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Transitivity of preferences: When does it matter?

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We define necessary and sufficient conditions on prices and incomes under which quantity choices can violate SARP (strong axiom of revealed preference) but not WARP (weak axiom of revealed preference). As SARP extends WARP by additionally imposing transitivity on the revealed preference relation, this effectively defines the conditions under which transitivity adds bite to the empirical analysis. For finite data sets, our characterization takes the form of a triangular condition that must hold for all three-element subsets of normalized prices, and which is easy to verify in practice. For infinite data sets, we formally establish an intuitive connection between our characterization and the concept of Hicksian aggregation. We demonstrate the practical use of our conditions through two empirical illustrations.

Keywords. Revealed preferences, WARP, SARP, transitive preferences, testable implications, Hicksian aggregation.

JEL classification. C14, D01, D11, D12.

1. Introduction

For demand behavior under linear budget constraints, it is well established that the weak axiom of revealed preference (WARP) implies the strong axiom of revealed preference (SARP) as long as there are no more than two goods.[1] Rose (1958) provided a

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[1] Samuelson (1938) originally introduced the WARP as a basic consistency requirement on consumption behavior: if a consumer chooses a first bundle over a second one in a particular choice situation (characterized by a linear budget constraint), then (s)he cannot choose this second bundle over the first one in a different choice situation. Houthakker (1950) defined SARP as the extension of WARP with a transitive revealed preference relation. See Chambers and Echenique (2016) for more discussion about WARP, SARP,

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first formal statement of this fact. As SARP extends WARP by (only) imposing the additional requirement that the revealed preference (RP) relation must be transitive, this effectively implies that transitivity itself does not add bite to the empirical revealed preference analysis.[2]

This equivalence between WARP and SARP has an intuitive analogue in terms of testable properties of Slutsky matrices, which are typically studied in differential analysis of continuous demand. Specifically, Slutsky symmetry is always satisfied by construction in situations with two goods and, thus, only negative semi-definiteness of the Slutsky matrix can be tested empirically in such instances. This directly complies with the two classic results of Samuelson (1938) and Houthakker (1950): Samuelson showed that demand is consistent with WARP only if compensated demand effects are negative, whereas Houthakker showed that a consumer behaves consistent with utility maximization (implying Slutsky symmetry in addition to Slutsky negativity) if and only if demand is consistent with SARP.[3] In a two-goods setting, the equivalence between WARP and SARP translates into nontestability of Slutsky symmetry (in contrast to negativity).

Contribution. We can conclude that the (lack of) empirical content of transitivity of the RP relation with two goods is well understood by now. However, the question remains regarding under which conditions WARP implies SARP when there are more than two goods. In this respect, an intuitive starting point relates to the possibility of dimension reduction that is based on Hicksian aggregation.[4] A set of goods can be represented by a Hicksian aggregate if the goods' relative prices remain fixed over decision situations. Thus, by verifying the empirical validity of constant relative prices, we can check whether the demand for multiple goods can be studied in terms of two Hicksian aggregates. If this happens to be the case, it immediately follows from the results of Rose (1958) that WARP and SARP will be empirically equivalent.

Clearly, the condition of constant relative prices is not met in most real life settings, which provides the core motivation for our current study. Specifically, we establish the empirical conditions on prices and incomes that characterize the empirical bite of transitivity of the RP relation in a general situation with multiple goods. These conditions are necessary and sufficient for WARP and SARP to be equivalent. In other words, if (and only if) the conditions are met, then dropping the transitivity condition will lead to exactly the same empirical conclusions. The fact that our conditions are defined in terms of budget sets, without requiring quantity information, is particularly convenient from a practical point of view. It makes it possible to check, on the basis of given prices and incomes, whether it suffices to (only) check WARP (instead of SARP) to verify consistency with utility maximization. Conversely, it characterizes the budget conditions under which transitivity restrictions can potentially add value to the empirical analysis.

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and other revealed preference axioms such as the (weak) generalized axiom of revealed preferences (which we consider in Section 6).

[2]See Quah (2006) for more discussion on rationalizalibity in terms of nontransitive preferences.

[3]See also Kihlstrom et al. (1976) for related discussion.

[4]See, for example, Varian (1992) for a general discussion on Hicksian aggregation. Lewbel (1996) presents related results on commodity aggregation under specific assumptions.

Interestingly, we can show that our general characterization generates the conclusion of Rose (1958) in the specific instance with two goods. Furthermore, we can establish an intuitive relationship between our characterization and the Hicksian aggregation argument that we gave above. Specifically, when applying our characterization result to a continuous setting (with infinitely many price–income regimes), we obtain a condition that basically states that all prices must lie in a common two-dimensional plane. We show that this is formally equivalent to a setting where goods can be linearly aggregated into two composite commodities, which we can interpret as two Hicksian aggregates. As an implication, this also establishes that (in a continuous setting) Slutsky negativity entails symmetry if and only if prices satisfy this particular type of Hicksian aggregation.

Relationship with the literature. The questions of whether and under what conditions WARP and SARP are empirically distinguishable has attracted considerable attention in the theoretical literature. Shortly after the result of Rose (1958) on the equivalence between WARP and SARP for two goods, Gale (1960) constructed a counterexample showing that WARP and SARP may differ in settings with more than two goods. Since then, various authors have presented further clarifications and extensions of Gale's basic result (see, e.g., Shafer 1977, Peters and Wakker 1994, Heufer 2014). In a similar vein, Uzawa (1989) showed that if a demand function satisfies WARP together with some regularity condition, then it also satisfies SARP. However, Bossert (1993) put this result into perspective by demonstrating that, for continuous demand functions, Uzawa's regularity condition alone already implies SARP.

A main difference with our current contribution is that these previous studies typically exemplified the distinction between WARP and SARP by constructing hypothetical "demand" functions (i.e., functions of prices and income) or data sets (containing prices, incomes, and consumption quantities) that satisfy WARP but violate SARP. Such functions or data sets, however, might never be encountered in reality. In this sense, it leaves open the question of whether the possibility to distinguish SARP from WARP is merely a theoretical curiosity or also an empirical regularity. Moreover, the data sets that are constructed do not define general conditions on budget sets (i.e., prices and incomes, without quantities) under which SARP and WARP are empirically equivalent.

Finally, the strong axiom of revealed preference is closely related to the so-called generalized axiom of revealed preference (GARP). As shown by Afriat (1967), GARP gives necessary and sufficient conditions on a finite data set for consistency with utility maximizing behavior.[5] In contrast to SARP, GARP allows for consumers with flat indifference curves. As a variation on the Rose (1958) result, Banerjee and Murphy (2006) showed that in a two-goods setting, the pairwise version of GARP, which they call the weak generalized axiom of revealed preference (WGARP), is equivalent to GARP. We will show that, when restricting consumption quantities to be strictly positive, the conditions on prices and budgets that equate SARP with WARP are also necessary and sufficient for GARP to be equivalent to WGARP.

[5]The term GARP was introduced by Varian (1982) as an alternative name for the cyclical consistency condition of Afriat (1967).

Outline. The remainder of this paper unfolds as follows. Section 2 first introduces some notation and basic definitions, and subsequently presents our main result. Section 3 provides some further discussion of this main result. Section 4 shows the connection between our characterization and Hicksian aggregation when the set of possible budgets becomes infinite. Section 5 shows the practical use of our theoretical findings through two empirical illustrations. Section 6 extends our SARP- and WARP-based results to (W)GARP. The Appendix contains the proofs of our main results.

2. When WARP equals SARP

Notation and terminology. We consider a setting where we observe n budget sets for m goods. This defines an original data set $\{(\hat{\mathbf{p}}_t, \hat{x}_t)\}_{t=1,\dots,n}$ with price (row) vectors $\hat{\mathbf{p}}_t \in \mathbb{R}^m_{++}$ and expenditure levels $\hat{x}_t \in \mathbb{R}_{++}$. To facilitate our further discussion and to simplify the notation, we summarize the budgets $(\hat{\mathbf{p}}_t, \hat{x}_t)$ in terms of normalized price vectors $\mathbf{p}_t = (\hat{\mathbf{p}}_t/\hat{x}_t)$, which divide each observed price vector $\hat{\mathbf{p}}_t$ by the associated expenditure level $\hat{x}_t$. By construction, these normalized prices $\mathbf{p}_t$ correspond to an expenditure level $x_t = 1$ for all observations t. Using this, we can summarize all the relevant information on the observed budgets by the data set $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ containing the normalized price vectors.

Let $\{\mathbf{q}_t\}_{t=1,\dots,n}$ be a collection of quantity (column) vectors on the budget lines defined by the normalized price vectors in $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$. In other words, for all observations $t = 1, \dots, n$, $\mathbf{q}_t \in \mathbb{R}^m_+$, and $\mathbf{p}_t\mathbf{q}_t = 1$. We denote by $\mathcal{Q}(P)$ the collection of all such quantity vectors, i.e., for $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ we set

$$\mathcal{Q}(P) = \big\{\{\mathbf{q}_t\}_{t=1,\dots,n} : \mathbf{q}_t \in \mathbb{R}^m_+, \mathbf{p}_t\mathbf{q}_t = 1\big\}.$$

The collection $\mathcal{Q}(P)$ gives us all possible consumption vectors that a consumer may choose when confronted with the normalized price vectors in P. We can now define the basic revealed preference concepts.

Definition 1. Let $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ be a set of normalized price vectors and let $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$. Then, for all $t, v \leq n$, the quantity vector $\mathbf{q}_t$ is *revealed preferred* to the bundle $\mathbf{q}_v$ if $\mathbf{p}_t\mathbf{q}_t (= 1) \geq \mathbf{p}_t\mathbf{q}_v$. We denote this as $\mathbf{q}_t \, R \, \mathbf{q}_v$.

In words, $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$ if $\mathbf{q}_v$ was cheaper than $\mathbf{q}_t$ at the normalized prices observed at t. Then we have the following definitions of WARP and SARP.

Definition 2. Let $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ be a set of normalized price vectors and let $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$. Then $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates *WARP* if R has a cycle of length 2, i.e.,

$$\mathbf{q}_t \, R \, \mathbf{q}_v \, R \, \mathbf{q}_t$$

for some observation t, v and $\mathbf{q}_t \neq \mathbf{q}_v$.

Definition 3. Let $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ be a set of normalized price vectors and let $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$. Then $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates *SARP* if R has a cycle, i.e.,

$$\mathbf{q}_t \; R \; \mathbf{q}_v \; R \; \mathbf{q}_s \dots R \; \mathbf{q}_k \; R \; \mathbf{q}_t$$

for some sequence of observations $t, v, s, \dots, k$ and not all bundles $\mathbf{q}_t, \dots, \mathbf{q}_k$ are identical.

It is clear from the definitions that SARP consistency implies WARP consistency, i.e., if $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates WARP, then it also violates SARP. We are interested in the reverse relationship: under which conditions does a violation of SARP also imply a violation of WARP. Given this specific research question, we consider settings in which the empirical analyst does not necessarily observe the quantity choices, but only the normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$. In other words, what are the conditions on the normalized price vectors in P such that for all possible quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$, WARP is equivalent to SARP for all subsets of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$. To this end, we use the following definition.

Definition 4. A set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is said to be *WARP-reducible* if, for any set of quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$, a subset of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates SARP only if it also violates WARP.

Main result. To set the stage, we first repeat the original result of Rose (1958), which says that WARP is always equivalent to SARP if the number of goods equals two (i.e., $m = 2$). Recently, Chambers and Echenique (2016) presented an insightful geometric proof of Rose's result. We phrase this result in the terminology that we introduced above.

Proposition 1. *If there are only two goods (i.e., $m = 2$), then any set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is WARP-reducible.*

Our main result provides a generalization of Proposition 1. It makes use of the concept of a triangular configuration.

Definition 5. A set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is a *triangular configuration* if, for any three normalized price vectors $\mathbf{p}_t$, $\mathbf{p}_v$, and $\mathbf{p}_k$ in P, there exists a number $\lambda \in [0, 1]$ and a permutation $\sigma : \{t, v, k\} \to \{t, v, k\}$ such that the following condition holds:

$$\mathbf{p}_{\sigma(t)} \leq \lambda \mathbf{p}_{\sigma(v)} + (1 - \lambda)\mathbf{p}_{\sigma(k)} \quad \text{or} \quad \mathbf{p}_{\sigma(t)} \geq \lambda \mathbf{p}_{\sigma(v)} + (1 - \lambda)\mathbf{p}_{\sigma(k)}.$$

We note that the inequalities in this definition are vector inequalities. As such, Definition 5 states that, for any three price vectors, we need that there is a convex combination of two of the three price vectors that is either smaller or larger than the third price vector. Checking whether a set of price vectors is a triangular configuration merely requires verifying the linear inequalities in Definition 5 for any possible combination of three price vectors. Clearly, this is easy to do in practice, even if the number of observations (i.e., n) gets large. In particular, given that there are only $n(n-1)(n-2)/6$ possible combinations of price vectors that need to be considered, the triangular condition in Definition 5 can be checked in polynomial ($O(n^3)$) time.

