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Efficient cooperation by exchanging favors

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We study chip-strategy equilibria in two-player repeated games. Intuitively, in these equilibria, players exchange favors by taking individually suboptimal actions if these actions create a "gain" for the opponent larger than the player's "loss" from taking them. In exchange, the player who provides a favor implicitly obtains from the opponent a chip that entitles the player to receiving a favor at some future date. Players are initially endowed with a number of chips, and a player who runs out of chips is no longer entitled to receive any favors until she provides a favor to the opponent, in which case she receives one chip back.

We show that such simple chip strategies approximate efficient outcomes in a class of repeated symmetric games with incomplete information, in which each player has two possible types, when discounting vanishes. This class includes many important applications, studied in numerous previous papers, such as the favor-exchange model of Möbius (2001), repeated auctions, and the repeated version of Spulber duopolies of Athey and Bagwell (2001), among others. We also show the limitation of chip strategies. For example, if players have more than two types, then such simple chip strategies may not approximate efficient outcomes even in symmetric games.

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JEL classification. C73, D43, D44, D61.

1. Introduction

Favor exchange is a simple principle that is seen in everyday cooperation. Examples abound in various spheres of human relations, such as employees replacing fellow employees in performing some duty or neighbors tolerating each other being occasionally in a bad mood and exhibiting minor offense. This kind of behavior arises in repeated interactions, where favor providers expect reciprocity when the roles happen to be reversed. Usually, it is unlikely to observe long sequences of favors performed in one direction, as favor providers would break cooperation at some point.

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The literature on repeated games or, more generally, dynamic games is very successful in explaining cooperation. Yet this literature emphasizes equilibrium payoffs more than equilibrium behavior. Cooperation is quite often (i) supported by trigger strategies, which penalize any misbehavior by breaking cooperation, or (ii) attained in strategies constructed by self-generation techniques, which are powerful for characterizing payoffs, but typically less useful for characterizing behavior.

In this paper, we formally study cooperation in a simple form of favor exchange, which is often called *chip strategies*. Some version of chip strategies was introduced in the context of a (two-player) favor-exchange model by Möbius (2001). Intuitively, according to these strategies, a player takes an individually suboptimal action if that action creates a "gain" for the opponent that is larger than the player's "loss" from taking it. In exchange, the player implicitly obtains from the opponent a chip that entitles the player to receiving this kind of favor at some future date. Players are initially endowed with a number of chips, and a player who runs out of chips is no longer entitled to receive any favors until she provides a favor to her opponent, in which case the player receives one chip back.

We show that *simple chip strategies*, in which one favor is exchanged for one chip, are capable of approximating efficient outcomes (as the discount factor tends to 1) in a class of games in which players have private information, and this information evolves over time according to a Markov process. We view this contribution as providing a positive model of playing some repeated games with incomplete information.[1] This class of games includes several models studied extensively in the existing literature and has a large array of applications, including (i) the favor-exchange model studied by Möbius (2001) (more precisely, a discrete version of this model studied by Abdulkadiroğlu and Bagwell, Abdulkadiroğlu and Bagwell (2012, 2013), (ii) the repeated version of Spulber (1995) duopolies studied by Athey and Bagwell (2001) and several other authors, and (iii) repeated auctions studied by Skrzypacz and Hopenhayn (2004) and others. We show that in these models, the efficient outcome can be approximated in simple chip strategies, when the discount factor converges to 1; moreover, the proofs of these results are engagingly simple and potentially applicable to other settings, even those not directly covered by our analysis.

Perhaps unsurprisingly, we require strong assumptions on the stage game for the simple chip strategies to be approximately efficient. The stage game must be symmetric, it must be played by two players, and each player's private information must be captured by two possible types. These are essential limitations on the possibility of approximating efficient outcomes in simple chip strategies. We show in Section 6 that if a stage game is asymmetric, one must augment chip strategies with a public randomization device, which allows players to exchange one favor for receiving some chips with some probability, in other words, to introduce divisible chips. This is what we call *random*

[1] In addition, we generalize the existing results obtained in some applications by allowing players' private types to evolve over time according to a Markov process.

chip strategies.[2] Furthermore, in Section 7, we show that if players have more than two types, even these random chip strategies may not approximate efficient outcomes, even in symmetric stage games.

The trade-off between efficiency and simplicity is not surprising. More complicated settings clearly require more elaborate incentive schemes. This suggests another approach to studying chip strategies, parallel to a large volume of recent research in computer science. Namely, simple or random chip strategies may not be efficient in richer settings, but may still capture a sizable fraction of the maximum total surplus. For example, simple chip strategies capture a nontrivial fraction of the maximum total surplus in the setting studied in Section 6, and so do random chip strategies in the setting studied in Section 7. But simple chip strategies cannot attain more surplus than repetitions of stage-game equilibria in the latter case.

In the case of more than two players, an additional difficulty arises if a favor is provided by more than one player and more than one player benefits from this favor. It must be decided then who should issue a chip (or what fraction of it) and who should obtain the chip. This implies that more involved strategies are required to approximate the efficient outcome. In a companion paper, Olszewski and Safronov (forthcoming), we provide a more complicated version of chip strategies, constructed by imitating the Arrow (1979) and d'Aspremont and Gérard-Varet (1979) mechanism. These chip strategies approximate the efficient outcomes for a very large class of two-player games with any number of types, and, under somewhat more restrictive conditions, for a class of games with more than two players.

The rest of the paper is organized as follows. The rest of this section contains a literature review. In Section 2, we present the applications of chip strategies to favor-exchange model of Möbius (2001) and to repeated auctions. The analysis of these applications contains the key arguments behind our efficiency results, putting aside some more involved, but rather technical issues. Section 3 contains the general result for symmetric games in which players have two types, and Section 4 contains a discussion of chip strategies and a discussion of possible extensions of the result. Section 5 provides some additional applications, including to the repeated duopoly model of Athey and Bagwell (2001). Sections 6 and 7 study asymmetric games and games in which players have more than two types. Technically demanding proofs are relegated to Appendices A and B.

1.1 *Related literature*

To date, chip strategies have appeared in the existing literature only in the context of specific applications (which we discuss shortly).

Möbius (2001) analyzed a model of voluntary favor exchange between two players. In his model, favor opportunities arrive according to a Poisson process, and the benefit of receiving a favor exceeds the cost of providing it. In the chip strategies studied by Möbius (which are somewhat different and less efficient than those studied in our paper), cooperation breaks down when a player issues a certain number of chips.

[2]We still require though that the exchange rate between chips and favors does not change with the number of chips.

Hauser and Hopenhayn (2008) suggest two improvements to chip strategies that enhance the efficiency of equilibria: exchanging favors and chips at different rates (i.e., not one to one), and appreciation and depreciation of chips. Solving the model numerically, they demonstrate that for a set of parameter values, the efficiency gains are quite large. As already mentioned earlier, Abdulkadiroğlu and Bagwell, Abdulkadiroğlu and Bagwell (2012, 2013) studied a discrete-time version of Möbius' model. Their 2012 paper is closely related to our paper, because they analyze chip strategies in the same form as we do here. They consider a fixed discount factor, identify the optimal number of chips, and compare this optimal chip mechanism with a more sophisticated favor-exchange relationship in which the size of a favor owed may decline over time. For any given discount factor, the equilibria in chip strategies obviously cannot be fully efficient, because incentive compatibility imposes a limit on the number of chips that can be used. None of these papers shows (explicitly or implicitly) that any kind of chip strategies attain efficient outcomes when the discount factor converges to 1. With no restriction on strategies, the possibility of attaining efficient cooperation when types are independent and identically distributed (i.i.d.) follows from the folk theorem for games with adverse selection established in Fudenberg et al. (1994), and when types are Markov, follows from the folk theorem established in Escobar and Toikka (2013).

The repeated version of Spulber (1995) duopolies (more generally, oligopolies) is an important application of the present results. These models with i.i.d. types were studied in Athey and Bagwell (2001), Athey et al. (2004), Hörner and Jamison (2007), and—with more general Markov types—by Athey and Bagwell (2008) and Escobar and Toikka (2013). Even though some strategies used by Athey and Bagwell resemble our one-chip-strategy profiles,[3] and some elements of chip strategies appear in Hörner and Jamison (2007), it remains the case that the primary focus of these papers is not on chip strategies per se. Compared to the strategies used by these authors (even for i.i.d. types), the chip strategies used in the present paper seem simpler.

Repeated auctions are another important application of our methods. Some papers on repeated auctions (e.g., Aoyagi 2003, 2007 and Rachmilevitch 2013) explore strategies that share some common features with the chip strategies used in this paper and in Olszewski and Safronov (forthcoming). Compared to the strategies used by these authors, the chip strategies used in this paper also seem simpler. In addition, our result on repeated auctions does not follow from the existing results.

In addition, the idea of chip strategies is related, although less directly, to the idea that money can provide "memory" of past trade (see, e.g., Kocherlakota 1998), to the idea that a finite amount of continuous money can be used to support incentive compatibility in a trading relationship (see, e.g., Athey and Miller 2007), and to the idea that tokens can facilitate transmission of information on networks (see Wolitzky 2015).

2. Applications

We begin our exposition with two applications. The analysis of these two application contains the key arguments behind our efficiency results, putting aside some technical issues that appear in the more general results.

[3] See also the strategies used in the discussion of their paper in Mailath and Samuelson (2006).

2.1 *Discrete model of favor exchange*

We first study a two-player repeated game introduced in Abdulkadiroğlu and Bagwell (2012), which in turn is a discrete version of the model of favor exchange in Möbius (2001). In the stage game, either player 1 is given an income of \$1 or player 2 is given an income of \$1, or neither player is given any income. The former two events occur with probability $p \in (0, 1/2)$ each, and the latter event occurs with probability $1 - 2p$. Each player is privately informed as to whether or not she receives income. Thus, if a player does not receive income, then she does not observe whether the opponent received any income. If a player receives income, then the player may transfer the income to the other player. The transferred income is worth $\gamma > 1$ to the receiver, making it value-enhancing.

This game is played repeatedly, states are i.i.d., and the players have a common discount factor δ. The payoffs are normalized by the factor of $1 - \delta$. The players cannot store income, that is, income must be either transferred or consumed in the period it is received. The *efficient (total, ex ante) payoff*, that is, the maximum of the sum of the players' payoffs, is achieved if the income received by any player is transferred to the other player. This payoff is equal to

$$v = 2p\gamma.$$

2.1.1 *Description of efficient chip strategies* Consider the following strategies. At the beginning of each period, each player i holds $k_i \in \{0, \dots, 2n\}$ chips, where $k_1 + k_2 = 2n$. If player i obtains an income of \$1 and $k_i < 2n$, then player i gives the income to player j and j gives i (implicitly) one chip in return. If $k_i = 2n$, i.e., when i already holds all the chips, then i consumes the \$1 herself. No chip is given in this case. At the beginning of period 1, each player has n chips.

We obtain the following result, proved in the next two sections.

Proposition 1. *For every $\lambda > 0$, there exist $\underline{\delta} < 1$ and n such that if the players' discount factor is $\delta > \underline{\delta}$, then the chip-strategy profile with n chips is an equilibrium of the repeated game and the ex ante payoff of each player in this equilibrium exceeds $v/2 - \lambda$.*

2.1.2 *Continuation payoffs* In this section, we analyze the first-order approximation of players' payoffs.[4] Assuming that both players play the prescribed chip strategies, denote the continuation payoff of player 1 with k chips by V_k. By the symmetry of our model, the continuation payoff of player 2 can be examined analogously. These continuation payoffs are computed before the players learn about their income in the current period. For $k \in \{1, \dots, 2n - 1\}$, we have

$$V_k = p\big\{(1-\delta)\gamma + \delta V_{k-1}\big\} + p\delta V_{k+1} + (1-2p)\delta V_k. \tag{1}$$

The first component of the right-hand side corresponds to the payoff contingent on player 2 receiving an income of \$1 in the current period; the remaining two components

[4]As noticed by a referee, we do so by solving explicitly (in approximation) what is called, in the literature on Markov decision processes, the average cost optimality equation. (This equation also plays the key role in some related papers, e.g., Hörner et al. 2015 and Escobar and Llanes 2016.)

correspond to player 1 receiving an income of \$1 and no player receiving any income, respectively.

For $k = 0$ and $2n$, we have

$$V_0 = p\delta V_0 + p\delta V_1 + (1-2p)\delta V_0, \tag{2}$$

$$V_{2n} = p\{(1-\delta)\gamma + \delta V_{2n-1}\} + p\{(1-\delta) + \delta V_{2n}\} + (1-2p)\delta V_{2n}. \tag{3}$$

2.1.3 *Payoff efficiency and incentive constraints* We can now demonstrate the efficiency of the prescribed strategies.

Lemma 1. *For any given n and $\lambda > 0$, there is a $\underline{\delta}_1 > 0$ such that for every $\delta > \underline{\delta}_1$, we have*

$$V_k > \frac{2n-1}{2n+1} p\gamma - \lambda$$

for all $k = 0, \dots, 2n$.

Proof. The strategies induce a stochastic Markov chain over states $k = 0, \dots, 2n$. By the ergodic theorem (see, for example, Chapter 1, Section 12, Theorem 1 in Shiryaev 1996) there exists a probability distribution over states $\{\pi_k : k = 0, \dots, 2n\}$ such that the probability of being in state k after a sufficiently large number of periods is arbitrarily close to π_k, independent of the initial state.[5] This probability distribution is an eigenvector corresponding to eigenvalue 1 of the transition matrix

$$\begin{bmatrix} 1-p & p & 0 & \cdot & \cdot & \cdot & 0 \\ p & 1-2p & p & \cdot & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & p & 1-2p & p \\ 0 & \cdot & \cdot & \cdot & 0 & p & 1-p \end{bmatrix},$$

in which the entry in row i and column j is equal to the probability of transiting from state j to state i. It is easy to verify that the eigenvector corresponding to eigenvalue 1 must have all its coordinates equal to $1/(2n+1)$.

