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A tractable model of monetary exchange with ex post heterogeneity

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We construct a continuous-time new-monetarist economy that displays an endogenous, nondegenerate distribution of money holdings. Properties of equilibria are obtained analytically and equilibria are solved in closed form in a variety of cases. Lump-sum transfers financed with money creation are welfare-enhancing when labor productivity is low whereas regressive transfers approach first best when labor productivity is high and agents are not too impatient. We introduce illiquid government bonds and draw implications for liquidity-trap equilibria. We also study transitional dynamics under quadratic preferences and the velocity of money under heterogeneous preference shocks.

Keywords. Money, inflation, risk-sharing, liquidity traps.

JEL classification. E40, E50.

1. Introduction

We analyze a continuous-time new-monetarist economy, based on the competitive version of the Lagos and Wright (2005) model developed by Rocheteau and Wright (2005;

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LRW thereafter).[1] As in LRW, agents use a medium of exchange to finance random consumption opportunities and to make endogenous labor supply decisions. In contrast to LRW, but similar to Lucas (1980) and Bewley (1980, 1983), preferences allow for wealth effects, leading to a continuous distribution of money holdings and distributional effects for monetary policy. The existence of such distributional effects of inflation has been documented extensively, e.g., Doepke and Schneider (2006). Our first contribution is methodological: we show that the model remains tractable despite the unharnessed ex post heterogeneity in money holdings. We characterize the properties of equilibria, including policy functions, value functions, and distributions, and we solve the model in closed form in a variety of cases. Our second contribution is to study the effects of money creation on output and welfare for different productivity levels. We design incentive-compatible transfer schemes that approach first best and explain how the optimal design varies with productivity. Our third contribution consists in adding illiquid government bonds as a policy instrument. We provide conditions for the existence of liquidity traps and draw policy implications.

While we study a version of our model with money and bonds at the end of the paper, we focus most of our attention on a pure-currency economy as it is the most transparent benchmark in which to study the fundamental monetary policy trade-off between enhancing the rate of return of currency and providing risk-sharing. In our model, ex ante identical households, who enjoy consumption and leisure flows, have the possibility to trade continuously in competitive spot markets. At some random times, households receive idiosyncratic preference shocks that generate utility for lumps of consumption. These represent large shocks that cannot be paid for by a contemporaneous income flow, such as health shocks. Following Kocherlakota (1998), lack of enforcement and anonymity prevent households from borrowing to finance these spending shocks, thereby creating a role for money. Because shocks are independent across households, the model generates heterogeneous individual histories and, hence, a nondegenerate distribution of money holdings.

We provide a detailed characterization of households' consumption and saving decisions under smooth preferences that allow for wealth effects. We show that households have a target for their real balances that depends on their rate of time preference, the inflation rate, and the frequency of consumption opportunities. They approach this target gradually over time by saving a decreasing fraction of their labor income flow. When they are hit by a preference shock for lumpy consumption, households deplete their money holdings in full if their wealth is below a threshold or deplete partially otherwise. Given the households' optimal consumption–saving behavior, we can characterize the stationary distribution of real money holdings in the population, and we solve for the value of money, thereby establishing the existence and uniqueness of an equilibrium. Next, we study in detail the special case where households have linear preferences over

[1] The market structure with sequential competitive markets is analogous to that in Rocheteau and Wright (2005). It could be readily reinterpreted as one where households meet sellers bilaterally and at random, and terms of trade are determined by bargaining. See our companion paper, Rocheteau et al. (2015), for such a reinterpretation.

consumption and labor flows. In this case, households save their full labor income until they reach their target for real balances in finite time. This version showcases the tractability of the model, as the equilibrium can be characterized in closed form. Finally, we formally connect our model to earlier work in the literature by showing that it admits, as a limit when labor productivity grows large, the new-monetarist model of LRW.

We study monetary policy in the form of stationary transfer schemes financed by money creation. We emphasize the distributional and labor supply effects through which money creation affects output and welfare. We focus on comparative statics with respect to a single parameter—labor productivity—that determines the speed at which households insure themselves against preference shocks, and that parametrizes the policy trade-off between providing risk-sharing and promoting self-insurance. We first restrict transfers to be lump sum, e.g., as in Kehoe et al. (1992), and we show that when labor productivity is low, a positive but moderate inflation rate implements the first-best level of aggregate output and raises ex ante welfare. We then go beyond lump-sum transfers and characterize transfer schemes that generate allocations arbitrarily close to the first best. If labor productivity is large, an optimal transfer scheme is approximated by a step function that gives a proportional transfer above a cutoff level for real balances and no transfer below that level. If labor productivity is low and preferences are linear with a satiation point, then the optimal transfer is lump sum.

For quadratic utility functions, we are able to characterize transitional dynamics following an unanticipated money injection. A one-time increase in the money supply leads to a one-time increase in the price level and a decrease of the spread of the distribution of real balances, which raises society's welfare. We also consider preferences where the utility function of lumpy consumption is linear with random slope. Spending behavior is characterized by a simple optimal stopping rule whereby households spend all their real balances when their marginal utility of lumpy consumption is above an endogenous threshold. As inflation increases, households reduce this threshold, which is a manifestation of the "hot potato" effect of inflation.

Finally, we extend our pure-currency economy by adding another asset—illiquid nominal government bonds—that can bear interest. In equilibrium, the poorest households hold money only, which creates an endogenous segmentation of asset markets: only households with sufficient wealth participate in the bonds market. We show an equivalence between liquidity-trap equilibria where the nominal interest rate is zero and equilibria of the pure-currency economy where a fraction of households do not deplete their real balances following a preference shock. We use this equivalence to determine conditions for the existence of liquidity-trap equilibria and to draw implications for policy in terms of money growth and bond supply.

Literature

Our approach is closely related to incomplete-market models where households self-insure against idiosyncratic income risk by accumulating assets: fiat money in Bewley (1980, 1983) and Lucas (1980), physical capital in Aiyagari (1994), and private IOUs in

Huggett (1993).[2] We contribute to this literature by analyzing a tractable continuous-time model with a type of idiosyncratic risk that is reminiscent of that in random matching models and with nontrivial labor supply decisions.[3] While incomplete markets are most often solved by way of numerical methods, a few papers have developed analytically tractable frameworks. In particular, Scheinkman and Weiss (1986), Algan et al. (2011), and Lippi et al. (2015) study Bewley economies with quasi-linear preferences, with special attention to logarithmic preferences for consumption.[4] Our model differs in a number of ways, allowing for a comprehensive study of the welfare and output effects of inflation and the policy trade-off of monetary policy.[5] Among Bewley models that work with numerical methods, İmrohoroğlu (1992) and Dressler (2011) study the welfare cost of inflation.

Search-theoretic monetary models with nondegenerate distributions include Camera and Corbae (1999), Zhu (2005), Molico (2006), and Chiu and Molico (2010, 2011), all in discrete time. We avoid the intricacies due to bargaining by assuming competitive prices.[6] Green and Zhou (1998, 2002) and Zhou (1999) assume price posting, undirected search, and indivisible goods, which lead to a continuum of steady states. In contrast, the laissez-faire monetary equilibrium of our model is unique. Menzio et al. (2013) assume directed search and free entry of firms and characterize the monetary steady state under general preferences and a constant money supply. Their main result is to show existence and uniqueness of a steady state and neutrality of money (Theorem 4.2). They only briefly discuss how one would solve their model numerically with money growth (Section 5). In contrast, the tractability of our model allows us to characterize equilibria with money growth analytically, to characterize the manner in which the output effect and welfare effect of inflation depend on labor productivity, and to design transfer schemes financed with money creation that approach first best. Moreover, we extend our model to have money and illiquid bonds, we draw implications for liquidity traps, and we study examples in which transitional dynamics can be solved in closed form.

Our work also contributes to a recent literature developing continuous-time methods to analyze general equilibrium models with incomplete markets. Recently, Achdou

[2]See Ljungqvist and Sargent (2004, Chapters 16 and 17) and Heathcote et al. (2009) for surveys.

[3]In contrast to the seminal paper by Lucas (1980), we do not assume a cash-in-advance constraint (i.e., his Condition 1.4), since agents can finance their flow consumption with their current labor income, and we rule out credit arrangements from first principles. The nature of the idiosyncratic risk is also different: it takes the form of random arrivals of opportunities to consume lumpy amounts of consumption, which is analogous to the idiosyncratic liquidity risk in continuous-time random matching models. Also in contrast to Lucas (1980), we allow for endogenous labor supply, we study money creation financed with various transfer schemes, and we obtain closed-form solutions in some natural cases.

[4]See also Sun and Zhou (2016), who use an exogenous upper bound on real balances.

[5]We characterize our model under smooth concave preferences and consider quasi-linear preferences only as a special case. Even for this special case, our model differs in important ways from the Scheinkman–Weiss model: risk is idiosyncratic and arises from lumpy consumption opportunities, instead of an aggregate risk on agents' ability to work, and we impose a bound, $\bar{h}$, on flow labor supply that plays a key role for our normative analysis.

[6]Rocheteau et al. (2015) study a discrete-time version of the model with search and bargaining, and alternating market structures. The model remains tractable and can be used to study transitional dynamics following one-time money injections.

et al. (2017) proposed numerical tools based on mean-field-games techniques to study a wide class of heterogeneous-agent models in continuous time, with Huggett (1993) as their baseline. Our idiosyncratic lumpy consumption opportunities are similar to the uncertain lumpy expenditures in the Baumol–Tobin model of Alvarez and Lippi (2013). Our model in Section 5 with money and bonds is related, with the following differences: we assume no cost to liquidate assets and we do not take the consumption path (both in terms of flows and jump sizes) as exogenous; neither do we assume that labor income is exogenous.

2. The environment

Time, $t \in \mathbb{R}_+$, is continuous and goes on forever. The economy is populated with a unit measure of infinitely lived households who discount the future at rate $r > 0$. There is a single perishable consumption good produced according to a linear technology that transforms h units of labor into h units of output. Households have a finite endowment of labor per unit of time, $\bar{h} < \infty$. Alternatively, one can normalize labor endowment to 1 and interpret $\bar{h}$ as labor productivity.

Households value consumption, c, and leisure flows, ℓ, according to an instantaneous utility function, $u(c, \ell)$. In addition to consuming and producing in flows, households receive preference shocks that generate lumps of utility for the consumption of discrete quantities of the good. Lumpy consumption opportunities represent large shocks (e.g., health events) that require immediate spending.[7] These shocks occur at Poisson arrival times, $\{T_n\}_{n=1}^{\infty}$, with intensity α. The utility of consuming y units of goods at time T_n is given by a function, $U(y)$. Taken together, the lifetime expected utility of a household can be written as

$$\mathbb{E}\left[\int_0^{+\infty} e^{-rt} u(c_t, \bar{h} - h_t)\, dt + \sum_{n=1}^{\infty} e^{-rT_n} U(y_{T_n})\right],$$

given some adapted and left-continuous processes for c_t, h_t, and y_t. We impose the following regularity conditions on utility functions. First, we assume that both consumption and leisure are normal goods, and that $u(c, \ell)$ is increasing, bounded above, i.e., $\sup_{c \geq 0} u(c, \bar{h}) \equiv \|u\| < \infty$, and bounded below so that we can normalize $u(0, 0) = 0$. Moreover, $u(c, \ell)$ is strictly concave, and twice continuously differentiable, and it satisfies Inada conditions with respect to both arguments, i.e., $u_c(0, \ell) = \infty$ and $u_c(\infty, \ell) = 0$ for all $\ell > 0$, $u_\ell(c, 0) = \infty$ for all $c > 0$. We refer to $u(c, \ell)$ satisfying these assumptions as SI preferences. Second, $U(y)$ is bounded, strictly increasing, strictly concave, and twice continuously differentiable; it also satisfies the Inada condition $U'(0) = +\infty$ and the normalization $U(0) = 0$.

So as to make money essential, we assume that households cannot commit and that there is no monitoring or enforcement technology (Kocherlakota 1998). As a result, households cannot borrow to finance lumpy consumption, since otherwise they

[7] One could also interpret the preference shocks as random consumption opportunities in a decentralized goods market with search-and-matching frictions. For such an interpretation, see Rocheteau et al. (2015).

would default on their debt. The only asset in the economy is fiat money: a perfectly recognizable, durable, and intrinsically worthless object. The supply of money, denoted M_t, grows at a constant rate, $\pi \geq 0$, through lump-sum transfers to households. We consider alternative transfer schemes in Section 4.3. In accordance with the definition of a pure-currency economy (e.g., Wallace 2014), the absence of an enforcement technology prevents direct taxation and, hence, negative inflation rates.[8] Trades of money and goods take place in spot competitive markets. The price of money in terms of goods is denoted ϕ_t.

For the purpose of studying policy and welfare, our first-best benchmark is the full-insurance allocation, $(c^{\mathrm{FI}}, h^{\mathrm{FI}}, y^{\mathrm{FI}})$, that equalizes the marginal utilities of flow consumption, of leisure, and of lumpy consumption, i.e.,

$$u_c\left(c^{\mathrm{FI}}, \bar{h} - h^{\mathrm{FI}}\right) = u_\ell\left(c^{\mathrm{FI}}, \bar{h} - h^{\mathrm{FI}}\right) = U'\left(y^{\mathrm{FI}}\right).$$

3. Stationary monetary equilibrium

In this section, we study stationary monetary equilibria where aggregate real balances, $\phi_t M_t$, are constant. It implies the rate of return on money, $\dot{\phi}_t/\phi_t$, is constant and equal to the negative of the inflation rate, i.e., $\phi_t = \phi_0 e^{-\pi t}$, where the time-zero value of money, ϕ_0, is determined in equilibrium.

3.1 *The household's problem*

We analyze the household's problem given any constant inflation rate, $\pi \geq 0$, and given any real lump-sum transfer $\Upsilon = \pi \phi_0 M_0$. Let $W(z)$ denote the maximum attainable lifetime utility of a household holding z units of real balances. As is standard, $W(z)$ is a solution to the Bellman equation,

$$W(z) = \sup \mathbb{E}_{T_1}\left[\int_0^{T_1} e^{-rt} u(c_t, \bar{h} - h_t)\, dt + e^{-rT_1}\left[U(y_{T_1}) + W(z_{T_1} - y_{T_1})\right]\right], \tag{1}$$

with respect to left-continuous plans for $\{c_t, h_t, y_t\}$, a piecewise continuously differentiable plan for z_t, and subject to

$$z_0 = z, \tag{2}$$

$$0 \leq y_t \leq z_t, \tag{3}$$

$$\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon. \tag{4}$$

The random variable, T_1, is the arrival time of the next preference shock for lumpy consumption. It is exponentially distributed with mean $1/\alpha$. The first term on the right side of (1) is the discounted sum of utility flows until the preference shock occurs. The second term is the discounted utility at the time of the preference shock, T_1. It is composed of

[8]From a technical viewpoint, our analysis would go through with negative inflation rates provided that π is above a threshold. However, on logical grounds, the presence of an enforcement technology might open up the possibility of private IOUs and might make money inessential.

$U(y_{T_1})$ from consuming a lump of y_{T_1} units of consumption good and the continuation utility $W(z_{T_1} - y_{T_1})$ from keeping $z_{T_1} - y_{T_1}$ real balances.

Equation (2) is the initial condition for real balances, and (3) is a feasibility constraint stating that real balances must remain positive before and after a preference shock. In particular, the constraint that $y_t \leq z_t$ follows from the absence of enforcement and monitoring technologies that prevent households from issuing debt. Finally, (4) is the law of motion for real balances. The rate of change in real balances is equal to the household's output flow net of consumption, $h_t - c_t$, plus the negative flow return on currency, $-\pi z_t$, and a flow lump-sum transfer of real balances, Υ.

Using that T_1 is exponentially distributed, and after changing the order of integration, the objective (1) can be reexpressed as

$$W(z) = \sup \int_0^\infty e^{-(r+\alpha)t}\{u(c_t, \bar{h} - h_t) + \alpha[U(y_t) + W(z_t - y_t)]\}\, dt.$$

The effective discount factor, $e^{-(r+\alpha)t}$, is the product of the time discount factor, e^{-rt}, and the probability that no preference shock occurs during the time interval $[0, t)$, i.e., $\Pr(T_1 \geq t) = e^{-\alpha t}$. It multiplies the expected flow value of time-t decisions conditional on $T_1 \geq t$.

Theorem 1. *Given* Υ, *(1) has a unique bounded solution,* $W(z)$. *It is strictly increasing, strictly concave, and continuously differentiable over* $[0, \infty)$. *It is twice continuously differentiable over* $(0, \infty)$ *except perhaps at one point where the second derivatives has left and right limits. Moreover,*

$$W'(0) \leq \frac{r+\alpha}{\bar{h}+\Upsilon}\left(\frac{\|u\|}{r} + \alpha\frac{\|U\|}{r}\right), \qquad \lim_{z\to 0} W''(z) = -\infty, \quad \textit{and} \quad \lim_{z\to\infty} W'(z) = 0.$$

Finally, W *solves the Hamilton–Jacobi–Bellman (HJB) equation,*

$$rW(z) = \max\{u(c, \bar{h} - h) + \alpha[U(y) + W(z-y) - W(z)] + W'(z)(h - c - \pi z + \Upsilon)\}, \quad (5)$$

with respect to (c, h, y) *and subject to* $c \geq 0$, $0 \leq h \leq \bar{h}$, *and* $0 \leq y \leq z$.

The main technical challenge in Theorem 1 is to establish that $W(z)$ admits continuous derivatives of sufficiently high order. To that end, we adapt classical mathematical arguments (e.g., Bardi and Capuzzo-Dolcetta 1997) to show that the value function of the household is a viscosity solution of the HJB equation. Based on this result, we are able to establish the desired smoothness properties of the value function.

A perhaps surprising result is that $W'(0) < \infty$ even though $U'(0) = \infty$. Intuitively, a household with depleted money balances, $z = 0$, has a finite marginal utility for real balances because it has some positive time to accumulate real balances before the next opportunity for lumpy consumption, $\mathbb{E}[T_1] = 1/\alpha > 0$.

The HJB equation, (5), has a standard interpretation as an asset-pricing condition. If we think of $W(z)$ as the price of an asset, the opportunity cost of holding that asset is $rW(z)$. The asset yields a utility flow, $u(c, \ell)$, and a capital gain, $U(y) + W(z-y) - W(z)$, in the event of a preference shock with Poisson arrival rate α. Finally, the value of the asset changes over time due to the accumulation of real balances, represented by the last term on the right side of (5), $W'(z_t)\dot{z}_t$.

Optimal lumpy consumption From (5), a household chooses its optimal lumpy consumption so as to solve

$$V(z) \equiv \max_{0 \leq y \leq z} \{U(y) + W(z-y)\}. \tag{6}$$

In words, a household chooses its level of consumption so as to maximize the sum of its current utility, $U(y)$, and its continuation utility with $z-y$ real balances, $W(z-y)$. Because $U'(0) = \infty$ but $W'(0) < \infty$, a household always finds it optimal to choose strictly positive lumpy consumption, $y(z) > 0$, for all $z > 0$. Hence, the first-order condition of (6) is

$$U'(y) \geq W'(z-y), \tag{7}$$

with an equality if $y < z$. The following proposition provides a detailed characterization of the solution to (7).

Proposition 1 (Optimal lumpy consumption function). *The unique solution to (7), $y(z)$, admits the following properties:*

(i) The solution $y(z)$ is continuous and strictly positive for any $z > 0$.

(ii) Both $y(z)$ and $z - y(z)$ are increasing and satisfy $\lim_{z\to\infty} y(z) = \lim_{z\to\infty} z - y(z) = \infty$.

(iii) We have $y(z) = z$ if and only if $z \leq \bar{z}_1$, where $\bar{z}_1 > 0$ solves $U'(\bar{z}_1) = W'(0)$.

Finally, $V(z)$ is strictly increasing, strictly concave, and continuously differentiable over $(0, \infty)$ *with* $V'(z) = U'[y(z)]$.

Proposition 1 shows that as long as real balances are below some threshold $\bar{z}_1$, the household finds it optimal to deplete its real balances in full upon receiving a preference shock. This follows because the utility derived from spending a small amount of real balances, $U'(0) = \infty$, is larger than the benefit from holding onto it, $W'(z) \leq W'(0) < \infty$. This result—the fact that liquidity constraints bind over a nonempty interval of the support of the wealth distribution—is in contrast to the standard incomplete-market model in continuous time where liquidity constraints never bind in the interior of the state space (Achdou et al. 2017), and it plays a key role for the tractability of our model.

