

Efficient dynamic mechanisms in environments with interdependent valuations: The role of contingent transfers

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This paper addresses the problem of implementing socially efficient allocations in dynamic environments with interdependent valuations and evolving private information. In the case where the agents' information is correlated across time, we construct efficient and incentive compatible direct dynamic mechanisms. Unlike the mechanisms with history-independent transfers in the existing literature, these mechanisms feature history-dependent transfers. Moreover, they are reminiscent of the classical Vickrey–Clarke–Grove (VCG) mechanism, even though the latter is not incentive compatible with interdependent valuations. We further show that the VCG aspect of the direct mechanisms suggests natural ways for implementation in some repeated auctions.

KEYWORDS. Dynamic mechanism, interdependent valuation, intertemporal correlation.

JEL CLASSIFICATION. C73, D61, D82.

1. INTRODUCTION

This paper studies efficient mechanism design in dynamic allocation problems with interdependent valuations. A canonical real-world example of such problems is the following. Periodically, the U.S. government uses auctions to sell licenses for the right to drill for oil in adjacent offshore areas. Bidders in these auctions are oil firms. Presumably, these firms conduct geological surveys to estimate the amount of oil in each area before bidding in each auction, so that the information obtained by one firm is also valuable for the other firms. The efficient allocation of licenses depends on the evolving private information of the firms, so the government should carefully design the auctions to induce truthful revelation by the firms in every period. More abstractly, in the problems of interest, a sequence of decisions needs to be made over time: in each period an allocation is to be made among a group of agents, who have time-varying, payoff-relevant

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private information. Efficient mechanism design is the question of how to truthfully implement socially efficient allocations, i.e., how to handle the incentive compatibility constraints implied by the evolving private information.

Following the literature, we restrict ourselves to the case of quasi-linear preferences and private information that follows a general Markov decision process whose evolution depends on allocations. In this environment, and under the assumption that valuations are private, i.e., not interdependent, Bergemann and Välimäki (2010) and Athey and Segal (2013) have successfully addressed this question by means of dynamic extensions of the classic VCG (Vickrey 1961, Clarke 1971, and Groves 1973) and AGV (d'Aspremont and Gérard-Varet 1979 and Arrow 1979) mechanisms.¹ However, with interdependence, it is well known that the VCG mechanism and its dynamic extensions are not incentive compatible without additional strong assumptions. The key insight of the VCG mechanism—making each agent a residual claimant—is not applicable when an agent's information affects others' utilities. In fact, in generic environments with multidimensional and statistically independent private information, Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001) have shown that no efficient mechanism, VCG or not, is Bayesian incentive compatible.² Alternatively, with correlated private information, the lottery mechanism of Crémer and McLean (1988) is efficient and Bayesian incentive compatible. Yet in dynamic environments, a period-by-period extension of Crémer and McLean's mechanism may not be incentive compatible, because agents have more opportunities to deviate.³

But notice that long-term interactions offer a richer family of transfer schemes compared to the static case; in particular, transfers can be made history-dependent. With such transfers, an agent's current report affects not only her current payoff but also the entire stream of future transfers. Therefore, one might be able to restore incentive compatibility with a careful choice of intertemporal trade-offs. We show that this is indeed the case. For the above-mentioned dynamic allocation problems, we construct efficient and incentive compatible dynamic mechanisms, provided that information is correlated over time, as we explain below. In addition, the mechanisms ensure that each agent becomes a residual claimant, as in the VCG mechanism. That is, in each period and regardless of the history, an agent's expected continuation payoff equals the continuation social surplus when all agents truthfully report their private information. In other words, not only do we provide a solution to the dynamic incentive compatibility issue with interdependence, but also the solution shares some of the main features of the VCG mechanism.⁴ Furthermore, as in the private-valuation case, the constructed dynamic mechanisms satisfy a strong incentive compatibility requirement—periodic ex post incentive compatibility, which requires truth-telling to be a best response of an agent at every stage, irrespective of the past messages and allocations and other agents' current

¹Also see Parkes and Singh (2003).

²Jehiel et al. (2006) further prove that only constant allocation rules are ex post incentive compatible in generic models with multidimensional signals.

³See the example in Section 3.1.

⁴From a practical viewpoint, the constructed history-dependent transfers also point toward a new way to link information that has been largely ignored in the design of various economic mechanisms.

private information.⁵ Constructing a periodic ex post incentive compatible dynamic mechanism is not only of theoretical interest; it also suggests natural ways to implement the direct mechanism with dynamic auctions. In Section 4, in a class of repeated allocation problems where no static auction format is efficient, we define a dynamic format with contingent transfers that has an efficient symmetric equilibrium in monotone strategies.

The intertemporal correlation that is required for our results resembles the correlation conditions in Crémer and McLean (1988) when the state space of the Markov decision process is finite. That is, we require convex or linear independence conditions on the associated transition matrices.⁶ In Section 5, we extend the results to the infinite-signal case. Generalizing the convex and linear independence conditions, we construct efficient dynamic mechanisms that are approximately incentive compatible.⁷ Moreover, under stronger correlation conditions, there are mechanisms with contingent transfers that are periodic ex post incentive compatible. Therefore, the results in the dynamic mechanisms contrast sharply with those in the static counterparts, where one can only achieve approximate incentive compatibility or approximate surplus extraction.⁸

Finally, in the Supplemental Material (available in a supplementary file on the journal website, <http://econtheory.org/sup/2234/supplement.pdf>), we address the issues of budget balance and surplus extraction. Specifically, by modifying the transfers, we construct (i) an average externality mechanism that balances the budget,⁹ and (ii) a lottery-augmented mechanism à la Crémer and McLean (1988) and McAfee and Reny (1992) that extracts all the surplus of the agents in the finite case and virtually all the surplus in the infinite case. While the main results require intertemporal correlation, in the Supplemental Material, we also study the case where each agent's private information evolves independently. We focus on settings with one-dimensional private information and construct transfers that are the dynamic counterparts of the generalized VCG mechanism (cf. Crémer and McLean 1985, Jehiel and Moldovanu 2001, Bergemann and Välimäki 2002). In the private-valuation special case, these transfers reduce to the dynamic pivot mechanism constructed by Bergemann and Välimäki (2010). In the general interdependence case, we identify dynamic single-crossing conditions that ensure incentive compatibility.

⁵Athey and Miller (2007), Bergemann and Välimäki (2010), and Athey and Segal (2013) introduce the notion of ex post incentive compatibility in every period in the study of dynamic mechanisms with private valuation. In this paper, we follow Bergemann and Välimäki (2010) and call it *periodic ex post incentive compatibility*.

⁶These conditions are related to, but different from, those in Crémer and McLean (1988) for static mechanisms with correlated signals. Specifically, we do not impose any restriction on the information structure within a period.

⁷The convex independence condition is similar to McAfee and Reny's extension (cf. McAfee and Reny 1992) or Crémer and McLean 1988.

⁸See McAfee and Reny (1992) and Miller et al. (2007).

⁹The mechanism is related to the balanced team mechanism constructed in Athey and Segal (2013), which generalizes the AGV mechanism introduced by Arrow (1979) and d'Aspremont and Gérard-Varet (1979) to dynamic environments with independent private valuations.

1.1 Related literature

Efficient mechanisms with interdependent valuations In addition to the papers mentioned above, our dynamic mechanisms are also related to the two-stage VCG mechanism in Mezzetti (2004, 2007).¹⁰ Mezzetti provides one way to bypass the above impossibility results, under the assumptions that agents can observe their realized utilities and that transfers can be made based on the reported utilities. From an applied perspective, these are strong assumptions. More importantly, in Mezzetti's mechanism, agents are indifferent among all messages when they report their utilities. If it is costly to report utilities, then agents would rather walk away from the mechanism at this stage. In comparison, we consider direct mechanisms that ask agents to report their private signals in each period, in which truth-telling constitutes a perfect equilibrium. Furthermore, for each agent and each signal profile, there are messages that yield different expected payoffs in every period.

Dynamic mechanism design Most of the recent literature on dynamic mechanisms assumes independent private valuations (e.g., Bergemann and Välimäki 2010, Athey and Segal 2013, Said 2012, and Pavan et al. 2014), with an exception of Gershkov and Moldovanu (2009). Gershkov and Moldovanu consider a problem of sequential allocations of objects to myopic agents who arrive over time.¹¹ In their model, the time horizon is finite, valuations are private, and signals are one-dimensional. They show that if the distribution of signals is unknown, then interdependence arises endogenously as a result of learning, which may prevent efficient implementation with online mechanisms.^{12,13} Since agents are impatient in Gershkov and Moldovanu's model, the incentive problems are static. They identify single-crossing conditions on the underlying uncertainty that ensure the existence of efficient mechanisms. Related to the history-dependent mechanisms in this paper, they also point out that efficient mechanisms in their model exist if all transfers can be delayed to the last period.

Two closely related papers are Hörner et al. (2015) and Noda (2018). Independent to this paper, Hörner et al. (2015) also study the role of intertemporal correlation in dynamic Bayesian games with communication. They consider the case in which signal spaces are finite and the evolution of signals is stationary. Additionally, they study truthful Bayes Nash equilibria of the infinitely repeated game with private information.¹⁴ In the case with correlated signals and interdependent values, they extend the insight of Crémer and McLean (1988) (and also the static budget-balanced mechanism in Kosenok and Severinov 2008) to dynamic games and identify an intertemporal full-rank condition that is sufficient to obtain a folk theorem in truthful equilibria. They show that even

¹⁰See Deb and Mishra (2014) for a related recent study.

¹¹See also Gershkov and Moldovanu Gershkov and Moldovanu (2010, 2012) for studies of related questions.

¹²Segal (2003) also emphasizes this feature in a static model.

¹³The term "online mechanism" is mostly used in the algorithmic game theory literature to study allocation problems with arrivals and departures; it requires that allocations and transfers of an agent are made when she is present.

¹⁴Truthful Bayes Nash equilibria, defined by Hörner et al. (2015), generalize perfect public equilibria in repeated games with imperfect public monitoring.

in repeated games (where transfers are not allowed), they use continuation payoffs as effective transfers, thereby bridging the gap between dynamic games and mechanism design. By contrast, we consider a dynamic mechanism design setting with transferable utilities and interdependent valuations, where the evolution of private information can vary over time; our results cover both the finite and infinite signal space cases and emphasize the VCG feature of history-dependent transfers, which is absent from their game-theoretic analysis. Moreover, since the solution concept adopted in this paper—periodic ex post incentive compatibility—is stronger than their truthful equilibria in the case with interdependent values, we identify stronger intertemporal full-rank conditions. Finally, in the case with independent signals, Hörner et al. (2015) restrict attention to the private-valuation settings, whereas we consider the general setting with interdependent valuations and extend the existing positive results in the static environments to dynamic environments.

Noda (2018) also studies a question similar to ours, assuming signal spaces are finite. Noda (2018) generalizes the convex independence condition in Crémer and McLean (1988) to dynamic settings that guarantees implementability and surplus extraction. Different from Noda's work, this paper considers both finite- and infinite-signal spaces, gives sufficient conditions for the existence of periodic ex post incentive compatible mechanisms, and constructs the corresponding contingent transfers. For the case where signal spaces are finite, the intertemporal convex independence condition in Noda (2018) is weaker than that identified in this paper, although both conditions are generically satisfied in the finite-horizon case. Moreover, we also generalize the spanning conditions in Crémer and McLean (1988) to dynamic environments, whereas Noda (2018) only studies convex independence.

2. MODEL

2.1 *The environment*

We consider a dynamic interdependent valuation environment with N ($N \geq 2$) agents. Time is discrete, indexed by $t \in \{1, 2, \dots, T\}$, where $T \leq \infty$.¹⁵ In each period t , each agent $i \in \{1, 2, \dots, N\}$ privately observes a payoff-relevant signal $\theta_t^i \in \Theta_t^i$, where Θ_t^i is a *finite* set. The extension to the infinite signal space case is studied in the Supplemental Material. The signal space in period t is $\Theta_t = \prod_{i=1}^N \Theta_t^i$ with a generic element $\theta_t = (\theta_t^1, \dots, \theta_t^N)$. For each i and t , denote the private information held by agents other than i in period t by $\theta_t^{-i} = (\theta_t^1, \dots, \theta_t^{i-1}, \theta_t^{i+1}, \dots, \theta_t^N) \in \prod_{j \neq i} \Theta_t^j$.