The expected flow payoff of each player is $p\gamma$ in any state other than 0 and $2n$. When δ is sufficiently close to 1, each player's continuation payoff is bounded below by any number lower than $p\gamma(2n-1)/(2n+1)$, where $(2n-1)/(2n+1)$ is the limit occupation probability of states other than 0 and $2n$. □

It remains to show the incentive compatibility of the prescribed strategies. In this application, and throughout the rest of the paper, the issue concerns only deterring "on equilibrium path deviations." A player can only deviate when she has income, by consuming the income herself rather than transferring it to the opponent. The gain of this

[5]One can easily verify that the assumptions of the ergodic theorem are satisfied; indeed, each state is reached from any other state in $2n$ periods with a positive probability.

deviation, compared to playing as prescribed, is $1-\delta$. In turn, the gain from playing the prescribed strategy, compared to deviating, is that the player will have one more chip in the next period. We now prove that this gain is larger than $1-\delta$.

Lemma 2. *For every n, there is a $\underline{\delta}_2 < 1$ and $\kappa > 1$ such that for every $\delta > \underline{\delta}_2$, we have that $\Delta_k := V_k - V_{k-1} > \kappa(1-\delta)$ for all $k = 1, \dots, 2n$.*

Proof. From (1)–(3), by subtracting the equation for V_{k-1} from the equation for V_k, we obtain

$$\Delta_k = p\delta\Delta_{k-1} + p\delta\Delta_{k+1} + (1-2p)\delta\Delta_k$$

for $k = 2, \dots, 2n-1$,

$$\Delta_1 = p(1-\delta)\gamma + p\delta\Delta_2 + (1-2p)\delta\Delta_1$$

and

$$\Delta_{2n} = p(1-\delta) + p\delta\Delta_{2n-1} + (1-2p)\delta\Delta_{2n}.$$

For $\delta = 1$, this system of linear equations is satisfied when all Δs are equal to 0. For $\delta < 1$, the system is harder to solve, so we evaluate Δs in approximation. By the implicit function theorem, Δs are differentiable functions of δ. By taking the derivatives of the equations for Δs with respect to δ, and plugging in $\delta = 1$ and $\Delta_k = 0$ for all k, we obtain a system of equations for the derivatives of Δs at $\delta = 1$. In the matrix notation and after we divide by p, this new system of linear equations can be expressed as

$$\begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & \cdot & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & -1 & 2 & -1 \\ 0 & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \partial\Delta_{2n}/\partial\delta \\ \cdot \\ \\ \cdot \\ \\ \cdot \\ \partial\Delta_1/\partial\delta \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -\gamma \end{bmatrix}.$$

This system of linear equations can be solved easily by the Gauss–Jordan elimination method; the unique solution is given by

$$\frac{\partial\Delta_k}{\partial\delta} = -\frac{(2n-k+1)\gamma + k}{2n+1} < -1$$

for all k. Let $\kappa \in (1, [2n+\gamma]/(2n+1))$. Then $\Delta_k > \kappa(1-\delta)$ for all $k = 1, \dots, 2n$ and δ close to 1. □

Notice that this system of linear equations is nonsingular, which is equivalent to the uniqueness of our solution. This validates the use of the implicit function theorem. We could not use the implicit function theorem to approximate Vs directly, because the matrix of the system of equations (1)–(3) linearized at $\delta = 1$ would be singular.

2.2 *Repeated auctions*

In this section, we study the model of repeated auctions, in which the exchange of cheap-talk messages is allowed. In every period, two players participate in a first-price auction;[6] ties are resolved by a fifty–fifty lottery. At the end of each period, the identity of the winner, but not the bids, is revealed. Each player receives a private signal about the object. These signals can take one of two values, H or L; they are i.i.d. over time but may be correlated across the players. We restrict attention to symmetric signal structures, that is, to those in which the probability distribution over the signals is exhibited in the table

$$\text{player 2's signal}\;\underset{\text{player 1's signal}}{\begin{array}{c|cc} H & q & 2p \\ L & 1-2p-2q & q \\ \hline & L & H \end{array}},$$

where $p, q \geq 0$ and $p + q < 1/2$.

A player's valuation of the object is a function of both signals: that of the player herself and that of her opponent. We restrict attention to a symmetric case in which the players have the same valuation function v. The valuations are strictly positive and increasing in each signal, and each player's valuation increases in her own signal by more than in the signal of her opponent. That is, $v(H, H) > v(H, L) > v(L, H) > v(L, L) > 0$.

The efficient winner of the object is depicted in the table

$$\text{player 2's signal}\;\underset{\text{player 1's signal}}{\begin{array}{c|cc} H & \text{player 2} & \text{any player} \\ L & \text{any player} & \text{player 1} \\ \hline & L & H \end{array}}.$$

Collusion in repeated auctions has been studied by Skrzypacz and Hopenhayn (2004), Blume and Heidhues (2008), Rachmilevitch (2013), and Aoyagi (2003). The most advanced result in terms of generated payoffs was obtained by Aoyagi (2007). Allowing for mediated communication, he showed by modifying self-generation techniques that efficiency can be attained in a large class of repeated-auction settings.

In our example, types (i.e., signals) can be correlated and values (i.e., payoffs) are not necessarily private. Thus, the existence of efficient equilibria does not follow from Escobar and Toikka, because they assume private values and independent types. It does not follow from Aoyagi (2007) either, because his Assumption 4 is violated. In addition, if the correlation between players' types is negative, even Aoyagi's construction does not deliver an efficient equilibrium, since players no longer have incentives to report their types truthfully. Additionally, the result does not follow from Hörner et al. (2015),

[6] We assume this particular auction format only for concreteness. It is easy to check that all our arguments apply to any auction in which (a) the player who makes a higher bid wins the object, (b) the payment of a player who bids 0 is 0, and (c) the payment is continuous at 0, that is, the payment of a player who bids close to 0 must also be close to 0.

[7] because the monitoring structure (only the identity of the winner is revealed) violates the standard identifiability assumptions. (Furthermore, Hörner, Takahashi, and Vieille study only equilibria in which players truthfully reveal their types, both on and off the equilibrium path.)

2.2.1 *Description of efficient chip strategies* At the beginning of each period, each player i holds $k_i \in \{0, \dots, 2n\}$ chips, where $k_1 + k_2 = 2n$, and the repeated game begins with $k_1 = k_2 = n$. If $k_i \neq 0, 2n$, the players report their signals truthfully in the cheap-talk communication. If the two players report the same signal, they determine by a fifty–fifty lottery who the winner in the current period will be.[8] At the bidding stage, (a) a player bids ρ if her report is H and the opponent's report is L, (b) she also bids ρ when the players reported equal signals, and she was determined to be the winner in the lottery, (c) she bids 0 if her report is L and the opponent's report is H, and (d) she also bids 0 when the players reported equal signals and her opponent was determined to be the winner of the lottery. These actions closely approximate the desired efficient outcome if ρ is sufficiently small. A player with all $2n$ chips bids ρ, and a player with 0 chips bids 0, regardless of the signals. After each period, the number of chips held by the winner decreases by one and number of chips held by the loser increases by one.

If a player wins the auction when she was not supposed to (which cannot happen if the players play the prescribed strategies), the players switch to playing a bad symmetric stage-game Bayesian Nash equilibrium described at the end of this section in which each player's payoff is strictly less than half of the efficient payoff.

We offer the following result, which is proved in the next two subsections.

Proposition 2. *For every $\lambda > 0$, there exist $\underline{\delta} < 1$ and n such that if the players' discount factor is $\delta > \underline{\delta}$, then the chip-strategy profile with n chips is an equilibrium of the repeated auction, and the total payoff in this equilibrium does not fall short of the efficient total payoff by more than λ.*

2.2.2 *Continuation payoffs* Assuming that both players play the prescribed strategies, denote by V_k the continuation payoff of player 1, where $k \in \{0, \dots, 2n\}$ is the number of chips she holds. We focus on player 1. The analysis of player 2's continuation payoffs is analogous by symmetry. These continuation payoffs are computed before the players learn their current signals. For $k \in \{1, \dots, 2n-1\}$, we have

$$V_k = (1-\delta)C + \delta V_{k-1}/2 + \delta V_{k+1}/2,$$

where

$$C = pv(H,H) + qv(H,L) + (1/2 - p - q)v(L,L)$$

is the per-period efficient payoff.

[7] As far as we know, the first version of Hörner et al. (2015) was composed subsequent to the first version of our paper.

[8] This lottery can be performed by means of cheap-talk messages.

For $k = 0$ and $2n$, we have

$$V_0 = \delta V_1$$

and

$$V_{2n} = (1-\delta)D + \delta V_{2n-1},$$

where

$$D = 2pv(H,H) + qv(H,L) + (1-2p-2q)v(L,L) + qv(L,H).$$

2.2.3 *Payoff efficiency and incentive constraints* We can now demonstrate the approximate efficiency and incentive compatibility of the prescribed strategies.

Lemma 3. *For any n and any $\lambda > 0$, there is a $\underline{\delta}_1 > 0$ such that for every $\delta > \underline{\delta}_1$ and all $k = 0, \ldots, 2n$, we have that V_k does not fall short of a $(2n-1)/4n$ share of half of the efficient total payoff C by more than λ.*

Proof. As in the case of favor exchange from Section 2.1, the lemma follows from the ergodic theorem. Indeed, the ergodic probabilities π_k, $k = 0, \ldots, 2n$, must satisfy the system of linear equations

$$\begin{bmatrix} 1 & -1/2 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 1 & -1/2 & \cdot & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & & & \cdot & -1/2 & 1 & -1 \\ 0 & \cdot & \cdot & \cdot & 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \cdot \\ \\ \cdot \\ \\ \cdot \\ \pi_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix}.$$

This yields $\pi_0 = \pi_{2n} = 1/4n$ and $\pi_1 = \cdots = \pi_{2n-1} = 1/2n$. Thus, the players play inefficient actions only with a probability close to $1/2n$. $\square$

Lemma 4. *For every n, there is a $\underline{\delta}_2 < 1$ such that for every $\delta > \underline{\delta}_2$, the prescribed chip-strategy profile is an equilibrium of the repeated game.*

Proof. We begin by showing that the players have incentives to report their types truthfully. Consider first what they gain or lose in terms of the payoff in the current period by misreporting. Player 1 loses

$$(1-\delta)\left\{\frac{p}{2p+q}v(H,H) + \frac{q/2}{2p+q}v(H,L)\right\}$$

by reporting L instead of H, and gains

$$(1-\delta)\left\{\frac{q/2}{1-2p-q}v(L,H) + \frac{1/2-p-q}{1-2p-q}v(L,L)\right\}$$

by reporting H instead of L. Indeed, consider the first formula. When the player's signal is H, she assigns probability $2p/(2p+q)$ to her opponent having signal H; in this case, she loses $v(H,H)$ with probability $1/2$ by reporting L instead of H. Similarly, she assigns probability $q/(2p+q)$ to her opponent having signal L; in this case, she loses $v(H,L)$ with probability $1/2$ by reporting L instead of H.

Player 1 increases her chance of having $k+1$ chips (instead of having $k-1$ chips) by

$$\frac{p+q/2}{2p+q}=\frac{1}{2}$$

if she reports L instead of H, and she decreases her chance of having $k+1$ chips (instead of having $k-1$ chips) by

$$\frac{1/2-p-q/2}{1-2p-q}=\frac{1}{2}$$

if she reports H instead of L. Thus, we need to show that

$$(1-\delta)\left\{\frac{p}{2p+q}v(H,H)+\frac{q/2}{2p+q}v(H,L)\right\}>\delta\frac{1}{2}(V_{k+1}-V_{k-1}) \tag{4}$$

and

$$(1-\delta)\left\{\frac{q/2}{1-2p-q}v(L,H)+\frac{1/2-p-q}{1-2p-q}v(L,L)\right\}<\delta\frac{1}{2}(V_{k+1}-V_{k-1}) \tag{5}$$

for $k=1,\dots,2n$.

Let $\Delta_k=V_k-V_{k-1}$ for $k=1,\dots,2n$. Then for $k=2,\dots,2n-1$, we have

$$\Delta_k=\delta\Delta_{k-1}/2+\delta\Delta_{k+1}/2.$$

At $\delta=1$, we have $\Delta_k=0$ and

$$\frac{\partial\Delta_k}{\partial\delta}=\frac{1}{2}\frac{\partial\Delta_{k-1}}{\partial\delta}+\frac{1}{2}\frac{\partial\Delta_{k+1}}{\partial\delta}.$$

For $k=1,2n$, we have

$$\Delta_1=(1-\delta)C+\delta\Delta_2/2-\delta\Delta_1/2$$

and so

$$\begin{aligned}&\frac{\partial\Delta_1}{\partial\delta}=-\frac{2C}{3}+\frac{1}{3}\frac{\partial\Delta_2}{\partial\delta};\\&\Delta_{2n}=(1-\delta)[D-C]+\delta\Delta_{2n-1}/2-\delta\Delta_{2n}/2\end{aligned}$$

and so

$$\frac{\partial\Delta_{2n}}{\partial\delta}=-\frac{2[D-C]}{3}+\frac{1}{3}\frac{\partial\Delta_{2n-1}}{\partial\delta}.$$

This implies that $\partial\Delta_k/\partial\delta$ is a weighted average of C and $D-C$ for $k=1,\dots,2n$. By the implicit function theorem, this is also true, in approximation, for the expressions on the right-hand side (RHS) of (4) and (5) multiplied by 2. Recalling the values of C and D, by (4) and (5) it suffices to show that

$$\begin{aligned}&\left[\frac{q}{1-2p-q}v(L,H)+\frac{1-2p-2q}{1-2p-q}v(L,L)\right]\\&\quad<\left[2pv(H,H)+2qv(H,L)+(1-2p-2q)v(L,L)\right],\\&\quad\left[2pv(H,H)+2qv(L,H)+(1-2p-2q)v(L,L)\right]\end{aligned}$$

and

$$<\left[\frac{2p}{2p+q}v(H,H)+\frac{q}{2p+q}v(H,L)\right].$$

To see this, notice that each of the four expressions is a weighted average of $v(H,H)$, $v(H,L)$, $v(L,H)$, and $v(L,L)$, where $v(H,H)>v(H,L)>v(L,H)>v(L,L)$. Notice next that the weights of the first expression are first-order stochastically dominated by both the weights of the second expression (the first one on the second line) and the weights of the third expression (the second one on the second line). In turn, both the weights of the second expression and the weights of the third expression are first-order stochastically dominated by the weights of the fourth expression.