By induction, we can construct a sequence of thresholds for real balances, $\{\bar{z}_n\}_{n=1}^{+\infty}$, such that if a household's real balances belong to the interval $[\bar{z}_n, \bar{z}_{n+1})$, the post-trade real balances of the household following a preference shock, $z - y(z)$, belong to the adjacent interval, $[\bar{z}_{n-1}, \bar{z}_n)$. The properties of lumpy consumption, $y(z)$, and post-trade real balances, $z - y(z)$, are illustrated in Figure 1.

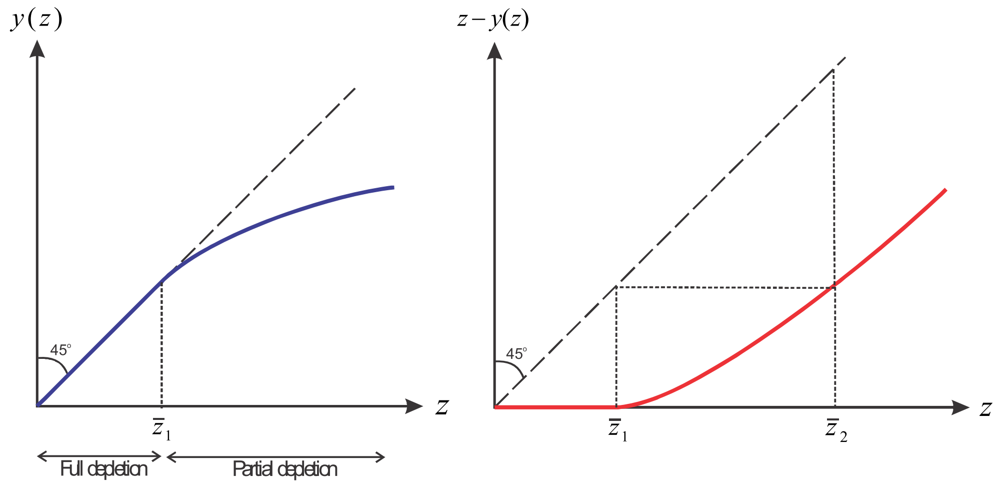

Figure 1. Left panel: Lumpy consumption. Right panel: Post-trade real balances.

Optimal saving Next, we characterize the household's optimal saving function, $s(z) \equiv h(z) - c(z) - \pi z + \Upsilon$, where (h, c) solves (5).

Proposition 2 (Optimal saving function). *The saving function is strictly decreasing over* $(0, \infty)$*, strictly positive near* $z = 0$*, and admits a unique* $z^\star \in (0, \infty)$ *such that* $s(z^\star) = 0$*. It is continuously differentiable except perhaps at* $z^\star$*, where it admits left and right limits.*

Proposition 2 highlights three general properties of households' saving behavior. The first states that households save less when they hold larger real balances. Indeed, the first-order conditions for an interior choice of consumption and leisure are

$$u_c(c, \ell) = u_\ell(c, \ell) = W'(z). \tag{8}$$

Given that flow consumption and leisure are assumed to be normal goods, it follows that c and ℓ increase with z. The second property of $s(z)$ is that it is strictly positive near zero. The third property is that households have a target, $z^\star < \infty$, for their real balances.

Next, we study the time path of a household's real balances conditional on not receiving a preference shock, namely, the solution to the initial value problem

$$\dot{z}_t = s(z_t) \quad \text{with } z_0 = 0. \tag{9}$$

Given the unique solution to (9), we can state the following proposition.

Proposition 3 (Optimal path of real balances). *The initial value problem (9) has a unique solution. This solution is strictly increasing for* $t \in [0, +\infty)$ *and it approaches* $z^\star$ *asymptotically.*

A household accumulates real money balances toward a target $z^\star$ by saving $s(z_t)$ over time; hence, z_t is increasing. The target, which is only reached asymptotically, provides the household with its desired level of self-insurance, taking into account the cost of holding real balances due to inflation and discounting. Figure 2 illustrates z_t.

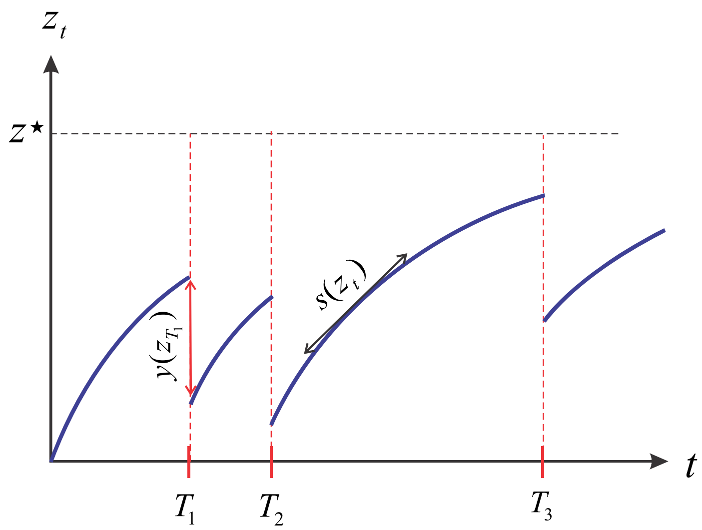

Figure 2. Optimal path of real balances.

3.2 *The stationary distribution of real balances*

We now show that the household's policy functions, $y(z)$ and $s(z)$, induce a unique stationary distribution of real balances over the support $[0, z^\star]$.

To this end, we first define the time to reach z from $z_0 = 0$,

$$\mathcal{T}(z) \equiv \inf\{t \geq 0 : z_t \geq z \mid z_0 = 0\},$$

with the convention that $\mathcal{T}(z) = \infty$ if the set is empty. From the function $\mathcal{T}(z)$, we obtain the minimal time that it takes for a household with z real balances at the time of a preference shock to accumulate strictly more than z' real balances following that shock:

$$\Delta(z, z') \equiv \max\{\mathcal{T}(z') - \mathcal{T}[z - y(z)], 0\}$$

for $z, z' \in [0, z^\star]$. Notice that $\Delta(z, z^\star) = \infty$ since the household never accumulates more than the target. Let $F(z)$ denote the stationary distribution of real balances. Given Υ, it must solve the fixed-point equation

$$1 - F(z') = \int_0^\infty \alpha e^{-\alpha u} \int_0^\infty \mathbb{I}_{\{u \geq \Delta(z, z')\}}\, dF(z)\, du = \int_0^\infty e^{-\alpha \Delta(z, z')}\, dF(z), \tag{10}$$

where the second equality is obtained by changing the order of integration. The right side of (10) calculates the measure of households with real balances strictly greater than z'. First, it partitions the population into cohorts indexed by the date of their last preference shock. There is a density measure, $\alpha e^{-\alpha u}$, of households who had their last preference shocks u periods ago. Second, in each cohort there is a fraction $dF(z)$ of households who held z real balances immediately before the shock. Those households consumed $y(z)$, which left them with $z - y(z)$ real balances. If $u \geq \Delta(z, z')$, then sufficient time has elapsed since the preference shock for their current holdings to be strictly greater than z'.

A key observation is that the fixed-point problem in (10) is equivalent to the problem of finding a stationary distribution for the Markov process $\{z_{T_n}\}_{n=1}^{\infty}$ that samples real balances at the discrete lumpy consumption times $\{T_n\}_{n=1}^{\infty}$. Indeed the transition probability function for that discretely sampled Markov process is

$$\begin{aligned} Q\big(z, [0, z']\big) &= \Pr\big(z_{T_{n+1}} \leq z' \mid z_{T_n} = z\big) \\ &= \Pr\big(T_{n+1} - T_n \leq \Delta\big(z, z'\big) \mid T_n\big) = 1 - e^{-\alpha\Delta(z,z')}. \end{aligned} \tag{11}$$

Hence, the equation for a stationary distribution of $\{z_{T_n}\}_{n=1}^{\infty}$ is $F(z') = \int_0^{\infty} Q(z, [0, z'])\, dF(z)$. From the definition of Q, and keeping in mind that $\int_0^{\infty} dF(z) = 1$, it is immediate that this equation is the same as (10).

This observation allows us to apply standard results for the existence and uniqueness of stationary distribution of discrete-time Markov processes. We obtain the following proposition.

Proposition 4 (Stationary distribution of real balances). *Given* Υ, *the fixed-point problem, (10), admits a unique solution,* $F(z)$. *This solution is continuous in the lump-sum transfer parameter,* Υ, *in the sense of weak convergence.*

In addition to obtaining existence and uniqueness of a stationary distribution, Proposition 4 shows that F is continuous in Υ because all policy functions are appropriately continuous in that parameter. This continuity property is helpful to establish the existence of a steady-state equilibrium, as it ensures that the market-clearing condition is continuous in the price of money, ϕ_0.

3.3 *The real value of money*

Equating the aggregate supply of real balances, $\phi_0 M_0$, with the aggregate demand of real balances as measured by the mean of the distribution F, we obtain the market-clearing condition

$$\phi_0 M_0 = \int_0^{\infty} z\, dF(z \mid \pi\phi_0 M_0), \tag{12}$$

where the right side makes it explicit that the stationary distribution depends on ϕ_0 via the lump-sum transfer, $\Upsilon = \pi\phi_0 M_0$. From (12), money is neutral at a steady state in the sense that aggregate real balances are determined independently of M_0.[9] We now define an equilibrium.

Definition 1. A stationary monetary equilibrium is composed of a value function, $W(z \mid \pi\phi_0 M_0)$, and associated policy functions that solve the household's optimization problem (1), a distribution of real balances, $F(z \mid \pi\phi_0 M_0)$, that solves the condition for a stationary distribution (10), and a price, $\phi_0 > 0$, that solves the market-clearing condition (12).

[9] In Rocheteau et al. (2015), we show in a discrete-time version of our model that one-time money injections are not neutral in the short run.

In the definition, our notations emphasize that the value function and the distribution of real balances depend on ϕ_0 via the real lump-sum transfer, $\pi\phi_0 M_0$.

Theorem 2 (Existence and uniqueness). *For all $\pi \geq 0$, there exists a unique stationary monetary equilibrium.*

Under the maintained assumption that $U'(0) = +\infty$, Proposition 2 establishes that a monetary equilibrium exists for all inflation rates. To prove this result, we study (12) at its boundaries. As ϕ_0 approaches zero, the left side of (12) goes to zero, but the right side remains strictly positive because, from Proposition 2, households accumulate strictly positive real balances even when the real value of the lump-sum transfer is zero, $\Upsilon = \pi\phi_0 M_0 = 0$. Indeed, from Proposition 2, $s(z) > 0$ near $z = 0$, since the marginal value of lumpy consumption is very large for y close to 0, which guarantees that there is a positive aggregate demand for real balances for all $\pi > 0$. As ϕ_0 tends to infinity, the left side of (12) becomes larger than the right side because the lump-sum transfer becomes so large that households only consume and stop working. Finally, Proposition 4 established that the stationary distribution, F, is continuous in $\Upsilon = \pi\phi_0 M_0$. Hence, we can apply the intermediate value theorem to prove existence.

The proof of uniqueness is straightforward if $\pi = \Upsilon = 0$, because in that case the equilibrium has a simple recursive structure. From Theorem 1, the value and policy functions are uniquely determined independently of F. From Proposition 4, F is uniquely determined given the policy functions.

The proof of uniqueness in the general case $\pi > 0$ is more challenging because the real value of the transfer of money depends on F, which breaks down the recursive structure of the equilibrium. To address this issue, we rewrite (12) as

$$\frac{\Upsilon}{\pi} = \int z\, dF(z \mid \Upsilon) \tag{13}$$

and we establish that, for any $\Upsilon' > \Upsilon$,

$$\frac{\Upsilon' - \Upsilon}{\pi} > \int z\, dF\big(z \mid \Upsilon'\big) - \int z\, dF(z \mid \Upsilon). \tag{14}$$

From (14), the slope of the left side of (13) as a function of Υ is always larger than the slope of the right side, which guarantees uniqueness of a solution Υ to (13). The left side of (14), $(\Upsilon' - \Upsilon)/\pi$, is the amount by which a household's stationary real balances would increase if it decided to save the entire increase in transfer, $\Upsilon' - \Upsilon$, at every point in time. Hence, the strict inequality in (14) means that such behavior is suboptimal, i.e., the household chooses optimally to consume some of the transfers it receives, so that stationary real balances increase by less than $(\Upsilon' - \Upsilon)/\pi$.[10]

[10]It is usually challenging to derive sufficient conditions for uniqueness in incomplete-market models because the equilibrating variable is the real interest rate, which has opposite income and substitution effects on current consumption, potentially resulting in multiple equilibria (see Açikgöz 2018 in discrete time and Achdou et al. 2017 in continuous time). In monetary models, the rate of return of money is constant and the equilibrating variable is the real value of lump-sum transfers, Υ, which intuitively should only have

3.4 *Equilibria with full depletion of real balances*

In this section, we study the class of equilibria with *full depletion*, in which households find it optimal to spend all their money holdings whenever a preference shock occurs, i.e., $y(z) = z$ for all $z \in [0, z^\star]$. In this case, our model is very tractable and it lends itself to a tight characterization of decision rules and distributions. We also show that full depletion occurs under appropriate parameter restrictions. These results are used in the following sections.

The optimal path for real balances under full depletion The ordinary differential equation (ODE) for the optimal path of real balances, (4), can be rewritten as

$$\dot{z}_t = h(\lambda_t) - c(\lambda_t) - \pi z_t + \Upsilon, \tag{15}$$

where $\lambda_t \equiv W'(z_t)$ is the marginal value of real balances, while $h(\lambda_t)$ and $c(\lambda_t)$ are the solutions to

$$\max_{c \geq 0, h \leq \bar{h}} \{u(c, \bar{h} - h) + \lambda(h - c - \pi z + \Upsilon)\}. \tag{16}$$

To solve for λ_t, we apply the envelope condition to differentiate the HJB (5) with respect to z along the optimal path of money holdings. This leads to the ODE

$$r\lambda_t = \alpha[U'(z_t) - \lambda_t] - \pi\lambda_t + \dot{\lambda}_t, \tag{17}$$

where we assume full depletion and use $V'(z_t) = U'[y(z_t)] = U'(z_t)$ from Proposition 1. The pair (z_t, λ_t) solves a system of two ODEs, (15) and (17).[11] We can show that the stationary point of this system is a saddle point and the optimal solution to the household's problem is the associated saddle path.

The stationary distribution of real balances under full depletion Under full depletion, $y(z) = z$, the time it takes for a household to accumulate z' real balances following a preference shock is $\Delta(z, z') = \mathcal{T}(z')$. Hence, from (11), the transition probability function,

$$Q(z, [0, z']) = 1 - e^{-\alpha\mathcal{T}(z')}, \tag{18}$$

does not depend on z. In words, the probability that a household holds less than z' is independent of its real balances just before its last lumpy consumption opportunity, z. This result is intuitive since households "restart from zero" after a lumpy consumption opportunity. It follows that the stationary probabilities coincide with the transition probabilities, i.e.,

$$F(z') = Q(z, [0, z']) = 1 - e^{-\alpha\mathcal{T}(z')}. \tag{19}$$

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a positive income effect. However, Hu and Shmaya (2017) argue in the context of a discrete-time Bewley economy that there remains a serious technical challenge in establishing monotonicity of the goods market-clearing equation. We are able to establish monotonicity using standard techniques by working with a different state variable and equilibrium condition than those in Hu and Shmaya (2017).

[11] A similar system of ODEs holds under partial depletion, where $U'(z_t)$ is replaced by $U'[y(z_t)]$. Hence, to solve for this system, one also needs to solve for the unknown function $y(z)$.

Finally, the equilibrium equation for the price level, (12), simplifies as well:

$$\phi_0 M_0 = \int_0^\infty z\, dF(z \mid \pi\phi_0 M_0) = \int_0^\infty \big[1 - F(z \mid \pi\phi_0 M_0)\big]\, dz = \int_0^{z^\star} e^{-\alpha\mathcal{T}(z \mid \pi\phi_0 M_0)}\, dz, \quad (20)$$

where our notation highlights that the time to accumulate real balances, $\mathcal{T}$, is a function of the real lump-sum transfer, $\Upsilon = \pi\phi_0 M_0$.

Verifying full depletion From the first-order condition, (7), households find it optimal to deplete their money holdings in full when a lumpy consumption opportunity occurs; $y(z) = z$ for all $z \in [0, z^\star]$ if and only if

$$U'\big(z^\star\big) \geq W'(0) = \lambda_0. \quad (21)$$

To verify this condition, one must solve for the equilibrium price, ϕ_0, and the associated real transfer, $\Upsilon = \pi\phi_0 M_0$. We turn to this task in the following proposition.

Proposition 5 (Sufficient conditions for full depletion). *There exists a threshold for the inflation rate, π_F, such that, for all $\pi \geq \pi_F$, all stationary monetary equilibria feature full depletion.*

There is full depletion of real balances if households choose to self-insure for their next preference shock only. From 0 to T_1, they accumulate real balances to be spent at the time of the first shock, T_1; from T_1 to T_2, they accumulate real balances to be spent at the time of the second shock and so on. This behavior is optimal if the cost of holding real balances is sufficiently large to outweigh the benefits from insuring against shocks that will happen far away in the future. Under high inflation, the opportunity cost of holding real balance is so high that money holdings become hot potatoes that households want to get rid of when the first opportunity arises, at time T_1.

4. New monetarism in continuous time

We now consider preferences of the form $u(c, \ell) = \min\{c, \bar{c}\} + \ell$ for some $\bar{c} \geq 0$, which corresponds to the quasi-linear specification used in monetary theory since Lagos and Wright (2005).[12] In this class of models, distributions are degenerate and there are no wealth effects because of the additional assumption that households have a large (or even infinite) endowment of labor per unit of time. In our model, in contrast, distributions are nondegenerate and there are wealth effects because the flow labor endowment

[12] Lagos and Wright (2005) assume quasi-linear preferences of the form $u(c) + \ell$. See also Scheinkman and Weiss (1986) for similar preferences. The fully linear specification comes from Lagos and Rocheteau (2005). We add a satiation point for consumption, $\bar{c}$, to guarantee that value functions are bounded. The satiation point does not bind in a stationary equilibrium because households find it optimal to choose $c < \bar{c}$ at all times. One can achieve the same amount of tractability with the larger class of preferences studied in Wong (2016), including constant returns to scale, constant elasticity of substitution, and constant absolute risk aversion (CARA). Outside such preferences, equilibria of the LRW model would feature nondegenerate distributions because of wealth effects.

$\bar{h}$ is finite, implying that $h \leq \bar{h}$ binds for some agents in equilibrium. The full-insurance allocation is such that

$$h^{\mathrm{FI}} = \alpha y^{\mathrm{FI}} = \min\{\alpha y^\star, \bar{h}\} \tag{22}$$

and $c^{\mathrm{FI}} = 0$, where $y^\star$ is the solution to $U'(y^\star) = 1$. In the Supplemental Appendix, available in a supplementary file on the journal website, http://econtheory.org/supp/2821/supplement.pdf, we extend our dynamic programming analysis to this specification and show that, up to some small changes, most of the results outlined in the previous section go through.

4.1 *Laissez-faire equilibria*

We focus on laissez-faire equilibria ($\pi = 0$) with similar spending patterns as in LRW, i.e., households deplete their real balances in full when they receive a preference shock. From (16), households choose $\dot{z} = h \leq \bar{h}$ to maximize $\dot{z}(\lambda - 1)$, where $\lambda(z) = W'(z)$. The saving correspondence is

$$s(z) = \begin{cases} \bar{h} & \text{if } \lambda(z) > 1, \\ [-\bar{c}, \bar{h}] & \text{if } \lambda(z) = 1, \\ -\bar{c} & \text{if } \lambda(z) < 1. \end{cases}$$

At the target $z^\star$, $s(z^\star) = 0$. It follows that $\lambda(z^\star) = 1$. In words, a household who keeps its real balances constant must be indifferent between not working and working at a disutility cost of 1 so as to accumulate one unit of real balances worth λ.[13] From (17),

$$U'(z^\star) = 1 + \frac{r}{\alpha}.$$

The marginal utility of lumpy consumption is equal to the marginal disutility of labor augmented by a wedge, r/α, due to discounting. If households are more impatient or if preference shocks are less frequent, households reduce their targeted real balances. The time path for individual real balances is $z_t = \bar{h}t$ for all $t \leq \mathcal{T}(z^\star) = z^\star/\bar{h}$, where t is the length of time since the last preference shock.[14]

From (18) and (19), the steady-state distribution of real balances is a truncated exponential distribution

$$F(z) = 1 - e^{-\frac{\alpha z}{\bar{h}}}\mathbb{I}_{\{z<z^\star\}} \quad \text{for all } z \in \mathbb{R}_+. \tag{23}$$

It has a mass point at the targeted real balances, $1 - F(z^\star_-) = e^{-\alpha z^\star/\bar{h}}$, which is increasing with $\bar{h}$. From market clearing, (20), aggregate real balances are

$$\phi_0 M_0 = \frac{\bar{h}}{\alpha}\left(1 - e^{-\frac{\alpha z^\star}{\bar{h}}}\right). \tag{24}$$

[13] The problem $\dot{z}_t = s(z_t)$ is well defined in the sense that $s(z)$ is single-valued for all z except when $W'(z) = 1$. In this case, we choose the s that is closest to 0. As a result, if $z^\star$ is such that $W'(z^\star) = 1$, this ensures that real balances remain constant and equal to their stationary point.

