(X' \cup \{(d', h')\})]_{h'} \supseteq [\mathrm{Ch}(X' \cup \{(d, h), (d', h')\})]_{h'} \setminus \{(d, h)\}$. This contradicts $(d', h') \notin \mathrm{Ch}(X' \cup \{(d', h')\})$ and $(d', h') \in \mathrm{Ch}(X' \cup \{(d, h), (d', h')\})$.

Step 2. Consider any r such that $h' \in r$. Let $\tilde{\kappa}_r^{w'}$ and $\tilde{\kappa}_r^{w''}$ be as defined in the procedure to compute $\tilde{\mathrm{Ch}}(w')$ and $\tilde{\mathrm{Ch}}(w'')$, respectively. Let $r' \in S(r)$ be the subregion such that $h' \in r'$. Suppose $\tilde{\kappa}_{r'}^{w'} < \tilde{\kappa}_{r'}^{w''}$. We will show that $\tilde{\kappa}_r^{w'} < \tilde{\kappa}_r^{w''}$. To see this, suppose the contrary, i.e., that $\tilde{\kappa}_r^{w'} \geq \tilde{\kappa}_r^{w''}$. Let $v' := (v_{r''}^{w'})_{r'' \in S(r)}$ and $v'' := (v_{r''}^{w''})_{r'' \in S(r)}$. Since $w' \leq w''$ and $v_{r''}^{w}$ is weakly increasing in w for any region r'', it follows that $v' \leq v''$. This and substitutability of $\unrhd_r$ imply

$$\left[\tilde{\mathrm{Ch}}_r\left(v'; \tilde{\kappa}_r^{w'}\right)\right]_{r'} \geq \min\left\{\left[\tilde{\mathrm{Ch}}_r\left(v''; \tilde{\kappa}_r^{w''}\right)\right]_{r'}, v'_{r'}\right\}.$$

Since we assume $\tilde{\kappa}_{r'}^{w'} < \tilde{\kappa}_{r'}^{w''}$ or, equivalently,

$$\left[\tilde{\mathrm{Ch}}_r\left(v'; \tilde{\kappa}_r^{w'}\right)\right]_{r'} < \left[\tilde{\mathrm{Ch}}_r\left(v''; \tilde{\kappa}_r^{w''}\right)\right]_{r'},$$

this means $[\tilde{\mathrm{Ch}}_r(v'; \tilde{\kappa}_r^{w'})]_{r'} \geq v'_{r'}$. But then by $[\tilde{\mathrm{Ch}}_r(v'; \tilde{\kappa}_r^{w'})]_{r'} \leq v'_{r'}$ (from the definition of $\tilde{\mathrm{Ch}}$), we have $[\tilde{\mathrm{Ch}}_r(v'; \tilde{\kappa}_r^{w'})]_{r'} = v'_{r'}$. This contradicts the assumption that $(d', h') \notin \mathrm{Ch}(X' \cup \{(d', h')\})$, while d' is acceptable to h' (because $(d', h') \in \mathrm{Ch}(X' \cup \{(d, h), (d', h')\})$). Thus, we must have that $\tilde{\kappa}_r^{w'} < \tilde{\kappa}_r^{w''}$.

Step 3. Step 1 and an iterative use of Step 2 imply that $\tilde{\kappa}_H^{w'} < \tilde{\kappa}_H^{w''}$. But we specified $\tilde{\kappa}_H^{w}$ for any w to be equal to κ_H, so this is a contradiction.

(ii) To show that Ch satisfies the law of aggregate demand, let $X' \subseteq X$ and (d, h) be a contract such that $d \succ_h \varnothing$. We shall show that $|\mathrm{Ch}(X')| \leq |\mathrm{Ch}(X' \cup \{(d, h)\})|$. To show this, denote $w = w(X')$ and $w' = w(X' \cup \{(d, h)\})$. By definition of $w(\cdot)$, we have that $w'_h = w_h + 1$ and $w'_{h'} = w_{h'}$ for all $h' \neq h$. Consider the following cases.

Case 1. Suppose $\sum_{r' \in S(r)} v_{r'}^{w} \geq \kappa_r$ for some $r \in R$ such that $h \in r$. Then we have:

CLAIM 2. $v_{r'}^{w'} = v_{r'}^{w}$ *unless* $r' \subsetneq r$.

PROOF. Let r' be a region that does not satisfy $r' \subsetneq r$. First, note that if $r' \cap r = \varnothing$, then the conclusion holds by the definitions of $v_{r'}^w$ and $v_{r'}^{w'}$ because $w'_{h'} = w_{h'}$ for any $h' \notin r$. Second, consider r' such that $r \subseteq r'$ (since R is hierarchical, these cases exhaust all possibilities). Since $v_r^w = \min\{\sum_{r' \in S(r)} v_{r'}^w, \kappa_r\}$, the assumption $\sum_{r' \in S(r)} v_{r'}^w \geq \kappa_r$ implies $v_r(w) = \kappa_r$. By the same argument, we also obtain $v_r(w') = \kappa_r$. Thus, for any r' such that $r \subseteq r'$, we inductively obtain $v_{r'}^{w'} = v_{r'}^w$.   □

The relation $v_{r'}^{w'} = v_{r'}^w$ for all $r' \subsetneq r$ implies that, together with the construction of $\tilde{Ch}$,

$$[\tilde{Ch}(w')]_{h'} = [\tilde{Ch}(w)]_{h'} \quad \text{for any } h' \notin r. \tag{C.1}$$

To consider hospitals in r, first observe that r satisfies $\sum_{r' \in S(r)} v_{r'}^w \geq \kappa_r$ by assumption, so $v_r^w = \min\{\sum_{r' \in S(r)} v_{r'}^w, \kappa_r\} = \kappa_r$, and similarly $v_r^{w'} = \kappa_r$, so $v_r^w = v_r^{w'}$. Therefore, by construction of $\tilde{Ch}$, we also have $v_{r'}^w = v_{r'}^{w'}$ for any region r' such that $r \subseteq r'$. This implies $\tilde{\kappa}_r^w = \tilde{\kappa}_r^{w'}$, where $\tilde{\kappa}_r^w$ and $\tilde{\kappa}_r^{w'}$ are the assigned regional caps on r under weight vectors w and w', respectively, in the algorithm to construct $\tilde{Ch}$.