Thus, the players have incentives to report their types truthfully. The players have no incentive to deviate at the bidding stage either, because if such a deviation changed the auction outcome, it would be detected; the player who was supposed to be the loser would be the winner (or the other way around).[9] This would mean switching to a stage-game Bayesian Nash equilibrium in which the payoff of each player is strictly less than half of the efficient payoff. This outcome is worse than playing chip strategies where the continuation payoff V_k converges to half of the efficient payoff for all $k\in\{0,\dots,2n\}$ if the discount factor converges to 1 and n is sufficiently large. □

2.2.4 *Bad stage-game equilibrium* Finally, we need to describe a symmetric stage-game equilibrium in which the players obtain payoffs so low that they find it unprofitable to take any action not prescribed for their reported type profile.

The players are prescribed to play a babbling equilibrium in the stage of reporting their types. A player who received a low signal bids $v(L,L)$ with probability 1. In turn, a player with a high signal chooses a bid according to a distribution $G(b)$ with support $[v(L,L),E(v\mid H)-q[v(H,L)-v(L,L)]/(q+2p)]$, where

$$E(v\mid H)=qv(H,L)/(q+2p)+2pv(H,H)/(q+2p)$$

is the expected value of the object, contingent on receiving a high signal. If a player who received a high signal bids b, her payoff is

$$\frac{q}{q+2p}\left[v(H,L)-b\right]+\frac{2p}{q+2p}G(b)\left[v(H,H)-b\right],$$

[9]Here we disregard as profitable any deviations to bids that are greater than 0 but smaller than ρ.

because she wins with probability 1 against the opponent who received a low signal and wins with probability $G(b)$ against the opponent who received a high signal. Differentiating this expression with respect to b must yield 0 on the support of the equilibrium bidding strategy, because the player must be indifferent between all such bids. That is,

$$\frac{2p}{q+2p}G'(b)\big[v(H,H)-b\big]=\frac{q}{q+2p}+\frac{2p}{q+2p}G(b),$$

which yields

$$G(b)=\frac{c}{2p}\frac{1}{v(H,H)-b}-\frac{q}{2p}$$

for a constant c. Since $G(v(L,L))=0$,

$$c=q\big[v(H,H)-v(L,L)\big];$$

that is,

$$G(b)=\frac{q}{2p}\frac{v(H,H)-v(L,L)}{v(H,H)-b}-\frac{q}{2p},$$

and indeed $G(E(v\mid H)-q[v(H,L)-v(L,L)]/(q+2p))=1$.

The payoff of a player who received a low signal is zero in this equilibrium, and the payoff of a player who received a high signal is $q[v(H,L)-v(L,L)]/(q+2p)$. The expected payoff is $q[v(H,L)-v(L,L)]$, which is strictly less than half of the efficient payoff, $pv(H,H)+qv(H,L)+(1/2-p-q)v(L,L)$.

3. Result

We now introduce the general setting. Let G be a two-player game. Denote by A_i the set of actions and denote by T_i the set of types of player $i=1,2$; denote by $u_i: A_1\times A_2\times T_1\times T_2\to\mathbb{R}$ the payoff function of player i. In this section, we consider only *symmetric* games, i.e., we assume that $A_1=A_2$, $T_1=T_2$, and $u_1(a_1,a_2,t_1,t_2)=u_2(a_2,a_1,t_2,t_1)$. The players are expected-payoff maximizers and discount future payoffs by a common discount factor $\delta<1$. The payoffs are normalized by the factor of $1-\delta$. We allow the players to communicate, i.e., to send cheap-talk messages at the beginning of each period.[10]

We analyze only games in which each player's type set has exactly two elements, i.e., $|T_i|=2$ for $i=1,2$. We denote these two types by H and L. We assume that the type profile $t=(t_1,t_2)$ evolves according to a Markov process, i.e., the probability $p^t_{t^{+1}}$ of the type profile taking a certain value t^{+1} in the following period is determined by its value t in the current period. We impose the following two conditions on transition probabilities:

[10] In some games, including the applications to favor exchange or repeated oligopoly, communication is not necessary. However, in the general case, the players would not be able to coordinate on efficient actions without learning about the opponents' types.

- Full support: All the transition probabilities are positive.[11]
- Transition probabilities are symmetric across players, i.e.,

$$p^{L,L}_{L,H} = p^{L,L}_{H,L}, \qquad p^{H,H}_{L,H} = p^{H,H}_{H,L}, \qquad p^{L,H}_{L,H} = p^{H,L}_{H,L},$$
$$p^{L,H}_{H,L} = p^{H,L}_{L,H}, \qquad p^{L,H}_{L,L} = p^{H,L}_{L,L}, \qquad p^{L,H}_{H,H} = p^{H,L}_{H,H}.$$

Denote the six probabilities by ζ, η, φ, χ, μ, and ν, respectively. Denote also (for brevity) $p^{L,L}_{L,L}$ by p^L_L, $p^{L,L}_{H,H}$ by p^L_H, $p^{H,H}_{L,L}$ by p^H_L, and $p^{H,H}_{H,H}$ by p^H_H. The transition probabilities are exhibited in the following table:

$t \setminus t^{+1}$	L, L	L, H	H, L	H, H
L, L	p^L_L	ζ	ζ	p^L_H
L, H	μ	φ	χ	ν
H, L	μ	χ	φ	ν
H, H	p^H_L	η	η	p^H_H

In particular, our assumptions are satisfied when each player's type evolves according to a Markov process and types are independent across players.

By the ergodic theorem, there exists the limiting (also called stable or ergodic) distribution over type profiles t.

Proposition 3. *There exists a stable distribution over type profiles. If the probability of profile L, L is denoted as q_L, the probability of profile H, H is denoted as q_H, and the probability of each L, H and H, L (which are equal by symmetry) is denoted as q, then the stable probabilities satisfy the properties*

$$q_L + 2q + q_H = 1,$$
$$q_L = p^L_L \cdot q_L + 2\mu \cdot q + p^H_L \cdot q_H,$$
$$q = \zeta \cdot q_L + (\varphi + \chi) \cdot q + \eta \cdot q_H,$$
$$q_H = p^L_H \cdot q_L + 2\nu \cdot q + p^H_H \cdot q_H.$$

For convenience, we assume that the probability distribution over the type profiles at the beginning of period 1 coincides with stable distribution.

We impose no conditions on the set of actions, except the existence of action profiles with certain properties, which are introduced later. In particular, we assume that there exist efficient action profiles, i.e., action profiles that maximize the sum of payoffs for any given types. By the symmetry of our model, we can with no loss of generality assume that if (a_{LH}, a_{HL}) is an efficient action profile for type profile (L, H), then (a_{HL}, a_{LH})

[11]The main result of this section generalizes to some settings in which full support is violated such as that from Section 2.1. The only role of full support is to guarantee that the assumptions of the ergodic theorem are satisfied; more specifically, that there exist a number of periods T such that each state is reached from any other state with positive probability in T periods.

is an efficient action profile for type profile (H, L). We can also assume that for type profiles (H, H) and (L, L), the same actions are specified for both players in an efficient action profile. These actions are denoted by a_{HH} and a_{LL}, respectively.[12] We denote by v the ex ante efficient total payoff, that is,

$$v = 2q_L u_1(a_{LL}, a_{LL}, L, L) + 2q\big[u_1(a_{LH}, a_{HL}, L, H) + u_1(a_{HL}, a_{LH}, H, L)\big] \\ + 2q_H u_1(a_{HH}, a_{HH}, H, H).$$

The players would each achieve the payoff of $v/2$ if they both report their types truthfully and then take the efficient actions. We assume, however, that each player obtains a higher flow payoff if she reports one of the types—let us say L—no matter what the player's actual type is. More precisely, we assume that for any player 1's current type t_1 and any type profile t^{-1} in the previous period,

$$E_{t_2}\big[u_1(a_{Lt_2}, a_{t_2 L}, t_1, t_2) \mid t_1, t^{-1}\big] - E_{t_2}\big[u_1(a_{Ht_2}, a_{t_2 H}, t_1, t_2) \mid t_1, t^{-1}\big] \tag{6}$$

is strictly greater than 0. This inequality guarantees that player 1 always prefers reporting L to reporting H. This assumption makes the setting appropriate for using chip strategies. Indeed, each player would always prefer to play as if she was of type L. However, this is not what the other player wants. Therefore, every time a player reports her more preferred type L, but the opponent reports the other type, the opponent provides the player a favor.

Finally, we assume that there exist action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H) such that player 1 prefers $(b_1^{t_1}, b_2^{t_1})$ (out of the two profiles) to be played when her type is t_1; these action profiles "reward" player 1 at the expense of player 2, and are prescribed by chip strategies when player 2 runs out of chips. These reward actions are required to satisfy two assumptions. To formulate these assumptions, we now define the four quantities B, A, B', and A', which are equal to the payoff differences between taking the efficient and the reward actions. We begin with quantity B, defined as

$$\begin{aligned} B = {} & \varphi\big[u_1\big(b_1^H, b_2^H, H, L\big) - u_1(a_{HL}, a_{LH}, H, L)\big] \\ & + \nu\big[u_1\big(b_1^H, b_2^H, H, H\big) - u_1(a_{HH}, a_{HH}, H, H)\big] \\ & + \mu\big[u_1\big(b_1^L, b_2^L, L, L\big) - u_1(a_{LL}, a_{LL}, L, L)\big] \\ & + \chi\big[u_1\big(b_1^L, b_2^L, L, H\big) - u_1(a_{LH}, a_{HL}, L, H)\big]. \end{aligned}$$

This quantity is the difference in player 1's expected flow payoff between playing the action profiles that reward player 1 and playing the efficient action profiles, contingent

[12] This assumption is without any loss of generality. If the specified efficient action profile is asymmetric—take (a_1, a_2) as an example—then (a_2, a_1) is also efficient by symmetry, and the specified efficient action profile can be replaced with a fifty–fifty lottery over (a_1, a_2) and (a_2, a_1). Such a lottery does not even require access to any public randomization device. Instead, players can generate the required public randomization device in communication by randomizing simultaneously over two extraneous messages, with the interpretation that (a_1, a_2) is going to be played if the messages "coincide"and (a_2, a_1) is going to be played if the messages are different. (See Aumann et al. 1968 for more details on jointly controlled lotteries.)

on the type profile being (H, L) in the previous period. We define the second quantity A as

$$\begin{aligned}A = \varphi\big[&u_1(a_{LH}, a_{HL}, L, H) - u_1\big(b_2^H, b_1^H, L, H\big)\big] \\ &+ \nu\big[u_1(a_{HH}, a_{HH}, H, H) - u_1\big(b_2^H, b_1^H, H, H\big)\big] \\ &+ \mu\big[u_1(a_{LL}, a_{LL}, L, L) - u_1\big(b_2^L, b_1^L, L, L\big)\big] \\ &+ \chi\big[u_1(a_{HL}, a_{LH}, H, L) - u_1\big(b_2^L, b_1^L, H, L\big)\big].\end{aligned}$$

This quantity is the difference in player 1's expected flow payoff between playing the efficient action profiles and playing the action profiles that reward player 2, contingent on the type profile being (L, H) in the previous period. Note that $A \geq B$, since the total payoff when playing the reward actions is no higher than the total efficient payoff. Finally, the quantities B' and A' are defined as

$$\begin{aligned}B' = \zeta\big[&u_1\big(b_1^L, b_2^L, L, H\big) - u_1(a_{LH}, a_{HL}, L, H)\big] \\ &+ p_L^L\big[u_1\big(b_1^L, b_2^L, L, L\big) - u_1(a_{LL}, a_{LL}, L, L)\big] \\ &+ p_H^L\big[u_1\big(b_1^H, b_2^H, H, H\big) - u_1(a_{HH}, a_{HH}, H, H)\big] \\ &+ \zeta\big[u_1\big(b_1^H, b_2^H, H, L\big) - u_1(a_{HL}, a_{LH}, H, L)\big],\end{aligned}$$

which is the difference in player 1's expected flow payoff between playing the action profiles that reward player 1 and playing the efficient action profiles, contingent on the type profile being (L, L) in the previous period, and

$$\begin{aligned}A' = \eta\big[&u_1(a_{LH}, a_{HL}, L, H) - u_1\big(b_2^H, b_1^H, L, H\big)\big] \\ &+ p_L^H\big[u_1(a_{LL}, a_{LL}, L, L) - u_1\big(b_2^L, b_1^L, L, L\big)\big] \\ &+ p_H^H\big[u_1(a_{HH}, a_{HH}, H, H) - u_1\big(b_2^H, b_1^H, H, H\big)\big] \\ &+ \eta\big[u_1(a_{HL}, a_{LH}, H, L) - u_1\big(b_2^L, b_1^L, H, L\big)\big],\end{aligned}$$

which is the difference in player 1's expected flow payoff between playing the efficient action profiles and playing the action profiles that reward player 2, contingent on the type profile being (H, H) in the previous period.

We now make two assumptions.

Assumption I. *Expressions (6) for $t_1 = L$ and any t^{-1} are greater than $A/(1 + (\varphi - \chi))$, and expressions (6) for $t_1 = H$ and any t^{-1} are smaller than $B/(1 + (\varphi - \chi))$.*

Assumption II. *Expressions (6) for $t_1 = L$ and any t^{-1} are greater than B', and expressions (6) for $t_1 = H$ and any t^{-1} are smaller than A'.*

Assumptions I and II look complicated, but are actually not difficult to understand and interpret. It is easier to explain and comment on them after we introduce our last assumption and define chip strategies. For now, notice that the existence of action profiles

(b_1^L, b_2^L) and (b_1^H, b_2^H), which reward player 1 at the expense of player 2, is straightforward under our other assumption that expressions (6) are strictly positive. Indeed, one can take $(b_1^L, b_2^L) = (b_1^H, b_2^H) = (a_{LH}, a_{HL})$. If, in addition, player types are i.i.d. (which yields $\varphi = \chi$) and players' valuations are private, then these action profiles satisfy Assumptions I and II. When types are persistent (e.g., when φ is much larger than χ), it is easy to construct examples in which action profiles $(b_1^L, b_2^L) = (b_1^H, b_2^H) = (a_{LH}, a_{HL})$ (as well as any other action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H)) violate Assumptions I and II. Finally, it seems that in simple settings, checking for the existence of action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H), which reward player 1 at the expense of player 2, and satisfy Assumptions I and II, should not be an involved process. (Section 5 provides some examples.)

We make another assumption.