[14] Relative to Proposition 3, the target is reached in finite time because $s(z)$ is not continuously differentiable at $z^\star$.

Aggregate real balances are smaller than the target, $\phi_0 M_0 < z^\star$, and they are increasing with the household's labor endowment.

Integrating (17) over $[t, \mathcal{T}(z^\star)]$ and using the change of variable $z = \bar{h}t$, we obtain a closed-form expression for λ as a function of $z \in [0, z^\star]$:

$$\lambda(z) = 1 + \alpha \int_z^{z^\star} e^{-(\frac{r+\alpha}{\bar{h}})(x-z)} \left[\frac{U'(x) - U'(z^\star)}{\bar{h}} \right] dx. \tag{25}$$

The marginal value of real balances is equal to the marginal disutility of labor, 1, plus the discounted sum of the differences between the marginal utility of lumpy consumption on the path going from z to $z^\star$, $U'(z_t)$, and at the target, $U'(z^\star)$. It is easy to check that $\lambda'(z) < 0$, i.e., the value function is strictly concave, and as z approaches $z^\star$, the marginal value of real balances approaches 1. From (25), the condition for full depletion, (21), can be expressed as

$$\frac{r}{\alpha} \geq \alpha \int_0^{\mathcal{T}(z^\star)} e^{-(r+\alpha)t} \left[U'(z_t) - U'(z^\star) \right] dt. \tag{26}$$

The right side of (26) is monotone decreasing in $\bar{h}$ (since $\mathcal{T}(z^\star) = z^\star/\bar{h}$) and it approaches 0 as $\bar{h}$ tends to $+\infty$. Equation (26) allows us to go beyond Proposition 5 and derive further conditions on exogenous parameters such that the equilibrium features full depletion. For example, (26) shows that, given any $\pi \geq 0$, there exists a threshold for labor productivity, $\bar{h}_F^\pi$, such that the equilibrium features full depletion if and only if $\bar{h} \geq \bar{h}_F^\pi$. If labor productivity is large enough, then households spend all of their money holdings when a preference shock hits because they anticipate that they can rebuild their money inventories quickly. Equation (26) also shows that the equilibrium features full depletion if households are sufficiently impatient because the cost of holding money outweighs the insurance benefits from hoarding real balances.

The following proposition establishes that as $\bar{h}$ tends to infinity the equilibrium approaches an equilibrium with degenerate distribution and linear value function, analogous to that in LRW.

Proposition 6 (Convergence to LRW). *As $\bar{h} \to \infty$, the measure of households holding $z^\star$ tends to* 1*, the value of money approaches $z^\star/M_0$, and $W(z)$ converges to $z - z^\star + \alpha[U(z^\star) - z^\star]/r$.*

4.2 *Money growth*

We now follow Kehoe et al. (1992, Section 6) and study money growth through lump-sum transfers.[15] We go beyond their analysis because our economy features a distribution of money holdings with a continuous support (instead of two mass points) and endogenous labor supply.

[15] Kehoe et al. (1992) consider a discrete-time version of the Scheinkman and Weiss (1986) economy with two types of agents—buyers and sellers—that alternate through time, which leads to aggregate uncertainty. They focus on two-state Markov equilibria where sellers hold all the currency at the end of each period and transfers of money are lump sum. Section 6, which is the closest to what we do, specializes on logarithmic preferences.

We focus on equilibria with full depletion, $y(z^\star) = z^\star$. In the presence of money growth, $\pi > 0$, the target for real balances can take two expressions, depending on whether the feasibility constraint, $h(z^\star) \leq \bar{h}$, is slack or binding:

$$z^\star \equiv \min\{z_s, z_b\}, \tag{27}$$

where

$$z_s \equiv (U')^{-1}\left(1 + \frac{r+\pi}{\alpha}\right) \quad \text{and} \quad z_b \equiv \frac{\bar{h}}{\pi} + \phi_0 M_0.$$

The quantity z_s can be interpreted as the *ideal target* that households aim for: it equalizes the marginal utility of lumpy consumption, $U'(z)$, and the cost of holding real balances, $1+(r+\pi)/\alpha$. It is feasible to reach only if $\bar{h} + \Upsilon = \bar{h} + \pi\phi M \geq \pi z_s$ or, equivalently, $z_b \geq z_s$. The quantity z_b is the highest level of real balances feasible to accumulate, given households' finite labor endowment, $\bar{h}$, the inflation tax on real balances, πz, and the lump-sum transfer, Υ. Thus z_b is a *constrained target*. From (27) the *effective target*, $z^\star$, is the minimum of these two quantities.

From (15), the trajectory for individual real balances is $z_t = z_b(1 - e^{-\pi t})$. Given that the time since the last preference shock is exponentially distributed, the distribution of real balances is

$$F(z) = 1 - \left(\frac{z_b - z}{z_b}\right)^{\frac{\alpha}{\pi}} \quad \text{for all } z < z^\star$$

and $F(z) = 1$ for all $z \geq z^\star$. If $z_b \leq z_s$, then households reach $z^\star = \min\{z_b, z_s\}$ only asymptotically, and the distribution of real balances has no mass point. In contrast, if $z_b > z_s$, then households reach $z^\star$ in finite time and the distribution has a mass point at $z = z^\star$. Substituting the closed-form expressions for $\mathcal{T}(z \mid \pi\phi M) = -\log(1 - z/z_b)/\pi$ and z_b into the market-clearing condition (20), we find after a few lines of algebra that aggregate real balances solve

$$\frac{\phi_0 M_0}{\bar{h}/\pi + \phi_0 M_0} = \frac{\pi}{\alpha+\pi}\left[1 - \left(1 - \min\left\{1, \frac{z_s}{\bar{h}/\pi + \phi_0 M_0}\right\}\right)^{\frac{\alpha+\pi}{\pi}}\right]. \tag{28}$$

The left side is strictly increasing in ϕ and the right side is decreasing in ϕ. Hence, (28) has a unique solution and there is a unique candidate equilibrium with full depletion. Finally, the condition for full depletion of money balances is given by (26), where r is replaced with $r + \pi$ and $\mathcal{T}(z \mid \pi\phi_0 M_0) = -\log(1 - z/z_b)/\pi$.

We now define aggregate output and households' ex ante welfare by

$$\mathcal{H}(\pi, \bar{h}) \equiv \int h(z; \pi, \bar{h})\, dF(z; \pi, \bar{h}),$$

$$\mathcal{W}(\pi, \bar{h}) \equiv \int \left[-h(z; \pi, \bar{h}) + \alpha U(z)\right] dF(z; \pi, \bar{h}).$$

The pointwise limits for those quantities when labor productivity goes to infinity are denoted by $\mathcal{H}^\infty(\pi) \equiv \lim_{\bar{h}\to\infty} \mathcal{H}(\pi, \bar{h})$ and $\mathcal{W}^\infty(\pi) \equiv \lim_{\bar{h}\to\infty} \mathcal{W}(\pi, \bar{h})$. The following proposition shows that the effects of money growth on $\mathcal{H}$ and $\mathcal{W}$ are qualitatively different depending on the size of $\bar{h}$.

Proposition 7 (Output and welfare effects of inflation). *In the quasi-linear economy:*

(i) Large labor productivity. *Both* $\mathcal{H}^{\infty}(\pi)$ *and* $\mathcal{W}^{\infty}(\pi)$ *are decreasing with* π.

(ii) Low labor productivity. *If* $U(z)/[zU'(z)]$ *is bounded above near zero, then there exists some minimum inflation rate,* $\underline{\pi}$*, and a continuous function* $\bar{H}:[0,\infty)\to\mathbb{R}_+$ *with limits* $\lim_{\pi\to 0}\bar{H}(\pi)=\lim_{\pi\to\infty}\bar{H}(\pi)=0$*, such that, for all* $\pi\geq\underline{\pi}$ *and* $\bar{h}\in[0,\bar{H}(\pi)]$*, there exists an equilibrium with binding labor,* $h(z^\star)=\bar{h}$*, and full depletion. In this equilibrium,* $\mathcal{H}(\pi,\bar{h})$ *attains its first-best level,* $\bar{h}$*, and* $\mathcal{W}(\pi,\bar{h})$ *increases with* π.

(iii) Large inflation. *As* $\pi\to\infty$*,* $\mathcal{H}(\pi,\bar{h})\to 0$ *and* $\mathcal{W}(\pi,\bar{h})\to 0$.

Labor productivity, $\bar{h}$, determines the speed at which households can reach their targeted real balances and the extent of ex post heterogeneity across households. As a result, $\bar{h}$ proves to be a key parameter to determine the extent to which lump-sum transfers of money provide risk-sharing and deter self-insurance, and, ultimately, how they affect households' ex ante welfare.

With large labor productivity, $\bar{h}\to\infty$, there is no role for risk-sharing, as all households reach their target almost instantly. Money growth implemented with lump-sum transfers reduces the rate of return of money, which adversely affects the incentives to self-insure, as measured by $z^\star$. Hence, aggregate output, which is approximately $\alpha z^\star$, and social welfare, which is approximately $\alpha[U(z^\star)-z^\star]$, are decreasing with the inflation rate. These are the standard comparative statics in models with degenerate distributions (e.g., Lagos and Wright 2005).

With low labor productivity, risk-sharing considerations dominate because even though π reduces $z^\star$, it takes a long time for households to reach $z^\star$. Indeed, in the laissez-faire equilibrium, the time that it takes, in the absence of any shock, to reach the target, $\mathcal{T}(z^\star)=z^\star/\bar{h}$, can be arbitrarily large when $\bar{h}$ is small. Consider the regime where the equilibrium features both full depletion, $y(z^\star)=z^\star$, and binding labor, $h(z^\star)=\bar{h}$. This regime occurs when the inflation rate is not too low and the labor endowment is not too high, as illustrated by the grey area in Figure 3. Because households cannot reach their ideal target, z_s, they all supply $\bar{h}$ irrespective of their wealth, and thus aggregate output is constant and equal to $\bar{h}$. This output level is also the full-insurance level, $h^{\mathrm{FI}}=\bar{h}$. Indeed, the condition for the binding labor constraint is $\bar{h}/\pi+\bar{h}/\alpha\leq z_s<y^\star$, which implies $\bar{h}<\alpha y^\star$, and from (22), $h^{\mathrm{FI}}=\bar{h}$. In addition, aggregate consumption is equal to the first-best level of consumption, $\phi_0M_0=\bar{h}/\alpha$. Hence, the equilibrium real allocation differs from first best in only one dimension: equilibrium lumpy consumption is dispersed across households, while first-best lumpy consumption is constant. Hence, the only consideration for policy is to improve risk-sharing by reducing lumpy consumption dispersion. Moderate inflation implemented with lump-sum transfers redistributes real resources from cash-rich to cash-poor households, thereby reducing lumpy consumption dispersion across households and raising their ex ante welfare. The social benefits of positive inflation stem from its redistributional effects in the presence of idiosyncratic risk, and not from distortions due to market power, trading externalities, and so on (as is the case in other theories). In Figure 3, welfare is increasing with

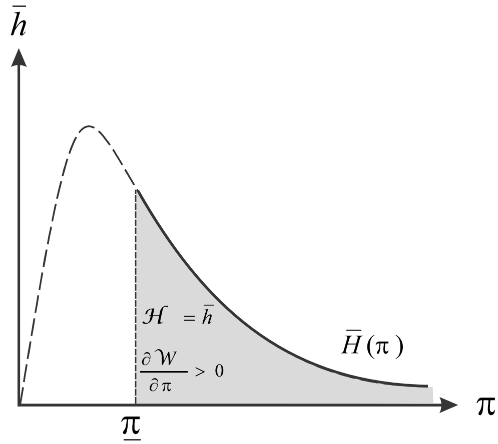

FIGURE 3. Regime with binding labor, $h(z^\star) = \bar{h}$, and full depletion, $y(z^\star) = z^\star$.

inflation in the grey area. While we cannot characterize the optimal inflation rate analytically, we found in numerical examples that for all $\bar{h} < \bar{H}(\underline{\pi})$, the optimal inflation rate is such that $\bar{h} = \bar{H}(\pi)$.[16]

4.3 *Beyond lump-sum transfers*

For $\bar{h}$ large enough, the equilibrium with money growth implemented through lump-sum transfers is such that $U'(z^\star) = 1 + (r + \pi)/\alpha > U'(y^\star) = 1$. Hence, consumption levels are bounded away from the first-best level, $y^\star$, and decrease with inflation. This means that money growth implemented with lump-sum transfers is welfare reducing. In the following proposition, we show that alternative incentive-compatible transfers financed with money creation can be designed to generate allocations approaching first best provided that agents are not too impatient and productivity is large enough.[17] When productivity is low, lump-sum transfers can implement first best for a class of preferences studied in the literature. In both cases, inflationary policies implement allocations near the first best even though the policymaker has no enforcement power and cannot use taxes.

PROPOSITION 8 (Near-efficient transfer schemes). *(i)* High labor productivity. *Assume that* $-ry^\star + \alpha[U(y^\star) - y^\star] > 0$. *Then there is some* $\underline{\pi}$ *such that, given any* $\pi > \underline{\pi}$, *as long as* $\bar{h}$ *is large enough, there exists a monetary equilibrium that features full*

[16]In our working paper (Rocheteau et al. 2017), we study a calibrated example with $r = 4\%$ and $U(y) = y^{1-a}/(1-a)$. The parameters, a, α, and $\bar{h}$, are calibrated to the distribution of the balances of transaction accounts in the 2013 Survey of Consumer Finance, which gives $a = 0.31$, $\alpha = 3.21$, and $\bar{h} = 6.26$. We compute numerically the optimal inflation rate under lump-sum transfers and show it is 0 for $\bar{h}$ above a threshold and it is positive otherwise.

[17]This result extends Wallace's (2014) conjecture according to which for generic pure-currency economies there exist transfer schemes financed with money creation that raise social welfare.

depletion of real balances with the scheme

$$\tau(z \mid \pi) = \begin{cases} 0 & \text{if } z < z^\star(\pi), \\ \pi z & \text{if } z \geq z^\star(\pi), \end{cases} \tag{29}$$

where $z^\star(\pi) = y^\star - \Delta/\bar{h}$ *and* $\Delta = 2[\pi y^\star + \alpha U(y^\star)]/|U''(y^\star)|$. *Moreover, the welfare loss in terms of first-best consumption is*

$$\delta = \frac{1}{\bar{h}} \times \frac{1}{y^\star} \int_0^{y^\star} \{[U(y^\star) - y^\star] - [U(z) - z]\}\, dz + o\left(\frac{1}{\bar{h}}\right). \tag{30}$$

(ii) Low labor productivity. *Assume* $U(y) = A \min\{y, \bar{y}\}$ *for some* $\bar{y} > 0$. *If* $\bar{h} < \alpha\bar{y}[\alpha(A - 1) - r]/(\alpha A - r)$, *then there is a monetary equilibrium under a lump-sum transfer scheme,* $\tau(z) = \pi\bar{h}/\alpha$ *for all* z *with* $\pi \in [\alpha\bar{h}/(\alpha\bar{y} - \bar{h}), \alpha(A - 1) - r]$, *that exactly implements a first best.*

When labor productivity is high, our nearly efficient transfer scheme takes the form of a step function. Households with real balances above $z^\star$ receive a proportional transfer that exactly compensates for the inflation tax. Households with less than $z^\star$ receive no transfer. When $z^\star$ is close to $y^\star$, this scheme rewards households who hold real balances close to the first best and punish those who hold too little real balances.[18] So the role of inflation is no longer redistributional: it is about providing incentives for self-insurance, which households can do easily when labor productivity is high, by making it costly to hold low real balances. The condition $-ry^\star + \alpha[U(y^\star) - y^\star] > 0$ is necessary for households to have incentives to hold real balances close to the first best, $y^\star$. It states that the opportunity cost of holding real balances, $ry^\star$, is less than the expected surplus from holding such real balances in the event of a preference shock for lumpy consumption, $\alpha[U(y^\star) - y^\star]$.

It should be noted that the transfer scheme (29) generates a net revenue to the government that corresponds to the inflation tax levied on households with $z < z^\star$. We assume this revenue is used to finance unproductive government consumption, which creates a welfare loss. Our efficiency result holds in spite of this loss.

For the transfer scheme (29) to be nearly efficient, there must be few households along the equilibrium path who hold less than $z^\star$, which requires $\bar{h}$ to be large. We measure the welfare loss relative to the first best as the fraction δ of the first-best consumption that a household would be willing to pay to move from the equilibrium to the first best. To a first-order approximation in $1/\bar{h}$, the welfare loss is equal to the average surplus that is lost by households with real balances strictly below the target, $z^\star$. Notice

[18]The transfer scheme in Proposition 8 is a generalization of that studied in Chiu and Wong (2015) and Bajaj et al. (2017) in the context of an economy with a degenerate distribution of money holdings. In these models with a degenerate distribution of money holdings, the proof of efficiency is easier: in equilibrium, households hold the target and never incur the inflation punishment. With a nondegenerate distribution of money, the punishment is incurred in equilibrium by households below the target. As shown by the estimate (30), this creates a first-order negative impact on social welfare that must be taken into account to establish near efficiency.

that the estimate of δ does not account for the welfare loss incurred by households at the target. This is because the target $z^\star$ is close to the first-best output, $y^\star$, and the first-best output already maximizes utilitarian welfare. Hence, the welfare loss incurred by households at $z^\star$ is of second order.

The second part of Proposition 8 considers the case where labor productivity is low. We specialize preferences further by assuming that the utility for lumpy consumption is linear with a satiation point, as in Kehoe et al. (1992, Section 5) and Green and Zhou (2005, Section 6). If $\bar{h} < \alpha\bar{y}$, then a first-best allocation is one where there is full employment, $h = \bar{h}$. We implement this outcome with positive money growth through lump-sum transfers. We construct an equilibrium in which inflation is sufficiently high so that households have to work full time just to maintain their targeted real balances at a level less than the satiation point $\bar{y}$. Because the marginal utility of lumpy consumption is constant and equal to A for all $z \in [0, \bar{y}]$, there is no welfare loss associated with households' ex post heterogeneity. Provided that the rate of time preference is not too large, there is a range of positive inflation rates that implement a first-best allocation. The inflation rate cannot be too low since otherwise households might find it optimal not to work when they reach $z^\star$. The inflation rate cannot be too high or households will not find it optimal to accumulate real balances.

5. Money and illiquid bonds

We now introduce illiquid nominal government bonds that can bear interest. With this extension, monetary policy can be implemented through both "helicopter drops" and open market operations by purchasing or selling government bonds in exchange for money. We show the equivalence between equilibria of the economy with money and bonds that feature a zero nominal interest rate—liquidity traps—and equilibria of the pure-currency economy that feature partial depletion. We use this equivalence and the analytical results from Section 4.3 to obtain insights for the existence of liquidity-trap equilibria and for policy.

In addition to fiat money, suppose there is a supply, B_t, of short-term pure-discount nominal bonds that pay one unit of money at the time of maturity that occurs according to independent Poisson processes with arrival rate $\mu > 0$. Bonds that expire are replaced with newly issued bonds. The supply of bonds is growing at the same rate, π, as the money supply to keep the ratio, B_t/M_t, constant and equal to B_0/M_0. The prices of money and bonds (in terms of goods) are denoted by ϕ_t and q_t. We focus on stationary equilibria in which $q_t B_t$ and $\phi_t M_t$ are constant, i.e., $\dot{\phi}_t/\phi_t = \dot{q}_t/q_t = -\pi$.

In a stationary equilibrium, the expected real rate of return on bonds, denoted by ϱ, is constant and solves

$$\varrho q_t = \mu(\phi_t - q_t) + \dot{q}_t \quad \Rightarrow \quad \varrho = \mu\left(\frac{\phi_0}{q_0} - 1\right) - \pi. \tag{31}$$

The rate of return on bonds has two components. First, with intensity μ, this bond matures into one unit of money, generating a capital gain of $\phi_0/q_0 - 1$. Second, the value of the bond, which is a claim on a unit of money, depreciates at the rate of inflation, π.