In each period t , the flow utility u^i of agent i is determined by the current signal profile θ_t , the current allocation $a_t \in A_t$, and the current monetary transfer $p_t^i \in \mathbb{R}$, where A_t is the *finite* set of social alternatives in period t . The flow utility of each agent is assumed to be quasi-linear in monetary transfers, and agents have a common discount factor $\delta \in (0, 1)$. Given sequences of signals $\{\theta_t\}_{t=1}^T$, allocations $\{a_t\}_{t=1}^T$, and monetary

¹⁵We study both the cases of finite and infinite horizon.

transfers $\{p_t^1, \dots, p_t^N\}_{t=1}^T$, the total payoff of each agent i is

$$\sum_{t=1}^T \delta^{t-1} [u^i(a_t, \theta_t) - p_t^i].$$

The agent's private signals evolve over time following a Markov decision process. Specifically, in the initial period, the signal profile θ_1 is drawn from a prior probability $\mu_1 \in \Delta(\Theta_1)$. In each period $t > 1$, the distribution of current signal profile θ_t is determined by the realized signal profile θ_{t-1} and the allocation decision a_{t-1} in the previous period, represented by a transition probability $\mu_t : A_{t-1} \times \Theta_{t-1} \rightarrow \Delta(\Theta_t)$. The utility functions u^i , the prior μ_1 , and the transition probabilities μ_t are assumed to be common knowledge.

In contrast to previous works that often assume independent prior and transitions across agents, here we specify a general Markov decision process for the evolution of signals, which allows correlation of private information. While in private-valuation environments the existence of efficient mechanisms does not depend on whether correlation is allowed or not, as shown by [Athey and Segal \(2013\)](#), it will be clear in [Section 3](#) how correlation makes a difference in dynamic settings with interdependent valuations.

2.2 Efficiency and mechanisms

A socially efficient allocation rule is a sequence of functions $\{a_t^* : \Theta_t \rightarrow A_t\}_{t=1}^T$ that solves the social program

$$\max_{\{a_t\}_{t=1}^T} \mathbb{E} \left[\sum_{t=1}^T \delta^{t-1} \sum_{i=1}^N u^i(a_t, \theta_t) \right],$$

where the expectation is taken with respect to the processes $\{\theta_t\}$ and $\{a_t\}$. Since the flow utility depends only on the current signal profile, which is assumed to be Markov, the social program can be written in the recursive form: for each $t \in \{1, 2, \dots, T\}$,

$$W_t(\theta_t) = \max_{a_t \in A} \sum_{i=1}^N u^i(a_t, \theta_t) + \delta \mathbb{E}[W_{t+1}(\theta_{t+1}) | a_t, \theta_t],$$

where $W_t(\theta_t)$ is the social surplus starting from period t given the realized signal profile θ_t , and $W_{T+1} \equiv 0$. By the principle of optimality, a_t^* solves the social program if and only if it is a solution to this recursive problem.

We focus on truthful equilibria of direct public mechanisms that implement the socially efficient allocations $\{a_t^*\}_{t=1}^T$. In [Section 4](#), we study indirect mechanisms that implement the direct mechanisms. In a direct public mechanism, in each period t , each agent i is asked to make a public report $r_t^i \in \Theta_t^i$ of her current private signal θ_t^i . Then a public allocation decision a_t and a transfer p_t^i for each agent i are made as functions of the current report profile $r_t = (r_t^i)_{i=1}^N$ and the period- t public history h_t . The period- t

public history contains all reports and allocations up to period $t - 1$, i.e.,

$$h_t = (r_1, a_1, r_2, a_2, \dots, r_{t-1}, a_{t-1}).^{16}$$

Let H_t denote the set of possible period- t public histories. Formally, an *efficient direct revelation mechanism* $\Gamma = \{\Theta_t, a_t^*, p_t\}_{t=1}^T$ consists of (i) Θ_t as the message space in each period t , (ii) a sequence of allocation rules $a_t^* : \Theta_t \rightarrow A$, and (iii) a sequence of monetary transfers $p_t : H_t \times \Theta_t \rightarrow \mathbb{R}^N$.

The period- t private history h_t^i of each agent i contains the period- t public history and the sequence of her realized private signals until period t , i.e.,

$$h_t^i = (r_1, a_1, \theta_1^i, r_2, a_2, \theta_2^i, \dots, r_{t-1}, a_{t-1}, \theta_{t-1}^i, \theta_t^i).$$

Let H_t^i denote the set of agent i 's possible period- t private histories. With a slight abuse of notation, a strategy for agent i is a sequence of mappings $r^i = \{r_t^i\}_{t=1}^T$, where $r_t^i : H_t^i \rightarrow \Theta_t^i$, that assign a report to each of her period- t private histories. A strategy for agent i is *truthful* if it always reports agent i 's private signal θ_t^i truthfully in each period t , regardless of her private history.

Given a mechanism $\Gamma = \{\Theta_t, a_t^*, p_t\}_{t=1}^T$ and a strategy profile $r = \{r^i\}_{i=1}^N$, agent i 's expected discounted payoff is

$$\mathbb{E} \sum_{t=1}^T \delta^{t-1} [u^i(a_t^*(r_t), \theta_t) - p_t^i(h_t, r_t)].^{17}$$

The equilibrium concept we adopt is periodic ex post equilibrium defined by Bergemann and Välimäki (2010) and Athey and Segal (2013). We say that the mechanism is *periodic ex post incentive compatible* or, equivalently, the truthful strategy profile is a *periodic ex post equilibrium* if for each agent and in each period, truth-telling is always a best response regardless of the private history and the current signals of other agents, given that other agents adopt truthful strategies. Formally, let $V_t^i(h_t^i)$ be agent i 's continuation payoff given period- t private history, given that other agents report truthfully. That is,

$$V_t^i(h_t^i) = \max_{r_t^i \in \Theta_t^i} \mathbb{E} [u^i(a_t^*(r_t^i, \theta_t^{-i}), \theta_t) - p_t^i(h_t, r_t^i, \theta_t^{-i}) + \delta V_{t+1}^i(h_{t+1}^i)].^{18}$$

The efficient mechanism is periodic ex post incentive compatible if for each i, t , and h_t^i ,

$$\theta_t^i \in \arg \max_{r_t^i \in \Theta_t^i} u^i(a_t^*(r_t^i, \theta_t^{-i}), \theta_t) - p_t^i(h_t, r_t^i, \theta_t^{-i}) + \delta \mathbb{E}[V_{t+1}^i(h_{t+1}^i) | a_t^*(r_t^i, \theta_t^{-i}), \theta_t]$$

¹⁶We assume that agents do not observe the realized per-period payoffs. Also note that since the mechanism is public, an agent can also infer the transfers for all other agents.

¹⁷In the infinite-horizon ($T = \infty$) case, we require that all agents' expected discounted payoffs are well defined under the mechanism Γ , that is, the expectation and the infinite sum in agents' payoffs are interchangeable.

¹⁸In the finite horizon case, we set $V_{T+1}^i \equiv 0$.

for each $\theta_t \in \Theta_t$. Define the period- t ex post continuation payoff to be

$$V_t^i(h_t^i; h_t, \theta_t^{-i}) = u^i(a_t^*(\theta_t^i, \theta_t^{-i}), \theta_t) - p_t^i(h_t, \theta_t^i, \theta_t^{-i}) + \delta \mathbb{E}[V_{t+1}^i(h_{t+1}^i) | a_t^*(\theta_t^i, \theta_t^{-i}), \theta_t].$$

As suggested by Bergemann and Välimäki (2010), ex post incentive compatibility notions need to be qualified within each period in a dynamic environment, since an agent may wish to change her report in some previous round based on the new information she has received in later periods. Given the fact that interdependent valuations render dominant strategy incentive compatibility impossible, periodic ex post incentive compatibility is the best we can hope for in the current setup.

Finally, the Vickrey–Clarke–Groves (VCG) mechanism is an efficient mechanism $\Gamma = \{\Theta_t, a_t^*, p_t\}_{t=1}^T$ under which each agent i 's continuation payoff is equal to the continuation social surplus net of a term that is independent of her current and future reports, i.e., for each i and t , there is a function $W_t^{-i}(\cdot)$ such that

$$V_t^i(h_t^i; h_t, \theta_t^{-i}) = W_t(\theta_t) - W_t^{-i}(\theta_1^{-i}, \dots, \theta_t^{-i})$$

for all h_t^i, h_t , and θ_t^{-i} .

3. EFFICIENT MECHANISM DESIGN

3.1 An example

Before presenting the general results, we present a two-period repeated auction example to explain the main ideas.¹⁹ Two firms, A and B , compete for licenses to drill for oil on two adjacent offshore areas. The two licenses are sold sequentially in two auctions ($t \in \{1, 2\}$) and the allocation in auction t is $a_t \in \{A, B\}$, where $a_t = i$ means that firm $i \in \{A, B\}$ obtains the license for the corresponding area. Each firm's payoff from obtaining a license depends on its drilling cost and the amount of oil s_t in that area:

$$u^A(s_t) = 2s_t - 1, \quad u^B(s_t) = 3s_t - 6.$$

Suppose that there is no discounting and each firm cares about its total profit from both auctions. Each firm $i \in \{A, B\}$ observes a private signal θ_t^i in auction t . Suppose that prior to the auctions each firm can perform a test in one of the areas. In particular, firm A 's private signal $\theta_1^A \in \{4, 6\}$ indicates the amount of oil in area 1, denoted $\theta_1^A = s_1$, and firm B learns privately from $\theta_2^B \in \{4, 6\}$ the expected amount of oil in area 2, denoted $\theta_2^B = s_2$. In addition, we assume that the joint distribution of θ_1^A and θ_2^B , denoted by $\mu(\theta_1^A, \theta_2^B)$, is

$$\begin{bmatrix} \mu(4, 4) & \mu(4, 6) \\ \mu(6, 4) & \mu(6, 6) \end{bmatrix} = \begin{bmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{bmatrix}$$

so that the conditional distribution of θ_2^B given θ_1^A , denoted by $\mu(\theta_2^B | \theta_1^A)$, is

$$\begin{bmatrix} \mu(4|4) & \mu(4|6) \\ \mu(6|4) & \mu(6|6) \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}.$$

¹⁹The example is adapted and extended from Dasgupta and Maskin (2000).

Finally, we assume that firm B does not learn any relevant information in the first auction, and neither does firm A in the second auction. That is, θ_1^A and θ_1^B are independently distributed, and so are θ_2^A and θ_2^B .

We first notice that efficiency and incentive compatibility are incompatible if only the first auction is conducted. To see this, note that efficiency in the first auction requires firm A to give up the license when it is more profitable, i.e.,

$$a_1^* = \begin{cases} A & \text{if } \theta_1^A = 4, \\ B & \text{if } \theta_1^A = 6. \end{cases}$$

This implies that firm A needs to be compensated for reporting $r_1^A = 6$ rather than $r_1^A = 4$. Specifically, we have the incentive compatibility conditions

$$\begin{aligned} 2 \times 4 - 1 - p_1^A(4) &\geq 0 - p_1^A(6), \\ 0 - p_1^A(6) &\geq 2 \times 6 - 1 - p_1^A(4). \end{aligned}$$

Summing up the two inequalities gives $7 \geq 11$. Thus, no incentive compatible transfer exists. Alternatively, when only the second license is being auctioned, firm B 's incentive constraint matters and it is straightforward to verify that the transfer for firm B ,

$$p_2^B = \begin{cases} 0 & \text{if } r_2^B = 4, \\ 11 & \text{if } r_2^B = 6, \end{cases}$$

truthfully implements the efficient allocation a_2^* in the second auction, where a_2^* is given by

$$a_2^* = \begin{cases} A & \text{if } \theta_2^B = 4, \\ B & \text{if } \theta_2^B = 6. \end{cases}$$

Now we show that by linking the two auctions, dynamic efficiency is implementable, despite the impossibility for static efficiency. The idea is to use the correlation between θ_1^A and θ_2^B and construct a history-dependent transfer for firm A in the *second* auction so that firm A is willing to report its true signal in the *first* auction. For instance, consider the transfer schedule $p_2^A(a_1, r_2^B)$ given by

$$p_2^A = \begin{cases} -4.5 & \text{if } a_1 = B, r_2^B = 4, \\ -14.5 & \text{if } a_1 = B, r_2^B = 6, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that the dynamic mechanism $\Gamma^{\text{link}} \equiv \{(a_1^*, a_2^*), (p_2^A, p_2^B)\}$ is ex post incentive compatible. Recall that truth-telling is optimal for firm B given p_2^B . Since the transfer p_2^A has no effect on firm B 's incentive constraints, under $\{p_2^A, p_2^B\}$, firm B is still willing to report its true signal in the second auction. Now consider firm A 's incentive constraints. Firm A , when reporting its signal, takes into account the fact that its future transfer depends on the current allocation a_1 and the opponent's report r_2^B in the next auction. As

a consequence, the incentive compatibility constraints are satisfied given the specified conditional distribution of signals:

$$2 \times 4 - 1 + 0 \geq 0 + \left(\frac{3}{4} \times 4.5 + \frac{1}{4} \times 14.5 \right),$$

$$0 + \left(\frac{1}{4} \times 4.5 + \frac{3}{4} \times 14.5 \right) \geq 2 \times 6 - 1 + 0.$$

The intuition for this mechanism is as follows. Note that by construction, the left-hand sides of the above two inequalities are equal to the social surplus given firm's A private signal. By exploiting the intertemporal correlation between θ_1^A and θ_2^B , the transfer p_2^A makes firm A a claimant of the social surplus in the first auction (without affecting any firm's incentive constraints in the second auction). Given that firm B adheres to truthful strategies, it is optimal for firm A to be truthful so as to maximize the social surplus and hence its own profit.