Assumption III. *There exists a "bad" dynamic-game equilibrium, that is, an equilibrium in which the payoff of each player is strictly lower than $v/2$.*

The *simple chip strategies* are defined as follows:

- There are $2n$ chips, which initially are distributed evenly between the two players.
- If player 1 has k chips and $k \neq 0$ or $2n$, then the players take the efficient action profile: (a_{LH}, a_{HL}), (a_{HL}, a_{LH}), (a_{HH}, a_{HH}), or (a_{LL}, a_{LL}). The truthfully reported type profile determines which of the four action profiles is taken.
- If either (a_{HH}, a_{HH}) or (a_{LL}, a_{LL}) is played, then the distribution of chips remains unaltered. If either (a_{LH}, a_{HL}) or (a_{HL}, a_{LH}) is played, then player 1 or player 2, respectively, gives a chip to the opponent.
- In the limit state $k = 2n$, the action profile (b_1^L, b_2^L) or (b_1^H, b_2^H) is played, depending on the report of player 1, and player 1 gives player 2 a chip. In the limit state $k = 0$, the action profile (b_2^L, b_1^L) or (b_2^H, b_1^H) is played, depending on the report of player 2, and player 2 gives player 1 a chip.
- The bad equilibrium is played off the equilibrium path.

Some additional remarks on action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H) and Assumptions I and II seem helpful. Assumptions I and II capture incentive constraints in terms of primitives of the model; more precisely, they guarantee that players are willing to receive a favor and give away a chip only when it is efficient to do so, which incentivizes truthtelling. To understand that Assumptions I and II indeed guarantee the right incentives, consider first the case when types evolve over time in the i.i.d. manner, with each player having either type with probability 1/2. In this case, expressions $A/(1 + (\varphi - \chi))$ and $B/(1 + (\varphi - \chi))$ reduce to simply A and B.

In addition, suppose that there are only two chips in the economy, and each player has exactly one chip. In this case, if player 1's type is L, she loses (6) in which $t_1 = L$ by reporting H, compared to reporting honestly. In turn, she gains A in the next period if player 2 happens to report H in the current period, and gains B' in the next period if player 2 happens to report L in the current period. So the next-period gain is a weighted average of A and B'. By the first parts of Assumptions I and II, the loss exceeds the gain. Similarly, if player 1's type is H, she gains (6) in which $t_1 = H$ by reporting L, compared to reporting honestly. In turn, she loses A' in the next period if player 2 happens to report H in the current period, and loses B in the next period if player 2 happens to report L in the current period. So, the next-period loss is a weighted average of A' and B. By the second parts of Assumptions I and II, the loss again exceeds the gain. The argument in the general case with $n > 1$ chips is more involved, because the future gain or loss is not immediate. However, this gain or loss still turns out to be a weighted average of A and B' or A' and B.

In the general Markov case, a factor of $1 + (\varphi - \chi)$ appears in Assumption I as a denominator for values A and B, but not for values A' and B'. The reason is that values A and B apply to the limit states with no chips or $2n$ chips, given that the previous type profile was (L, H) or (H, L), respectively. Then, even when a player with no chips (for example, player 1) obtains a chip back, the fact that the previous type profile was (L, H) indicates some persistence in returning to the state in which player 1 has no chips (when $\varphi > \chi$) or indicates some persistence in moving away from the state in which player 1 has no chips (when $\varphi < \chi$). This is so because the play returns (or moves away) from this state only when the reported type profile is (L, H) (or (H, L), respectively). In comparison, this factor does not appear in Assumption II, because it only applies after player 1 has misreported her type: (i) when the type profile in the previous period was (L, L), which moved the play to the state in which player 1 has $2n$ chips, and (ii) when the type profile in the previous period was (H, H), which moved the play to the state in which player 1 has no chips. Additionally, the fact that the previous type profile was (L, L) or (H, H) indicates no persistence.

Finally, note that Assumptions I and II are motivated by a specific idea or form of cooperation. Namely, players exchange favors for chips, and when a player runs out of chips, she must provide a favor to the opponent. In addition, we assumed that (a) this favor must be provided in the next period and (b) must depend only on the type of the player with all the chips. Assumptions I and II were chosen to guarantee the existence of such favors that provide the right incentives. Conditions (a) and (b) can be relaxed, which is discussed in the following section. More generally, one can design the play in the limit states by referring to a variety of other repeated-game strategies that reward one player at the expense of the other player. This would increase the number of settings in which such modified chip strategies attain an almost efficient outcome, but our equilibria would lose some of their "chip-strategy flavor," and we would depart from the goal of attaining efficient outcomes in simple strategies.

We obtain the following result.

Proposition 4. *For every $\lambda > 0$, there exist $\underline{\delta} < 1$ and n such that if the players' discount factor is $\delta > \underline{\delta}$, then the chip-strategy profile with n chips is an equilibrium and the ex ante payoff of each player in this equilibrium exceeds $v/2 - \lambda$.*

The proof can be found in Appendix A.

Remark 1. Assumption I is the weakest assumption that guarantees satisfying the incentive constraints and it cannot be relaxed. It follows from the proof that for any given n, the expected value for player i of having an additional chip changes with the current number of i's chips k, and is always (in approximation) a weighted average of $A/(1+(\varphi-\chi))$ and $B/(1+(\varphi-\chi))$ (given the play when $k=0$ or $2n$). When $k/2n \to 0$, that weighted average converges to $A/(1+(\varphi-\chi))$, and when $k/2n \to 1$, that weighted average converges to $B/(1+(\varphi-\chi))$. Thus, Assumption I would precisely require satisfying the incentive constraints to report truthfully if the value of having an additional chip was equal to one of these limit values.

At the same time, Assumption II could be somewhat weakened. This assumption incentivizes player i to report truthfully whenever (i) i has $2n-1$ chips and $t_i = L$, and (ii) i has 1 chip and $t_i = H$, and Assumption II does so *regardless* of i's beliefs on j's type. One could alternatively assume that player i is incentivized only for i's actual beliefs about j's type. However, those beliefs depend on i's type t_i as well as past type profile t^{-1}. So, the "new" Assumption II would be weaker but would consist in total of eight different conditions.

4. The play in limit states

When a limit state of $k=0$ or $2n$ chips is reached, simple chip strategies from Section 3 prescribe the action profiles (b_1^L, b_2^L) and (b_1^H, b_2^H) (or (b_2^L, b_1^L) and (b_2^H, b_1^H)) to be played for one period, after which the play leaves the limit state. These action profiles were assumed to reward the player with all chips at the expense of the player with no chips (compared to playing the efficient actions), to depend only on the report of the player with all chips, and to satisfy Assumptions I and II. This design of the play in limit states was sufficient for approximating efficient outcomes, but not necessary. In some games such action profiles may not exist, but by slightly modifying the play when the limit state is reached, we may construct equilibrium strategies that still approximate the efficient outcomes.

For example, in the favor-exchange model, we stay in a limit state until the player who ran out of chips has an opportunity to provide a favor to the opponent. Such an opportunity may not come in a single period, in which case the play stays in the limit state for longer. In other versions of the favor-exchange model, efficiency may be attained only by requiring the player who ran out of chips to provide more than one favor to the opponent before leaving the limit state (now, it is rather a limit phase), and this required number of favors may even be random if players can use a public randomization device.

More specifically, one can design a limit phase as follows. Let τ be a stopping time, which takes as input the type profiles reported by players in the limit phase, from its first period to the current period, and delivers as output the decision whether the play should leave the limit phase in the following period (i.e., the player who has no chip should be given a chip back). Suppose that until τ delivers the decision of leaving the limit phase, players are prescribed some actions, and these actions may depend on the reported types.

For example, we assumed in Section 3 that τ was always delivering the decision of leaving the limit phase in the following period, independently of the reported types, and the prescribed action profiles (b_1^L, b_2^L), (b_1^H, b_2^H), (b_2^L, b_1^L), or (b_2^H, b_1^H) depended only on the type (L or H) reported by the player having all chips. In the favor-exchange model, the decision to leave the limit phase in the following period is delivered only when the player running out of chips makes a transfer or reports being given a dollar.[13] The prescribed actions were that no player makes any transfer until this happened, and that the player running out of chips (and only this player) makes the transfer when she reports being given a dollar. (Notice that the prescribed actions depend in this case only on the type reported by the player having no chips.)

Of course, not all described designs of the play in the limit phases guarantee that chip strategies are an equilibrium and approximate the efficient outcome. The general idea behind the limit states (phases) is that the player with all the chips is rewarded at the expense of the player with no chips, compared to playing the efficient actions. This gives chips value. This value (which typically varies with the number of chips) can never be so low that a player with type H prefers reporting H to reporting L, and the value of a chip cannot be so high that a player with type L prefers reporting L to reporting H. Being in this right range of values is guaranteed in Section 3 by Assumptions I and II. Analogous conditions for some other strategies in limit phases can be derived by inspecting the proof of Proposition 4. However, the conditions in the general case are rather involved.

In addition, if the actions prescribed in the limit phase depend on players' types, players must have incentives to report their types truthfully. In Section 3, this is guaranteed by requiring actions to depend only on the report of the player with all chips, who can choose between two action profiles: (b_1^L, b_2^L) and (b_1^H, b_2^H) in the case of player 1 or (b_2^L, b_1^L) and (b_2^H, b_1^H) in the case of player 2. Finally, the limit phase must in expectation be sufficiently short, since it would otherwise create nonvanishing inefficiency.

5. Other applications

Our result has a number of applications. We now discuss several of them.

5.1 *Spulber's duopoly*

Consider the repeated version of Spulber's (1995) duopoly model, introduced to the literature by Athey and Bagwell (2001), in which two firms meet in periods $t = 1, 2, \ldots$. Each firm's cost of producing one unit of a good takes the value $c = \underline{c}$ or $\overline{c}$, and follows a first-order Markov process. The cost of a firm in the following period is equal to the current cost with probability $p \in [1/2, 1)$, and is different from the current cost with the remaining probability. The costs of the two firms are independent random processes.

In every period of this dynamic game, firms select prices simultaneously. A single consumer is willing to pay up to $r > \overline{c} > \underline{c}$ dollars for one unit of the good and buys from

[13] The favor-exchange model is somewhat specific, because the action space depends on an agent's type, and the reporting of types is not necessary.

So we refer here to an "isomorphic" model in which a transfer is assumed to be always feasible, but at an infinite (or sufficiently high) cost if a player has not been given the dollar. Then, making a transfer is equivalent to the player reporting being given a dollar, and making no transfer is equivalent to the player reporting not being given a dollar.

the firm that offers a lower price; if the two prices are equal, the consumer buys from each firm with a fifty–fifty chance.

Firms are expected-profit maximizers and discount future payoffs by a common discount factor $\delta < 1$. In period 0, the cost of each firm takes the value $\underline{c}$ or $\overline{c}$ with a fifty–fifty chance. Then the efficient, or most collusive, total payoff of the two firms is

$$v = r - \frac{3}{4}\underline{c} - \frac{1}{4}\overline{c}.$$

Our main result has the following implication.

Proposition 5. *For every $\lambda > 0$, there exist $\underline{\delta} < 1$ and n such that if the players' discount factor is $\delta > \underline{\delta}$, then the simple chip-strategy profile with n chips is an equilibrium, and the ex ante payoff of each firm in this equilibrium exceeds $v/2 - \lambda$.*

The simple chip strategies are constructed as follows. On the equilibrium path, if in some period, one firm charges a lower price than the other, then it serves the consumer, but gives an implicit chip to the other firm, and if one of the firms has no more chips, it lets the other firm serve the consumer for one period and receives one chip for this favor. Off the equilibrium path, the firms play a bad dynamic-game equilibrium. In terms of our general result, this means that the firms charge prices $a_{LH} = a_{LL} = r - \rho$, $a_{HL} = a_{HH} = r$, where ρ is a small number, and $(b_1^L, b_2^L) = (b_1^H, b_2^H) = (r - \rho, r)$. Assumptions I and II reduce to just saying that $(r - \underline{c})/2 > (r - \overline{c})/2 > 0$. Indeed, given the last period's type profile being (H, L), the difference in player 1's expected payoff between playing (b_1^L, b_2^L) and (b_1^H, b_2^H) and playing the efficient action profiles is

$$B = p^2[r - \overline{c}] + (1 - p)p\left[\frac{r - \underline{c}}{2}\right] + p(1 - p)\left[\frac{r - \overline{c}}{2}\right] \in \left(2p\frac{r - \overline{c}}{2}, 2p\frac{r - \underline{c}}{2}\right),$$

where ρ is taken to be 0. The difference in player 2's expected payoff between playing the efficient action profiles and playing (b_1^L, b_2^L) and (b_1^H, b_2^H) is

$$A = p^2[r - \underline{c}] + p(1 - p)\left[\frac{r - \underline{c}}{2}\right] + p(1 - p)\left[\frac{r - \overline{c}}{2}\right] \in \left(2p\frac{r - \overline{c}}{2}, 2p\frac{r - \underline{c}}{2}\right).$$

Similarly,

$$\begin{aligned} B' &= p^2\left[\frac{r - \underline{c}}{2}\right] + (1 - p)^2\left[\frac{r - \overline{c}}{2}\right] + p(1 - p)[r - \overline{c}] \\ &= p^2\left[\frac{r - \underline{c}}{2}\right] + \left(1 - p^2\right)\left[\frac{r - \overline{c}}{2}\right] \in \left(\frac{r - \overline{c}}{2}, \frac{r - \underline{c}}{2}\right) \end{aligned}$$

and

$$\begin{aligned} A' &= p(1 - p)[r - \underline{c}] + (1 - p)^2\left[\frac{r - \underline{c}}{2}\right] + p^2\left[\frac{r - \overline{c}}{2}\right] \\ &= \left(1 - p^2\right)\left[\frac{r - \underline{c}}{2}\right] + p^2\left[\frac{r - \overline{c}}{2}\right] \in \left(\frac{r - \overline{c}}{2}, \frac{r - \underline{c}}{2}\right). \end{aligned}$$

Off the equilibrium path, i.e., when a firm charges a price other than r or $r-\rho$, or does not charge the prescribed price in states $k=0$ or $2n$, the firms switch to playing a "bad" equilibrium, in which both firms obtain relatively low payoffs. The bad equilibrium used in this particular game can be, for example, the worst carrot-and-stick equilibrium from Athey and Bagwell (2008).

In Athey and Bagwell's carrot-and-stick equilibria, there are two states. In the war state, both firms choose a price γ lower than r, and in the reward state, both firms charge price r. The firms begin in the war state. In the war state, if both firms choose price $\gamma < r$, the firms switch to the reward state with a probability μ, and return to the war state with the remaining probability. In the reward state, if both firms choose price r, the firms remain in the reward state with probability 1. In each period, if any firm charges a price other than the prescribed price, the firms switch to the war state with probability 1.