If bonds can serve as means of payment, then money and bonds are perfect substitutes, $\phi_0 = q_0$, and so they generate the same rate of return, $\varrho = -\pi$. In what follows we assume that bonds are not as liquid as money: in the event of a preference shock, only fiat money can be used to finance lumpy consumption, e.g., because it is the only asset that can be authenticated instantly. However, households are free to trade bonds in between lumpy consumption opportunities.[19]

In Appendix SI in the Supplemental Appendix, we extend the analysis of the households' problem of Section 3 to the present environment. As in Theorem 1, we study the maximum attainable lifetime utility of a household with ω units of wealth, $W(\omega)$. We show that it is strictly increasing and strictly concave, with $W'(0) < +\infty$ and $W'(+\infty) = 0$, that it is continuously differentiable over $[0, \infty)$, and that it solves the Hamilton–Jacobi–Bellman equation

$$rW(\omega) = \max_{c,h,y,z} \{u(c, \bar{h} - h) + \alpha[U(y) + W(\omega - y) - W(\omega)] + W'(\omega)\dot{\omega}\} \tag{32}$$

subject to $c \geq 0$, $0 \leq h \leq \bar{h}$, $0 \leq y \leq z \leq \omega$, and $\dot{\omega} = h - c + \varrho(\omega - z) - \pi z + \Upsilon$, where, by the budget constraint of the government, $\Upsilon = \mu B_0(q_0 - \phi_0) + \pi(\phi_0 M_0 + q_0 B_0)$.[20] In any equilibrium, the nominal interest rate on government bonds is bounded below by zero, $\varrho + \pi \geq 0$. If the inequality is strict, then $y = z$ and households do not hold more money than what they intend to spend in case of a preference shock.

The first-order condition with respect to y is

$$\alpha[U'(y) - W'(\omega - y)] \geq (\varrho + \pi)W'(\omega), \tag{33}$$

with an equality if $y < \omega$. The left side is the same as in the pure-currency economy: it is the expected net utility of consuming a marginal unit of good at the time of a preference shock. The right side is new: it represents the expected opportunity cost of holding real balances instead of bonds until the next preference shock. Following the same reasoning as in Proposition 1, $y(\omega)$ is strictly positive and increasing with wealth. Because $W'(0) < +\infty$, the poorest households hold only money.

From the HJB equation, (32), the marginal value of wealth, $\lambda_t = W'(\omega_t)$, solves

$$(r + \pi)\lambda_t = \alpha\{U'[y(\omega_t)] - \lambda_t\} + \lambda'(\omega_t)\dot{\omega}_t. \tag{34}$$

Interestingly, (34) is identical to its version in the pure-currency economy. Intuitively, the choice of real balances is interior for all levels of wealth and, as a result, the marginal value of wealth coincides with the marginal value of real balances. If $y(\omega) \leq \omega$ does not bind, then the envelope theorem applied to (32) gives

$$(r - \varrho)\lambda(\omega) = \alpha[\lambda(\omega - y) - \lambda(\omega)] + \lambda'(\omega)\dot{\omega}. \tag{35}$$

[19]It takes an infinitesimal amount of time to authenticate bonds, but that delay is large enough to miss an opportunity to consume. The idea that assets are not acceptable because they lack recognizability is formalized in Lester et al. (2012), Li et al. (2012), and Hu (2013), among others.

[20]In the household's budget constraint, we assumed that the portfolio of bonds is fully diversified so that the return, $\varrho(\omega - z)$, is deterministic.

The left side is the opportunity cost of wealth measured by the difference between the rate of time preference and the rate of return on bonds. The first term on the right side is the change in the marginal value of wealth following an opportunity for lumpy consumption. The targeted wealth, $\omega^\star$, corresponds to the stationary solution to (35), $\dot{\lambda}_t = 0$. Together with (33) it implies

$$\left(1+\frac{r-\varrho}{\alpha}\right)W'(\omega^\star) \geq W'\big[\omega^\star - y(\omega^\star)\big]. \tag{36}$$

The strict concavity of $W(\omega)$ implies $r > \varrho$. Even though bonds are illiquid, in the sense that they cannot be used to finance lumpy consumption, they do provide insurance services by allowing households to replenish their holdings of liquid assets after a preference shock. By market clearing, the richest households must hold some bonds, so that (36) holds at equality.

From the policy functions, one can construct $\Delta(\omega, \omega')$, the minimal time that it takes for a household with wealth ω at the time of a preference shock to accumulate strictly more than ω'. A stationary distribution of wealth, $F(\omega)$, is a solution to

$$1 - F(\omega') = \int_0^\infty \alpha e^{-\alpha u} \int_0^\infty \mathbb{I}_{\{u \geq \Delta(\omega,\omega')\}}\, dF(\omega)\, du = \int_0^\infty e^{-\alpha\Delta(\omega,\omega')}\, dF(\omega). \tag{37}$$

By market clearing, bonds have to be held, which implies that the richest households do not deplete their wealth in full when a preference shock occurs. It is a key difference between equilibria of the economy with illiquid bonds and equilibria of the pure-currency economy: the former must feature partial depletion of wealth. Using that from (31), $q_0 = \mu\phi_0/(\varrho + \pi + \mu)$, the market-clearing conditions for real balances and bonds are

$$\frac{\mu\phi_0 B_0}{\varrho + \pi + \mu} = \int_0^\infty \big[\omega - z(\omega)\big]\, dF(\omega), \tag{38}$$

$$\phi_0 M_0 = \int_0^\infty z(\omega)\, dF(\omega), \tag{39}$$

where the right sides of (38) and (39) depend on the lump-sum transfer, Υ, and the real return on bonds, ϱ. A stationary monetary equilibrium is composed of a value function, $W(\omega)$, a distribution of wealth, $F(\omega)$, a price of money, $\phi_0 > 0$, and a real interest rate, ϱ, solving (32), (37), (38), and (39). The following proposition follows directly from market clearing (bonds have to be held) and the fact that $W'(0) < +\infty$ (the poorest households want to spend all their wealth).

Proposition 9 (Properties of equilibrium). *Any equilibrium has the following features:*

(i) Endogenous segmentation. *There is a threshold for wealth,* $\underline{\omega} \in (0, \omega^\star)$*, below which households hold all their wealth in the form of money, i.e., bonds are held by households with wealth above* $\underline{\omega}$*.*

(ii) Liquidity premium on bonds. *The real interest rate on illiquid bonds is less than the discount rate.*

Our model generates a form of segmentation that is similar to that in Grossman and Weiss (1983) and Alvarez et al. (2002). Only a fraction of households hold and trade bonds. In contrast to Alvarez et al. (2002), this segmentation does not require a fixed cost of participating in the bond market. An alternative interpretation is that our model generates poor and wealthy "hand-to-mouth" agents as in the life-cycle model with liquid and illiquid assets of Kaplan and Violante (2014). In our model, poor hand-to-mouth agents with $\omega < \underline{\omega}$ hold no interest-bearing assets and consume all their liquid wealth when they receive an opportunity to do so. Rich hand-to-mouth agents with $\omega > \underline{\omega}$ also consume all their liquid wealth when they receive an opportunity to consume, but they do hold interest-bearing assets.[21] A poor household who receives a dollar saves it in the form of money and spends it if a consumption opportunity arises. In contrast, the rich household saves a fraction of the dollar in money, the rest being saved in interest-bearing bonds, and consumes that fraction in the event of a preference shock.

The second result that bonds command a liquidity premium equal to $r - \varrho > 0$ is in contrast with LRW, where the rate of return of illiquid bonds is r.[22] Indeed, in LRW, illiquid bonds have no insurance value because the household can work sufficiently hard to replenish its real balances instantly after a preference shock. It is through this liquidity premium that changes in the supply of bonds can affect the real interest rate, ϱ.

We now focus on a subset of equilibria called *liquidity traps*, where the nominal interest rate on bonds reaches its lower bound, $\varrho + \pi = 0$. In such equilibria, money and bonds are perfect substitutes as savings vehicles.

Proposition 10 (Liquidity-trap equilibria: An equivalence result). *Consider a pure-currency economy with money growth rate, π, and initial money supply, $M_0 = 1$. Denote by $\{y^1(z), h^1(z), c^1(z), \phi^1, F^1(z)\}$ a steady-state equilibrium with partial depletion and let*

$$\beta \equiv \frac{\displaystyle\int_0^\infty \left[z - y^1(z)\right] dF^1(z)}{\displaystyle\int_0^\infty y^1(z)\, dF^1(z)} > 0.$$

Then, for all $B_0 \le \beta$, there is an equilibrium of the economy with initial money supply, $M_0 = 1$, and initial bonds supply, B_0: $\{y^2(\omega), h^2(\omega), c^2(\omega), z^2(\omega), \phi^2, \varrho^2, F^2(\omega)\}$,

[21] In contrast to Kaplan and Violante (2014), our model does not incorporate income shocks. In the discrete-time version with search and bargaining in Rocheteau et al. (2015), we show that our model can accommodate both income and spending shocks, and remains tractable. Also in contrast to Kaplan and Violante, our model does not incorporate a fixed cost to reallocate one's portfolio. As a result, we conjecture that our model would have a weaker response to fiscal stimulus payments. For instance, liquidity-trap equilibria are equivalent to equilibria of a one-asset economy, and it is known that for such economies, fiscal stimulus payments generate a weak response for realistic distributions.

[22] The idea that government bonds can pay a liquidity premium even if they are not used as a medium of exchange provided that they allow agents to reallocate liquidity in the presence of idiosyncratic preference shocks can be found in Kocherlakota (2003). See also Berentsen et al. (2007), Li and Li (2013), Lagos and Zhang (2015), and Geromichalos and Herrenbrueck (2016).

with $\varrho^2 = -\pi$, $F^2(\omega) = F^1(\omega)$, $y^2(\omega) = y^1(\omega)$, $h^2(\omega) = h^1(\omega)$, $c^2(\omega) = c^1(\omega)$, $z^2(\omega) \in [y^1(\omega), \omega]$, *and* $\phi_0^2 = \phi_0^1/(1+B_0)$. *Conversely, any liquidity-trap equilibrium of the two-asset economy corresponds to an equilibrium of the pure-currency economy that features partial depletion.*

Any equilibrium of the pure-currency economy with partial depletion is equivalent in terms of allocations, aggregate wealth, and welfare to a liquidity-trap equilibrium of the economy with money and bonds. To get some intuition for this equivalence result, note that in pure-currency economies, households accumulate money balances for two motives: $y(z)$ for a transaction motive and $z - y(z)$ for a precautionary motive. In equilibria that feature partial depletion, the second motive is active, i.e., $z - y(z) > 0$ for some z in the support of F. Suppose we introduce a small supply of illiquid nominal bonds in such an economy. If the nominal interest rate on nominal bonds is strictly positive, then households want to fulfill their precautionary motive with bonds only. However, if the supply of bonds is small enough, i.e., $B_0/M_0 \leq \beta$, the bond market would not clear. Hence, in equilibrium, the nominal interest rate must fall to zero.

We now use Proposition 10 and results from Section 4 to establish conditions for the existence of liquidity-trap equilibria.

Corollary 1 (Existence of liquidity-trap equilibria). *Consider the economy with constant money supply. There exists a liquidity-trap equilibrium if and only if* $\bar{h} < h_F^0$, *where* h_F^0 *solves*

$$\frac{r}{\alpha} = \alpha \int_0^{z^\star/h_F^0} e^{-(r+\alpha)\tau}\big[U'(z_\tau) - U'\big(z^\star\big)\big]\,d\tau$$

and $B_0/M_0 \leq \beta$.

Liquidity-trap equilibria exist when labor productivity is low and bonds are scarce. Indeed, when $\bar{h}$ is low, households have a high precautionary demand for assets because the pace of wealth accumulation is slow. If the bond supply is low, the bond yield is driven to zero, so that households are indifferent between holding money and bonds. One can show after some algebra that h_F^0 is increasing in α, which means liquidity traps occur when the idiosyncratic risk, measured by α, is high. In contrast, liquidity-trap equilibria do not exist for any bond–money ratio in times of high productivity and low idiosyncratic uncertainty, in which case $\varrho \in (-\pi, r)$.[23]

The condition under which a liquidity trap exists, namely, a low $\bar{h}$, is the same condition for which risk-sharing considerations matter the most and inflation is beneficial. So even though open-market operations, interpreted as changes in the ratio B_0/M_0, are ineffective in liquidity traps, anticipated inflation through a higher growth rate of governments' liabilities can raise welfare. Following the logic in Proposition 7, for very low

[23] Williamson (2012), Andolfatto and Williamson (2015), and Rocheteau et al. (2017) obtain liquidity-trap equilibria in models with degenerate distributions where bonds are partially acceptable as a means of payment and markets are segmented or liquidity is reallocated through intermediaries. Related to what we do, Guerrieri and Lorenzoni (2017) study liquidity traps in a heterogenous-agents incomplete-market model and characterize transitional dynamics following a one-time aggregate shock.

values of $\bar{h}$, welfare can be improved by raising inflation to a level that induces full depletion of real balances and by driving B/M to zero. So while an open-market operation alone is ineffective, a combination of open-market operations and helicopter drops is useful.

In our working paper (Rocheteau et al. 2017), we compute the optimal policy in terms of inflation and nominal interest rate for a calibrated example (see footnote 16). The optimal policy requires $\pi = 1\%$ and $i = 4.75\%$. The real interest rate on bonds is $\rho \approx 3.75\% < r = 4\%$, i.e., bonds pay a liquidity premium. The optimal supply of bonds is positive, which shows that interest-bearing illiquid bonds are socially beneficial. Our model can then be used to study how optimal monetary policy is affected by changes in productivity. If labor productivity is reduced by half, the optimal policy is $\pi = 2\%$ and $i = 5.75\%$. It is optimal to raise the inflation rate, which allows financing larger transfers and interest payments on nominal bonds.

6. Other applications

In the following discussion, we illustrate additional insights and other tractable cases of our pure-currency economy. We first provide an example with quadratic preferences, allowing us to characterize in closed form the transitional dynamics following a one-time money injection. Second, we assume general preferences over c and h but linear and stochastic preferences over lumpy consumption, so as to discuss the effects of inflation on households' spending behavior.

6.1 *Money in the short run*

Suppose now that preferences are quadratic: $U(y) = Ay - y^2/2$ and $u(c, \bar{h} - h) = \varepsilon c - c^2/2 - h^2/2$.[24] From (8), and assuming interiority, the optimal choices of consumption and labor in a steady state are $c_t = \varepsilon - \lambda_t$ and $h_t = \lambda_t$. Under full depletion of real balances, the stationary solution to the system of ODEs, (15)–(17), is $\lambda^\star = \varepsilon/2$ and $z^\star = A - (1 + r/\alpha)\varepsilon/2$. We assume that $A > (1 + r/\alpha)\varepsilon/2$ to guarantee $z^\star > 0$. Along the saddle-path trajectory,

$$\lambda(z) = \frac{\nu}{2}\big(z - z^\star\big) + \lambda^\star,$$

where $\nu = (r + \alpha - \sqrt{(r + \alpha)^2 + 8\alpha})/2 < 0$. It follows that the household's policy functions are

$$c(z) = \frac{\varepsilon - \nu\big(z - z^\star\big)}{2}, \tag{40}$$

$$h(z) = \frac{\varepsilon + \nu\big(z - z^\star\big)}{2}. \tag{41}$$

As households get richer, their marginal value of wealth decreases, their consumption flow increases, and their supply of labor decreases. The condition for full depletion is

[24] Notice that these preferences do not satisfy the Inada conditions imposed earlier. But previous results are not needed, as we are able to solve the equilibrium in closed form.

$A - z^\star > -\nu z^\star/2 + \lambda^\star$ and $c(z)$ is interior for all z if $c(0) \geq 0$. It can be shown that the set of parameter values for which these restrictions hold is nonempty.

The saddle path of (15)–(17) is such that $z_t = z^\star(1 - e^{\nu t})$, where t is the length of the time interval since the last preference shock. Given that t is exponentially distributed,

$$F(z) = 1 - \left(\frac{z^\star - z}{z^\star}\right)^{-\frac{\alpha}{\nu}} \quad \text{for all } z \leq z^\star.$$

Market clearing gives

$$\phi M = \int_0^{z^\star} \big[1 - F(z)\big]\, dz = \frac{\nu}{\nu - \alpha} z^\star. \tag{42}$$

As before, aggregate real balances depend on all preference parameters, r, ε, and A, but not on M: money is neutral in the long run.

We now turn to the transitional dynamics following a one-time increase in the money supply, from M to γM, where $\gamma > 1$. In general, one should take into account that the rate of return of money, $\dot{\phi}/\phi$, might vary along the transitional path. Here, however, we guess and verify the existence of an equilibrium where the value of money adjusts instantly to its new steady-state value, ϕ/γ. Along the equilibrium path, aggregate real balances, $Z = \phi M$, are constant. To check that our proposed equilibrium is indeed an equilibrium, we show that the goods market clears at any point in time. From (40) and (41), it is easy to check that aggregate consumption is $C \equiv \int c(z)\, dF_t(z) + \alpha \int z\, dF_t(z) = [\varepsilon - \nu(Z - z^\star)]/2 + \alpha Z$ while aggregate output is $H = \int h(z)\, dF_t(z) = [\varepsilon + \nu(Z - z^\star)]/2$. From (42), it follows that $C + \alpha Z = H$, i.e., the goods market clears. The predictions of the model for aggregate quantities are consistent with the quantity theory: the price level moves in proportion to the money supply and real quantities are unaffected. So, from an aggregate viewpoint, money is neutral in the short run.[25]

However, money affects the distribution of real balances and consumption levels across households, which is relevant for welfare under strictly concave preferences. We compute society's welfare at the time of the money injection as $\int W(z)\, dF_0(z)$, where

$$F_0(z) = F\big[\gamma z - (\gamma - 1)Z\big]. \tag{43}$$

According to (43), the measure of households who hold less than z real balances immediately after the money injection is equal to the measure of households who were holding less than $\gamma z - (\gamma - 1)Z$ just before the shock: they received a lump-sum transfer of size $(\gamma - 1)Z$ and their real wealth is scaled down by a factor γ^{-1} due to the increase in the price level. The value function, $W(z)$, being strictly concave, the reduction in the spread of the distribution leads to an increase in welfare.

[25]This result is not general, but it is a useful benchmark suggesting that the effects of a one-time money injection on aggregate real quantities crucially depends on preferences that determine the relationship between labor supply decisions and wealth. In Rocheteau et al. (2015), we study transitional dynamic following one-time money injections in a discrete-time version of our model with search and bargaining and quasi-linear preferences. We show that the money injection affects the rate of return of money, aggregate real balances, and output levels.

6.2 *Inflation and velocity*

Suppose now that $U(y) = Ay$, where A is an independent and identically distributed (i.i.d.) draw from some distribution $\Psi(A)$.[26] We use this version of the model to capture the common wisdom according to which households spend their real balances faster on less valuable commodities as inflation increases, thereby generating a misallocation of resources.[27]

We conjecture that $W(z)$ is linear with slope λ. The HJB equation, (5), becomes

$$rW(z) = \max_{c,h}\left\{u(c, \bar{h} - h) + \alpha \int V(z) + \lambda(h - c - \pi z + \Upsilon)\right\}, \tag{44}$$

where $V(z) \equiv \int V(z, A)\, d\Psi(A)$ with

$$V(z, A) \equiv \max_{0\leq y\leq z}\left\{Ay + W(z - y)\right\} = \max_{0\leq y\leq z}(A - \lambda)y + W(z). \tag{45}$$

From (45), the household spends all its real balances whenever $A > \lambda$. Differentiating (44) and using that $V'(z) - \lambda = \int_\lambda^{\bar{A}}(A - \lambda)\, d\Psi(A)$, λ solves

$$(r + \pi)\lambda = \alpha\left[\int_\lambda^{\bar{A}}(A - \lambda)\, d\Psi(A)\right] = \alpha \int_\lambda^{\bar{A}}\left[1 - \Psi(A)\right] dA. \tag{46}$$

Equation (46) has the interpretation of an optimal stopping rule. According to the left side of (46), by spending its real balances, the household saves the opportunity cost of holding money, as measured by $r + \pi$. According to the middle term in (46), if the household does not spend its real balances, then it must wait for the next preference shock with $A \geq \lambda$. Such a shock occurs with Poisson arrival rate $\alpha[1 - \Psi(\lambda)]$, in which case the expected surplus from spending one unit of real balances is $\mathbb{E}[A - \lambda \mid A \geq \lambda] = \int_\lambda^{\bar{A}}(A - \lambda)\, d\Psi(A)/[1 - \Psi(\lambda)]$. Finally, the right side of (46) is obtained by integration by parts. It is straightforward to check that there is a unique solution to (46), $\lambda^\star$, and this solution is independent of the household's real balances as initially guessed. As inflation increases, $\lambda^\star$ decreases and, in accordance with the hot potato effect, households spend their money holdings on goods for which they have a lower marginal utility of consumption. Given $\lambda^\star$, (8) describes the flow of consumption, $c^\star$, and hours, $h^\star$.