Now let us modify the example to illustrate the role of intertemporal correlation and its difference from within-period correlation (Cr mer and McLean 1988) in dynamic mechanisms. We remove the assumption that θ_1^A and θ_1^B are independent and suppose that before firms learn their payoff relevant signals, firm A has access to some private signal $\theta_0^A \in \{0, 1\}$ that determines the joint distribution $\mu(\theta_1^A, \theta_1^B | \theta_0^A)$ of θ_1^A and θ_1^B :

$$\begin{bmatrix} \mu(4, 4|0) & \mu(4, 6|0) \\ \mu(6, 4|0) & \mu(6, 6|0) \end{bmatrix} = \begin{bmatrix} 1/8 & 3/8 \\ 3/8 & 1/8 \end{bmatrix},$$

$$\begin{bmatrix} \mu(4, 4|1) & \mu(4, 6|1) \\ \mu(6, 4|1) & \mu(6, 6|1) \end{bmatrix} = \begin{bmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{bmatrix}.$$

That is, θ_1^A and θ_1^B are negatively correlated if $\theta_0^A = 0$ and are positively correlated if $\theta_0^A = 1$. Finally, the joint distribution of θ_1^A and θ_2^B remains the same and is assumed to be independent of θ_0^A .

Suppose that the auctioneer wants to exploit the correlation between θ_1^A and θ_1^B to incentivize firm A . This amounts to constructing lottery transfers for firm A based on firm B 's first period report r_1^B . However, for such lotteries to work, the auctioneer needs to know the joint distribution of θ_1^A and θ_1^B , which is firm A 's private information. Given a lottery scheme in the first auction, firm A may have an incentive to misreport its signal θ_0^A . To see this, suppose that the auctioneer believes that firm A 's initial report $r_0^A \in \{0, 1\}$ is truthful, and thus uses the following transfers $p_1^A(r_1^A, r_1^B; r_0^A)$ for firm A :

$$\begin{bmatrix} p_1^A(4, 4; 0) & p_1^A(4, 6; 0) \\ p_1^A(6, 4; 0) & p_1^A(6, 6; 0) \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} p_1^A(4, 4; 1) & p_1^A(4, 6; 1) \\ p_1^A(6, 4; 1) & p_1^A(6, 6; 1) \end{bmatrix} = \begin{bmatrix} 5 & 13 \\ 0 & 0 \end{bmatrix}.$$

Given the joint distributions, it is straightforward to check that under $p_1^A(r_1^A, r_1^B; r_0^A)$, if firm B reports its signals truthfully, then it is optimal for firm A to reveal θ_1^A and obtain

zero surplus in the first auction, *had it reported its initial private signal* θ_0^A *truthfully*. However, given $p_1^A(r_1^A, r_1^B; r_0^A)$, firm A could benefit from misreporting θ_0^A . For example, when $\theta_0^A = 0$, the following contingent deviation of firm A is profitable: it first reports $r_0^A = 1$ so that the transfer in the first auction is $p_1^A(r_1^A, r_1^B; 1)$; then after learning θ_1^A , it always reports the opposite $r_1^A \neq \theta_1^A$. When $\theta_1^A = 4$, firm A reports $r_1^A = 6$ and loses the first auction with no surplus,

$$0 - \frac{1}{4} \times p_1^A(6, 4; 1) - \frac{3}{4} \times p_1^A(6, 6; 1) = 0;$$

when $\theta_1^A = 6$, firm A wins by reporting $r_1^A = 4$ and receives a positive surplus,

$$2 \times 6 - 1 - \frac{3}{4} \times p_1^A(4, 4; 1) - \frac{1}{4} \times p_1^A(4, 6; 1) = 4.$$

Similar contingent deviations of firm A exist when $\theta_0^A = 1$.

Finally, we note that since the intertemporal correlation cannot be manipulated by either firm, the dynamic mechanism Γ^{link} constructed before remains ex post incentive compatible.²⁰

3.2 Main results

In this section, we construct periodic ex post incentive compatible efficient dynamic mechanisms under general transition dynamics. **Theorem 3.1** shows that under a generic intertemporal correlation condition and some restrictions on utility functions and signal spaces in the last period, such a dynamic mechanism always exists.²¹ In particular, we show that in each period t the correlation between θ_t^i and θ_{t+1}^{-i} can be used to construct history-dependent transfers such that agent i 's incentive is aligned with the social incentive. Moreover, the resulting transfers are reminiscent of both the VCG transfers and the lottery transfers in **Cr mer and McLean (1988)**. In **Theorem 3.2**, we show that a slightly stronger intertemporal correlation condition ensures dynamic efficiency with a sequence of ‘‘VCG-type’’ transfers.

We make the following assumptions on the utility functions and the evolution of private information.

ASSUMPTION 1 (Bounded payoffs). *For each agent i ,*

$$\max_{(a_t, \theta_t)_{t \geq 1}} \sum_{t=1}^T \delta^{t-1} |u^i(a_t, \theta_t)| < \infty.$$

²⁰In this example, the within-period rank condition of **Cr mer and McLean (1988)** fails, which implies that implementing the efficient allocations with static mechanisms is impossible. If we also assume that firm B receives a private signal in period 0 that is correlated with firm A 's period-0 signal, then a period-by-period Cr mer–McLean mechanism would implement the efficient allocations, but it is only Bayesian incentive compatible.

²¹For the infinite-horizon case, no such restrictions are imposed.

ASSUMPTION 2 (Convex independence). For each $1 \leq t \leq T$, $i \in N$, $a_t \in A_t$, and $\theta_t^{-i} \in \Theta_t^{-i}$, no column of the matrix

$$M_{t+1}^{-i}(a_t, \theta_t^{-i}) \equiv [\mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i})]_{|\Theta_{t+1}^{-i}| \times |\Theta_t^i|}$$

is a convex combination of other columns, i.e., for each θ_t^i ,

$$\mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) \notin \text{Conv}\{\mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i})\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}},$$

where $\text{Conv}\{\mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i})\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$ is the convex hull generated by the set of vectors $\{\mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i})\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$. Moreover, the transition probabilities satisfy

$$\inf_{t, i, a_t, \theta_t^{-i}, \theta_t^i} \text{dist}_2(\mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}), \text{Conv}\{\mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i})\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}) > 0.^{22}$$

ASSUMPTION 3 (Spanning condition). For each $1 \leq t \leq T$, $i \in N$, $a_t \in A_t$, and $\theta_t^{-i} \in \Theta_t^{-i}$, the column vectors of the matrix

$$M_{t+1}^{-i}(a_t, \theta_t^{-i}) \equiv [\mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i})]_{|\Theta_{t+1}^{-i}| \times |\Theta_t^i|}$$

are linearly independent, i.e., there does not exist a collection of real numbers $\{\eta^i(\theta_t^i)\}_{\theta_t^i \in \Theta_t^i}$, which are not all equal to zero, such that

$$\sum_{\theta_t^i \in \Theta_t^i} \eta^i(\theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) = 0$$

for all $\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}$. Moreover, if $T = \infty$, then there exist $\bar{D} \in \mathbb{R}_+$ and $\bar{T} \in \mathbb{N}_+$ such that for any $t \geq \bar{T}$, any i , a_t and θ_t^{-i} , the norm of the pseudo-inverse of the matrix $M_{t+1}^{-i}(a_t, \theta_t^{-i})$ satisfies

$$\|(M_{t+1}^{-i}(a_t, \theta_t^{-i}))^+\| \leq \bar{D}.^{23}$$

Assumption 1 says that the payoff function of each agent is well defined. This assumption is vacuous in the case where allocation and signal spaces are time-independent. Assumptions 2 and 3 require that transition probabilities exhibit intertemporal correlation among different agents' signals and the intertemporal correlation does not vanish in the infinite-horizon case.^{24,25} In particular, for each agent i and

²²The function $\text{dist}_2(\mu, C)$ is the Euclidean distance between a point μ and a set C . Noda (2018) imposes a similar condition in the investigation of surplus extraction mechanisms in the infinite-horizon case.

²³The pseudo-inverse A^+ of a full column-rank matrix A is defined as $A^+ = (A'A)^{-1}A'$, where A' is the transpose of A . The norm of a matrix A is defined as $\|A\| = \sup\{\|Ax\|_\infty : \|x\|_\infty = 1\}$.

²⁴Cr mer and McLean (1988) consider similar conditions in the study of static mechanism design with correlated information.

²⁵I thank an anonymous referee for pointing out an error in the previous version and suggesting strengthening of the assumptions for the infinite-horizon case. The nonvanishing intertemporal correlation condition in Assumption 2 is based on the analysis in Noda (2018). The corresponding condition in Assumption 3 is new.

in each period t , conditional on any a_t and θ_t^{-i} , agent i 's current private signal θ_t^i is correlated with other agents' signals θ_{t+1}^{-i} in the next period. Independent evolution of private information across agents is ruled out by these assumptions. The assumptions of nonvanishing intertemporal correlation will guarantee that agents' discounted payoffs are well defined in the infinite-horizon case under our dynamic mechanisms.²⁶ One example that the intertemporal correlation does not vanish is when the transition probabilities are stationary, i.e., for each t , $\Theta_t = \Theta_{t+1}$, $A_{t+1} = A_t$, and $\mu_{t+1}(\theta_{t+1}|a_t, \theta_t) = \mu(\theta_{t+1}|a_t, \theta_t)$.

To motivate the information correlation assumptions, suppose that there is an underlying state of nature ω_t with possible values in a set Ω in each period t . In addition, ω_t follows a hidden Markov process that evolves over time and is not observed by any agent. In each period t , the relationship between the state of nature ω_t and the agents' private information θ_t is described by a joint distribution ξ_t over $\Omega \times \Theta_t$. If each agent's private signal θ_t^i provides useful information about ω_t , i.e., the conditional $\xi_t(\omega_t|\theta_t^i)$ varies with θ_t^i , then as long as ω_t is not independently distributed, θ_t^i is correlated with θ_{t+1}^{-i} , even conditional on θ_t^{-i} and a_t .

In the finite-horizon case ($T < \infty$), we also impose the following ex post incentive compatibility assumption on the allocation rule a_T^* .

ASSUMPTION 4 (Ex post incentive compatibility in period T). *If $T < \infty$, then the efficient allocation in period T , a_T^* , is ex post incentive compatible.*

In our setup, the allocation problem in period T is essentially a static one. Thus, we can adopt a set of sufficient conditions from the existing literature (Bergemann and Välimäki 2002 in particular) on static mechanism design. The sufficient conditions for ex post incentive compatibility in static models are restrictive given the impossibility results in Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel et al. (2006). In particular, period- T signals have to be one-dimensional and the utility functions have to satisfy a single-crossing condition. We also emphasize that no such assumptions are imposed on the private signals and utility functions from period 1 to $T - 1$. We can think of a situation where agents trade a new asset with each other in multiple periods. Initially, each agent's private information may be multidimensional since there is much uncertainty about many aspects of the asset. As agents trade over time, they gradually learn more information about the asset. In the last period, each agent's signal is simply a real number that represents her estimation of the asset value.

Now we state the main results that generalize the idea of the example in Section 3.1. All the proofs of the results in Section 3 are relegated to Appendix A.

THEOREM 3.1. *Under Assumptions 1, 2, and 4, there exists a sequence of transfers*

$$p_{t+1}^i : \Theta_{t+1}^{-i} \times \Theta_t^i \times A_t \times \Theta_t^{-i} \rightarrow \mathbb{R} \quad \forall i, t < T,$$

such that the efficient dynamic mechanism $\{a_t^, p_t\}$ is periodic ex post incentive compatible.*

²⁶The uniform lower bound ϵ in Assumption 2 and the uniform upper bound \bar{D} in Assumption 3 can be further relaxed to allow for time-dependent bounds as long as agents' payoffs under the constructed mechanisms are well defined.