Now note that for any $r' \in R$, since $v_{r'}^w$ is defined as $\min\{\sum_{r'' \in S(r')} v_{r''}^w, \kappa_{r'}\}$ and all regional preferences are acceptant, the entire assigned regional cap $\tilde{\kappa}_{r'}^w$ is allocated to some subregion of r', that is, $\tilde{\kappa}_{r'}^w = \sum_{r'' \in S(r')} \tilde{\kappa}_{r''}^w$. Similarly we also have $\tilde{\kappa}_{r'}^{w'} = \sum_{r'' \in S(r')} \tilde{\kappa}_{r''}^{w'}$. This is the case not only for $r' = r$, but also for all subregions of r, their further subregions, and so forth. Going forward until this reasoning reaches the singleton sets, we obtain the relation

$$\sum_{h' \in r} [\tilde{Ch}(w')]_{h'} = \sum_{h' \in r} [\tilde{Ch}(w)]_{h'}. \tag{C.2}$$

By (C.1) and (C.2), we conclude that

$$|Ch(X')| = \sum_{h' \in H} [\tilde{Ch}(w)]_{h'} = \sum_{h' \in H} [\tilde{Ch}(w')]_{h'} = |Ch(X' \cup \{(d, h)\})|,$$

completing the proof for this case.

Case 2. Suppose $\sum_{r' \in S(r)} v_{r'}^w < \kappa_r$ for all $r \in R$ such that $h \in r$. Then the regional cap for r is not binding for any r such that $h \in r$, so we have

$$[\tilde{Ch}(w')]_h = [\tilde{Ch}(w)]_h + 1. \tag{C.3}$$

In addition, the following claim holds.

CLAIM 3. $[\tilde{Ch}(w')]_{h'} = [\tilde{Ch}(w)]_{h'}$, for all $h' \neq h$.

PROOF. First note that $v_r^{w'} = v_r^w + 1$ for all r such that $h \in r$ because the regional cap for r is not binding for any such r. Then consider the largest region H. By assumption, κ_H has not been reached under w, that is, $\sum_{r' \in S(H)} v_{r'}^w < \kappa_H$. Thus, since $\tilde{Ch}_H$ is acceptant, the entire vector $(v_{r'}(w))_{r' \in S(H)}$ is accepted by $\tilde{Ch}_H$, that

is, $\tilde{\kappa}^w_{r'} = v^w_{r'}$. Hence, for any $r' \in S(H)$ such that $h \notin r'$, both its assigned regional cap and all vs in their regions are identical under w and w', that is, $\tilde{\kappa}^w_{r'} = \tilde{\kappa}^{w'}_{r'}$ and $w'_{h'} = w_{h'}$ for all $h' \in r'$. So, for any hospital $h' \in r'$, the claim holds.

Now, consider $r \in S(H)$ such that $h \in r$. By the above argument, the assigned regional cap has increased by 1 in w' compared to w. But since r's regional cap κ_r has not been binding under w, all the v's in the subregions of r are accepted in both w and w'. This means that (i) for each subregion r' of r such that $h \notin r'$, it gets the same assigned regional cap and vs, so the conclusion of the claim holds for these regions, and (ii) for the subregion r' of r such that $h \in r'$, its assigned regional cap is increased by 1 in w' compared to w, and its regional cap $\kappa_{r'}$ has not been binding. And (ii) guarantees that we can follow the same argument inductively, so the conclusion holds for all $h \neq h'$. $\square$

By (C.3) and Claim 3, we obtain

$$\left| \mathrm{Ch}(X' \cup \{(d, h)\}) \right| = \sum_{h' \in H} \left[\tilde{\mathrm{Ch}}(w') \right]_{h'} = \sum_{h' \in H} \left[\tilde{\mathrm{Ch}}(w) \right]_{h'} + 1 = \left| \mathrm{Ch}(X') \right| + 1,$$

so we obtain $|\mathrm{Ch}(X' \cup \{(d, h)\})| > |\mathrm{Ch}(X')|$, completing the proof. $\square$

A subset X' of $X = D \times H$ is said to be *individually rational* if (i) for any $d \in D$, $|\{(d, h) \in X' | h \in H\}| \leq 1$, and if $(d, h) \in X'$, then $h \succ_d \varnothing$, and (ii) $\mathrm{Ch}(X') = X'$.

DEFINITION 4. A set of contracts $X' \subseteq X$ is a *stable allocation* if the following hold:

(i) It is individually rational.

(ii) There exists no hospital $h \in H$ and a doctor $d \in D$ such that $(d, h) \succ_d x$ and $(d, h) \in \mathrm{Ch}(X' \cup \{(d, h)\})$, where x is the contract that d receives at X' if any and is $\varnothing$ otherwise.

When condition (ii) is violated by some (d, h), we say that (d, h) is a *block* of X'.

Given any individually rational set of contracts X', define a *corresponding matching* $\mu(X')$ in the original model by setting $\mu_d(X') = h$ if and only if $(d, h) \in X'$ and $\mu_d(X') = \varnothing$ if and only if no contract associated with d is in X'. For any individually rational X', $\mu(X')$ is well defined because each doctor receives at most one contract at such X'.

PROPOSITION 2. *Suppose that $\unrhd_r$ is substitutable for all $r \in R$. If X' is a stable allocation in the associated model with contracts, then the corresponding matching $\mu(X')$ is a stable matching in the original model.*

PROOF. First, the following observation is straightforward.

