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On the manipulability of efficient exchange rules

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We identify a large subdomain, $\mathcal{D}$, of quasilinear economies on which any efficient exchange rule will be generically (in the Baire sense) manipulable. For generic economies outside of $\mathcal{D}$, we find rules that are locally non-manipulable. The interior of the set $\mathcal{D}$ consists of all economies in which competitive equilibrium would prescribe that all agents consume a positive quantity of money. Since we study quasilinear preferences, this is the domain of primary interest. Our locally non-manipulable rules rely on the existence of traders who are willing to sell all of their cash and absorb the imbalances in the trading of the commodity.

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It is well known that there is no efficient trading rule (mechanism) that makes true preference revelation a dominant strategy. Thus, in a world with private information, realizing all gains from trade is problematic. Obviously, we should explore rules with less attractive incentive properties, possibly by relying on higher order rationality (as in Nash equilibrium), by relying on knowledge of the belief structure (as in Bayes–Nash equilibrium), or by considering other, novel weakenings. We join a recent strand of the literature that seeks to *quantify the opportunities* to gain from deceiving a rule with desirable properties (Maus et al., 2007a,b, Andersson et al., 2014). The benefit of this approach is that it makes no behavioral assumptions, and so may be applicable to many different models of behavior. The cost is in determining, and then applying, a good statistic to report this information. Since we consider a classical model of trading in divisible goods, the space of preferences (agent types) is necessarily infinite dimensional. As there is no translation invariant, nontrivial measure and no satisfactory way to extend the notion of "Lebesgue measure zero" to an infinite dimensional space (Stinchcombe, 2001), we instead work with purely topological notions: denseness and Baire category. We review these notions later, but to preview our results, we find that the set of economies in which some agent can manipulate an efficient trading rule is generic in the set of all economies that are relevant to the problem. Recall that genericity implies denseness and is in fact strictly stronger. This is the first paper to derive the denseness of manipulability in the

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pure exchange setting, and the first in this literature to demonstrate genericity. We emphasize the importance of the latter approach: dense sets may nonetheless be quite small.

Hurwicz (1972) first showed that, for two agents, there is no *strategy-proof*, *efficient*, and *individually rational* rule in the classical fair allocation setting. Later work on the two-agent case dropped the individual rationality requirement and found that each strategy-proof and efficient rule is dictatorial (Zhou, 1991, Schummer, 1997, Hashimoto, 2008). For more than two agents, Kato and Ohseto (2002) found nondictatorial rules, but they also conjectured, with strong justification from other results below, that any strategy-proof and efficient rule must, at each economy, have one agent consuming at the origin. Characterizing the general structure of what is possible with an arbitrary population remains a difficult open question.

For the exchange setting, it is important that agents' endowments be respected as outside options. With this extra constraint, inroads have been made into the multiple-agent problem. Serizawa (2002), working with homothetic preferences, showed that no strategy-proof and efficient rule is individually rational. Serizawa and Weymark (2003) extended this to show that, in fact, for any positive lower bound on welfare, each strategy-proof and efficient rule will violate this bound for at least one agent at one economy. Goswami et al. (2014) showed the corresponding result for quasi-linear economies.

Thus, if we hope to implement trading rules with desirable properties, we must ask for less. Hurwicz and Walker (1990) address the question that is symmetric to ours: insisting on strategy-proofness, how often is the resulting rule inefficient? The answer is "densely," suggesting that the incentive constraint is to blame for the negative results cited above.[1] However, even among those who study strategy-proofness, many agree that it is unnecessarily robust. We do not really believe that agents will manipulate at every chance they get. This is not because agents are inherently honest, but rather because manipulation is costly; it requires information acquisition and strategizing. For example, while uniform-price auctions admit simple strategic manipulation via the exercise of market power, Keloharju et al. (2005) reject this hypothesis in an analysis of Finnish government bond sales. Thus, in the hope of regaining some flexibility to design trading mechanisms, we remove the global character of strategy-proofness. We study economies one at a time, and measure the "size" of the set of economies that can be manipulated. One might hope to find rules for which the manipulable economies are scattered across the domain and make up an insignificant part of it. This is unfortunately not the case.