Here we give a heuristic argument. Recall that in the private-valuation case, the history-independent transfers in the VCG mechanism (or team mechanism in [Athey and Segal 2013](#)),

$$p_t^i(\theta_t) = - \sum_{j \neq i} u^j(a_t^*(\theta_t), \theta_t) = - \sum_{j \neq i} u^j(a_t^*(\theta_t), \theta_t^j), \quad (1)$$

are incentive compatible. However, with interdependent valuations, transfers in (1) depend directly on agent i 's report, which creates an incentive for misreporting. To fix this problem, we consider general history-dependent transfers $p_t^i(h_t, \theta_t)$. It turns out that under [Assumptions 1, 2, and 4](#), it is enough to use transfers that depend on the history in the previous round. Specifically, we show that if $T = \infty$, there exist transfers $p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i})$ under which a truthful strategy profile is periodic ex post equilibrium. These history-dependent transfers work as follows. In each period t , the transfer p_t^i for agent i does not depend on her current report r_t^i , so agent i 's incentive in period t is unaffected by p_t^i . Instead, her transfer in the next period p_{t+1}^i depends on r_t^i and a_t , which means that a truth-telling incentive in period t is provided through p_{t+1}^i . Under the truth-telling strategy profile, in period $t + 1$ agent i receives the sum of period- t flow payoffs of all other agents, so agent i 's continuation payoff in period t is equal to the social surplus from period t onward. Furthermore, the transfer for agent i in period $t + 1$ is such that there will be no expected gain from lying in period t . Therefore, agent i has no incentive to deviate from truth-telling in period t .

The above argument also suggests the necessity of a boundary condition for the incentive problem in the last period (when T is finite). Since the allocation problem in period T is static and there is no available information afterward, [Assumption 4](#) is needed.²⁷

The next result shows that under a slightly stronger condition on the transition probabilities, the dynamic efficient allocations are incentive compatible with a sequence of "VCG-type" transfers for each agent in the sense that each agent's report in each period affects her payoff only through the determination of allocation.

THEOREM 3.2. *Under [Assumptions 1, 3, and 4](#), there exists a sequence of transfers*

$$\bar{p}_{t+1}^i : \Theta_{t+1}^{-i} \times A_t \times \Theta_t^{-i} \rightarrow \mathbb{R} \quad \forall i, t < T,$$

such that the efficient dynamic mechanism $\{a_t^, \bar{p}_t\}$ is periodic ex post incentive compatible.*

The efficient mechanism in [Theorem 3.2](#) shares another distinctive feature of the VCG mechanism: each agent's report affects her own transfers only through the impact on allocations. The intuition in this case is even simpler. The transfer \bar{p}_t^i for agent i does not depend on θ_t^i or θ_{t-1}^i . Instead, an incentive for truth-telling in period t is again guaranteed through \bar{p}_{t+1}^i : under \bar{p}_{t+1}^i , agent i 's continuation payoff in period t is equal to the social surplus from period t onward.

²⁷Bayesian incentive compatibility of a_t^* is not enough for our result to hold, as agents have the opportunity to manipulate the designer's period- T belief by misreporting in period $T - 1$.

In the above two theorems, there seems to be a gap between the infinite- and the finite-horizon cases, as the positive result in the latter requires more conditions than that in the former. However, the next corollary builds a connection between these cases: in the finite horizon case, by replacing the efficient allocation in the last period with a constant allocation (which is ex post incentive compatible but inefficient), efficiency can be achieved in all but the last period; moreover, as the time horizon grows to infinity, the inefficiency in the last period vanishes in the limit. The proof follows directly from that of [Theorem 3.1](#).

COROLLARY 3.3. *In the finite-horizon case ($T < \infty$), under Assumptions 1 and 2, there exists a sequence of transfers*

$$\bar{p}_{t+1}^i : \Theta_{t+1}^{-i} \times A_t \times \Theta_t^{-i} \rightarrow \mathbb{R} \quad \forall i, t < T,$$

such that the (almost efficient) dynamic mechanism $\{(a_t^*, \bar{p}_t)_{t < T}, \bar{a}_T\}$, where, for all θ_T , $\bar{a}_T(\theta_T) \equiv \bar{a}$ for some $\bar{a} \in A_T$, is periodic ex post incentive compatible.

REMARK 3.4. If $|\Theta_t^i| \leq |\Theta_{t+1}^{-i}|$ for each i and t , then Assumptions 2 and 3 are generically satisfied in the finite-horizon case even if in each period signals are independently distributed conditional on all the available information.²⁸ Accordingly, efficient dynamic mechanisms exist in a large class of dynamic environments provided that ex post incentive compatibility is achievable in the last period ([Assumption 4](#)). Moreover, if the time horizon is infinite, then [Assumption 4](#) has no bite. Therefore, instead of creating difficulties for efficient mechanisms as one would imagine, repeated interactions, in fact, facilitate the construction of incentive compatible transfers.

We also note that both Assumptions 2 and 3 rule out certain information environments that are relevant in applications. For instance, in the drilling example in [Section 3.1](#), both assumptions fail if a firm’s signal consists of a common value component (about the amount of oil) that is correlated across auctions and a firm-specific private cost component that is independently distributed. Nevertheless, if these two components are additively separable in a firm’s valuation, then we can have an efficient mechanism that merges the dynamic mechanisms constructed above and the dynamic VCG mechanism for the private valuations.²⁹

REMARK 3.5. We have considered sufficient conditions for the existence of history-dependent transfers that implement the efficient allocation. There exist other weaker conditions on the transition probabilities. For example, each agent i ’s period- t signal θ_t^i could be correlated with all future signals θ_s^{-i} ($s > t$) of other agents. Formally, for each i, t , and θ_t^{-i} , there exists $s > t$ such that for any sequence $(a_t, a_{t+1}, \dots, a_{s-1}) \in \prod_{\tau=t}^{s-1} A_\tau$, there does not exist a θ_t^i and a collection of real numbers $\{\xi^i(\tilde{\theta}_t^i)\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}$ such that

- (i) $\xi^i(\tilde{\theta}_t^i) \geq 0$ for all $\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}$ and

²⁸These two assumptions are also generic in the infinite-horizon case if the transition probabilities are stationary.

²⁹I thank an anonymous referee for suggesting this discussion.

(ii) $\mu_s^{-i}(\theta_s^{-i}|a_t, a_{t+1}, \dots, a_{s-1}, \theta_t) = \sum_{\tilde{\theta}_t^i \neq \theta_t^i} \xi^i(\tilde{\theta}_t^i) \mu_s^{-i}(\theta_s^{-i}|a_t, a_{t+1}, \dots, a_{s-1}, \tilde{\theta}_t^i, \theta_t^{-i})$ for all $\theta_s^{-i} \in \Theta_s^{-i}$,

where $\mu_s^{-i}(\theta_s^{-i}|a_t, a_{t+1}, \dots, a_{s-1}, \theta_t)$ is the conditional probability distribution of θ_s^{-i} , given θ_t and $a_t, a_{t+1}, \dots, a_{s-1}$, i.e.,

$$\begin{aligned} & \mu_s^{-i}(\theta_s^{-i}|a_t, a_{t+1}, \dots, a_{s-1}, \theta_t) \\ &= \sum_{\tilde{\theta}_t^i} \sum_{\theta_{t+1}, \dots, \theta_{s-1}} \mu_{t+1}(\theta_{t+1}|a_t, \theta_t) \cdots \mu_{s-2}(\theta_{s-2}|a_{s-2}, \theta_{s-2}) \mu_s(\tilde{\theta}_t^i, \theta_s^{-i}|a_{s-1}, \theta_{s-1}). \end{aligned}$$

If so, agent i 's truth-telling incentive in each period could be provided through all future reports of other agents. An alternative sufficient condition, which shares some similarities of Mezzetti's two-stage VCG mechanism and guarantees the construction of our VCG-type dynamic mechanism with history-dependent transfers, is that each agent's period- $t + 1$ signal generates an unbiased prediction of his realized utility in period t , i.e., for each i and t , there exists a function $b_{t+1}^i: \Theta_{t+1}^i \times \Theta_t^i \times \mathcal{A}_t \rightarrow \mathbb{R}$ such that

$$u^i(a_t, \theta_t) = \sum_{\theta_{t+1}^i} \mu_t^i(\theta_{t+1}^i|a_t, \theta_t) b_{t+1}^i(\theta_{t+1}^i, \theta_t^i, a_t).$$

A common feature in the above sufficient conditions is that the transition probabilities involve conditioning on *all* agents' private information in period t ; this is the critical condition for periodic ex post incentive compatibility.

REMARK 3.6. By replacing the sequence of efficient allocations with an arbitrary sequence of allocation functions, it can be shown straightforwardly that in the infinite-horizon case, under the intertemporal correlation assumption, *any* dynamic allocation is *periodic ex post* incentive compatible. Thus our possibility results for efficient design should be taken under the same caveat as the Crémer–McLean mechanism: the results are somewhat unrealistic and may suggest some limitations of the mechanism design theory. In this regard, our results could also be interpreted as stronger negative results in dynamic mechanism design: enough intertemporal correlation of different agents' information solves agents' incentive problems in a robust way.³⁰ Similar to the Crémer–McLean mechanism, our mechanisms rely on the assumptions that (i) the transition probabilities are common knowledge, (ii) there is no competition on the designer's side, and (iii) agents are risk-neutral, have unlimited liability, and cannot collude or default at the ex post stage in each period. Whether these assumptions are reasonable in dynamic environments depends on the particular applications. Nonetheless, our results point toward an important channel, namely intertemporal correlation of private information, through which the designer can fully exploit the benefits from long-term interactions among agents.

³⁰Note that periodic ex post incentive compatibility is weaker than ex post incentive compatibility. Thus our results do not contradict the negative result in Jehiel et al. (2006).

4. INDIRECT IMPLEMENTATION WITH AUCTIONS: AN EXAMPLE

In the previous section, we focused on direct dynamic mechanisms to address feasibility issues: the existence of efficient dynamic mechanisms that are periodic ex post incentive compatible. A natural question is whether there are indirect mechanisms, such as auctions, that implement the direct mechanisms. One difficulty of this is that the history-dependent transfers in our direct mechanisms are complex in general. Nevertheless, the VCG aspect of the direct mechanisms suggests a natural way for indirect implementation: static auctions combined with contingent transfers.

Here we present a repeated allocation problem in which no static auction format is efficient, but history-dependent transfers facilitate implementing our efficient direct mechanisms with familiar auction formats. In every period $t = 1, 2, \dots, \infty$, an indivisible object is to be allocated to a bidder $i \in \{1, 2, \dots, N\}$. The allocation $a_t \in \{1, 2, \dots, N\}$ determines which bidder gets the object in period t . We assume that bidder i 's valuation of the object in period t is symmetric and given by

$$v^i(\theta_t) = \theta_t^i + \gamma \sum_{j \neq i} \theta_t^j,$$

where $\gamma > 0$ is a measure of interdependence in valuations. We also assume that the allocation does not affect the evolution of agents' private information. This implies that it is efficient to allocate each object to the agent (with an arbitrary tie-breaking rule) whose valuation of the object is the highest. Finally, we assume that for each i, t , and θ_t , there exists a map $\eta^{-i} : \Theta_{t+1}^{-i} \rightarrow \mathbb{R}$ such that

$$\frac{1}{N} \sum_j \theta_t^j = \sum_{\theta_{t+1}^{-i}} \eta^{-i}(\theta_{t+1}^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | \theta_t). \tag{2}$$

Condition (2), which is stronger than Assumption 3, states that the average of all bidders' private signals today is an unbiased estimation of an index that aggregates all but one bidder's signal tomorrow. For instance, this condition holds when there is an unobserved state of the world ω_t that is a martingale process and agents' signals are identically distributed with marginal distribution $\mu_t(\theta_t^i | \omega_t)$ such that $\sum_{\theta_t^i \in \Theta_t^i} \theta_t^i \mu_t(\theta_t^i | \omega_t) = \omega_t$ and $\sum_i \theta_t^i / N = \omega_t$. In this case, we have

$$\eta^{-i}(\theta_{t+1}^{-i}) = \frac{1}{N-1} \sum_{j \neq i} \theta_{t+1}^j.$$

First note that when $\gamma \in (0, 1]$, the standard single-crossing condition on valuations is satisfied; this ensures that the symmetric equilibrium of a repeated sealed-bid second-price auction is efficient.³¹ Alternatively, when $\gamma > 1$, it is well known that no standard auction format is efficient.³² Applying the insight from the direct mechanisms

³¹The generalized VCG mechanism is also periodic ex post incentive compatible when $\gamma \in (0, 1]$.

