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Strategy-proof tie-breaking in matching with priorities

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A set of indivisible objects is allocated among agents with strict preferences. Each object has a weak priority ranking of the agents. A collection of priority rankings, a *priority structure*, is *solvable* if there is a *strategy-proof* mechanism that is *constrained efficient*, i.e., that always produces a *stable* matching that is not Pareto-dominated by another stable matching. We characterize all solvable priority structures satisfying the following two restrictions:

(A) Either there are no ties or there is at least one four-way tie.

(B) For any two agents i and j, if there is an object that assigns higher priority to i than to j, there is also an object that assigns higher priority to j than to i.

We show that there are at most three types of solvable priority structures: The *strict* type, the *house allocation with existing tenants* (HET) type, where, for each object, there is at most one agent who has strictly higher priority than another agent, and the *task allocation with unqualified agents* (TAU) type, where, for each object, there is at most one agent who has strictly lower priority than another agent. Out of these three, only HET priority structures are shown to admit a strongly group-strategy-proof and constrained efficient mechanism.

KEYWORDS. Weak priorities, stability, constrained efficiency, strategy-proofness.

JEL CLASSIFICATION. C78, D61, D78, I20.

1. INTRODUCTION

In this paper we consider various classes of priority-based allocation problems where a set of indivisible objects is allocated among a finite set of agents and no monetary transfers are permitted. Agents have privately known strict preferences over available objects. For any object there is an exogenously given weak priority ordering that specifies strict rankings and *ties*. We restrict attention to *strategy-proof* (direct) mechanisms that provide agents with dominant strategy incentives to report preferences truthfully.[1]

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[1] Strategy-proofness is the most widely used incentive compatibility requirement in the area of market design without monetary transfers (see Roth 2008, as well as Sönmez and Ünver 2011, for recent surveys).

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A *matching* (of agents to objects) is *stable* if (i) no agent is worse off than receiving no object (*individual rationality*), (ii) no agent strictly prefers an unassigned object to her assignment (*non-wastefulness*), and (iii) there is no agent i who strictly prefers an object o (over her assignment) that was assigned to another agent j who has strictly lower priority for o than i (*fairness*).[2] A matching is *constrained efficient* (or *agent-optimal*) if it is stable and not Pareto-dominated by another stable matching. Our goal is to characterize priority structures that are *solvable* in the sense of admitting constrained efficient and strategy-proof mechanisms.

Important real-life examples of the class of problems we analyze are school choice, where a student's priority for a school is determined by objective criteria such as distance or the existence of siblings already attending the school, the allocation of dorm rooms, where an existing tenant is usually guaranteed priority for her room over others, and (live-donor) kidney exchange, where a potential donor who is immunologically incompatible with her intended recipient is only willing to give her kidney to someone else if her intended recipient receives a compatible kidney in exchange.[3] These three problems share the feature that priorities are exogenous and commonly known. Furthermore, stability is an important allocative desideratum: For the school choice problem, an unstable assignment is susceptible to appeals by unhappy parents and may be detrimental to public acceptance of an admissions procedure given the absence of a clear rationale for rejections at over-demanded schools; in the dorm allocation or the kidney exchange problem, a violation of stability means that some existing tenants/patients would have been strictly better off not participating in the assignment procedure (staying in their old room in the former case and sparing their incompatible donor the pain of kidney extraction in the latter case). While efficiency losses due to stability constraints may thus be deemed acceptable, it is important to avoid any further efficiency losses and thus ensure constrained efficiency of the chosen matching. Given the private information that is inherent to the problems described above, whether a priority structure is solvable is an important and practically relevant question.

Prior to our research, the only known types of solvable priority structures were the *strict* type, where no two distinct agents can ever have the same priority for an object, and the *house allocation with existing tenants* (HET) type, where, for each object, there is at most one agent who has strictly higher priority than another agent. These positive results for two very different types of priority structures—one without any ties and one where, for each object, at least all but one agent have the same priority—may lead one to believe that there should be many solvable priority structures. Our main result shows that within a very general class of priority structures, there is at most *one* type of priority structure that could be solvable and that has not already been discovered by the existing

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See Abdulkadiroğlu et al. (2006) for a fairness rationale supporting strategy-proofness. Budish and Cantillon (2012) provide a critical perspective on the restriction to strategy-proof mechanisms.

[2] See Roth and Sotomayor (1990) for an excellent introduction to the theory and applications of stable matching mechanisms.

[3] See Abdulkadiroğlu and Sönmez (2003) for an analysis of school choice problems, Abdulkadiroğlu and Sönmez (1999) for an analysis of the dorm room allocation problem, and Roth et al. (2004) for an analysis of the kidney exchange problem.

literature. This gives clear-cut guidance to market designers: If some real-life application gives rise to a priority structure that does not belong to one of the three types that we identify in this paper, one has to give up on either constrained efficiency or strategy-proofness, and focus on designing a mechanism that achieves a reasonable compromise between the two conflicting goals.

We consider a very general class of priority structures that satisfies two natural restrictions. First, we require that there are either no ties at all or there is at least one four-way tie. This is likely to be satisfied in real-life applications, such as school choice, where indifference classes are typically either very small, e.g., when exact grade point averages (GPAs) and other criteria determine priorities, or very large, e.g., when schools distinguish only between students living within a certain radius around a school and those who do not. Second, we require *reversibility,* meaning that, for any two agents i and j, if there is an object that assigns higher priority to i than to j, there is also an object that assigns higher priority to j than to i. This second restriction ensures that possibility results do not depend on intricate assumptions about the correlation of priorities across objects. Our main result, Theorem 1, shows that within the just described class, there are at most three types of solvable priority structures: The strict type, the HET type, and the *task allocation with unqualified agents* (TAU) type, where, for each object, there is at most one agent who has strictly lower priority than another agent. As discussed above, solvability of strict and HET priority structures is well known. To the best of our knowledge, TAU priority structures have not been explicitly considered in the previous literature. We have not been able to rule out that TAU priority structures are solvable, but strongly suspect that they are not. To substantiate our suspicion we show how various approaches to resolving ties fail to give rise to a constrained efficient and strategy-proof mechanism. We then shift attention to the stronger incentive compatibility requirement of *strong group-strategy-proofness,* which requires that there should never be a group of agents who can, through a coordinated deviation from truth-telling, obtain an outcome that is weakly better for each and strictly better for at least one member of the group. Theorem 2 shows that among all priority structures satisfying reversibility, only HET priority structures permit a constrained efficient and strongly group-strategy-proof mechanism. This result only relies on the reversibility assumption and does not require us to assume that a nonstrict priority structure has at least one four-way tie.

Related literature

In recent years several important contributions have analyzed priority-based allocation problems with weak priority orders. Erdil and Ergin (2008) study priority-based allocation problems with arbitrary weak priority structures.[4] Their main result, which we use to prove our main results, is that whenever a stable matching is not constrained efficient, it is possible to increase agents' welfare via a cyclical exchange of assignments that respects stability constraints. Erdil and Ergin (2008) also provide a simple example showing that the introduction of a tie between two agents in a strict priority structure

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[4] Ehlers (2006) was the first to study stable and strategy-proof mechanism for priority-based allocation problems with weak priorities.

might result in an unsolvable priority structure. Prior to our research, it was not clear whether unsolvable priority structures are the norm or an exception. This is not a trivial question. Importantly, the two types of priority structures that have turned out to be extremely useful for applications, strict and HET, are highly specific. Our main result gives a precise sense in which these two classes of priority structures and TAU priority structures are the only ones that could be solvable without further information on priority rankings: Any other type of priority structure will be solvable only if it has very small indifference classes or if agents' priorities are highly correlated across different objects. In another important contribution, Abdulkadiroğlu et al. (2009) show that no strategy-proof mechanism can Pareto-dominate the deferred acceptance (DA) mechanism resulting from some exogenous, i.e., independent of submitted preferences, tie-breaking rule for all profiles of agents' preferences. The focus of our analysis is different since we investigate whether a constrained efficient and strategy-proof mechanism exists without requiring that the mechanism Pareto-dominates the mechanism induced by the DA algorithm. This distinction is important since, for example, for HET priority structures the well known top-trading cycles (TTC) mechanism achieves efficiency and (strong group) strategy-proofness, but does not Pareto-dominate any DA mechanism with exogenous tie-breaking.

From the literature on priority-based allocation problems with strict priority orders the most relevant paper is Ergin (2002). He characterizes the set of strict priority structures for which stability is compatible with efficiency by means of an acyclicity condition.[5] The result of Ergin (2002) has been extended to the case of weak priority structures in two ways: First, Ehlers and Erdil (2010) characterize the set of weak priority structures for which all constrained efficient matchings are guaranteed to be efficient. Second, Han (2018) characterizes the set of weak priority structures for which a stable and efficient matching is guaranteed to exist.[6] The characterizations in both of the just mentioned papers rely on different strengthenings of Ergin's acyclicity condition. The main difference between this line of research and our analysis is that the former is concerned with the compatibility of two allocative criteria that are known to be in conflict with each other for most strict priority structures. In contrast, we are interested in characterizing when one allocative criterion—that could always be satisfied if no additional criteria were imposed—is compatible with dominant strategy incentive compatibility.[7]

Finally, in an important contribution, Pápai (2000) characterizes the class of strongly group-strategy-proof, efficient, and reallocation-proof mechanisms for settings in which priorities are not primitives of the model (or, equivalently, where all agents have the same priorities for all objects). She shows that mechanisms satisfying the aforementioned properties work like TTC mechanisms in which agents iteratively exchange endowments that they receive according to a fixed hierarchical endowment

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[5] He shows that the very same condition characterizes the sets of strict priority structures for which the DA is strongly group-strategy-proof and consistent, respectively.

[6] Han (2018) also characterizes the classes of weak priority structures for which stable, efficient, and (group) strategy-proof mechanisms exist. We comment on these results below.

[7] Several other papers have investigated consequences of the structural properties of strict priority structures; see, e.g., Kesten (2006) and Ehlers and Klaus (2006).

structure. Hence, the combination of strong group-strategy-proofness, efficiency, and reallocation-proofness gives rise to *endogenously* determined priorities. However, for settings in which priorities are primitives of the model, as in those that we consider in this paper, TTC-like mechanisms usually fail to satisfy stability since they allow agents to freely trade their priorities, thus ignoring the veto power that stability constraints bestow upon agents. In a more recent contribution, Pycia and Ünver (2017) characterize the slightly larger class of mechanisms, compared to the class of mechanisms identified by Pápai (2000), that satisfy strong group-strategy-proofness and efficiency. We discuss the just mentioned contribution in more detail below, when we explain why (extensions of) the mechanisms described by Pycia and Ünver (2017) must fail either constrained efficiency or strategy-proofness for TAU priority structures.

Organization of the paper

The remainder of the paper is organized as follows: Section 2 introduces the basic priority-based allocation model and solvability. Section 2.1 describes different types of priority structures that play an important role for our characterization results and states the solvability of popular priority structures. In Section 3, we first introduce and motivate the two restrictions we place on priority structures, and then present our main result, Theorem 1, and outline its proof. Section 3.1 studies the concept of strong solvability. Section 4 concludes. The Appendix contains proofs of the main results and two key auxiliary results. The Supplemental Appendix, available in a supplementary file on the journal website, http://econtheory.org/supp/2547/supplement.pdf, contains proofs of further auxiliary results and a discussion of the assumptions underlying our main results.

2. PRIORITY-BASED ALLOCATION PROBLEMS

A *priority-based allocation problem* is a quadruple $(I, O, \succeq, R)$ that has the following components:

- A finite set of agents $I = \{1, \dots, N\}$, where $N \geq 1$.
- A finite set of objects O.
- A priority structure $\succeq = (\succeq_o)_{o \in O}$, where, for each $o \in O$, $\succeq_o$ is a (weak) priority ordering of I.
- A preference profile $R = (R_i)_{i \in I}$, where, for each $i \in I$, R_i is a strict preference relation on $O \cup \{i\}$.

We fix I, O, and $\succeq$ throughout, so that a problem is given by a (strict) preference profile. We denote by $i \succ_o j$ that agent i has higher priority for object o than agent j and denote by $i \sim_o j$ that i and j have equal priority for o. If $i \sim_o j$ and $i \neq j$, we say that there is a *tie* between i and j at o. An *indifference class* of $\succeq_o$ consists of a set of agents who are involved in a tie at o. We say that $\succeq$ is *strict* if, for all $o \in O$, $\succeq_o$ contains no tie. Given an object $o \in O$ and two nonempty disjoint subsets $J_1, J_2 \subseteq I$, we write $J_1 \succeq_o J_2$ if $i \succeq_o j$ for all $i \in J_1$ and all $j \in J_2$. Given two nonempty disjoint subsets $J_1, J_2 \subseteq I$, we write $J_1 \succeq J_2$ if $J_1 \succeq_o J_2$ for all $o \in O$. For a strict ranking R_i of $O \cup \{i\}$ and any two options $a, b \in O \cup \{i\}$, we denote by $a\ P_i\ b$ that i strictly prefers a to b, and denote by $a\ R_i\ b$ that either $a\ P_i\ b$ or $a = b$. We say that object o is *acceptable* for i if $o\ P_i\ i$ (and call o *unacceptable* otherwise). We use the convention to write $R_i : opq$ if $o\ P_i\ p\ P_i\ q\ P_i\ i$ and all objects in $O \setminus \{o, p, q\}$ are unacceptable. Let $\mathcal{P}^i$ denote the set of all strict preference rankings of $O \cup \{i\}$ and let $\mathcal{P}^I = \times_{i \in I} \mathcal{P}^i$ denote the set of all preference profiles (or problems).

A *matching* is a mapping $\mu : I \to I \cup O$ such that, for all $i \in I$, $\mu(i) \in O \cup \{i\}$, and for all distinct $i, j \in I$, $\mu(i) \neq \mu(j)$. Agent i is *unmatched* under μ if $\mu(i) = i$. Given a matching μ and $o \in O$, let $\mu(o) := \mu^{-1}(o)$ denote the agent matched to object o (where $\mu(o) = \varnothing$ if o is unassigned under μ). Let $\mathcal{M}$ denote the set of all matchings. A matching μ is *stable* for problem $R \in \mathcal{P}^I$ if it satisfies the following conditions:

(i) It is *individually rational*, that is, for all $i \in I$, $\mu(i)\ R_i\ i$.

(ii) It is *non-wasteful*, that is, there is no agent–object pair (i, o) such that $o\ P_i\ \mu(i)$ and $\mu(o) = \varnothing$.

(iii) It is *fair*, that is, there is no agent–object pair (i, o) such that $o\ P_i\ \mu(i)$ and, for some $j \in \mu(o)$, $i \succ_o j$.