The real balances of a household who depleted its money holdings t periods ago are $z_t = (h^\star - c^\star + \pi\phi M)(1 - e^{-\pi t})/\pi$. The probability that a household does not receive a preference shock with $A \geq \lambda^\star$ over a time interval of length t is $e^{-\alpha[1-\Psi(\lambda^\star)]t}$. Consequently,

$$F(z) = 1 - \left[\frac{h^\star - c^\star + \pi(\phi M - z)}{h^\star - c^\star + \pi\phi M}\right]^{\frac{\alpha[1-\Psi(\lambda)]}{\pi}} \quad \text{for all } z \leq \frac{h^\star - c^\star + \pi\phi M}{\pi}.$$

[26] For a significant extension of our model with linear utility for lumpy consumption, see Herrenbrueck (2014). The model is extended to account for quantitative easing and the liquidity channel of monetary policy.

[27] This wisdom has proved difficult to formalize in models with degenerate distributions. See Lagos and Rocheteau (2005), Ennis (2009), Liu et al. (2011), and Nosal (2011) for several attempts to generate the hot potato' effect in this class of models.

By market clearing, (12),

$$\phi M = \frac{h^\star - c^\star}{\alpha\big[1-\Psi\big(\lambda^\star\big)\big]}. \tag{47}$$

Aggregate real balances fall with inflation: because households save less, $h^\star - c^\star$ is lower, and because they spend their real balances more rapidly, $\alpha[1-\Psi(\lambda^\star)]$ increases. The velocity of money, denoted $\mathcal{V}$, is defined as nominal aggregate output divided by the stock of money. From (47),

$$\mathcal{V} \equiv \frac{h^\star}{\phi M} = \frac{\alpha\big[1-\Psi\big(\lambda^\star\big)\big]}{1-\dfrac{c^\star}{h^\star}}.$$

The velocity of money increases with inflation for two reasons: households spend their real balances more often following preference shocks, $1-\Psi(\lambda^\star)$ increases, and the saving rate, $(h^\star - c^\star)/h^\star$, decreases. A monetary equilibrium exists if $h^\star - c^\star > 0$, which holds if the inflation rate is not too large and the preference shocks are sufficiently frequent.

7. Conclusion

We constructed a continuous-time pure-currency economy in which households are subject to idiosyncratic preference shocks for lumpy consumption. We characterized steady-state equilibria under smooth preferences and solved in closed form a version with quasi-linear preferences resembling the new-monetarist framework of Lagos and Wright (2005) and Rocheteau and Wright (2005) that preserves wealth effects and ex post heterogeneity. We studied incentive-compatible transfers financed with money creation and their effects on output and welfare. We showed that a regressive transfer scheme implements allocations close to the first best when labor productivity is high. We extended our analysis to an environment with both money and nominal bonds, and showed that liquidity traps occur when labor productivity is low and idiosyncratic risk is high. In such equilibria, money growth through helicopter drops is welfare-enhancing. We also described transitional dynamics following an unanticipated monetary shock and we characterized the effects of inflation on the velocity of money under idiosyncratic and heterogeneous spending shocks.

We are working on extending our framework in several directions. In Rocheteau et al. (2015), we incorporate search and bargaining in a discrete-time version and study transitional dynamics following monetary shocks. The model remains tractable and delivers new insights for the short-run effects of money. For instance, a one-time money injection in a centralized market with flexible prices and perfect information generates an increase in aggregate real balances in the short run (price adjustments look sluggish), a decrease in the rate of return of money, and a redistribution of consumption levels across agents. These effects are non-monotone with the size of the money injection: Small injections can lead to short-run deflation, while large injections generate inflation. We are also working on incorporating idiosyncratic employment risk, e.g., by adding a frictional labor market, and private assets, such as capital and claims on firms' profits. The model will have implications for how the distribution of liquidity affects firms' entry, employment, and interest rates.

Appendix: Proofs of propositions

Appendix A: Proof of Theorem 1

A.1 *Solution W is bounded, positive, concave, strictly increasing, and continuous*

The Bellman equation (1) can be written as a functional equation $T[f]=f$, where

$$T[f](z)=\sup\int_0^\infty e^{-(r+\alpha)t}\{u(c_t,\bar{h}-h_t)+\alpha[U(y_t)+f(z_t-y_t)]\}\,dt,$$

with respect to left-continuous plans for flow consumption, flow labor, lumpy consumption, and real balances, subject to (2), (3), and (4). Standard arguments show that T is a contraction mapping the space of bounded functions into itself, equipped with the sup norm. Hence, T has a unique bounded fixed point, W.

The properties of W follow from the following observations. First, if f is positive and concave, then $T[f]$ is positive and concave as well. Second, if f is increasing, then $T[f]$ is strictly increasing. This is because any plan feasible given some initial real balances z remains feasible given some initial real balances $z'>z$, but allows us to increase lumpy consumption strictly. Finally, if f is concave and increasing, then $T[f]$ is continuous. Continuity at all $z\in(0,\infty)$ follows from the fact that $T[f]$ is concave and increasing (see, for example, Corollary 1, Chapter 7, in Luenberger 1969). To establish continuity at $z=0$, consider any sufficiently small $\varepsilon>0$. By working full time, $h_t=\bar{h}$, consuming nothing, and saving only in cash, the household can reach real balances ε at some time $T_\varepsilon=\varepsilon/(\bar{h}+\Upsilon)+o(\varepsilon)$, where $o(\varepsilon)$ is a function that satisfies $\lim_{\varepsilon\to 0}o(\varepsilon)/\varepsilon=0$. Clearly, since utility flows are bounded below by zero, it must be the case that $T[f](0)$ is greater than the value of working full time and consuming nothing until T_ε, and behaving optimally thereafter: $T[f](0)\geq e^{-(r+\alpha)T_\varepsilon}T[f(\varepsilon)]$. This implies in turn that $0\leq T[f](\varepsilon)-T[f](0)\leq(1-e^{-(r+\alpha)T_\varepsilon})T[f](\varepsilon)$. Since $T[f]$ is bounded and $\lim_{\varepsilon\to 0}T_\varepsilon=0$, continuity at zero follows.

A.2 *Solution W has strictly positive directional derivatives at all z and* $\lim_{z\to\infty}W'(z)=0$

The existence of left and right derivatives at all $z\in(0,\infty)$ and their strict positivity follows from the fact that W is concave and strictly increasing. To show that a left derivative exists at zero, we replace $W=T[W]$ in the inequality $0\leq T[f](\varepsilon)-T[f](0)\leq(1-e^{-(r+\alpha)T_\varepsilon})T[f](\varepsilon)$ that was derived at the end of the previous section. We obtain $0\leq W(\varepsilon)-W(0)\leq(1-e^{-(r+\alpha)T_\varepsilon})W(\varepsilon)$. Divide both sides by ε and let $\varepsilon\to 0$. Keeping in mind that $T_\varepsilon=\varepsilon/(\bar{h}+\Upsilon)+o(\varepsilon)$, we obtain that $W(z)$ has a right derivative at $z=0$ such that $W'(0)\leq\|W\|(r+\alpha)/(\bar{h}+\Upsilon)$. Taking norms on both sides of $W=T[W]$ implies, after appropriate applications of the triangle inequality, that $(r+\alpha)\|W\|\leq\|u\|+\alpha(\|U\|+\|W\|)$, so that $\|W\|\leq(\|u\|+\alpha\|U\|)/r$. The upper bound for $W'(0)$ stated in Theorem 1 follows.

Finally, to show that $\lim_{z\to\infty} W'(z)=0$, recall that $W(z)$ is concave and bounded below so that $W(z)-W(0)\geq W'_{-}(z)z\geq 0$. The result follows by taking the $z\to\infty$ limit, keeping in mind that $W(z)$ is bounded above.

A.3 *Solution W is a viscosity solution of the Hamilton–Jacobi–Bellman equation*

We first establish that the value function is a viscosity solution of the HJB equation (5).

Proposition 11. *For all $z\geq 0$ and all $\lambda\in[W'_{+}(z), W'_{-}(z)]$,*

$$(r+\alpha)W(z)\leq \sup\{u(c,\bar{h}-h)+\alpha\big[U(y)+W(z-y)\big]+\lambda[h-c-\pi z+\Upsilon]\}, \tag{48}$$

with respect to $c\geq 0$, $h\in[0,\bar{h}]$, $y\in[0,z]$, and with the convention that $W'_{-}(0)=+\infty$. Moreover, if $W'_{+}(z)=W'_{-}(z)$, then (5) holds with equality.

We provide a detailed proof below because our result does not follow directly from theorems found in existing applied mathematics textbooks on viscosity solutions of Hamilton–Jacobi–Bellman equations. This is so for at least two reasons. First, existing theorems assume that the rate of change of the state variable is bounded (for example, Assumption A1 in Chapter 3 of Bardi and Capuzzo-Dolcetta 1997). This assumption fails in our model since a household can deplete its real balances very quickly, if it chooses arbitrarily large consumption flows. Second, existing theorems do not consider optimization problems in which some decisions are made at Poisson arrival times. This creates a technical difficulty because the "flow reward" of the optimal control problem now depends on $W(z)$, the function whose smoothness we seek to establish.

A.3.1 *Changing z too rapidly is strictly suboptimal* We first establish that depleting money balances too rapidly cannot be optimal.

Lemma 1. *For all $z>0$ and all $\theta>0$, there is some $k>0$ such that, for all $\delta>0$ and any feasible controls c_t, h_t, and y_t starting at z, $z_\delta\leq z-k\delta$ implies that*

$$W(z)>\theta\delta+\int_0^\delta\{u(c_t,\bar{h}-h_t)+\alpha\big[U(y_t)+W(z_t-y_t)\big]\}e^{-(r+\alpha)t}\,dt+W(z_\delta)e^{-(r+\alpha)\delta}.$$

Indeed, consider any feasible control starting at z. We have that

$$\begin{aligned}W(z)&-\int_0^\delta\{u(c_t,\bar{h}-h_t)+\alpha\big[U(y_t)+W(z_t-y_t)\big]\}e^{-(r+\alpha)t}\,dt-W(z_\delta)e^{-(r+\alpha)\delta}\\&\geq W(z)-W(z_\delta)-\delta\big[\|u\|+\alpha\|U\|+\alpha\|W\|\big]\geq\lambda(z-z_\delta)-\delta\big[\|u\|+\alpha\|U\|+\alpha\|W\|\big]\end{aligned}$$

for any $\lambda\in[W'_{+}(z),W'_{-}(z)]$, since W is concave. Given that left and right derivatives are strictly positive, $\lambda>0$ and so the result follows by choosing some $k>(\theta+\|u\|+\alpha\|U\|+\alpha\|W\|)/\lambda$.

A.3.2 *An equicontinuity property for the path of real balances* Next, we show the following lemma.

Lemma 2. *Consider any $k > 0$ and any $\varepsilon > 0$. Then there exists some $\delta > 0$ such that, for any feasible control over $[0, \delta]$, starting at z_0, $z_\delta \geq z_0 - k\delta \Rightarrow |z_t - z_0| \leq \varepsilon$ for all $t \in [0, \delta]$.*

Indeed, since $\pi \geq 0$, it follows that $z_t - z_0 \leq (\bar{h} + \Upsilon)t \leq (\bar{h} + \Upsilon + k)\delta$. Applying the same argument over $[\delta, t]$ gives $z_\delta - z_t \leq (\bar{h} + \Upsilon)(\delta - t) \leq (\bar{h} + \Upsilon)\delta$. Substituting in $z_\delta \geq z_0 - k\delta$, we obtain that $z_t - z_0 \geq -(\bar{h} + \Upsilon + k)\delta$. Taken together, the two inequalities show that $|z_t - z_0| \leq (\bar{h} + \Upsilon + k)\delta$, and the result follows.

A.3.3 *Proof of the HJB inequality (48)* Toward a contradiction, suppose that there is some $\lambda \in [W'_+(z), W'_-(z)]$ and some $\theta > 0$ such that

$$(r + \alpha)W(z) > \theta + \sup\{u(c, \bar{h} - h) + \alpha[U(y) + W(z - y)] + \lambda[h - c - \pi z + \Upsilon]\}, \tag{49}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$, and $y \in [0, z]$. By continuity, there exists $\varepsilon > 0$ such that this inequality holds for all $\hat{z}$ such that $|z - \hat{z}| \leq \varepsilon$. Given this $\varepsilon > 0$ and the k constructed in Lemma 1, pick the δ constructed in Lemma 2. This implies that, for any feasible control such that $z_\delta \geq z - k\delta$, we have that $|z_t - z| \leq \varepsilon$ for all $t \in [0, \delta]$ and, combining with (49), we obtain that for any feasible control such that $z_\delta \geq z - k\delta$, $(r + \alpha)W(z_t) > \theta + u(c_t, \bar{h} - h_t) + \alpha[U(y_t) + W(z_t - y_t)] + \lambda[h_t - c_t - \pi z_t + \Upsilon]$ for all $t \in [0, \delta]$. Now let $\varphi(\hat{z}) \equiv W(z) + \lambda(\hat{z} - z)$. By construction, $W(z) = \varphi(z)$, and by concavity, $\varphi(\hat{z}) \geq W(\hat{z})$ for all $\hat{z}$. Therefore,

$$\begin{aligned}
&\varphi(z_\delta)e^{-(r+\alpha)\delta} - W(z_\delta)e^{-(r+\alpha)\delta} \geq \varphi(z) - W(z) = 0 \\
\Leftrightarrow \quad & W(z) \geq -\int_0^\delta \frac{d}{dt}\left[\varphi(z_t)e^{-(r+\alpha)t}\right] dt + W(z_\delta)e^{-(r+\alpha)\delta} \\
\Leftrightarrow \quad & W(z) \geq \int_0^\delta \left\{(r + \alpha)\varphi(z_t) - \varphi'(z_t)\dot{z}_t\right\} e^{-(r+\alpha)t}\, dt + W(z_\delta)e^{-(r+\alpha)\delta}.
\end{aligned}$$

But $\varphi(z_t) \geq W(z_t)$, $\varphi'(z) = \lambda$, and $\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon$. Plugging these into the above equations and substituting in inequality (49) evaluated at time t, we obtain

$$\begin{aligned}
W(z) \geq\ & \theta \int_0^\delta e^{-(r+\alpha)t}\, dt \\
& + \int_0^\delta \left\{u(c_t, \bar{h} - h_t) + \alpha\left[U(y_t) + W(z_t - y_t)\right]\right\} e^{-(r+\alpha)t}\, dt \\
& + W(z_\delta)e^{-(r+\alpha)\delta}
\end{aligned} \tag{50}$$

for any control such that $z_\delta \geq z - k\delta$.

If $z = 0$, all controls satisfy $z_\delta \geq z - k\delta$, so the inequality (50) holds for all feasible controls. If $z > 0$, given that $\delta > \int_0^\delta e^{-(r+\alpha)t}$, Lemma 1 shows that inequality (50) also holds for any control such that $z_\delta \leq z - k\delta$. Hence, it holds for all feasible controls. Therefore, for all $z \geq 0$, we can take the supremum over all feasible controls and we obtain a contradiction of the maximum principle.

A.3.4 *Proof that the HJB inequality (48) holds with equality if W is differentiable* This proof follows from the textbook proof: see, for example, step (P1) in the proof of Theorem 10.1 in Bressan (2001).

A.4 *Solution W is continuously differentiable*

Following standard practice, it is convenient to write the HJB inequality of (48) as $(r+\alpha)W(z) \leq H(z, \lambda)$, where, for any $(z, \lambda) \in [0, \infty) \times (0, \infty)$,

$$H(z, \lambda) \equiv \max\bigl\{u(c, \bar{h} - h) + \alpha\bigl[U(y) + W(z - y)\bigr] + \lambda[h - c - \pi z + \Upsilon]\bigr\} \tag{51}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$, and $y \in [0, z]$. Denote by $c(\lambda)$, $h(\lambda)$ the optimal flow consumption, flow labor, and lumpy consumption for this optimization problem. It is easy to show that $H(z, \lambda)$ is continuous in (z, λ) and strictly convex in λ, that $c(\lambda)$ is continuous and strictly decreasing, and that $h(\lambda)$ is continuous and increasing.

Now let us turn to the proof of continuous differentiability. We already know that the value function is differentiable at $z = 0$ in that it has a right derivative. Now consider any $z > 0$. Since W is concave, it is differentiable almost everywhere, and so there exists an increasing sequence $\underline{z}_n \uparrow z$ such that W is differentiable at $\underline{z}_n$. Note that, by concavity, $W'(\underline{z}_n)$ is decreasing and positive, and so has a limit $\bar{\lambda} \geq W'_-(z)$. Going to the limit in the Hamilton–Jacobi–Bellman equation of (48), which holds with equality at every point of the sequence, we obtain $(r+\alpha)W(z) = H(z, \bar{\lambda})$ for some $\bar{\lambda} \geq W'_-(z)$. Proceeding in the same way to the right of z, we obtain $(r+\alpha)W(z) = H(z, \underline{\lambda})$ for some $\underline{\lambda} \leq W'_+(z)$. Since $H(z, \lambda)$ is strictly convex, this implies that $(r+\alpha)W(z) > H(z, \lambda)$ for all λ in the open interval $(W'_+(z), W'_-(z))$. But according to (48), the opposite inequality must hold. Therefore, the open interval $(W'_+(z), W'_-(z))$ must be empty, i.e., $W'_+(z) = W'_-(z)$ and the value function is differentiable at z. Continuous differentiability follows from the continuity properties of the directional derivatives of concave functions (see, for example, Theorem 24.1 in Rockafellar 1970).

A.5 *Solution W is a classical solution of the HJB equation*

We now establish that W solves the HJB equation of Theorem 1 or, equivalently, that the HJB inequality (48) holds with equality. For $z > 0$, this follows from Proposition 11 and from the result, proved in the previous subsection, that W is differentiable. All we need to show is that the equality holds at $z = 0$. To see this, recall that $W'(z)$ is continuous and that $H(z, \lambda)$ is continuous in (z, λ). Therefore, it follows that $H[z, W'(z)]$ is continuous in $z \geq 0$. Going to the limit as $z \to 0$ leads to $(r+\alpha)W(0) = H[0, W'(0)]$, which completes the proof.

A.6 *Solution W has a strictly decreasing derivative*

Suppose that there exists some $0 \leq a_0 < b_0$ such that $W'(a_0) = W'(b_0) = \lambda_0$. Then, since $W'(z)$ is decreasing, $W'(z) = \lambda_0$ for all $z \in [a_0, b_0]$. Plugging this back into the HJB equation, we obtain that

$$(r+\alpha)W(z)=\sup\{u(c,\bar{h}-h)+\alpha[U(y)+W(z-y)]+\lambda_0[h-c-\pi z+\Upsilon]\}$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$, and $y \in [0, z]$.

Let $\bar{z}$ solves $\alpha U'(\bar{z}) = (r+\pi+\alpha)W'(0)$. One easily sees by taking first-order conditions that, for all $z \leq \bar{z}$, $y = z$ solves the lumpy consumption problem. Suppose that $a_0 < \bar{z}$. Plugging $y = z$ back into the right side of the HJB equation, one obtains that $(r+\alpha)W(z) - U(z)$ is weakly concave. But since $U(z)$ is strictly concave, this implies that $W(z)$ is also strictly concave, contradicting the premise that $W'(z) = \lambda_0$ for all $z \in [a_0, b_0]$. Hence, $a_0 \geq \bar{z}$.