³²Similarly, there is no efficient and periodic ex post incentive compatible static mechanism.

with history-dependent transfers, we consider the following dynamic winner-pay auction format: Step 1. Bidder i submits a sealed bid $b_t^i \in \mathbb{R}$ in period t . Step 2. The object is then allocated to the bidder who submitted the lowest bid (with an arbitrary tie-breaking rule),

$$a_t(b_t^1, \dots, b_t^N) = \min\{i \in \{1, \dots, N\} : b_t^i \leq b_t^j, \forall j \neq i\}.$$

Step 3. The winner in period t pays the second lowest bid in this period; other bidders does not pay. Step 4. The winner also pays a contingent transfer in period $t + 1$ that depends on all other bidders' bids in both period t and $t + 1$. Formally, if bidder i wins in period t , he pays $b_t^j = \min\{b_t^k : k \neq i\}$ in period t and r_{t+1}^i in period $t + 1$, which is given by

$$r_{t+1}^i(b_{t+1}^{-i}, b_t^{-i}) = \frac{N\gamma}{\delta} \left[\eta^{-i} \left(\frac{b_{t+1}^{-i}}{1 + \gamma(N-1)} \right) - \frac{b_t^j}{1 + \gamma(N-1)} \right].$$

It is straightforward to verify that a symmetric and monotone equilibrium in the constructed auction is, for all i and t , $b_t^i(\theta_t^i) = (1 + \gamma(N-1))\theta_t^i$. Moreover, this symmetric strategy profile remains an equilibrium of the dynamic auction irrespective of the bids or winners that the auctioneer may choose to disclose to some bidders.

REMARK 4.1. In the above implementation result, we have assumed symmetry in bidder's valuations so as to obtain a symmetric equilibrium. The logic extends to the asymmetric valuation case, although there is no symmetric equilibrium. For instance, in the example in Section 3.1, the single-crossing condition is violated in the first auction; consequently, to have an efficient equilibrium, firm A pays an amount that is independent of its bid in the first auction (in this case, it is zero since only firm A needs to submit a nontrivial bid), and the incentive to follow the equilibrium strategy is provided from the contingent bonuses based on firm B 's bid in the second period.

5. INFINITE SIGNAL SPACES

In this section, we study the case where agents' signal spaces are infinite and focus on the infinite-horizon setting ($T = \infty$). We first identify conditions on the transition probabilities under which there exist mechanisms that are approximately periodic ex post incentive compatible, thereby establishing infinite-signal versions of Theorems 3.1 and 3.2 under a weaker solution concept. We then show that under stronger conditions there are mechanisms that are periodic ex post incentive compatible.

Suppose for each i and t , Θ_t^i is the unit interval $[0, 1]$ endowed with the Borel sigma algebra, $A_t = A$, where A is a finite set, and $u^i(a_t, \cdot)$ is continuous in θ_t for each $a_t \in A$.³³ In addition, we assume that the transition probability $\mu(\theta_{t+1}|a_t, \theta_t)$ is stationary (independent of t) and has a continuous density representation $f(\theta_{t+1}|a_t, \theta_t)$. The marginal density on Θ_{t+1}^{-i} is denoted by $f^{-i}(\theta_{t+1}^{-i}|a_t, \theta_t)$.

³³The results in this section hold when each Θ_t^i is a compact and convex subset of an Euclidean space.

5.1 *Approximate periodic ex post incentive compatibility*

First consider a weakening of periodic ex post equilibrium, which requires that after any history, truth-telling is “almost” a best response if all other agents report truthfully. Formally, for any $\varepsilon > 0$, we say that the mechanism $\{a_t^*, p_t\}_{t \geq 1}$ is ε -periodic ex post incentive compatible if for each t, i, h_t^i , and θ_t^i ,

$$u^i(a_t^*(\theta_t^i, \theta_t^{-i}), \theta_t) - p_t^i(h_t, \theta_t^i, \theta_t^{-i}) + \delta \mathbb{E}[V^i(h_{t+1}^i) | a_t^*(\theta_t^i, \theta_t^{-i}), \theta_t] \geq u^i(a_t^*(r_t^i, \theta_t^{-i}), \theta_t) - p_t^i(h_t, r_t^i, \theta_t^{-i}) + \delta \mathbb{E}[V^i(h_{t+1}^i) | a_t^*(r_t^i, \theta_t^{-i}), \theta_t] - \varepsilon$$

for any $r_t^i \in \Theta_t^i$, where $V^i(h_{t+1}^i)$ is the continuation payoff of agent i if all agent report truthfully from period $t + 1$ onward. The condition implies that after any history, any one-shot deviation from truth-telling would yield an agent at most ε improvement in his continuation payoff. Note that because of discounting, if a mechanism is ε -periodic ex post incentive compatible, then truth-telling consists of a (contemporaneous) $\varepsilon(1 - \delta)^{-1}$ -perfect ex post equilibrium.

In the following two lemmas, we identify conditions on the transition densities $f^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t)$ such that for every $\varepsilon > 0$, there exist transfer schedules p_t that are ε -periodic ex post incentive compatible.

LEMMA 5.1. *Fix any i, t, a_t , and θ_t^{-i} . If for every θ_t^i and every $\mu^i \in \Delta(\Theta_t^i)$,*

$$f^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) = \int_{\Theta_t^i} f^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) \mu^i(d\tilde{\theta}_t^i) \Rightarrow \mu^i(\{\theta_t^i\}) = 1, \tag{3}$$

then for any $\varepsilon > 0$, there exist transfers that are $p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i})$ measurable in θ_t^i and continuous in θ_{t+1}^{-i} and θ_t^{-i} such that

$$\max_{\theta_t^i \in \Theta_t^i} \left| - \sum_{j \neq i} u^j(a_t, \theta_t) - \delta \int_{\Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) f^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t) d\theta_{t+1}^{-i} \right| \leq \varepsilon \tag{4}$$

and

$$\begin{aligned} & \int_{\Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) f^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t) d\theta_{t+1}^{-i} \\ & \leq \int_{\Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i}) f^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t) d\theta_{t+1}^{-i} \end{aligned} \tag{5}$$

for any $r_t^i \in \Theta_t^i$.

LEMMA 5.2. *Fix any i, t, a_t , and θ_t^{-i} . If there does not exist a nonzero signed measure η^i on the Borel subsets of Θ_t^i such that*

$$\int_{\Theta_t^i} f^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) \eta^i(d\tilde{\theta}_t^i) = 0, \tag{6}$$

then for any $\varepsilon > 0$, there exist continuous transfers $p_{t+1}^i(\theta_{t+1}^{-i}; a_t, \theta_t^{-i})$ such that

$$\max_{\theta_t^i \in \Theta_t^i} \left| -\sum_{j \neq i} w^j(a_t, \theta_t) - \delta \int_{\Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}; a_t, \theta_t^{-i}) f^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t) d\theta_{t+1}^{-i} \right| \leq \varepsilon. \quad (7)$$

The proofs of the results in this section are relegated to [Appendix B](#). Condition (3) in [Lemma 5.1](#) is a direct extension of [McAfee and Reny \(1992\)](#) to the dynamic case. Following the proof of [Theorem 3.1](#), it implies that there are ε -periodic ex post incentive compatible transfers of the form $p_{t+1}^i : \Theta_{t+1}^{-i} \times \Theta_t^i \times A_t \times \Theta_t^{-i} \rightarrow \mathbb{R}$. The spanning condition in [Lemma 5.2](#) is new. It guarantees the existence of ε -periodic ex post incentive compatible transfers of the form $p_{t+1}^i : \Theta_{t+1}^{-i} \times A_t \times \Theta_t^{-i} \rightarrow \mathbb{R}$. Similar to the mechanisms presented in [Section 3](#), each agent is *almost* a residual claimant and, hence, never gains by more than ε from misreporting in any period. Finally, we note that without further restrictions on the utility functions and transition probabilities, solutions to either the inequality system (4) and (5) or (7) may not exist for $\varepsilon = 0$.³⁴ In other words, in general it is unlikely to achieve 0-periodic ex post incentive compatibility with contingent transfers considered in [Lemmas 5.1](#) and [5.2](#). Intuitively, there may not be enough variation of θ_{t+1}^{-i} with respect to θ_t^i in the density $f^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t)$ to account for the variation of θ_t^i in $-\sum_{j \neq i} h^j(a_t, \theta_t)$. However, our results show that under either condition (3) or (6), the sets of expected values of all these contingent transfers are dense in the set of possible utility functions, which delivers ε -periodic ex post incentive compatibility.

5.2 Periodic ex post incentive compatibility

The lemmas in [Section 5.1](#) generalize the main results in [Section 3](#). However, they are not very satisfactory, especially in the dynamic environments. That is, agents may well deviate from truth-telling under ε -periodic ex post incentive compatibility, yet they evaluate their continuation payoffs assuming others are always truthful. In this section, we strengthen the results to (full) periodic ex post incentive compatibility under stronger correlation conditions.

Note that the contingent transfers that deliver ε -periodic ex post incentive compatibility in [Section 5.1](#) depend on the reports one period ahead, whereas in principle they could depend on reports in the more distant future (see [Remark 3.5](#)). Therefore, we consider the contingent transfers

$$p_t^i : \Theta_t^i \times \Theta_t^{-i} \times A_t \times \prod_{\tau > t} (\Theta_\tau^{-i} \times A_\tau) \rightarrow \mathbb{R}.$$

Intuitively, if agent i 's current private signal θ_t^i is correlated with other agents' future signals $\{\theta_\tau^{-i}\}_{\tau > t}$, then provided that other agents always report truthfully, it is possible to use the entire sequence, $\{\theta_\tau^{-i}\}_{\tau > t}$, to provide incentive for agent i to report θ_t^i truthfully. To put it differently, we might fill the gap in ε -incentive compatibility with an infinite sequence of correlated signals. We formalize this intuition in the next two propositions.

³⁴For instance, when $\varepsilon = 0$, (7) reduces to a Fredholm integral equation of the first kind, which may not have solutions.

For each i, t , and $\tau > t$, let $f_\tau^{-i}(\theta_\tau^{-i}|a_t, \dots, a_{\tau-1}, \theta_t)$ denote the marginal density on Θ_τ^{-i} given any a_t, \dots, a_τ and θ_t .

PROPOSITION 5.3. *Fix any i, t , and θ_t^{-i} . If for every $\tau > t$, $(a_t, \dots, a_\tau) \in A_t \times \dots \times A_\tau$, $\theta_t^i \in \Theta_t^i$, and $\mu_\tau^i \in \Delta(\Theta_t^i)$,*

$$f_\tau^{-i}(\cdot|a_t, \dots, a_{\tau-1}, \theta_t^i, \theta_t^{-i}) = \int_{\Theta_t^i} f_\tau^{-i}(\cdot|a_t, \dots, a_{\tau-1}, \tilde{\theta}_t^i, \theta_t^{-i}) \mu^i(d\tilde{\theta}_t^i)$$

$$\Rightarrow \mu_\tau^i(\{\theta_t^i\}) = 1,$$

then there exists a sequence of transfers $(p_\tau^i(\theta_\tau^{-i}, \theta_t^i; a_t, \dots, a_{\tau-1}, \theta_t^{-i}))_{\tau>t}$ measurable in θ_t^i and continuous in θ_τ^{-i} and θ_t^{-i} such that

$$-\sum_{j \neq i} u^j(a_t, \theta_t) = \sum_{\tau=t+1}^\infty \delta^{\tau-t} \int_{\Theta_\tau^{-i}} p_\tau^i(\theta_\tau^{-i}, \theta_t^i; a_t, \dots, a_{\tau-1}, \theta_t^{-i}) f_\tau^{-i}(\theta_\tau^{-i}|a_t, \dots, a_{\tau-1}, \theta_t) d\theta_\tau^{-i}$$

and

$$\int_{\Theta_\tau^{-i}} p_\tau^i(\theta_\tau^{-i}, \theta_t^i; a_t, \dots, a_{\tau-1}, \theta_t^{-i}) f_\tau^{-i}(\theta_\tau^{-i}|a_t, \dots, a_{\tau-1}, \theta_t) d\theta_\tau^{-i}$$

$$\leq \int_{\Theta_\tau^{-i}} p_\tau^i(\theta_\tau^{-i}, r_t^i; a_t, \dots, a_{\tau-1}, \theta_t^{-i}) f_\tau^{-i}(\theta_\tau^{-i}|a_t, \dots, a_{\tau-1}, \theta_t) d\theta_\tau^{-i}$$

for any $r_t^i \in \Theta_t^i$ and $\tau > t$.