The off-equilibrium payoff of each firm, when the discount factor converges to 1, is bounded by $r/2-\underline{c}/2-\overline{c}/2$, regardless of the current cost profile, which is less than the efficient payoff.[14] Notice that in this application, chip strategies require no explicit communication, because types are revealed through actions.

5.2 *Taking turns*

Suppose two individuals have to perform an unpleasant duty, such as cleaning their shared apartment. This task must be performed in every single period. The cost of performing the task is $c_i=\underline{c}$ or $\overline{c}$, $i=1,2$, where $\underline{c}<\overline{c}\in(1,2)$. The single player's payoff from having the task performed is 0, and the payoff is equal to -1 otherwise. The costs are independent across individuals and Markov over time. It is efficient if the task is always performed by an individual with lower cost.

This model was studied by Leo (2015), who assumed that the costs are i.i.d. over time, but allowed them to have more than two values; for example, to be continuously distributed on $(1,2)$. He showed numerically that the total payoff achieved by the simple chip mechanism (with a sufficiently large number of chips and for the parameter values assumed in the numeric exercise) converges to an appropriately defined second-best outcome.

The approximate efficiency and incentive compatibility of the simple chip strategies in Leo's (2015) model with two possible cost values follow from our result. The efficient actions are a_{LH} is to perform or volunteer, a_{HL} is to do nothing, a_{HL} and a_{LH} are the symmetric actions, and $(a_{LL},a_{LL})=(a_{HH},a_{HH})$ is the action profile in which each individual performs the duty with probability $1/2$.[15] Finally, the action profile $(b_1^L,b_2^L)=(b_1^H,b_2^H)$ requires individual 2 to perform the task no matter what the cost profile, while the action profile $(b_2^L,b_1^L)=(b_2^H,b_1^H)$ requires individual 1 to perform the task no matter what the cost profile. The bad equilibrium is not cleaning the apartment.

[14] The bound follows from the fact that strategies are independent of the firms' costs.

[15] This action profile is feasible with a public randomization device, but does not require access to any public randomization device, since players can generate the fifty–fifty lottery by communication.

5.3 *Other versions of favor exchange*

In Section 2, we analyzed the version of the favor-exchange model introduced by Abdulkadiroğlu and Bagwell (2012). The efficiency of chip strategies for this version of the model does not follow directly from our main result (Proposition 3).[16] One can, however, formulate an alternative favor-exchange model that captures the same idea as that from Abdulkadiroglu and Bagwell, and that is covered by our more general setting in Section 3. Suppose that in each period, each of two players has some good, let us say an apple, that can be consumed by one of the players. The private type, high or low, of a player $i = 1, 2$ is the value of consuming any apple; "high" means that a player is hungry and "low" means that she is not hungry. That is, player i's utility from consuming each apple in a given period is equal to her type in that period. In each period, each player announces her type, and the player with the higher type consumes the two apples; if both players announce the same type, they consume their apples themselves. A bad repeated-game equilibrium in this setting means the players always consume their apples themselves. Now, it follows from Proposition 3 that simple chip equilibria approximate the efficient outcome.

5.4 *Dynamic cheap talk*

Suppose a sender recommends to a receiver one of two actions, say, to invest in a risky or in a safe asset. The sender is equally likely to obtain the signal that the receiver should invest in the risky asset as well as the signal that the receiver should invest in the safe asset. The sender always gets an additional payoff if the receiver invests in the risky asset; this additional payoff is higher conditional on the signal that the receiver should indeed invest in the risky asset. The efficiency requires the "right" action of the receiver, and the receiver will invest in the safe asset in the absence of any advice from the sender.

Since the game is asymmetric, our results do not apply directly. However, the approximate efficiency and incentive compatibility follow from analogous arguments. In such a simple chip equilibrium, the sender recommends the right investment and the receiver follows the sender's advice. The receiver gives a chip to the sender if she invests in the safe asset and takes a chip from the sender otherwise. The receiver always takes the risky action if the sender collects all the chips and always takes the safe action if the receiver collects all the chips.

6. Asymmetric games

So far, we have assumed the symmetry of a stage game and the symmetry of transition probabilities. In this section, we use an example to show that the efficient outcome may not be approximated in simple chip strategies in asymmetric settings. Then we introduce a slightly more general class of *random chip strategies* and show that the efficient outcome in this example can be approximated in this more general class. Random chip

[16] Indeed, the simple chip strategies are defined somewhat differently for $k = 0$ and $2n$. A player with no chips obtains a chip from the opponent only when she provides a favor.

strategies differ from simple chip strategies in that they allow for different exchange rates between favors and chips for different players. These different rates are implemented by allowing for chips and favors to be provided or transferred only with some probability. We still require the exchange rate between favors and chips to be independent of the current state (i.e., the number of chips owned by each player) as long as the play has not reached a limit state.

Consider the game in which the signal of player 1 determines which actions are efficient. This is a feature of numerous settings, including some asymmetric versions of favor-exchange models and repeated auctions. More specifically, suppose that in each period, players have an indivisible good, such as an apple, to share. If player 1's signal is high, then player 1 values the apple more; if player 1's signal is low, player 2 values the apple more. (For example, the value of the apple is constant for player 2, but varies over time for player 1.) Player 1 obtains signal high with probability θ and obtains signal low with the remaining probability. Suppose further that player 1 suggests who should consume the apple (e.g., by saying "mine" or "yours"), and player 2 can agree or disagree. If player 2 disagrees, no player is allowed to consume the apple. Otherwise, the apple is allocated according to player 1's suggestion.

The efficient outcome is attained when player 1 consumes the apple contingent on signal high and player 2 consumes the apple contingent on signal low. If $\theta = 1/2$, the efficient outcome can be approximated in equilibrium by the following simple chip strategies. Players begin with n chips each. Player 1 consumes the apple and gives player 2 a chip when player 1 says "mine." In turn, player 2 consumes the apple and gives player 1 a chip when player 1 says "yours." This is contingent on the number of chips not reaching a limit value. In a limit state, player 1 says "mine" or "yours," depending on whether she or her opponent has all the chips and independent of her signal, and player 2 agrees with this suggestion. Any off-equilibrium action triggers the phase in which no player is ever allowed to consume an apple.

By arguments analogous to those used in Section 2.1, when $\theta = 1/2$, one can show that for every $\lambda > 0$, if players' discount factor exceeds some cutoff and the initial number of chips n is large enough, then the simple chip-strategy profile with n chips is an equilibrium of the repeated game, and the total payoff in this equilibrium falls below the efficient total payoff by at most λ. (We omit the details of this proof.)

The above result, however, is no longer true when $\theta \neq 1/2$, as shown by the result below.

Proposition 6. *Suppose that $\theta \neq 1/2$. Then there exists a bound $K > 0$ such that for all $\delta < 1$, the ex ante expected total payoff of any simple chip strategies is lower than the efficient total payoff at least by K.*

Proof. Let $\theta > 1/2$. Denote by k the number of chips that player 1 has. Then the transition probability from the state with k chips to the state with $k-1$ chips is $\theta > 1/2$, while that to the state with $k+1$ chips is $1-\theta < 1/2$. Since the transition probability from the state with $2n$ chips to the state with $2n-1$ chips and from the state with 0 chips to the state with 1 chip is equal to 1, the ergodic probability distribution over states π_k, $k = 0, 1, \dots, 2n$, must satisfy the system of linear equations

$$\begin{bmatrix} 0 & 1-\theta & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 1-\theta & \cdot & & & \cdot \\ 0 & \theta & 0 & 1-\theta & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \theta & 0 & 1-\theta & 0 \\ \cdot & & & \cdot & \theta & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & \theta & 0 \end{bmatrix} \begin{bmatrix} \pi_{2n} \\ \cdot \\ \\ \cdot \\ \\ \cdot \\ \pi_0 \end{bmatrix} = \begin{bmatrix} \pi_{2n} \\ \cdot \\ \\ \cdot \\ \\ \cdot \\ \pi_0 \end{bmatrix}.$$

This system can be solved by the Gauss–Jordan method. This yields

$$\pi_{2n} = (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{2n-2}\left(\frac{1}{\theta}\right)$$
$$\cdot\left\{1 + \left(\frac{1}{\theta}\right)\left[1 + \left(\frac{1-\theta}{\theta}\right) + \cdots + \left(\frac{1-\theta}{\theta}\right)^{2n-2}\right] + (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{2n-2}\left(\frac{1}{\theta}\right)\right\}^{-1}.$$

Thus the probability of being in an inefficient state $2n$ is

$$\pi_{2n} \to_{n\to\infty} \frac{2\theta - 1}{2\theta} > 0.$$

This yields the required bound, since the efficient total payoff could be approximated in simple chip strategies only when the probability of being in an inefficient state is limited to 0. □

Remark 2. It can be shown that equilibria in simple chip strategies approximate outcomes in which $1/2\theta$ of the efficient total payoff is attained. We omit the proof of this result.

We now show that the efficient outcome can be approximated by *random chip strategies*. These strategies allow for using a public randomization device. Suppose that $\theta > 1/2$. When player 1 says "mine," she obtains the apple, but gives player 2 a chip only with probability p such that $p\theta = 1 - \theta$. Player 2 obtains the apple and gives player 1 a chip (with probability 1) when player 1 says "yours." This is contingent on the number of chips not reaching a limit value. For a limit value, the apple is allocated independently of player 1's signal. Player 1 obtains the apple with a (presumably large) probability r_{2n} if player 2 has no chips, and player 1 obtains the apple with a (presumably small) probability r_0 if player 1 has no chips. With probability $1 - \theta$, one chip is returned to the player who currently has no chips. Off the equilibrium path, no player is ever allowed to consume an apple.

Proposition 7. *For every $\lambda > 0$, there exist $\underline{\delta} < 1$ and n such that if the players' discount factor is $\delta > \underline{\delta}$, then a random chip-strategy profile with n chips is an equilibrium, and the ex ante expected total payoff in this equilibrium is no lower than the efficient total payoff by more than λ.*

Proof. The fact that the total payoff from the random chip strategies converges to the efficient total payoff (as the number of chips $2n$ diverges to ∞) follows from the ergodic theorem by the same arguments we used previously. In fact, the transition matrix is the same as that in the proof of Lemma 1 if we replace p in the matrix from that proof with $1-\theta$.

To show that incentive constraints are satisfied, we first evaluate the continuation payoff of player 1; this payoff is denoted by V_k. For $k \in \{1, \dots, 2n-1\}$, we have

$$V_k = \theta\big\{(1-\delta)A_H + \delta\big(pV_{k-1} + (1-p)V_k\big)\big\} + (1-\theta)\delta V_{k+1},$$

where A_t is the value of an apple for type t of player 1 being equal to high or low. For $k = 0$ and $2n$, we have

$$V_0 = r_0\big[(1-\delta)\theta A_H + (1-\delta)(1-\theta)A_L\big] + \theta\delta V_0 + (1-\theta)\delta V_1,$$
$$V_{2n} = r_{2n}\big[(1-\delta)\theta A_H + (1-\delta)(1-\theta)A_L\big] + \theta\delta V_{2n} + (1-\theta)\delta V_{2n-1}.$$

It follows that for the difference $\Delta_k = V_k - V_{k-1}$,

$$\Delta_k = \theta p\delta\Delta_{k-1} + \theta(1-p)\delta\Delta_k + (1-\theta)\delta\Delta_{k+1}$$

for $k = 2, \dots, 2n-1$, which yields (since $1-\theta = p\theta$)

$$\frac{\partial\Delta_k}{\partial\delta} = \frac{1}{2}\frac{\partial\Delta_{k-1}}{\partial\delta} + \frac{1}{2}\frac{\partial\Delta_{k+1}}{\partial\delta}, \tag{7}$$

where the derivatives are evaluated at $\delta = 1$. In addition, note that $\Delta_k = 0$ for $\delta = 1$.

Similarly,

$$\Delta_1 = (1-r_0)(1-\delta)\theta A_H - r_0(1-\delta)(1-\theta)A_L + (1-\theta)\delta\Delta_2 + \theta\delta(1-p)\Delta_1,$$
$$\frac{\partial\Delta_1}{\partial\delta} = \frac{-(1-r_0)\theta A_H + r_0(1-\theta)A_L}{2-2\theta} + \frac{1}{2}\frac{\partial\Delta_2}{\partial\delta}$$

and

$$\Delta_{2n} = -(1-r_{2n})(1-\delta)\theta A_H + r_{2n}(1-\delta)(1-\theta)A_L + (1-\theta)\delta\Delta_{2n-1} + \theta\delta(1-p)\Delta_{2n},$$
$$\frac{\partial\Delta_{2n}}{\partial\delta} = \frac{(1-r_{2n})\theta A_H - r_{2n}(1-\theta)A_L}{2-2\theta} + \frac{1}{2}\frac{\partial\Delta_{2n-1}}{\partial\delta}.$$

Take r_0 and r_{2n} such that

$$\frac{-(1-r_0)\theta A_H + r_0(1-\theta)A_L}{2-2\theta} = \frac{-1}{2(1+p)}A_H,$$
$$\frac{(1-r_{2n})\theta A_H - r_{2n}(1-\theta)A_L}{2-2\theta} = \frac{-1}{2(1+p)}A_L.$$

Such values of $r_0, r_{2n} \in (0,1)$ exist, since the left-hand side (LHS) falls below the RHS for $r_0 = 0$, but the LHS exceeds the RHS for $r_0 = 1$; also, the LHS exceeds the RHS for $r_{2n} = 0$, but the LHS falls below the RHS for $r_{2n} = 1$. Thus,

$$\frac{\partial \Delta_1}{\partial \delta} = \frac{-1}{2(1+p)} A_H + \frac{1}{2} \frac{\partial \Delta_2}{\partial \delta} \tag{8}$$

and

$$\frac{\partial \Delta_{2n}}{\partial \delta} = \frac{-1}{2(1+p)} A_L + \frac{1}{2} \frac{\partial \Delta_{2n-1}}{\partial \delta}. \tag{9}$$

Equations (7)–(9) imply that Δ_k for $k = 1, \ldots, 2n$, for a sufficiently large discount factor and in approximation, is a convex combination of $(1-\delta)\frac{1}{1+p} A_L$ and $(1-\delta)\frac{1}{1+p} A_H$. This in turn implies that player 1 has the right incentives. Indeed, by "reporting" high instead of low, player 1 gains A_L, but loses one chip with probability $1-p$ and loses two chips with probability p; by reporting low instead of high, player 1 loses A_H, but gains one chip with probability $1-p$ and gains two chips with probability p. □

Propositions 6 and 7 suggest a number of conclusions and conjectures. Note that the symmetry of payoffs is inessential for approximating the efficient outcome in simple chip strategies. Indeed, for $\theta = 1/2$, the simple chip strategies are (approximately) efficient, no matter by how much the value of an apple for player 1 exceeds (or by how much it falls below) that for player 2 contingent on high (respectively, low) signal. This is intuitive since neither efficiency nor incentives depends on the symmetry of gains and losses across players. What matters for the inefficiency result in Proposition 6 is the asymmetry in the frequency with which each player needs a favor (represented by $\theta \neq 1/2$). Simple chip strategies treat players' needs symmetrically and fail to counteract this asymmetry, which results in a positive frequency of being in an inefficient state in which one of the players is running out of chips. Random chip strategies restore efficiency by carefully selecting the probability of issuing chips to exactly offset the asymmetry in the frequencies of favor needs.