Next, the first-order condition with respect to y is $\alpha U'[y(z)] - \alpha W'[z - y(z)] - \psi = 0$, where ψ is the multiplier for the constraint $y \leq z$. An application of the envelope theorem (for example, Corollary 5 in Milgrom and Segal 2002) shows, alternatively, that $(r+\alpha)W'(z) = \alpha W'[z-y(z)] - \pi\lambda_0 + \psi$. Substituting in $W'(z) = \lambda_0$ and using the first-order condition to eliminate ψ, we obtain $(r+\alpha+\pi)\lambda_0 = \alpha U'[y(z)]$, which implies that y is constant over $z \in [a_0, b_0]$, equal to some y_0. Since the constraint $y_0 \leq z$ must hold at the bottom of the interval for $z = a_0$, it must be slack for all other z in the interval, $z \in (a_0, b_0]$. Therefore, if $z \in (a_0, b_0]$, $\psi = 0$ and, from the envelope condition,

$$W'(z-y_0)=\left(1+\frac{r+\pi}{\alpha}\right)\lambda_0 \equiv \lambda_1$$

for all $z \in (a_0, b_0]$. Clearly, this remains true by continuity at $z = a_0$. Thus, we have found a new interval, $[a_1, b_1]$, where $a_1 = a_0 - y_0$ and $b_1 = b_0 - y_0$, such that $W'(z) = \lambda_1$ for all $z \in [a_1, b_1]$. As before $a_1 \geq \bar{z}$. By induction, we obtain a decreasing sequence $a_k \geq \bar{z}$ such that

$$W'(a_k)=\left(1+\frac{r+\pi}{\alpha}\right)^k\lambda_0.$$

But we know from that $W'(z)$ is bounded above. Since $r + \pi > 0$, we have reached a contradiction.

A.7 *Solution W is twice continuously differentiable in $z > 0$ when savings are nonzero*

Consider any $z > 0$ and let $\lambda \equiv W'(z)$. Define savings as $h[W'(z)] - c[W'(z)] - \pi z + \Upsilon$, where $c(\lambda)$ and $h(\lambda)$ are the optimal flow consumption and labor in the optimization program (51). Recall that $h(\lambda)$ is increasing, $c(\lambda)$ is strictly decreasing, and $W'(z)$ is strictly decreasing. Therefore, savings are nonzero except perhaps at one point.

Now consider some $z > 0$ such that savings are nonzero. Since $z > 0$, there exists $\tilde{y}$ such that $0 < \tilde{y} < \hat{z}$ for all $(\hat{z}, \hat{\lambda})$ in a neighborhood of (z, λ). Hence, an application of Corollary 5 in Milgrom and Segal (2002) implies that $H(\hat{z}, \hat{\lambda})$ is differentiable in a neighborhood of (z, λ) with partial derivatives:

$$\frac{\partial H}{\partial \hat{\lambda}}=h(\hat{\lambda})-c(\hat{\lambda})-\pi\hat{z}+\Upsilon \quad \text{and} \quad \frac{\partial H}{\partial \hat{z}}=\alpha U'[y(\hat{z})]-\pi\hat{\lambda}.$$

One easily sees that the partial derivatives are continuous in $(\hat{z}, \hat{\lambda})$. Taken together with the HJB equation, the above discussion implies that the equation $-(r+\alpha)W(\hat{z}) + H(\hat{z}, \hat{\lambda}) = 0$ is continuously differentiable with respect to $(\hat{z}, \hat{\lambda})$ in a neighborhood of (z, λ), and is solved by $\hat{\lambda} = \lambda$ when $\hat{z} = z$. Moreover, $\partial H/\partial\lambda \neq 0$ at (z, λ) by our assumption that savings are nonzero. Hence, an application of the implicit function theorem (for example, Theorem 13.7 in Apostol 1974) shows that this equation has a unique solution in some neighborhood of z, and that this unique solution can be written as a continuously differentiable function of $\hat{z}$. But the HJB equation implies that $W'(\hat{z})$ also solves this equation and, by continuity, must lie in the same neighborhood of z for $\hat{z}$ close enough to z. Hence, $W'(\hat{z})$ must coincide with the continuously differentiable function obtained by the above application of the implicit function theorem. Moreover, the implicit function theorem also provides an explicit expression for the derivative of $W'(z)$, i.e., for the second derivative of $W(z)$,

$$W''(z) = -\frac{(r+\alpha)W'(z) - \partial H/\partial z}{-\partial H/\partial\lambda} = \frac{-\alpha U'\big[y(z)\big] + (r+\alpha+\pi)W'(z)}{s(z)}, \tag{52}$$

where $s(z) \equiv h[W'(z)] - c[W'(z)] - \pi z + \Upsilon$ denotes savings at z.

A.8 *The first derivative of W when savings are zero*

Consider any point $z^\star > 0$ such that savings are zero, i.e., $h^\star - c^\star - \pi z^\star + \Upsilon = 0$, where $h^\star$ and $c^\star$ denote the optimal flow labor and consumption at $z^\star$. Let $y^\star$ denote the optimal lumpy consumption at $z^\star$. Then, for any z close enough to $z^\star$, $c^\star$, $h^\star$, and $y^\star + z - z^\star$ are feasible for the optimization problem in the HJB equation at z, and generate savings equal to $-\pi(z - z^\star)$. Therefore,

$$(r+\alpha)W(z) \geq u\big(c^\star, \bar{h} - h^\star\big) + \alpha\big[U\big(y^\star + z - z^\star\big) + W\big(z^\star - y^\star\big)\big] - \pi W'(z)\big(z - z^\star\big),$$

with an equality if $z = z^\star$. Subtracting the equality at $z^\star$ from the inequality at z, we obtain

$$(r+\alpha)\big[W(z) - W\big(z^\star\big)\big] \geq \alpha\big[U\big(y + z - z^\star\big) - U\big(y^\star\big)\big] - \pi W'(z)\big(z - z^\star\big).$$

After dividing by $z - z^\star > 0$, letting $z \downarrow z^\star$, and keeping in mind that $W'(z)$ is continuous, we obtain that $W'(z^\star) \geq \alpha U'(y)/(r+\alpha+\pi)$. Dividing by $z - z^\star < 0$ and letting $z \uparrow z^\star$, we obtain the reverse inequality. Altogether, we obtain that

$$W'\big(z^\star\big) = \frac{\alpha U'\big(y^\star\big)}{r+\alpha+\pi}.$$

A.9 *The second derivative of W goes to minus infinity at zero*

We know from the previous section that if the saving function is zero for some $z > 0$, then $(r+\alpha+\pi)W'(z) = \alpha U'[y(z)] \geq \alpha U'(z) \to \infty$ if $z \to 0$. But we have shown that $W'(z)$ is bounded. Therefore, savings are nonzero for all z close enough to zero and so W is twice continuously differentiable. In the expression (52) for $W''(z)$, the numerator goes to $-\infty$ as $z \to 0$ because $y(z) \le z$ and because $W'(z)$ remains bounded. At the same time, concavity implies that $W''(z) \le 0$. Therefore, the denominator cannot be negative; hence, $s(z) > 0$. Since savings are decreasing, they must remain bounded away from zero for all z close enough to zero. Since the numerator of (52) goes to $-\infty$, the result follows.

A.10 *The second derivative of W when savings are zero*

The previous section showed that savings are strictly positive near zero, while the section before showed that savings are strictly decreasing in z. Hence, there exists a unique $z^\star > 0$ such that $s(z^\star) = 0$. At $z^\star$, the expression (52) for $W''(z)$ is not well defined because both the numerator and the denominator are equal to zero. In the Supplemental Appendix, Section SIII.2, available in a supplementary file on the journal website, http://econtheory.org/supp/2821/supplement.pdf, we use a saddle point argument to show that $W''(z)$ has left and right limits at $z^\star$.[28] Given that left and right limits exist, an application of l'Hôspital rule to (52) show that these left and right limits must be the negative solutions of

$$\begin{aligned}
&\{h'[W'(z^\star)+] - c'[W'(z^\star)+]\}x^2 - (r+\alpha+2\pi)x + \alpha V''(z^\star) = 0,\\
&\{h'[W'(z^\star)-] - c'[W'(z^\star)-]\}x^2 - (r+\alpha+2\pi)x + \alpha V''(z^\star) = 0,
\end{aligned}$$

where $V(z) \equiv \max_{y\in[0,z]}\{U(y) + W(z-y)\}$. The above equation presumes that $V(z)$ is twice continuously differentiable in a neighborhood of $z^\star$, a result we also establish in the Supplemental Appendix, Section SIII.2, available in a supplementary file on the journal website, http://econtheory.org/supp/2821/supplement.pdf. Alternatively, since the flow labor constraint may bind, we must allow for the possibility that $h(\lambda)$ and $c(\lambda)$ are not differentiable at $W'(z^\star)$.

A.11 *An integral formula for the marginal value*

We conclude the analysis with the derivation of an integral formula for marginal value.

Lemma 3. *Let $\mathcal{T}(x)$ denote the time to reach $x \in [0, z^\star)$ real balances starting from zero. Then $z \in [0, z^\star)$:*

$$W'(z) = \frac{\alpha}{r+\alpha+\pi}\int_z^{z^\star} U'[y(x)]\,dG(x \mid z), \quad \textit{where } G(x \mid z) \equiv 1 - e^{-(r+\alpha+\pi)[\mathcal{T}(x)-\mathcal{T}(z)]}.$$

[28]Specifically consider the real balance at time t, z_t, starting from some $z_0 \neq z^\star$. Let the marginal value of real balances at time t be $\lambda_t = W'(z_t)$. By construction, $z_t \to z^\star$ and $\lambda_t \to \lambda^\star \equiv W'(z^\star)$ as $t \to \infty$. From (52), it is clear that (z_t, λ_t) solves the initial value problem $\dot{z}_t = h(\lambda_t) - c(\lambda_t) - \pi z_t + \Upsilon$ and $\dot{\lambda}_t = (r+\alpha+\pi)\lambda_t - \alpha V'(z_t)$, starting with initial condition (z_0, λ_0). One can then verify that $z^\star$ and $\lambda^\star = W'(z^\star)$ is a saddle point of this problem; hence, (z_t, λ_t) must coincide with the saddle path since it converges to $(z^\star, \lambda^\star)$. A standard first-order approximation of the saddle path near the saddle point $(z^\star, \lambda^\star)$ shows that $\dot{\lambda}_t/\dot{z}_t$ has a limit as $t \to \infty$, which corresponds to the limit of the second derivative of W at $z^\star$.

For the proof, recall from Theorem 1 that of SI preferences, the value function is twice continuously differentiable over $(0,\infty)$ except perhaps at $z^\star$. Hence, we can take derivatives on the right side of the HJB equation along the path of real balances z_t. Applying the envelope condition, we obtain that $(r+\alpha+\pi)W'(z_t) = \alpha U'[y(z_t)] + W''(z_t)\dot{z}_t$ if $z_t < z^\star$. We can integrate this formula forward over the time interval $[t,\infty]$ and we obtain that $W'(z_t) = \int_t^\infty \alpha U'[y(z_s)]e^{-(r+\alpha+\pi)(s-t)}\,ds$. Keeping in mind the definition of $G(x \mid z)$, we obtain that the integral can be written as

$$\begin{aligned}\int_t^\infty \alpha U'\big[y(z_s)\big]e^{-(r+\alpha+\pi)(s-t)}\,ds &= \frac{\alpha}{r+\alpha+\pi}\int_t^\infty U'\big[y(z_s)\big]\,d\big[G(z_s \mid z_t)\big] \\ &= \frac{\alpha}{r+\alpha+\pi}\int_z^{z^\star} U'\big[y(x)\big]\,dG(x \mid z),\end{aligned}$$

where the second equality follows by an application of the change of variable formula for the Lebesgue–Stieltjes integral (see Carter and Van Brunt 2000, Theorem 6.2.1).

Appendix B: Proof of Proposition 1

The results follow directly because $U(y)$ and $W(z)$ are both strictly concave and continuously differentiable because $U'(0)=\infty$ while $W'(0)<\infty$ and because $U'(\infty) = W'(\infty) = 0$. The expression for the derivative follows from an application of the envelope theorem, for example, Corollary 5 in Milgrom and Segal (2002).

Appendix C: Proof of Proposition 2

All results follow directly from Theorem 1, except for the strict positivity of $s(z)$ near $z=0$, which follows from arguments shown in Appendix A.10.

Appendix D: Proof of Proposition 3

The saving functions is continuously differentiable in $z>0$. However, since the initial value problem (9) starts at $z=0$, we cannot directly use a standard existence theorem because $W''(0)=\infty$ and so $s(z)$ fails to be Lipschitz with respect to z at $z=0$. However, it is easy to construct a solution by running the ODE forward and backward starting at some $\hat{z} \in (0, z^\star)$. The details are provided in Rocheteau et al. (2017, Section III.3).

Appendix E: Proof of Proposition 4

The proof follows from a direct application of Theorems 12.12 and 12.13 in Stokey et al. (1989). To show existence and uniqueness, the key step is to verify monotonicity, i.e., a household with higher real balances just before a lumpy consumption opportunity tends to have higher real balances at the next consumption opportunity. This follows because $z-y(z)$ is increasing in z, and a higher starting real balance shifts up the subsequent path of real balances. To show continuity with respect to Υ, in the sense of weak convergence, the key step is to establish that the household's decision rules are appropriately continuous in Υ. The details are provided in the Supplemental Appendix, Sections SII, SIII.3, and SV, available in a supplementary file on the journal website, http://econtheory.org/supp/2821/supplement.pdf.

Appendix F: Proof of Theorem 2

F.1 *Existence*

Note first that the stationary distribution cannot be concentrated at $z = 0$, since $Q(z, \{0\}) = 0$ for all z. Hence, when $\phi = 0$, the left side of (12) is 0 and so is less than the right side, which is strictly positive. When $\phi \to \infty$, we have from the upper bound of Theorem 1 that $W'(z) \to 0$ for all $z \in [0, \infty)$. This implies that labor supply is 0 and consumption is strictly positive for all $z \in [0, z^\star]$; hence, the saving rate is $s(z) < -\pi z + \Upsilon$. Plugging in $s(z^\star) = 0$, it follows that $z \leq z^\star < \Upsilon/\pi$ for all real balances z in the support of the stationary distribution, $[0, z^\star]$, implying that the right side of (12) is less than the left side. Finally, note that (12) is continuous in ϕ because, by Proposition 4, the stationary distribution is continuous in ϕ in the sense of weak convergence. The result follows from an application of the intermediate value theorem.

F.2 *Uniqueness in the laissez-faire case, $\pi = 0$*

This follows directly from the observation that the household's optimization problem does not depend on any endogenous variable. In the laissez-faire case, $\pi = 0$, uniqueness follows directly.

F.3 *Uniqueness in the inflationary case, $\pi > 0$*

As a preliminary result, we first show in the Supplemental Appendix, Section SIII.3, available in a supplementary file on the journal website, http://econtheory.org/supp/2821/supplement.pdf that the marginal value of real balances, $W'(z \mid \Upsilon)$, is strictly decreasing in lump-sum transfer. The intuition is simply that future lump-sum transfers and current real balances are imperfect substitutes.

F.3.1 *A change of variable*

Lemma 4. *Consider the stochastic process for real balances adjusted for money transfers $\hat{z}_t(\Upsilon) \equiv z_t(\Upsilon) - \Upsilon/\pi$, any $\Upsilon' > \Upsilon$, and any initial conditions $\hat{z}_0(\Upsilon') \leq \hat{z}_0(\Upsilon)$. Then holding the sequence of lumpy consumption times the same for Υ' and Υ, $\hat{z}_t(\Upsilon') < \hat{z}_t(\Upsilon)$ at all $t > 0$.*

For the proof, write the law of motion for adjusted real balances as

$$t \in (T_{n-1}, T_n): \quad \dot{\hat{z}}_t(\Upsilon) = h\left[\hat{z}_t(\Upsilon) + \frac{\Upsilon}{\pi} \,\middle|\, \Upsilon\right] - c\left[\hat{z}_t(\Upsilon) + \frac{\Upsilon}{\pi} \,\middle|\, \Upsilon\right] - \pi \hat{z}_t(\Upsilon),$$

$$t = T_n: \quad \hat{z}_{t+}(\Upsilon) = \hat{z}_t(\Upsilon) - y\left[\hat{z}(\Upsilon) + \frac{\Upsilon}{\pi} \,\middle|\, \Upsilon\right],$$

where $h(z \mid \Upsilon)$, $c(z \mid \Upsilon)$, and $y(z \mid \Upsilon)$ are, respectively, the decision rules for optimal flow labor, flow consumption, and lumpy consumption implied by the Hamilton–Jacobi–Bellman equation. Since we already know from Theorem 1 that $W'(z \mid \Upsilon)$ is strictly decreasing in z, and since $W'(z \mid \Upsilon)$ is strictly decreasing in Υ, it follows directly from taking first-order conditions in the HJB equation that $h(\hat{z}+\Upsilon/\pi \mid \Upsilon)$ is decreasing in $\hat{z}$ and Υ, that $c(\hat{z}+\Upsilon/\pi \mid \Upsilon)$ is strictly increasing in $\hat{z}$ and Υ, that $y(\hat{z}+\Upsilon/\pi \mid \Upsilon)$ is strictly increasing in $\hat{z}$ and increasing in Υ, and that $\hat{z}-y(\hat{z}+\Upsilon/\pi \mid \Upsilon)$ is increasing in $\hat{z}$ and strictly decreasing in Υ. Note that the last property, that $\hat{z}-\hat{y}(\hat{z}+\Upsilon/\pi \mid \Upsilon)$ is strictly decreasing in Υ, follows because $y(z \mid \Upsilon)$ is strictly increasing in z.

Now, to prove the lemma, we first note that by the monotonicity of the decision rules, if $\hat{z}_0(\Upsilon') = \hat{z}_0(\Upsilon)$, then $\dot{\hat{z}}_0(\Upsilon') < \dot{\hat{z}}_0(\Upsilon)$, implying that, for t close enough to 0, the Υ' path of real balances is below the Υ path, $\hat{z}_t(\Upsilon') < \hat{z}_t(\Upsilon)$. If $\hat{z}_0(\Upsilon') < \hat{z}_0(\Upsilon)$, then the same conclusion obtains by continuity. Now consider the first time $t < T_1$, if any, such that the Υ' path crosses the Υ path, i.e., such that $z_t(\Upsilon) = z_t(\Upsilon')$. Since the Υ' path of real balance is below the Υ path of real balances before time t, it follows that at time t, the Υ' path crosses the Υ path from below, that is, $\dot{\hat{z}}_t(\Upsilon') \geq \dot{\hat{z}}_t(\Upsilon)$. But this is impossible given that, holding real balances the same, flow labor is smaller and flow consumption is strictly larger under Υ' than under Υ. Therefore, the Υ' path never crosses the Υ path over $(0, T_1)$. In particular, at time T_1, we have that $\hat{z}_{T_1}(\Upsilon') < \hat{z}_{T_1}(\Upsilon)$. Given that $\hat{z} - y(\hat{z}+\Upsilon/\pi \mid \Upsilon)$ is increasing in $\hat{z}$ and strictly decreasing in Υ, it follows that adjusted real balances after lumpy consumption satisfy $\hat{z}_{T_1+}(\Upsilon') < \hat{z}_{T_1+}(\Upsilon)$. Proceeding by induction over successive lumpy consumption opportunities, we obtain the desired property.

F.3.2 *The distribution of adjusted balances is decreasing* Let $\hat{F}(\hat{z} \mid \Upsilon)$ denote the stationary distribution of the adjusted real balances process. Below we establish two key monotonicity properties for that stationary distribution.

Lemma 5. *For all* $\Upsilon < \Upsilon'$ *and any increasing function* $g(\hat{z})$, $\int g(\hat{z})\, d\hat{F}(\hat{z} \mid \Upsilon') \leq \int g(\hat{z})\, d\hat{F}(\hat{z} \mid \Upsilon)$.

Start from any initial conditions $\hat{z}_0(\Upsilon') \leq \hat{z}_0(\Upsilon)$. By an application of Lemma 4, we have that, for any increasing function $g(z)$, $g[\hat{z}_{T_n}(\Upsilon')] \leq g[\hat{z}_{T_n}(\Upsilon)]$, where T_n is the arrival of the nth lumpy consumption opportunity. Taking expectations on both sides gives $\mathbb{E}[g(\hat{z}_{T_n}(\Upsilon')) \mid \hat{z}_0(\Upsilon')] \leq \mathbb{E}[g(\hat{z}_{T_n}(\Upsilon)) \mid \hat{z}_0(\Upsilon)]$. Theorem 12.12 in Stokey et al. (1989), which we used to establish the existence of a stationary distribution, allows us to take the limit as long as $g(z)$ is continuous, and the result follows.

Lemma 6. *For all* $\Upsilon < \Upsilon'$, $\int \hat{z}\, d\hat{F}(\hat{z} \mid \Upsilon') < \int \hat{z}\, d\hat{F}(\hat{z} \mid \Upsilon)$.

