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Dynamic objective and subjective rationality

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We characterize prior-by-prior Bayesian updating using a model proposed by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) that jointly considers objective and subjective rationality. These rationality concepts are subject to the Bewley unanimity rule and maxmin expected utility, respectively, with a common set of priors and the same utility over consequences. We use this setup with two preference relations to develop a novel rationale for full Bayesian updating of maxmin expected utility preferences.

Keywords. Dynamic consistency, full Bayesian updating, incomplete preferences, objective rationality, subjective rationality, maxmin expected utility, multiple priors.

JEL classification. D81.

1. Introduction

The behavioral foundation of objective and subjective rationality proposed by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) (henceforth, GMMS) shows how the Knightian decision model of Bewley (2002) and the maxmin expected utility model of Gilboa and Schmeidler (1989) are complementary and can be studied in a single model.[1] Recall that an act f is a mapping from state space S to a set of possible consequences X. Given two acts f and g, it is objectively rational to choose f over g when this ranking appears uncontroversial to the decision maker (DM). Intuitively, the DM can convince

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[1]Motivated by Knight (1921), Bewley (2002) axiomatizes the unanimity rule in a multiple priors model by relaxing only completeness in the approach used by Anscombe and Aumann (1963) to axiomatize expected utility preferences. Using the same framework but inspired by the well known Ellsberg (1961) paradox, Gilboa and Schmeidler axiomatize the maxmin expected utility model by relaxing the independence axiom (see also Schmeidler 1989 and Chateauneuf 1991).

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others of the correctness of this declared ranking, and this power to convince stems from some sort of hard evidence that f is at least as good as g. However, in many situations, some courses of action cannot be objectively justified, but decisions must be made nonetheless. When these situations occur, subjectively rational choices represent situations in which others cannot convince the DM of being wrong. Both notions of rationality are considered and captured by a pair of preference relations that are intended to demonstrate the ability either to convince others or to stand by one's choice.[2]

In an Anscombe–Aumann setting, GMMS assume that objective preferences satisfy axioms that ensure a Bewley-type representation, while subjective preferences may satisfy only the minimal conditions of completeness, transitivity, and continuity. One of their main results shows that if a subjective preference represents the completion of an objective preference and these preferences jointly satisfy what they call *default to certainty*, the former is a maxmin expected utility preference. This result provides, inter alia, a novel foundation of maxmin expected utility preferences.

The GMMS approach does not address how DMs update their objective and subjective preferences in response to new information they obtain over time about possible events that may occur before uncertainty is completely resolved. Our main goal is to propose a dynamic version of the GMMS model that provides an axiomatic foundation for full Bayesian updating. The GMMS model specifies a set of prior probabilities, which should then be updated when there is new and relevant information about future contingencies. Our behavioral contribution provides two novel axioms on the interplay of unconditional objective preferences and conditional subjective preferences, which we call *intertemporal consistency* and *intertemporal default to certainty*. Weak conditions on subjective rationality are imposed; in fact, we assume that ex ante and ex post subjective preferences are complete, transitive, and continuous relations.

Essentially, our main result shows that intertemporal consistency and intertemporal default to certainty constitute an axiomatic foundation for conditional subjective relations to be maxmin expected utility preferences.[3] Most notably, the priors characterizing conditional subjective preference are derived by the prior-by-prior Bayesian updating of the priors characterizing the unconditional objective preference. We assume that standard dynamic consistency is a compelling requirement only for the ranking of acts based on hard evidence captured by ex ante and ex post objective preferences. Dynamic consistency entails the same full prior-by-prior updating rule for deriving the conditional objective preferences.

Related Literature. We provide a dynamic version of the GMMS model in which unconditional beliefs are updated prior-by-prior. To the best of our knowledge, ours is the first theory of belief updating in a model with two preference relations.

[2]See Gilboa (2009) and Postlewaite and Schmeidler (2012) for extensive discussions on rationality and uncertainty that consider both objective and subjective preferences.

[3]This result means that subjective preferences are sequentially consistent, as proposed by Sarin and Wakker (1998): Sequentially consistent means that if DMs have committed to a family of preferences, then they use the same family after conditioning on any (relevant) event. Note that this notion is completely independent of any form of dynamic consistency.

Ghirardato, Maccheroni, and Marinacci (GMM 2004, 2008) consider a model with a single binary relation that may exhibit nonneutrality to ambiguity and induce a subrelation called *unambiguous preference* that has a Bewley representation. By requiring dynamic consistency and consequentialism, GMM (2008) show that an unambiguous preference must be updated in accordance with full Bayesian updating. The equivalence between dynamic consistency and the prior-by-prior updating rule in Bewley's model was previously discussed by Bewley (1987) and Epstein and Le Breton (1993) as a kind of "folk theorem." Formally, the relationship between our model and GMMS is similar to the relationship between GMM (2008) and GMM (2004). We also note that while GMM (2008) assume consequentialism (as does Ghirardato 2002), we derive it from our mild assumptions (Proposition 2 in the Appendix). Furthermore, there seems to be no straightforward way to apply the arguments in GMM (2008) to derive foundations for prior-by-prior updating of maxmin expected utility in the two-preference setting (Theorem 1).