Discussing the size of a set in an infinite dimensional space necessarily brings technical complications. There is an expanding list of notions of largeness and smallness. We work with Baire category because it has been studied extensively and relies only on topological properties. Competing notions of smallness would require very different techniques to study, and so are beyond the scope of this paper. For example, a notion

[1] They get "open and dense" by adding a continuity requirement.

that has been of interest in related literature is *shyness*, which requires algebraic structure: for a set to by shy, its translation must also be shy.[2] Each notion also has a weakness. Shyness in particular has a strange feature discovered by Stinchcombe (2001): each probability measure on the space assigns full probability to at least one shy set. Thus, even though it was developed to extend measure theoretic smallness to infinite dimensional spaces, it does not make perfect predictions about what a measure will conclude about a given set.

A standard notion of smallness for a subset of a topological space is one that identifies every set whose closure contains no open sets as negligible. In a complete metric space, the class of countable unions of negligible sets is a σ-ideal, whose elements are called the *meager* sets. We call a set generic if its complement is meager.[3] Generic subsets of any complete metric space are dense, and it is easy to see that in a Polish space, denseness does not imply genericity.[4] Genericity in this sense has been used as an alternative to "measure zero" in many applications when a focal probability measure is unavailable (Jehiel et al., 2006, Ely and Pęski, 2011, Chen et al., 2012).

Denote by $\mathcal{D}$ the closure of the set of those economies for which the Walrasian allocation of money is, for all agents, positive. We show that *any* (Baire-) measurable and efficient trading rule will be generically manipulable on $\mathcal{D}$. Moreover, for every economy in the complement of $\mathcal{D}$, if the economy is replicated sufficiently many times, then there is an efficient trading rule that is non-manipulable in a neighborhood of the replicated economy.

Whether my results are considered negative or positive depends on the importance of the set $\mathcal{D}$, which in turn depends on the application. I study the "partial equilibrium" case: there is a divisible good and money, and preferences are quasi-linear in money. The classical motivation for studying the partial equilibrium model is that the commodity in question makes up a small portion of each agent's expenditure. Given such a motivation, $\mathcal{D}$ is in fact *all* of the economies of interest, as claimed above, and thus my result is completely negative. That said, in much of the mechanism design literature, quasi-linearity is taken as a primitive, without the classical motivation.

Finally, we should contrast our results with the case when agents contribute money to provide a public good, instead of using money to facilitate trade of a private good. Beviá and Corchón (1995), working also with quasi-linear preferences, found that efficient and individually rational rules are densely manipulable on the entire domain. In our case, dense manipulability no longer extends to the entire domain; we show in Section 3 that, outside of $\mathcal{D}$, we can construct efficient and individually rational rules that are *locally* non-manipulable. The presence of a public good thus imposes more stringent incentive constraints.

[2]Here is an important difference with Jehiel et al. (2006): they study the class of rules that are *globally* incentive compatible and measure the set of type spaces (i.e., functions from signal to valuation) that admit a nontrivial rule. Also, they study common values.

[3]We follow Kechris (1995) in this terminology. Frequently seen synonyms are *co-meager*, *residual*, and *typical*.

[4]A Polish space is a separable, completely metrizable space. Let $\{x_n : n \in \mathbb{N}\}$ be a dense set in a Polish space. Since each $\{x_n\}$, is closed, $\{x_n : n \in \mathbb{N}\}$ is the countable union of closed and nowhere dense sets, and therefore is meager.