OBSERVATION 1. *Suppose that R is a hierarchy. Then a matching μ is stable if and only if it is feasible, individually rational, and if (d, h) is a blocking pair then there exists $r \in R$ with $h \in r$ such that (i) $|\mu_r| = \kappa_r$, (ii) $d' \succ_h d$ for all doctors $d' \in \mu_h$, and (iii) either $\mu_d \notin r$*

or $(w_{r'})_{r' \in S(SC(h, \mu_d))} \unrhd_{SC(h, \mu_d)} (w'_{r'})_{r' \in S(SC(h, \mu_d))}$, *where* $w_{r'} = \sum_{h' \in r'} |\mu_{h'}|$ *for all* $r' \in S(r)$
and $w'_{r_h} = w_{r_h} + 1$, $w'_{r_d} = w_{r_d} - 1$, *and* $w'_{r'} = w_{r'}$ *for all other* $r' \in S(r)$, *where* r_h *and* r_d *are*
subregions of r *such that* $h \in r'_h$, *and* $\mu_d \in r_d$.

Suppose that X' is a stable allocation in the associated model with contracts and
denote $\mu := \mu(X')$. Individual rationality of μ is obvious from the construction of μ.
Suppose that (d, h) is a blocking pair of μ. By the above observation, it suffices to show
that there exists a region r that includes h such that the following conditions (C.4), (C.5),
and $\mu_d \notin r$ hold, or (C.4), (C.5), (C.6), and $\mu_d, h \in r$ hold:

$$|\mu_r| = \kappa_r, \tag{C.4}$$

$$d' \succ_h d \text{ for all } d' \in \mu_h, \tag{C.5}$$

$$(w_{r''})_{r'' \in S(SC(h, \mu_d))} \unrhd_{SC(h, \mu_d)} (w'_{r''})_{r'' \in S(SC(h, \mu_d))}, \tag{C.6}$$

where for any region r' we write $w_{r''} = \sum_{h' \in r''} |\mu_{h'}|$ for all $r'' \in S(r')$, and $w'_{r_h} = w_{r_h} + 1$,
$w'_{r_d} = w_{r_d} - 1$, and $w'_{r''} = w_{r''}$ for all other $r'' \in S(r')$, where $r_h, r_d \in S(r)$, $h \in r_h$, and $\mu_d \in r_d$.
Let $w = (w_h)_{h \in H}$.

For each region r that includes h, let $w''_{r'} = w_{r'} + 1$ for r' such that $h \in r'$ and $w''_{r''} = w_{r''}$
for all other $r'' \in S(r)$. Let $w'' = (w''_h)_{h \in H}$.

CLAIM 4. *Condition (C.5) holds, and there exists* r *that includes* h *such that condition*
(C.4) holds.

PROOF. First note that the assumption that $h \succ_d \mu_d$ implies that $(d, h) \succ_d x$, where x
denotes the (possibly empty) contract that d signs under X'.

(i) Assume, to the contrary, that condition (C.5) is violated, that is, $d \succ_h d'$ for some
   $d' \in \mu_h$. First note that $[\tilde{\mathrm{Ch}}(w'')]_h \geq [\tilde{\mathrm{Ch}}(w)]_h$. That is, weakly more contracts in-
   volving h are signed at $X' \cup (d, h)$ than at X'. This is because for any r and $r' \in S(r)$
   such that $h \in r'$,

$$\left[\tilde{\mathrm{Ch}}_r\left((v_{r''}^{w''})_{r'' \in S(r)}; \tilde{\kappa}_r\right)\right]_{r'} \geq \left[\tilde{\mathrm{Ch}}_r\left((v_{r''}^{w})_{r'' \in S(r)}; \tilde{\kappa}'_r\right)\right]_{r'} \quad \text{if } \tilde{\kappa}_r \geq \tilde{\kappa}'_r. \tag{C.7}$$

To see this, first note that $[\tilde{\mathrm{Ch}}_r((v_{r''}^w)_{r'' \in S(r)}; \tilde{\kappa}_r)]_{r'} \geq [\tilde{\mathrm{Ch}}_r((v_{r''}^w)_{r'' \in S(r)}; \tilde{\kappa}'_r)]_{r'}$ by sub-
stitutability of $\unrhd_r$. Also, by consistency of $\tilde{\mathrm{Ch}}_r$ and $v_{r''}^{w''} \geq v_{r''}^w$ for every region r'', the
inequality

$$\left[\tilde{\mathrm{Ch}}_r\left((v_{r''}^{w''})_{r'' \in S(r)}; \tilde{\kappa}_r\right)\right]_{r'} \geq \left[\tilde{\mathrm{Ch}}_r\left((v_{r''}^{w})_{r'' \in S(r)}; \tilde{\kappa}_r\right)\right]_{r'}$$

follows,[43] showing condition (C.7). An iterative use of condition (C.7) gives us the desired result that $[\tilde{Ch}(w'')]_h \geq [\tilde{Ch}(w)]_h$. This property, together with the assumptions that $d \succ_h d'$ and that $(d', h) \in X'$ imply that $(d, h) \in Ch(X' \cup (d, h))$.[44] Thus, together with the above-mentioned property that $(d, h) \succ_d x$, (d, h) is a block of X' in the associated model of matching with contracts, contradicting the assumption that X' is a stable allocation.