None of the papers discussed in GMMS (2010) that consider a pair of preference relations addresses updating (see Nehring 2001, 2009, Rubinstein 1988, Mandler 2005, Danan 2008, and Kopylov 2009). More recently, Giarlotta and Greco (2013), Karni and Vierø (2013), Lehrer and Teper (2014), Cerreia-Vioglio (2016), and Cerreia-Vioglio, Giarlotta, Greco, Maccheroni, and Marinacci (forthcoming) also consider a pair of binary relations, but again, these authors do not study updating. Finally, the reader is referred to the following sections in which we elaborate on other connections between our present work and the literature.

2. Framework

Consider a set S of states of nature endowed with a σ-algebra Σ of subsets called *events* and a nonempty set X of consequences. A function $f : S \to X$ is simple if $f(S) := \{f(s) : s \in S\}$ is finite. A simple function $f : S \to X$ is Σ-measurable if $\{s \in S : f(s) = x\} \in \Sigma$ for all $x \in X$. The set $\mathcal{F}$ denotes the collection of all simple and Σ-measurable functions, and each $f \in \mathcal{F}$ is called an *act*. The set $B_0(\Sigma)$ denotes the collection of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The sup norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = \sup_{s\in S} |a(s)|$, and $B(\Sigma)$ denotes the sup norm closure of $B_0(\Sigma)$.[4]

Given a mapping $u : X \to \mathbb{R}$, the function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s))$ for all $s \in S$. We note that $u(f) \in B_0(\Sigma)$ whenever f belongs to $\mathcal{F}$.

Let x be a consequence in X; abusing notation, we define $x \in \mathcal{F}$ as the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set of constant acts in $\mathcal{F}$. We assume that the set of consequences X is the convex subset of a vector space; e.g., in the Anscombe–Aumann setting, as restated by Fishburn (1970), X is the set of

[4]The set $B_0(\Sigma)$ can also be regarded as the vector space generated by the indicator functions of the elements of Σ, endowed with the sup norm (for further details, see Dunford and Schwartz (1988), Chapter 5, Section 5). The set $ba(\Sigma)$ denotes the Banach space of all bounded and finitely additive set functions on Σ endowed with the total variation norm and is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Note also that the weak* topology $\sigma(ba(\Sigma), B_0(\Sigma))$ of $ba(\Sigma)$ coincides with the eventwise convergence topology. Throughout this article, we assume that any subset of $ba(\Sigma)$ is endowed with the topology inherited from the weak* topology, which includes the subset of all probability measures.

all simple lotteries on a fixed outcome space. Recall that $u : X \to \mathbb{R}$ is affine when for all $x, y \in X$ and $\alpha \in [0, 1]$, $u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y)$. Moreover, the utility index u_1 is cardinally equivalent to the utility index u_2 if and only if the former is a positive affine transformation of the latter.

Due to the linear structure of X, we define for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act

$$\alpha f + (1-\alpha)g : S \to X$$
$$\big(\alpha f + (1-\alpha)g\big)(s) = \alpha f(s) + (1-\alpha)g(s).$$

Given two acts $f, g \in \mathcal{F}$ and an event $E \in \Sigma$, we denote by fEg the act that delivers the consequences $f(s)$ in E and $g(s)$ in $E^c := S \backslash E$.

The set $\Delta := \Delta(\Sigma)$ denotes the collection of all (finitely additive) probability measures $p : \Sigma \to [0, 1]$. Given an act $f \in \mathcal{F}$, a utility index u on X, and a probability measure $p \in \Delta$, the expected utility of f is denoted by $\int u(f)\, dp$.

Given a binary relation $\succsim$ on $\mathcal{F}$, also called a preference relation, the symmetric and asymmetric components of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. For each event $E \in \Sigma$, the ex ante preference relation $\succsim$ is associated with the ex post preference relation $\succsim_E$ on $\mathcal{F}$, which is the conditional preference that emerges after learning that event E has occurred (if $E = S$, we obtain $\succsim_S \equiv \succsim$).

An event $E \in \Sigma$ is *relevant* (with respect to $\succsim$) if there exist two consequences such that $x \succ y$, $xEy \succ y$, and $x \succ xEy$. The set $\mathcal{R}_{\succsim} \subseteq \Sigma$ denotes the family of all relevant events. For instance, suppose that $\succsim$ is an expected utility preference on $\mathcal{F}$ represented by the utility function $V(f) = \int u(f)\, dp$; we obtain $E \in \mathcal{R}_{\succsim}$ if and only if $p(E) > 0$. A more general conclusion holds if we assume that the underlying preference $\succsim$ is a Bewley preference or a maxmin expected utility preference: For all $E \in \mathcal{R}_{\succsim}$, given the corresponding set of multiple priors C, $p(E) > 0$ for all $p \in C$.