A matching μ is *fully efficient* for problem $R \in \mathcal{P}^I$ if there is no other matching ν such that, for all $i \in I$, $\nu(i)\ R_i\ \mu(i)$, and, for at least one $j \in I$, $\nu(j)\ P_j\ \mu(j)$. As shown by Ergin (2002), stability is often incompatible with full efficiency. However, given that the set of stable matchings is finite, there always exists at least one stable matching that is not Pareto-dominated by any other stable matching. More formally, we call a matching μ *constrained efficient* (or *agent-optimal stable*) for problem $R \in \mathcal{P}^I$ if (i) μ is stable and (ii) there is no other stable matching ν such that, for all $i \in I$, $\nu(i)\ R_i\ \mu(i)$, and, for at least one $j \in I$, $\nu(j)\ P_j\ \mu(j)$. We denote the set of all constrained efficient matchings by $\mathrm{CE}^{\succeq}(R)$. Erdil and Ergin (2008) develop an algorithm for finding constrained efficient matchings that is based on the observation that whenever a stable matching μ is not constrained efficient, it is possible to increase agents' welfare via a cyclical exchange that respects stability constraints. Formally, fix a problem $R \in \mathcal{P}^I$, let μ be an arbitrary stable matching, and say that agent i *desires object o at μ* if $o\ P_i\ \mu(i)$, and, for each o, let $D_o(\mu)$ denote the set of highest $\succeq_o$-priority agents among those who desire o at μ. A *stable improvement cycle* (SIC) of μ at $R \in \mathcal{P}^I$ consists of m distinct agents $i_1, \ldots, i_m$ such that for all $l = 1, \ldots, m$, $i_l \in D_{\mu(i_{l+1})}(\mu)$ (where $m + 1 := 1$). Erdil and Ergin (2008) show that μ is constrained efficient at $R \in \mathcal{P}^I$ if and only if μ admits no SIC of μ at R.

A *(matching) mechanism* is a function $f : \mathcal{P}^I \to \mathcal{M}$ that, for each problem $R \in \mathcal{P}^I$, chooses one matching $f(R)$. To avoid confusion, we sometimes include the priority structure $\succeq$ in the description of a mechanism and write $f^{\succeq}$. Given $i \in I$ and $R \in \mathcal{P}^I$, we write $f_i(R)$ for i's assignment at $f(R)$. Mechanism f has the following properties:

- It is *stable* if, for all $R \in \mathcal{P}^I$, $f(R)$ is stable.
- It is *constrained efficient* if, for all $R \in \mathcal{P}^I$, $f(R) \in \mathrm{CE}^{\succeq}(R)$.

- It is *strategy-proof* if, for all $R \in \mathcal{P}^I$ and all $i \in I$, there does not exist a manipulation $\tilde{R}_i \in \mathcal{P}^i$ such that $f_i(\tilde{R}_i, R_{-i})\ P_i\ f_i(R)$.

A priority structure $\succeq$ is *solvable* if there exists a strategy-proof and constrained efficient mechanism f for $\succeq$ and is *unsolvable* otherwise. It sometimes is useful to think about the existence of constrained efficient and strategy-proof mechanisms for situations where not all agents are present simultaneously. For this purpose, given some subset $J \subseteq I$ and a weak priority structure $\succeq$, we denote by $\succeq|_J$ the restriction of $\succeq$ to the agents in J. We say that $\succeq|_J$ is solvable if there exists a mechanism that is strategy-proof and constrained efficient when the set of agents is J, the set of available objects is O, the priority structure is $\succeq|_J$, and each agent $j \in J$ is allowed to report any preference relation in $\mathcal{P}^j$.

By means of an example,[8] Erdil and Ergin (2008) have shown that unsolvable priority structures do exist. Apart from Erdil and Ergin's example, not much is known about unsolvable priority structures prior to our research. Our main goal is to characterize the classes of solvable and unsolvable priority structures.

2.1 *A taxonomy of priority structures*

The next definition introduces three classes of priority structure that play a key role in our analysis.

Definition 1. A priority structure $\succeq$ has the following properties:

(i) It is strict if there is no object $o \in O$ such that $i \sim_o j$ for two distinct $i, j \in I$.

(ii) It is in the house allocation with existing tenants (HET) class if, for any object $o \in O$, either $i \sim_o j$ for all agents $i, j \in I$ or there exists exactly one agent $i(o)$ such that $i(o) \succ_o j \sim_o k$ for all $j, k \in I \setminus \{i(o)\}$.

(iii) It is in the task allocation with unqualified agents (TAU) class if, for any object $o \in O$, either $i \sim_o j$ for all agents $i, j \in I$ or there exists exactly one agent $i(o)$ such that $j \sim_o k \succ_o i(o)$ for all $j, k \in I \setminus \{i(o)\}$.

Note that according to Definition 1, a trivial (no-) priority structure $\succeq$ such that, for all $o \in O$ and all $i, j \in I$, $i \sim_o j$ belongs to both the HET and the TAU class. We say that $\succeq$ is a *nontrivial HET/TAU priority structure* if $\succeq$ is not the trivial (no-) priority structure and belongs to the HET/TAU class. Note that for HET/TAU priority structures, we allow for the possibility that a given agent has the highest/lowest priority for multiple objects, i.e., it is possible that $i(o) = i(p)$ for distinct objects o and p. We now discuss the three classes introduced in Definition 1 in turn and summarize important findings from the previous literature.

First, for strict priority structures, it is well known that, for any problem $R \in \mathcal{P}^I$, there is a unique constrained efficient matching that can be found by the *agent-proposing deferred acceptance* (DA) algorithm of Gale and Shapley (1962):

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[8] Example 2 in Section IV of Erdil and Ergin (2008).

Step 1. Each agent proposes to her most preferred acceptable object. Each object tentatively accepts the highest priority agent from its proposals and rejects all other agents.

Step k. Any agent who is not tentatively accepted proposes to her most preferred acceptable object among those that have not rejected her. Each object tentatively accepts the highest priority agent from its new proposals and the tentatively accepted one (if any), and rejects all other agents.

The DA algorithm stops when each agent has either proposed to all acceptable objects or been tentatively accepted by some object. At this point, tentative assignments become final matches and agents who are not tentatively accepted remain unmatched. Dubins and Freedman (1981) and Roth (1982a) have established that the direct mechanism induced by the DA is strategy-proof, so that, in particular, any strict priority structure is solvable. For strict priority structures, it is well known that the existence of a fully efficient stable matching can be guaranteed only if the priority structure satisfies a strong acyclicity condition Ergin (2002). Finally, it is worth mentioning that the properties of the DA for strict priorities imply that strategy-proofness and stability are always compatible: If we arbitrarily break all ties in $\succeq$ while maintaining all strict priority rankings, the DA for the resulting strict priority structure $\succeq'$ is guaranteed to produce a matching that is stable with respect to the original priority structure $\succeq$ and induces a strategy-proof direct mechanism Abdulkadiroğlu et al. (2009).

Second, for HET priority structures, constrained efficient matchings can be found by means of the *top-trading cycles* (TTC) algorithm of Abdulkadiroğlu and Sönmez (1999). To describe this algorithm, it is convenient to think of the objects as houses. We say that house o is *occupied* if there is an agent $i(o)$, the *owner of* o, such that, for all $j \in I \setminus \{i(o)\}$, $i(o) \succ_o j$, and that o is *vacant* otherwise.

Step 1. Each agent i points to her most preferred option in $O \cup \{i\}$, each occupied house points to its owner, and each vacant house points to the highest numbered agent.[9] For each cycle,[10] assign each agent to the house he is pointing to and remove all agents and houses belonging to the cycle from the procedure. Let I_1 denote the set of remaining agents and let O_1 denote the set of remaining houses.

Step $k \geq 2$. Each agent $i \in I_{k-1}$ points to her most preferred option in $O_{k-1} \cup \{i\}$, each occupied house $o \in O_{k-1}$ for which $i(o) \in I_{k-1}$ points to $i(o)$, and all other houses point to the highest numbered agent in I_{k-1}. For each cycle, assign each agent to the house he is pointing to and remove all agents and houses belonging to the cycle from the procedure. Let I_k denote the set of remaining agents and let O_k denote the set of remaining houses.

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[9] Recall that all agents have the same priority for each vacant house in a HET priority structure. It is easy to see that any procedure to decide where vacant houses point that does not depend on submitted preferences gives rise to a constrained efficient and strategy-proof mechanism. For simplicity, we focus on a version of the TTC in which all vacant houses point to the same agent.

[10] A cycle is either a sequence $i^1, o^1, \ldots, i^M, o^M$ such that $M \geq 1$ and, for each $m \leq M$, i^m points to o^m and o^m points to i^{m+1} (where $M + 1 \equiv 1$) or an agent who points to himself because there are no acceptable houses for him.

The TTC algorithm ends when all agents are assigned. Abdulkadiroğlu and Sönmez (1999) have established that the TTC algorithm always produces an efficient matching and that the mechanism that picks the TTC outcome for each problem is strategy-proof. Since it is evident that the TTC algorithm is guaranteed to respect all stability constraints induced by a HET priority structure, the result of Abdulkadiroğlu and Sönmez (1999) shows that HET priority structures are always solvable.[11] In case all houses are vacant, the TTC algorithm reduces to a serial dictatorship mechanism.[12] For the case where each agent owns exactly one house and there are no vacant houses,[13] Ma (1994) showed that the TTC mechanism is the only mechanism that satisfies *individual rationality for owners* (i.e., no agent is ever worse off than staying in the house that he was already occupying), full efficiency, and strategy-proofness. For HET priority structures, it is easy to show that individual rationality for owners and full efficiency are equivalent to constrained efficiency. Thus, by Ma's result, if any house is owned by exactly one agent, then the mechanism induced by the TTC algorithm is the only strategy-proof and constrained efficient mechanism. In particular, stability and efficiency are compatible with each other for HET priority structures.

Third, TAU priority structures have not been explicitly considered in the previous literature. To interpret TAU priority structures. think of objects as representing "tasks" that have to be performed by the agents. Our definition of a TAU priority structure then requires that, for each task o, either all agents are *qualified* to perform the task (and have equal priority) or there is a unique agent $i(o)$ who is *unqualified* to perform o and should only be allocated o if none of the qualified agents (who all have equal priority for o) is willing to perform this task. In general, the stable improvement cycles algorithm of Erdil and Ergin (2008) can be used to find constrained efficient matchings for a TAU priority structure (because that algorithm produces constrained efficient matchings for any priority structure). However, the stable improvement cycles algorithm does not necessarily induce a strategy-proof mechanism and it is not known whether TAU priority structures are solvable. Finally, for TAU priority structures, it is easy to show that there always exists at least one fully efficient and stable matching.[14]

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[11] To be more precise, our definition of the HET class is slightly more general than that in Abdulkadiroğlu and Sönmez (1999), since they assume that each agent owns at most one house, i.e., that $i(o) \neq i(q)$ for any distinct houses o and q. It is straightforward to check that the arguments in Abdulkadiroğlu and Sönmez (1999) apply to our setting as well.

[12] This variant is typically called the *house allocation problem* and was first studied by Hylland and Zeckhauser (1979). For the house allocation problem, the full class of strategy-proof and efficient mechanisms is large and has not been characterized yet. See Svensson (1999), Pápai (2000), and Pycia and Ünver (2017) for characterizations of different subclasses of the class of strategy-proof and efficient mechanisms in the deterministic setting, and see Bogomolnaia and Moulin (2001) for random assignment in the house allocation problem.

[13] This variant is typically called the *housing market* and was first studied by Shapley and Scarf (1974), who also proposed the first version of the TTC algorithm. In housing markets, each agent is endowed with one object and Gale's TTC algorithm finds for each problem its unique Roth and Postlewaite (1977) core matching. Roth (1982b) was the first to show that the associated direct mechanism is strategy-proof.

[14] To see this, consider a variant of the DA in which, in each round, each object that gets a proposal from a qualified agent is (randomly) allocated among applying qualified agents and removed from the procedure.

3. SOLVABLE PRIORITY STRUCTURES

The main purpose of this section is to characterize solvable priority structures. We restrict attention to environments that satisfy the following two assumptions.

ASSUMPTION 1. *(A) Strict/four-way tie. If $\succeq$ is not strict, then there exist $o \in O$ and four distinct agents $i_1, i_2, i_3, i_4 \in I$ such that $i_1 \sim_o i_2 \sim_o i_3 \sim_o i_4$.*

(B) Reversibility. For any pair $i, j \in I$, either there exist objects $p, q \in O$ such that $i \succ_p j$ and $j \succ_q i$ or $i \sim_o j$ for all $o \in O$.

Our first main result is a partial characterization of solvable priority structures within the class of priority structures that satisfy both parts of Assumption 1.[15] Before presenting our result, we now motivate our assumption. Assumption 1(A) is likely to be satisfied in real-life applications, such as school choice, where indifference classes are typically either very small, e.g., when exact GPAs and other criteria determine priorities, or very large, e.g., when schools only distinguish between students living within a certain radius around a school and those who do not. Assumption 1(B) ensures that possibility results do not depend on assumptions about the correlation of priorities across objects. This approach is in line with much of the literature on matching theory, where attention is often restricted to domains that have a (Cartesian) product structure, i.e., domains described by conditions that can be checked independently for each object.[16] In the Supplemental Appendix, we show that Assumption 1 is crucial for our results. Hence, we cannot completely rule out that there are interesting solvable priority structures that do not satisfy Assumption 1. However, the results in our earlier working paper Ehlers and Westkamp (2011) suggest that one needs a very strong degree of correlation in priorities across objects to find solvable priority structures that are not covered by our main result below.[17] Note that Assumption 1(B) allows for situations in which two or more agents have equal priority for all objects. The following theorem is our first main result.

THEOREM 1. *Let $N \geq 6$ and $\succeq$ be a priority structure satisfying Assumption 1. If $\succeq$ is solvable, then $\succeq$ must be either a strict, HET, or TAU priority structure.*

In particular, an object remains open for applications in later rounds only if it receives either no applications or exactly one application from the agent who is unqualified for the task. It is easy to show that the procedure just outlined is fully efficient; details are available upon request.

[15] Note that Assumption 1 covers only those nontrivial HET priority structures in which all agents own at least one house and only those nontrivial TAU priority structures in which all agents are unqualified for at least one of the tasks. We discuss below how to weaken Assumption 1 so as to cover all HET and all TAU priority structures.

[16] Two notable exceptions are Ostrovsky (2008) and Pycia (2012).

[17] More specifically, in Ehlers and Westkamp (2011), we characterize the class of all solvable priority structures for which priorities are only allowed at the bottom of priority rankings. In that paper, we do not impose Assumption 1. The main results in Ehlers and Westkamp (2011) show that for a priority structure that is neither of the strict nor the HET type (both of which satisfy the "ties only at the bottom" assumption) to be solvable, agents' priorities can vary by at most two ranks across all objects (e.g., an agent who has the unique highest priority for one of the objects must have at least third highest priority for all other objects). These earlier results make us doubt that there are interesting solvable priority structures different from strict, HET, or TAU priority structures that do not satisfy Assumption 1(B).

One way to visualize Theorem 1 is the following: A weak priority order $\succeq_o$ assigns to each agent exactly one of at most $|I|$ different "priority levels" (or bins). We use the convention that higher priority for an object is associated with a lower priority level. Each rule for assigning priorities induces a set of possible collections of *level sets* of $\succeq$. For example, a HET priority structure only allows for two possibilities: Either all agents have the same priority level or exactly one agent has the first priority level and everyone else has the second priority level. Theorem 1 establishes for which collections of level sets a constrained efficient and strategy-proof mechanism could potentially be guaranteed to exist, irrespective of who occupies which priority bin at which object. We see that we are almost exclusively confined to the two, by now classical, examples of strict and HET priority structures. This is a dramatic reduction from the set of all possible priority structures satisfying Assumption 1. The added value of our first main result is to give market designers clear-cut guidance: If, for a particular application, rules for assigning priorities do not give rise to one of the three priority structures in Theorem 1, then the existence of a constrained efficient and strategy-proof mechanism cannot be guaranteed, and one has to settle for a compromise between efficiency and incentive properties. The next corollary presents one special case of an unsolvable priority structure that seems particularly relevant for applications to school choice, where multiple non-singleton indifference classes are very common (e.g., because of walk-zone priority).[18]

Corollary 1. *Let $N \geq 6$ and $\succeq$ be a priority structure satisfying Assumption 1. If there are two or more objects that each have two or more non-singleton indifference classes, then $\succeq$ is unsolvable.*

Before proceeding to a sketch of the proof of Theorem 1, we mention one straightforward extension of our first main result to settings where Assumption 1 is not satisfied. The extension rests on the following simple observation.