At this point, we want to emphasize once more that in practical applications the triangular condition in Definition 5 involves both observed prices and observed expenditures. As explained above, we work with normalized price vectors $\mathbf{p}_t = (\hat{\mathbf{p}}_t/\hat{x}_t)$ for some given data set $\{(\hat{\mathbf{p}}_t, \hat{x}_t)\}_{t=1,\dots,n}$. Thus, in terms of the original data we can rephrase the inequalities in Definition 5 as

$$\frac{\hat{\mathbf{p}}_{\sigma(t)}}{\hat{x}_{\sigma(t)}} \leq \lambda \frac{\hat{\mathbf{p}}_{\sigma(v)}}{\hat{x}_{\sigma(v)}} + (1-\lambda)\frac{\hat{\mathbf{p}}_{\sigma(k)}}{\hat{x}_{\sigma(k)}} \quad \text{or} \quad \frac{\hat{\mathbf{p}}_{\sigma(t)}}{\hat{x}_{\sigma(t)}} \geq \lambda \frac{\hat{\mathbf{p}}_{\sigma(v)}}{\hat{x}_{\sigma(v)}} + (1-\lambda)\frac{\hat{\mathbf{p}}_{\sigma(k)}}{\hat{x}_{\sigma(k)}}.$$

We can show that the triangular condition in Definition 5 is necessary and sufficient for WARP and SARP to be equivalent.

Proposition 2. *A set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is WARP-reducible if and only if it is a triangular configuration.*

It is instructive to briefly sketch the main steps of the proof of this result (the full proof is provided in the Appendix). For the necessity part, we assume three price vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ that do not satisfy the triangular condition. An application of Farkas' lemma (theorem of the alternative) then establishes the existence of three quantity vectors $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\} \in \mathcal{Q}(\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\})$ such that $\{(\mathbf{p}_1, \mathbf{q}_1), (\mathbf{p}_2, \mathbf{q}_2), (\mathbf{p}_3, \mathbf{q}_3)\}$ satisfies WARP and violates SARP.

Next, we prove the sufficiency part by contradiction. In particular, we assume that P is a triangular configuration but is not WARP-reducible. This implies that there exist quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$ such that a subset of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates SARP but not WARP. Without loss of generality, we let this subset be $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,J}$ ($J \leq n$) and we assume it is minimal (i.e., does not contain another subset violating SARP). In other words, we have a minimal revealed preference cycle

$$\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \dots R \, \mathbf{q}_J \, R \, \mathbf{q}_1.$$

Subsequently, we consider for any $j \leq J$ the six possible cases for the triangular inequalities that involve $\mathbf{p}_j, \mathbf{p}_{j+1}$ and $\mathbf{p}_{j+2}$ (i.e., three consecutive elements in the SARP cycle). For each of these six cases, we can show that satisfying the associated triangular inequality in Definition 5 leads to a contradiction. To do so, we exploit either that the length of the SARP cycle is minimal or that WARP is satisfied.

3. Further discussion

To further interpret our characterization in Proposition 2, we clarify the specific relationship between our main result and the original result of Rose (1958). Subsequently, we sharpen the intuition of our triangular condition through a specific example with

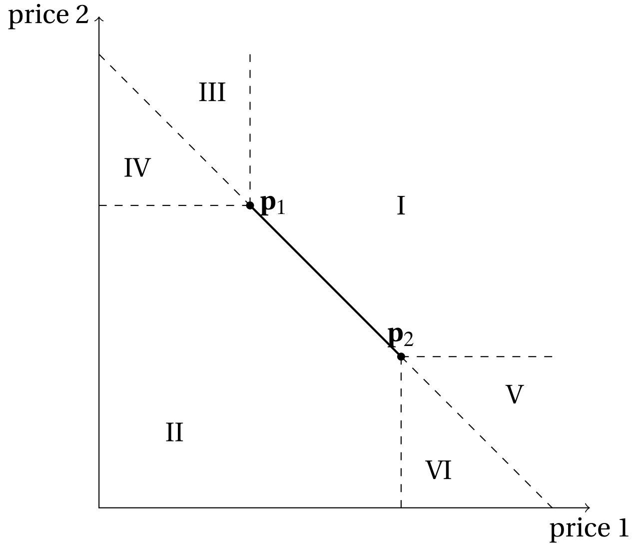

FIGURE 1. The triangular condition in a two-goods setting.

three normalized price vectors. Finally, we discuss the possibility of using our triangular condition to bound the length of (potential) SARP cycles.

It is fairly easy to verify that Proposition 2 generalizes Rose's result in Proposition 1. In particular, it suffices to show that if the number of goods is equal to two, then any set of price vectors is a triangular configuration. To see this, consider three normalized price vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for two goods (i.e., $m = 2$). Obviously, if $\mathbf{p}_1 \geq \mathbf{p}_2$ or $\mathbf{p}_2 \geq \mathbf{p}_1$, we have that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a triangular configuration. Let us then consider the more interesting case where $\mathbf{p}_1$ and $\mathbf{p}_2$ are not ordered, which we illustrate in Figure 1.

The price vector $\mathbf{p}_3$ should then fall into one of the six regions, which are numbered I–VI. For any of these six possible scenarios, the triangular condition in Definition 5 is met. To show this, we first consider the case where $\mathbf{p}_3$ lies in region I. In that case, $\mathbf{p}_3$ is obviously larger than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. Similarly, if $\mathbf{p}_3$ lies in region II, it is smaller than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. Next, if $\mathbf{p}_3$ lies in region III, then $\mathbf{p}_1$ is smaller than a convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ and, conversely, $\mathbf{p}_1$ is larger than a convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ if $\mathbf{p}_3$ lies in region IV. Finally, if $\mathbf{p}_3$ lies in region V, there is a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$ that dominates $\mathbf{p}_2$, and if $\mathbf{p}_3$ lies in region VI, then $\mathbf{p}_2$ is larger than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$. We can thus conclude that any possible set of price vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is WARP-reducible.

The following example provides some further intuition for the result in Proposition 2. In this example, we focus on cycles of length 3, and show that the triangular configuration implies that each SARP violation of length 3 must contain a WARP violation.

EXAMPLE 1. Consider a set of three normalized price vectors $P = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ that is a triangular configuration. Without loss of generality, we may assume that one of the inequalities $\mathbf{p}_1 \leq \lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$ or $\mathbf{p}_1 \geq \lambda\mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$ holds for some $\lambda \in [0, 1]$.

Let us first consider $\mathbf{p}_1 \leq \lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$. Assume that there exists a SARP violation with a cycle of length 3. With three observations, there are only two possibilities for cycles of length 3: $\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_1$ or $\mathbf{q}_1 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_1$. If $\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_1$, then it must be that

$$1 = \mathbf{p}_2\mathbf{q}_2 \geq \mathbf{p}_2\mathbf{q}_3 \quad \text{and} \quad 1 = \mathbf{p}_3\mathbf{q}_3.$$

Together with our triangular inequality this implies that

$$1 \geq \big(\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3\big)\mathbf{q}_3 \geq \mathbf{p}_1\mathbf{q}_3.$$

As such, we can conclude that $\mathbf{q}_1 \, R \, \mathbf{q}_3$, which gives $\mathbf{q}_1 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_1$, i.e., a violation of WARP. Similar reasoning holds for the second possibility (i.e., $\mathbf{q}_1 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_1$), which shows that in this first case each violation of SARP implies a WARP violation.

For the second case, $\mathbf{p}_1 \geq \lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$, we must consider the same two possible SARP violations. The reasoning is now slightly different. In particular, let us assume that there is no violation of WARP. For the SARP violation $\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_1$, this requires $1 < \mathbf{p}_3\mathbf{q}_2$ (i.e., not $\mathbf{q}_3 \, R \, \mathbf{q}_2$). Since $1 = \mathbf{p}_2\mathbf{q}_2$, we obtain that if $\lambda < 1$, then

$$1 < \big(\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3\big)\mathbf{q}_2 \leq \mathbf{p}_1\mathbf{q}_2.$$

This clearly contradicts $\mathbf{q}_1 \, R \, \mathbf{q}_2$ (i.e., $1 \geq \mathbf{p}_1\mathbf{q}_2$). If $\lambda = 1$, we have $\mathbf{p}_1 \geq \mathbf{p}_2$ and, thus,

$$1 = \mathbf{p}_1\mathbf{q}_1 \geq \mathbf{p}_2\mathbf{q}_1.$$

This again yields a contradiction, as it implies the WARP violation $\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_1$. A similar reasoning holds for the second possibility (i.e., $\mathbf{q}_1 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_1$), which shows that also for this case, any SARP violation implies a WARP violation. ◊

Finally, an obvious question pertains to the possibility of using our characterization in Proposition 2 to bound the length of (potential) SARP cycles.[6] In particular, assume that we have a set of n normalized price vectors that form a triangular configuration, and suppose that we add a $(n+1)$th price vector such that the extended set of prices is no longer a triangular configuration. Is it possible to bound the length of the (potential) SARP cycles for this extended set of normalized prices?

In Appendix B, we show that the answer to this question is negative: the extended price set can imply a SARP cycle of maximal length (i.e., all observations are involved). We obtain this conclusion in three steps. In the first step, we construct a set of n normalized price vectors $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ that is a triangular configuration and, therefore, WARP-reducible. In the second step, we construct a corresponding set of n quantity vectors $(\mathbf{q}_1, \dots, \mathbf{q}_n)$ to define a data set $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\dots,n}$ for which we characterize the associated revealed preference relations. In the third and final step, we introduce a $(n+1)$th price vector $(\mathbf{p}_0)$ and a corresponding $(n+1)$th quantity vector $(\mathbf{q}_0)$ that obtains a SARP cycle of length $n+1$ for the data set $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=0,\dots,n}$, without there being any smaller SARP cycle. At a general level, this leads us to conclude that, for a set of n price vectors that is WARP-reducible, it is possible to add a $(n+1)$th price vector that obtains a SARP cycle of any length.

[6]We thank Mark Dean for pointing this question out to us.

4. Connection with Hicksian aggregation

So far we have assumed a finite data set with n normalized price vectors (i.e., budget sets). This corresponds to a typical situation in empirical demand analysis, when the empirical analyst can only use a finite number of observations. In this section, we consider the theoretical situation with a continuum of (normalized) price vectors. This establishes a formal connection between our triangular condition and the notion of Hicksian aggregation. Specifically, we show that when the set of price vectors becomes infinite, our conditions lead to the requirement that the demand for multiple (i.e., m) goods can be summarized in terms of two Hicksian aggregates. In a sense, it establishes our characterization in Proposition 2 as a finite sample version of the Hicksian aggregation requirement for WARP to be equivalent to SARP.

To formalize the argument, we assume that the infinite set of normalized price vectors P is a cone, that is, for all vectors $\mathbf{p} \in P$ and all $\gamma > 0$, $\gamma\mathbf{p} \in P$. We remark that because we focus on normalized price vectors (with total expenditures equal to unity), the price vector $\gamma\mathbf{p}$ equivalently corresponds to a situation with (nonnormalized) price vector $\mathbf{p}$ and total expenditures $1/\gamma$. In other words, our condition on the set P actually allows us to consider any possible expenditure level for a given specification of (nonnormalized) price vectors. Likewise, it gives the set of possible normalized price vectors when we do not have any prior information on the total expenditure level. We can derive the following result.

Proposition 3. *Let the set of normalized price vectors P be a cone. Define the $(m-1)$-dimensional simplex $\Delta = \{\mathbf{p} \in \mathbb{R}^m_{++} \mid \sum_{i=1}^m (\mathbf{p})_i = 1\}$.*[7] *If $P \cap \Delta$ is closed, then any three price vectors of P satisfy the triangular condition if and only if there exist two vectors $\mathbf{r}_1, \mathbf{r}_2 \in P$ such that for all $\mathbf{p} \in P$, $\mathbf{p}$ is a linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$. In particular, there are numbers $\alpha, \beta \in \mathbb{R}_+$ not both zero such that*

$$\mathbf{p} = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2.$$

Basically, this result requires that all price vectors $\mathbf{p} \in P$ must lie in a common two-dimensional plane. The additional requirement that $P \cap \Delta$ is closed is a technical condition guaranteeing that $\mathbf{r}_1$ and $\mathbf{r}_2$ belong to P.