Let p be a probability measure in Δ and let $E \in \Sigma$ be such that $p(E) > 0$, the probability measure p^E denotes its Bayesian updating or its conditional probability (with respect to E), which is defined by

$$p^E(F) = \frac{p(E \cap F)}{p(E)} \quad \forall F \in \Sigma.$$

The corresponding conditional expected utility of an act $f \in \mathcal{F}$ is given by

$$\int_S u(f)\, dp^E = \frac{1}{p(E)} \int_E u(f)\, dp.$$

We extend the definition of Bayesian updating to sets of probability measures as follows. Let C be a set of probability measures and let $E \in \Sigma$ be such that $p(E) > 0$ for all $p \in C$. The set C^E is defined by the full Bayesian rule

$$C^E = \{p^E : p \in C\}.$$

3. Model and results

Objective and subjective rationality theory builds on two binary relations that represent the dual perspective of rationality as proposed by GMMS. The relations $\succsim^*$ and $\succsim^\#$ denote the objective and subjective preference relations, respectively. Suppose that the DM is informed that the true state of nature lies in a relevant event $E \in \mathcal{R}_{\succsim^*}$; the corresponding updating objective and subjective preferences are denoted by $\succsim^*_E$ and $\succsim^\#_E$, respectively. The following diagram illustrates the timing of our model:

$$\begin{array}{ccc} \succsim^* & \text{unconditional preferences} & \succsim^\# \\ \downarrow & \text{the event } E \text{ contains the true state} & \downarrow \\ \succsim^*_E & \text{conditional preferences} & \succsim^\#_E \end{array}.$$

The basic conditions are intended to capture the minimal properties that objective and subjective relations should satisfy.[5]

Basic Conditions. In this study, a binary relation $\succsim$ on $\mathcal{F}$ satisfies basic conditions if the following statements hold:

(i) The relation $\succsim$ is nontrivial, i.e., there exist acts $f, g \in \mathcal{F}$ such that $f \succ g$.

(ii) The relation $\succsim$ is reflexive, i.e., for any act $f \in \mathcal{F}$, $f \succsim f$.

(iii) The relation $\succsim$ is transitive, i.e., given $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

(iv) The relation $\succsim$ is mixture-continuous, i.e., given any $f, g, h \in \mathcal{F}$, the sets

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\} \quad \text{and} \quad \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$$

are closed in $[0, 1]$.

3.1 *Objective preferences*

We assume that the unconditional objective preference $\succsim^*$ satisfies the basic conditions and the following axioms:

Monotonicity. For all $f, g \in \mathcal{F}$, if $f(s) \succsim^* g(s)$ $\forall s \in S$, then $f \succsim^* g$.

C-Completeness. For all $x, y \in X$, either $x \succsim^* y$ or $y \succsim^* x$.

Independence. For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$,

$$f \succsim^* g \quad \Leftrightarrow \quad \alpha f + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h.$$

Monotonicity is a common property that can also be interpreted as state independence. The other conditions imposed by GMMS are natural properties of objective rationality. For instance, C-Completeness implies that if the objective preference is incomplete, then this is unrelated to any difficulties that DMs might have when determining their preferences under certainty. Independence follows the standard argument as discussed in GMMS (p. 757).

[5] For a detailed discussion of the rationale for the basic conditions, see GMMS (p. 759).

Given an objectively relevant event $E \in \mathcal{R}_{\succsim^*}$, consider the conditional objective preference $\succsim^*_E$. We call $\succsim^*_E$ the *objective dynamic consistent update* of the ex ante objective preference $\succsim^*$ if, for all acts $f, g \in \mathcal{F}$,

$$fEg \succsim^* g \quad \Leftrightarrow \quad f \succsim^*_E g.$$

When the pairing $(\succsim^*, \succsim^*_E)$ satisfies dynamic consistency as above, for any two acts h_1 and h_2 that may differ only across an event E, any novelty that excludes scenarios in E^c does not change the DM's ability to compare such acts or change the ranking between them. Moreover, the ex ante ranking $h_1 \succsim^* h_2$ can be fully justified by the conjecture that $E := \{s : h_1(s) \neq h_2(s)\}$ will occur when the corresponding ex post ranking reveals $h_1 \succsim^*_E h_2$.

Another standard and uncontroversial property is that individuals should not be concerned with states that they know will not occur.

Objective Consequentialism. For all acts $f, g \in \mathcal{F}$, $f \sim^*_E fEg$.

It would make sense for the objective justification of any declared ranking $f \succsim_E g$ to be unaffected by counterfactual arguments based on the outcomes in states in E^c. In fact, we can derive objective consequentialism from even simpler conditions: If a reflexive ex post objective preference $\succsim^*_E$ is the objective dynamic consistent update of the ex ante objective preference $\succsim^*$, then the relation $\succsim^*_E$ satisfies consequentialism[6] (see Proposition 2 in the Appendix).

3.2 *Conditional subjective preferences*

We now analyze how ex ante objective rationality supports and narrows down the conditional subjective preferences. Given an objectively relevant event $E \in \mathcal{R}_{\succsim^*}$, the subjective preference $\succsim^\#_E$ is assumed to satisfy the basic conditions and the following condition to avoid unresolved rankings.