(ii) Assume, to the contrary, that condition (C.4) is violated, so that $|\mu_r| \neq \kappa_r$ for every r that includes h. Then, for such r, since $|\mu_r| \leq \kappa_r$ by the construction of μ and the assumption that X' is individually rational, it follows that $|\mu_r| < \kappa_r$. Then $(d, h) \in Ch(X' \cup (d, h))$ because of the following:

(a) $d \succ_h \varnothing$ by assumption,

(b) Since $\sum_{r' \in S(r)} w_{r'} = \sum_{h \in r} |\mu_h| = |\mu_r| < \kappa_r$, it follows that $\sum_{r' \in S(r)} w''_{r'} = \sum_{r' \in S(r)} w_{r'} + 1 \leq \kappa_r$. This property and the fact that $\tilde{Ch}_r$ is acceptant and the definition of the function $v_{r'}$ for regions r' imply that $\tilde{Ch}(w'') = w''$. In particular, this implies that every contract $(d', h) \in X' \cup (d, h)$ such that $d' \succ_h \varnothing$ is chosen at $Ch(X' \cup (d, h))$.

Thus, together with the above-mentioned property that $(d, h) \succ_d x$, (d, h) is a block of X' in the associated model of matching with contracts, contradicting the assumption that X' is a stable allocation.   □

To finish the proof of the proposition, suppose, to the contrary, that there is no r that includes h such that (C.4), (C.5), and $\mu_d \notin r$ hold, and that condition (C.6) fails. That is, we suppose $(w'_{r''})_{r'' \in S(SC(h,\mu_d))} \rhd_{SC(h,\mu_d)} (w_{r''})_{r'' \in S(SC(h,\mu_d))}$. Then it must be the case that $[\tilde{Ch}_r((v''_{r''})_{r'' \in S(SC(h,\mu_d))}; \tilde{\kappa}^{w''}_{SC(h,\mu_d)})]_{r'} = w''_{r'} = w_{r'} + 1 = |\mu_h| + 1$, where $h \in r'$ and $\tilde{\kappa}^{w''}_{SC(h,\mu_d)}$ is as defined in the procedure to compute $\tilde{Ch}(w'')$.[45] Note that for all r' such that $h \in r'$ and $r' \subsetneq SC(h, \mu_d)$, it follows that $\mu_d \notin r'$. Also note that (C.5) is satisfied by Claim 4. Therefore we have $|\mu_{r'}| < \kappa_{r'}$ for all $r' \subsetneq SC(h, \mu_d)$ that includes h by assumption and,

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[43]To show this claim, let $v = (v^w_{r''})_{r'' \in S(r)}$ and $v'' = (v^{w''}_{r''})_{r'' \in S(r)}$ for notational simplicity and assume, to the contrary, that $[\tilde{Ch}_r(v''; \tilde{\kappa}_r)]_{r'} < [\tilde{Ch}_r(v; \tilde{\kappa}_r)]_{r'}$. Then $[\tilde{Ch}_r(v''; \tilde{\kappa}_r)]_{r'} < [\tilde{Ch}_r(v; \tilde{\kappa}_r)]_{r'} \leq v_{r'}$. Moreover, since $v''_{r''} = v_{r''}$ for every $r'' \neq r'$ by the construction of v'', it follows that $[\tilde{Ch}_r(v'')]_{r''} \leq v''_{r''} = v_{r''}$. Combining these inequalities, we have that $\tilde{Ch}_r(v'') \leq v$. Also we have $v \leq v''$ by the definition of v'', so it follows that $\tilde{Ch}_r(v'') \leq v \leq v''$. Thus, by consistency of $\tilde{Ch}_r$, we obtain $\tilde{Ch}_r(v'') = \tilde{Ch}_r(v)$, a contradiction to the assumption $[\tilde{Ch}_r(v'')]_{r'} < [\tilde{Ch}_r(v)]_{r'}$.

[44]The proof of this claim is as follows: $Ch(X')$ induces hospital h to select its $[\tilde{Ch}(w)]_h$ most preferred contracts while $Ch(X' \cup (d, h))$ induces h to select a weakly larger number $[Ch(w'')]_h$ of its most preferred contracts. Since (d', h) is selected as one of the $[\tilde{Ch}(w)]_h$ most preferred contracts for h at X' and $d \succ_h d'$, we conclude that (d, h) must be one of the $[Ch(w'')]_h (\geq [\tilde{Ch}(w)]_h)$ most preferred contracts at $X' \cup (d, h)$ and, thus, selected at $X' \cup (d, h)$.

[45]To show this claim, assume, to the contrary, that $[\tilde{Ch}_{SC(h,\mu_d)}((v^{w''}_{r''})_{r'' \in S(SC(h,\mu_d))}; \tilde{\kappa}^{w''}_{SC(h,\mu_d)})]_{r'} \leq w_{r'}$, where $h \in r'$. Let $v := (v^w_{r''})_{r'' \in S(SC(h,\mu_d))}$ and $v'' := (v^{w''}_{r''})_{r'' \in S(SC(h,\mu_d))}$. Since $w''_{r''} = w_{r''}$ for any $r'' \neq r'$ by the definition of w'', it follows that

$$\tilde{Ch}_{SC(h,\mu_d)}(v''; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}) \leq (w_{r''})_{r'' \in SC(h,\mu_d)} \leq (w''_{r''})_{r'' \in SC(h,\mu_d)}.$$

hence, $|\mu_{r'}| + 1 \le \kappa_{r'}$ for all such r'. Moreover we have $d \succ_h \varnothing$ and, thus,

$$(d, h) \in \mathrm{Ch}(X' \cup (d, h)).$$

This relationship, together with the assumption that $h \succ_d \mu_d$ and, hence, $(d, h) \succ_d x$, is a contradiction to the assumption that X' is stable in the associated model with contracts.   □

REMARK 6. The definition of stability in this paper is based on Pareto improvement for multiple regions. For the case of hierarchies, we provide an alternative interpretation here. The idea of condition (iii) in Observation 1 is to invoke a region's preferences when a doctor moves within a region whose regional cap is binding (region r in the definition). However, when r is a strict superset of $SC(h, \mu_d)$, we do not invoke region r's regional preferences, but the preferences of $SC(h, \mu_d)$. The use of preferences of $SC(h, \mu_d)$ reflects the following idea: if the regional cap at r is binding, then holding fixed the number of doctors matched in r but not in $SC(h, \mu_d)$, there is essentially a binding cap for $SC(h, \mu_d)$. This motivates our use of the regional preferences of $SC(h, \mu_d)$. The reason for not using preferences of r (or any region between r and $SC(h, \mu_d)$) is that the movement of a doctor within the region $SC(h, \mu_d)$ does not affect the distribution of doctors on which preferences of r (or regions of any smaller depth than $SC(h, \mu_d)$) are defined.