Interestingly, Proposition 3 allows us to interpret our triangular condition (under infinitely many prices) in terms of Hicksian quantity aggregation. Specifically, Hicksian aggregation requires that all prices in a subset of goods change proportionally to some common price vector (i.e., $\mathbf{p} = \alpha\mathbf{r}$ for all t, with $\mathbf{r} \in \mathbb{R}^m_+$ and scalar $\alpha > 0$). In our case, we can, for any bundle $\mathbf{q}_t$, construct a new "quantity vector" $\mathbf{z}_t$ of two goods where $(\mathbf{z}_t)_1 = \mathbf{r}_1\mathbf{q}_t$ and $(\mathbf{z}_t)_2 = \mathbf{r}_2\mathbf{q}_t$. Correspondingly, we can construct new "price vectors" $\mathbf{w}_t = [\alpha_t, \beta_t]$. Then, for any two observations t and v, we have

$$1 \geq \mathbf{p}_t\mathbf{q}_v = (\alpha_t\mathbf{r}_1 + \beta_t\mathbf{r}_2)\mathbf{q}_v = \alpha_t\mathbf{r}_1\mathbf{q}_v + \beta_t\mathbf{r}_2\mathbf{q}_v = \mathbf{w}_t\mathbf{z}_v.$$

[7] For $(\mathbf{p})_i$ representing the ith component of $\mathbf{p}$ (i.e., the price of good i).

In other words, we obtain $\mathbf{q}_t \; R \; \mathbf{q}_v$ for the set of quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$ if and only if $\mathbf{z}_t \; R \; \mathbf{z}_v$ for the set of quantity vectors $\{\mathbf{z}_t\}_{t=1,\dots,n}$. This implies that $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates SARP (resp. WARP) if and only if the data set $\{(\mathbf{w}_t, \mathbf{z}_t)\}_{t=1,\dots,n}$ violates SARP (resp. WARP). Moreover, the data set $\{(\mathbf{w}_t, \mathbf{z}_t)\}_{t=1,\dots,n}$ only contain two goods, so Proposition 1 implies that WARP is equivalent to SARP, and this equivalence carries over to the set of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$. Basically, this defines the possibility to construct two Hicksian aggregates as a necessary and sufficient condition for WARP to be equivalent to SARP when there are infinitely many price vectors.

By building further on this intuition, we can also directly interpret Proposition 3 in terms of utility maximizing behavior. To see this, we start by considering a rational (i.e., SARP-consistent) individual with indirect utility function $v(\mathbf{p})$, which defines the maximal attainable utility given the normalized price vector $\mathbf{p}$. By construction, this function $v(\mathbf{p})$ is quasi-convex, is decreasing, and satisfies Roy's identity, i.e., the m-dimensional demand functions are given by $\mathbf{q} = \nabla_{\mathbf{p}} v(\mathbf{p})/(\mathbf{p}\nabla_{\mathbf{p}} v(\mathbf{p}))$. By using our above notation, if the Hicksian aggregation property in Proposition 3 is satisfied, we can write $v(\mathbf{p}) = v(\alpha \mathbf{r}_1 + \beta \mathbf{r}_2) \equiv \tilde{v}(\alpha, \beta) = \tilde{v}(\mathbf{w})$. It is easy to verify that also $\tilde{v}(\mathbf{w})$ is quasi-convex, is decreasing, and satisfies Roy's identity, which in this case states that the two-dimensional demand functions satisfy $\mathbf{z} = \nabla_{\mathbf{w}} \tilde{v}(\mathbf{w})/(\mathbf{w}\nabla_{\mathbf{w}} \tilde{v}(\mathbf{w}))$.

5. Empirical illustrations

To show the practical relevance of our triangular condition, we present empirical applications that make use of two different types of household data sets that have been the subject of empirical revealed preference analysis in recent studies. They illustrate alternative possible uses of our characterization in Proposition 2.

Panel data. Our first application considers household data that are drawn from the Spanish survey ECPF (Encuesta Continua de Presupestos Familiares), which has been used in various SARP-based empirical analyses.[8] In what follows, we specifically focus on the data set that was studied by Beatty and Crawford (2011). This data set contains a time series of 8 observations for 1585 households, on 15 nondurable goods. Importantly, different households can be characterized by other price regimes, which results in the empirical content of our triangular condition varying over households.

We begin by verifying whether the household-specific price series satisfies the conditions for two-dimensional Hicksian aggregation as we define them in Section 4 (Proposition 3). As discussed before, these conditions are sufficient (but not necessary) for WARP to be equivalent to SARP in the case of infinite data sets. It turns out that none of the 1585 household data sets satisfies the conditions. This shows that the Hicksian aggregation criteria are very stringent from an empirical point of view. More generally, it suggests that for finite data sets, there is little hope that Hicksian aggregation arguments provide an effective basis to justify a WARP-based empirical analysis instead of a SARP-based analysis.

[8] See, for example, Crawford (2010), Beatty and Crawford (2011), Demuynck and Verriest (2013), Adams et al. (2014), and Laurens and Thomas (2015).

By contrast, if we check the triangular condition in Definition 5, we conclude that no less than 69.34% of the data sets satisfies these requirements. For these data sets, a WARP-based analysis is as equally informative as a SARP-based analysis. In view of the computational burden associated with the transitivity requirement that is captured by SARP, we see this as quite a comforting conclusion from a practical point of view. It also indicates that the (necessary and sufficient) triangular condition provides a substantially more useful basis than the (sufficient) Hicksian aggregation conditions to empirically support a WARP-based analysis. Even though the two types of conditions converge for infinitely large data sets, their empirical implications for finite data sets can differ considerably.

Repeated cross-sectional data. Our second application uses the data from the British Family Expenditure Survey (FES) that have been analyzed by Blundell et al. (2003, 2008, 2015). These authors developed methods to combine Engel curves with revealed preference axioms to obtain tight bounds on cost of living indices and demand responses. These methods become substantially more elaborate when considering SARP instead of WARP. This makes it directly relevant to check whether WARP and SARP are equivalent for the budget sets taken up in the analysis.[9]

More specifically, the data set is a repeated cross section that contains 25 yearly observations (1975–1999) for three product categories (food, other nondurables, and services). As in the original studies, we focus on mean income for each observation year. When checking our triangular condition for all triples of (normalized) price vectors, we conclude that 2.39% of these triples violate these conditions. This indicates that WARP and SARP are not fully equivalent for these data. However, for a fraction as low as 2.39%, it is also fair to conclude that the subset of price vectors that may induce differences between WARP and SARP is quite small.

6. When WGARP equals GARP

As a final step of our analysis, we extend our SARP-based result in Proposition 2 to apply to the generalized axiom of revealed preference (GARP). GARP generalizes SARP by allowing for linear parts in the indifference curves of the consumer, i.e., multi-valued demand correspondences. In this GARP-based setting, we use the following modifications of Definitions 1–4.

Definition 6. Let $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ be a set of normalized prices and let $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$. Then, for all $t, v \leq n$, the quantity vector $\mathbf{q}_t$ is *strictly revealed preferred* to the bundle $\mathbf{q}_v$ if $\mathbf{p}_t\mathbf{q}_t(=1) > \mathbf{p}_t\mathbf{q}_v$. We denote this as $\mathbf{q}_t \; P_R \; \mathbf{q}_v$.

[9] In this respect, Kitamura and Stoye (2013) use the same FES data in their application of so-called stochastic axioms of revealed preference, which form the population analogues of the more standard revealed preference axioms such as WARP and SARP (see McFadden 2005 for an overview). In a stochastic revealed preference setting, the verification of WARP is relatively easy from a computational point of view (see, for example, Hoderlein and Stoye 2014 and Cosaert and Demuynck 2018), while the verification of SARP is known to be difficult (i.e., nondeterministic polynomial (NP) hard). As a direct implication, the knowledge that WARP is empirically equivalent to SARP can have a huge impact on the computation time.

Definition 7. Let $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ be a set of normalized price vectors and let $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$. Then $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates *WGARP* if there are observations $t, v \leq n$ such that $\mathbf{q}_t \, P_R \, \mathbf{q}_v \, R \, \mathbf{q}_t$.

Definition 8. Let $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ be a set of normalized price vectors and let $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$. Then $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates *GARP* if there is a revealed preference cycle with at least one strict revealed preference comparison, i.e.,

$$\mathbf{q}_t \, P_R \, \mathbf{q}_v \, R \, \mathbf{q}_s \dots R \, \mathbf{q}_k \, R \, \mathbf{q}_t$$

for some sequence of observations $t, v, s, \dots, k$.

Definition 9. A set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is said to be *WGARP-reducible* if, for any set of quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$ and $\mathbf{q}_t \gg \mathbf{0}$ for all t, if a subset of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ violates GARP, then the same subset also violates WGARP.

In Definition 9, the restriction $\mathbf{q}_t \gg \mathbf{0}$ imposes that every element in the quantity vector $\mathbf{q}_t$ should be strictly positive, i.e., all goods should be consumed with strictly positive amounts. We clarify the relevance for this additional constraint after presenting our main result in Proposition 5.

Before stating Proposition 5, we recapture the result of Banerjee and Murphy (2006), which provided the GARP-based extension of Proposition 1 derived by Rose (1958).

Proposition 4. *If there are only two goods (i.e., $m = 2$), then any set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is WGARP-reducible.*

When using the concept of triangular configuration in Definition 5, we can derive the following generalization of Banerjee and Murphy's result.

Proposition 5. *A set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is WGARP-reducible if and only if it is a triangular configuration.*

At this point, it is worth remarking that the requirement $\mathbf{q}_t \gg \mathbf{0}$ in Definition 9 is crucial for this result to hold. We show this by means of Example 2, which presents a set of three normalized prices that is a triangular configuration. For these prices, we can define quantity bundles with zero entries that satisfy WGARP but not GARP.[10]

Example 2. Consider the following set of three normalized price vectors (which is presented graphically in Figure 2):

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix}, \qquad \mathbf{p}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{p}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

[10]We note that the fact that we have three goods is crucial for the construction in Example 2. Such a construction is not possible if there are only two goods.

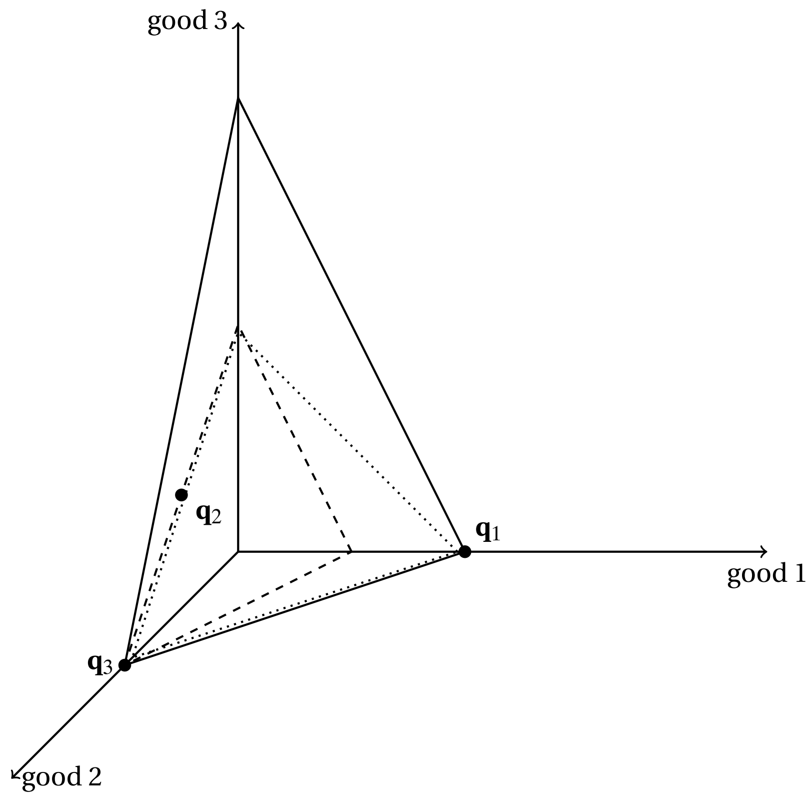

Figure 2. Illustration of the counterexample. Budget 1 is the solid hyperplane. Budget 2 is the dashed hyperplane. Budget 3 is the dotted hyperplane.