Lemma 1. *If I_1 and I_2 are two nonempty disjoint subsets of I such that $I_1 \cup I_2 = I$ and $I_1 \succeq I_2$, then $\succeq$ is solvable if and only if $\succeq|_{I_1}$ and $\succeq|_{I_2}$ are both solvable.*

Proof. Assume first that $\succeq$ is solvable. Let f be a constrained efficient and strategy-proof mechanism for an economy with set of agents $I_1 \cup I_2$. If we restrict f to those profiles of preferences for which all agents in I_2 rank all objects as unacceptable, then we obtain a constrained efficient and strategy-proof mechanism for an economy with set of agents I_1. Analogously, if we restrict f to those profiles of preferences for which all agents in I_1 rank all objects as unacceptable, then we obtain a constrained efficient and strategy-proof mechanism for an economy with set of agents I_2.

Now assume that $\succeq|_{I_t}$ is solvable for $t = 1, 2$. Let f_t be a constrained efficient and strategy-proof mechanism for an economy with set of agents I_t for $t = 1, 2$. Consider

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[18] We thank an anonymous referee for suggesting we include Corollary 1.

a mechanism f that is obtained from f_1 and f_2 as follows: For any preference profile, first allocate objects among agents in I_1 according to f_1 and then allocate the remaining objects among agents in I_2 according to f_2.[19] It is clear that f inherits the strategy-proofness of its component mechanisms. Constrained efficiency of the combined mechanism follows from the constrained efficiency of the component mechanisms and the assumption that all agents in I_1 have weakly higher priority than all agents in I_2. □

Lemma 1 immediately implies that we can extend Theorem 1 as follows.

Corollary 2. *Assume that there exists a partition* $\{I_1, \dots, I_T\}$ *of* I *such that, for all* $t = 1, \dots, T$, $|I_t| \geq 6$, $I_t \succeq I \setminus (I_1 \cup \dots \cup I_t)$, *and* $\succeq|_{I_t}$ *satisfies Assumption 1. If* $\succeq$ *is solvable, then, for all* $t = 1, \dots, T$, $\succeq|_{I_t}$ *must be either a strict, HET, or TAU priority structure.*

In words, if the set of agents can be partitioned into a sequence of sets that are ordered by agents' priorities and satisfy Assumption 1, then the solvability of a priority structure implies that it must be a succession of strict, HET, and TAU priority structures. Note that Corollary 2 allows for the possibility that some parts of the priority structure have a different structure than others (e.g., the priority structure for the highest priority agents is HET, while the priority structure for lower priority agents is strict). The corollary nests HET priority structures in which some agents are *existing tenants*, i.e,. initially occupy at least one of the objects, and others are *newcomers*, i.e., do not initially occupy any of the objects. To see why, note that if I_1 is the set of existing tenants and I_2 is the set of newcomers, then the partition $\{I_1, I_2\}$ satisfies the assumptions of Corollary 2. For the case of HET priority structures, our main result can be interpreted as focusing only on existing tenants since these are the only agents who impose stability constraints on the system. Similarly, for general TAU priority structures, we can let I_2 be the set of agents who are unqualified for at least one task and take $I_1 = I \setminus I_2$ to obtain a partition that satisfies the assumptions of Corollary 2.

We now provide a sketch of the proof of Theorem 1. Fundamental to our proof is the following lemma that identifies two tie-breaking decisions that *any* constrained efficient and strategy-proof mechanism has to respect.[20]

Lemma 2. *(a) Let* $o, p \in O$ *and* $1, 2, 3 \in I$ *be such that* $3 \succ_o 2 \sim_o 1$ *and* $1 \succ_p 3$. *Consider a preference profile* R *such that*

R	R_1	R_2	R_3
	o	o	p
	$\vdots$	$\vdots$	o

---

[19] More formally, given a preference profile R, we first allocate objects among agents in I_1 according to f_1. We then allocate objects among agents in I_2 according to f_2 using a preference profile $\tilde{R}$ that is obtained from R by having agents in I_2 rank all objects already assigned to agents in I_1 as unacceptable.

[20] More precisely, Lemma 2 is a simplified version of Lemmas 3, 4, which are stated and proved in the Appendix.

and such that no agent in $I \setminus \{1, 2, 3\}$ *ranks* p *as acceptable. If* f *is constrained efficient and strategy-proof, then we must have* $f_2(R) \neq o$.

(b) Let $o, p \in O$ *and* $1, 2, 3 \in O$ *be such that* $1 \sim_o 2 \sim_o 3$ *and* $\{1, 2\} \succ_p 3$. *Consider a preference profile* R *such that*

R	R_1	R_2	R_3
	o	o	o
	p	p	$\vdots$

and such that no agent in $I \setminus \{1, 2, 3\}$ *ranks* p *as acceptable. If* f *is constrained efficient and strategy-proof, then we must have* $f_3(R) \neq o$.

Lemma 2 allows us to uncover a simple necessary condition for the solvability of a priority structure:

> There cannot be four distinct agents 1, 2, 3, 4 and three distinct objects o, p, q such that either
>
> $$1 \succ_p 3, \qquad 2 \succ_q 4 \quad \& \quad \{3, 4\} \succ_o 1 \sim_o 2$$
>
> or
>
> $$1 \sim_o 2 \sim_o 3 \sim_o 4, \qquad 1 \succ_p 3 \succ_p 2 \quad \& \quad 2 \succ_q 4 \succ_q 1.$$

To see why a priority structure with property (a) is unsolvable, consider a preference profile R such that

R	R_1	R_2	R_3	R_4
	o	o	p	q
			o	o

and such that no agent in $I \setminus \{1, \dots, 4\}$ ranks o, p, or q as acceptable. By the first part of Lemma 2, if f is constrained efficient and strategy-proof, then $f_1(R) \neq o$ *and* $f_2(R) \neq o$. But then either o, p, or q must remain unassigned and $f(R)$ is wasteful, i.e., f cannot be constrained efficient. A similar argument can be used to show that the second part of Lemma 2 implies that a priority structure with property (b) is also unsolvable.

Our proof of Theorem 1 uses the necessary condition that a solvable priority structure cannot have property (a) or (b) as a basic building block. We show first that a solvable priority structure cannot have ties below the second priority level (Step 1). By Assumption 1(A), a priority structure $\succeq$ that is not strict has at least one four-way tie $i_1 \sim_o i_2 \sim_o i_3 \sim_o i_4$. We show that the restriction of $\succeq$ to $\{i_1, i_2, i_3, i_4\}$ can have at most two priority levels (Steps 2 and 3) and then, that $\succeq$ must be either a HET or a TAU priority structure (Steps 4 and 5). In the Supplemental Appendix, we provide counterexamples that show that we cannot dispense with either part of Assumption 1. More specifically, we provide counterexamples of solvable priority structures with an arbitrary number of agents that are not of the strict/HET/TAU type when (i) there is a four-way tie but reversibility is not satisfied and (ii) reversibility is satisfied but there are no four-way ties.[21]

As mentioned before, it is widely known that strict and HET priority structures are both solvable. To the best of our knowledge, the full class of TAU priority structures has not been analyzed in the previous literature. This is perhaps not entirely surprising given that, in contrast to HET priority structures, it seems unlikely that one will be able to find real-world applications that match the key characteristics of a TAU priority structure. It is easy to see that constrained efficient and strategy-proof mechanisms exist as long as there are at most two unqualified agents across all objects/tasks.[22] We have not been able to answer the question of whether TAU priority structures with three or more unqualified agents are solvable, but strongly suspect such priority structures to be unsolvable. To substantiate our suspicion, we now outline why two approaches from the previous literature do not work.

Example 1. Consider a priority-based allocation problem with three agents 1, 2, and 3, and four objects o, p_1, p_2, and p_3, where priorities are

$\succeq_o$	$\succeq_{p_1}$	$\succeq_{p_2}$	$\succeq_{p_3}$
1, 2, 3	2, 3	1, 3	1, 2
	1	2	3

Consider first an *exogenous tie-breaking* rule that randomly picks a strict priority structure $\succeq'$ that respects all strict priority rankings in $\succeq$ and then chooses the outcome of DA with respect to $\succeq'$ for each preference profile. Given the symmetries of the example, we can assume without loss of generality (w.l.o.g.) that $1 \succ'_o 2 \succ'_o 3$. Now consider a preference profile $R = (R_1, R_2, R_3)$ such that $R_1 : p_1 o$, $R_2 : o$, and $R_3 : o p_1$. The DA with respect to $\succeq'$ and R assigns o to 1 and p_1 to 3, which is not constrained efficient.

Next, consider the *trading cycles mechanisms* introduced by Pycia and Ünver (2017). A trading cycles mechanism can be described by a *control rights structure* that, for each possible *submatching*[23] μ and, for each unassigned object o, determines which unassigned agent *controls o at μ*. Given a control rights structure and a preference profile, the outcome of the corresponding trading cycles mechanism is determined sequentially by allowing agents to trade control rights and updating controls according to the control rights structure after each round. In contrast to the usual TTC mechanism, the mechanisms of Pycia and Ünver (2017) allow for a simple constraint on trading: In each round, there can be at most one remaining object o that is *brokered* by the agent who controls it in the sense that the agent is allowed to trade away her control right for o for some other object p, but is not allowed to consume o. One may hope that we can

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[21] In addition, we show in the Supplemental Appendix that any priority structure that (A) has ties only at the top of priority rankings and (B) never assigns equal priority to more than two agents is always solvable.

[22] To see this, fix a TAU priority structure $\succeq$ in which i_1 and i_2 are the only agents who are unqualified for some of the tasks. Let $\succeq'$ be any strict priority structure such that (i) for all $o \in O$ and all $j \in I \setminus \{i_1, i_2\}$, $j \succ'_o i_1$, $j \succ'_o i_2$, $i_1 \succ'_o i_2$ if $i_1 \succ_o i_2$ and $i_2 \succ'_o i_1$ if $i_2 \succeq_o i_1$, and (ii) for $o, p \in O$, $\succeq'_o |_{I \setminus \{i_1, i_2\}} = \succeq'_p |_{I \setminus \{i_1, i_2\}}$. Clearly, the DA with respect to $\succeq'$ is guaranteed to yield constrained efficient outcomes.

[23] A submatching is a matching μ that leaves at least one agent and at least one object unassigned.

construct a constrained efficient and strategy-proof mechanism for TAU priority structures by making agents brokers of the tasks that they are unqualified to perform. However, when there are more than two agents who are unqualified for some task, there is no control rights structure that induces a constrained efficient and strategy-proof trading cycles mechanism. To see this, consider a preference profile $R' = (R'_1, R_2, R'_3)$ such that $R'_1 : o$ and $R'_3 : o$. Consider a trading cycles mechanism f such that $f_1(R') = o$. Now consider the preference profile $R'' = (R'_1, R'_2, R_3)$, where $R'_2 : op_1$ (and, as defined above, $R_3 : op_1$). Since control rights structures can only condition on submatchings, we must have $f_1(R'') = o$. Given that $\{2, 3\} \succ_{p_1} 1$, the second part of Lemma 2 immediately implies that f cannot be constrained efficient and strategy-proof. Analogous arguments show that no trading cycles mechanism that assigns o to 2 or 3 at R' can be constrained efficient and strategy-proof. This implies that there is no constrained efficient and strategy-proof trading cycles mechanism in the example we consider here. $\diamondsuit$

The preceding example suggests that if TAU priority structures are solvable, one probably has to rely on intricate tie-breaking mechanisms to ensure constrained efficiency and strategy-proofness.[24]

3.1 *Strongly solvable priority structures*

In some situations, it is conceivable that agents are able to engage in coordinated deviations from truth-telling. Therefore, it may be desirable to design mechanisms that are not only nonmanipulable by individuals, but also nonmanipulable by groups of agents. In this subsection, we show how such a stronger incentive compatibility notion further narrows the class of solvable priority structures.

We begin by introducing two notions of group-strategy-proofness:

- Mechanism f is *group-strategy-proof* if, for all $J \subseteq I$ and all R, there does not exist a joint manipulation $\tilde{R}_J = (\tilde{R}_j)_{j \in J}$ such that, for all $j \in J$, $f_j(\tilde{R}_J, R_{-J}) \, P_j \, f_j(R)$.
- Mechanism f is *strongly group-strategy-proof* if, for all $J \subseteq I$ and all P, there does not exist a joint manipulation $\tilde{R}_J = (\tilde{R}_j)_{j \in J}$ such that, for all $j \in J$, $f_j(\tilde{R}_J, R_{-J}) \, R_j \, f_j(R)$, and, for at least one $j^* \in J$, $f_{j^*}(\tilde{R}_J, R_{-J}) \, P_{j^*} \, f_{j^*}(R)$.[25]

A priority structure $\succeq$ is *strongly solvable* if there exists a strongly group-strategy-proof and constrained efficient mechanism f for $\succeq$. Our second main result shows that there is only one type of strongly solvable priority structure among all priority structures satisfying reversibility. For this result, we do *not* need to rely on Assumption 1(A).

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[24] In related work, Han (2018) shows that if there are at least four agents, then HET priority structures are the only priority structures for which stability, efficiency, and strategy-proofness are compatible. As we have mentioned previously, stability and efficiency are usually in conflict with each other, while constrained efficiency can always be satisfied.

[25] Barberà et al. (2010, 2016) show that for many relevant resource allocation problems, including the priority-based allocation problem we study in this paper, strategy-proofness and group-strategy-proofness are equivalent. It is well known that this equivalence does not extend to strong group-strategy-proofness.

Theorem 2. *Let $N \geq 4$. If a strongly solvable priority structure $\succeq$ satisfies reversibility, then $\succeq$ must be a HET priority structure.*

One implication of Theorem 2 is that TAU priority structures are not strongly solvable. Note that $N \geq 4$ is necessary in Theorem 2 because Ehlers (2006) shows that for three agents and three objects, TAU structures are strongly solvable. Theorem 2 is related to Theorem 4 in Han (2018), which shows that HET is the only type of priority structure for which stability, efficiency, and strong group-strategy-proofness are compatible. Our results show that, conditional on reversibility, constrained efficiency and strong group-strategy-proofness are already sufficient to be left with only HET priority structures.