1. Model

1.1 *Environment and preferences*

There are two divisible goods: a commodity and money. With abuse of notation, the commodity is indexed as x, and a typical allocation of commodity is also written as x. Similarly, money is written m. There is a finite set N of agents. Each agent $i \in N$ has a commonly known endowment $e_i := (\overline{x}_i, \overline{m}_i) > 0$ of the two goods. The point $E = (X, M) := \sum e_i > 0$ is the total supply of goods in the economy.[5]

The endowments are to be reallocated such that each agent i receives a bundle $(x_i, m_i) \in \mathbb{R}^2_+$. An *allocation* is therefore a list of bundles $((x_i, m_i))_{i\in N} \in (\mathbb{R}^2_+)^N$, though we interchangeably view these vectors as elements $(x, m) \in \mathbb{R}^N \times \mathbb{R}^N$. An allocation is *feasible* if $\sum_{i\in N}(x_i, m_i) \leq E$. The set of feasible allocations is denoted Z, with typical element denoted z.

Let $\mathcal{U} \subseteq \mathcal{C}^k[0, X]$ consist of all the (weakly) concave and increasing functions, where $k \in \{0, 1, 2\}$. View $\mathcal{U}$ as a subspace of $\mathcal{C}^k[0, X]$ with its usual topology, which has the compatible metric $d(f, g) = \sum_{m=0}^{k} \sup_{x_i\in X} |f^{(m)}(x_i) - g^{(m)}(x_i)|$.[6,7] For each $i \in N$, there is $u_i \in \mathcal{U}$ such that i's preferences can be represented by the function

$$U_i(x_i, m_i) := u_i(x_i) + m_i,$$

where U_i is always the utility function induced by u_i, U_i' is induced by u_i', etc. Endowment remains fixed throughout; therefore, an economy is identified by its preference profile, which is in turn identified by an element $u := (u_i)_{i\in N} \in \mathcal{U}^N$. A *social choice rule*, hereafter simply called a *rule*, is a function $\varphi = (\xi, \mu) : \mathcal{U}^N \to Z$, where ξ is the allocation rule for commodity and μ is the rule for money.

1.2 *Min-stability*

We are primarily interested in the following two properties.

Property 1 (Voluntary participation). An agent $i \in N$ boycotts bundle (x_i, m_i) if $u_i(x_i) + m_i < u_i(\overline{x}_i) + \overline{m}_i$. A rule φ satisfies *voluntary participation* if, for each economy $u \in \mathcal{U}^N$, there are no agents who boycott $\varphi(u)$.

Property 2 (Efficiency). A feasible allocation (x, m) is *efficient* for economy u if there exists no feasible allocation (x', m') such that, for each $i \in N$, $u_i(x_i') + m_i' \geq u_i(x_i) + m_i$, and for at least one agent, the inequality is strict. A rule φ is efficient if, for each $\mathcal{U}^N$, $\varphi(u)$ is efficient for u.

If a rule fails either Property 1 or Property 2, it may be undermined in practice. Even if an agent believed that, in expectation, a rule would improve his welfare, he might

[5] For $\{x, y\} \subseteq \mathbb{R}^K$, we write $x > y$ only if, for each coordinate k, $x_k > y_k$.

[6] The term $f^{(m)}$ denotes the mth derivative of f.

[7] Note that $\{f \in \mathcal{C}^k[0, X] : f(0) = 0\} = \bigcap_{n\in\mathbb{N}}\{f \in \mathcal{C}^k[0, X] : |f(0)| < 1/n\}$ is a denumerable intersection of open sets, called a G_δ set. The set of increasing and strictly concave functions is open. Thus $\mathcal{U}$ is a G_δ subset of $\mathcal{C}^k[0, X]$ and, therefore, is Polish, because G_δ subsets of Polish spaces are themselves Polish spaces.

refuse ex post to execute a trade that would make him worse off. Alternatively, if a rule is not efficient, agents might seek further trades after it is executed. The expectation of further trades would change the entire model, perhaps undermining the very goals of the rule. Thus, we define

Property 3 (Minimal stability). A rule is *min-stable* if it is efficient and satisfies voluntary participation.

A necessary condition for efficiency is material balance: $\sum_{i\in N} x_i = X$ and $\sum_{i\in N} m_i = M$. Furthermore, each efficient allocation is supported by at least one price vector $(p, 1)$, where, as usual, we normalize so the price of money is 1. Because preferences need not be differentiable, depending on the choice of k, it remains the case that the *normalized* supporting price need not be unique.