Because $\mathbf{p}_1, \mathbf{p}_3 \leq \mathbf{p}_2$, we easily obtain that the triangular condition is satisfied. However, if we allow for zero quantities, we can construct quantity bundles that violate GARP but not WGARP for the given prices. For example, this applies to

$$\mathbf{q}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{q}_2 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \mathbf{q}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

These quantities imply a GARP violation because $\mathbf{q}_1 \; P_R \; \mathbf{q}_2 \; R \; \mathbf{q}_3 \; R \; \mathbf{q}_1$. However, $1 < \mathbf{p}_2\mathbf{q}_1 = 2$, so $\{(\mathbf{p}_1, \mathbf{q}_1), (\mathbf{p}_2, \mathbf{q}_2)\}$ does not form a WGARP cycle. $\diamondsuit$

Thus, we conclude that the triangular condition does not characterize the sets of normalized price vectors that are WGARP-reducible without the restriction that $\mathbf{q}_t \gg \mathbf{0}$ for all t. From a technical perspective, the requirement $\mathbf{q} \gg \mathbf{0}$ allows us to go from a strict vector inequality $\mathbf{p} > \widetilde{\mathbf{p}}$ to the inequality $\mathbf{p}\mathbf{q} > \widetilde{\mathbf{p}}\mathbf{q}$. By contrast, when $\mathbf{q}$ can contain zero entries, we may well have $\mathbf{p} > \widetilde{\mathbf{p}}$ and $\mathbf{p}\mathbf{q} = \widetilde{\mathbf{p}}\mathbf{q}$.

When allowing for zero quantities, it is possible to define a sufficient (but not always necessary) condition for WGARP reducibility that may be interpreted as a "strict" version of the triangular condition in Definition 9. In particular, we can use the following concept.

Definition 10. A set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is a *strict triangular configuration* if, for any three normalized price vectors $\mathbf{p}_t, \mathbf{p}_v$, and $\mathbf{p}_k$ in P, there exists a number $\lambda \in [0, 1]$ and a permutation $\sigma : \{t, v, k\} \to \{t, v, k\}$ such that the following condition holds:

$$\mathbf{p}_{\sigma(t)} \ll \lambda \mathbf{p}_{\sigma(v)} + (1-\lambda)\mathbf{p}_{\sigma(k)} \quad \text{or} \quad \mathbf{p}_{\sigma(t)} \gg \lambda \mathbf{p}_{\sigma(v)} + (1-\lambda)\mathbf{p}_{\sigma(k)}.$$

Essentially, this definition replaces the weak inequalities $\leq$ and $\geq$ in Definition 5 by the strict element-wise inequalities $\ll$ and $\gg$. A simple adaptation of the sufficiency part of our proof of Proposition 2 then obtains the following result.[11]

Corollary 1. *If a set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ satisfies the strict triangular condition, then it is WGARP-reducible even if we allow for zero entries of the quantity vectors in Definition 9.*

Some remarks are in order. First, in practice there is almost no finite data set that satisfies the triangular condition in Definition 5 but not the strict triangular condition in Definition 10. In other words, the empirical content of the two conditions coincides in most real life settings. Next, we note that in a two-goods setting, the strict triangular configuration is violated only if the three normalized price vectors are collinear, i.e., one of the normalized price vectors is a convex combination of the other two.

Finally, Reny (2015) recently showed that any finite or infinite data set can be rationalized by a utility function if and only if the data set satisfies GARP.[12] Given this, our triangular condition makes sense even in nonfinite settings: for any finite or infinite data set (with strictly positive consumption bundles), if the triangular condition in Definition 5 is satisfied, then the data set can be rationalized by a utility function if and only if WGARP is satisfied.[13]

Appendix A: Proofs

A.1 *Proof of Proposition 2*

Sufficiency. Consider a set of price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ that is a triangular configuration. Toward a contradiction, assume that P is not WARP-reducible. This means that there exists a set of quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}(P)$ and a subset of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ such that SARP is violated but WARP is satisfied.

[11] Intuitively, this result builds on the fact that, for any nonzero vector $\mathbf{q} \geq \mathbf{0}$, we have $\mathbf{pq} > \widetilde{\mathbf{p}}\mathbf{q}$ when $\mathbf{p} \gg \widetilde{\mathbf{p}}$, so that strict (price) vector inequalities translate into strict (price × quantity) vector product inequalities.

[12] Rationalization by a utility function means that there exists a utility function $u : \mathbb{R}^m \to \mathbb{R}$ such that for every observation $(\mathbf{p}_t, \mathbf{q}_t)$, $\mathbf{p}_t\mathbf{q} \leq \mathbf{p}_t\mathbf{q}_t$ implies $u(\mathbf{q}) \leq u(\mathbf{q}_t)$ and $\mathbf{p}_t\mathbf{q} < \mathbf{p}_t\mathbf{q}_t$ implies $u(\mathbf{q}) < u(\mathbf{q}_t)$.

[13] We thank an anonymous referee for pointing this out.

Let $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t\in J}$ ($J \subseteq \{1, \dots, n\}$) be such a subset that is minimal with respect to set inclusion (such a set exists by the fact that n is finite). Although the set J is not necessarily unique, this minimality property implies that (i) $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t\in J}$ cannot have a smaller subset that also violates SARP, (ii) all elements in $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t\in J}$ must be involved in the SARP cycle, and (iii) all vectors in $\{\mathbf{q}_j\}_{j\in J}$ are distinct. These three features are crucial to establish the contradiction. In what follows, we use that feature (ii) implies, for the set J, that the shortest SARP cycle has (minimal) length $|J|$.[14]

Without loss of generality, let us re-index the observations in J such that the SARP violation is given by the cycle $\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_3 \dots R \, \mathbf{q}_{|J|} \, R \, \mathbf{q}_1$, i.e.,

$$
\begin{aligned}
1 &\geq \mathbf{p}_1\mathbf{q}_2, \\
1 &\geq \mathbf{p}_2\mathbf{q}_3, \\
&\vdots \\
1 &\geq \mathbf{p}_{|J|-1}\mathbf{q}_{|J|}, \\
1 &\geq \mathbf{p}_{|J|}\mathbf{q}_1.
\end{aligned}
$$

For a number $j \geq 1$, we denote by $\lfloor j \rfloor$ the number $j \mod |J|$.

Given that P is a triangular configuration, we have that, for any $j \leq |J|$, there must exist a $\lambda \in [0, 1]$ such that one of the following inequalities holds:

$$\mathbf{p}_{\lfloor j+1\rfloor} \leq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}, \tag{1}$$

$$\mathbf{p}_j \leq \lambda\mathbf{p}_{\lfloor j+1\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}, \tag{2}$$

$$\mathbf{p}_{\lfloor j+2\rfloor} \leq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}, \tag{3}$$

$$\mathbf{p}_j \geq \lambda\mathbf{p}_{\lfloor j+1\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}, \tag{4}$$

$$\mathbf{p}_{\lfloor j+1\rfloor} \geq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}, \tag{5}$$

$$\mathbf{p}_{\lfloor j+2\rfloor} \geq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}. \tag{6}$$

In the remainder of this sufficiency proof, we show that none of these inequalities can hold, which gives us the desired contradiction.

Lemma A1.1. *Inequalities (4), (5), and (6) cannot hold.*

Proof. There are three similar cases to consider.

Case 1. Assume $\mathbf{p}_j \geq \lambda\mathbf{p}_{\lfloor j+1\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}$ holds. We first note that if $\lambda = 1$, then we have that $\mathbf{p}_j \geq \mathbf{p}_{\lfloor j+1\rfloor}$. Multiplying both sides by $\mathbf{q}_j$ gives

$$1 = \mathbf{p}_j\mathbf{q}_j \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_j.$$

[14] We remark that, for $|J| = 3$, a SARP cycle with length smaller than $|J|$ actually implies a WARP violation. For compactness, we do not consider this case separately in what follows. But it is easily verified that it is implicitly included in our further argument.

Since $\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+1\rfloor}$ also equals 1, this obtains the WARP violation

$$\mathbf{q}_j \; R \; \mathbf{q}_{\lfloor j+1\rfloor} \; R \; \mathbf{q}_j.$$

This is a contradiction. As such, we can assume that $\lambda < 1$. Then multiplying both sides of the inequality by $\mathbf{q}_{\lfloor j+1\rfloor}$ gives

$$\mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor} \geq \lambda\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+1\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\rfloor j+1\rfloor}.$$

Since $\mathbf{p}_j\mathbf{q}_j(=1) \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor}$, this implies

$$\begin{aligned}
&1 \geq \lambda + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+1\rfloor} \\
&\quad\Leftrightarrow\quad 1-\lambda \geq (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+1\rfloor} \\
&\quad\Leftrightarrow\quad 1 \geq \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\rfloor j+1\rfloor}.
\end{aligned}$$

This obtains again a WARP violation

$$\mathbf{q}_{\lfloor j+1\rfloor} \; R \; \mathbf{q}_{\lfloor j+2\rfloor} \; R \; \mathbf{q}_{\lfloor j+1\rfloor}.$$

Case 2. Assume $\mathbf{p}_{\lfloor j+1\rfloor} \geq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}$ holds. We first note that if $\lambda = 0$, then we have that $\mathbf{p}_{\lfloor j+1\rfloor} \geq \mathbf{p}_{\lfloor j+2\rfloor}$. After similar reasoning as in Case 1, we then obtain a WARP violation by multiplying both sides by $\mathbf{q}_{\lfloor j+1\rfloor}$.

As such, we can assume that $\lambda > 0$. Then multiplying both sides of the inequality by $\mathbf{q}_{\lfloor j+2\rfloor}$ gives

$$\begin{aligned}
&1 \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+2\rfloor} \geq \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+2\rfloor} \\
&\quad\Leftrightarrow\quad 1 \geq \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda) \\
&\quad\Leftrightarrow\quad 1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor}.
\end{aligned}$$

This implies that $\mathbf{q}_j \; R \; \mathbf{q}_{\lfloor j+2\rfloor}$. As such, we can remove $\lfloor j+1\rfloor$ from J to obtain a SARP cycle with length smaller than $|J|$. But this contradicts the minimality property of the set J.

Case 3. Assume $\mathbf{p}_{\lfloor j+2\rfloor} \geq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}$ holds. We first note that if $\lambda = 1$, then we have that $\mathbf{p}_{\lfloor j+2\rfloor} \geq \mathbf{p}_j$. By multiplying both sides of the inequality by $\mathbf{q}_{\lfloor j+2\rfloor}$, we obtain that we can remove $\lfloor j+1\rfloor$ from J, which obtains the same case as under Case 2.

As such, we can assume that $\lambda < 1$. Multiplying both sides of the inequality by $\mathbf{q}_{\lfloor j+3\rfloor}$ gives

$$\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+3\rfloor} \geq \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+3\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+3\rfloor}.$$

If $\lfloor j+3\rfloor = j$, then $1 \geq \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_j \geq \lambda + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_j$. So $1 \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_j$ which gives the WARP cycle

$$\mathbf{q}_j \; R \; \mathbf{q}_{\lfloor j+1\rfloor} \; R \; \mathbf{q}_j.$$

As such, assume that $\lfloor j+3 \rfloor \neq j$. Then, given that the left hand side (which is smaller than or equal to 1) must be larger than a convex combination of two positive numbers, it must be larger than at least one of them. If $1 \geq \mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+3 \rfloor} \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+3 \rfloor}$, we can remove $\lfloor j+1 \rfloor$ and $\lfloor j+2 \rfloor$ from J to obtain a SARP cycle with length smaller than $|J|$. Similarly, if $1 \geq \mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+3 \rfloor} \geq \mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+3 \rfloor}$, we can remove $\lfloor j+2 \rfloor$ from J to obtain a SARP cycle with length smaller than $|J|$. In each situation, we get a contradiction with the minimality property of J. ◁

The following lemma considers inequalities (2) and (3).

Lemma A1.2. *Inequalities (2) and (3) cannot hold.*

Proof. There are two similar cases to consider.

Case 1. Assume $\mathbf{p}_j \leq \lambda\mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}$ holds. Multiplying both sides by $\mathbf{q}_{\lfloor j+2 \rfloor}$ gives

$$\mathbf{p}_j\mathbf{q}_{\lfloor j+2 \rfloor} \leq \lambda\mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+2 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+2 \rfloor}.$$

The right hand side is a weighted average of two numbers that are less than or equal to 1, so this number is also less than or equal to 1. As such,

$$1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+2 \rfloor}.$$

This implies that $\mathbf{q}_j \; R \; \mathbf{q}_{\lfloor j+2 \rfloor}$ and we can remove $\lfloor j+1 \rfloor$ from J to obtain a SARP cycle with length smaller than $|J|$, which contradicts with the minimality property of the set J.