Subjective Completeness. For all acts $f, g \in \mathcal{F}$, either $f \succsim^\#_E g$ or $g \succsim^\#_E f$.

The postulate of completeness for any relevant event E justifies the role of $\succsim^\#_E$ as summarizing the ranking of the DM if he has to make a choice after learning E.

Our first axiom about the link between unconditionally objective and conditionally subjective preferences is a natural extension of the consistency axiom of GMMS in our framework.

Intertemporal Consistency. Given two acts $f, g \in \mathcal{F}$, if $fEg \succsim^* g$, then $f \succsim^\#_E g$.

Consider two acts h_1 and h_2 that have the same consequences for all states outside of an objective relevant event E. If the DM can find an ex ante objective proof that h_1 is at least as good as h_2, then these acts must have the same ranking after learning that the

[6]Machina (1989) argues in favor of relaxing consequentialism in nonexpected utility theory in terms of its nonseparability. Hanany and Klibanoff (2007) study the problem of updating multiple prior preferences without consequentialism, where there is a dependence on the considered feasible set of acts and the act chosen from that set.

true state will belong to E. Clearly, if unconditionally objective and subjective preferences coincide,[7] then intertemporal consistency captures the "if" direction of dynamic consistency.[8] When $E = S$, we obtain consistency: For all $f, g \in \mathcal{F}$, if $f \succsim^* g$, then $f \succsim^\# g$. This is one of the main properties in the GMMS model.[9] Consistency also appears in Nehring (2001, 2009), where it is referred to as compatibility.

The next axiom complements our proposed link between ex ante objective preferences and ex post subjective preferences, and can be viewed as a natural extension of the default to certainty axiom proposed by GMMS.

Intertemporal Default to Certainty. For any act $f \in \mathcal{F}$ and consequence $x \in X$,

$$\text{if not } fEx \succsim^* x, \text{ then } x \succ^\#_E f.$$

Consider a partial resolution of uncertainty described by an event $E \in \mathcal{R}_{\succsim^*}$ to compare act f to constant act x. The DM may first check whether there are ex ante compelling reasons to prefer fEx to x. If the DM can conclude that $fEx \succsim^* x$, then $fEx \succsim^\#_E x$ from intertemporal consistency. Moreover, if $\succsim^\#_E$ satisfies consequentialism (a property that will be a consequence of our main result), then the DM also concludes that $f \succsim^\#_E x$. However, if there are no ex ante reasons to prefer fEx to x, then the DM will opt for the constant act rather than the uncertain act after learning about E.[10] Clearly, when $E = S$, we obtain default to certainty, as proposed in the GMMS model.[11]

3.3 *The main result*

Next, we present and discuss the main result of our paper.

THEOREM 1. *Let $\succsim^*$ and $\succsim^\#$ be two binary relations on $\mathcal{F}$, and consider an objectively relevant event $E \in \mathcal{R}_{\succsim^*}$. The following conditions are equivalent:*

(i) The preference relation $\succsim^*$ satisfies the basic conditions, C-completeness, monotonicity, and independence; $\succsim^\#_E$ satisfies the basic conditions and completeness; and the two relations $\succsim^*$ and $\succsim^\#_E$ jointly satisfy intertemporal consistency and intertemporal default to certainty.

[7]In the GMMS model, both relations coincide if, and only if, there is a single prior that represents objective rationality, which means that the objective relation is complete and, therefore, a subjective expected utility preference.

[8]Ghirardato (2002) calls this direction of dynamic consistency *consistency of implementation*. The "only if" direction of dynamic consistency is called *information is valuable*.

[9]We note that according to Theorem 14 in Faro (2015, p. 714), the GMMS characterization still works under a weaker notion of consistency.

[10]Clearly, since $\succsim^\#_E$ is complete, the intertemporal default to certainty can be rewritten as *if* $f \succsim^\#_E x$, *then* $fEx \succsim^* x$. Thus, if the DM cannot be ex post convinced of being wrong in choosing f over x, then there must be ex ante hard evidence for concluding that fEx is at least as good as x.

[11]This refers to the emphasis of GMMS on an axiom called *caution*, which is weaker than default to certainty: If not $f \succsim^* x$, then $x \succsim^\# f$. We note that under GMMS's multiple priors *representation*, default to certainty and caution are equivalent: Take a pair (u, C) that represents a DM in the sense of GMMS; then if $f \succsim^* x$ does not hold, there exists $q \in C$ such that $\int u(f)\,dq < u(x)$. Hence, $u(x) > \int u(f)\,dq \geq \min_{p\in C} \int u(f)\,dp$, i.e., $x \succ^\# f$.