We conjecture that in asymmetric games with two players of two types each, random chip strategies attain approximate efficiency under assumptions that are similar to those in Section 3. Note also that the inefficiency of simple chip strategies, represented in the limit by the probability $(2\theta - 1)/2\theta$ of being in an inefficient state, monotonically vanishes as the asymmetry disappears (i.e., as $\theta \to 1/2$). We conjecture that this property of vanishing inefficiency of simple chip strategies is true in a more general setting, such as that studied in Section 3.

7. More than two types, more than two players

The case of more than two types turns out to be even more challenging. We demonstrate below that even random chip strategies may not approximate efficient outcomes. Consider the model in which two players have an apple in each period. Suppose that there are three possible, equally likely values of the apple, 1, 2, and 3, that are i.i.d. over time and independent across players.

In symmetric random chip strategies,[17] each player begins the game with n chips. If player i announces type $t \in \{1, 2, 3\}$ that is higher than type $s \in \{1, 2, 3\}$ announced by player j, then player i consumes the apple and gives player j a chip, but only with some probability that depends on the two announcements. That is, if $t > s$, then player i gives player j a chip with probability $p_{s,t}$. If the players announce the same type, then the state (i.e., the number of chips owned by each player) is unchanged and each player consumes the apple with probability $1/2$. If a player reaches 0 chips, the player's opponent consumes the apple with probability q (presumably large) and gives the player a chip with probability p_r.

The efficient outcome is attained when, in each period, a player with higher type consumes the apple. So the total efficient payoff is

$$\frac{5}{9} \cdot 3 + \frac{3}{9} \cdot 2 + \frac{1}{9} \cdot 1 = \frac{22}{9}.$$

We show that the following statement holds.

Proposition 8. *There exist $\underline{\delta} < 1$ and $V < 22/9$ such that for all $\delta > \underline{\delta}$, if the payoff of a random chip-strategy profile exceeds V, then that profile is not incentive compatible.*

The proof of this result can be found in Appendix B.

Proposition 8 concerns strategies such that the probabilities $p_{s,t}$ may depend on the number of states (that is, they may depend on n), but are not allowed to depend on the current state (i.e., the number of chips owned by each player). One may wonder whether the efficient outcome could be attained if we allow them to depend on the current state. We conjecture, but have not proved formally, that the answer is positive. The intuition is that we could imitate the strategies used in our companion paper (Olszewski and Safronov forthcoming), which in turn imitate the mechanism from Arrow (1979) and d'Aspremont and Gérard-Varet (1979).

The case of more than two players raises yet an additional difficulty, namely, groups of players may provide favors to other groups of players, and the contribution of different players from the former groups, as well as the benefits to the players from the latter groups, may be different. This must be reflected in the transition of chips among the players.

Overcoming these difficulties calls for more involved strategies, which is somewhat in conflict with the objective of providing a positive model of playing dynamic games. In the companion paper (Olszewski and Safronov forthcoming), we suggest a more complicated version of chip strategies, constructed by imitating the mechanisms in Arrow (1979) and d'Aspremont and Gérard-Varet (1979), which approximate the efficient outcomes for a large class of games (even with more than two players).

Remark 3. An inspection of the proof of Proposition 8 shows that the only equilibrium in simple chip strategies is the repetition of the bad stage-game equilibrium, in which each player consumes the apple with probability 1/2, independent of players'

[17] Since the game is symmetric, assuming the symmetry of strategies is without loss of generality.

types, whereas some random chip strategies attain a higher fraction of the maximum total surplus.

Appendix A

The simple chip-strategy profile has $4(2n+1)-6$ states. Each state is described by a number $k \in \{0, 1, \dots, 2n\}$ and the profile of players' types $t = (t_1, t_2) \in \{L, H\}^2$ as reported in the previous period, with the exception that $k = 0$ implies that $t = (L, H)$ and $k = 2n$ implies that $t = (H, L)$. The number k stands for player 1 having k chips.

A.1 *Continuation payoffs*

As in Section 2.1.2, we analyze players' continuation payoffs using the average cost optimality equations. Assuming that both players play the prescribed strategies, we analyze the continuation payoffs of player 1; the continuation payoff (of player 1) in state k, t is denoted by $V_{k,t}$. By the symmetry of our model, the continuation payoff of player 2 can be analyzed analogously. These continuation payoffs are computed before the players learn about their types in the current period. For $k \in \{1, \dots, 2n-1\}$, we have

$$\begin{aligned} V_{k,L,H} = {} & (1-\delta)\bigl\{\varphi u_1(a_{LH}, a_{HL}, L, H) + \mu u_1(a_{LL}, a_{LL}, L, L) \\ & + \nu u_1(a_{HH}, a_{HH}, H, H) + \chi u_1(a_{HL}, a_{LH}, H, L)\bigr\} \\ & + \delta\{\varphi V_{k-1,L,H} + \mu V_{k,L,L} + \nu V_{k,H,H} + \chi V_{k+1,H,L}\}, \end{aligned} \tag{10}$$

$$\begin{aligned} V_{k,L,L} = {} & (1-\delta)\bigl\{\zeta u_1(a_{LH}, a_{HL}, L, H) + p_L^L u_1(a_{LL}, a_{LL}, L, L) \\ & + p_H^L u_1(a_{HH}, a_{HH}, H, H) + \zeta u_1(a_{HL}, a_{LH}, H, L)\bigr\} \\ & + \delta\bigl\{\zeta V_{k-1,L,H} + p_L^L V_{k,L,L} + p_H^L V_{k,H,H} + \zeta V_{k+1,H,L}\bigr\}, \end{aligned} \tag{11}$$

$$\begin{aligned} V_{k,H,H} = {} & (1-\delta)\bigl\{\eta u_1(a_{LH}, a_{HL}, L, H) + p_L^H u_1(a_{LL}, a_{LL}, L, L) \\ & + p_H^H u_1(a_{HH}, a_{HH}, H, H) + \eta u_1(a_{HL}, a_{LH}, H, L)\bigr\} \\ & + \delta\bigl\{\eta V_{k-1,L,H} + p_L^H V_{k,L,L} + p_H^H V_{k,H,H} + \eta V_{k+1,H,L}\bigr\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} V_{k,H,L} = {} & (1-\delta)\bigl\{\chi u_1(a_{LH}, a_{HL}, L, H) + \mu u_1(a_{LL}, a_{LL}, L, L) \\ & + \nu u_1(a_{HH}, a_{HH}, H, H) + \varphi u_1(a_{HL}, a_{LH}, H, L)\bigr\} \\ & + \delta\{\chi V_{k-1,L,H} + \mu V_{k,L,L} + \nu V_{k,H,H} + \varphi V_{k+1,H,L}\}. \end{aligned} \tag{13}$$

For $k = 0$ and $2n$, we have

$$\begin{aligned} V_{0,L,H} = {} & (1-\delta)\bigl\{\varphi u_1\bigl(b_2^H, b_1^H, L, H\bigr) + \mu u_1\bigl(b_2^L, b_1^L, L, L\bigr) \\ & + \nu u_1\bigl(b_2^H, b_1^H, H, H\bigr) + \chi u_1\bigl(b_2^L, b_1^L, H, L\bigr)\bigr\} \\ & + \delta\{\varphi V_{1,L,H} + \mu V_{1,L,L} + \nu V_{1,H,H} + \chi V_{1,H,L}\}, \end{aligned}$$

$$\begin{aligned}V_{2n,H,L} = (1-\delta)\{&\chi u_1(b_1^L, b_2^L, L, H) + \mu u_1(b_1^L, b_2^L, L, L) \\ &+ \nu u_1(b_1^H, b_2^H, H, H) + \varphi u_1(b_1^H, b_2^H, H, L)\} \\ &+ \delta\{\chi V_{2n-1,L,H} + \mu V_{2n-1,L,L} + \nu V_{2n-1,H,H} + \varphi V_{2n-1,H,L}\}.\end{aligned}$$

A.1.1 *Payoff efficiency of prescribed strategies* We can now demonstrate the approximate efficiency of the prescribed strategies.

Proposition 9. *For every $\lambda > 0$, there exist $\underline{\delta} < 1$ and n_0 such that for every $\delta > \underline{\delta}$ and $n > n_0$, we have that*

$$\left|V_{k,t} - \frac{v}{2}\right| < \lambda$$

for all $k = 0, \dots, 2n$ and $t \in \{L, H\}^2$ ($t = (L, H)$ when $k = 0$ and $t = (H, L)$ when $k = 2n$).

Proof. The strategies induce a stochastic Markov chain over states k, t. It is easy to check that in $2n + 2$ periods, each state is reached from any other state with positive probability. Thus, by the ergodic theorem there exists a probability distribution over states $\{\pi_{k,t} : k, t\}$ such that the probability of being in state k, t after a sufficiently large number of periods is arbitrarily close to $\pi_{k,t}$. This probability distribution is an eigenvector of the transition matrix corresponding to eigenvalue 1.

To estimate the probabilities $\{\pi_{k,t} : k, t\}$, consider first an auxiliary Markov chain with $8n$ states in which, instead of the states with $k = 0$ or $k = 2n$, we have four states $\{0, 2n\}$, t for $t \in \{L, H\}^2$. One should think of the "number" $\{0, 2n\}$ as being between the 1 and $2n - 1$. The transitions are as in the Markov chain induced by our strategies, except states $\{0, 2n\}$, t. In state $\{0, 2n\}$, (L, H), the chain transits to 1, (H, L) with probability χ, transits to $2n - 1$, (L, H) with probability φ, and transits to $\{0, 2n\}$, (L, L) and $\{0, 2n\}$, (H, H) with probabilities μ and ν, respectively. In state $\{0, 2n\}$, (H, L), the chain transits to 1, (H, L) with probability φ, transits to $2n - 1$, (L, H) with probability χ, and transits to $\{0, 2n\}$, (L, L) and $\{0, 2n\}$, (H, H) with probabilities μ and ν, respectively. In state $\{0, 2n\}$, (L, L), the chain transits to 1, (H, L) and to $2n-1$, (L, H) with probability ζ each, transits to $\{0, 2n\}$, (L, L) with probability p_L^L, and transits to $\{0, 2n\}$, (H, H) with probability p_H^L. In state $\{0, 2n\}$, (H, H), the chain transits to 1, (H, L) and to $2n-1$, (L, H) with probability η each, transits to $\{0, 2n\}$, (L, L) with probability p_L^H, and transits to $\{0, 2n\}$, (H, H) with probability p_H^H.

The ergodic theorem still applies to the new chain. For any $t = (t_1, t_2) \in \{H, L\}^2$, this new chain is completely symmetric across states k, where $k = 1, \dots, 2n - 1, \{0, 2n\}$, and, therefore, for any k, the sum of ergodic probabilities $\overline{\pi}_{k,t}$ over $t \in \{L, H\}^2$ is equal to $1/2n$.

Claim 1. *We show that ergodic probabilities $\overline{\pi}_{k,t}$ are proportional to the corresponding ergodic probabilities of the Markov process over type profiles. More precisely, $\overline{\pi}_{k,(L,L)} = q_L/2n$, $\overline{\pi}_{k,(H,H)} = q_H/2n$, and $\overline{\pi}_{k,(L,H)} = \overline{\pi}_{k,(H,L)} = q/2n = (1 - q_H - q_L)/4n$ for all k.*

Indeed, the current state can be equal to k, (H,H) when (a) the previous state was equal to k, (L,L) and then the Markov process transits to k, (H,H) with probability p_H^L, (b) the previous state was equal to k, (H,H) and then the Markov process transits to k, (H,H) with probability p_H^H, (c) the previous state was equal to $k-1$, (H,L) or $k+1$, (L,H) and in each of the two cases, the Markov process transits to k, (H,H) with probability ν. This yields

$$\overline{\pi}_{k,(H,H)} = \nu\overline{\pi}_{k,(L,H)} + p_H^L\overline{\pi}_{k,(L,L)} + p_H^H\overline{\pi}_{k,(H,H)} + \nu\overline{\pi}_{k,(H,L)}.$$

Similarly,

$$\overline{\pi}_{k,(L,L)} = \mu\overline{\pi}_{k,(L,H)} + p_L^L\overline{\pi}_{k,(L,L)} + p_L^H\overline{\pi}_{k,(H,H)} + \mu\overline{\pi}_{k,(H,L)}.$$

In turn, the current state can be equal to k, (H,L) only when the previous state was $k-1$, t. (Recall that $k=\{0,2n\}$ is a number that is one above $k=2n-1$ and one below $k=1$.) This happens with probability ζ if $t=(L,L)$, with probability η if $t=(H,H)$, with probability φ if $t=(H,L)$, and with probability χ if $t=(L,H)$. This yields

$$\overline{\pi}_{k,(H,L)} = \chi\overline{\pi}_{k-1,(L,H)} + \zeta\overline{\pi}_{k-1,(L,L)} + \eta\overline{\pi}_{k-1,(H,H)} + \varphi\overline{\pi}_{k-1,(H,L)}.$$

Similarly,

$$\overline{\pi}_{k,(L,H)} = \varphi\overline{\pi}_{k+1,(L,H)} + \zeta\overline{\pi}_{k+1,(L,L)} + \eta\overline{\pi}_{k+1,(H,H)} + \chi\overline{\pi}_{k+1,(H,L)}.$$

By the symmetry of our auxiliary Markov chain, we must have that $\overline{\pi}_{k,t}$ does not depend on k and $\overline{\pi}_{k,(L,H)} = \overline{\pi}_{k,(H,L)}$. Therefore, such a solution $\overline{\pi}_{k,t}$ of this system of four equations must coincide with the solution of the equations from Proposition 3. This gives the claim.