Case 2. Assume $\mathbf{p}_{\lfloor j+2 \rfloor} \leq \lambda\mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1 \rfloor}$ holds. Now multiply both sides by $\mathbf{q}_{\lfloor j+1 \rfloor}$:

$$\mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor} \leq \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+1 \rfloor} + (1-\lambda).$$

The right hand side is again a weighted average of two numbers that are smaller than or equal to 1, so the left hand side is also smaller than or equal to 1. This gives the WARP cycle

$$\mathbf{q}_{\lfloor j+2 \rfloor} \; R \; \mathbf{q}_{\lfloor j+1 \rfloor} \; R \; \mathbf{q}_{\lfloor j+2 \rfloor}.$$ ◁

Lemmata A1.1 and A1.2 show that we can conclude that condition (1) must hold for all $j \leq |J|$. That is, for all j, there exists a $\lambda_j \in [0, 1]$ such that

$$\mathbf{p}_{\lfloor j+1 \rfloor} \leq \lambda_j\mathbf{p}_j + (1-\lambda_j)\mathbf{p}_{\lfloor j+2 \rfloor},$$

or, in other words, there must exist $\lambda_1, \ldots, \lambda_{|J|} \in [0, 1]$ that solve the system of inequalities

$$\lambda_1\mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3 \geq \mathbf{p}_2,$$

$$\lambda_2\mathbf{p}_2 + (1-\lambda_2)\mathbf{p}_4 \geq \mathbf{p}_3,$$

$$\vdots$$

$$\lambda_{|J|-1}\mathbf{p}_{|J|-1} + (1-\lambda_{|J|-1})\mathbf{p}_1 \geq \mathbf{p}_{|J|},$$
$$\lambda_{|J|}\mathbf{p}_{|J|} + (1-\lambda_{|J|})\mathbf{p}_2 \geq \mathbf{p}_1.$$

We show that this system of inequalities cannot have a solution for the λ_j. As a first step, we note that none of the λ_j can be equal to 0 or 1. Specifically, if $\lambda_j = 0$, then we have $\mathbf{p}_{\lfloor j+1 \rfloor} \leq \mathbf{p}_{\lfloor j+2 \rfloor}$. Multiplying both sides by $\mathbf{q}_{\lfloor j+3 \rfloor}$ then gives

$$\mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+3 \rfloor} \leq \mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+3 \rfloor} \leq 1,$$

which implies that $\mathbf{q}_{\lfloor j+1 \rfloor} \, R \, \mathbf{q}_{\lfloor j+3 \rfloor}$, so that we can remove $\lfloor j+2 \rfloor$ from J to obtain a subset $J \setminus \{\lfloor j+2 \rfloor\}$ that satisfies WARP and violates SARP. This contradiction with the minimality property of J shows that $\lambda_j > 0$. Alternatively, if $\lambda = 1$, then $\mathbf{p}_{\lfloor j+1 \rfloor} \leq \mathbf{p}_j$. Multiplying both sides by $\mathbf{q}_j$ then gives

$$\mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_j \leq \mathbf{p}_j\mathbf{q}_j = 1,$$

which implies that $\mathbf{q}_j \, R \, \mathbf{q}_{\lfloor j+1 \rfloor} \, R \, \mathbf{q}_j$, a violation of WARP.

Thus, we must have $\lambda_j \in (0,1)$ for all $j = 1, \dots, |J|$. Now, for any good i, let us define j such that (for $(\mathbf{p})_i$ representing the ith component of $\mathbf{p}$)

$$(\mathbf{p}_{\lfloor j+1 \rfloor})_i = \max_{t \in J}(\mathbf{p}_t)_i = M_i.$$

Then the inequality $\mathbf{p}_{\lfloor j+1 \rfloor} \leq \lambda_j \mathbf{p}_j + (1-\lambda_j)\mathbf{p}_{\lfloor j+2 \rfloor}$ implies

$$(\mathbf{p}_j)_i = (\mathbf{p}_{j+2})_i = (\mathbf{p}_{j+1})_i = M_i,$$

because $\lambda_j \in (0,1)$. We can repeat the same reasoning for $j+1, j+2, \dots$ to obtain

$$(\mathbf{p}_t)_i = M_i \quad \text{for all } t \in J.$$

By replicating this argument for all goods $i = 1, \dots, m$, we get that the price vectors $\mathbf{p}_t$ are identical for all $t \in J$. But this makes it impossible that the set $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,|J|}$ violates SARP (and WARP), which gives the desired contradiction.

This finishes the sufficiency part of our proof and we can conclude that the set of price vectors $P = \{\mathbf{p}_t\}_{t=1,\dots,n}$ is WARP-reducible if it is a triangular configuration.

Necessity. To show the reverse, let us consider a set of price vectors P that is not a triangular configuration. In particular, let $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \subseteq P$ be a set of three distinct price vectors such that none of the vector inequalities for the triangular configuration is satisfied.

Our aim is to show the existence of vectors $\{\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}\} \in \mathcal{Q}(\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\})$ such that $\{(\mathbf{p}_1, \mathbf{q}_1), (\mathbf{p}_2, \mathbf{q}_2), (\mathbf{p}_3, \mathbf{q}_3)\}$ violates SARP but not WARP.

To obtain the result, let us first show that there exists a vector $\widetilde{\mathbf{q}}_1 \in \mathbb{R}^m_+$ and a number $M > 0$ such that the following system of inequalities has a solution:

$$\mathbf{p}_1\widetilde{\mathbf{q}}_1 = M,$$
$$\mathbf{p}_3\widetilde{\mathbf{q}}_1 \leq M,$$
$$\mathbf{p}_2\widetilde{\mathbf{q}}_1 > M.$$

By introducing the slack variables a and b, the feasibility of this system of linear inequalities is equivalent to the existence of a vector $\widetilde{\mathbf{q}}_1 \geq 0$ and numbers $M, a, b \geq 0$ such that the following system of linear equalities has a solution:

$$\begin{bmatrix} \mathbf{p}_1^T & -1 & 0 & 0 \\ \mathbf{p}_3^T & -1 & 1 & 0 \\ \mathbf{p}_2^T & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{q}}_1 \\ M \\ a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that in the last equation we use that the inequality was strict. As such, the right side should contain a strictly positive number, which we can assume to be 1 (since rescaling is always possible). Also, the last restriction requires $\widetilde{\mathbf{q}}_1 \neq \mathbf{0}$, so the first restriction automatically guarantees that $M > 0$.

To show that this last system has a solution, we make use of Farkas' lemma. Therefore, we need to show that there do not exist numbers α, β, and γ that solve the set of inequality constraints (i.e., the dual system)

$$\begin{aligned} \alpha\mathbf{p}_1 + \beta\mathbf{p}_3 + \gamma\mathbf{p}_2 &\geq 0, \\ \alpha + \beta + \gamma &\leq 0, \\ \beta &\geq 0, \\ \gamma &< 0. \end{aligned}$$

Let us first assume that $\alpha > 0$ (and thus $\alpha + \beta > 0$). Then we have

$$\frac{\alpha}{\alpha+\beta}\mathbf{p}_1 + \frac{\beta}{\alpha+\beta}\mathbf{p}_3 \geq \frac{-\gamma}{\alpha+\beta}\mathbf{p}_2 \geq \mathbf{p}_2,$$

where the second inequality follows from the fact that $\frac{-\gamma}{\alpha+\beta} \geq 1$. This implies that the triangular condition is satisfied, which is a contradiction.

As such, we may assume that $\alpha \leq 0$ (and thus $-\alpha - \gamma > 0$). Note that since the three price vectors are strictly positive, we must have that β is also strictly positive. Then we have

$$\mathbf{p}_3 \geq \frac{\beta}{-\alpha-\gamma}\mathbf{p}_3 \geq \frac{-\gamma}{-\alpha-\gamma}\mathbf{p}_2 + \frac{-\alpha}{-\alpha-\gamma}\mathbf{p}_1,$$

where the first inequality follows from the fact that $\beta/(-\alpha - \gamma) \leq 1$. Once more this implies that the triangular condition is satisfied.

We conclude that the dual system has no solution and, thus, Farkas' lemma states that the original system does have a solution. That is, there exists a vector $\widetilde{\mathbf{q}}_1 \in \mathbb{R}^m_+$ and a number $M > 0$ such that

$$\begin{aligned} \mathbf{p}_1\widetilde{\mathbf{q}}_1 &= M, \\ \mathbf{p}_3\widetilde{\mathbf{q}}_1 &\leq M, \\ \mathbf{p}_2\widetilde{\mathbf{q}}_1 &> M. \end{aligned}$$

Given that $M > 0$, we can divide both sides by M and define $\mathbf{q}_1 = \widetilde{\mathbf{q}}_1/M$ to obtain

$$\mathbf{p}_1\mathbf{q}_1 = 1,$$
$$\mathbf{p}_3\mathbf{q}_1 \le 1,$$
$$\mathbf{p}_2\mathbf{q}_1 > 1.$$

By simply exchanging the indices, we can repeat the above reasoning to show the existence of $\mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}^m_+$ satisfying

$$\begin{aligned} &\mathbf{p}_1\mathbf{q}_1 = 1, \qquad &&\mathbf{p}_2\mathbf{q}_2 = 1, \qquad &&\mathbf{p}_3\mathbf{q}_3 = 1, \\ &1 \ge \mathbf{p}_1\mathbf{q}_2, &&1 \ge \mathbf{p}_2\mathbf{q}_3, &&1 \ge \mathbf{p}_3\mathbf{q}_1, \\ &1 < \mathbf{p}_1\mathbf{q}_3, &&1 < \mathbf{p}_2\mathbf{q}_1, &&1 < \mathbf{p}_3\mathbf{q}_2. \end{aligned}$$

Thus, we obtain three distinct vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ for which

$$\mathbf{q}_1 \, R \, \mathbf{q}_2 \, R \, \mathbf{q}_3 \, R \, \mathbf{q}_1,$$

which gives a SARP violation. Moreover, the last row of inequalities shows that there are no WARP violations. □

A.2 *Proof of Proposition 3*

Necessity. Assume that P satisfies the triangular condition. We need to show the existence of two vectors $\mathbf{r}_1, \mathbf{r}_2 \in P$ such that, for all $\mathbf{p} \in P$,

$$\mathbf{p} = \alpha\mathbf{r}_1 + \beta\mathbf{r}_2,$$

where $\alpha, \beta \ge 0$ are not both 0.

Take any $\mathbf{p} \in P$. Since $\mathbf{p} \in \mathbb{R}^m_{++}$, we have that $\gamma = 1/(\sum_i (\mathbf{p})_i) > 0$ and we can define $\widetilde{\mathbf{p}} \equiv \gamma\mathbf{p} \in \Delta \cap P$. If $P \cap \Delta$ is a singleton, say $\mathbf{r}_1$, then we have that $\gamma\mathbf{p} = \mathbf{r}_1$, which obtains the desired result. If $P \cap \Delta$ is not a singleton, then it contains at least two vectors, say $\mathbf{p}_1$ and $\mathbf{p}_2$, and there exists a $j \le m$ such that the vectors are not equal in the jth component (i.e., $(\mathbf{p}_1)_j \neq (\mathbf{p}_2)_j$). Let

$$\mathbf{r}_1 \in \arg\min_{\mathbf{p}\in\Delta\cap P} (\mathbf{p})_j \quad \text{and} \quad \mathbf{r}_2 \in \arg\max_{\mathbf{p}\in\Delta\cap P} (\mathbf{p})_j,$$

where $\mathbf{r}_1$ is the vector in $\Delta \cap P$ whose component $(\mathbf{r})_j$ is minimal. Likewise, $\mathbf{r}_2$ is the vector in $\Delta \cap P$ whose component $(\mathbf{r}_2)_j$ is maximal.

The compactness of $\Delta \cap P$ (i.e., Δ is bounded and $\Delta \cap P$ is closed by assumption) assures that $\mathbf{r}_1$ and $\mathbf{r}_2$ are well defined. Furthermore, by definition we have

$$(\mathbf{r}_1)_j \le (\widetilde{\mathbf{p}})_j \le (\mathbf{r}_2)_j \quad \text{and} \quad (\mathbf{r}_1)_j < (\mathbf{r}_2)_j.$$

Since $\tilde{\mathbf{p}}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ belong to P, we know that the triangular condition holds. Moreover, the inequality is actually an equality since $\tilde{\mathbf{p}}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ belong to the simplex Δ.