(ii) There exist a nonconstant and affine function $u: X \to \mathbb{R}$ *and a nonempty, closed, and convex set* $C \subseteq \Delta$ *such that for all* $f, g \in \mathcal{F}$,

$$f \succsim^* g \quad \Leftrightarrow \quad \int u(f)\,dq \geq \int u(g)\,dq \quad \text{for all } q \in C$$

and

$$f \succsim^{\#}_E g \quad \Leftrightarrow \quad \min_{q \in C^E} \int u(f)\,dq \geq \min_{q \in C^E} \int u(g)\,dq.$$

Moreover, C is unique, and u is cardinally unique. Finally, $\succsim^*_E$ *is the objective dynamic consistent update of* $\succsim^*$ *if and only if for all* $f, g \in \mathcal{F}$,

$$f \succsim^*_E g \quad \Leftrightarrow \quad \int u(f)\,dq \geq \int u(g)\,dq \quad \text{for all } q \in C^E.$$

Theorem 1 provides an axiomatic foundation for a dynamic GMMS model of choices under uncertainty. As in GMMS, the ex ante objective preference has a Bewley representation with a utility index u and a set of priors C. The ex ante subjective preference has a maximum expected utility (MEU) representation with the same u and C, and is a completion of the ex ante objective preference. Our model extends GMMS by incorporating dynamics and updating into their two-preference model: The collection of all relevant events $\mathcal{R}_{\succsim^*}$ gives rise to a unique family of ex post MEU preferences represented by the same utility index u and sets of multiple priors C^E. Each of these sets is derived from the priors associated with ex ante preference by prior-by-prior updating.

In particular, our main result also allows us to conclude that subjective preferences are sequentially consistent in the sense of Sarin and Wakker (1998): Conditional subjective preferences belong to the same class as the ex ante subjective preference. In her axiomatization of prior-by-prior updating, Pires (2002) assumes that ex post preferences satisfy the Gilboa and Schmeidler (1989) axioms.[12] In Theorem 1, the fact that ex post preferences are MEU is a conclusion not an assumption. We note that setting $E = S$ in Theorem 1 yields GMMS's main result (see Theorem 4 in GMMS, p. 762, and Proposition 2 in Cerreia-Vioglio 2016, p. 533).

Our model describes a DM whose preferences are inherently incomplete; the DM will become an MEU maximizer by invoking subjective criteria only if forced to make a choice. As such, we would argue that our DM need not satisfy dynamic consistency. There is also a practical argument for relaxing dynamic consistency: In general, it is inconsistent with MEU. Epstein and Schneider (2003) identify a condition—rectangularity—that ensures the dynamic consistency of prior-by-prior updating of MEU preferences.[13] However, they note that the rectangularity condition does not hold

[12] Epstein and Schneider (2003) provide an axiomatic foundation for both *recursive* multiple priors and a prior-by-prior updating rule. This approach requires a special structure called *rectangularity* for the sets of priors. Similar to Pires (2002), they assume that each conditional preference satisfies adapted versions of the Gilboa–Schmeidler axioms.

[13] Gumen and Savochkin (2013) study the rectangularity condition used by Epstein and Schneider (2003) to characterize dynamic consistency maxmin preferences and show that under weak conditions, rectangularity cannot be achieved.

in the three-color Ellsberg urn experiment. Siniscalchi (2011) provides additional arguments indicating that rectangularity is too stringent a condition for many applications because it rules out ambiguity aversion.[14]

The final implication of Theorem 1 is that the objective dynamically consistent updates $\succsim_E^*$ are Bewley preferences with priors also derived from the priors of the ex ante objective (Bewley) preference by full Bayesian (i.e., prior-by-prior) updating. In particular, Theorem 1 also yields that the two ex post preference relations $\succsim_E^*$ and $\succsim_E^\#$ jointly satisfy consistency and default to certainty. Following GMMS, we interpret the set C of initial priors as representing ex ante expert opinions (based on hard evidence), and the updated set C^E is simply the collection of updated expert opinions after learning that the true state belongs to E. The idea of using prior-by-prior updating to ensure dynamic consistency is due to Bewley (1987) (see the discussion of Definition 4.2 in Bewley 1987, pp. 12–13). In a different framework, GMM (2008, Theorem 1) provide an axiomatic foundation for prior-by-prior updating of Bewley preferences by imposing consequentialism and dynamic consistency. Epstein and Le Breton (1993) also note the connection between dynamic consistency and prior-by-prior updating of Bewley preferences as a kind of "folk theorem" (see their Section 3.1 about incomplete preferences).

4. Conclusion

We propose and axiomatize a model of updating for the objective and subjective rationality theory of GMMS. In particular, this study provides a novel foundation for the full Bayesian updating of the maxmin expected utility preferences introduced by Gilboa and Schmeidler (1989). To the best of our knowledge, ours is the first theory for updating beliefs in a model of choice under uncertainty with a pair of preference relations. In Theorem 1, we assume that unconditional objective preferences are Bewley preferences and that conditional subjective preferences are complete relations satisfying mild conditions. We show that intertemporal consistency and intertemporal default to certainty constitute an axiomatic justification for the sequential consistency of maxmin subjective preferences. Most notably, these conditions also imply full Bayesian updating. To summarize, we show how the model of GMMS offers a novel rationale for full Bayesian updating of maxmin expected utility preferences.