REMARK 7. Each step of the flexible deferred acceptance algorithm corresponds to a step of the cumulative offer process (Hatfield and Milgrom 2005), that is, at each step, if doctor d proposes to hospital h in the flexible deferred acceptance algorithm, then at the same step of the cumulative offer process, contract (d, h) is proposed. Moreover, the set of doctors accepted for hospitals at a step of the flexible deferred acceptance algorithm corresponds to the set of contracts held at the corresponding step of the cumulative offer process. Therefore, if X' is the allocation that is produced by the cumulative offer process, then $\mu(X')$ is the matching produced by the flexible deferred acceptance algorithm.

PROOF OF THEOREM 1. By Proposition 1, the choice rule $\mathrm{Ch}(\cdot)$ satisfies the substitutes condition and the law of aggregate demand in the associated model of matching with

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But $\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v; \tilde{\kappa}^w_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$ because X' is a stable allocation in the associated model of matching with contracts, which in particular implies $v = (w_{r''})_{r'' \in SC(h,\mu_d)}$. Since $v \le v''$, this means that

$$\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v''; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}) \le v \le v''.$$

Thus by consistency of $\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}$, we obtain

$$\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v''; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}) = \tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}).$$

But again by $\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v; \tilde{\kappa}^w_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$, by substitutability we obtain $\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$; thus, $\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v''; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}) = (w_{r''})_{r'' \in SC(h,\mu_d)}$. This is a contradiction because $(w'_{r''})_{r'' \in SC(h,\mu_d)} \le (w_{r''})_{r'' \in SC(h,\mu_d)} = v''$ and $(w'_{r''})_{r'' \in SC(h,\mu_d)} \rhd_{SC(h,\mu_d)} (w_{r''})_{r'' \in SC(h,\mu_d)}$ while $\tilde{\mathrm{Ch}}_{SC(h,\mu_d)}(v''; \tilde{\kappa}^{w''}_{SC(h,\mu_d)}) \in \arg\max_{\ge_{SC(h,\mu_d)}} \{(w'''_{r''})_{r'' \in SC(h,\mu_d)} | (w'''_{r''})_{r'' \in SC(h,\mu_d)} \le v'', \sum_{r'' \in S(SC(h,\mu_d))} w'''_{r''} \le \tilde{\kappa}^{w''}_{SC(h,\mu_d)} \}$.

contracts. By Hatfield and Milgrom (2005), Hatfield and Kojima (2009), and Hatfield and Kominers (2012), the cumulative offer process with choice rules satisfying these conditions produces a stable allocation and is (group) strategy-proof.[46] The former fact, together with Remark 7 and Proposition 2, implies that the outcome of the flexible deferred acceptance algorithm is a stable matching in the original model. The latter fact and Remark 7 imply that the flexible deferred acceptance mechanism is (group) strategy-proof for doctors.   □

APPENDIX D: PROOF OF THEOREM 2

Fix R, and suppose that it is not a hierarchy. Then there exist $r, r' \in R$ and $h_1, h_2, h_3 \in H$ such that $h_1 \in r \setminus r'$, $h_2 \in r \cap r'$, and $h_3 \in r' \setminus r$. We show that there does not exist a mechanism that is stable and strategy-proof for doctors.

To see this, first pick $r_1 \in R$ such that (i) $\{h_1, h_2\} \subseteq r_1 \subseteq r$ and (ii) there is no $\tilde{r} \in R$ with the property that $\{h_1, h_2\} \subseteq \tilde{r} \subsetneq r_1$. Similarly, pick $r_2 \in R$ such that (i) $\{h_2, h_3\} \subseteq r_2 \subseteq r'$ and (ii) there is no $\tilde{r} \in R$ with the property that $\{h_2, h_3\} \subseteq \tilde{r} \subsetneq r_2$.

By the construction of r_1, there exist $S_1 \subseteq R$ and $\hat{r}_1, \hat{r}_2 \in R$ such that (i) S_1 is a largest partition of r_1, (ii) $\hat{r}_1 \in S_1$ and $h_1 \in \hat{r}_1$, and (iii) $\hat{r}_2 \in S_1$ and $h_2 \in \hat{r}_2$. Similarly, by the construction of r_2, there exist $S_2 \subseteq R$ and $\tilde{r}_2, \tilde{r}_3 \in R$ such that (i) S_2 is a largest partition of r_2, (ii) $\tilde{r}_2 \in S_2$ and $h_2 \in \tilde{r}_2$, and (iii) $\tilde{r}_3 \in S_2$ and $h_3 \in \tilde{r}_3$.

Let $\hat{w}^1$ be a vector of nonnegative integers over the set S_1 such that the coordinate corresponding to $\hat{r}_1$ is 1 and other coordinates are 0. Also, let $\hat{w}^2$ be a vector of nonnegative integers over the set S_1 such that the coordinate corresponding to $\hat{r}_2$ is 1 and other coordinates are 0. Suppose that $\hat{w}^1 \rhd_{r_1, S_1} \hat{w}^2$. Similarly, let $\tilde{w}^2$ be a vector of nonnegative integers over the set S_2 such that the coordinate corresponding to $\tilde{r}_2$ is 1 and other coordinates are 0. Also, let $\tilde{w}^3$ be a vector of nonnegative integers over the set S_2 such that the coordinate corresponding to $\tilde{r}_3$ is 1 and other coordinates are 0. Suppose that $\tilde{w}^2 \rhd_{r_2, S_2} \tilde{w}^3$.