Now consider again any initial condition $\hat{z}_0(\Upsilon) = \hat{z}_0(\Upsilon') = \hat{z} \in [-\Upsilon/\pi, \infty)$. From Lemma 4, we have that $\mathbb{E}[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}] < \mathbb{E}[\hat{z}_{T_1}(\Upsilon) \mid \hat{z}_0(\Upsilon) = \hat{z}]$. We then integrate both sides against the stationary distribution of $\hat{z}(\Upsilon)$, $\hat{F}(\hat{z} \mid \Upsilon)$, over the interval

$[-\Upsilon/\pi, \infty)$, and we obtain

$$\int_{-\frac{\Upsilon}{\pi}}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}\big]\, d\hat{F}(\hat{z} \mid \Upsilon) < \int_{-\frac{\Upsilon}{\pi}}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon) \mid \hat{z}_0(\Upsilon) = \hat{z}\big]\, d\hat{F}(\hat{z} \mid \Upsilon)$$
$$\Leftrightarrow \quad \int_{-\infty}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}\big]\, d\hat{F}(\hat{z} \mid \Upsilon) < \int_{-\infty}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon) \mid \hat{z}_0(\Upsilon) = \hat{z}\big]\, d\hat{F}(\hat{z} \mid \Upsilon)$$
$$\Leftrightarrow \quad \int_{-\infty}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}\big]\, d\hat{F}(\hat{z} \mid \Upsilon) < \int_{-\infty}^{+\infty} \hat{z}\, d\hat{F}(\hat{z} \mid \Upsilon).$$

On both sides of the second line, we replaced an integral over $[-\Upsilon/\pi, \infty)$ by an integral over $(-\infty, +\infty)$ because $\hat{F}(\hat{z} \mid \Upsilon)$ puts no mass on $\hat{z} < -\Upsilon/\pi$. On the right side of the third line, we used that $\hat{F}(\hat{z} \mid \Upsilon)$ is the stationary distribution of the sampled stochastic process $\hat{z}_{T_n}(\Upsilon)$ for adjusted real balances. Now, since the Markov process for adjusted real balances is monotone, and since the transition probability function satisfies the Feller property, it follows that $\hat{z} \mapsto \mathbb{E}[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}]$ is an increasing and continuous function. Therefore, an application of Lemma 5 implies that

$$\int_{-\infty}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}\big]\, d\hat{F}\big(\hat{z} \mid \Upsilon'\big) \leq \int_{-\infty}^{+\infty} \mathbb{E}\big[\hat{z}_{T_1}(\Upsilon') \mid \hat{z}_0(\Upsilon') = \hat{z}\big]\, d\hat{F}(\hat{z} \mid \Upsilon).$$

But since $\hat{F}(\hat{z} \mid \Upsilon')$ is the stationary distribution for $\hat{z}_t(\Upsilon')$, the left side is equal to $\int \hat{z}\, d\hat{F}(\hat{z} \mid \Upsilon')$ and the result follows.

F.3.3 *The equilibrium is unique* In term of the nonadjusted process for real balances, Lemma 6 can be written as

$$\pi \int z\, dF\big(z \mid \Upsilon'\big) - \pi \int z\, dF(z \mid \Upsilon) < \Upsilon' - \Upsilon$$

for any $\Upsilon < \Upsilon'$. Now take Υ to be the smallest solution of the equilibrium equation. Then rearranging terms above leads to

$$\pi \int z\, dF\big(z \mid \Upsilon'\big) - \Upsilon' < \pi \int z\, dF(z \mid \Upsilon) - \Upsilon = 0,$$

where the equality to 0 on the right side follows because Υ satisfies the equilibrium equation. The uniqueness result follows.

Appendix G: Proof of Proposition 5

Note that, at the target $z^\star$, $h^\star - c^\star - \pi z^\star + \Upsilon = 0$, where $(c^\star, h^\star)$ are optimal consumption and labor choices when $z = z^\star$. Since, in equilibrium, $\Upsilon = \pi \int_0^{z^\star} z\, dF(z) < \pi z^\star$, we obtain that $h^\star - c^\star > 0$. One easily shows that $h(\lambda) - c(\lambda)$ is strictly increasing, and goes to $-\infty$ as $\lambda \to 0$ and to $\bar{h}$ as $\lambda \to \infty$. Since $h^\star - c^\star = h[W'(z^\star)] - c[W'(z^\star)] > 0$, we obtain that $W'(z^\star) \geq \underline{\lambda} > 0$, where $\underline{\lambda}$ solves $h(\lambda) - c(\lambda) = 0$. Moreover, $\underline{\lambda}$ does not depend on π since $h(\lambda)$ and $c(\lambda)$ do not depend on π. Next, we use the result of Appendix A.9: $(r + \alpha + \pi)W'(z^\star) = U'[y(z^\star)]$. Since $W'(z^\star) \geq \underline{\lambda}$, this implies that $\lim_{\pi \to \infty} y(z^\star) = 0$. Finally, since we have established in Theorem 1 that $W'(0) \leq (r+\alpha)/\bar{h} \times (\|u\| + \alpha\|U\|)/r$, we obtain that $W'(0) < U'[y(z^\star)]$ for π large enough. Therefore, the solution of the optimal lumpy consumption problem is $y(z^\star) = z^\star$, i.e., there is full depletion. We conclude that $\lim_{\pi\to\infty} z^\star = \lim_{\pi\to\infty} y(z^\star) = 0$.

Appendix H: Proof of Proposition 6

From (23), $\lim_{\bar{h}\to\infty} F(z) = 0$ for all $z < z^\star$ and $F(z) = 1$ for all $z \geq z^\star$. From (24), $\lim_{\bar{h}\to\infty} \phi M = z^\star$. Finally, from (25), we compute the value function in closed form:

$$\begin{aligned} W(z) &= z + W(z^\star) - z^\star \\ &\quad - \frac{\alpha}{r+\alpha}\int_z^{z^\star}\left[1 - e^{-\frac{(r+\alpha)(u-z)}{\bar{h}}}\right]\left[U'(u) - U'(z^\star)\right] du \quad \forall z < z^\star, \end{aligned} \tag{53}$$

$$W(z^\star) = \frac{\alpha}{r}\left\{U(z^\star) - z^\star - \frac{\alpha}{r+\alpha}\int_0^{z^\star}\left[1 - e^{-\frac{(r+\alpha)u}{\bar{h}}}\right]\left[U'(u) - U'(z^\star)\right] du\right\}. \tag{54}$$

From (53), $\lim_{\bar{h}\to\infty} W(z) = z + W(z^\star) - z^\star$, and from (54), $\lim_{\bar{h}\to\infty} W(z^\star) = \alpha[U(z^\star) - z^\star]/r$.

Appendix I: Proof of Proposition 7

I.1 *Part (i): Large labor endowment*

Fix some $\pi \geq 0$. We first note that $y(z^\star) \leq z^\star \leq z_s$; hence, equilibrium aggregate demand is bounded by αz_s independently of $\bar{h}$. Equilibrium aggregate supply can be written as

$$F(z^\star_-)\bar{h} + \left[1 - F(z^\star_-)\right]h^\star.$$

To remain bounded as $\bar{h} \to \infty$, it must be the case that $\lim_{\bar{h}\to\infty} F(z^\star_-) = 0$. This also implies that, for $\bar{h}$ large enough, there is an atom at $z^\star$, so that $W'(z^\star) = 1$ and $z^\star = z_s$. Because F converges to a Dirac distribution concentrated at $z^\star = z_s$, we have that $\lim_{\bar{h}\to\infty} \phi M = z_s$.

Next we argue that, as $\bar{h}$ is large enough, $y(z^\star) = y(z_s) = z_s$, i.e., all equilibria must feature full depletion. For this we use the expression for $W'(z)$ derived in Lemma 3. This expression was derived in the case of SI preferences, but in Rocheteau et al. (2017, Lemma 2), we extend it to the case of LRW preferences. We obtain

$$\begin{aligned} W'(0) &= \frac{\alpha}{r+\alpha+\pi}\int_0^{z^\star} U'\left[y(z)\right] dG(z \mid 0) \\ &\leq \frac{\alpha}{r+\alpha+\pi}\int_0^{z^\star} \max\left\{U'(z), W'(0)\right\} dG(z \mid 0) \\ &\leq \frac{\alpha}{r+\alpha+\pi}\left[G\left(z_s^- \mid 0\right)\int_{z\in[0,z_s)} \max\left\{U'(z), W'(0)\right\}\frac{dG(z \mid 0)}{G\left(z_s^- \mid 0\right)}\right. \\ &\quad \left. + \left[1 - G\left(z_s^- \mid 0\right)\right]\max\left\{U'(z_s), W'(0)\right\}\right], \end{aligned}$$

as long as $\bar{h}$ is large enough. To obtain the inequality of the first line, we have used that $U'[y(z)] = U'(z)$ if there is full depletion, while $U'[y(z)] = W'[z - y(z)] \leq W'(0)$ if there is partial depletion. To obtain the second line, we have used that $z^\star = z_s$ as long as $\bar{h}$ is large enough. Substituting the expression for $\mathcal{T}(z \mid \pi\phi M)$ into the definition of $G(z \mid 0)$, we obtain that

$$G(z \mid 0) = \begin{cases} 1 - \left(1 - \dfrac{z}{z_b}\right)^{1+\frac{r+\alpha}{\pi}} & \text{if } z < z_s, \\ 1 & \text{if } z = z_s. \end{cases}$$

Given that z_b goes to infinity as $\bar{h}$ goes to infinity, one sees that $G(z \mid 0)$ converges weakly to a Dirac distribution concentrated at z_s. We also have

$$\frac{G'(z \mid 0)}{G(z_s^- \mid 0)} = \left(1 + \frac{r+\alpha}{\pi}\right) \frac{\dfrac{1}{z_b}\left(1 - \dfrac{z}{z_b}\right)^{\frac{1+\alpha}{\pi}}}{1 - \left(1 - \dfrac{z_s}{z_b}\right)^{1+\frac{1+\alpha}{\pi}}} \leq \frac{\left(1 + \dfrac{r+\alpha}{\pi}\right)\dfrac{1}{z_b}}{1 - \left(1 - \dfrac{z_s}{z_b}\right)^{1+\frac{1+\alpha}{\pi}}} \to \frac{1}{z_s}$$

as $\bar{h}$ goes to infinity (since it implies $z_b \to \infty$). Thus, the conditional probability distribution, $G(z \mid 0)/G(z_s^- \mid 0)$, has a density that can be bounded uniformly in $\bar{h}$. Finally, our bound for $W'(0)$ in Theorem 1 can be written, in the case of linear preferences, as

$$W'(0) \leq \frac{r+\alpha}{\bar{h}}\left(\frac{\bar{h}+\bar{c}}{r} + \alpha\frac{\|U\|}{r}\right) \to 1 + \frac{\alpha}{r}$$

as $\bar{h} \to \infty$. Taken together, these observations imply that

$$\int_{z\in[0,z_s)} \max\{U'(z), W'(0)\}\frac{dG(z \mid 0)}{G(z_s^- \mid 0)} \leq \frac{1}{z_s}\int_0^{z_s} \max\left\{U'(z), 1 + \frac{\alpha}{r}\right\} dz + \varepsilon$$

for some $\varepsilon > 0$ as long as $\bar{h}$ is large enough (note that the integral on the right side is well defined since $U(z) = \int_0^z U'(z)\,dx$). Together with the fact that $G(z_s^- \mid 0) \to 0$ as $\bar{h} \to \infty$, we obtain that

$$G(z_s^- \mid 0)\int_{z\in[0,z_s)} \max\{U'(z), W'(0)\}\frac{dG(z \mid 0)}{G(z_s^- \mid 0)} \to 0 \quad \text{and} \quad 1 - G(z_s^- \mid 0) \to 1$$

as $\bar{h} \to \infty$. Hence, for any $\varepsilon > 0$, $W'(0) \leq \alpha/(r + \alpha + \pi)\max\{U'(z_s), W'(0)\} + \varepsilon$ as long as $\bar{h}$ is large enough. Picking $\varepsilon < (r + \alpha)U'(z_s)/(r + \alpha + \pi)$, we obtain that $W'(0) < \max\{W'(0), U'(z_s)\}$, which implies that $W'(0) < U'(z_s)$ for $\bar{h}$ large enough, i.e., there is full depletion.

Because $H = \alpha\phi M$ under full depletion, and because the distribution of real balances converges toward a Dirac distribution concentrated at z_s, we obtain that $\lim_{\bar{h}\to\infty} H = H^\infty(\pi) = \alpha z_s$, which is decreasing in π. Aggregate welfare can be written as $\alpha\int U(z)\,dF(z) - H$, the average utility enjoyed from lumpy consumption net of the average disutility of supplying labor. As $\bar{h} \to \infty$, F converges weakly to a Dirac distribution concentrated at z_s, and H converges to αz_s. It follows that welfare converges to $\mathcal{W}^\infty = \alpha[U(z_s) - z_s]$, which is decreasing with π.

I.2 *Part (ii): Small labor endowment*

We have shown that there exists a unique candidate equilibrium with full depletion. In this candidate equilibrium, the condition for binding labor is that $z_s \geq z_b$ or, using the definition of z_s,

$$U'(z_b) \leq 1 + \frac{r+\pi}{\alpha}.$$

Recall that $z_b = \bar{h}/\pi + \bar{h}/\alpha$ is an increasing function of $\bar{h}$. Since marginal utility is decreasing, the condition for binding labor can be written as

$$\bar{h} \in \big[0, \bar{H}(\pi)\big], \quad \text{where } \bar{H}(\pi) = \frac{\alpha\pi}{\alpha+\pi}\big(U'\big)^{-1}\left(1 + \frac{r+\pi}{\alpha}\right).$$

One immediately sees that $\lim_{\pi\to 0} \bar{H}(\pi) = \lim_{\pi\to\infty} \bar{H}(\pi) = 0$.

Next, we turn to the sufficient condition for full depletion. Using Lemma 3 we have, in the candidate equilibrium with full depletion,

$$W'(0) = \frac{\alpha}{r+\alpha+\pi}\int_0^{z_b} U'(z)\, dG(z), \quad \text{where } G(z) = 1 - \left(1 - \frac{z}{z_b}\right)^{1+\frac{r+\alpha}{\pi}}.$$

Substituting the expression for $G(z)$ into the integral, we obtain

$$W'(0) = \frac{\alpha}{\pi}\frac{1}{z_b}\int_0^{z_b} U'(z)\left(1 - \frac{z}{z_b}\right)^{\frac{r+\alpha}{\pi}} dz \leq \frac{\alpha}{\pi}\frac{U(z_b)}{z_b},$$

where the inequality follows by using $(1 - z/z_b)^{\frac{r+\alpha}{\pi}} \leq 1$, integrating, and keeping in mind that $U(0) = 0$. Full depletion obtains if $W'(0) \leq U'(z_b)$. Using the above upper bound for $W'(0)$, we obtain that a sufficient condition for full depletion is

$$\frac{\pi}{\alpha} \geq \frac{U(z_b)}{z_b U'(z_b)}.$$

Note that $z_b \leq (U')^{-1}(1)$, that the function $z \mapsto [U(z) - U(0)]/[zU'(z)]$ is continuous over $(0, (U')^{-1}(1)]$ and, by our maintained assumption in the lemma, bounded near zero. Hence, it is bounded over the closed interval $[0, (U')^{-1}(1)]$. Therefore, the condition for full depletion is satisfied if

$$\pi \geq \underline{\pi} \equiv \alpha \times \sup_{z\in[0,(U')^{-1}(1)]} \frac{U(z)}{zU'(z)}.$$

Output effect of inflation In the regime with binding labor, $h(z) = \bar{h}$ for all $z \in \operatorname{supp}(F)$. Hence, for all $\bar{h} \in [0, \hat{h}]$ and for all $\pi \in [\underline{\pi}, \bar{\pi}]$, $H = \bar{h}$.

Welfare effect of inflation From (28) in the regime with binding labor, $\phi M = \bar{h}/\alpha$. Hence, an increase in the money growth rate through lump-sum transfers is a mean-preserving reduction in the distribution of real balances. In this regime, social welfare is measured by

$$\mathcal{W} = \int \left[-h(z) + \alpha U(z)\right] dF(z) = -\bar{h} + \alpha \int U(z)\, dF(z).$$

Given the strict concavity of $U(y)$, money growth leads to an increase in welfare.

I.3 *Part (iii): Large inflation*

From (27), as $\pi \to \infty$, $z^\star \to 0$, $\phi M \to 0$, $H \to 0$, and $\mathcal{W} \to 0$.

Appendix J: Proof of Proposition 8

J.1 *Large labor productivity*

With linear preferences, equilibrium welfare is $W = \int\{\min\{\bar{c}, c(z)\} + \bar{h} - h(z) + \alpha U[y(z)]\}\, dF(z)$. The first-best welfare is, for large enough $\bar{h}$, $W^\star = \bar{h} + \alpha[U(y^\star) - y^\star]$. The welfare loss relative to the first-best allocation is then defined as the $\delta \in [0, 1]$ solving

$$\bar{h} + \alpha\left\{U\left[y^\star(1-\delta)\right] - y^\star\right\} = \int \left\{\min\left\{\bar{c}, c(z)\right\} + \bar{h} - h(z) + \alpha U\left[y(z)\right]\right\} dF(z).$$

We also provide an explicit formula for the threshold inflation $\underline{\pi}$ appearing in the proposition. We start by fixing some $\varepsilon > 0$ small enough such that

$$\alpha\left[U\left(y^\star\right) - U(\varepsilon)\right] - (r+\alpha)y^\star > 0. \tag{55}$$

The existence of such an ε is guaranteed by the maintained assumption that $\alpha U(y^\star) - (r+\alpha)y^\star > 0$. Given such an ε, we choose

$$\underline{\pi} \equiv \frac{(r+\alpha)y^\star}{\varepsilon}. \tag{56}$$

With these preliminaries in mind, we turn to the proof of the proposition.

J.1.1 *Sufficient optimality conditions for the household's problem*

We guess that, under the conditions stated in the proposition, the household optimal policy is as follows.

- For $z < z^\star$: flow consume $c = 0$, flow work $h = \bar{h}$, and lumpy consume $y = z$.
- For $z = z^\star$: flow consume $c = 0$, flow work $h = 0$, and lumpy consume $y = z$.
- For $z > z^\star$: flow consume $c = \bar{c}$, flow work $h = 0$, and lumpy consume some $y(z) > z^\star$.

The corresponding equations for the value function are

$$\text{for } z < z^\star, \quad (r+\alpha)W(z) = \alpha\left[U(z) + W(0)\right] + W'(z)(\bar{h} - \pi z), \tag{57}$$

$$\text{for } z = z^\star, \quad (r+\alpha)W\left(z^\star\right) = \bar{h} + \alpha\left[U\left(z^\star\right) + W(0)\right], \tag{58}$$

$$\text{for } z > z^\star, \quad (r+\alpha)W(z) = \bar{h} + \bar{c} + \alpha \max_{y\in[0,z]}\left[U(y) + W(z-y)\right] - W'(z)\bar{c}. \tag{59}$$

Below we solve for $W(z)$ explicitly for $z \leq z^\star$, and we provide an implicit construction of $W(z)$ for $z > z^\star$. A nonstandard feature of $W(z)$, which is a consequence of the discontinuity of the transfer scheme, is that it is not differentiable at $z^\star$, with a concave kink. This means in particular that the standard optimality verification argument, which assumes continuous differentiability, does not apply directly here. In the Supplemental Appendix, available in a supplementary file on the journal website, http://econtheory.org/supp/2821/supplement.pdf, we follow Bressan and Hong (2007) and Aguiar et al. (2013) so as to extend that argument to our case and prove sufficiency. That is, the stated optimal consumption–saving policy is optimal if the function $W(z)$ is bounded, continuously differentiable for $z \neq z^\star$, and satisfies two Hamilton–Jacobi–Bellman equations. First, for $z \neq z^\star$,

$$(r+\alpha)W(z) = \max\bigl\{u(c, \bar{h}-h) + \alpha\bigl[U(y) + W(z-y)\bigr] + W'(z)\bigl[h - c - \pi z + \tau(z)\bigr]\bigr\},$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$, and $y \in [0, z]$. Second, for $z = z^\star$,

$$(r+\alpha)W(z) = \max\bigl\{u(c, \bar{h}-h) + \alpha\bigl[U(y) + W(z-y)\bigr]\bigr\},$$

with respect to $c \geq 0$, $h \in [0, \bar{h}]$, and $y \in [0, z]$, and subject to $h - c - \pi z + \tau(z) = 0$.