A future agenda for the GMMS model might include a generalization of Theorem 1 to the variational preferences introduced by Maccheroni, Marinacci, and Rustichini (2006).[15] Building on Cerreia-Vioglio (2016), this generalization could be made by weakening our intertemporal default to certainty. However, such a generalization seems difficult, since it would require additional assumptions regarding the conditional subjective preferences and even more assumptions to ensure that a suitable notion of full Bayesian updating for variational preferences could be satisfied.

[14] Kajii and Ui (2009) study interim efficiency with maxmin preferences and rectangular prior sets; they find a condition that is only sufficient for nonspeculative trade. Martins-da-Rocha (2010, Example 8.1, p. 2009) comprehensively evaluates this issue and shows that rectangularity is not a necessary condition for interim efficiency.

[15] This could be achieved by considering the more general case of uncertainty-averse preferences introduced by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), which include the variational preferences and confidence preferences introduced by Chateauneuf and Faro (2009), to name but a few.

Appendix

Next, we elaborate on the result related to consequentialism, as discussed in the presentation of objective dynamic consistent update relations.

Proposition 2. *Given a relevant event $E \in \mathcal{R}_{\succsim^*}$, assume that $\succsim^*$ is reflexive and that the pairing $(\succsim^*, \succsim_E^*)$ satisfies dynamic consistency. Then $\succsim_E^*$ satisfies objective consequentialism.*

Proof. Given $E \in \mathcal{R}_{\succsim^*}$, we need to show that for all $f, g \in \mathcal{F}$,

$$f \sim_E^* fEg.$$

Recall that dynamic consistency implies that $\succsim_E^*$ follows the rule

$$f \succsim_E^* g \quad \Leftrightarrow \quad fEg \succsim^* g.$$

Since $\succsim^*$ is reflexive, we obtain that

$$f \succsim^* f \quad \Leftrightarrow \quad (fEg)Ef \succsim^* f \quad \Leftrightarrow \quad fEg \succsim_E^* f$$

and

$$fEg \succsim^* fEg \quad \Leftrightarrow \quad fE(fEg) \succsim^* fEg \quad \Leftrightarrow \quad f \succsim_E^* fEg.$$

That is, $f \sim_E^* fEg$. □

Proof of the results in the main text

Proof of Theorem 1. By a simple computation, we can show that if $\succsim^*$ is a Bewley preference represented by (u, C) and $\succsim_E^\#$ is a maxmin expected utility preference represented by (u, C^E), then intertemporal consistency and intertemporal default to certainty hold for the pairing $(\succsim^*, \succsim_E^\#)$.

Conversely, note that by Theorem 1 in GMMS (p. 760), an objective preference $\succsim^*$ satisfies the basic conditions—monotonicity, C-completeness, and independence—if and only if $\succsim^*$ is represented by an affine utility index $u : X \to \mathbb{R}$ (cardinally unique) and a unique (weak*) closed and convex set of probabilities C, where for all $f, g \in \mathcal{F}$,

$$f \succsim^* g \quad \Leftrightarrow \quad \int u(f)\,dp \geq \int u(g)\,dp \quad \text{for all } p \in C.$$

Given an arbitrary objectively relevant event $E \in \Sigma$, we note that $\succsim^*$ and $\succsim_E^\#$ are the same relations over the subset of constant acts. In fact, by applying the definition of an objectively relevant event and intertemporal consistency, we obtain

$$x \succsim^* y \quad \Rightarrow \quad xEy \succsim^* y \quad \Rightarrow \quad x \succsim_E^\# y$$

and

$$x \succ^* y \quad \Rightarrow \quad x \succ^* yEx \quad \Rightarrow \quad x \succ_E^\# y.$$

Therefore, $\succsim^*$ and $\succsim_E^\#$ coincide in $X \times X$.

Note that monotonicity of $\succsim^*$ and intertemporal consistency imply that $\succsim_E^{\#}$ satisfies monotonicity in E. Let $f, g \in \mathcal{F}$ be such that $f(s) \succsim_E^{\#} g(s)$ for any $s \in E$. The act $h := fEg$ is such that $h(s) \succsim^* g(s)$ for all $s \in S$. Now, the monotonicity of $\succsim^*$ yields $fEg \succsim^* g$; therefore, by applying intertemporal consistency, we obtain $f \succsim_E^{\#} g$.

Since $\succsim_E^{\#}$ satisfies monotonicity and mixture continuity, for any act $f \in \mathcal{F}$, we can find $x_f \in X$, which is a certainty equivalent of f with respect to $\succsim_E^{\#}$. We note that if $fEx_f \succsim^* x_f$ does not hold, then by intertemporal default to certainty, we obtain $f \sim_E^{\#} x_f \succ_E^{\#} f$, a contradiction. Hence, $fEx_f \succsim^* x_f$, i.e., for any $p \in C$,

$$
\begin{aligned}
\int_S u(fEx_f)\,dp \geq u(x_f) \quad &\Rightarrow \quad \int_E u(f)\,dp + p\big(E^c\big)u(x_f) \geq u(x_f) \\
&\Rightarrow \quad \frac{1}{p(E)} \int_E u(f)\,dp \geq u(x_f) \\
&\Rightarrow \quad \int_E u(f)\,dp^E \geq u(x_f).
\end{aligned}
$$