4. Conclusion

We characterized the class of priority structures that are solvable in the sense of admitting a constrained efficient and strategy-proof mechanism. Within a large class of priority structures—that, in our opinion, contain most priority structures that could potentially be useful for practical market design purposes—we have shown that there are at most three types of solvable priority structures: strict, the house allocation with existing tenants (HET) type, where, for each object, at most one agent has strictly higher priority than another agent, and the task allocation with unqualified agents (TAU) type, where, for each object, at most one agent has strictly lower priority than another agent. Hence, apart from at most three isolated points in the vast space of possible priority structures, imposing strategy-proofness *and* constrained efficiency comes at a strictly higher welfare cost than imposing only strategy-proofness *and* stability. Out of the three potentially solvable types of priority structures, only HET type structures are strongly solvable in the sense of admitting a constrained efficient and strongly group-strategy-proof mechanism.

Appendix A: Proof of Theorem 1

A.1 *Preliminaries*

In this subsection, we derive several tie-breaking rules that constrained efficient and strategy-proof mechanisms always have to respect. The results in this subsection apply to all priority structures, not just those that satisfy Assumption 1.

Definition 2. Fix a weak priority structure $\succeq$, let $i, j \in I$ be two distinct agents, and let $o, p \in O$ be two objects. An $(i, j; o, p)$ *path* consists of $M + 1$ distinct agents $i \equiv i^0, i^1, \ldots, i^M \in I \setminus \{j\}$ and $M \geq 0$ distinct objects $p^1, \ldots, p^M \in O \setminus \{o, p\}$ such that

(i) $i^m \succ_{p^{m+1}} i^{m+1}$ for all $m \in \{0, \ldots, M-1\}$.

(ii) $i^M \succ_p j$.

We write $i \rightarrow_{p^1} i^1 \rightarrow_{p^2} i^2 \cdots \rightarrow_{p^M} i^M \rightarrow_p j$ to denote the $(i, j; o, p)$ path.

Note that an $(i, j; o, p)$ path is connected to object o only insofar that $o \notin \{p^1, \dots, p^M\}$. Note also that Definition 2 allows for the case of $o = p$. If $o = p$, we write $(i, j; o)$ instead of $(i, j; o, p)$, and we often use the convention $p^{M+1} \equiv p$. Finally, note that, for the case $M = 0$, an $(i, j; o, p)$ path just specifies that $i \succ_p j$. The next lemma uses the concept of paths to derive a simple first tie-breaking rule that *any* stable mechanism has to follow.

Lemma 3. *Fix a weak priority structure* $\succeq$, *let* $i, j \in I$ *be two distinct agents, and let* $o \in O$ *be an object such that* $i \sim_o j$. *Assume that there is an* $(i, j; o)$ *path* $i \to_{p^1} i^1 \cdots \to_{p^M} i^M \to_o j$ *and let* R *be a preference profile such that*[26]

R	R_i	R_{i^m}	R_j	R_z
	o	p^m	o	$\vdots$
	p^1	p^{m+1}	$\vdots$	
	$\vdots$	$\vdots$	$\vdots$	

If f *is stable, then* $f_j(R) \neq o$.

Proof. Suppose to the contrary that $f_j(R) = o$. Since o can only be allocated to one agent, we must have $f_i(R) \neq o$. Then stability together with $i = i^0$ and $i^0 \succ_{p^1} i^1$ implies $f_{i^1}(R) \neq p^1$. Proceeding inductively, assume that, for some $M' \geq 1$ and all $m \in \{1, \dots, M' - 1\}$, $f_{i^m}(R) \neq p^m$. The definition of an $(i, j; o)$ path, the construction of R, and the stability of $f(R)$ imply $f_{i^{M'}}(R) \neq p^{M'}$ given that $i^{M'-1} \succ_{p^{M'}} i^{M'}$. In particular, $f_{i^M}(R) \neq p^M$ and $o \, R_{i^M} \, f_{i^M}(R)$ (by $p^{M+1} = o$). Given that $i^M \succ_o j$, $o \, R_{i^M} \, f_{i^M}(R)$ is compatible with stability only when $f_j(R) \neq o$. This contradiction completes the proof. $\square$

Next, we derive a rule for breaking three-way ties. For this, we need the following notion of compatibility between two paths in the priority structure.

Definition 3. Fix a weak priority structure $\succeq$, let $i, j, k \in I$ be three distinct agents, and let $o, p \in O$ be two objects. An $(i, k; o, p)$ path $i \to_{p^1} i^1 \cdots \to_{p^M} i^M \to_p k$ is *compatible* with a $(j, k; o, p)$ path $j \to_{q^1} j^1 \cdots \to_{q^N} j^N \to_p k$ if there exist $m^* \leq M$ and $n^* \leq N$ such that

(i) $\{i, p^1, i^1, \dots, p^{m^*}, i^{m^*}\} \cap \{j, q^1, j^1, \dots, q^{n^*}, j^{n^*}\} = \varnothing$.

(ii) $M - m^* = N - n^*$.

(iii) for all $t \in \{1, \dots, M - m^*\}$, $(p^{m^*+t}, i^{m^*+t}) = (q^{n^*+t}, j^{n^*+t})$.

Roughly speaking, compatibility of two paths requires that the paths coincide from the first point at which they intersect. This includes the case in which the paths are disjoint, i.e., when $m^* = M$ and $n^* = N$. The simplest possible example of compatible paths

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[26] In the preference profile R, m runs from 1 through M and z runs through all agents in $I \setminus \{i^1, \dots, i^M, i, j\}$. This convention applies to all preference profiles used in the Appendix.

is when $i \succ_p k$ and $j \succ_p k$ (where $M = N = 0$). We l now use the concept of compatible paths to derive a second tie-breaking rule that *any* constrained efficient and strategy-proof mechanism has to respect.

Lemma 4. *Fix a weak priority structure $\succeq$, let $i, j, k \in I$ be three distinct agents, and let $o \in O$ be an object such that $i \sim_o j \sim_o k$. Let $p \in O \setminus \{o\}$ and assume that there exists an $(i, k; o, p)$ path $i \rightarrow_{p^1} i^1 \cdots \rightarrow_{p^M} i^M \rightarrow_p k$ that is compatible with the $(j, k; o, p)$ path $j \rightarrow_{q^1} j^1 \cdots \rightarrow_{q^N} j^N \rightarrow_p k$. Let R be a preference profile such that*

R	R_i	R_j	R_k	R_{i^m}	R_{j^n}
	o	o	o	p^m	q^n
	p^1	q^1	$\vdots$	p^{m+1}	q^{n+1}

and such that for all $z \in I \setminus \{i, j, k, i^1, \dots, i^M, j^1, \dots, j^N\}$ and all $q \in \{o, p, p^1, \dots, p^M, q^1, \dots, q^N\}$ for which $z \succeq_q l$ for some $l \in \{i, j, k, i^1, \dots, i^M, j^1, \dots, j^N\}$, $z \, P_z \, q$. If f is constrained efficient and strategy-proof, then $f_k(R) \neq o$.

Proof. Note that the compatibility of the two paths ensures that R is well defined: For any $l \in \{i^1, \dots, i^M\} \cap \{j^1, \dots, j^N\}$, there exist m^*, n^*, and t such that $l = i^{m^*+t} = j^{n^*+t}$ and $p^{m^*+t} = q^{n^*+t}$ as well as $p^{m^*+t+1} = q^{n^*+t+1}$.

Now let f be an arbitrary constrained efficient and strategy-proof mechanism. Suppose that, contrary to what we want to show, $f_k(R) = o$. We establish that f cannot be constrained efficient and strategy-proof. Throughout the proof, we specify only the preferences of agents in $\{i, j, k, i^1, \dots, i^M, j^1, \dots, j^N\}$ over objects in $\{o, p, p^1, \dots, p^M, q^1, \dots, q^N\}$.[27]

Consider first the preference profile

R^1	R_i	R_j	R_k^1	R_{i^m}	R_{j^n}
	o	o	p	p^m	q^n
	p^1	q^1	o	p^{m+1}	q^{n+1}

We claim that $f_k(R) = o$ implies $f_k(R^1) = o$. Suppose to the contrary that $f_k(R^1) \neq o$. Since $f_k(R) = o$, strategy-proofness then requires $f_k(R^1) = p$. Since there is only one copy of o, we must have either $f_i(R^1) \neq o$ or $f_j(R^1) \neq o$. Suppose the former, i.e., $f_i(R^1) \neq o$. Then, by constrained efficiency, we must have $f_j(R^1) = o$. By the definition of an $(i, k; o, p)$ path and the construction of R, stability implies that, for all $m = 0, \dots, M$, $f_{i^m}(R^1) = p^{m+1}$. Given that $p^{M+1} = p$ and $i^M \neq k$, we obtain a contradiction to our assumption that $f_k(R^1) = p$. The argument is completely symmetric in the case $f_j(R^1) \neq o$. Since $f_k(R^1) = p$ necessarily leads to a contradiction, we must have $f_k(R^1) = o$.

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[27] This implicitly assumes that the preferences of agents in $I \setminus \{i, j, k, i^1, \dots, i^M, j^1, \dots, j^N\}$ are fixed at the preferences these agents have in the profile R. This convention applies to all proofs in the Appendix.

We now complete the proof of Lemma 4 by showing that no constrained efficient and strategy-proof mechanism can assign o to k at R^1. The following diagram summarizes our proof:[28]

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{array}{c|ccccc} R^1 & R_i & R_j & R_k^1 & R_{i^m} & R_{j^n} \\ \hline & o & o & p & p^m & q^n \\ & p^1 & q^1 & \boxed{o} & p^{m+1} & q^{n+1} \end{array} \arrow[r] \arrow[d] \& \begin{array}{c|ccccc} R^2 & R_i^1 & R_j & R_k^1 & R_{i^m} & R_{j^n} \\ \hline & o & \boxed{o} & p & p^m & q^n \\ & & q^1 & o & p^{m+1} & q^{n+1} \end{array} \arrow[d] \\
\begin{array}{c|ccccc} R^3 & R_i & R_j^1 & R_k^1 & R_{i^m} & R_{j^n} \\ \hline & \boxed{o} & o & p & p^m & q^n \\ & p^1 & & o & p^{m+1} & q^{n+1} \end{array} \arrow[r] \& \begin{array}{c|ccccc} R^4 & R_i^1 & R_j^1 & R_k^1 & R_{i^m} & R_{j^n} \\ \hline & \boxed{o} & \boxed{o} & p & p^m & q^n \\ & & & o & p^{m+1} & q^{n+1} \end{array}
\end{tikzcd}$$

We show first that $f_k(R^1) = o$ implies $f_j(R^2) = o$. Suppose to the contrary that $f_j(R^2) \neq o$. Since f is strategy-proof and $f_i(R^1) \neq o$, we must have $f_i(R^2) \neq o$ as well. Next, note that since $i \rightarrow_{p^1} i^1 \cdots \rightarrow_{p^M} i^M \rightarrow_p k$ is an $(i, k; o, p)$ path and $j \rightarrow_{q^1} j^1 \cdots \rightarrow_{q^N} j^N \rightarrow_p k$ is a $(j, k; o, p)$ path, we must have $o \notin \{p^1, \dots, p^M, q^1, \dots, q^N, p\}$ given that $o \neq p$. Note that by definition of R and R^2, for all $m \in \{1, \dots, M\}$, $f_{i^m}(R^2) \in \{p^m, p^{m+1}\}$, and for all $n \in \{1, \dots, N\}$, $f_{j^n}(R^2) \in \{q^n, q^{n+1}\}$. Hence, again by definition of R^2, object o is assigned at R^2 to agent i, j, or k. But then it has to be the case that $f_k(R^2) = o$ if $f_j(R^2) \neq o$ and $f_i(R^2) \neq o$ since $f(R^2)$ would not be non-wasteful otherwise. Since $f_j(R^2) \neq o$, stability requires that, for all $n = 0, \dots, N$, $f_{j^n}(R^2) = q^{n+1}$ given that $j^n \succ_{q^{n+1}} j^{n+1}$ (where $j^{N+1} \equiv k$). But then $j = j^0, \dots, j^{N+1} = k$ form a stable improvement cycle of $f(R^2)$ at R^2,[29] contradicting constrained efficiency of f. Hence, we must have $f_j(R^2) = o$.

A completely symmetric argument shows that $f_k(R^1) = o$ implies $f_i(R^3) = o$. We omit the details.

However, if f is strategy-proof, $f_j(R^2) = o$ implies $f_j(R^4) = o$ and $f_i(R^3) = o$ implies $f_i(R^4) = o$. Since there is only one copy of o and $i \neq j$, $f_j(R^4) = f_i(R^4) = o$ is impossible. Hence, f cannot be a constrained efficient and strategy-proof mechanism if $f_k(R) = o$. □

The next lemma lists three further tie-breaking rules that are important for the proof of Theorem 1. The proof involves a series of straightforward but tedious implications of Lemmas 3, 4, and we relegate the full details to the Supplemental Appendix.

Lemma 5. *Fix a weak priority structure $\succeq$.*

(a) *Let $i, j, k \in I$ be three distinct agents and let $o, p \in O$ be two distinct objects such that $i \sim_o j \sim_o k$ and $i \succ_p k \succ_p j$. Let R be a preference profile such that*

$$\begin{array}{ccc} R_i & R_j & R_k \\ \hline o & o & p \end{array}$$

---

[28] Here and in all proofs that follow, arrows indicate how we move between profiles and boxes indicate object assignments.

[29] Note that the definition of compatible paths and stability imply that, for all $m \in \{1, \dots, M\}$ such that $i^m \notin \{j^1, \dots, j^N\}$, $f_{i^m}(R^2) = p^m$.

and such that for all $z \in I \setminus \{i, j, k\}$ *and all* $\tilde{o} \in \{o, p\}$ *for which* $z \succeq_{\tilde{o}} \tilde{i}$ *for some* $\tilde{i} \in \{i, j, k\}$, $z \, P_z \, \tilde{o}$. *If* f *is constrained efficient and strategy-proof, then* $f_i(R) = o$.

(b) *Let* $i, j, k, l \in I$ *be four distinct agents and let* $o, p, q \in O$ *be three distinct objects such that* $i \sim_o j \sim_o k \sim_o l$, $\{i, j\} \succ_p k \succ_p l$, *and* $i \succeq_q l \succ_q j$. *Let* R *be a preference profile such that*

R_i	R_j	R_k	R_l
o	o	p	q

and such that for all $z \in I \setminus \{i, j, k, l\}$ *and all* $\tilde{o} \in \{o, p, q\}$ *for which* $z \succeq_{\tilde{o}} \tilde{i}$ *for some* $\tilde{i} \in \{i, j, k, l\}$, $z \, P_z \, \tilde{o}$. *If* f *is constrained efficient and strategy-proof, then* $f_i(R) = o$.

(c) *Let* $i, j, k, l \in I$ *be four distinct agents and let* $o, p, q \in O$ *be three distinct objects such that* $i \sim_o j \sim_o k$, $i \succ_p l \succ_p k$, *and* $k \succ_q l \succ_q j$. *Let* R *be a preference profile such that*

R_i	R_j	R_k	R_l
o	o	p	q

and such that for all $z \in I \setminus \{i, j, k, l\}$ *and all* $\tilde{o} \in \{o, p, q\}$ *for which* $z \succeq_{\tilde{o}} \tilde{i}$ *for some* $\tilde{i} \in \{i, j, k, l\}$, $z \, P_z \, \tilde{o}$. *If* f *is constrained efficient and strategy-proof, then* $f_i(R) = o$.

Next, we use Lemmas 3, 4, 5 to derive two simple conditions for the nonexistence of a constrained efficient and strategy-proof mechanism. Again, we relegate the details to the Supplemental Appendix.