Indeed, suppose that there exists $\lambda \in [0,1] : \mathbf{p} \leq \lambda \mathbf{r}_1 + (1-\lambda)\mathbf{r}_2$. If this inequality were strict, then we obtain the contradiction

$$1 = \sum_{i=1}^{m} (\widetilde{\mathbf{p}})_i < \lambda \sum_{i=1}^{m} (\mathbf{r}_1)_i + (1-\lambda) \sum_{i=1}^{m} (\mathbf{r}_2)_i = 1.$$

Obviously, a similar reasoning holds for the other inequalities captured by the triangular condition.

This shows that the triangular condition implies that there exists a $\lambda \in [0,1]$ such that one of the following three conditions holds:

$$\widetilde{\mathbf{p}} = \lambda \mathbf{r}_1 + (1-\lambda)\mathbf{r}_2,$$

$$\mathbf{r}_1 = \lambda \widetilde{\mathbf{p}} + (1-\lambda)\mathbf{r}_2,$$

$$\mathbf{r}_2 = \lambda \widetilde{\mathbf{p}} + (1-\lambda)\mathbf{r}_1.$$

Note that if $\lambda = 0$ or $\lambda = 1$, these conditions imply that either $\tilde{\mathbf{p}} = \mathbf{r}_1$, $\tilde{\mathbf{p}} = \mathbf{r}_2$, or $\mathbf{r}_1 = \mathbf{r}_2$. The latter contradicts the definition of $\mathbf{r}_1$ and $\mathbf{r}_2$, while in the first two cases, we obtain what we needed to prove.

Let us then show that the last two conditions can never hold if $0 < \lambda < 1$. Assume that $\mathbf{r}_1 = \lambda \widetilde{\mathbf{p}} + (1-\lambda)\mathbf{r}_2$ holds. Then $(\mathbf{r}_1)_j \leq (\widetilde{\mathbf{p}})_j$ implies

$$\lambda(\widetilde{\mathbf{p}})_j + (1-\lambda)(\mathbf{r}_2)_j \leq (\widetilde{\mathbf{p}})_j.$$

This implies that $(\mathbf{r}_2)_j \leq (\widetilde{\mathbf{p}})_j \leq (\mathbf{r}_2)_j$ or, equivalently, $(\widetilde{\mathbf{p}})_j = (\mathbf{r}_2)_j$. Then

$$(\mathbf{r}_1)_j = \lambda(\widetilde{\mathbf{p}})_j + (1-\lambda)(\mathbf{r}_2)_j = (\mathbf{r}_2)_j.$$

This contradicts the definition of $\mathbf{r}_1$ and $\mathbf{r}_2$. A similar reasoning holds for the last condition.

We conclude that $\widetilde{\mathbf{p}} = \lambda \mathbf{r}_1 + (1-\lambda)\mathbf{r}_2$ and, thus,

$$\mathbf{p} = \frac{\lambda}{\gamma}\mathbf{r}_1 + \frac{1-\lambda}{\gamma}\mathbf{r}_2.$$

Both coefficients λ/γ and $(1-\lambda)/\gamma$ are positive and at least one is different from 0.

Sufficiency. Take any three vector $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ and assume that

$$\mathbf{p}_1 = \alpha_1 \mathbf{r}_1 + \beta_1 \mathbf{r}_2,$$

$$\mathbf{p}_2 = \alpha_2 \mathbf{r}_1 + \beta_2 \mathbf{r}_2,$$

$$\mathbf{p}_3 = \alpha_3 \mathbf{r}_1 + \beta_3 \mathbf{r}_2.$$

We need to show that the triangular condition is satisfied. Assume that $(\alpha_i, \beta_i > 0,\ i = 1, 2, 3)$. If one or more of these coefficients is 0, the reasoning is similar but the equations have to be somewhat adjusted. From the first two equations it follows that

$$\alpha_2\mathbf{p}_1 - \alpha_1\mathbf{p}_2 = (\beta_1\alpha_2 - \beta_2\alpha_1)\mathbf{r}_2,$$
$$\beta_2\mathbf{p}_1 - \beta_1\mathbf{p}_2 = (\beta_2\alpha_1 - \beta_1\alpha_2)\mathbf{r}_1.$$

If $\beta_1\alpha_2 = \beta_2\alpha_1$, then $\mathbf{p}_1$ is proportional to $\mathbf{p}_2$ and, thus, the triangular condition is satisfied. Otherwise, we obtain

$$\frac{\alpha_2\mathbf{p}_1 - \alpha_1\mathbf{p}_2}{\beta_1\alpha_2 - \beta_2\alpha_1} = \mathbf{r}_2$$

and, similarly,

$$\frac{\beta_1\mathbf{p}_2 - \beta_2\mathbf{p}_1}{\beta_1\alpha_2 - \beta_2\alpha_1} = \mathbf{r}_1.$$

Substituting this last equation into the third equation above gives

$$\mathbf{p}_3 = \alpha_3\left(\frac{\beta_1\mathbf{p}_2 - \beta_2\mathbf{p}_1}{\beta_1\alpha_2 - \beta_2\alpha_1}\right) + \beta_3\left(\frac{\alpha_2\mathbf{p}_1 - \alpha_1\mathbf{p}_2}{\beta_1\alpha_2 - \beta_2\alpha_1}\right)$$
$$\Leftrightarrow \quad (\beta_1\alpha_2 - \beta_2\alpha_1)\mathbf{p}_3 = (\alpha_3\beta_1 - \beta_3\alpha_1)\mathbf{p}_2 + (\beta_3\alpha_2 - \alpha_3\beta_2)\mathbf{p}_1.$$

Using $\alpha_i, \beta_i > 0$ $(i = 1, 2, 3)$, we can exclude that the inequalities $\beta_2\alpha_1 > \beta_1\alpha_2$, $\beta_1\alpha_3 > \beta_3\alpha_1$, and $\beta_3\alpha_2 > \beta_2\alpha_3$ hold simultaneously (multiplying all left hand sides and right hand sides together gives, $1 > 1$), and that the inequalities $\beta_2\alpha_1 < \beta_1\alpha_2$, $\beta_1\alpha_3 < \beta_3\alpha_1$, and $\beta_3\alpha_2 < \beta_2\alpha_3$ hold simultaneously. Therefore, we can always rearrange the above equality such that all the coefficients are positive. Therefore, without loss of generality, we can assume that there exist $\gamma_1, \gamma_2, \gamma_3 \geq 0$ (and two of the three distinct from 0) such that

$$\gamma_3\mathbf{p}_3 + \gamma_2\mathbf{p}_2 = \gamma_1\mathbf{p}_1.$$

If we divide by $(\gamma_3 + \gamma_2)$, we get

$$\frac{\gamma_3}{\gamma_3 + \gamma_2}\mathbf{p}_3 + \frac{\gamma_2}{\gamma_3 + \gamma_2}\mathbf{p}_2 = \frac{\gamma_1}{\gamma_3 + \gamma_2}\mathbf{p}_1.$$

If $\gamma_1/(\gamma_3 + \gamma_2) \geq 1$, then $\mathbf{p}_1$ is smaller than some convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$. Otherwise, $\mathbf{p}_1$ is larger than some convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$. □

A.3 *Proof of Proposition 5*

The following proof is similar to the proof of Proposition 2, but more care should be taken with strict versus weak inequalities and with the exact position of the strict revealed preference relation in the GARP cycle.

Sufficiency. Consider a set of normalized price vectors $P = \{\mathbf{p}_t\}_{t=1,\ldots,n}$ that is a triangular configuration. Assume, to the contrary, that P is not WGARP-reducible. This means that there exists a set of quantity vectors $\{\mathbf{q}_t\}_{t=1,\dots,n} \in \mathcal{Q}$, with $\mathbf{q}_t \gg \mathbf{0}$ for all t, and a subset of $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t=1,\dots,n}$ such that GARP is violated but WGARP is satisfied.

Let $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t\in J}$ ($J \subseteq \{1, \dots, n\}$) be such a subset that is minimal with respect to set inclusion (such a set exists by the fact that n is finite). Although J is not necessarily unique, this minimality property implies that (i) $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t\in J}$ cannot have a smaller subset that also violates GARP, (ii) all elements in $\{(\mathbf{p}_t, \mathbf{q}_t)\}_{t\in J}$ must be involved in the GARP cycle, and (iii) all vectors in $\{\mathbf{q}_j\}_{j\in J}$ are distinct. These three features are crucial to establish the contradiction. In what follows, we use that feature (ii) implies, for the set J, that the shortest GARP cycle has (minimal) length $|J|$.[15]

Without loss of generality, let us re-index the observations in J such that the GARP violation is given by the cycle $\mathbf{q}_1 \, P_R \, \mathbf{q}_2 \, R \, \mathbf{q}_3 \dots R \, \mathbf{q}_{|J|} \, R \, \mathbf{q}_1$, i.e.,

$$
\begin{aligned}
&1 > \mathbf{p}_1\mathbf{q}_2, \\
&1 \geq \mathbf{p}_2\mathbf{q}_3, \\
&\ \vdots \\
&1 \geq \mathbf{p}_{|J|-1}\mathbf{q}_{|J|}, \\
&1 \geq \mathbf{p}_{|J|}\mathbf{q}_1.
\end{aligned}
$$

For a number $j \geq 1$, let us again denote $\lfloor j \rfloor$ for $(j \mod |J|)$.

Now consider all three element subsets $\{\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}, \mathbf{p}_{\lfloor j+2 \rfloor}\}$ for $j \leq |J|$. Given that P is a triangular configuration, we have that, for all j, there is a $\lambda \in [0, 1]$ such that one of the following inequalities holds:

$$\mathbf{p}_{\lfloor j+1 \rfloor} \leq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{7}$$

$$\mathbf{p}_j \leq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{8}$$

$$\mathbf{p}_{\lfloor j+2 \rfloor} \leq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1 \rfloor}, \tag{9}$$

$$\mathbf{p}_j \geq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{10}$$

$$\mathbf{p}_{\lfloor j+1 \rfloor} \geq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{11}$$

$$\mathbf{p}_{\lfloor j+2 \rfloor} \geq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1 \rfloor}. \tag{12}$$

Below, we give six lemmata. In Lemma A3.1, we show that (7) cannot hold for $\lambda \in \{0, 1\}$. In Lemmata A3.2–A3.6, we show that (8)–(12) cannot hold. From these results, it follows that, for all j, there must exist a number $\lambda_j \in (0, 1)$ such that

$$\mathbf{p}_{\lfloor j+1 \rfloor} \leq \lambda_j \mathbf{p}_j + (1-\lambda_j)\mathbf{p}_{\lfloor j+2 \rfloor}.$$

In the proof of Proposition 2 we already showed that this system of inequalities has a solution only if all prices $\mathbf{p}_j$, $j = 1, \dots |J|$, are identical. However, this shows that the

[15]We remark that, for $|J| = 3$, a GARP cycle with length smaller than $|J|$ actually implies a WGARP violation. For compactness, we do not consider this case separately in what follows. But it is easily verified that it is implicitly included in our further argument.

inequalities also hold for $\lambda_1 = \cdots = \lambda_{|J|} = 1$, which gives the desired contradiction and demonstrates the sufficiency part of Proposition 5.

LEMMA A3.1. *If (7) holds, then* $\lambda \in (0, 1)$.

PROOF. We only consider the case $\lambda = 1$, since the proof for $\lambda = 0$ is readily analogous. Assume, by contradiction, that $\lambda = 1$. Then

$$\mathbf{p}_j \geq \mathbf{p}_{\lfloor j+1 \rfloor}.$$

There are three cases to consider. Either $\mathbf{p}_1 = \mathbf{p}_j$ (i.e., $1 = j$), $\mathbf{p}_1 = \mathbf{p}_{\lfloor j+1 \rfloor}$ (i.e., $1 = \lfloor j+1 \rfloor$), or $\mathbf{p}_1 \notin \{\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}\}$ (i.e., $1 \neq j$, $\lfloor j+1 \rfloor$).

Case 1: $1 = j$. Then we have

$$\mathbf{p}_1 \geq \mathbf{p}_2.$$

This gives $1 > \mathbf{p}_1\mathbf{q}_2 \geq \mathbf{p}_2\mathbf{q}_2 = 1$, a contradiction.