1.3 *Manipulability*

Once we have found a desirable min-stable rule, we would hope to implement it. To do so, we need some form of incentive compatibility. We study dominant strategy incentive compatibility, but not in the traditional sense. Rather than insisting on incentive compatibility for the entire domain of economies, we seek to measure the set of economies in which incentive compatibility fails.

Property 4 (Manipulability of φ at u). There exist an agent $i \in N$ and a preference relation $u_i' \in \mathcal{U}$ such that

$$U_i\big(\varphi_i(u_i', u_{-i})\big) > U_i\big(\varphi_i(u)\big).$$

Collect such profiles in set $\mathcal{M}^\varphi$.

1.4 *Baire category*

Recall that a subset A of a topological space is dense in the subset B if each open set $U \subseteq B$ has $A \cap U \neq \varnothing$. While the denseness of A implies that it is pervasive, it does *not* imply it is large—in cardinality or in topological substance—relative to B. For example, the rational numbers are dense in the reals and yet $\mathbb{R} \setminus \mathbb{Q}$ is still most of the real numbers.

Throughout the paper, the modifier *nowhere* means "in no open set." So A is nowhere dense in B if there does not exist an open set $U \subseteq B$ with A dense in U. A set is *meager* if it is a countable union of sets, each one closed and nowhere dense. As the countable union of singletons, the rational numbers are meager. The Cantor set is meager. Somewhat surprisingly, the functions that are differentiable at some point in their domain form a meager subset of continuous functions. A set whose complement is meager is called *generic*, and so genericity requires more than denseness.

The smallness of meager sets is highlighted by the Baire category theorem, which implies that any complete metric space must be built from *at least uncountably many* meager sets. Note that $\mathcal{U}$ has the cardinality of the continuum because it is uncountable and Polish, so assuming the continuum hypothesis, partitioning $\mathcal{U}$ into meager subsets results in a quotient of the same cardinality that we started with.

Theorem (The Baire category theorem). *A complete metric space is not meager in itself.*

2. Manipulability results

As mentioned in the Introduction, we partition the universe of all economies, $\mathcal{U}^N$, into two sets, one in which manipulability is pervasive and another in which local incentive compatibility can be achieved. In this section, we present our results on the former element of the partition, which includes the large class of economies in which there is a Walrasian allocation with all agents consuming a positive quantity of money. For a fixed profile of preferences, augmenting the endowment of money for the relevant agents is guaranteed to produce such an outcome. Since we maintain a fixed endowment profile, we instead state the condition as a function of preferences. Denote the Walrasian correspondence W. Let

$$\mathcal{D}_o := \left\{u \in \mathcal{U}^N : \exists z \in W(u) \text{ s.t. } \forall i \in N, z_{im}(u) > 0\right\}$$

and so define $\mathcal{D} := \overline{\mathcal{D}_o}$. Note that its interior is not empty.

Theorem 1. *Assume φ is a min-stable rule. Then $\mathcal{M}^\varphi$ is dense in $\mathcal{D}$.*

This result follows from two results, one of which is original to this paper. The original finding is a series of lemmas that characterize the Pareto set and are shown in Appendix A.1. Our results here generalize those of Goswami et al. (2014). We use the lemmas to find that, in fact, any allocation that does not maximize utilitarian welfare admits a successful manipulation by an agent reporting almost-linear preferences. However, maximizing utilitarian welfare puts us in the framework of classical mechanism design, and so we can appeal to the characterization of budget-balanced Vickrey–Clarke–Groves (VCG) mechanisms provided in Holmström (1977) (and also discussed in Milgrom, 2004).

Theorem 1 shows that the manipulable economies are pervasive, yet this does not mean they are large in the Baire sense. The following theorem yields this.