We now return to the analysis of original chain induced by our strategies.

Claim 2. *We show that the ratio $\pi_{k,t}/\pi_{1,(L,H)}$, for $k=0,1$ and all t, is independent of n, and so is the ratio $\pi_{k,t}/\pi_{2n-1,(H,L)}$ for $k=2n-1,2n$ and all t.*

Similarly to the proof of the previous claim, we obtain the "ergodic" equations

$$\begin{aligned}
\pi_{0,(L,H)} &= \varphi\pi_{1,(L,H)} + \zeta\pi_{1,(L.L)} + \eta\pi_{1,(H,H)} + \chi\pi_{1,(H,L)},\\
\pi_{1,(H,H)} &= \nu\pi_{1,(L,H)} + p_H^L\pi_{1,(L,L)} + p_H^H\pi_{1,(H,H)} + \nu\pi_{1,(H,L)} + \nu\pi_{0,(L,H)},\\
\pi_{1,(L,L)} &= \mu\pi_{1,(L,H)} + p_L^L\pi_{1,(L,L)} + p_L^H\pi_{1,(H,H)} + \mu\pi_{1,(H,L)} + \mu\pi_{0,(L,H)},
\end{aligned}$$

and

$$\pi_{1,(H,L)} = \chi\pi_{0,(L,H)}.$$

Divide each equation by $\pi_{1,(L,H)}$. This yields a system of equations with variables $\pi_{k,t}/\pi_{1,(L,H)}$ for $k=0,1$ and $t\neq(L,H)$. This system has a unique solution, independent of n, and this yields the first part of the claim. Obviously, the second part can be proved in an analogous way.

Claim 3. *We show that ergodic probabilities $\pi_{k,t}$ are proportional to the corresponding ergodic probabilities of the Markov process over type profiles from Proposition 3; that is,*

$$\pi_{k,(L,L)}/q_L = \pi_{k,(H,H)}/q_H = \pi_{k,(L,H)}/q = \pi_{k,(H,L)}/q$$

for $k = 2, \dots, 2n-2$.

Indeed, notice first that by symmetry $\pi_{2,(H,L)} = \pi_{2n-2,(L,H)}$. Next observe that except states 2, (H, L) and $2n-2$, (L, H), the ergodic equations for states k, t, where $k = 2, \dots, 2n-2$, include no probability $\pi_{0,d}$, $\pi_{1,d}$, $\pi_{2n-1,d}$, or $\pi_{2n,d}$, for any $d \in \{L, H\}^2$. This means that for any given $\pi_{2,(H,L)}$ and $\pi_{2n-2,(L,H)}$, the remaining probabilities $\pi_{k,t}$, where $k = 2, \dots, 2n-2$, are determined by these ergodic equations.

The original chain induced by our strategies can be obtained from the chain used in Claim 1 by renaming state $\{0, 2n\}$, (L, H) as 0, (L, H), renaming state $\{0, 2n\}$, (H, L) as $2n$, (H, L), changing appropriately the transition probabilities in these two states, and removing states $\{0, 2n\}$, (L, L) and $\{0, 2n\}$, (H, H). This means that for any given $\overline{\pi}_{2,(H,L)} = \overline{\pi}_{2n-2,(L,H)}$, the remaining probabilities $\overline{\pi}_{k,t}$, where $k = 2, \dots, 2n-2$, are determined by the same ergodic equations as in the case of the original chain induced by our strategies. In addition, since by Claim 1, $\overline{\pi}_{k,(L,L)}/q_L = \overline{\pi}_{k,(H,H)}/q_H = \overline{\pi}_{k,(L,H)}/q = \overline{\pi}_{k,(H,L)}/q = \overline{\pi}_{2,(H,L)}/q = \overline{\pi}_{2n-2,(L,H)}/q$ for $k = 2, \dots, 2n-2$, the same must be true when $\overline{\pi}$s are replaced with πs. □

To complete the proof of Proposition 9, notice that the ergodic equations for 2, (H, L) and $2n-2$, (L, H) imply that $\pi_{2,(H,L)}$ is a weighted average of $\pi_{1,t}$ for $t \in \{L, H\}^2$ and $\pi_{2n-2,(L,H)}$ is a weighted average of $\pi_{2n-1,t}$ for $t \in \{L, H\}^2$. This together with Claims 2 and 3 implies that any two probabilities $\pi_{k,t}$, where $k = 0, 1, 2n-1, 2n$ and $t \in \{L, H\}^2$, are proportional to $\pi_{2,(H,L)}$, and the coefficients of this proportionality depend only on p, that is, are independent of n. Thus, all probabilities converge to zero as n diverges to infinity.

Since our strategies are inefficient only in states 0, (L, H) and $2n$, (H, L), this means that as $1-\delta$ is sufficiently close to 0, the inefficiency is approximately proportional to the sum of the ergodic probabilities of states 0, (L, H) and $2n$, (H, L). Therefore, it disappears when n diverges to infinity. By symmetry of the strategies, the payoff of each player is close to half of the efficient payoff.

A.1.2 *Incentive constraints* We now turn to verifying the incentive constraints. The constraints for the states in which $k = 0$ or $2n$ are immediate, since no player wants to trigger the bad equilibrium, the action profiles b^L and b^H are independent of the report of the player with no chips, and the player with all chips can choose between b^L and b^H. Thus, consider the states in which k is such that $0 < k < 2n$. For each of these $8n-4$ states, there are two constraints to check: one for the player with type L and one for the player with type H.

Suppose that the play is in state k, t. We check the constraints for player 1. (By symmetry, this implies the constraints for player 2.) Consider first the effect of playing the prescribed strategies, compared to a deviation, on the state in the following period. If

$t_1 = L$, by reporting honestly, player 1 will be in a state with one fewer chip compared to reporting state H, but the distribution over the next period's type profiles $t^{+1} \in \{L, H\}^2$ will be exactly the same under the two possible reports. Similarly, if $t_1 = H$, player 1 will be in a state with one more chip by reporting honestly compared to reporting L, but the distribution over $t^{+1} \in \{L, H\}^2$ will be exactly the same. This is so because the state in the next period depends on the actual type of the player, not the reported type.

Let $\Delta_{k,t} := V_{k,t} - V_{k-1,t}$ for all k and t. The continuation payoff of player 1 of type L with k chips decreases in expectation when reporting truthfully compared to lying about her type. This loss in continuation payoff is a weighted average of $\Delta_{k,(L,H)}$ and $\Delta_{k+1,(L,L)}$, with weights depending on the type profile t^{-1} in the previous period. (For example, for $t^{-1} = (L, L)$, the loss of continuation payoff equals $\Delta_{k,(L,H)} \cdot \zeta/(\zeta + p_L^L) + \Delta_{k+1,(L,L)} \cdot p_L^L/(\zeta + p_L^L)$.) Additionally, player 1 of type H gains by reporting truthfully (in terms of the continuation payoff) a weighted average of $\Delta_{k,(H,H)}$ and $\Delta_{k+1,(H,L)}$. In turn, when reporting truthfully rather than lying, the player of type L gains expression (6) for $t_1 = L$ as a payoff in the current period and the player of type H loses expression (6) for $t_1 = H$.

Proposition 10. *For all $k = 1, \ldots, 2n-1$ and t, player 1 has incentives to report her type honestly.*

Proof. We first establish the relationships between various Δs. For $k = 2, \ldots, 2n-1$, by applying (10)–(13), we obtain

$$\Delta_{k,L,H} = \delta\{\varphi\Delta_{k-1,L,H} + \mu\Delta_{k,L,L} + \nu\Delta_{k,H,H} + \chi\Delta_{k+1,H,L}\}, \tag{14}$$

$$\Delta_{k,L,L} = \delta\bigl\{\zeta\Delta_{k-1,L,H} + p_L^L\Delta_{k,L,L} + p_H^L\Delta_{k,H,H} + \zeta\Delta_{k+1,H,L}\bigr\}, \tag{15}$$

$$\Delta_{k,H,H} = \delta\bigl\{\eta\Delta_{k-1,L,H} + p_L^H\Delta_{k,L,L} + p_H^H\Delta_{k,H,H} + \eta\Delta_{k+1,H,L}\bigr\}, \tag{16}$$

and

$$\Delta_{k,H,L} = \delta\{\chi\Delta_{k-1,L,H} + \mu\Delta_{k,L,L} + \nu\Delta_{k,H,H} + \varphi\Delta_{k+1,H,L}\}. \tag{17}$$

In turn, for $k = 1$ and $t = (L, H)$, and $k = 2n$ and $t = (H, L)$, we obtain

$$\Delta_{1,L,H} = (1-\delta)A + \delta\{-\varphi\Delta_{1,L,H} + \chi\Delta_{2,H,L}\} \tag{18}$$

and

$$\Delta_{2n,H,L} = (1-\delta)B + \delta\{\chi\Delta_{2n-1,L,H} - \varphi\Delta_{2n,H,L}\}. \tag{19}$$

Finally, we must also introduce the terms $\Delta_{1,H,H}$ and $\Delta_{2n,L,L}$ because, in states with $k = 1$, chip player 1 of type H may deviate by reporting L, and in states with $k = 2n-1$, chips player 1 of type L may deviate by reporting H. This results in moving to a state with 0 or $2n$ chips, respectively, but at profile $t = (H, H)$ or (L, L). We have that

$$\Delta_{1,H,H} = \delta\{-\eta\Delta_{1,L,H} + \eta\Delta_{2,H,L}\} + (1-\delta)\cdot A'$$

and

$$\Delta_{2n,L,L} = \delta\{\zeta\Delta_{2n-1,L,H} - \zeta\Delta_{2n,H,L}\} + (1-\delta)\cdot B'. \tag{20}$$

For $\delta = 1$, this system of linear equations is satisfied by all Δs being equal to 0. For $\delta < 1$, we evaluate Δs in approximation by referring to the implicit function theorem. By taking the derivatives of the equations for Δs with respect to δ, and plugging in $\delta = 1$ and $\Delta_{k,t} = 0$ for all k and t, we obtain a system of equations for the derivatives of Δs at $\delta = 1$. That is, if we replace each $\Delta_{k,t}$ by its derivative $\partial\Delta_{k,t}/\partial\delta$, then our system of linear equations must be satisfied for $\delta = 1$, and $(1-\delta)A$ and $(1-\delta)B$ must be replaced with $-A$ and $-B$, respectively. We now show how to solve this new system of linear equations of variables $\partial\Delta_{k,t}/\partial\delta$. This enables us to evaluate $\Delta_{k,t} \approx (1-\delta)\cdot(-\partial\Delta_{k,t}/\partial\delta)$. (In particular, we see that the solution of this new system is unique, which validates the use of the implicit function theorem as well as guarantees that all Δs being equal to 0 is a unique solution of the system for $\delta = 1$.)

First, add together the equations for $\partial\Delta_{k,t}/\partial\delta$'s corresponding to (14)–(17), with weights equal to the ergodic probabilities of stable distribution in Proposition 3. That is, one sums up the equations for $\partial\Delta_{k,t}/\partial\delta$ corresponding to (14) and (17) with weight q each, the equations for $\partial\Delta_{k,t}/\partial\delta$ corresponding to (15) with weight q_L, and the equations for $\partial\Delta_{k,t}/\partial\delta$ corresponding to (16) with weight q_H. As a result, we obtain

$$\begin{aligned} &q\cdot\partial\Delta_{k,L,H}/\partial\delta + q_L\cdot\partial\Delta_{k,L,L}/\partial\delta + q_H\cdot\partial\Delta_{k,H,H}/\partial\delta + q\cdot\partial\Delta_{k,H,L}/\partial\delta \\ &\quad = q\cdot\partial\Delta_{k-1,L,H}/\partial\delta + q_L\cdot\partial\Delta_{k,L,L}/\partial\delta + q_H\cdot\partial\Delta_{k,H,H}/\partial\delta + q\cdot\partial\Delta_{k+1,H,L}/\partial\delta. \end{aligned}$$

This yields

$$\partial\Delta_{k,L,H}/\partial\delta + \partial\Delta_{k,H,L}/\partial\delta = \partial\Delta_{k-1,L,H}/\partial\delta + \partial\Delta_{k+1,H,L}/\partial\delta.$$

This equation can also be expressed as $\partial\Delta_{k,H,L}/\partial\delta - \partial\Delta_{k-1,L,H}/\partial\delta = \partial\Delta_{k+1,H,L}/\partial\delta - \partial\Delta_{k,L,H}/\partial\delta$, which means that the value of $\partial\Delta_{k,H,L}/\partial\delta - \partial\Delta_{k-1,L,H}/\partial\delta$ is the same for all k. To simplify notation, denote this value by ρ.

Subtracting the equations for $\partial\Delta_{k,t}/\partial\delta$ corresponding to (14) from that corresponding to (17), we obtain that

$$\partial\Delta_{k,H,L}/\partial\delta - \partial\Delta_{k,L,H}/\partial\delta = \frac{2(\varphi-\chi)}{1+(\varphi-\chi)}\rho,$$

from which we derive that

$$\partial\Delta_{k+1,H,L}/\partial\delta - \partial\Delta_{k,H,L}/\partial\delta = \partial\Delta_{k+1,L,H}/\partial\delta - \partial\Delta_{k,L,H}/\partial\delta = \frac{1-(\varphi-\chi)}{1+(\varphi-\chi)}\rho. \tag{21}$$

The last equation shows that the difference between $\partial\Delta_{k,t}/\partial\delta$s for two consecutive values of k, given type profiles H, L or L, H, is independent of k. The difference between $\partial\Delta_{k,t}/\partial\delta$s for two consecutive values of k, given type profiles L, L or H, H is equal to that given H, L or L, H by the equations for $\partial\Delta_{k,t}/\partial\delta$ corresponding to (15) or (16). In particular, this means that the values of $\partial\Delta_{k,t}/\partial\delta$ change linearly with k.