Case 2: $1 = \lfloor j+1 \rfloor$. Then we have

$$\mathbf{p}_{|J|} \geq \mathbf{p}_1.$$

If $\mathbf{p}_{|J|} = \mathbf{p}_1$, then $\mathbf{p}_{|J|}\mathbf{q}_2 = \mathbf{p}_1\mathbf{q}_2 < 1$, so we obtain $\mathbf{q}_{|J|}\ P_R\ \mathbf{q}_2$. This gives us a shorter GARP cycle (with length smaller than $|J|$), which contradicts minimality of set J. If $\mathbf{p}_{|J|} > \mathbf{p}_1$, then $1 \geq \mathbf{p}_{|J|}\mathbf{q}_1 > \mathbf{p}_1\mathbf{q}_1 = 1$, a contradiction.

Case 3: $1 \neq j$, $\lfloor j+1 \rfloor$. Then

$$\mathbf{p}_j \geq \mathbf{p}_{\lfloor j+1 \rfloor}.$$

If $\mathbf{p}_j = \mathbf{p}_{\lfloor j+1 \rfloor}$, then $\mathbf{p}_j\mathbf{q}_{\lfloor j+2 \rfloor} = \mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+2 \rfloor} \leq 1$. Thus, $\mathbf{q}_j\ R\ \mathbf{q}_{\lfloor j+2 \rfloor}$, which shows that there exists a shorter GARP cycle (with length smaller than $|J|$). We conclude that $\mathbf{p}_j > \mathbf{p}_{\lfloor j+1 \rfloor}$, but then $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+1 \rfloor} > \mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor} = 1$, a contradiction. ◁

LEMMA A3.2. *Condition (8) does not hold.*

PROOF. Assume that

$$\mathbf{p}_j \leq \lambda\mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}.$$

Note that if $\lambda = 0$, we obtain

$$\mathbf{p}_{\lfloor j+2 \rfloor} \geq \mathbf{p}_j.$$

There are four cases to consider. Either $\mathbf{p}_1 = \mathbf{p}_j$ (i.e., $1 = j$), $\mathbf{p}_1 = \mathbf{p}_{\lfloor j+1 \rfloor}$ (i.e., $1 = \lfloor j+1 \rfloor$), $\mathbf{p}_1 = \mathbf{p}_{\lfloor j+2 \rfloor}$ (i.e., $1 = \lfloor j+2 \rfloor$), or $\mathbf{p}_1 \notin \{\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}, \mathbf{p}_{\lfloor j+2 \rfloor}\}$ (i.e., $1 \neq$, j, $\lfloor j+1 \rfloor$, $\lfloor j+2 \rfloor$).

Case 1: $1 = j$. Then

$$\mathbf{p}_3 \geq \mathbf{p}_1.$$

If $\mathbf{p}_3 = \mathbf{p}_1$, then $\mathbf{p}_3\mathbf{q}_2 = \mathbf{p}_1\mathbf{q}_2 < 1$, which means that we get a WGARP violation $\mathbf{q}_2 \; R \; \mathbf{q}_3 \; P_R \; \mathbf{q}_2$. If $\mathbf{p}_3 > \mathbf{p}_1$, then $1 = \mathbf{p}_3\mathbf{q}_3 > \mathbf{p}_1\mathbf{q}_3$. So $\mathbf{q}_1 \; P_R \; \mathbf{q}_3$ and we obtain a shorter GARP cycle (with length smaller than $|J|$).

Case 2: $1 = \lfloor j+1 \rfloor$. Then

$$\mathbf{p}_2 \geq \mathbf{p}_{|J|}.$$

If $\mathbf{p}_2 = \mathbf{p}_{|J|}$, then $\mathbf{p}_2\mathbf{q}_1 = \mathbf{p}_{|J|}\mathbf{q}_1 \leq 1$, so we obtain the WGARP violation $\mathbf{q}_1 \; P_R \; \mathbf{q}_2 \; R \; \mathbf{q}_1$. If $\mathbf{p}_2 > \mathbf{p}_{|J|}$, then $1 = \mathbf{p}_2\mathbf{q}_2 > \mathbf{p}_{|J|}\mathbf{q}_2$, so we obtain that $\mathbf{q}_{|J|} \; P_R \; \mathbf{q}_2$, which gives a shorter GARP cycle (with length smaller than $|J|$).

Case 3: $1 = \lfloor j+2 \rfloor$. Then

$$\mathbf{p}_1 \geq \mathbf{p}_{|J|-1}.$$

This implies $1 = \mathbf{p}_1\mathbf{q}_1 \geq \mathbf{p}_{|J|-1}\mathbf{q}_1$, so $\mathbf{q}_{|J|-1} \; R \; \mathbf{q}_1$ and we obtain a shorter GARP cycle (with length smaller than $|J|$).

Case 4: $1 \neq j, \lfloor j+1 \rfloor, \lfloor j+2 \rfloor$. Then

$$1 \geq \mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+2 \lfloor} \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+2 \rfloor}.$$

This implies that we get a shorter GARP cycle (with length smaller than $|J|$), since $\mathbf{q}_j \; R \; \mathbf{q}_{\lfloor j+2 \rfloor}$.

If $\lambda = 1$, we obtain that $\mathbf{p}_{\lfloor j+1 \rfloor} \geq \mathbf{p}_j$, and this complies with the case $\lambda = 0$ of Lemma A3.1.

Thus, we conclude that there must be a $\lambda \in (0, 1)$ such that

$$\mathbf{p}_j \leq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}.$$

Again, we need to consider four cases: $\mathbf{p}_1 = \mathbf{p}_j$ (i.e., $1 = j$), $\mathbf{p}_1 = \mathbf{p}_{\lfloor j+1 \rfloor}$ (i.e., $1 = \lfloor j+1 \rfloor$), $\mathbf{p}_1 = \mathbf{p}_{\lfloor j+2 \rfloor}$ (i.e., $1 = \lfloor j+2 \rfloor$), and $\mathbf{p}_1 \notin \{\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}, \mathbf{p}_{\lfloor j+2 \rfloor}\}$ (i.e., $1 \neq, j, \lfloor j+1 \rfloor, \lfloor j+2 \rfloor$).

Case 1: $1 = j$. This gives

$$\mathbf{p}_1 \leq \lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3.$$

If $\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3 = \mathbf{p}_1$, then $\lambda + (1-\lambda)\mathbf{p}_3\mathbf{q}_2 = \mathbf{p}_1\mathbf{q}_2 < 1$. This shows that $1 > \mathbf{p}_3\mathbf{q}_2$, so we get the WGARP violation $\mathbf{q}_2 \; R \; \mathbf{q}_3 \; P_R \; \mathbf{q}_2$. If $\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3 > \mathbf{p}_1$, then

$$1 \geq \lambda \mathbf{p}_2\mathbf{q}_3 + (1-\lambda) > \mathbf{p}_1\mathbf{q}_3,$$

where the first inequality follows from the fact that the rights hand side is a convex combination of two numbers smaller than or equal to 1. This shows that $\mathbf{q}_1 \; P_R \; \mathbf{q}_3$, so that we obtain a shorter GARP cycle (with length smaller than $|J|$).

Case 2: $1 = \lfloor j+1 \rfloor$. This gives

$$\mathbf{p}_{|J|} \leq \lambda \mathbf{p}_1 + (1-\lambda)\mathbf{p}_2.$$

Then, as before, we obtain

$$1 > \lambda \mathbf{p}_1 \mathbf{q}_2 + (1-\lambda) \geq \mathbf{p}_{|J|}\mathbf{q}_2,$$

which shows that $\mathbf{q}_{|J|} \; P_R \; \mathbf{q}_2$. This gives a shorter GARP cycle (with length smaller than $|J|$).

Case 3: $1 = \lfloor j+2 \rfloor$. This gives

$$\mathbf{p}_{|J|-1} \leq \lambda \mathbf{p}_{|J|} + (1-\lambda)\mathbf{p}_1.$$

Then

$$1 \geq \lambda \mathbf{p}_{|J|}\mathbf{q}_1 + (1-\lambda) \geq \mathbf{p}_{|J|-1}\mathbf{q}_1,$$

which shows that $\mathbf{q}_{|J|-1} \; R \; \mathbf{q}_1$. So we obtain a shorter GARP cycle (with length smaller than $|J|$).

Case 4: $1 \neq j, \lfloor j+1 \rfloor, \lfloor j+2 \rfloor$. Then

$$1 \geq \lambda \mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+2 \rfloor} + (1-\lambda) \geq \mathbf{p}_j \mathbf{q}_{\lfloor j+2 \rfloor},$$

which means that $\mathbf{q}_j \; R \; \mathbf{q}_{\lfloor j+2 \rfloor}$. So we again obtain a shorter GARP cycle (with length smaller than $|J|$). $\lhd$

Lemma A3.3. *Condition (9) cannot hold.*

We omit this proof since it is readily analogous to the proof of Lemma A3.2.

Lemma A3.4. *Condition (10) cannot hold.*

Proof. First note that we only need to consider the strict inequality, since Lemma A3.2 shows that the equality cannot hold. Toward a contradiction, assume that

$$\mathbf{p}_j > \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}.$$

If $\lambda = 1$, then $\mathbf{p}_j > \mathbf{p}_{\lfloor j+1 \rfloor}$. This was shown to lead to a violation in the proof of Lemma A3.1. If $\lambda = 0$, then $\mathbf{p}_j > \mathbf{p}_{\lfloor j+2 \rfloor}$. This was shown to lead to a violation in the proof of Lemma A3.3. As such, the above inequality should hold with $\lambda \in (0,1)$. Then

$$1 \geq \mathbf{p}_j \mathbf{q}_{\lfloor j+1 \rfloor} > \lambda + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor}.$$

This implies $1 > \mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor}$ and gives the WGARP violation $\mathbf{q}_{\lfloor j+1 \rfloor} \; R \; \mathbf{q}_{\lfloor j+2 \rfloor} \; P_R \; \mathbf{q}_{\lfloor j+1 \rfloor}$. $\lhd$

LEMMA A3.5. *Condition (11) does not hold.*

We omit this proof since it is readily analogous to the proof of Lemma A3.4.

LEMMA A3.6. *Condition (12) does not hold.*

We omit this proof since it is readily analogous to the proof of Lemma A3.4.

Necessity. Let us consider a set of prices P that is not a triangular configuration. In particular, let $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \subseteq P$ be a set of three distinct price vectors such that none of the vector inequalities for the triangular configuration is satisfied.

Our aim is to show the existence of vectors $\{\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}\} \in \mathcal{Q}(\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\})$, with $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \gg \mathbf{0}$, such that $\{(\mathbf{p}_1, \mathbf{q}_1), (\mathbf{p}_2, \mathbf{q}_2), (\mathbf{p}_3, \mathbf{q}_3)\}$ violates GARP but not WGARP.

To obtain the result, let us first show that there exists a vector $\widetilde{\mathbf{q}}_1 \in \mathbb{R}^m_{++}$ and a number $M > 0$ such that the following system of inequalities has a solution:

$$\begin{aligned}
\mathbf{p}_1\widetilde{\mathbf{q}}_1 &= M, \\
\mathbf{p}_3\widetilde{\mathbf{q}}_1 &< M, \\
\mathbf{p}_2\widetilde{\mathbf{q}}_1 &> M.
\end{aligned}$$

By rescaling, the feasibility of this system is equivalent to the existence of a vector $\widetilde{\mathbf{q}}_1 \geq 0$, a vector $\mathbf{w} \geq 0$, and numbers $M, a, b \geq 0$ such that the system

$$\begin{bmatrix} \mathbf{p}_1^T & -1 & 0 & 0 & \mathbf{0} \\ \mathbf{p}_3^T & -1 & 1 & 0 & \mathbf{0} \\ \mathbf{p}_2^T & -1 & 0 & -1 & \mathbf{0} \\ I & 0 & 0 & 0 & -I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{q}}_1 \\ M \\ a \\ b \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ \mathbf{1} \end{bmatrix}$$

has a solution, where I is the m-dimensional unit matrix and $\mathbf{1}$ is the m-dimensional vector of 1s.