Theorem 2. *Assume φ is a min-stable rule. Then $\mathcal{M}^\varphi$ is nowhere meager in $\mathcal{D}$.*

This is proven via the fact that $\mathcal{D} \setminus \mathcal{M}^\varphi$ is nowhere generic in $\mathcal{D}$, which in turn follows from a collection of lemmas that amount to a "revenue equivalence"-type result for min-stable rules: if there is a strategy-proof, min-stable rule defined on a separably dense set, then it must be a VCG-type rule. Separable denseness requires that a set A is not only dense, but that its denseness can be separated across dimensions analogous to how a measure is separated into marginals: there is a dense set of "lines" $\{l^\nu\}$ in the space, all parallel to a fixed axis, such that each $l^\nu \cap A$ is dense in l^ν. A non-manipulable rule on a separably dense set can be extended to an open set, yielding a contradiction with Theorem 1 and showing that $\mathcal{D} \setminus \mathcal{M}^\varphi$ is nowhere separably dense. The Kuratowski–Ulam theorem, which is the Baire category analogue of Fubini's theorem, then implies that $\mathcal{D} \setminus \mathcal{M}^\varphi$ is nowhere generic.

Non-meager sets are also known as *sets of second category*. The Baire category theorem implies that any countable partition of $\mathcal{U}$ must contain at least one non-meager set. Theorem 2 would easily imply that $\mathcal{M}^\varphi$ is generic, if we knew that $\mathcal{M}^\varphi$ had the Baire property.

Property 5 (Baire property (BP)). There is an open V such that the symmetric difference $\mathcal{M}^\varphi \setminus V \cup V \setminus \mathcal{M}^\varphi$ is meager.

The BP is the topological analogue of measurability. Letting $\mathfrak{B} \subseteq 2^{\mathcal{U}^N}$ denote the class of sets with this property, $\mathfrak{B}$ is the smallest σ-algebra containing all open and all meager sets. A function f is called Baire-measurable if for each open set U in the range, the pre-image $f^{-1}(U)$ has the BP. Since open sets have the BP, $\mathfrak{B}$ contains the Borel sets, and it follows that every Borel-measurable function is Baire-measurable. Thus, by imposing a technical assumption that is no stronger than Borel measurability, we can deduce a more powerful result.

Theorem 3. *Assume φ is a Baire-measurable, min-stable rule. Then $\mathcal{M}^\varphi$ is generic in $\mathcal{D}$.*

Proof. The theorems and definitions referenced here can be found in Kechris (1995) and the numbers in square brackets refer to the numbering in that reference. First we show that Baire measurability is preserved under composition, which is elementary, but not shown in any reference text I could find. A set B having the Baire property is the union of a G_δ set, $\bigcap U_\nu$, and a meager set $\bigcup V_\nu$ [8.23], where the U_ν are open and the V_ν are closed and nowhere dense. If $f : \mathcal{X} \to \mathcal{Y}$ is Baire-measurable, then since each $f^{-1}(U_\nu) \in \mathfrak{B}$ and $\mathfrak{B}$ is a σ-algebra, $f^{-1}(\bigcap_{\nu\in\mathbb{N}} U_\nu) = \bigcap_{\nu\in\mathbb{N}} f^{-1}(U_\nu) \in \mathfrak{B}$. Since $f^{-1}(V_\nu) = \mathcal{X} \setminus f^{-1}(\mathcal{Y} \setminus V_\nu)$ and each $f^{-1}(V_\nu) \in \mathfrak{B}$, again we deduce $f^{-1}(\bigcup_{\nu\in\mathbb{N}} V_\nu) = \bigcup_{\nu\in\mathbb{N}} f^{-1}(V_\nu) \in \mathfrak{B}$. Therefore,

$$f^{-1}(B) = f^{-1}\Big(\bigcap U_\nu \cup \bigcup V_\nu\Big) = f^{-1}\Big(\bigcap U_\nu\Big) \cup f^{-1}\Big(\bigcup V_\nu\Big) \in \mathfrak{B}.$$

It follows that if g is Baire-measurable and U is open, then $(g \circ f)^{-1}(U) \in \mathfrak{B}$.