Let $\kappa_{r_1} = \kappa_{r_2} = 1$ and $\kappa_{\bar{r}} = |D| + 1$ for all $\bar{r} \in R \setminus \{r_1, r_2\}$. Fix two doctors d_1 and d_2 in D. Finally, assume that preferences of doctors and hospitals are

$$\succ_{d_1} : h_3, \qquad \succ_{d_2} : h_2, h_1,$$

$$\succ_{h_1} : d_2, d_1, \qquad \succ_{h_2} : d_1, d_2, \qquad \succ_{h_3} : d_2, d_1,$$

the capacities of h_1, h_2, and h_3 are sufficiently large so that they never bind,[47] and all doctors in $D \setminus \{d_1, d_2\}$ regard all hospitals unacceptable.[48]

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[46] Aygün and Sönmez (2012) point out that a condition called path-independence (Fleiner 2003) or irrelevance of rejected contracts (Aygün and Sönmez 2012) is needed for these conclusions. Aygün and Sönmez (2012) show that the substitutes condition and the law of aggregate demand imply this condition. Since the choice rules in our context satisfy the substitutes condition and the law of aggregate demand, the conclusions go through.

[47] For instance, let $q_{h_1} = q_{h_2} = q_{h_3} = |D| + 1$.

[48] Preferences for hospitals in $H \setminus \{h_1, h_2, h_3\}$ can be arbitrary.

By inspection, it is straightforward to see that the following two are the only stable matchings given the above preferences:

$$\mu = \begin{pmatrix} h_1 & h_2 & h_3 & \text{other hospitals} & \varnothing & \varnothing \\ \varnothing & d_2 & \varnothing & \cdots \varnothing \cdots & d_1 & \text{other doctors} \end{pmatrix},$$

$$\mu' = \begin{pmatrix} h_1 & h_2 & h_3 & \text{other hospitals} & \varnothing \\ d_2 & \varnothing & d_1 & \cdots \varnothing \cdots & \text{other doctors} \end{pmatrix}.$$

We consider two cases.

   Case 1. Suppose that a mechanism produces μ given the above preference profile. Consider d_1's preferences

$$\succ'_{d_1} : h_1, h_2, h_3.$$

Under the preference profile $(\succ'_{d_1}, \succ_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, it is straightforward to check that μ' is a unique stable matching. Note that d_1 is matched to h_3 under this new preference profile, which is strictly better under $\succ_{d_1}$ than d_1's match $\varnothing$ under the original preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1} \succ_{h_2}, \succ_{h_3})$. This implies that if a mechanism is stable and produces μ given preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, then it is not strategy-proof for doctors.

   Case 2. Suppose that a mechanism produces μ' given the above preference profile. Consider d_2's preferences

$$\succ'_{d_2} : h_2, h_3, h_1.$$

Under the preference profile $(\succ_{d_1}, \succ'_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, it is straightforward to check that μ is a unique stable matching. Note that d_2 is matched to h_2 under this new preference profile, which is strictly better under $\succ_{d_2}$ than d_2's match h_1 under the original preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1} \succ_{h_2}, \succ_{h_3})$. This implies that if a mechanism is stable and produces μ' given preference profile $(\succ_{d_1}, \succ_{d_2}, \succ_{h_1}, \succ_{h_2}, \succ_{h_3})$, then it is not strategy-proof for doctors.

Appendix E: Additional discussions

E.1 *Alternative definition of stability*

Given a matching μ, denote by $\mu^{d \to h, d'}$ the matching such that $\mu^{d \to h, d'}_{d''} = \mu_{d''}$ for all $d'' \in D \setminus \{d, d'\}$, $\mu^{d \to h, d'}_d = h$, and $\mu^{d \to h, d'}_{d'} = \varnothing$. We say that a pair (d, h) satisfies *condition* (*) at μ if there exists $r \in R$ with $\mu_d \notin r$, $h \in r$, and $|\mu_r| = \kappa_r$ such that for all d' with $\mu_{d'} \in r$, $\mu^{d \to h, d'}$ is not Pareto superior to μ for $\{r' \in R | \mu_d, h \in r' \text{ and } r' \subseteq r\}$.

   A matching μ is *stable** if it is feasible, individually rational, and if (d, h) is a blocking pair then $d' \succ_h d$ for all doctors $d' \in \mu_h$ and (d, h) either (i) satisfies infeasibility and condition (*) at μ or (ii) is illegitimate at μ.[49] A mechanism φ is said to be *stable** if, for each $(\kappa, \trianglerighteq)$ and a preference profile $\succ$, the matching $\varphi^{\kappa, \trianglerighteq}(\succ)$ is *stable**.

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[49]In fact, one can show that condition (*) implies infeasibility. Here, we listed the two conditions so as to make the comparison with stability transparent.

COROLLARY 1*. *Fix D, H, and a set of regions R. The following statements are equivalent.*

   (i) *R is a hierarchy.*

 (ii*) *There exists a mechanism that is stable* and strategy-proof for doctors.*

Since stability* is stronger than stability, it is immediate that (ii*) implies (i) by Corollary 1. The converse direction can be seen by investigating the proof of Proposition 2.

E.2 *Group strategy-proofness*

The statement of Corollary 1 holds when we strengthen the incentive compatibility requirement. A mechanism φ is said to be *group strategy-proof for doctors* if there are no $(\kappa, \rhd)$, preference profile $\succ$, a subset of doctors $D' \subseteq D$, and a preference profile $(\succ'_{d'})_{d' \in D'}$ of doctors in D' such that

$$\varphi_d^{\kappa, \rhd}\left((\succ'_{d'})_{d' \in D'}, (\succ_i)_{i \in D \cup H \setminus D'}\right) \succ_d \varphi_d^{\kappa, \rhd}(\succ) \quad \text{for all } d \in D'.$$

That is, no subset of doctors can jointly misreport their preferences to receive a strictly preferred outcome for every member of the coalition under the mechanism. Clearly, this property is stronger than strategy-proofness for doctors. The proof in the Appendix shows that the statement of Corollary 1 holds when we replace strategy-proofness for doctors with group strategy-proofness for doctors.