PROPOSITION 5.4. *Fix any i, t , and θ_t^{-i} . If for every $\tau > t$, $(a_t, \dots, a_\tau) \in A_t \times \dots \times A_\tau$, there does not exist a nonzero signed measure η_τ^i on the Borel subsets of Θ_t^i such that*

$$\int_{\Theta_t^i} f_\tau^{-i}(\cdot|a_t, \dots, a_{\tau-1}, \tilde{\theta}_t^i, \theta_t^{-i}) \eta_\tau^i(d\tilde{\theta}_t^i) = 0,$$

then there exists a sequence of transfers $(p_\tau^i(\theta_\tau^{-i}, \theta_t^i; a_t, \dots, a_{\tau-1}, \theta_t^{-i}))_{\tau>t}$ measurable in θ_t^i and continuous in θ_τ^{-i} and θ_t^{-i} such that

$$-\sum_{j \neq i} u^j(a_t, \theta_t)$$

$$= \sum_{\tau=t+1}^\infty \delta^{\tau-t} \int_{\Theta_\tau^{-i}} p_\tau^i(\theta_\tau^{-i}, \theta_t^i; a_t, \dots, a_{\tau-1}, \theta_t^{-i}) f_\tau^{-i}(\theta_\tau^{-i}|a_t, \dots, a_{\tau-1}, \theta_t) d\theta_\tau^{-i}.$$

Propositions 5.3 and 5.4 imply that there are contingent transfers under which an agent becomes a residual claimant as in the VCG mechanism when her current signal is correlated with others' signals in the entire future. To provide an intuition of the results, first note that the convex independence condition in Lemma 5.1 implies that the closure of the set of functions generated by all one-period-ahead contingent transfers equals the set of all continuous functions on the unit interval. Therefore, for any $g \in C[0, 1]$ and

$\varepsilon > 0$, there is an infinite sequence of continuous functions $\{h_n\}_{n=1}^\infty$ such that for each n , there exists a measurable function $p_n(s, t)$ with $h_n(s) = \int_T p_n(s, t) f(t|s) dt$, and

$$\sup_{s \in [0,1]} \left| g(s) - \sum_{m=1}^n h_m(s) \right| \leq \frac{\varepsilon}{2^n}.$$

Since g is bounded, the infinite sum $\sum_{n=1}^\infty h_n$ is well defined and equals g . Hence, for any fixed sequence of allocations and in any period, we can find a sequence of contingent transfers, which are used to provide incentives for agents to report truthfully in that period. One subtle difference between our construction and Crémer and McLean's mechanism is that we use the assumption that for any given allocation a_t , the utility functions are continuous in agents' signals, whereas in Crémer and McLean's mechanism, agents get zero payoff if being truthful and get negative payoff if lying.

6. CONCLUDING REMARKS

Dynamic mechanism design features a richer family of history-dependent transfers compared with the static counterpart. This paper has taken a first step toward understanding the implications of such richness on efficient implementations in general environments with interdependent valuations. In particular, we have shown how intertemporal correlation of private information leads to contingent transfers that resemble dynamic VCG mechanisms. We also emphasize that while the theoretical possibility results in this paper serve as a benchmark for the design of efficient mechanisms, the practicality of contingent transfers may vary with specific economic problems.

We conclude by noting that the model can be extended to accommodate the possibility of arrival and departure of potential agents. In particular, the intertemporal correlation condition can be generalized straightforwardly to this case. Several new issues need to be addressed. First, with interdependent valuations, agents' arrival and departure would change both the information structure and the utility functions, since each active agent holds information that directly affects other agents' payoffs. Second, agents are required to make contingent transfers in the dynamic mechanisms. Thus, transfers to an agent may occur even if she is no longer active. This may be problematic in some situations where monetary transfers have to be made along with the physical allocations. Third, the arrival (or departure) times may also be agents' private information.³⁵ Moreover, there may be uncertainty in arrival (or departure) rates, which further complicates the incentive compatibility constraints.

APPENDIX A: PROOFS OF THE RESULTS IN SECTION 3

Theorems 3.1 and 3.2 consider both the infinite-horizon and the finite-horizon cases. We prove Theorem 3.1 for the infinite-horizon case, using the one-shot deviation principle. Then we prove Theorem 3.2 for the finite horizon case, using backward induction. The proofs of the other two cases (the finite-horizon case in Theorem 3.1 and the

³⁵See Gershkov et al. (2015) and Mierendorff (2016) for examples.

infinite-horizon case in [Theorem 3.2](#)) follow similar lines and, therefore, are relegated to the Supplemental Material.

PROOF OF THEOREM 3.1. Here we prove the infinite-horizon case; the proof for finite horizon case is given in Section S5.1 of the Supplemental Material. The proof consists of three lemmas.

LEMMA A.1. *Suppose that Assumptions 1 and 2 hold. For each i and t , there exists a transfer function $p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i})$ such that, for each a_t and θ_t^{-i} , the following two conditions are satisfied:*

(i) For each θ_t^i ,

$$-\sum_{j \neq i} u^j(a_t, \theta_t^i, \theta_t^{-i}) = \delta \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}).$$

(ii) For each θ_t^i and r_t^i ,

$$\begin{aligned} & \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) \\ & \leq \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}), \end{aligned}$$

where $\mu_{t+1}^{-i}(\cdot | a_t, \theta_t)$ is the marginal of $\mu_{t+1}(\cdot | a_t, \theta_t)$ on Θ_{t+1}^{-i} .

PROOF. First note that the first part of [Assumption 2](#) is equivalent to the following condition: for each i, t, a_t , and θ_t^{-i} , and for each $\pi_i : \Theta_t^i \rightarrow \Delta(\Theta_t^i)$,

$$\begin{aligned} & \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) \quad \forall \theta_t^i \\ & \Rightarrow \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i) = 1 \quad \forall \theta_t^i. \end{aligned} \tag{8}$$

To see this, suppose [Assumption 2](#) holds. If $\pi : \Theta_t^i \rightarrow \Delta(\Theta_t^i)$ satisfies

$$\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}),$$

then $\pi_i(\theta_t^i | \theta_t^i) = 1$ and $\pi_i(\tilde{\theta}_t^i | \theta_t^i) = 0$ for all θ_t^i and $\tilde{\theta}_t^i \neq \theta_t^i$. Therefore, $\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i) = 1$. Conversely, suppose condition (9) holds but the first part of [Assumption 2](#) is violated. That is, there exists a θ_t^i and a collection of nonnegative numbers $\{\xi^i(\tilde{\theta}_t^i)\}_{\tilde{\theta}_t^i \neq \theta_t^i}$ such that

$$\mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t) = \sum_{\tilde{\theta}_t^i \neq \theta_t^i} \xi^i(\tilde{\theta}_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i})$$

for all $\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}$. Let $\pi_i(\theta_t^i | \theta_t^i) = 1/2$ and $\pi_i(\tilde{\theta}_t^i | \theta_t^i) = \xi^i(\tilde{\theta}_t^i)/2$ for all $\tilde{\theta}_t^i \neq \theta_t^i$. Moreover, for all $\tilde{\theta}_t^i \neq \theta_t^i$, let $\pi_i(\tilde{\theta}_t^i | \tilde{\theta}_t^i) = 1$ and $\pi_i(\theta_t^i | \tilde{\theta}_t^i) = 0$. Then $\pi : \Theta_t^i \rightarrow \Delta(\Theta_t^i)$ satisfies

$$\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i})$$

and $\sum_{\tilde{\theta}_t^i} \pi(\theta_t^i | \tilde{\theta}_t^i) = 1/2$, a contradiction.

Then we show that under **Assumption 1**, the above condition is equivalent to the existence of a transfer $p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i})$ satisfying the two requirements in the lemma.³⁶

We first show that the two requirements in the lemma are equivalent to the following relaxed condition.

CONDITION 1. There exists a transfer $p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i})$ such that

(a) for each θ_t^i ,

$$-\sum_{j \neq i} u^j(a_t, \theta_t^i, \theta_t^{-i}) \geq \delta \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i});$$

(b) for each θ_t^i and r_t^i ,

$$\begin{aligned} & \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) \\ & \leq \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}). \end{aligned}$$

To see this, suppose **Condition 1** holds with $p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i})$. Let

$$\begin{aligned} k(a_t, \theta_t^i, \theta_t^{-i}) &= -\sum_{j \neq i} u^j(a_t, \theta_t^i, \theta_t^{-i}) \\ & - \delta \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) \geq 0. \end{aligned}$$

Then $\hat{p}_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i})$, defined by

$$\hat{p}_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \equiv p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) + \frac{1}{\delta} k(a_t, \theta_t^i, \theta_t^{-i}),$$

satisfies the two requirements in the lemma.

For each i, t, a_t , and θ_t^{-i} , and for each $u^i(a_t, \theta_t^i)$ satisfying **Assumption 1**, by the theorem of alternatives (see **Rockafellar 1970, Section 22, Theorem 22.1**), either **Condition 1** holds or the following condition (**Condition 2**) holds, but not both:

³⁶The technique of constructing transfers from the theorem of alternatives first appears in **Kandori and Matsushima (1998)**. The proof here follows closely the argument in **Rahman (2010)**.

CONDITION 2. There exists $\eta : \Theta_t^i \rightarrow \mathbb{R}$ and $\lambda : \Theta_t^i \times \Theta_t^i \rightarrow \mathbb{R}_+$, such that for each $\theta_t^i, \theta_{t+1}^{-i}$ and θ_{t+1}^{-i} ,

$$\begin{aligned} & \eta(\theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) \\ &= \sum_{\tilde{\theta}_t^i \in \Theta_t^i} [\lambda(\tilde{\theta}_t^i, \theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) - \lambda(\theta_t^i, \tilde{\theta}_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i})] \end{aligned}$$

and $-\eta(\theta_t^i) \sum_{j \neq i} u^j(a_t, \theta_t^i, \theta_t^{-i}) > 0$.

Therefore, Condition 1 holds if and only if for each $\eta : \Theta_t^i \rightarrow \mathbb{R}$ and $\lambda : \Theta_t^i \times \Theta_t^i \rightarrow \mathbb{R}_+$ that satisfy

$$\begin{aligned} & \eta(\theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}) \\ &= \sum_{\tilde{\theta}_t^i \in \Theta_t^i} [\lambda(\tilde{\theta}_t^i, \theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) - \lambda(\theta_t^i, \tilde{\theta}_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i})] \end{aligned} \tag{9}$$

for all θ_t^i and θ_{t+1}^{-i} , we must have $\eta(\theta_t^i) \equiv 0$.

Now we show that condition (9) is equivalent to condition (8). Suppose first that for each pair (η, λ) with $\lambda \geq 0$, if for each θ_t^i ,

$$\begin{aligned} & \eta(\theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) \\ &= \sum_{\tilde{\theta}_t^i \in \Theta_t^i} [\lambda(\tilde{\theta}_t^i, \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) - \lambda(\theta_t^i, \tilde{\theta}_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i})], \end{aligned} \tag{10}$$

then $\eta(\theta_t^i) \equiv 0$. Fix any $\pi_i : \Theta_t^i \rightarrow \Delta(\Theta_t^i)$ satisfying, for each θ_t^i ,

$$\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}). \tag{11}$$

We want to show that $\pi_i(\theta_t^i | \theta_t^i) = 1$ for each θ_t^i . Note that (11) implies

$$\mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) - \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = 0. \tag{12}$$

Define $\eta(\theta_t^i) = 1 - \sum_{\tilde{\theta}_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i)$. Then condition (12) is equivalent to

$$\left[\eta(\theta_t^i) + \sum_{\tilde{\theta}_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i) \right] \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) - \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = 0.$$

Since π_i is nonnegative by definition, it follows from condition (10) that for each θ_t^i , $\eta(\theta_t^i) = 0$ or, equivalently, $\sum_{\tilde{\theta}_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i) = 1$, which establishes condition (8).