Lemma 6. *Fix a weak priority structure* $\succeq$.

(a) *Let* $i, j \in I$ *be two distinct agents and let* $o \in O$ *be an object such that* $i \sim_o j$. *If there is an* $(i, j; o)$ *path* $i \rightarrow_{p^1} i^1 \cdots \rightarrow_{p^M} i^M \rightarrow_o j$ *that is compatible with a* $(j, i; o)$ *path* $j \rightarrow_{q^1} j^1 \cdots \rightarrow_{q^N} j^N \rightarrow_o i$, *then* $\succeq$ *is unsolvable.*

(b) *Let* $i, j, k, l \in I$ *be four distinct agents and let* $o \in O$ *be an object such that* $i \sim_o j \sim_o k \sim_o l$. *If there exist two objects* $p, q \in O$ *such that* $i \succ_p k \succ_p j$ *and* $j \succ_q l \succ_q i$, *then* $\succeq$ *is unsolvable.*

Finally, the next lemma points out six basic unsolvable priority structures. We use these structures as building blocks for the proof of Theorem 1. The proof is provided in the Supplemental Appendix.

Lemma 7. *Let* i_1, i_2, i_3, i_4, *and* i_5 *be five distinct agents and let* o_1, o_2, o_3, o_4, *and* o_5 *be five distinct objects. Each of the following priority structures is unsolvable:*

$$\boxed{\begin{array}{rl} i_1 & \sim_{o_1} i_2 \sim_{o_1} i_3 \sim_{o_1} i_4 \\ \{i_1, i_2\} & \succ_{o_2} i_3 \succ_{o_2} i_4 \\ \{i_1, i_2\} & \succ_{o_3} i_3 \\ i_2 & \succeq_{o_4} i_4 \succ_{o_4} i_1 \\ i_1 & \succeq_{o_5} i_4 \succ_{o_5} i_2 \end{array}} \qquad (1*)$$

$$\boxed{\begin{array}{llllllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 & \sim_{o_1} & i_4 \\ \{i_1, i_2\} & \succ_{o_2} & i_3 & \succ_{o_2} & i_4 & & \\ i_4 & \succ_{o_3} & \{i_2, i_5\} & & & & \\ i_2 & \succ_{o_4} & i_5 & \succ_{o_4} & i_1 & & \end{array}} \tag{2$*$}$$

$$\boxed{\begin{array}{lllllllllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 & & & & & & \\ i_1 & \sim_{o_2} & i_4 & \sim_{o_2} & i_5 & \succ_{o_2} & i_2 & \succ_{o_2} & i_3 \\ i_4 & \succ_{o_3} & i_5 & \succ_{o_3} & i_1 & & & & \\ \{i_2, i_3\} & \succ_{o_4} & i_4 & & & & & & \end{array}} \tag{3$*$}$$

$$\boxed{\begin{array}{lllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ \{i_2, i_3\} & \succ_{o_2} & i_4 & & \\ i_4 & \succ_{o_3} & i_1 & \succ_{o_3} & \{i_2, i_3\} \end{array}} \tag{4$*$}$$

$$\boxed{\begin{array}{lllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ i_1 & \succ_{o_2} & i_4 & & \\ i_2 & \succ_{o_3} & i_5 & & \\ \{i_2, i_4\} & \succ_{o_4} & i_3 & \succ_{o_4} & i_1 \\ \{i_1, i_5\} & \succ_{o_5} & i_3 & \succ_{o_5} & i_2 \end{array}} \tag{5$*$}$$

$$\boxed{\begin{array}{lllll} i_1 & \sim_{o_1} & i_2 & \sim_{o_1} & i_3 \\ \{i_1, i_2\} & \succ_{o_2} & i_5 & \succ_{o_2} & i_3 \\ i_2 & \succ_{o_3} & i_4 & & \\ i_3 & \succ_{o_4} & i_5 & \succ_{o_4} & i_1 \\ \{i_3, i_4\} & \succ_{o_5} & i_5 & \succ_{o_5} & i_2 \end{array}} \tag{6$*$}$$

A.2 *Proof of Theorem 1*

Throughout the proof, we fix a solvable weak priority structure $\succeq$ that satisfies Assumption 1.

STEP 1. There cannot exist four distinct agents $1, 2, 3, 4 \in I$ and an object $o \in O$ such that $\{3, 4\} \succ_o 1 \sim_o 2$.

Suppose to the contrary. Since $3 \succ_o 1$ and $4 \succ_o 2$, Assumption 1(B) guarantees that there exist two objects q_1 and q_2 such that $1 \succ_{q_1} 3$ and $2 \succ_{q_2} 4$.

If $q_1 = q_2$, then we must have either $\{1, 2\} \succ_{q_1} 3$ (if $4 \succeq_{q_1} 3$) or $\{1, 2\} \succ_{q_1} 4$ (if $3 \succeq_{q_1} 4$), say $\{1, 2\} \succ_{q_1} 3$. Then $1 \rightarrow_{q_1} 3 \rightarrow_o 2$ is an $(1, 2; o)$ path that is compatible with the $(2, 1; o)$ path $2 \rightarrow_{q_1} 3 \rightarrow_o 1$. But then the first part of Lemma 6 implies that $\succeq$ is unsolvable. Hence, we must have $q_1 \neq q_2$.

If $q_1 \neq q_2$, then $1 \rightarrow_{q_1} 3 \rightarrow_o 2$ is a $(1, 2; o)$ path that is compatible with the $(2, 1; o)$ path $2 \rightarrow_{q_2} 4 \rightarrow_o 1$. The first part of Lemma 6 again implies that $\succeq$ is unsolvable. □

STEP 2. There cannot exist four distinct agents $1, 2, 3, 4 \in I$ and three distinct objects $o, p, q \in O$ such that $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{1, 2\} \succ_p 3 \succ_p 4$, and $\{2, 4\} \succ_q 3 \succ_q 1$.

Suppose to the contrary. We derive a contradiction through a series of six claims.

Claim 1. *We have* $2 \succeq_p 1$ *and* $2 \succeq_q 4$.

Proof. We argue first that $2 \succeq_p 1$. If $1 \succ_p 2$, then the second part of Lemma 6 implies that $\succeq$ is unsolvable since $1 \succ_p 2 \succ_p 4$ and $4 \succ_q 3 \succ_q 1$. The argument for $2 \succeq_q 4$ is analogous. □

Before proceeding to the second claim, note that by Claim 1 and the priority rankings assumed in Step 2, we must have $2 \succeq_p 1 \succ_p 3 \succ_p 4$ and $2 \succeq_q 4 \succ_q 3 \succ_q 1$.

Claim 2. *There exists an object* $q' \in O \setminus \{o, p, q\}$ *such that* $1 \sim_{q'} 4 \succ_{q'} 2$ *and* $3 \succ_{q'} 2$.

Proof. Since $2 \succ_p 4$, Assumption 1(B) implies that there exists an object q' such that $4 \succ_{q'} 2$. By Claim 1 and the properties of $\succeq$ that were already specified, we must have $q' \in O \setminus \{o, p, q\}$.

We argue first that we must have $1 \sim_{q'} 4$. If $1 \succ_{q'} 4 \succ_{q'} 2$, then the second part of Lemma 6 implies that $\succeq$ is unsolvable since we also have that $2 \succ_q 3 \succ_q 1$. Similarly, if $4 \succ_{q'} 1 \succ_{q'} 2$ (or $4 \succ_{q'} 2 \succ_{q'} 1$), then the second part of Lemma 6 implies that $\succeq$ is unsolvable since we also have that $2 \succ_p 3 \succ_p 4$ (or $1 \succ_p 3 \succ_p 4$). If $1 \sim_{q'} 2$, the first part of Lemma 6 implies that $\succeq$ is unsolvable since $\{1, 2\} \succ_p 4 \succ_{q'} 1 \sim_{q'} 2$. Since we have now exhausted all possible cases, we must have $1 \sim_{q'} 4 \succ_{q'} 2$.

Next, assume that, contrary to what we want to show, $2 \succeq_{q'} 3$. Since $\{1, 4\} \succ_{q'} 2$, Step 1 implies $2 \succ_{q'} 3$. Thus, $1 \sim_{q'} 4 \succ_{q'} 2 \succ_{q'} 3$. By Assumption 1(B) and the properties of $\succeq$ that were already specified, there has to exist an object $\tilde{q} \in O \setminus \{o, p, q, q'\}$ such that $3 \succ_{\tilde{q}} 2$. If $4 \succ_{\tilde{q}} 2$, the arguments used to show that $1 \sim_{q'} 4$ are easily seen to imply $1 \sim_{\tilde{q}} 4$. But then we have $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{1, 2\} \succ_p 3 \succ_p 4$, $\{1, 2\} \succ_{q'} 3$, $2 \succeq_q 4 \succ_q 1$, and $1 \succeq_{\tilde{q}} 4 \succ_{\tilde{q}} 2$, so that, by Lemma 7, $\succeq$ is unsolvable because it is of the form in (1∗). If $2 \sim_{\tilde{q}} 4$, the first part of Lemma 6 implies that $\succeq$ is unsolvable since $\{2, 4\} \succ_q 3 \succ_{\tilde{q}} 2 \sim_{\tilde{q}} 4$. Hence, we must have $2 \succ_{\tilde{q}} 4$. A completely symmetric argument establishes that $2 \succ_{\tilde{q}} 1$. Since $\{1, 4\} \succ_{q'} 2 \succ_{q'} 3$ and $3 \succ_{\tilde{q}} 2 \succ_{\tilde{q}} \{1, 4\}$, the first part (if $1 \sim_{\tilde{q}} 4$) or the second part (if $1 \nsim_{\tilde{q}} 4$) of Lemma 6 again implies that $\succeq$ must be unsolvable. Thus, we must have $3 \succ_{q'} 2$ and this completes the proof of Claim 2. □

Claim 3. *We have* $2 \sim_p 1$ *and* $2 \sim_q 4$.

Proof. We show that $2 \sim_p 1$ (the arguments to establish $2 \sim_q 4$ are completely analogous). By Claim 1, $2 \nsim_p 1$ implies $2 \succ_p 1$. By Claim 2, there exists a $q' \in O \setminus \{o, p, q\}$ such that $\{1, 3, 4\} \succ_{q'} 2$. Since $1 \sim_o 3 \sim_o 4$ and $2 \succ_p 1 \succ_p \{3, 4\}$, $\succeq$ is of the form in (4∗) (where 2 is in the role of i_4) and, hence, by Lemma 7, unsolvable. The only possible case is thus $2 \sim_p 1$. □

Claim 4. *There does not exist an agent* $5 \in I \setminus \{1, 2, 3, 4\}$ *such that either* $1 \succ_p 5 \succ_p 4$ *or* $4 \succ_q 5 \succ_q 1$.

PROOF. Suppose to the contrary that there is an agent $5 \in I \setminus \{1, 2, 3, 4\}$ such that $1 \succ_p 5 \succ_p 4$ (the argument in the case $4 \succ_q 5 \succ_q 1$ is completely analogous).

We argue first that we must have $4 \succ_o 5$. Since $\{1, 2\} \succ_p 5$ and $1 \sim_o 2$, the first part of Lemma 6 implies that $\succeq$ is unsolvable if $5 \succ_o 1 \sim_o 2$. Thus, it has to be the case that $1 \sim_o 2 \sim_o 3 \sim_o 4 \succeq_o 5$. If $1 \sim_o 2 \sim_o 3 \sim_o 4 \sim_o 5$, then the second part of Lemma 6 implies that $\succeq$ is unsolvable since we also have $4 \succ_q 3 \succ_q 1$ and $1 \succ_p 5 \succ_p 4$. Hence, we must have $4 \succ_o 5$.

For the remainder of the proof of Claim 4, let $q' \in O \setminus \{o, p, q\}$ be such that $1 \sim_{q'} 4 \succ_{q'} 2$ and $3 \succ_{q'} 2$. Remember that the existence of such an object follows from Claim 2.

Next, note that the solvability of $\succeq$ implies $4 \sim_{q'} 5$: If $4 \succ_{q'} 5$, Lemma 7 implies that $\succeq$ is unsolvable because it is of the form in (2∗) given that we also have $\{2, 4\} \succ_q 3 \succ_q 1$, $1 \succ_{q'} \{2, 5\}$, and $2 \succ_p 5 \succ_p 4$; if $5 \succ_{q'} 4$, the first part of Lemma 6 implies that $\succeq$ is unsolvable given that $\{1, 4\} \succ_o 5 \succ_{q'} 4 \sim_{q'} 1$. Since $3 \succ_{q'} 2$, the only remaining options are $1 \sim_{q'} 4 \sim_{q'} 3 \sim_{q'} 5$ and $3 \succ_{q'} 1 \sim_{q'} 4 \sim_{q'} 5 \succ_{q'} 2$. In the first case, the second part of Lemma 6 implies that $\succeq$ is unsolvable since $1 \succ_p 5 \succ_p 4$ and $4 \succ_q 3 \succ_q 1$. In the second case, the first part of Lemma 6 implies that $\succeq$ is unsolvable since $\{1, 4\} \succ_o 5 \succ_p 4 \succ_q 3 \succ_{q'} 1 \sim_{q'} 4$. This completes the proof of Claim 4. □

CLAIM 5. *There does not exist an agent* $5 \in I \setminus \{1, 2, 3, 4\}$ *such that either* $4 \succ_p 5$ *or* $1 \succ_q 5$.

PROOF. Suppose to the contrary that there is an agent $5 \in I \setminus \{1, 2, 3, 4\}$ such that $4 \succ_p 5$ (the argument for $1 \succ_q 5$ is completely symmetric). Since $\{1, 2, 3, 4\} \succ_p 5$ and $1 \sim_o 2 \sim_o 3 \sim_o 4$, the first part of Lemma 6 implies that $\succeq$ is unsolvable if $5 \succ_o 1$. Hence, we must have $4 \succeq_o 5$.

First, we show that there is no object $\tilde{q}$ such that $4 \succ_{\tilde{q}} 5 \succ_{\tilde{q}} 2$; otherwise, Lemma 7 implies that $\succeq$ is unsolvable because it is of the form in (2∗) given that $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{2, 4\} \succ_q 3 \succ_q 1$, $4 \succ_{\tilde{q}} 5 \succ_{\tilde{q}} 2$, and $1 \succ_p \{4, 5\}$.

Second, we argue that there is no object $\tilde{q}$ such that $5 \succ_{\tilde{q}} \{2, i\}$ for some $i \in \{1, 4\}$. Assume to the contrary that there is an object $\tilde{q}$ such that $5 \succ_{\tilde{q}} \{1, 2\}$ (the argument in the case $5 \succ_{\tilde{q}} \{2, 4\}$ is completely symmetric). If $5 \succ_{\tilde{q}} \{1, 2, 4\}$, then by $\{1, 2, 4\} \succ_p 5$ and the first part of Lemma 6, we must have $1 \nsim_{\tilde{q}} 2$, $1 \nsim_{\tilde{q}} 4$, and $2 \nsim_{\tilde{q}} 4$. But then Lemma 7 implies that $\succeq$ is unsolvable because it is of the form in (4∗) given that we also have $1 \sim_o 2 \sim_o 4$ and $\{1, 2, 4\} \succ_p 5$. If $4 \succeq_{\tilde{q}} 5 \succ_{\tilde{q}} \{1, 2\}$, the second part of Lemma 6 implies that $\succeq$ is unsolvable given that $\{1, 2\} \succ_p 3 \succ_p 4$ and given that Step 1 implies that we cannot have $1 \sim_{\tilde{q}} 2$.