Define $\phi_i : \mathcal{U}^N \times \mathcal{U}^N \to \mathbb{R}$ for each $i \in N$ by

$$\phi_i(u, u') = U_i(\varphi_i(u_i, u'_{-i})) - U_i(\varphi_i(u'))$$

and note that the manipulable set is the projection of $\bigcup_{i\in N} \phi_i^{-1}(\mathbb{R}_{++}) \in \mathfrak{B}$ onto the second factor set. Since φ is Baire-measurable, there is a generic G_δ set $G \subseteq \mathcal{U}^N$ such that $\varphi|_G$ is continuous [8.38]. It follows that $\phi_i|_G^{-1}(\mathbb{R}_{++})$ is open relative to G and, therefore, also to G_δ. Since N is finite, $\bigcup_{i\in N} \phi_i^{-1}(\mathbb{R}_{++})$ is G_δ and, therefore, Borel. Thus, being a projection of a Borel set in a Polish product space, $\mathcal{M}^\varphi$ is an analytic set [14.1] and, therefore, has the BP [21.6].

Since $\mathcal{U}^N$ is a Polish space and since $\mathcal{M}^\varphi \in \mathfrak{B}$, for each open $U \subseteq \mathcal{U}^N$, either $\mathcal{M}^\varphi \cap U$ is meager or there is a further open set $U' \subseteq U$ such that $\mathcal{M}^\varphi$ is generic in U' [8.26].[8] By

[8]The theorem only requires that $\mathcal{U}^N$ be a Baire space. Every Polish space is a Baire space.

Theorem 2, the latter must be true for all $U \subseteq \mathcal{D}$. It follows that the set of economies u with a neighborhood V with $\mathcal{M}^\varphi$ *not* generic in V is nowhere dense. So there is an open dense set $A \subseteq \mathcal{D}$ such that, for each $u \in A$, there is a neighborhood $V \ni u$ with $\mathcal{M}^\varphi$ generic in V. Obviously these neighborhoods cover A so $\mathcal{M}^\varphi$ is generic in A and, therefore, is generic in $\mathcal{D}$. □

3. Non-manipulability result

We may wonder whether $\mathcal{M}^\varphi$ is dense on a set beyond $\mathcal{D}$. It might be, but the result of this section shows that it cannot extend very far. We construct an algorithm that yields, for generic $u \in \mathcal{U}^N \setminus \mathcal{D}$, an efficient rule that is immune to *local* manipulations—lies that are close to the truth—by making the agents who consume at the boundary absorb trade imbalances among those agents who consume in the interior. They can do this because, being at the boundary, their subgradient set is not a singleton. How much excess or deficit these agents can absorb of course depends on the fine details of the economy, and so any attempt to extend Theorem 1 would have to confront this. By replicating the economy, there will be enough boundary traders to achieve non-manipulability by *any* report, near or far from one's true preference, for any true economy in a neighborhood of u. For many economies outside $\mathcal{D}$, replication is unnecessary.

Before stating the proposition, we must first recall the notion of *replica economy*. An economy may be replicated a natural number $\nu \in \mathbb{N}$ times. This creates a new set of agents—denoted $\nu * N$—each of whom is identical to some agent in N, in preferences and endowment. In other words, for each $i \in N$, there is a set $[i] \subseteq \nu * N$ such that, for each $j \in [i]$, $u_j = u_i$ and $e_j = e_i$. Since each original agent induces ν copies, each i has $|[i]| = \nu$. The replicated endowments define the new feasible set $\nu * Z$ in the natural way:

$$\nu * Z := \left\{(x, m) : \sum_{i \in \nu * N} (x_i, m_i) \leq \nu E\right\}.$$

We denote the resulting economy $\nu * u$, and so implicitly we have defined a "copying operator" $*$.