E.3 *Further discussion on substitutability*

The following examples show that conditions (1) and (2) of substitutability are independent.

EXAMPLE 2 (Regional preferences that violate (1) while satisfying (2)). There is a region r in which two hospitals h_1 and h_2 reside. $S = \{\{h_1\}, \{h_2\}\}$ is the unique largest partition of r. The capacity of each hospital is 2. Region r's preferences are as follows.

$$\rhd_{r,S} : (2,2), (2,1), (1,2), (2,0), (0,2), (1,1), (1,0), (0,1), (0,0).$$

One can check by inspection that condition (2) and consistency are satisfied. To show that (1) is not satisfied, observe first that there is a unique associated choice rule (since preferences are strict), and denote it by $\tilde{\text{Ch}}_{r,S}$. The above preferences imply that $\tilde{\text{Ch}}_{r,S}((1,2);2) = (0,2)$ and $\tilde{\text{Ch}}_{r,S}((2,2);2) = (2,0)$. But this is a contradiction to (1) because $(1,2) \leq (2,2)$ but $\tilde{\text{Ch}}_{r,S}((1,2);2) \geq \tilde{\text{Ch}}_{r,S}((2,2);2) \wedge (1,2)$ does not hold (the left hand side is $(0,2)$ while the right hand side is $(1,0)$). ◇

EXAMPLE 3 (Regional preferences that violate (2) while satisfying (1)). There is a region r in which three hospitals h_1, h_2, and h_3 reside. $S = \{\{h_1\}, \{h_2\}, \{h_3\}\}$ is the unique largest partition of r. The capacity of each hospital is 1. Region r's preferences are as follows.

$$\rhd_{r,S} : (1,1,1), (1,1,0), (1,0,1), (0,1,1), (0,0,1), (0,1,0), (1,0,0), (0,0,0).$$

One can check by inspection that condition (1) (and, hence, consistency by Claim 1) are satisfied. To show that (2) is not satisfied, observe first that there is a unique associated choice rule (since preferences are strict), and denote it by $\tilde{\text{Ch}}_{r,S}$. The above preferences imply that $\tilde{\text{Ch}}_{r,S}((1,1,1);1) = (0,0,1)$ and $\tilde{\text{Ch}}_{r,S}((1,1,1);2) = (1,1,0)$. But this is a contradiction to (2) because $1 \leq 2$ but $\tilde{\text{Ch}}_{r,S}((1,1,1);1) \leq \tilde{\text{Ch}}_{r,S}((1,1,1);2)$ does not hold (the left hand side is $(0,0,1)$ while the right hand side is $(1,1,0)$).   ◇

References

Abdulkadiroğlu, Atila (2005), "College admissions with affirmative action." *International Journal of Game Theory*, 33, 535–549. [763]

Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth (2005), "The New York City high school match." *American Economic Review Papers and Proceedings*, 95, 364–367. [763]

Abdulkadiroğlu, Atila and Tayfun Sönmez (2003), "School choice: A mechanism design approach." *American Economic Review*, 93, 729–747. [763, 767]

Abraham, David J., Robert W. Irving, and David F. Manlove (2007), "Two algorithms for the student-project allocation problem." *Journal of Discrete Algorithms*, 5, 73–90. [763]

Alkan, Ahmet (2001), "On preferences over subsets and the lattice structure of stable matchings." *Review of Economic Design*, 6, 99–111. [768]

Alkan, Ahmet (2002), "A class of multipartner matching markets with a strong lattice structure." *Economic Theory*, 19, 737–746. [778, 780]

Alkan, Ahmet and David Gale (2003), "Stable schedule matching under revealed preferences." *Journal of Economic Theory*, 112, 289–306. [768, 778, 780]

Aygün, O. and T. Sönmez (2012), "Matching with contracts: The critical role of irrelevance of rejected contracts." Working paper, No 804, Boston College Working Papers in Economics. [778, 787]

Biró, Petér, Tamás Fleiner, Robert W. Irving, and David F. Manlove (2010), "The college admissions problem with lower and common quotas." *Theoretical Computer Science*, 411, 3136–3153. [763]

Blair, C. (1988), "The lattice structure of the set of stable matchings with multiple partners." *Mathematics of operations research*, 13, 619–628. [778]

Budish, Eric, Yeon-Koo Che, Fuhito Kojima, and Paul Milgrom (2013), "Designing random allocation mechanisms: Theory and applications." *American Economic Review*, 103, 585–623. [762]

Crawford, Vincent P. and Elsie Marie Knoer (1981), "Job matching with heterogeneous firms and workers." *Econometrica*, 49, 437–450. [763]

Dur, Umut, Parag A. Pathak, and Tayfun Sönmez (2016), "Explicit vs. statistical preferential treatment in affirmative action: Theory and evidence from Chicago's Exam Schools." Working paper, National Bureau of Economic Research, Working Paper No. 22109. [763]

Echenique, Federico and M. Bumin Yenmez (2015), "How to control controlled school choice." *American Economic Review*, 105, 2679–2694. [763]

Ehlers, Lars, Isa E. Hafalir, M. Bumin Yenmez, and Muhammed A. Yildirim (2014), "School choice with controlled choice constraints: Hard bounds versus soft bounds." *Journal of Economic Theory*, 153, 648–683. [763]

Ergin, Haluk and Tayfun Sönmez (2006), "Games of school choice under the Boston mechanism." *Journal of Public Economics*, 90, 215–237. [763]

Fleiner, Tamás (2003), "A fixed-point approach to stable matchings and some applications." *Mathematics of Operations Research*, 28, 103–126. [763, 778, 787]