Given that $u(c, \bar{h}-c) = \min\{c, \bar{c}\} + \bar{h} - h$ and given the transfer scheme $\tau(z \mid \pi)$, we obtain that the stated consumption–saving policy is optimal if there exists a bounded function $W(z)$, continuously differentiable for $z \neq z^\star$, satisfying (57)–(59) as well as

$$W'(z) < U'\bigl(z^\star\bigr) \quad \text{for } z < z^\star, \tag{60}$$

$$W'(z) > 1 \quad \text{for } z < z^\star, \tag{61}$$

$$W'(z) \leq 1 \quad \text{for } z > z^\star. \tag{62}$$

Condition (60) is sufficient for full depletion, that is, $z = \arg\max_{y \in [0,z]}\{U(y) + W(z-y)\}$.

J.1.2 *A closed-form expression for $W(z)$ for $z \leq z^\star$* We use (57) and (58) to construct a guess for the function $W(z)$. First, taking the limit in (57) as $z \uparrow z^\star$ and comparing with (58), we obtain that, for $W(z)$ to be continuous, it must be that $W'(z^\star-) = \bar{h}/(\bar{h} - \pi z^\star)$. Second, taking the derivative in (57), we obtain that

$$(r+\alpha+\pi)\lambda(z) = \alpha U'(z) + \lambda'(z)(\bar{h} - \pi z),$$

where we used our notation $\lambda(z) \equiv W'(z)$. After integration, using the terminal condition derived above, $\lambda(z^\star-) = \bar{h}/(\bar{h} - \pi z^\star)$, we obtain

$$\lambda(z) = \frac{\bar{h}}{\bar{h} - \pi z}\left[\frac{\alpha}{\bar{h}}\int_z^{z^\star} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}} dy + \left(\frac{\bar{h} - \pi z^\star}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}}\right]. \tag{63}$$

Hence,

$$W(z) = W(0) + \int_0^z \lambda(y)\, dy,$$

where $W(0)$ is obtained by equating the limit at $z \uparrow z^\star$ with (58):

$$rW(0) = \bar{h} + \alpha U(z^\star) - (r+\alpha)\int_0^{z^\star} \lambda(z)\, dz.$$

Conversely, one easily shows that the function $W(z)$ thus constructed satisfies (57) and (58). One can also check that this function is convex near $z^\star$ with a slope strictly larger than one in a left neighborhood of $z^\star$.

J.1.3 *Verification of the full time work condition, (61), for all $\bar{h}$ large enough* We first derive a uniform lower bound for $\lambda(z)$ over the interval $z \in [\varepsilon, 1]$ and show that it is greater than 1 for large enough $\bar{h}$. Clearly, in (63), the multiplicative term and the second term in the square bracket are both minimized at $z = \varepsilon$ over $z \in [\varepsilon, 1]$. The integral term is clearly positive. Therefore,

$$\begin{aligned}\lambda(z) &\geq \frac{\bar{h}}{\bar{h} - \pi\varepsilon}\left(\frac{\bar{h} - \pi y^\star}{\bar{h} - \pi\varepsilon}\right)^{\frac{r+\alpha}{\pi}} = 1 + \frac{1}{\bar{h}}\left[\pi\varepsilon - (r+\alpha)(y^\star - \varepsilon)\right] + o\left(\frac{1}{\bar{h}}\right) \\ &> 1 + \frac{1}{\bar{h}}\left[\pi\varepsilon - (r+\alpha)y^\star\right] + o\left(\frac{1}{\bar{h}}\right).\end{aligned}$$

Therefore, condition (56) ensures that as long as $\pi > \underline{\pi}$, this expression is strictly greater than 1 for all $\bar{h}$ large enough.

Next, we derive a uniform lower bound for $\lambda(z)$ over the interval $[0, \varepsilon]$ and we show that this lower bound is strictly greater than 1. The multiplicative term in (63) is greater than 1. The derivative of the integral between the square brackets is

$$\begin{aligned}\frac{d}{dz}\int_z^{z^\star} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}} dy &= -U'(z) + \frac{r+\alpha}{\bar{h} - \pi z}\int_z^{z^\star} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}} dy \\ &\leq -U'(\varepsilon) + \frac{r+\alpha}{\bar{h} - \pi\varepsilon} U(y^\star) < 0,\end{aligned}$$

as long as $\bar{h}$ is large enough. The second term in the square bracket of (63) is increasing in z and decreasing in $z^\star$. Therefore, as long as $\bar{h}$ is large enough, we obtain that

$$\lambda(z) \geq \frac{\alpha}{\bar{h}}\int_\varepsilon^{y^\star - \frac{\Delta}{\bar{h}}} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi\varepsilon}\right)^{\frac{r+\alpha}{\pi}} dy + \left(\frac{\bar{h} - \pi y^\star}{\bar{h}}\right)^{\frac{r+\alpha}{\pi}}.$$

By an application of the dominated convergence theorem, one easily sees that the integral on the right side converges toward $U(y^\star) - U(\varepsilon)$ as $\bar{h} \to \infty$. The second term on the right side can be written $1 - (r+\alpha)y^\star/\bar{h} + o(1/\bar{h})$. Taken together, we obtain

$$\lambda(z) \geq 1 + \frac{1}{\bar{h}}\left\{\alpha\left[U(y^\star) - U(\varepsilon)\right] - (r+\alpha)y^\star\right\} + o\left(\frac{1}{\bar{h}}\right).$$

Therefore, condition (55) ensures that the right side is strictly greater than 1 for all $\bar{h}$ large enough.

J.1.4 *Verification of the full depletion condition, (60)* We obtain a uniform upper bound for $\lambda(z)$ as follows. For $z \leq z^\star \leq y^\star$, we have that $\bar{h}/(\bar{h} - \pi z)$ is less than $\bar{h}/(\bar{h} - \pi y^\star)$. In the square bracket, both $(\bar{h} - \pi y)/(\bar{h} - \pi z)$ and $(\bar{h} - \pi z^\star)/(\bar{h} - \pi z)$ are less than 1. Therefore,

$$\lambda(z) \leq \frac{\bar{h}}{\bar{h} - \pi y^\star}\left[\frac{\alpha}{\bar{h}}U(y^\star) + 1\right] = 1 + \frac{1}{\bar{h}}\left[\pi y^\star + \alpha U(y^\star)\right] + o\left(\frac{1}{\bar{h}}\right).$$

Alternatively, with $z^\star = y^\star - \Delta/\bar{h}$, $U'(z^\star) = U'(y^\star) - U''(y^\star)\Delta/\bar{h} + o(1/\bar{h}) = 1 + |U''(y^\star)|\Delta/\bar{h} + o(1/\bar{h})$, since $U''(z) < 0$ and $U'(y^\star) = 1$ by assumption. Our choice of Δ ensures that $\Delta > [\pi y^\star + \alpha U(y^\star)]/|U''(y^\star)|$, and so (62) holds for all $\bar{h}$ large enough.

J.1.5 *Verification of the full consumption condition, (62), for all $\bar{h}$ large enough* We provide an implicit construction of the value function for $z \geq z^\star$ and show that the full consumption condition, $\lambda(z) \leq 1$, holds for all $z \geq z^\star$. Specifically, we show that, starting from the above construction of $W(z)$ for $z \in [0, z^\star]$, there exists $W(z)$, $V(z)$, and $\lambda(z)$ for $z \geq z^\star$ satisfying

$$W(z) = W(z^\star) + \int_{z^\star}^{z} \lambda(x)\, dx, \tag{64}$$

$$(r+\alpha)\lambda(z) = \alpha V'(z) - \bar{c}\lambda'(z) \quad \text{almost everywhere}, \tag{65}$$

$$\lambda(z^\star) = 1 \quad \text{and} \quad \lambda(z) \in [0,1] \text{ for } z > z^\star, \tag{66}$$

$$V(z) = \max_{y \in [0,z]}\left\{U(y) + W(z-y)\right\}. \tag{67}$$

A solution to this problem does solve the HJB equation (59) for $z \geq z^\star$, i.e., $(r+\alpha)W(z) = \bar{h} + \bar{c} + \alpha V(z) - \bar{c}W'(z)$. Indeed, a candidate value function solves the HJB holds if (i) it solves the HJB equation at $z^\star$ and (ii) its derivative satisfies the derivative of the HJB. Precisely,

$$(r+\alpha)W(z^\star) = \bar{h} + \bar{c} + \alpha V(z^\star) - \bar{c}W'(z^\star_+),$$

$$(r+\alpha)W'(z) = \alpha V'(z) - \bar{c}W''(z).$$

The first condition is satisfied because $W'(z^\star_+) = \lambda(z^\star) = 1$ by construction, and because $W(z^\star)$ and $V(z^\star)$ satisfy the Bellman equation (58) at $z^\star$. The second condition is satisfied because it is equivalent to (65), given that (64) ensures that $W'(z) = \lambda(z)$.

We now construct a solution to the problem (64)–(67) as follows. Suppose that we have constructed a solution over some interval $[z^\star, Z]$, where $Z \geq z^\star$. We first observe that

$$U'(z^\star) > U'(y^\star) = 1 \geq \sup_{x \in [0,z^\star]} \lambda(x) = \sup_{x \in [0,Z]} \lambda(x), \tag{68}$$

where the first equality and the first inequality follow from our construction of $W(z)$ and $\lambda(z)$ over $[0, z^\star]$, and the last equality follows because $\lambda(z) \leq 1$ for $z \in [z^\star, Z]$. We now show how to extend this solution over the interval $[Z, Z + z^\star]$. First, we let

$$\tilde{V}(z) \equiv \max_{y \in [z-Z, z]} U(y) + W(z-y), \tag{69}$$

which is well defined for all $z \in [Z, Z + z^\star]$, given that we have constructed $W(z)$ for all $z \le Z$ and since $z - y \le Z$ by the choice of our constraint set. Note that, in principle, the function $\tilde{V}(z)$ differs from $V(z)$ because it imposes the constraint that $y \ge z - Z$. Our goal is to show that, nevertheless, $\tilde{V}(z) = V(z)$. Precisely, if one extends $\lambda(z)$ over $[Z, Z + z^\star]$ using (65) and defines $W(z)$ using (64), then the household never finds it optimal to choose $y < z^\star$, implying that the additional constraint we imposed to define $\tilde{V}(z)$ is not binding.

We first establish that $\tilde{V}(z)$ is absolutely continuous and $\tilde{V}'(z) \le U'(z^\star)$. Consider first $z \in [Z, Z + z^\star/2]$. Given (68), it follows that the solution to (69) must be greater than $z^\star$. By implication, since $z - Z \le z^\star/2$, the solution y to (69) must be greater than $z - Z + z^\star/2$. Given this observation and after making the change of variable $x = z - y$, we obtain that

$$\tilde{V}(z) \equiv \max_{x \in [0, Z - z^\star/2]} U(z - x) + W(x).$$

The objective is continuously differentiable with respect to z, and its partial derivative is $U'(z - x) \le U'(z^\star/2)$ given that $z \ge Z$ and $x \le Z - z^\star/2$. Proceeding to the interval $z \in [Z + z^\star/2, Z + z^\star]$, we make the change of variable $x = z - y$ in (69) and obtain that $\tilde{V}(z) = \max_{x \in [0,Z]} U(z - x) + W(x)$. Again, the objective is continuously differentiable with a partial derivative with respect to z equal to $U'(z-x) \le U'(z^\star/2)$, since $z \ge Z + z^\star/2$ and $x \le Z$. Hence, in both cases, given that the objective has a bounded partial derivative with respect to z, we can apply Theorem 2 in Milgrom and Segal (2002): $\tilde{V}(z)$ is absolutely continuous and the envelope condition holds, i.e., $\tilde{V}'(z) = U'[y(z)]$ whenever this derivative exists. By condition (68), it follows that $y(z) \ge z^\star$; hence, $\tilde{V}'(z) \le U'(z^\star)$, as claimed.

Next, we construct a solution over $[Z, Z + z^\star]$. Given that the function $\tilde{V}(z)$ constructed above is absolutely continuous, we can integrate the ODE (65) with $\tilde{V}'(z)$ and we obtain a candidate solution:

$$\tilde{\lambda}(z) = \lambda(Z) e^{-\frac{r+\alpha}{\bar{c}}(z-Z)} + \frac{\alpha}{\bar{c}} \int_Z^z \tilde{V}'(x) e^{-\frac{r+\alpha}{\bar{c}}(z-x)}\, dx.$$

Notice that $U(y^\star)/y^\star = 1 + r/\alpha$ implies, by strict concavity, that $U'(y^\star) < 1 + r/\alpha$ and, hence, $U'(z^\star) < 1 + r/\alpha$ for $\bar{h}$ large enough. Keeping in mind that $\lambda(Z) \le 1$ and $\tilde{V}'(x) \le U'(z^\star)$, one obtains after direct integration the upper bound $\tilde{\lambda}(z) \le 1 \le U'(z^\star)$ for all $z \in [Z, Z + z^\star]$. Now let

$$\tilde{W}(z) = W(Z) + \int_Z^z \tilde{\lambda}(x)\, dx.$$

We now show that if we extend $W(z)$ by $\tilde{W}(z)$, $\lambda(z)$ by $\tilde{\lambda}(z)$, and $V(z)$ by $\tilde{V}(z)$ over the interval $[Z, Z + z^\star]$, we obtain a solution of the problem (64) and (65) over $[Z, Z + z^\star]$: indeed, we have just shown that $\tilde{\lambda}(z) = \tilde{W}'(z) \le U'(z^\star)$ for all $z \in [Z, Z + z^\star]$, implying that the constraint $y \ge z - Z$ we imposed in the definition of $\tilde{V}(z)$ is not binding. That is,

$$V(z) = \max_{y \in [0,z]} \{U(y) + W(z - y)\} = \max_{y \in [z-Z,Z]} \{U(y) + W(z - y)\} = \tilde{V}(z).$$

Hence, we have extended the solution from $[z^\star, Z]$ to $[Z, Z+z^\star]$. Notice that the argument does not depend on Z: we can start with $Z = z^\star$ and repeat this extension until we obtain a solution defined over $[z^\star, \infty)$.

Finally, we show that $W(z)$ is bounded. By construction we have

$$\lambda(z) = \lambda(z^\star)e^{-\frac{r+\alpha}{\bar{c}}(z-z^\star)} + \frac{\alpha}{\bar{c}}\int_{z^\star}^{z} V'(x)e^{-\frac{r+\alpha}{\bar{c}}(x-z^\star)}\,dx,$$

$$W(z) = W(z^\star) + \int_{z^\star}^{z} \lambda(y)\,dy.$$

Plugging the first equation into the second, keeping in mind that $\lambda(z^\star) = 1$, and changing the order of integration, we obtain

$$\begin{aligned}
W(z) &= W(z^\star) + \frac{\bar{c}}{r+\alpha}\left(1 - e^{-\frac{r+\alpha}{\bar{c}}(z-z^\star)}\right) + \frac{\alpha}{r+\alpha}\int_{z^\star}^{z} V'(x)\left[1 - e^{-\frac{r+\alpha}{\bar{c}}(z-x)}\right]dy \\
&\le W(z^\star) + \frac{\bar{c}}{r+\alpha} + \frac{\alpha}{r+\alpha}\left[V(z) - V(z_0^\star)\right] \\
&\le W(z^\star) + \frac{\bar{c}}{r+\alpha} + \frac{\alpha}{r+\alpha}\left[W(z) + \|U\| - W(z^\star)\right],
\end{aligned}$$

where the first inequality follows because $1 - e^{-\frac{r+\alpha}{\bar{c}}(z-x)} \le 1$ for all $x \in [z^\star, z]$, and the second inequality follows because $W(z) \le V(z) \le W(z) + \|U\|$. Rearranging and simplifying, we obtain that

$$W(z) \le W(z^\star) + \frac{\bar{c} + \alpha\|U\|}{r}.$$

J.2 *Low labor productivity*

We construct an equilibrium featuring full depletion and such that $z^\star < \bar{y}$. The ODE for the marginal value of real balances, (17), becomes

$$(r+\pi)\lambda_t = \alpha(A - \lambda_t) + \dot{\lambda}_t \quad \forall t \in \left[0, \mathcal{T}(z^\star)\right],$$

with $\dot{\lambda}_{\mathcal{T}(z^\star)} = 0$. With Poisson arrival rate, α, the household spends all its real balances, which generates a marginal surplus equal to $A - \lambda$. The solution is $\lambda_t = \mathbb{E}[e^{-(r+\pi)T_1}A] = \alpha A/(r+\pi+\alpha)$ for all $t \in \mathbb{R}_+$. It is straightforward to check that $A > \lambda_0 = \alpha A/(r+\pi+\alpha)$, which guarantees that full depletion is optimal. From $U'(\bar{y}^-) = A \ge 1 + (r+\pi)/\alpha$, the unconstrained target is such that $z_s \ge \bar{y}$. The condition $\pi \ge \alpha\bar{h}/(\alpha\bar{y} - \bar{h})$ implies $z_b = \bar{h}(1/\pi + 1/\alpha) \le \bar{y}$. So $z^\star = z_b = \bar{h}(1/\pi + 1/\alpha)$ and the equilibrium features full employment, $h = \bar{h}$. Finally, $\bar{h} < \alpha\bar{y}[\alpha(A-1) - r]/(\alpha A - r)$ implies $\bar{h} < \alpha\bar{y}$, which implies that the first best is such that $h = \bar{h}$.

Appendix K: Proof of Proposition 10

Let us first recall the problem of the household in the pure-currency economy under a lump-sum transfer scheme, Υ^1,

$$rW(z) = \max\left\{u(c, \bar{h} - h) + \alpha\left[U(y) + W(z-y) - W(z)\right] + W'(z)\dot{z}\right\}, \tag{70}$$

with respect to (c, h, y), subject to $c \geq 0$, $0 \leq h \leq \bar{h}$, $0 \leq y \leq z$, and $\dot{z} = h - c + \Upsilon^1 - \pi z$. Policy functions are denoted $y^1(z)$, $h^1(z)$, and $c^1(z)$. The distribution of real balances across households is $F^1(z)$. We compare equilibria of the pure-currency economy to equilibria of an economy with money and bonds such that $\varrho = -\pi$, i.e., money and bonds have the same rate of return. The household problem, (32), becomes

$$rW(\omega) = \max_{c,h,y}\{u(c, \bar{h} - h) + \alpha[U(y) + W(\omega - y) - W(\omega)] + W'(\omega)\dot{\omega}\}, \tag{71}$$

subject to $c \geq 0$, $0 \leq h \leq \bar{h}$, $0 \leq y \leq \omega$, and $\dot{\omega} = h - c - \pi\omega + \Upsilon^2$. The household problem (71) is formally equivalent to (70) provided that $\Upsilon^2 = \Upsilon^1$. If this condition holds, $y^2(\omega) = y^1(\omega)$, $h^2(\omega) = h^1(\omega)$, $c^2(\omega) = c^1(\omega)$, and the distributions of wealth across the two economies are the same, $F^2(\omega) = F^1(\omega)$. So as to check that $\Upsilon^2 = \Upsilon^1$, we use the budget constraint of the government,

$$\Upsilon^2 = [(\mu + \pi)q_0 - \mu\phi_0^2]B_0 + \pi\phi_0^2 M_0 = \pi(\phi_0^2 B_0 + \phi_0^2),$$

where we used that $q_0 = \phi_0^2$, B_0 is the initial supply of bonds, and $M_0 = 1$ is the initial supply of money. By market clearing,

$$\pi(\phi_0^2 B_0 + \phi_0^2 M_0) = \pi\int_0^\infty \omega\, dF^2(\omega) = \pi\int_0^\infty z\, dF^1(z) = \pi\phi_0^1 M_0 = \pi\phi_0^1,$$

which implies

$$\phi_0^2 = \frac{\phi_0^1}{1 + B_0} \tag{72}$$

and $\Upsilon^2 = \Upsilon^1$. From (38), the clearing of the bonds market requires that there is a $z^2(\omega) \in [y^2(\omega), \omega]$ such that

$$\phi_0^2 B_0 = \int_0^\infty [\omega - z^2(\omega)]\, dF^2(\omega).$$

Such a function exists provided that

$$\phi_0^2 B_0 \leq \int_0^\infty [\omega - y^2(\omega)]\, dF^2(\omega) = \int_0^\infty [z - y^1(z)]\, dF^1(z).$$

Substituting ϕ_0^2 by its expression given by (72), we rewrite this inequality as

$$B_0 \leq \beta \equiv \frac{\displaystyle\int_0^\infty [z - y^1(z)]\, dF^1(z)}{\displaystyle\int_0^\infty y^1(z)\, dF^1(z)}.$$

The proof of the converse, namely, any liquidity-trap equilibrium of a two-asset economy corresponds to an equilibrium of the pure-currency economy that features partial depletion, is analogous and is therefore omitted.

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