Theorem 4. *Let $u \in \mathcal{U}^N \setminus \mathcal{D}$ be such that there is $z \in W(u)$ with $U_i(z_i) > U_i(e_i)$ for all $i \in N$. Then there exist a natural number $\nu \in \mathbb{N}$, a min-stable rule $\varphi^u : \mathcal{U}^{\nu * N} \to \nu * Z$, and a neighborhood $V \ni \nu * u$ such that φ^u is non-manipulable in V.*

In the rule constructed by our algorithm, the agents who would consume no money under Walras are made to be local residual claimants of *commodity*. Locally, the shadow price of the commodity is kept fixed at the Walrasian price of the target economy u; interior traders have $\hat{u}_i'(x_i) = p^*$, where p^* is the Walrasian price at economy u. Since the market clearing price at the true economy $\hat{u}$ will not necessarily be the same as at u, there will be imbalances. The imbalance in commodity is absorbed by the boundary traders and the imbalance in money is absorbed by the interior traders. Correcting these imbalances generally does not result in an outcome that could be generated by a fixed-price trading rule, as the redistribution of money is allowed to differ among interior traders. Their marginal utilities can remain constant only because they experience no wealth effect in their preferences. In sum, the rules we construct are novel and highly calibrated in the immediate region of the input economy. They then blend into the Walrasian correspondence outside of the target neighborhood.

4. Conclusion

We have shown that efficiency and voluntary participation bring with them plentiful opportunities for agents to manipulate. It is worth noting as well that the manipulations available to agents are, in a sense that we do not make formal, simple. Two types of manipulations suffice: to declare an almost-linear preference relation that prefers one's endowment to one's current allocation or to make an arbitrarily small deviation.

It is also worth noting, however, that our result makes full use of the large preference domain and the fact that Vickrey–Clarke–Groves (VCG) mechanisms are not budget balanced in general. There may be subdomains in which budget-balanced, *voluntary* VCG rules exist. On such a domain, our result would not hold. Nonetheless, such a domain would be a meager subset of whichever smoothness class it belongs to.

Appendix A: Proofs of manipulability results

A.1 *A characterization of the efficient set*

We begin with a characterization of the efficient set, which is denoted, for each $u \in \mathcal{U}$, by $\mathcal{E}(u)$. Referencing the second welfare theorem, we may partition the efficient set into allocations in which each agent's individual money constraint is slack and allocations in which some agent would prefer to consume negative money; such a partition is not trivial. This approach leads us to one of our main tools, Lemma 10, which discovers that if $u \in \mathcal{D}$ and if φ is not manipulable at u, then $\varphi(u)$ must be in the former set of the partition—money consumption must be unconstrained for all agents. This then permits us to invoke some of the standard tools of auction theory; however, these tools alone do not take us all the way to Theorem 2. For that result, some more work is necessary and is provided in Appendix A.3. The sum of these technical contributions amounts to a revenue equivalence result, though it is not ordered in terms of generality with the Green–Laffont–Holmström (GLH) theorem (Holmström, 1979). We have imposed voluntary participation, where the GLH theorem does not, but we assume the decision rule is efficient only on a dense subset of the preference domain, rather than the full domain.

Consider the allocations available when the feasibility constraint for money is ignored. For each economy $u \in \mathcal{U}^N$ we study the program

$$\begin{aligned} \mathbf{V}_N(u) := \max_x \sum_{i \in N} u_i(x_i) & \\ \text{s.t.} \quad X - \sum_{i \in N} x_i \geq 0 & \\ \forall i \in N, x_i \geq 0. & \end{aligned}$$

Since the objective is continuous and the constraint set is compact, the maximum is attained. Since Slater's constraint qualification is obviously satisfied, it is both necessary and sufficient to study the saddle points of the Lagrangian,

$$L^0(x, p, \lambda) := \sum_{i\in N} u_i(x_i) + p\left(X - \sum_{i\in N} x_i\right) + \sum_{i\in N} \lambda_i x_i. \tag{A.1}$$

For economy u, denote by $\mathcal{S}(u)$ the saddle points of expression (A.1). Denote by $\mathcal{X}^*(u)$ the projection of $\mathcal{S}(u)$ on the $x \in \mathbb{R}^N$ variable and by $\mathcal{P}^*(u)$ the projection on the p variable.