Conversely, suppose that for each $\pi_i : \Theta_t^i \rightarrow \Delta(\Theta_t^i)$, if

$$\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\tilde{\theta}_t^i | \theta_t^i) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) = \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i}), \tag{13}$$

then for each θ_t^i , $\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i) = 1$. Fix any pair (η, λ) satisfying $\lambda \geq 0$ and for each θ_t^i ,

$$\begin{aligned} & \eta(\theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) \\ &= \sum_{\tilde{\theta}_t^i \in \Theta_t^i} [\lambda(\tilde{\theta}_t^i, \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) - \lambda(\theta_t^i, \tilde{\theta}_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i})]. \end{aligned} \tag{14}$$

We want to show that $\eta(\theta_t^i) \equiv 0$. Condition (14) implies

$$\left[\eta(\theta_t^i) + \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\theta_t^i, \tilde{\theta}_t^i) \right] \mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) = \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\tilde{\theta}_t^i, \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i}) \tag{15}$$

and

$$\eta(\theta_t^i) = \sum_{\tilde{\theta}_t^i \in \Theta_t^i} [\lambda(\tilde{\theta}_t^i, \theta_t^i) - \lambda(\theta_t^i, \tilde{\theta}_t^i)], \tag{16}$$

where condition (16) follows from integration over θ_{t+1}^{-i} of condition (15). Therefore, we have

$$\eta(\theta_t^i) + \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\theta_t^i, \tilde{\theta}_t^i) = \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\tilde{\theta}_t^i, \theta_t^i). \tag{17}$$

Note that $\lambda(\theta_t^i, \tilde{\theta}_t^i) > 0$ can be chosen arbitrarily without affecting condition (14). Therefore, conditions (15) and (17) imply that

$$\mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) = \frac{\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\tilde{\theta}_t^i, \theta_t^i) \mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i})}{\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\tilde{\theta}_t^i, \theta_t^i)}.$$

Moreover, we can set $\lambda(\theta_t^i, \tilde{\theta}_t^i) > 0$ such that for each θ_t^i , $\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\tilde{\theta}_t^i, \theta_t^i) = C$, where C is a positive constant. For each pair θ_t^i and $\tilde{\theta}_t^i$, define $\pi_i(\tilde{\theta}_t^i | \theta_t^i) = \lambda(\tilde{\theta}_t^i, \theta_t^i) / C$. Then π_i is a mapping from Θ_t^i to $\Delta(\Theta_t^i)$. It then follows from condition (13) that $\sum_{\tilde{\theta}_t^i} \pi_i(\theta_t^i | \tilde{\theta}_t^i) = 1$ for each θ_t^i . Therefore, we have

$$\sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\theta_t^i, \tilde{\theta}_t^i) = \sum_{\tilde{\theta}_t^i \in \Theta_t^i} \lambda(\tilde{\theta}_t^i, \theta_t^i)$$

and, hence, by condition (16), $\eta(\theta_t^i) \equiv 0$. This proves condition (9). ◁

Note that Assumption 2 implies that if $T = \infty$, then there exist $\epsilon \in \mathbb{R}_+$ and $\tilde{T} \in \mathbb{N}_+$ such that for any $t \geq \tilde{T}$, and any i, a_t, θ_t^{-i} , and θ_t^i ,

$$\text{dist}_2(\mu_{t+1}^{-i}(\cdot | a_t, \theta_t^i, \theta_t^{-i}) - \text{Conv}\{\mu_{t+1}^{-i}(\cdot | a_t, \tilde{\theta}_t^i, \theta_t^{-i})\}_{\tilde{\theta}_t^i \in \Theta_t^i \setminus \{\theta_t^i\}}) \geq \epsilon,$$

where $\|\cdot\|_2$ is the Euclidean norm. The next lemma shows that since the intertemporal

correlation does not vanish as t goes to infinity, there exists an upper bound on the size of the transfers.

LEMMA A.2. *Under Assumptions 1 and 2, for each $t \geq \tilde{T}$, the transfer $p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i})$ constructed in Lemma A.1 satisfies*

$$\max |p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i})| \leq \frac{1}{\delta} \left(1 + \frac{4}{\epsilon}\right) \cdot \max_{a_t, \theta_t} \left| \sum_{j \neq i} u^j(a_t, \theta_t) \right|.$$

For the proof, see Section S5.1 in the Supplemental Material.

LEMMA A.3. *If for each i and t , there exists a transfer function $p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i})$ that satisfies the three conditions*

(i) *for each a_t, θ_t^{-i} and θ_t^i ,*

$$-\sum_{j \neq i} u^j(a_t, \theta_t^i, \theta_t^{-i}) = \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu(\theta_{t+1} | a_t, \theta_t^i, \theta_t^{-i}),$$

(ii) *for each $a_t, \theta_t^{-i}, \theta_t^i$ and r_t^i ,*

$$\begin{aligned} & \sum_{\theta_{t+1} \in \Theta_{t+1}} p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t, \theta_t^{-i}) \mu(\theta_{t+1} | a_t, \theta_t^i, \theta_t^{-i}) \\ & \leq \sum_{\theta_{t+1} \in \Theta_{t+1}} p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i}) \mu(\theta_{t+1} | a_t, \theta_t^i, \theta_t^{-i}), \end{aligned}$$

(iii) *there exists $D \in \mathbb{R}_+$ such that for any $t \geq \tilde{T}$,*

$$\max |p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t, \theta_t^{-i})| \leq \frac{1}{\delta} \left(1 + \frac{4}{\epsilon}\right) \cdot \max_{a_t, \theta_t} \left| \sum_{j \neq i} u^j(a_t, \theta_t) \right|,$$

then the dynamic efficient allocation $\{a_t^\}$ can be implemented in a periodic ex post equilibrium.*

PROOF. First note that under condition (iii) of this lemma, agent i 's discounted payoffs are always well defined under the mechanism $\{a_t^*, \{p_{t+1}^i\}_{i=1}^N\}_{t \geq 1}$. To see this, for any sequence $(a_t, \theta_t)_{t \geq 1}$, we have

$$\begin{aligned} & \sum_{t=1}^{\infty} \delta^{t-1} |u^i(a_t, \theta_t) - p_t^i(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i})| \\ & = \sum_{t=1}^{\tilde{T}} \delta^{t-1} |u^i(a_t, \theta_t) - p_t^i(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i})| \\ & \quad + \sum_{t=\tilde{T}}^{\infty} \delta^t |u^i(a_{t+1}, \theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}; a_t, \theta_t^{-i})| \end{aligned}$$

$$\begin{aligned} &\leq L^i + \sum_{t=\bar{T}}^{\infty} \delta^t \left[|u^i(a_{t+1}, \theta_{t+1})| + \frac{1}{\delta} \left(1 + \frac{4}{\epsilon} \right) \cdot \max_{a_t, \theta_t} \left| \sum_{j \neq i} u^j(a_t, \theta_t) \right| \right] \\ &\leq L^i + \frac{1}{\delta} \left(1 + \frac{4}{\epsilon} \right) \cdot \left(\sum_{j=1}^N \max_{(a_t, \theta_t)_{t \geq 1}} \sum_{t=1}^{\infty} \delta^{t-1} |u^j(a_t, \theta_t)| \right), \end{aligned}$$

where $L^i = \max_{(a_t, \theta_t)_{t=1}^{\bar{T}}} \sum_{t=1}^{\bar{T}} \delta^{t-1} |u^i(a_t, \theta_t) - p_t^i(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i})| < \infty$. That is, there is a uniform upper bound on agent i 's realized discounted payoff under transfers $\{p_t^i\}_{t \geq 1}$.

Assume all agents other than i report their signals truthfully and focus on agent i 's incentive problem. Fix a socially efficient allocation rule a_t^* . By the one-shot deviation principle, we only need to show that after any public history up to period t , agent i does not benefit from deviating to $r_t^i \neq \theta_t^i$ and $r_s^i = \theta_s^i$ for $s > t$.³⁷

If agent i reports truthfully in period t , i.e., $r_t^i = \theta_t^i$, her continuation payoff is

$$\begin{aligned} &u^i(a_t^*(\theta_t), \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\ &\quad + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t^*(\theta_t), \theta_t^{-i})] \mu(\theta_{t+1} | a_t^*(\theta_t), \theta_t) \\ &= u^i(a_t^*(\theta_t), \theta_t) + \sum_{j \neq i} u^j(a_t^*(\theta_t), \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\ &\quad + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} W(\theta_{t+1}) \mu(\theta_{t+1} | a_t^*(\theta_t), \theta_t) \\ &= W(\theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i). \end{aligned}$$

Suppose agent i deviates to a message r_t^i such that $a_t^*(r_t^i, \theta_t^{-i}) = a_t^*(\theta_t)$. Then her continuation payoff satisfies

$$\begin{aligned} &u^i(a_t^*(r_t^i, \theta_t^{-i}), \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\ &\quad + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t^*(r_t^i, \theta_t^{-i}), \theta_t^{-i})] \mu(\theta_{t+1} | a_t^*(r_t^i, \theta_t^{-i}), \theta_t) \\ &= u^i(a_t^*(\theta_t), \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\ &\quad + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t^*(\theta_t), \theta_t^{-i})] \mu(\theta_{t+1} | a_t^*(\theta_t), \theta_t) \\ &\leq u^i(a_t^*(\theta_t), \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\ &\quad + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a_t^*(\theta_t), \theta_t^{-i})] \mu(\theta_{t+1} | a_t^*(\theta_t), \theta_t) \\ &= W(\theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i), \end{aligned}$$

³⁷Under the constructed mechanism, each agent's payoff function is well defined and, hence, is continuous at infinity, which justifies the application of the one-shot deviation principle.

where the inequality follows from condition (ii) in this lemma. Thus, deviating to a message r_t^i without changing the allocation is not profitable.

Finally, if agent i deviates to a message r_t^i such that $a_t^*(r_t^i, \theta_t^{-i}) = a' \neq a_t^*(\theta_t)$, then her continuation payoff satisfies

$$\begin{aligned}
 & u^i(a_t^*(r_t^i, \theta_t^{-i}), \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\
 & + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a_t^*(r_t^i, \theta_t^{-i}), \theta_t^{-i})] \mu(\theta_{t+1} | a_t^*(r_t^i, \theta_t^{-i}), \theta_t) \\
 & = u^i(a', \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\
 & + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, r_t^i; a', \theta_t^{-i})] \mu(\theta_{t+1} | a', \theta_t) \\
 & \leq u^i(a', \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\
 & + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} [W(\theta_{t+1}) - p_{t+1}^i(\theta_{t+1}^{-i}, \theta_t^i; a', \theta_t^{-i})] \mu(\theta_{t+1} | a', \theta_t) \\
 & = u^i(a', \theta_t) + \sum_{j \neq i} u^j(a', \theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i) \\
 & + \delta \sum_{\theta_{t+1} \in \Theta_{t+1}} W(\theta_{t+1}) \mu(\theta_{t+1} | a', \theta_t) \\
 & \leq W(\theta_t) - p_t^i(\theta_t^{-i}, r_{t-1}^i; a_{t-1}, \theta_{t-1}^i),
 \end{aligned}$$

where the first inequality is by condition (ii) in this lemma, the second inequality is by condition (i) in this lemma, and the second inequality is by the definition of a_t^* . Thus, deviating to a message r_t^i , which changes the allocation, is not profitable either. Therefore, we conclude that truth-telling consists of a periodic ex post equilibrium. \square

PROOF OF THEOREM 3.2. Here we prove the finite-horizon case; the proof of the infinite-horizon case is given in Section S5.2 of the Supplemental Material. The proof consists of two lemmas.

LEMMA A.4. *Under Assumptions 1 and 3, for each i and $t < T$, there exists a transfer function $p_{t+1}^i : \Theta_{t+1}^{-i} \times A_t \times \Theta_t^{-i} \rightarrow \mathbb{R}_+$ such that*

$$- \sum_{j \neq i} u^j(a_t, \theta_t^i, \theta_t^{-i}) = \delta \sum_{\theta_{t+1}^{-i} \in \Theta_{t+1}^{-i}} p_{t+1}^i(\theta_{t+1}^{-i}; a_t, \theta_t^{-i}) \mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t) \tag{18}$$

for every a_t, θ_t^{-i} and $\theta_t^i \in \Theta_t^i$.