Thus, for all $q \in C^E$, $\int_E u(f)\,dq \geq u(x_f)$, and we obtain

$$
u(x_f) \leq \min_{q \in C^E} \int_E u(f)\,dq.
$$

If strict inequality holds, then a $y \in X$ exists such that

$$
u(x_f) < u(y) < \min_{q \in C^E} \int_E u(f)\,dq,
$$

which implies that for all $p \in C$,

$$
u(x_f) < u(y) < \int_S u(fEy)\,dp.
$$

That is, $fEy \succsim^* y$ and $y \succ_E^{\#} x_f$.

By intertemporal consistency, $f \succsim_E^{\#} y$ and $y \succ_E^{\#} x_f$. Since $\succsim_E^{\#}$ is transitive, we obtain $f \succ_E^{\#} x_f$, which is impossible. Hence,

$$
u(x_f) = \min_{q \in C^E} \int_E u(f)\,dq,
$$

i.e., $\succsim_E^{\#}$ is a maxmin expected utility preference represented by (u, C^E).

The proof that $\succsim_E^*$ is the objective dynamically consistent update of $\succsim^*$ if and only if $\succsim_E^*$ is a Bewley-type preference with the set of priors given by C^E is quite easy and follows the same logical argument used by GMM (2008) from the equivalence of (i) and (ii) in their Theorem 1. For the sake of completeness, we note that the pairing $(\succsim^*, \succsim_E^*)$

satisfies dynamic consistency if and only if $\forall f, g \in \mathcal{F}$, $f \succsim_E^* g \Leftrightarrow fEg \succsim^* g$. By the representation of $\succsim^*$,

$$\begin{aligned} f \succsim_E^* g \quad &\Leftrightarrow \quad \int_S u(fEg)\,dp \geq \int_S u(g)\,dp \quad \text{for all } p \in C \\ &\Leftrightarrow \quad p(E)^{-1} \int_E u(f)\,dp \geq p(E)^{-1} \int_E u(g)\,dp \quad \text{or all } p \in C \\ &\Leftrightarrow \quad \int u(f)\,dq \geq \int u(g)\,dq \quad \text{for all } q \in C^E. \end{aligned}$$

□

Remark 3. Imposing GMM's (2004) monotone continuity axiom would ensure the countable additivity of each prior.[16] Hence, according to Theorem 1, given an event $E \in \mathcal{R}_{\succsim^*}$, if $\succsim^*$ satisfies the monotone continuity axiom, then any subjective preference $\succsim_E^\#$ has a monotone continuous multiple prior representation of Chateauneuf, Maccheroni, Marinacci, and Tallon (2005).

References

Anscombe, Francis J. and Robert J. Aumann (1963), "A definition of subjective probability." *Annals of Mathematical Statistics*, 34, 199–205. [1]

Bewley, Truman F. (1987), "Knightian decision theory, part II: Intertemporal problems." Working paper, Cowles Foundation Discussion Paper 835. [3, 9]

Bewley, Truman F. (2002), "Knightian decision theory: Part I." *Decisions Economics and Finance*, 25, 79–110. [1]

Cerreia-Vioglio, Simone (2016), "Objective rationality and uncertainty averse preferences." *Theoretical Economics*, 11, 523–545. [3, 8, 9]

Cerreia-Vioglio, Simone, Alfio Giarlotta, Salvatore Greco, Fabio Maccheroni, and Massimo Marinacci (forthcoming), "Rational preference and rationalizable choice." *Economic Theory*. [3]

Cerreia-Vioglio, Simone, Fabio Maccheroni, Massimo Marinacci, and Luigi Montrucchio (2011), "Uncertainty averse preferences." *Journal of Economic Theory*, 146, 1275–1330. [9]

Chateauneuf, Alain (1991), "On the use of capacities in modeling uncertainty aversion and risk aversion." *Journal of Mathematical Economics*, 20, 343–369. [1]

Chateauneuf, Alain and José H. Faro (2009), "Ambiguity through confidence functions." *Journal of Mathematical Economics*, 45, 535–558. [9]

Chateauneuf, Alain, Fabio Maccheroni, Massimo Marinacci, and Jean-Marc Tallon (2005), "Monotone continuous multiple priors." *Economic Theory*, 26, 973–982. [12]

Danan, Eric (2008), "Revealed preference and indifferent selection." *Mathematical Social Sciences*, 55, 24–37. [3]

[16]Monotone continuity axiom: For all $x, y, z \in X$, if $A_n \downarrow \varnothing$ and $y \succ^* z$, then $y \succsim^* xA_nz$ for some $n \in \mathbb{N}$.