Fragiadakis, Daniel and Peter Troyan (2017), "Improving matching under hard distributional constraints." *Theoretical Economics*, 12, 863–908. [775]

Gale, David and Lloyd S. Shapley (1962), "College admissions and the stability of marriage." *American Mathematical Monthly*, 69, 9–15. [761, 762, 777]

Goto, Masahiro, Naoyuki Hashimoto, Atsushi Iwasaki, Yujiro Kawasaki, Suguru Ueda, Yosuke Yasuda, and Makoto Yokoo (2014b), "Strategy-proof matching with regional minimum quotas." In *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems* (Alessio Lomuscio, Paul Scerri, Ana Bazzan, and Michael Huhns, eds.), 1225–1232, International Foundation for Autonomous Agents and Multiagent Systems. [764, 775]

Goto, Masahiro, Atsushi Iwasaki, Yujiro Kawasaki, Yosuke Yasuda, and Makoto Yokoo (2014a), "Improving fairness and efficiency in matching with distributional constraints: An alternative solution for the Japanese medical residency match." Working Paper, Munich Personal RePEc Archive, Paper No. 53409. [764, 775]

Hafalir, Isa E., M. Bumin Yenmez, and Muhammed A. Yildirim (2013), "Effective affirmative action in school choice." *Theoretical Economics*, 8, 325–363. [763]

Hatfield, J. W. and S. D. Kominers (2016), "Hidden Substitutes." Working paper, Harvard University. [781]

Hatfield, John W. and Fuhito Kojima (2009), "Group incentive compatibility for matching with contracts." *Games and Economic Behavior*, 67, 745–749. [764, 772, 787]

Hatfield, John W. and Fuhito Kojima (2010), "Substitutes and stability for matching with contracts." *Journal of Economic Theory*, 145, 1704–1723. [764]

Hatfield, John W., Scott D. Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp (2013), "Stability and competitive equilibrium in trading networks." *Journal of Political Economy*, 121, 966–1005. [764]

Hatfield, John W. and Paul R. Milgrom (2005), "Matching with contracts." *American Economic Review*, 95, 913–935. [763, 764, 768, 771, 772, 778, 780, 786, 787]

Hatfield, John William and Scott Duke Kominers (2011), "Contract design and stability in matching markets." Unpublished paper, Harvard University. [764]

Hatfield, John William and Scott Duke Kominers (2012), "Matching in networks with bilateral contracts." *American Economic Journal: Microeconomics*, 4, 176–208. [764, 772, 787]

Hatfield, John William, Scott Duke Kominers, and Alexander Westkamp (2015), "Stability, strategy-proofness, and cumulative offer mechanisms." Unpublished paper, Harvard Business School. [764]

Hirata, Daisuke and Yusuke Kasuya (2017), "On stable and strategy-proof rules in matching markets with contracts." *Journal of Economic Theory*, 168, 27–43. [764]

Kamada, Yuichiro and Fuhito Kojima (2015a), "Efficient matching under distributional constraints: Theory and applications." *American Economic Review*, 105, 67–99. [762, 764, 769, 770, 771, 776]

Kamada, Yuichiro and Fuhito Kojima (2015b), "General theory of matching under distributional constraints." Unpublished paper, Haas School of Business, University of California, Berkeley. [761, 762]

Kamada, Yuichiro and Fuhito Kojima (2017), "Stability concepts in matching under distributional constraints." *Journal of Economic Theory*, 168, 107–142. [764, 770]

Kelso, Alexander S. and Vincent P. Crawford (1982), "Job matching, coalition formation, and gross substitutes." *Econometrica*, 50, 1483–1504. [764]

Kojima, Fuhito and Mihai Manea (2010), "Axioms for deferred acceptance." *Econometrica*, 78, 633–653. [768]

Kojima, Fuhito, Akhisa Tamura, and Makoto Yokoo (2017), "Designing matching mechanisms under constraints: An approach from discrete convex analysis." Working Paper, Stanford University. [764, 775]

Kominers, Scott Duke and Tayfun Sönmez (2016), "Matching with slot-specific priorities: Theory.", volume 11, 683–710. [764]

Milgrom, Paul (2009), "Assignment messages and exchanges." *American Economic Journal: Microeconomics*, 1, 95–113. [762]

Ministry of Health, Labour and Welfare, Japan (2008), "Measures to address regional imbalance and specialty imbalance of doctors (in Japanese)." http://www.mhlw.go.jp/shingi/2008/08/dl/s0823-3e.pdf. [762]

NTT Com Research (2007), "Obstetricians are in short supply! Footsteps of obstetrics breakdown." URL: http://research.nttcoms.com/database/data/000687/. [762]

Roth, Alvin E. (1982), "The economics of matching: Stability and incentives." *Mathematics of Operations Research*, 7, 617–628. [771]

Roth, Alvin E. (1985), "The college admission problem is not equivalent to the marriage problem." *Journal of Economic Theory*, 36, 277–288. [766]

Roth, Alvin E. (1991), "A natural experiment in the organization of entry level labor markets: Regional markets for new physicians and surgeons in the United Kingdom." *American Economic Review*, 81, 415–440. [763]

Roth, Alvin E. and Marilda A. O. Sotomayor (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, volume 18 of Econometric Society Monographs. Cambridge University Press, Cambridge, United Kingdom. [768]

Sönmez, Tayfun (2013), "Bidding for army career specialties: Improving the ROTC branching mechanism." *Journal of Political Economy*, 121, 186–219. [764, 781]

Sönnez, Tayfun and Tobias B. Switzer (2013), "Matching with (branch-of-choice) contracts at the United States military academy." *Econometrica*, 81, 451–488. [764, 781]

Westkamp, Alexander (2013), "An analysis of the German university admissions system." *Economic Theory*, 53, 561–589. [763]