PROOF. Fix any a_t and θ_t^{-i} . Equality (18) is a system of linear equations. Since the transition matrix $\mu_{t+1}^{-i}(\theta_{t+1}^{-i} | a_t, \theta_t^i, \theta_t^{-i})$ from θ_t^i to θ_{t+1}^{-i} has full rank under Assumption 3, the system of equations has a solution given by

$$p_{t+1}^i(\cdot; a_t, \theta_t^{-i}) = \frac{1}{\delta} (M_{t+1}^{-i}(a_t, \theta_t^{-i}))^+ \mathbf{u}^{-i}(\cdot; a_t, \theta_t^{-i}),$$

where $\mathbf{p}_{t+1}^i(\cdot; a_t, \theta_t^{-i}) = (p_{t+1}^i(\theta_{t+1}^{-i}; a_t, \theta_t^{-i}))_{\theta_{t+1}^{-i}}$ and $\mathbf{u}^{-i}(\cdot; a_t, \theta_t^{-i}) = (-\sum_{j \neq i} u^j(a_t, \theta_t^j, \theta_t^{-i}))_{\theta_t^i}$ are column vectors. ◁

LEMMA A.5. *Under Assumptions 1, 3, and 4, there exists a sequence of transfers $\bar{p}_t : H_t \times \Theta_t \rightarrow \mathbb{R}^N$ such that the efficient dynamic mechanism $\{a_t^*, \bar{p}_t\}_{t=1}^T$ is periodic ex post incentive compatible.*

PROOF. Let $W_t(\theta_t)$ denote the expected period- t continuation social surplus given signal profile θ_t , i.e.,

$$W_t(\theta_t) = \mathbb{E} \left[\sum_{s=t}^T \delta^{s-t} \sum_{i=1}^N u^i(a_t^*(\theta_t), \theta_t) \mid \theta_t \right].$$

First consider the problem in period T . By **Assumption 4**, there exists an ex post incentive compatible transfer $p_T : \Theta_T \rightarrow \mathbb{R}^N$ that implements the efficient allocation a_T^* . Given (a_T^*, p_T) , the payoff V_T^i for each agent i in the truth-telling equilibrium is given by

$$V_T^i(\theta_T) = u^i(a_T^*(\theta_T), \theta_T) - p_T^i(\theta_T)$$

for each θ_T .

Next consider agent i 's incentive problem in period $T - 1$ with an arbitrary public history $h_{T-1} = (r_1, a_1, r_2, a_2, \dots, r_{t-1}, a_{t-1})$. Suppose that agents other than i always report truthfully. For each pair (a_{T-1}, θ_{T-1}) , define

$$\pi_{T-1}^i(a_{T-1}, \theta_{T-1}) = \sum_{j \neq i} u^j(a_{T-1}, \theta_{T-1}) + \delta \mathbb{E}[W(\theta_T) - V_T^i(\theta_T) \mid a_{T-1}, \theta_{T-1}].$$

By **Lemma A.4** there exists a function $\tilde{p}_T^i(\theta_T^{-i}; a_{T-1}, \theta_{T-1}^{-i})$ such that for every $a_{T-1}, \theta_{T-1}^{-i}$, and θ_{T-1}^i ,

$$\pi_{T-1}^i(a_{T-1}, \theta_{T-1}) = \delta \sum_{\theta_T \in \Theta_T} \tilde{p}_T^i(\theta_T^{-i}; a_{T-1}, \theta_{T-1}^{-i}) \mu_T(\theta_T \mid a_{T-1}, \theta_{T-1}).$$

Define a new period- T transfer $\bar{p}_T^i : \Theta_{T-1}^{-i} \times A_{T-1} \times \Theta_T \rightarrow \mathbb{R}$ for agent i as

$$\bar{p}_T^i(\theta_{T-1}^{-i}; a_{T-1}, \theta_T) = p_T^i(\theta_T) - \tilde{p}_T^i(\theta_T^{-i}; a_{T-1}, \theta_{T-1}^{-i}).$$

Note that \tilde{p}_T^i is independent of θ_T^i , so agent i still finds it optimal to report truthfully in period T under this new transfer \bar{p}_T^i . Suppose agent i reports r_{T-1}^i in period $T - 1$. Then for any realized signal profile θ_{T-1} , her expected continuation payoff from $T - 1$ on is equal to

$$\begin{aligned} & u^i(a_{T-1}^*(r_{T-1}^i, \theta_{T-1}^{-i}), \theta_{T-1}) + \delta \mathbb{E}[V^i(\theta_T) \mid a_{T-1}^*(r_{T-1}^i, \theta_{T-1}^{-i}), \theta_{T-1}] \\ & + \pi_{T-1}^i(a_{T-1}^*(r_{T-1}^i, \theta_{T-1}^{-i}), \theta_{T-1}) \\ & = \sum_{i=1}^N u^i(a_{T-1}^*(r_{T-1}^i, \theta_{T-1}^{-i}), \theta_{T-1}) + \delta \mathbb{E}[W_T(\theta_T) \mid a_{T-1}^*(r_{T-1}^i, \theta_{T-1}^{-i}), \theta_{T-1}]. \end{aligned}$$

By definition, the allocation rule $a_{T-1}^* : \Theta_{T-1} \rightarrow A_{T-1}$ maximizes the social surplus from period $T - 1$ onward. Given that other agents always report truthfully, it follows that for every realized signal θ_{T-1}^i , it is optimal for agent i to report $r_{T-1}^i = \theta_{T-1}^i$. Also note that for every signal profile θ_{T-1} , agent i 's continuation payoff V_{T-1}^i in the truth-telling equilibrium is

$$V_{T-1}^i(\theta_{T-1}) = W_{T-1}(\theta_{T-1}).$$

Now for any $t < T$, suppose that there exist transfer schedules $\{\bar{p}_{s+1}^i\}_{s=t}^{T-1}$ for each agent i such that truth-telling consists of a periodic ex post equilibrium from any period $s = t, \dots, T$ and each agent i 's continuation payoff in the truth-telling equilibrium is $V_t^i(\theta_t) = W_t(\theta_t)$ for all θ_t .

We would like to construct a transfer $\bar{p}_t^i : \Theta_{t-1}^{-i} \times A_{t-1} \times \Theta_t \rightarrow \mathbb{R}$ for each agent i such that

$$-\sum_{j \neq i} u^j(a_{t-1}, \theta_{t-1}) = \delta \sum_{\theta_t \in \Theta_t} \bar{p}_t^i(\theta_t^{-i}; a_{t-1}, \theta_{t-1}^{-i}) \mu_t(\theta_t | a_{t-1}, \theta_{t-1})$$

for all a_{t-1} , θ_{t-1}^{-i} , and θ_{t-1}^i . The existence of \bar{p}_t^i again follows from Lemma A.4. Since \bar{p}_t^i is independent of θ_t^i , incentive constraints for truth-telling in periods $s = t, \dots, T$ still hold.

For each realized signal profile θ_{t-1} , suppose agent i reports r_{t-1}^i . Then her expected continuation payoff from $t - 1$ on is

$$\sum_{i=1}^N u^i(a_{t-1}^*(r_{t-1}^i, \theta_{t-1}^{-i}), \theta_{t-1}) + \delta \mathbb{E}[W_t(\theta_t) | a_{t-1}^*(r_{t-1}^i, \theta_{t-1}^{-i}), \theta_{t-1}].$$

By the definition of a_{t-1}^* , for each agent i , any report $r_{t-1}^i \in \Theta_{t-1}^i$ in period $t - 1$ other than θ_{t-1}^i is suboptimal under \hat{p}_{t-1} and $\{\bar{p}_s\}_{s=t}^T$. Finally, note that in period $t - 1$, agent i 's continuation payoff in the truth-telling equilibrium is

$$V_{t-1}^i(\theta_{t-1}) = W_{t-1}(\theta_{t-1})$$

for all signal profiles θ_{t-1} .

Inducting on t backwards, we have a sequence of transfers $\{\bar{p}_t\}_{t=1}^T$, where $\bar{p}_1^i \equiv 0$ for each i . Therefore, truth-telling consists of a periodic ex post equilibrium under the efficient dynamic mechanism $\{a_t^*, \bar{p}_t\}_{t=1}^T$. □

APPENDIX B: PROOFS OF THE RESULTS IN SECTION 5

In this section, we first state and prove two lemmas, which are the infinite-signal versions of the convex independence (Lemma B.1) and spanning (Lemma B.2) conditions, respectively. Applying the measurable “measurable choice” theorem in Mertens (2003) to establish measurability of the transfers, Lemma 5.1 and Proposition 5.3 follow from Lemma B.1, and Lemma 5.2 and Proposition 5.3 follow from Lemma B.2.³⁸

³⁸See Barelli and Duggan (2014) for an application of Mertens’s theorem in stochastic games.

Let $C[0, 1]$ denote the set of continuous functions on $[0, 1]$. Let $f(s|t)$ be a continuous conditional density function of $s \in [0, 1]$, given $t \in [0, 1]$. Define the sets

$$C(f) = \left\{ \pi : \exists p : [0, 1]^2 \rightarrow \mathbb{R} \text{ s.t. } \begin{array}{l} \forall t, p(\cdot, t) \in C[0, 1], \forall s, p(s, \cdot) \text{ is Borel measurable,} \\ \forall t, t', \pi(t) = \int_0^1 p(s, t) f(s|t) ds \leq \int_0^1 p(s, t') f(s|t) ds \end{array} \right\}$$

and

$$S(f) = \left\{ \pi : \exists p(s) \in C[0, 1] \text{ s.t. } \forall t \in [0, 1], \pi(t) = \int_0^1 p(s) f(s|t) ds \right\}.$$

Note that $S(f)$ is a linear subspace of $C[0, 1]$ and we have $S(f) \subset C(f)$. We consider the supnorm $\|\pi\| = \max_{t \in [0, 1]} |\pi(t)|$ and denote the closure of $C(f)$ under this norm by $\bar{C}(f)$. Similarly, $\bar{S}(f)$ is the closure of $S(f)$ under the same norm. In the next two lemmas, we identify conditions on the conditional density $f(s|t)$ such that either $\bar{C}(f) = C[0, 1]$ or $\bar{S}(f) = C[0, 1]$.

LEMMA B.1. *We have $\bar{C}(f) = C[0, 1]$ if and only if the following condition holds: for each $t \in [0, 1]$ and each $\eta \in \Delta([0, 1])$,*

$$f(\cdot|t) = \int_0^1 f(\cdot|\tilde{t}) \eta(d\tilde{t}) \quad \Rightarrow \quad \eta(t) = 1.$$

The proof follows directly from Theorem 2 in McAfee and Reny (1992, pp. 404–406).

LEMMA B.2. *We have $\bar{S}(f) = C[0, 1]$ if and only if the following condition holds: there does not exist a regular, nonzero signed measure ξ on the Borel sets of $[0, 1]$ such that*

$$\int_0^1 f(\cdot|t) \xi(dt) = 0.$$

PROOF. For the only if part, suppose, to the contrary, that there is a regular, nonzero signed measure ξ on the Borel sets of $[0, 1]$ such that $\int_0^1 f(\cdot|t) \xi(dt) = 0$. Since $\bar{S}(f) = C[0, 1]$ for any $\varepsilon > 0$ and any $\pi \in C[0, 1]$, there exists a $\tilde{\pi} \in S(f)$ such that $\|\pi - \tilde{\pi}\| < \varepsilon$. Then we have

$$\int_0^1 \tilde{\pi}(t) \xi(dt) = \int_0^1 \left[\int_0^1 p(s) f(s|t) ds \right] \xi(dt)$$

for some $p(s) \in C[0, 1]$ by the definition of $S(f)$. By Fubini's theorem,

$$\int_0^1 \left[\int_0^1 p(s) f(s|t) ds \right] \xi(dt) = \int_0^1 p(s) \left[\int_0^1 f(s|t) \xi(dt) \right] ds = 0.$$

That is, $\int_0^1 \tilde{\pi}(t) \xi(dt) = 0$. Hence, $\xi = 0$, which is a contradiction.

For the if part, suppose, to the contrary, that $\bar{S}(f) \neq C[0, 1]$. Then there exists $\tilde{\pi} \in C[0, 1]$ such that $\tilde{\pi} \notin \bar{S}(f)$. Since $\bar{S}(f)$ is closed and convex, by the separating hyperplane theorem (see Aliprantis and Border 2006, Theorem 5.79, p. 207), there is a nonzero

continuous linear functional on $C[0, 1]$ separating $\bar{S}(f)$ and $\bar{\pi}$. Since $\bar{S}(f)$ is a linear subspace of $C[0, 1]$, it follows from the Riesz representation theorem (see Aliprantis and Border 2006, Corollary 14.15, p. 498) that there exists a regular, nonzero signed measure ξ on the Borel sets of $[0, 1]$ such that for each $\pi \in \bar{S}(f)$,

$$\int_0^1 \pi(t) \xi(dt) = 0.$$

By the definition of $S(f)$, we then have

$$\int_0^1 \left[\int_0^1 p(s) f(s|t) ds \right] \xi(dt) = 0$$

for each $p(s) \in C[0, 1]$. It then follows from Fubini's theorem that

$$\int_0^1 p(s) \left[\int_0^1 f(s|t) \xi(dt) \right] ds = 0$$

for each $p(s) \in C[0, 1]$. Therefore, $\int_0^1 f(\cdot|t) \xi(dt) = 0$, which is a contradiction. \square

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