We can now find the values of $\partial\Delta_{k,t}/\partial\delta$ explicitly by using the equations for the states with extreme numbers of chips $k=1$ and $2n$. By the equation corresponding to (18),

$$\partial\Delta_{1,L,H}/\partial\delta = -A + \{-\varphi\partial\Delta_{1,L,H}/\partial\delta + \chi\partial\Delta_{2,H,L}/\partial\delta\},$$

and since $\partial\Delta_{2,H,L}/\partial\delta - \partial\Delta_{1,L,H}/\partial\delta = \rho$, this is equivalent to

$$\partial\Delta_{1,L,H}/\partial\delta = \frac{-A+\chi\rho}{1+(\varphi-\chi)}. \tag{22}$$

Similarly by the equation corresponding to (19), as $\partial\Delta_{2n,H,L}/\partial\delta - \partial\Delta_{2n-1,L,H}/\partial\delta = \rho$, we obtain that

$$\partial\Delta_{2n,H,L}/\partial\delta = \frac{-B-\chi\rho}{1+(\varphi-\chi)}. \tag{23}$$

Applying $2n-2$ times (21), we have that

$$\partial\Delta_{2n-1,L,H}/\partial\delta = \frac{-A+\chi\rho}{1+(\varphi-\chi)} + (2n-2)\frac{1-(\varphi-\chi)}{1+(\varphi-\chi)}\rho$$

and

$$\partial\Delta_{2n,H,L}/\partial\delta = \frac{-A+\chi\rho}{1+(\varphi-\chi)} + (2n-2)\frac{1-(\varphi-\chi)}{1+(\varphi-\chi)}\rho + \rho. \tag{24}$$

Combining (24) and (23), we obtain that

$$\rho = \frac{A-B}{1+\varphi+\chi+(2n-2)\big(1-(\varphi-\chi)\big)}.$$

Using this value of ρ, one can find all $\partial\Delta_{k,t}/\partial\delta$s for profiles (L,H) and (H,L) by starting with (22) or (23), respectively, and then recursively applying (21). So, one can find the (approximate) values of $\Delta_{k,t}$ for profiles (L,H) and (H,L). In particular, we can immediately see that $\frac{\Delta_{k,t}}{1-\delta}$ for $k=1,\dots,2n$ and types H, L and L, H are weighted averages of $\frac{A}{1+(\varphi-\chi)}$ and $\frac{B}{1+(\varphi-\chi)}$, and by (15) or (16), so are $\frac{\Delta_{k,t}}{1-\delta}$ for $k=2,\dots,2n-1$ and types L, L and H, H. The values of $\frac{\Delta_{k,t}}{1-\delta}$ for $k=1$ are the closest to $\frac{A}{1+(\varphi-\chi)}$, and the values of $\frac{\Delta_{k,t}}{1-\delta}$ for $k=2n$ are the closest to $\frac{B}{1+(\varphi-\chi)}$.

We can now show that all incentive constraints are satisfied for every given n, provided the discount factor is large enough.[18] By Assumption I from Section 3, the observation from the previous paragraph shows that player 1 has incentives to report her type truthfully, except in two cases: (i) when her type is L and she has $2n-1$ chips; (ii) when her type is H and she has one chip. These two cases are exceptional because the analysis of incentives involves $\Delta_{2n,(L,L)}$ and $\Delta_{1,(H,H)}$, respectively. So they must be considered separately.

Consider case (i). By deviating and reporting H, player 1 gains, compared to reporting truthfully, a weighted average of $\Delta_{2n-1,(L,H)}$ and $\Delta_{2n,(L,L)}$ (with weights depending

[18]For larger values of n, the threshold for the discount factor above which the incentive constraints are satisfied is typically larger.

on previous type profile t^{-1}), but loses expression (6) for $t_1 = L$ and previous type profile t^{-1}. By Assumption I, $\Delta_{2n-1,(L,H)}$ is smaller than the loss, and by Assumption II together with (20), $\Delta_{2n,(L,L)}$ is smaller than the loss, which prevents player 1 from deviating. Case (ii) follows from analogous arguments. □

Appendix B

In this appendix, we prove Proposition 8. Recall that the strategies considered in that proposition specify a (presumably large) probability q such that when a player has no chips, the player's opponent consumes the apple with this probability q. Throughout the proof, we assume that $q = 1$, which simplifies notation; we comment at the end regarding why allowing for $q < 1$ does not affect the result.

Denote by V_k the continuation payoff of player 1 with k chips. Then V_k, $0 < k < 2n$, is equal to

$$\begin{aligned}
V_k = {} & \frac{1}{3}\Big[1(1-\delta)\frac{1}{6} + \delta\Big(\frac{1}{3}V_k + \frac{1}{3}\big[p_{1,2}V_{k+1} + (1-p_{1,2})V_k\big] \\
& + \frac{1}{3}\big[p_{1,3}V_{k+1} + (1-p_{1,3})V_k\big]\Big)\Big] \\
& + \frac{1}{3}\Big[2(1-\delta)\frac{3}{6} + \delta\Big(\frac{1}{3}\big[p_{1,2}V_{k-1} + (1-p_{1,2})V_k\big] \\
& + \frac{1}{3}V_k + \frac{1}{3}\big[p_{2,3}V_{k+1} + (1-p_{2,3})V_k\big]\Big)\Big] \\
& + \frac{1}{3}\Big[3(1-\delta)\frac{5}{6} + \delta\Big(\frac{1}{3}\big[p_{1,3}V_{k-1} + (1-p_{1,3})V_k\big] \\
& + \frac{1}{3}\big[p_{2,3}V_{k-1} + (1-p_{2,3})V_k\big] + \frac{1}{3}V_k\Big)\Big].
\end{aligned}$$

The value function V_k is represented as the sum of three terms; each term corresponds to one of the three player 1 types 1, 2, and 3. For example, the first of these terms represents the case when the type of player 1 is equal to 1. In this case, player 1 values the apple at 1 and obtains it with probability 1/6. With probability 1/3, player 2 is also of type 1 and, therefore, the number of chips remains the same. With probability 1/3, player 2's type is 2 and then player 1 obtains a chip with probability $p_{1,2}$. Finally, with probability 1/3, player 2's type is 3, and then player 1 obtains a chip with probability $p_{1,3}$.

We derive recursive equations for $V_{k+1} - V_k$, for $k = 0, \dots, 2n-1$, estimate $(V_1 - V_0)/(V_{2n} - V_{2n-1})$, and by referring to this estimate, we conclude that incentive constraints cannot be satisfied for all k.

The expression above can be simplified to

$$V_k = (1-\delta)\frac{22}{18} + \delta\big[pV_{k-1} + (1-2p)V_k + pV_{k+1}\big], \tag{25}$$

where

$$p = \frac{1}{9}(p_{1,2} + p_{1,3} + p_{2,3}).$$

For $k = 2n$, one has

$$V_{2n} = (1-\delta)2 + \delta\big[p_r V_{2n-1} + (1-p_r)V_{2n}\big].$$

Indeed, if a player has all the chips, she obtains the apple, no matter what the types, which yields the expected payoff of 2. With probability p_r, the opponent obtains a chip back, and with the remaining probability, the distribution of chips remains unaltered.

When δ goes to 1,

$$V_{2n} = \frac{(1-\delta)2 + \delta p_r V_{2n-1}}{(1-\delta)(1-p_r) + p_r} = \frac{(1-\delta)2}{p_r} + \left[1 - \frac{(1-\delta)}{p_r}\right]V_{2n-1} + o(1-\delta), \tag{26}$$

where $o(1-\delta)$ stands for an expression that goes to 0 (when δ goes to 1) faster than $1-\delta$. To check the second equality, multiply it (omitting term $o(1-\delta)$) by $(1-\delta)(1-p_r)+p_r$ and remove each term containing $(1-\delta)^2$. Similarly, for $k=0$, one has

$$V_0 = \left[1 - \frac{(1-\delta)}{p_r}\right]V_1 + o(1-\delta). \tag{27}$$

From now on, we omit all terms $o(1-\delta)$, that is, all equalities and equations should be understood as holding up to such a term. We now derive the recursive formula

$$V_k = (1-\delta)\alpha_k + \big[1 - (1-\delta)\beta_k\big]V_{k+1} \tag{28}$$

for some coefficients α_k and β_k; these coefficients are also determined. By (27), this formula holds for $k = 0$ with $\alpha_0 = 0$ and $\beta_0 = 1/p_r$. Assume that it holds for k. Plug the expression for V_k from (28) into (25) with k replaced with $k+1$ to compute that

$$V_{k+1} = (1-\delta)\left(\frac{22}{18p} + \alpha_k\right) + \left[1 - (1-\delta)\left(\frac{1}{p} + \beta_k\right)\right]V_{k+2},$$

which gives the recursive equations

$$\alpha_{k+1} = \alpha_k + \frac{22}{18p},$$

$$\beta_{k+1} = \beta_k + \frac{1}{p}.$$

Since $\alpha_0 = 0$ and $\beta_0 = 1/p_r$,

$$\alpha_{2n-1} = (2n-1)\frac{22}{18p}$$

and

$$\beta_{2n-1} = \frac{1}{p_r} + (2n-1)\frac{1}{p}.$$

By using (28) for $k = 2n-1$ and (26), compute that

$$V_{2n} = \frac{2s + (2n-1)\dfrac{22}{18}}{2s + (2n-1)}, \tag{29}$$

where

$$s = \frac{p}{p_r}.$$

Thus, V_{2n} is a weighted average of 22/18—the efficient per-player payoff—and 1—the expected per-player payoff in inefficient states (in which the player with $2n$ chips obtains on average the payoff of $(1/3)1 + (1/3)2 + (1/3)3 = 2$, and the opponent obtains the payoff of 0). To approximate the efficient outcome, s must be much lower than $2n$. (Recall that the probabilities $p_{s,t}$ are allowed to depend on n.)

First, estimate $V_{k+1} - V_k$, the value of an extra chip for a player with k chips. By (28),

$$V_{k+1} - V_k = -(1-\delta)\alpha_k + (1-\delta)\beta_k V_{k+1}.$$

It also follows from (28) that V_{k+1} for $k = 0, \ldots, 2n-1$ is equal to (up to a term of order $(1-\delta)$) the same weighted average as V_{2n} in (29). By plugging in this weighted average for V_{k+1} and plugging in the formulas for α_k, β_k, we obtain that

$$V_{k+1} - V_k = -(1-\delta)k\frac{22}{18p} + (1-\delta)\left[\frac{1}{p_r} + k\frac{1}{p}\right]\frac{2s + (2n-1)\dfrac{22}{18}}{2s + (2n-1)}. \tag{30}$$

Thus, $V_{k+1} - V_k$ is linear in k. At $k = 0$, one has

$$V_1 - V_0 = (1-\delta)\frac{1}{p_r}\frac{2s + (2n-1)\dfrac{22}{18}}{2s + (2n-1)}.$$

In turn, at $k = 2n-1$, one has

$$V_{2n} - V_{2n-1} = -(1-\delta)(2n-1)\frac{22}{18sp_r} + (1-\delta)\left[\frac{1}{p_r} + (2n-1)\frac{1}{sp_r}\right]\frac{2s + (2n-1)\dfrac{22}{18}}{2s + (2n-1)}.$$

This enables us to compute the ratio of $V_1 - V_0$ to $V_{2n} - V_{2n-1}$ and, using the fact that $2n-1$ must be much larger than s, we obtain that this ratio is close to 11/7.

We can now show that incentive constraints cannot be satisfied for all k. Player 1 with type t_i obtains the flow payoff of

$$(1-\delta)t_i\frac{1}{6};$$

by reporting type 1; if the current continuation payoff is V_k, this report changes her continuation payoff by

$$\left[\frac{1}{3}p_{1,2} + \frac{1}{3}p_{1,3}\right](V_{k+1} - V_k).$$

We have omitted the factor δ, because $V_{k+1} - V_k$ is of order $1 - \delta$, and all our quantities are evaluated up to terms $o(1 - \delta)$.

Therefore, the total effect on payoff of reporting type 2 is

$$(1-\delta)t_i\frac{3}{6} + \frac{1}{3}p_{1,2}(V_{k-1} - V_k) + \frac{1}{3}p_{2,3}(V_{k+1} - V_k),$$

and the effect on player 1's payoff of reporting type 3 is

$$(1-\delta)t_i\frac{5}{6} + \left[\frac{1}{3}p_{1,3} + \frac{1}{3}p_{2,3}\right](V_{k-1} - V_k).$$

The strategies are incentive compatible when every type $t_i \in \{1, 2, 3\}$ of player 1 has incentives to report this type honestly; in particular, types 2 and 3 cannot prefer to mimic one another, which means that

$$(1-\delta)\frac{4}{6} \leq \frac{1}{3}p_{2,3}(V_{k+1} - V_k) + \frac{1}{3}(-p_{1,2} + p_{1,3} + p_{2,3})(V_k - V_{k-1}) \leq (1-\delta)\frac{6}{6}. \tag{31}$$

Indeed, the leftmost term in (31) is the stage-game gain of type 2 from reporting type 3, the rightmost term is the stage-game loss of type 3 from reporting type 2, and the middle term represents the difference in continuation payoffs contingent on reporting types 2 and 3.

Since $V_{k+1} - V_k$ is linear in k (see (30)), when n is large, the ratio of the middle term from (31) for $k = 2$ and the middle term from (31) for $k = 2n-1$ is close to $(V_1 - V_0)/(V_{2n} - V_{2n-1})$. We have established that this last expression is close to 11/7. So (31) can be satisfied only when the ratio of $6(1-\delta)/6$ and $4(1-\delta)/6$ is at least 11/7. Since $3/2 < 11/7$, the strategies cannot be incentive compatible for all k.

Remark 4. Note that the property that the ratio of the rightmost term of (31) to the leftmost term of (31) is lower than $(V_1 - V_0)/(V_{2n} - V_{2n-1})$ is crucial for the conclusion that the random chip strategies violated incentive constraints. Were it not the case, one could fairly easily find probabilities $p_{1,2}$, $p_{1,3}$, $p_{2,3}$, and p_r such that the strategies would be incentive compatible and would attain an almost-efficient outcome for large enough discount factors.

Note also that if we allowed for "probabilistic consumption" of the apple in the limit states (i.e., for $q < 1$), then both the values $(V_1 - V_0)$ and $(V_{2n} - V_{2n-1})$ would decrease by the same amount, which would make their ratio even greater than 11/7, and the result would still hold true.

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