Third, note that for any $\tilde{q}$ such that $5 \succ_{\tilde{q}} 2$, we must have $1 \sim_{\tilde{q}} 4 \sim_{\tilde{q}} 5 \succ_{\tilde{q}} 2$: By the previous arguments in the proof of Claim 5, we must have $4 \sim_{\tilde{q}} 5 \succ_{\tilde{q}} 2$ and $1 \succeq_{\tilde{q}} 5 \succ_{\tilde{q}} 2$. If $1 \succ_{\tilde{q}} 4$, we have $1 \succ_{\tilde{q}} 4 \succ_{\tilde{q}} 2$ and the second part of Lemma 6 implies that $\succeq$ is unsolvable since we also have that $2 \succ_q 3 \succ_q 1$.

Finally, we show that if $\succeq$ is solvable and $4 \succ_p 5$, there cannot exist $\hat{q} \in O \setminus \{o, p\}$ such that $5 \succ_{\hat{q}} 2$. Hence, Assumption 1(B) must be violated given that $2 \succeq_o 5$ and $2 \succ_p 5$. This then completes the proof of Claim 5. Suppose to the contrary that there exists $\hat{q}$ such that $5 \succ_{\hat{q}} 2$. We show that $\succeq$ must be unsolvable. The previous arguments in the proof of Claim 5 immediately imply that $1 \sim_{\hat{q}} 4 \sim_{\hat{q}} 5 \succ_{\hat{q}} 2$. We now distinguish four cases according to the priority of agent 3 for $\hat{q}$ and show that $\succeq$ must be unsolvable for each of these cases.

Case 1. $2 \succeq_{\hat{q}} 3$. If $2 \sim_{\hat{q}} 3$, then by Step 1, $\succeq$ is unsolvable. Suppose that $2 \succ_{\hat{q}} 3$. Note that $\{1,4\} \succ_{\hat{q}} 2$ implies $\hat{q} \in O \setminus \{o, p, q\}$. Since $1 \sim_{\hat{q}} 4 \succ_{\hat{q}} 2$, the arguments in the proof of Claim 2 are easily seen to imply that $\succeq$ is unsolvable if $2 \succ_{\hat{q}} 3$.

Case 2. $1 \sim_{\hat{q}} 3 \sim_{\hat{q}} 4 \sim_{\hat{q}} 5 \succ_{\hat{q}} 2$. By Assumption 1(B), there must exist an object $\hat{q}' \in O \setminus \{o, p, \hat{q}\}$ such that $5 \succ_{\hat{q}'} 1$. If $\hat{q}' = q$, Lemma 7 implies that $\succeq$ is unsolvable because it is of the form in (4∗) given that $3 \sim_{\hat{q}} 4 \sim_{\hat{q}} 5$, $1 \succ_p 3 \succ_p \{4,5\}$, and $\{4,5\} \succ_q 1$. Hence, we must have $1 \succeq_q 5$ and $\hat{q}' \neq q$. Since $1 \succ_p 3 \succ_p 4 \succ_p 5$ and $4 \succ_q 3 \succ_q 1 \succeq_q 5$, $\succeq$ is unsolvable if $5 \succ_{\hat{q}'} 4$: If $5 \succ_{\hat{q}'} 4 \sim_{\hat{q}'} 1$, then the first part of Lemma 6 implies that $\succeq$ is unsolvable given that we also have $\{4,1\} \succ_p 5$; if $5 \succ_{\hat{q}'} 4 \succ_{\hat{q}'} 1$, then the second part of Lemma 6 implies that $\succeq$ is unsolvable given that we also have $1 \succ_p 3 \succ_p 5$; if $5 \succ_{\hat{q}'} 1 \succ_{\hat{q}'} 4$, then the second part of Lemma 6 implies that $\succeq$ is unsolvable given that we also have $4 \succ_q 3 \succ_q 5$. Hence, it has to be the case that $4 \succeq_{\hat{q}'} 5 \succ_{\hat{q}'} 1$.

But then Lemma 7 implies that $\succeq$ is not solvable since it is of the form in (4∗) given that $3 \sim_{\hat{q}} 4 \sim_{\hat{q}} 5$, $1 \succ_p 3 \succ_p \{4,5\}$, and $\{4,5\} \succ_{\hat{q}'} 1$.

Case 3. $3 \succ_{\hat{q}} 1 \sim_{\hat{q}} 4 \sim_{\hat{q}} 5 \succ_{\hat{q}} 2$. The first part of Lemma 6 implies that $\succeq$ is unsolvable since $\{1,4\} \succ_p 5$ and since there exists a $\hat{q}' \in O \setminus \{p, \hat{q}\}$ such that $5 \succ_{\hat{q}'} 3$.

Case 4. $1 \sim_{\hat{q}} 4 \sim_{\hat{q}} 5 \succ_{\hat{q}} 3 \succ_{\hat{q}} 2$. Note first that we must have $4 \succeq_q 5$; otherwise, Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (4∗) given that $1 \sim_o 3 \sim_o 4$, $\{1,3,4\} \succ_p 5$, and $5 \succ_q 4 \succ_q \{1,3\}$. But then Assumption 1(B) implies that there exists an object $\hat{q}' \in O \setminus \{o, p, q, \hat{q}\}$ such that $5 \succ_{\hat{q}'} 4$. If $1 \succ_{\hat{q}'} 4$, Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (5∗) given that we also have $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{2,4\} \succ_q 3 \succ_q 1$, $\{1,4,5\} \succ_{\hat{q}} 3 \succ_{\hat{q}} 2$, and $2 \succ_p 5$. Hence, it has to be the case that $5 \succ_{\hat{q}'} 4 \succeq_{\hat{q}'} 1$. If $4 \sim_{\hat{q}'} 1$, then the first part of Lemma 6 implies that $\succeq$ is unsolvable since $\{4,1\} \succ_p 5$. Thus, we are left to consider the possibility that $5 \succ_{\hat{q}'} 4 \succ_{\hat{q}'} 1$. By the previous arguments in the proof of Claim 5, $5 \succ_{\hat{q}'} \{1,4\}$ implies $2 \succeq_{\hat{q}'} 5$. But then the second part of Lemma 6 implies that $\succeq$ is unsolvable since $1 \succ_{\hat{q}} 3 \succ_{\hat{q}} 2$ and $2 \succ_{\hat{q}'} 4 \succ_{\hat{q}'} 1$. ☐

Claim 6. *A priority structure of the type specified in Step 2 is unsolvable.*

Proof. For this part of the proof, fix an object $q' \in O \setminus \{o, p, q\}$ such that $1 \sim_{q'} 4 \succ_{q'} 2$ and $3 \succ_{q'} 2$. Such an object exists by Claim 2.

Next, note that Claims 4 and 5 imply that, for all $j \in I \setminus \{1,2,3,4\}$, either $j \succeq_p 1$ or $j \sim_p 4$, and either $j \succeq_q 4$ or $j \sim_q 1$. By Step 1, $j \sim_p 4$ and $j \sim_q 1$ are both impossible. Hence, for all $j \in I \setminus \{1,2,3,4\}$, $j \succeq_p 1$ and $j \succeq_q 4$.

We now show that, for all $j \in I \setminus \{1,2,3,4\}$, $4 \succ_o j$. If there is a $j \in I \setminus \{1,2,3,4\}$ such that $j \succ_o 1$, the first part of Lemma 6 implies that $\succeq$ is unsolvable since $j \succ_o 1 \sim_o 2$, $\{1,2\} \succ_p 3$, and there exists a $\tilde{q} \in O \setminus \{o, p, q\}$ such that $3 \succ_{\tilde{q}} j$. If there is a $j \in I \setminus \{1,2,3,4\}$ such that $j \sim_o 1$, Assumption 1(B) requires that there exists an object $\tilde{q} \in O \setminus \{o, p, q\}$ such that $1 \succ_{\tilde{q}} j$. If $\tilde{q} = q'$, the second part of Lemma 6 implies that $\succeq$ is unsolvable since

$\{2, j\} \succ_q 3 \succ_q 1$ and $1 \succ_{\tilde{q}} \{2, j\}$. Hence, we must have $\tilde{q} \neq q'$. But then Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (5∗) given that $1 \sim_o 3 \sim_o 4$, $\{1, 2\} \succ_p 3 \succ_p 4$, $\{4, j\} \succ_q 3 \succ_q 1$, $4 \succ_{q'} 2$, and $1 \succ_{\tilde{q}} j$. Since we have exhausted all possible cases, we must have $4 \succ_o j$.

Next, we argue that, for all $j \in I \setminus \{1, 2, 3, 4\}$, $j \sim_p 1$ and $j \sim_q 4$. Suppose to the contrary that there is a $j \in I \setminus \{1, 2, 3, 4\}$ such that $j \succ_p 1$ (the argument in the case $j \succ_q 4$ is completely analogous). By Claim 3, we have $1 \sim_p 2$. By the previous arguments, $\{1, 2\} \succ_o j \succ_p 1 \sim_p 2$ and $\succeq$ is not solvable by the first part of Lemma 6.

For the remainder of the proof, let 5 and 6 be two distinct agents in $I \setminus \{1, 2, 3, 4\}$.[30] By the above, we can assume without loss of generality that $1 \sim_o 2 \sim_o 3 \sim_o 4 \succ_o 5 \succ_o 6$ (where again $5 \sim_o 6$ is impossible by Step 1), $1 \sim_p 2 \sim_p 5 \sim_p 6 \succ_p 3 \succ_p 4$, and $2 \sim_q 4 \sim_q 5 \sim_q 6 \succ_q 3 \succ_q 1$.

We now show that we must have $5 \succ_{q'} 2$. Suppose to the contrary that $2 \succeq_{q'} 5$. By Step 1 and $\{1, 4\} \succ_{q'} 2$, we must have $2 \not\sim_{q'} 5$. If $2 \succ_{q'} 5$, Assumption 1(B) implies that there exists a $\hat{q} \in O \setminus \{o, p, q, q'\}$ such that $5 \succ_{\hat{q}} 2$. If either $1 \succ_{\hat{q}} 2$ or $4 \succ_{\hat{q}} 2$, Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (5∗) given that $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{1, 2, 5, 6\} \succ_p 3 \succ_p 4$, $\{2, 4, 5, 6\} \succ_q 3 \succ_q 1$, and $\{1, 4\} \succ_{q'} 2 \succ_{q'} 5$. Next, note that the first part of Lemma 6 and $5 \succ_{\hat{q}} \{1, 2, 4\}$ imply $1 \not\sim_{\hat{q}} 2$, $1 \not\sim_{\hat{q}} 4$, and $2 \not\sim_{\hat{q}} 4$. But then Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (4∗) given that we also have $1 \sim_o 2 \sim_o 4$ and $\{1, 2, 4\} \succ_{q'} 5$. Since we have shown that $\succeq$ is unsolvable whenever $2 \succeq_{q'} 5$, we must have $5 \succ_{q'} 2$.

Next, we argue that $1 \sim_{q'} 4 \sim_{q'} 5 \succ_{q'} 2$. If $5 \succ_{q'} 1 \sim_{q'} 4 \succ_{q'} 2$, then $1 \sim_o 4 \succ_o 5$ together with the first part of Lemma 6 implies that $\succeq$ is unsolvable. Given that $5 \succ_{q'} 2$, the only other possibility in the case of $5 \not\sim_{q'} 1$ is $1 \sim_{q'} 4 \succ_{q'} 5 \succ_{q'} 2$. But then Claim 4 implies that $\succeq$ is unsolvable if $1 \sim_{q'} 4 \succ_{q'} 3 \succ_{q'} 2$ (substitute q' for q and switch the roles of 1 and 2 in the statement of Claim 4). Furthermore, if $3 \succ_{q'} 1$, then the first part of Lemma 6 implies that $\succeq$ is unsolvable since $\{1, 4\} \succ_o 5$, $5 \succ_p 3$, and $3 \succ_{q'} 1 \sim_{q'} 4$. By $3 \succ_{q'} 2$ and Step 1, the only remaining option is $1 \sim_{q'} 3 \sim_{q'} 4 \succ_{q'} 5 \succ_{q'} 2$. If $1 \succ_{q'} 6$, then, again by Step 1, $6 \not\sim_{q'} 5$ and $6 \not\sim_{q'} 2$. Hence, if $1 \succ_{q'} 6$, Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (4∗) given that $2 \sim_p 5 \sim_p 6$, $\{2, 5, 6\} \succ_q 1$, $1 \succ_{q'} \{2, 5, 6\}$, and that $\succ_{q'} |_{\{2,5,6\}}$ is strict. By the first part of Lemma 6 and $\{1, 3, 4\} \succ_o 6$, the only remaining option is that $1 \sim_{q'} 3 \sim_{q'} 4 \sim_{q'} 6 \succ_{q'} 5 \succ_{q'} 2$. But then Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (3∗) given that $1 \sim_{q'} 3 \sim_{q'} 6$, $4 \sim_q 5 \sim_q 6 \succ_q 3 \succ_q 1$, $4 \succ_o 5 \succ_o 6$, and $\{1, 3\} \succ_p 4$. Since $1 \sim_{q'} 4 \succ_{q'} 5 \succ_{q'} 2$ necessarily implies that $\succeq$ is unsolvable, we must have $1 \sim_{q'} 4 \sim_{q'} 5 \succ_{q'} 2$.

Now by Assumption 1(B), there exists $\tilde{q} \in O \setminus \{o, p, q\}$ such that $6 \succ_{\tilde{q}} 5$. If $\tilde{q} = q'$, then $\{1, 4\} \succ_o 6 \succ_{q'} 1 \sim_{q'} 4$ and $\succeq$ is unsolvable by the first part of Lemma 6. Thus, $\tilde{q} \neq q'$. If $1 \succ_{\tilde{q}} 5$ or $4 \succ_{\tilde{q}} 5$, Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (5∗) given that we also have $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{1, 2, 5, 6\} \succ_p 3 \succ_p 4$, $\{2, 4, 5, 6\} \succ_q 3 \succ_q 1$, and $\{1, 4\} \succ_{q'} 2$. Hence, we must have $5 \succeq_{\tilde{q}} 1$ and $5 \succeq_{\tilde{q}} 1$, so that $6 \succ_{\tilde{q}} 5$ implies $6 \succ_{\tilde{q}} \{1, 4, 5\}$. By the first part of Lemma 6 and $\{1, 4, 5\} \succ_o 6$, we must have $1 \not\sim_{\tilde{q}} 4$, $1 \not\sim_{\tilde{q}} 5$, and $4 \not\sim_{\tilde{q}} 5$. But then Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (4∗) given that $1 \sim_{q'} 4 \sim_{q'} 5$, $\{1, 4, 5\} \succ_o 6$, $6 \succ_{\tilde{q}} \{1, 4, 5\}$, and that $\succeq_{\tilde{q}} |_{\{1,4,5\}}$ is strict.

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[30] This is the only place in the proof where we rely on the existence of six distinct agents.