We prove feasibility of the system by contradiction. If the system is not feasible, then by Farkas' lemma (theorem of the alternative) there must exist numbers α, γ, β, and μ_i ($i \leq m$) such that the following set of inequalities is feasible:

$$\begin{aligned}
\alpha\mathbf{p}_1 + \gamma\mathbf{p}_3 + \beta\mathbf{p}_2 + I\boldsymbol{\mu} &\geq 0, \\
\alpha + \gamma + \beta &\leq 0, \\
\gamma &\geq 0, \\
\beta &\leq 0, \\
\mu_i &\leq 0, \\
-\gamma + \beta + \sum_i \mu_i &< 0.
\end{aligned}$$

Let us first show that either γ or α is strictly positive. If not, then $\gamma = 0$, so the last condition tells us that $\beta + \sum_i \mu_i < 0$. If $\beta < 0$, then if we add up the first condition over all goods m and set $\gamma = 0$, we get

$$\alpha \sum_i (\mathbf{p}_1)_i \geq -\beta \sum_i (\mathbf{p}_2)_i - \sum_i \mu_i \geq -\beta \sum_i (\mathbf{p}_2)_i > 0,$$

which shows that $\alpha > 0$, a contradiction. If $\beta = 0$, then $\sum_i \mu_i < 0$ and we get

$$\alpha \sum_i (\mathbf{p}_1)_i \geq -\sum_i \mu_i > 0,$$

which gives again the contradiction $\alpha > 0$.

Next, observe that the first and fifth conditions together imply

$$\alpha \mathbf{p}_1 + \gamma \mathbf{p}_3 + \beta \mathbf{p}_2 \geq 0.$$

Given that either $\alpha > 0$ or $\gamma > 0$, we can distinguish three cases.

Case 1: $\gamma > 0$, $\alpha \geq 0$, $\beta \leq 0$. Then

$$\frac{\alpha}{\alpha+\gamma}\mathbf{p}_1 + \frac{\gamma}{\alpha+\gamma}\mathbf{p}_3 \geq \frac{-\beta}{\alpha+\gamma}\mathbf{p}_2 \geq \mathbf{p}_2,$$

which shows that the triangular condition holds. This is a contradiction.

Case 2: $\gamma > 0$, $\alpha < 0$, $\beta \leq 0$. Then

$$\mathbf{p}_3 \geq \frac{\gamma}{-\alpha-\beta}\mathbf{p}_3 \geq \frac{-\alpha}{-\alpha-\beta}\mathbf{p}_1 + \frac{-\beta}{-\alpha-\beta}\mathbf{p}_2,$$

which again shows that the triangular condition holds.

Case 3: $\gamma = 0$, $\alpha > 0$, $\beta \leq 0$. Then

$$\mathbf{p}_1 \geq \frac{-\beta}{\alpha}\mathbf{p}_2 \geq \mathbf{p}_2.$$

Once again, the triangular condition is satisfied.

In all cases, we conclude that the triangular condition should be satisfied. We can therefore conclude that the dual system has no solution, which means that the original system does have a solution, i.e., there is a vector $\tilde{\mathbf{q}}_1 \gg \mathbf{0}$ and a number $M > 0$ such that

$$\mathbf{p}_1\tilde{\mathbf{q}}_1 = M,$$

$$\mathbf{p}_3\tilde{\mathbf{q}}_1 < M,$$

$$\mathbf{p}_2\tilde{\mathbf{q}}_1 > M.$$

Given that $M > 0$, we can divide both sides by M and define $\mathbf{q}_1 = \tilde{\mathbf{q}}_1/M \gg 0$, which obtains

$$\begin{aligned}\mathbf{p}_1\mathbf{q}_1 &= 1,\\ \mathbf{p}_3\mathbf{q}_1 &< 1,\\ \mathbf{p}_2\mathbf{q}_1 &> 1.\end{aligned}$$

By simply exchanging the indices, we can repeat the above reasoning to show the existence of $\mathbf{q}_2, \mathbf{q}_3 \in \mathbb{R}^m_{++}$ that satisfy

$$\begin{array}{ccc}\mathbf{p}_1\mathbf{q}_1 = 1, & \mathbf{p}_2\mathbf{q}_2 = 1, & \mathbf{p}_3\mathbf{q}_3 = 1,\\ 1 > \mathbf{p}_1\mathbf{q}_2, & 1 > \mathbf{p}_2\mathbf{q}_3, & 1 > \mathbf{p}_3\mathbf{q}_1,\\ 1 < \mathbf{p}_1\mathbf{q}_3, & 1 < \mathbf{p}_2\mathbf{q}_1, & 1 < \mathbf{p}_3\mathbf{q}_2.\end{array}$$

Thus, we obtain three distinct vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ for which

$$\mathbf{q}_1 \; P_R \; \mathbf{q}_2 \; P_R \; \mathbf{q}_3 \; P_R \; \mathbf{q}_1.$$

That is, $\{(\mathbf{p}_1, \mathbf{q}_1), (\mathbf{p}_2, \mathbf{q}_2), (\mathbf{p}_3, \mathbf{q}_3)\}$ violates GARP. Moreover, the last row of inequalities shows that there are no WGARP violations. □

Appendix B: Bounding the length of SARP cycles

Our following example obtains the general conclusion that, for a set of n price vectors that is WARP-reducible, it is possible to add a $(n+1)$th price vector that obtains a SARP cycle of any length. As a preliminary note, we remark that our set of prices and quantities may seem somewhat artificial. We emphasize that this is mainly for mathematical convenience and not crucial for the core of our argument. It is possible to perturb the prices and quantities to make them “more realistic.” However, this would complicate the computations and, more importantly, it would make the argument substantially less transparent.

Step 1. We start by defining a set of n normalized price vectors $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ that is a triangular consideration and, therefore, WARP-reducible. In particular, we consider the set of normalized price vectors

$$\begin{aligned}\mathbf{p}_1 &= \left(\frac{1}{3}, \frac{2}{3}, 1\right),\\ \mathbf{p}_2 &= \left(\frac{1}{5}, \frac{6}{5}, 1\right),\\ &\vdots\\ \mathbf{p}_t &= \left(\frac{1}{2t+1}, \frac{t(t+1)}{2t+1}, 1\right),\end{aligned}$$

$$\vdots$$

$$\mathbf{p}_n = \left(\frac{1}{2n+1}, \frac{n(n+1)}{2n+1}, 1\right).$$

Since the price of the third good equals unity for all $t = 1, \dots, n$, these prices form a triangular configuration if the triangular inequalities hold for the price vectors restricted to the first two goods. However, from Section 3 we know that in a two-goods setting, the triangular inequality is always satisfied by construction. Therefore, we can conclude that this set of normalized prices is a triangular configuration.

Step 2. For the given set of n normalized price vectors, we next construct a corresponding set of n quantity vectors $(\mathbf{q}_1, \dots, \mathbf{q}_n)$ and, for the resulting data set $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,\dots,n}$, we characterize the revealed preference relations. Specifically, we consider the set of quantities

$$\begin{aligned}
\mathbf{q}_1 &= (1, 1, 0), \\
\mathbf{q}_2 &= (2, 1/2, 0), \\
\mathbf{q}_3 &= (3, 1/3, 0), \\
&\vdots \\
\mathbf{q}_t &= (t, 1/t, 0), \\
&\vdots \\
\mathbf{q}_n &= (n, 1/n, 0).
\end{aligned}$$

Observe that total expenditure in each observation t is equal to unity, as required for our definition of normalized prices. Specifically,

$$\mathbf{p}_t\mathbf{q}_t = \frac{1}{2t+1}t + \frac{t(t+1)}{2t+1}\frac{1}{t} = \frac{2t+1}{2t+1} = 1.$$

Next, for any s and t we have

$$\mathbf{p}_t\mathbf{q}_s = \frac{1}{2t+1}s + \frac{t(t+1)}{2t+1}\frac{1}{s}.$$

From this we can derive that $\mathbf{q}_t\, R\, \mathbf{q}_{t+1}$, while there are no other direct revealed preference relations between any two different t and s (i.e., we have $\mathbf{p}_t\mathbf{q}_s > 1$ for any $s \neq t, t+1$).

To see this last result, we first note that the above product $\mathbf{p}_t\mathbf{q}_s$ is a strictly convex function in s. The minimum of this function is reached when

$$\begin{aligned}
&\frac{1}{2t+1} = \frac{t(t+1)}{2t+1}\frac{1}{s^2} \\
\Leftrightarrow \quad & s^2 = t(t+1) \\
\Rightarrow \quad & s = \sqrt{t(t+1)},
\end{aligned}$$

which gives a number between t and $t+1$. Thus, for integer s, the minimal values are obtained for the values $s=t$ and $s=t+1$. For $s=t$, we simply have $\mathbf{p}_t\mathbf{q}_t=1$, as verified above. For $s=t+1$, we obtain

$$\mathbf{p}_t\mathbf{q}_{t+1}=\frac{1}{2t+1}(t+1)+\frac{t(t+1)}{2t+1}\frac{1}{t+1}=\frac{t+1+t}{2t+1}=1.$$

From this we can conclude $\mathbf{q}_t \; R \; \mathbf{q}_{t+1}$. Moreover, because the convex function $\mathbf{p}_t\mathbf{q}_s$ reaches a minimum at t and $t+1$, we also have that $\mathbf{p}_t\mathbf{q}_s>1$ for any $s\neq t, t+1$. As such we obtain consistency with WARP and SARP.

Step 3. We now add a $(n+1)$th price vector $\mathbf{p}_0$ and a corresponding $(n+1)$th quantity vector $\mathbf{q}_0$ that obtains a SARP cycle of length $n+1$ for the data set $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=0,\dots,n}$, without there being any smaller SARP cycle. In particular, we consider the $(n+1)$th price vector

$$\mathbf{p}_0=(2/3, 1/3, 1/2)$$

and the $(n+1)$th quantity vector

$$\mathbf{q}_0=\left(\frac{6n+3}{8n+1}, 0, \frac{8n-2}{8n+1}\right).$$

We obtain our result in five steps.

(i) As required, $\mathbf{p}_0\mathbf{q}_0=1$:

$$\begin{aligned}\mathbf{p}_0\mathbf{q}_0&=\frac{2}{3}\frac{6n+3}{8n+1}+\frac{1}{2}\frac{8n-2}{8n+1}\\&=\frac{4n+2+4n-1}{8n+1}=1.\end{aligned}$$

(ii) We have a first extra revealed preference relation $\mathbf{q}_n R\mathbf{q}_0$ (i.e., $1\geq\mathbf{p}_n\mathbf{q}_0$):

$$\begin{aligned}\mathbf{p}_n\mathbf{q}_0&=\frac{1}{2n+1}\frac{6n+3}{8n+1}+\frac{8n-2}{8n+1}\\&=\frac{3+8n-2}{8n+1}=1.\end{aligned}$$

(iii) There is also a second extra revealed preference relation $\mathbf{q}_0 \; R \; \mathbf{q}_1$ (i.e., $1\geq\mathbf{p}_0\mathbf{q}_1$):

$$\mathbf{p}_0\mathbf{q}_1=\frac{2}{3}+\frac{1}{3}=1.$$

All this obtains the SARP violation $\mathbf{q}_0R\mathbf{q}_1R\mathbf{q}_2R\dots R\mathbf{q}_nR\mathbf{q}_0$.

(iv) The bundle $\mathbf{q}_0$ is not directly revealed preferred to any bundle $\mathbf{q}_s$, with $s\neq 0, 1$ (i.e., $\mathbf{p}_0\mathbf{q}_s>1$ for all $s=2,\dots,n$):

$$\mathbf{p}_0\mathbf{q}_s=\frac{2}{3}s+\frac{1}{3}\frac{1}{s}.$$

Here the right hand side is strictly increasing in s for all $s>1$ and is equal to unity for $s=1$.

(v) Finally, no quantity bundle $\mathbf{q}_s$, with $s \neq 0, n$, is directly revealed preferred to the bundle $\mathbf{q}_0$ (i.e., $\mathbf{p}_s\mathbf{q}_0 > 1$ for all $s = 1, \ldots, n-1$). We prove this by contradiction. Specifically, assume

$$
\begin{aligned}
1 \geq \mathbf{p}_s\mathbf{q}_0 &= \frac{1}{2s+1}\frac{6n+3}{8n+1} + \frac{8n-2}{8n+1} \\
\Leftrightarrow \quad 1 &\geq \frac{1}{8n+1}\left(\frac{1}{2s+1}(6n+3) + 8n-2\right) \\
\Leftrightarrow \quad 3 &\geq \frac{6n+3}{2s+1} \\
\Leftrightarrow \quad 2s+1 &\geq 2n+1 \\
\Leftrightarrow \quad s &\geq n,
\end{aligned}
$$

which gives a contradiction. These last two steps show that there does not exist a shorter SARP cycle. In particular, the data set $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=0,\ldots,n}$ satisfies WARP but not SARP.

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