Dunford, Nelson and Jacob T. Schwartz (1988), *Linear Operators, Part I: General Theory*. Wiley, Hoboken, New Jersey. [3]

Ellsberg, Daniel (1961), "Risk, ambiguity, and the Savage axioms." *Quarterly Journal of Economics*, 75, 643–669. [1]

Epstein, Larry G. and Michel Le Breton (1993), "Dynamically consistent beliefs must be Bayesian." *Journal of Economic Theory*, 61, 1–22. [3, 9]

Epstein, Larry G. and Martin Schneider (2003), "Recursive multiple-priors." *Journal of Economic Theory*, 113, 1–31. [8]

Faro, José Heleno (2015), "Variational Bewley preferences." *Journal of Economic Theory*, 157, 699–729. [7]

Fishburn, Peter C. (1970), *Utility Theory for Decision Making*. Wiley, New York. [3]

Ghirardato, Paolo (2002), "Revisiting Savage in a conditional world." *Economic Theory*, 20, 83–92. [3, 7]

Ghirardato, Paolo, Fabio Maccheroni, and Massimo Marinacci (2004), "Differentiating ambiguity and ambiguity attitude." *Journal of Economic Theory*, 118, 133–173. [3, 12]

Ghirardato, Paolo, Fabio Maccheroni, and Massimo Marinacci (2008), "Revealed ambiguity and its consequences: Updating." In *Advances in Decision Making Under Risk and Uncertainty*, 3–18, Springer-Verlag, Berlin. [3, 9, 11]

Giarlotta, Alfio and Salvatore Greco (2013), "Necessary and possible preferences structures." *Journal of Mathematical Economics*, 49, 163–172. [3]

Gilboa, Itzhak (2009), *Theory of Decision Under Uncertainty*, volume 45 of Econometric Society Monographs. Cambridge University Press, New York, New York. [2]

Gilboa, Itzhak, Fabio Maccheroni, Massimo Marinacci, and David Schmeidler (2010), "Objective and subjective rationality in a multiple prior model." *Econometrica*, 78, 755–770. [1, 3]

Gilboa, Itzhak and David Schmeidler (1989), "Maxmin expected utility with a non-unique prior." *Journal of Mathematical Economics*, 18, 141–153. [1, 8, 9]

Gumen, Anna and Andrei Savochkin (2013), "Dynamically stable preferences." *Journal of Economic Theory*, 148, 1487–1508. [8]

Hanany, Eran and Peter Klibanoff (2007), "Updating preferences with multiple priors." *Theoretical Economics*, 2, 261–298. [6]

Kajii, Atsushi and Takashi Ui (2009), "Interim efficient allocations under uncertainty." *Journal of Economic Theory*, 144, 337–353. [9]

Karni, Edi and Marie-Louise Vierø (2013), "Reverse Bayesianism: A choice-based theory of growing awareness." *American Economic Review*, 103, 2790–2810. [3]

Knight, Frank H. (1921), *Risks, Uncertainty and Profit*. Houghton-Mifflin, Boston. [1]

Kopylov, Igor (2009), "Choice deferral and ambiguity aversion." *Theoretical Economics*, 4, 199–225. [3]

Lehrer, Ehud and Roee Teper (2014), "Extension rules or what would the sage do?" *American Economic Journal: Microeconomics*, 6, 5–22. [3]

Maccheroni, Fabio, Massimo Marinacci, and Aldo Rustichini (2006), "Ambiguity aversion, robustness, and the variational representation of preferences." *Econometrica*, 74, 1447–1498. [9]

Machina, Mark J. (1989), "Dynamic consistency and non-expected utility models of choice under uncertainty." *Journal of Economic Literature*, 27, 1622–1668. [6]

Mandler, Michael (2005), "Incomplete preferences and rational intransitivity of choice." *Games and Economic Behavior*, 50, 255–277. [3]

Martins-da-Rocha, Victor F. (2010), "Interim efficiency with MEU-preferences." *Journal of Economic Theory*, 145, 1987–2017. [9]

Nehring, Klaus (2001), "Ambiguity in the context of probabilistic beliefs." Unpublished paper, University of California, Davis. [3, 7]

Nehring, Klaus (2009), "Imprecise probabilistic beliefs as a context for decision-making under ambiguity." *Journal of Economic Theory*, 144, 1054–1091. [3, 7]

Pires, Cesaltina P. (2002), "A rule for updating ambiguous beliefs." *Theory and Decision*, 53, 137–152. [8]

Postlewaite, Andrew and David Schmeidler (2012), "Rationality and uncertainty." *Rivista Internazionale di Scienze Sociali*, 102, 289–293. [2]

Rubinstein, Ariel (1988), "Similarity and decision-making under risk (is there a utility theory resolution to the Allais paradox?)." *Journal of Economic Theory*, 46, 145–153. [3]

Sarin, Rakesh and Peter P. Wakker (1998), "Dynamic choice and nonexpected utility." *Journal of Risk and Uncertainty*, 17, 87–119. [2, 8]

Schmeidler, David (1989), "Subjective probability and expected utility theory without additivity." *Econometrica*, 57, 571–587. [1]

Siniscalchi, Marciano (2011), "Dynamic choice under ambiguity." *Theoretical Economics*, 6, 379–421. [9]

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