The preceding arguments show that a priority structure $\succeq$ such that $1 \sim_o 2 \sim_o 3 \sim_o 4 \succ_o 5 \succ_o 6$, $1 \sim_p 2 \sim_p 5 \sim_p 6 \succ_p 3 \succ_p 4$, and $2 \sim_q 4 \sim_q 5 \sim_q 6 \succ_q 3 \succ_q 1$ must be unsolvable. This completes the proof of Claim 6. $\square$

STEP 3. If 1, 2, 3, 4 are four distinct agents such that, for some object o, $1 \sim_o 2 \sim_o 3 \sim_o 4$, then $\succeq|_{\{1,2,3,4\}}$ has at most two priority levels.

Suppose to the contrary that there exists an object p such that $1 \succeq_p 2 \succeq_p 3 \succeq_p 4$ and $\succeq_p|_{\{1,2,3,4\}}$ contains more than two priority levels. By Step 1, $3 \sim_p 4$ is impossible and $3 \succ_p 4$. We show that without loss of generality, we may suppose $\{1, 2\} \succ_p 3 \succ_p 4$.

If not, then $1 \succ_p 2 \sim_p 3 \succ_p 4$. By Assumption 1(B), there must exist $q \in O \setminus \{o, p\}$ such that $4 \succ_q 1$. By the second part of Lemma 6, $4 \succ_q 2 \succ_q 1$ and $4 \succ_q 3 \succ_q 1$ are impossible. If $\{2, 3\} \succ_q 1$, then by $1 \succ_p 2 \sim_p 3$ and the first part of Lemma 6, $\succeq$ is unsolvable, a contradiction. Thus, $4 \succ_q 1 \succeq_q 2$ (or $4 \succ_q 1 \succeq_q 3$). If $4 \succ_q 1 \sim_q 2$, then by $\{1, 2\} \succ_p 4$ and the first part of Lemma 6, $\succeq$ is unsolvable, a contradiction. Thus, $4 \succ_q 1 \succ_q 2$. The same argument yields a contradiction if $1 \sim_q 3$. By Step 1, $3 \sim_q 2$ is impossible. Thus, $\{4, 3\} \succ_q 1 \succ_q 2$ or $\{4, 1\} \succ_q 2 \succ_q 3$ or $\{4, 1\} \succ_q 3 \succ_q 2$. In all cases we may choose q instead of p (with the appropriate relabeling of the agents).

Thus, let $\{1, 2\} \succ_p 3 \succ_p 4$. Let q_1 and q_2 be such that $4 \succ_{q_1} 1$ and $4 \succ_{q_2} 2$. Since $\{1, 2\} \succ_p 3 \succ_p 4$, we cannot have $4 \succ_{q_1} 1 \sim_{q_1} 2$ (since the first part of Lemma 6 implies that $\succeq$ is unsolvable in this case) or $4 \succ_{q_1} \{1, 2\}$ and $1 \nsim_{q_1} 2$ (since the second part of Lemma 6 implies that $\succeq$ is unsolvable in this case). Thus, $2 \succeq_{q_1} 4 \succ_{q_1} 1$ and, using similar arguments, $1 \succeq_{q_2} 4 \succ_{q_2} 2$.

We show first that $3 \succeq_{q_1} 1$ and $3 \succeq_{q_2} 2$. Assume to the contrary that $1 \succ_{q_1} 3$ (the arguments to establish that $2 \succ_{q_2} 3$ leads to a contradiction are completely analogous). Then $2 \succeq_{q_1} 4 \succ_{q_1} 1 \succ_{q_1} 3$ and, by Assumption 1(B), there must exist an object $q_3 \in O \setminus \{o, p, q_1\}$ such that $3 \succ_{q_3} 1$. If $q_3 = q_2$, then the second part of Lemma 6 implies that $\succeq$ is unsolvable given that $3 \succ_{q_2} 1 \succeq_{q_2} 4 \succ_{q_2} 2$ and $2 \succeq_{q_1} 4 \succ_{q_1} 1 \succ_{q_1} 3$. Thus, $q_3 \neq q_2$. If $4 \succ_{q_3} 1$, we can use the same arguments used to establish $2 \succeq_{q_1} 4 \succ_{q_1} 1$ to show that we must have $2 \succeq_{q_3} 4 \succ_{q_3} 1$ as well. But then Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (1∗) given that $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{1, 2\} \succ_p 3 \succ_p 4$, $\{1, 2\} \succ_{q_1} 3$, $1 \succeq_{q_2} 4 \succ_{q_2} 2$, and $2 \succeq_{q_3} 4 \succ_{q_3} 1$. Hence, we must have $3 \succ_{q_3} 1 \succeq_{q_3} 4$. Since $\{2, 4\} \succ_{q_1} 1 \succ_{q_1} 3$, the first part of Lemma 6 implies $1 \nsim_{q_3} 4$. Hence, we must have $3 \succ_{q_3} 1 \succ_{q_3} 4$. If $3 \succ_{q_3} 2 \succ_{q_3} 4$ or $3 \succ_{q_3} 1 \succ_{q_3} 4 \succ_{q_3} 2$, the second part of Lemma 6 implies that $\succeq$ is unsolvable given that we also have $2 \succeq_{q_1} 4 \succ_{q_1} 1 \succ_{q_1} 3$. Furthermore, since $\{1, 3\} \succ_{q_3} 4$, Step 1 implies $4 \nsim_{q_3} 2$. Hence, we are left to consider the case of $2 \succeq_{q_3} 3 \succ_{q_3} 1 \succ_{q_3} 4$. But in this case Step 2 is easily seen to imply that $\succeq$ is unsolvable since $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{2, 4\} \succ_{q_1} 1 \succ_{q_1} 3$, and $\{2, 3\} \succ_{q_3} 1 \succ_{q_3} 4$.[31] Since we have shown that $\succeq$ is unsolvable if $1 \succ_{q_1} 3$, we must have $3 \succeq_{q_1} 1$.

Next, we argue that $3 \succeq_{q_1} 4$ and $3 \succeq_{q_2} 4$. If $4 \succ_{q_1} 3$, then by $2 \succeq_{q_1} 4 \succ_{q_1} 1$ and $3 \succeq_{q_1} 1$, Step 1 implies $3 \nsim_{q_1} 1$. Thus, $\{2, 4\} \succ_{q_1} 3 \succ_{q_1} 1$. Given that $\{1, 2\} \succ_p 3 \succ_p 4$, Step 2 implies

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[31] One just needs to switch the roles of 1 and 3, replace p with q_3, and replace q with q_1 in the arguments used to establish Step 2.

that $\succeq$ is unsolvable. The argument for $3 \succeq_{q_2} 4$ is completely analogous. Thus, $\{2, 3\} \succeq_{q_1} 4 \succ_{q_1} 1$ and $\{1, 3\} \succeq_{q_2} 4 \succ_{q_2} 2$.

Now by Assumption 1(B), there must exist an object $q_3 \in O \setminus \{o, p, q_1, q_2\}$ such that $4 \succ_{q_3} 3$. If $\{1, 2\} \succ_{q_3} 3$, Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (1∗) given that we also have $1 \sim_o 2 \sim_o 3 \sim_o 4$, $\{1, 2\} \succ_p 3 \succ_p 4$, $2 \succeq_{q_1} 4 \succ_{q_1} 1$, and $1 \succeq_{q_2} 4 \succ_{q_2} 2$. Thus, $3 \succeq_{q_3} 2$ or $3 \succeq_{q_3} 1$. Since $\{1, 2, 3\} \succ_p 4 \succ_{q_3} 3$, the first part of Lemma 6 implies $3 \succ_{q_3} 2$ if $3 \succeq_{q_3} 2$ and $3 \succ_{q_3} 1$ if $3 \succeq_{q_3} 1$. If $4 \succ_{q_3} 3 \succ_{q_3} \{1, 2\}$, the second part of Lemma 6 implies that $\succeq$ is unsolvable given that we also have $\{1, 2\} \succ_p 3 \succ_p 4$. Hence, we can assume w.l.o.g. that $\{2, 4\} \succ_{q_3} 3 \succ_{q_3} 1$. Given that we also have $\{1, 2\} \succ_p 3 \succ_p 4$, Step 2 implies that $\succeq$ is unsolvable.

Step 4. Let 1, 2, 3, and 4 be four distinct agents and let o be an object such that $1 \sim_o 2 \sim_o 3 \sim_o 4$. If there exists an object p such that, for some $i \in \{1, 2, 3, 4\}$ and two distinct $j, k \in \{1, 2, 3, 4\} \setminus \{i\}$, $i \succ_p \{j, k\}$, then $\succeq$ is a HET priority structure.

Assume that there exists an object p such that $4 \succ_p \{1, 2\}$. Since $\succeq |_{\{1,2,3,4\}}$ has at most two priority levels, Step 1 implies that we must have $4 \succ_p 1 \sim_p 2 \sim_p 3$.

We argue first that $\succeq_o$ satisfies the requirements of a HET priority structure. Given that $1 \sim_o 2 \sim_o 3 \sim_o 4$, Step 1 immediately implies that there exists at most one agent $j \in I \setminus \{1, 2, 3, 4\}$ such that $j \succ_o 1$. Next, note that, for any $j \in I \setminus \{1, 2, 3, 4\}$, we must have $j \succeq_o 1$; otherwise, $\{1, 2\} \succ_o j$ and the first part of Lemma 6 implies that $\succeq$ is unsolvable since there exists $q' \in O \setminus \{o\}$ such that $j \succ_{q'} 4$ and since we also have that $4 \succ_p 1 \sim_p 2$.

Second, we show that, for all $j \in I \setminus \{1, 2, 3, 4\}$, $j \sim_p 1$. By Step 1 and our assumption that $4 \succ_p 1 \sim_p 2$, we cannot have $j \succ_p 1$. Hence, $j \nsim_p 1$ implies $4 \succ_p 1 \sim_p 2 \succ_p j$. By the first part of Lemma 6, $\succeq$ is unsolvable if $j \succ_o 1 \sim_o 2 \succ_p j$. Since $j \succeq_o 1$, $4 \succ_p 1 \sim_p 2 \succ_p j$ thus implies $1 \sim_o 2 \sim_o 4 \sim_o j$. But then Step 3 implies that $\succeq |_{\{1,2,4,j\}}$ can have at most two priority levels, contradicting $4 \succ_p 1 \sim_p 2 \succ_p j$.

Finally, let $q \in O \setminus \{o, p\}$ be an arbitrary object. We show that $\succeq_q$ also satisfies the requirements of a HET priority structure. Suppose to the contrary that there exist three distinct agents $i_1, i_2, i_3 \in I$ such that $\{i_1, i_2\} \succ_q i_3$. Note first that we can assume w.l.o.g. that $i_1 = 1$ and $i_2 = 2$: If there is only one agent $i' \in \{1, 2, 3\}$ such that $i' \succ_q i_3$, say $i' = 1$, Step 3 implies that $\{i_1, i_2, i'\} \succ_q i_3 \succeq_q 2 \sim_q 3$ so that $\succeq$ is unsolvable by Step 1. Now if $i_3 = 4$, the first part of Lemma 6 immediately implies that $\succeq$ is not solvable since $\{1, 2\} \succ_q 4 \succ_p 1 \sim_p 2$. If $i_3 \neq 4$ and $4 \succ_q i_3$, the first part of Lemma 6 again implies that $\succeq$ is unsolvable given that $\{1, 2\} \succ_q i_3$ and given that either $i_3 \succ_o 1 \sim_o 2$ or, by Assumption 1(B), there is an object $q' \in O \setminus \{o, p, q\}$ such that $i_3 \succ_{q'} 4$.

Step 5. Let 1, 2, 3, and 4 be four distinct agents and let o be an object such that $1 \sim_o 2 \sim_o 3 \sim_o 4$. If there does not exist an object p such that, for some $i \in \{1, 2, 3, 4\}$ and two distinct $j, k \in \{1, 2, 3, 4\} \setminus \{i\}$, $i \succ_p \{j, k\}$, then $\succeq$ is a TAU priority structure.

Suppose to the contrary that $\succeq$ is not a TAU priority structure. Then there exist an object q and three distinct agents i, j, and k such that $i \succ_q \{j, k\}$. We show that $\succeq$ is unsolvable. Throughout the proof, we use the fact that $\succeq |_{\{1,2,3,4\}}$ has at most two priority levels (which follows from Step 3 since $1 \sim_o 2 \sim_o 3 \sim_o 4$).

First, we show that we can assume w.l.o.g. $i \in \{1, 2, 3, 4\}$. Assume that $i \in I \setminus \{1, 2, 3, 4\}$ and $i \succ_q \{1, 2, 3, 4\}$. Because $\succeq |_{\{1,2,3,4\}}$ contains at most two priority levels, there must be a tie between two agents in $\{1, 2, 3, 4\}$ at $\succeq_q$, say $1 \sim_q 2$. By Assumption 1(B) there exists $q' \in O \setminus \{q\}$ such that $1 \succ_{q'} i$. By the first part of Lemma 6 and $i \succ_q 1 \sim_q 2$, we cannot have $\{1, 2\} \succ_{q'} i$. Hence, $i \succeq_{q'} 2$ and $1 \succ_{q'} i \succeq_{q'} 2$ so that we have found an object with the desired properties.

For the remainder of Step 5, we assume w.l.o.g. that $1 \succ_q \{j, k\}$. By the assumptions of Step 5, we must have either $j \in \{2, 3, 4\}$ and $k \in I \setminus \{1, 2, 3, 4\}$, or $j, k \in I \setminus \{1, 2, 3, 4\}$.

Assume first that $j \in \{2, 3, 4\}$, say $j = 4$, and that $k \in I \setminus \{1, 2, 3, 4\}$, say $k = 5$. Since $\succeq |_{\{1,2,3,4\}}$ has at most two priority levels, we must have $1 \sim_q 2 \sim_q 3 \succ_q \{4, 5\}$. By Assumption 1(B), there is an object q_1 such that $5 \succ_{q_1} 1$. We argue next that $\{2, 3, 4\} \succ_{q_1} 1$. Since $1 \sim_q 2 \sim_q 3 \succ_q \{4, 5\}$, the first part of Lemma 6 implies that $\succeq$ is unsolvable if either $2 \sim_{q_1} 1$ or $3 \sim_{q_1} 1$. If $1 \succ_{q_1} 2$, Step 1 implies that $3 \nsim_{q_1} 2$ since $\{1, 5\} \succ_{q_1} 2$. But then $\succeq_{q_1} |_{\{1,2,3\}}$ must be strict and we obtain a contradiction to Step 3. An analogous argument shows that $3 \succ_{q_1} 1$. But if $\{2, 3\} \succ_{q_1} 1$, Step 1 implies that $1 \nsim_{q_1} 4$ and $\succeq_{q_1} |_{\{1,2,3,4\}}$ has at least three priority levels if $4 \not\succ_{q_1} 1$—another contradiction to Step 3. Hence, we must have $\{2, 3, 4\} \succ_{q_1} 1$. By Step 3, $\succeq |_{\{1,2,3,4\}}$ has at most two priority levels and we obtain that $2 \sim_{q_1} 3 \sim_{q_1} 4$. If $4 \sim_{q_1} 5$, we obtain an immediate contradiction to Step 3 since $\{2, 3\} \succ_p \{4, 5\}$ and, by Step 1, $4 \nsim_p 5$. If $5 \succ_{q_1} 2$, the first part of Lemma 6 implies that $\succeq$ is unsolvable given that $\{2, 3\} \succ_q 5$ and $5 \succ_{q_1} 2 \sim_{q_1} 3$. Hence, we must have $\{2, 3, 4\} \succ_{q_1} 5 \succ_{q_1} 1$. Proceeding analogously, there must also exist two objects q_2 and q_3 such that $\{1, 3, 4\} \succ_{q_2} 5 \succ_{q_2} 2$ and $\{1, 2, 4\} \succ_{q_3} 5 \succ_{q_3} 3$. But then Lemma 7 implies that $\succeq$ is unsolvable since it is of the form in (6∗).

Hence, we are left to consider the case where there are two distinct agents $j, k \in I \setminus \{1, 2, 3, 4\}$, say $j = 5$ and $k = 6$, such that $\{2, 3, 4\} \succeq_q 1 \succ_q \{5, 6\}$. Assumption 1(B) implies that there exists an object q_1 such that $5 \succ_{q_1} 1$. By the arguments from the previous paragraph (which only depended on the fact that $\{1, 2, 3\} \succ_q 5$), we must have that $2 \sim_{q_1} 3 \sim_{q_1} 4 \succeq_{q_1} 5 \succ_{q_1} 1$. If $4 \succ_{q_1} 5$, an argument analogous to that showing that we cannot have $j = 4$ and $k = 5$ can be used to show that $\succeq$ is unsolvable. If $2 \sim_{q_1} 3 \sim_{q_1} 4 \sim_{q_1} 5$, we can again use analogous arguments as in the case of $j = 4$ and $k = 5$ to show that $\succeq$ must be unsolvable.[32]

APPENDIX B: PROOF OF THEOREM 2

Let $|I| \geq 4$. If, for some $o \in O$, $\succeq_o$ contains three or more priority levels, then there exist $i, j, k \in I$ such that $i \succ_o j \succ_o k$. By Assumption 1(B) there exists $q \in O$ such that $k \succ_q i$. Obviously, $q \neq o$ and the strict part of $\succeq$ contains a cycle à la Ergin (2002). Then his arguments can be used to show that no strongly group-strategy-proof and stable mechanism exists, and, thus, $\succeq$ is not strongly solvable.

Thus, for all $o \in O$, $\succeq_o$ contains at most two priority levels. If for some $i, j, k, l \in I$ and some $o \in O$, $\{i, j\} \succ_o k \sim_o l$, then by Step 1, which only relies on Assumption 1(B), of the proof of Theorem 1, $\succeq$ is unsolvable, a contradiction. Similarly, by the first part

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[32] This follows since $\{2, 3, 4\} \succ_q \{5, 6\}$ so that we can use the four-way tie $2 \sim_{q_1} 3 \sim_{q_1} 4 \sim_{q_1} 5$ instead of the four-way tie $1 \sim_o 2 \sim_o 3 \sim_o 4$ in the above arguments.

of Lemma 6, if for some $i, j, k \in I$ and some $o, p \in O$, we have $k \succ_o i \sim_o j$ and $\{i, j\} \succ_p k$, then $\succeq$ is unsolvable.

Thus, if $\succeq$ is not HET, then there exists $o \in O$ such that for some $i(o) \in I$, we have $I \setminus \{i(o)\} \succ_o i(o)$, and for all $i, j \in I \setminus \{i(o)\}$, we have $i \sim_o j$. But then, given that $\succeq$ contains at most two priority levels, $\succeq$ is a TAU structure.

Using $|I| \geq 4$ and the fact that $\succeq$ is not HET, it is easy to see that Assumption 1(B) implies the existence of three distinct agents, 1, 2, and 3, and four distinct objects, o, p_1, p_2, and p_3 such that

$$\begin{array}{c} 1 \sim_o 2 \sim_o 3 \\ 2 \sim_{p_1} 3 \succ_{p_1} 1 \\ 1 \sim_{p_2} 3 \succ_{p_2} 2 \\ 1 \sim_{p_3} 2 \succ_{p_3} 3 \end{array}$$

We show that $\succeq$ is not strongly solvable. We assume throughout that the preferences of agents in $I \setminus \{1, 2, 3\}$ are fixed at some profile at which they do not rank any of the objects in $\{o, p_1, p_2, p_3\}$ as acceptable. The proof revolves around the preference profile

R	R_1	R_2	R_3
	o	o	o
	p_1	p_2	p_3

For the following discussion, let f be an arbitrary strongly group-strategy-proof and constrained efficient mechanism. We show by contradiction that $f_1(R) = o$ is impossible. Since all three agents play completely symmetric roles in the preference profile R, completely analogous arguments then show that we can also not have $f_2(R) = o$ or $f_3(R) = o$. Hence, f has to be wasteful at R and $\succeq$ can therefore not be strongly solvable.

Assume that $f_1(R) = o$ and consider the preference profile

R^1	R_1	R_2^1	R_3
	o	o	o
	p_1	p_1	p_3

By strategy-proofness, we must have $f_2(R^1) \neq o$. Hence, either $f_1(R^1) = o$ or $f_3(R^1) = o$. We show that both cases necessarily lead to a contradiction.

Case 1. $f_1(R^1) = o$. Consider the profile

$R^{1,1}$	R_1	R_2^1	R_3^0
	o	o	o
	p_1	p_1	p_1

By $f_1(R^1) = o$, we have $f_3(R^1) \neq o$. From strategy-proofness for 3, we can infer that $f_3(R^{1,1}) \neq o$. Since $2 \rightarrow_{p_1} 1$ is an $(2, 1; o, p_1)$ path that is compatible with the $(3, 1; o, p_1)$ path $3 \rightarrow_{p_1} 1$ and since $1 \sim_o 2 \sim_o 3$, Lemma 4 implies $f_1(R^{1,1}) \neq o$. Hence, we must have $f_2(R^{1,1}) = o$. By $3 \succ_{p_1} 1$, we have $f_3(R^{1,1}) = p_1$.

Next, we derive a few implications from our initial assumption $f_1(R) = o$:

$$\begin{array}{ccc}
\begin{array}{c|ccc} R & R_1 & R_2 & R_3 \\ \hline & \boxed{o} & o & o \\ & p_1 & \boxed{p_2} & \boxed{p_3} \end{array} & \rightarrow & \begin{array}{c|ccc} R^{1,2} & R_1^1 & R_2 & R_3 \\ \hline & \boxed{o} & o & o \\ & p_3 & \boxed{p_2} & \boxed{p_3} \end{array} \\
 & & \downarrow \\
\begin{array}{c|ccc} R^{1,4} & R_1 & R_2^2 & R_3 \\ \hline & \boxed{o} & o & o \\ & p_1 & \boxed{p_3} & p_3 \end{array} & \leftarrow & \begin{array}{c|ccc} R^{1,3} & R_1^1 & R_2^2 & R_3 \\ \hline & \boxed{o} & o & o \\ & p_3 & \boxed{p_3} & p_3 \end{array}
\end{array}$$

In moving from R to $R^{1,2}$ we have used $f_1(R) = o$ and strategy-proofness for 1 to infer $f_1(R^{1,2}) = o$. Given that $f_2(R^{1,2}) \neq o$, strategy-proofness for 2 implies $f_2(R^{1,3}) \neq o$. Since $1 \rightarrow_{p_3} 3$ is an $(1, 3; o, p_3)$ path that is compatible with the $(2, 3; o, p_3)$ path $2 \rightarrow_{p_3} 3$ and since $1 \sim_o 2 \sim_o 3$, Lemma 4 implies $f_3(R^{1,3}) \neq o$. Hence, we must have $f_1(R^{1,3}) = o$. Strategy-proofness for 1 then yields $f_1(R^{1,4}) = o$. Given that $2 \succ_{p_3} 3$, stability requires that $f_2(R^{1,4}) = p_3$ and $f_3(R^{1,4}) = 3$.

Finally, consider the preference profile

$$\begin{array}{c|ccc} R^{1,5} & R_1 & R_2^2 & R_3^1 \\ \hline & o & o & o \\ & p_1 & p_3 & p_3 \\ & & & p_1 \end{array}$$

By strategy-proofness and the just established fact that $f_1(R) = o$ implies $f_1(R^{1,4}) = o$ and $f_3(R^{1,4}) = 3$, we must have $f_3(R^{1,5}) \notin \{o, p_3\}$. This is compatible with constrained efficiency only if $f_1(R^{1,5}) = o$, $f_2(R^{1,5}) = p_3$, and $f_3(R^{1,5}) = p_1$. Given that 2 and 3 can obtain $R^{1,1}$ from $R^{1,5}$ by means of a coordinated deviation to (R_2^1, R_3^0) and given that $f_2(R^{1,1}) = o$ as well as $f_3(R^{1,1}) = f_3(R^{1,5}) = p_1$, f cannot be strongly group-strategy-proof. This completes the proof for Case 1.

Case 2. Strategy-proofness for 3 implies $f_3(R^{1,1}) = o$ (where $R^{1,1}$ is the profile defined at the beginning of Case 1). By $2 \succ_{p_1} 1$, we have $f_2(R^{1,1}) = p_1$. Switching the roles of 2 and 3 and of p_2 and p_3 in the arguments used in Case 1, we find that $f_1(R) = o$ implies

$$\begin{array}{c|ccc} \tilde{R} & R_1 & \tilde{R}_2 & \tilde{R}_3 \\ \hline & \boxed{o} & o & o \\ & p_1 & p_2 & \boxed{p_2} \\ & & \boxed{p_1} & \end{array}$$

We again obtain a contradiction to the strong group-strategy-proofness of f given that 2 and 3 can obtain $R^{1,1}$ from $\tilde{R}$ by means of a coordinated deviation to (R_2^1, R_3^0) and given that $f_2(R^{1,1}) = f_2(\tilde{R}) = p_1$ as well as $f_3(R^{1,1}) = o$.

References

Abdulkadiroğlu, Atila, Parag Pathak, Alvin E. Roth, and Tayfun Sönmez (2006), "Changing the Boston school choice mechanism: Strategy-proofness as equal access." Unpublished paper, Harvard University. [1010]

Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth (2009), "Strategyproofness versus efficiency in matching with indifferences: Redesigning the NYC high school match." *American Economic Review*, 99, 1954–1978. [1012, 1016]

Abdulkadiroğlu, Atila and Tayfun Sönmez (1999), "House allocation with existing tenants." *Journal of Economic Theory*, 88, 233–260. [1010, 1016, 1017]

Abdulkadiroğlu, Atila and Tayfun Sönmez (2003), "School choice: A mechanism design approach." *American Economic Review*, 93, 729–747. [1010]

Barberà, Salvador, Dolors Berga, and Bernardo Moreno (2010), "Individual versus group strategy-proofness: When do they coincide?" *Journal of Economic Theory*, 145, 1648–1674. [1023]

Barberà, Salvador, Dolors Berga, and Bernardo Moreno (2016), "Group strategy-proofness in private good economies." *American Economic Review*, 106, 1073–1099. [1023]

Bogomolnaia, Anna and Hervé Moulin (2001), "A new solution to the random assignment problem." *Journal of Economic Theory*, 100, 295–328. [1017]

Budish, Eric and Estelle Cantillon (2012), "The multi-unit assignment problem: Theory and evidence from course allocation at Harvard." *American Economic Review*, 102, 2237–2271. [1010]

Dubins, Lester E. and David A. Freedman (1981), "Machiavelli and the Gale–Shapley algorithm." *American Mathematical Monthly*, 88, 485–494. [1016]

Ehlers, Lars (2006), "Respecting priorities when assigning students to schools." CIREQ Discussion Paper 2006-04, Département de sciences économiques, Université de Montréal. [1011, 1024]

Ehlers, Lars and Aytek Erdil (2010), "Efficient assignment respecting priorities." *Journal of Economic Theory*, 145, 1269–1282. [1012]

Ehlers, Lars and Bettina Klaus (2006), "Efficient priority rules." *Games and Economic Behavior*, 55, 372–384. [1012]

Ehlers, Lars and Alexander Westkamp (2011), "Strategy-proof tie-breaking." CIREQ Discussion Paper 2011-07, Département de sciences économiques, Université de Montréal. [1018]

Erdil, Aytek and Haluk Ergin (2008), "What's the matter with tie-breaking? Improving efficiency in school choice." *American Economic Review*, 98, 669–689. [1011, 1014, 1015, 1017]

Ergin, Haluk (2002), "Efficient resource allocation on basis of priorities." *Econometrica*, 70, 2489–2497. [1012, 1014, 1016, 1036]

Gale, David and Lloyd S. Shapley (1962), "College admissions and the stability of marriage." *American Mathematical Monthly*, 69, 9–15. [1015]

Han, Xiang (2018), "Stable and efficient resource allocation under weak priorities." *Games and Economic Behavior*, 107, 1–20. [1012, 1023, 1024]

Hylland, Aanund and Richard Zeckhauser (1979), "The efficient allocation of individuals to positions." *Journal of Political Economy*, 87, 293–314. [1017]

Kesten, Onur (2006), "On two competing mechanisms for priority-based allocation problems." *Journal of Economic Theory*, 127, 155–171. [1012]

Ma, Jinpeng (1994), "Strategy-proofness and the strict core in a market with indivisibilities." *International Journal of Game Theory*, 23, 75–83. [1017]

Ostrovsky, Michael (2008), "Stability in supply chain networks." *American Economic Review*, 98, 897–923. [1018]

Pápai, Szilvia (2000), "Strategyproof assignment by hierarchical exchange." *Econometrica*, 68, 1403–1433. [1012, 1013, 1017]

Pycia, Marek (2012), "Stability and preference alignment in matching and coalition formation." *Econometrica*, 80, 323–362. [1018]

Pycia, Marek and Utku Ünver (2017), "Incentive compatible allocation and exchange of discrete resources." *Theoretical Economics*, 12, 287–329. [1013, 1017, 1022]

Roth, Alvin E. (1982a), "The economics of matching: Stability and incentives." *Mathematics of Operations Research*, 7, 617–628. [1016]

Roth, Alvin E. (1982b), "Incentive compatibility in a market with indivisible goods." *Economics Letters*, 9, 127–132. [1017]

Roth, Alvin E. (2008), "Deferred acceptance algorithms: History, theory, practice, and open questions." *International Journal of Game Theory*, 36, 537–569. [1009]

Roth, Alvin E. and Andrew Postlewaite (1977), "Weak versus strong domination in a market with indivisible goods." *Journal of Mathematical Economics*, 4, 131–137. [1017]

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver (2004), "Kidney exchange." *Quarterly Journal of Economics*, 119, 457–488. [1010]

Roth, Alvin E. and Marilda A. O. Sotomayor (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, volume 18 of Econometric Society Monographs. Cambridge University Press, Cambridge. [1010]

Shapley, Lloyd S. and Herbert E. Scarf (1974), "On cores and indivisibility." *Journal of Mathematical Economics*, 1, 23–37. [1017]

Sönmez, Tayfun and Utku Ünver (2011), "Matching, allocation, and exchange of discrete resources." In *Handbook of Social Economics* (Jess Benhabib, Alberto Bisin, and Matthew O. Jackson, eds.), 781–852, North-Holland, Amsterdam. [1009]

Svensson, Lars-Gunnar (1999), “Strategy-proof allocation of indivisible goods.” *Social Choice and Welfare*, 16, 557–567. [1017]

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