Since each u_i is concave, at each $x_i \in \mathbb{R}_+$, the set $Du_i(x_i)$ of subderivatives is well defined and is an interval. In particular, for $x_i > 0$, denote by $d^- u_i(x_i)$ and $d^+ u_i(x_i)$ the left and right hand derivatives, respectively, and note that $Du_i(x_i) = [d^+ u_i(x_i), d^- u_i(x_i)]$. Set $d^- u_i(0) := \infty$ and $d^+ u_i(X) := 0$.

Lemma 5. *Let $p \in \mathcal{P}^*(u)$ and $x \in \mathcal{X}^*(u)$. Then for each $i \in N$, $p \in Du_i(x_i)$. Thus, for each $x \in \mathcal{X}^*(u)$, $\mathcal{P}^*(u) \subseteq \bigcap_{i\in N} Du_i(x_i)$.*

Proof. Let $p^1 \in \mathcal{P}^*(u)$ and $x^0 \in \mathcal{X}^*(u)$. There are (x^1, p^1, λ^1) and $(x^0, p^0, \lambda^0) \in \mathcal{S}(u)$. By definition, for each $x' \in \mathbb{R}^N$ and $(p', \lambda') \in \mathbb{R} \times \mathbb{R}^N$,

$$L^0(x^1, p', \lambda') \geq L^0(x^1, p^1, \lambda^1) \geq L^0(x', p^1, \lambda^1).$$

Since x^0 and x^1 are both solutions to the problem, $\sum_{i\in N} u_i(x_i^1) = \sum_{i\in N} u_i(x_i^0)$. Since preferences are increasing, $\sum_{i\in N} x_i^1 = X = \sum_{i\in N} x_i^0$. Thus, expanding the Lagrangians and making replacements, we write

$$\begin{aligned}
&\sum_{i\in N} u_i(x_i^0) + \cancelto{0}{p'\left(X - \sum_{i\in N} x_i^0\right)} + \sum_{i\in N} \lambda_i' x_i^1 \\
&\quad \geq \sum_{i\in N} u_i(x_i^0) + \cancelto{0}{p^1\left(X - \sum_{i\in N} x_i^0\right)} + \cancelto{0 \text{ (complementary slackness)}}{\sum_{i\in N} \lambda_i^1 x_i^1} \\
&\quad \geq \sum_{i\in N} u_i(x_i') + p^1\left(X - \sum_{i\in N} x_i'\right) + \sum_{i\in N} \lambda_i^1 x_i'.
\end{aligned}$$

Since the inequalities hold for arbitrary x', we may replace x' with x^0. The second inequality then yields $\sum_{i\in N} \lambda_i^1 x_i^0 \leq 0$. Since $x^0 \geq 0$ and $\lambda^1 \geq 0$, $\sum_{i\in N} \lambda_i^1 x_i^0 = 0$. Thus,

$$\begin{aligned}
&\sum_{i\in N} u_i(x_i^0) + p'\left(X - \sum_{i\in N} x_i^0\right) + \sum_{i\in N} \lambda_i' x_i^0 \\
&\quad \geq \sum_{i\in N} u_i(x_i^0) + p^1\left(X - \sum_{i\in N} x_i^0\right) + \sum_{i\in N} \lambda_i^1 x_i^0 \\
&\quad \geq \sum_{i\in N} u_i(x_i') + p^1\left(X - \sum_{i\in N} x_i'\right) + \sum_{i\in N} \lambda_i^1 x_i',
\end{aligned}$$

where the first inequality comes from the fact that $L^0(x^0, p', \lambda') \geq L^0(x^0, p^0, \lambda^0) = \sum_{i\in N} u_i(x_i^0)$. We have deduced that

$$L^0(x^0, p', \lambda') \geq L^0(x^0, p^1, \lambda^1) \geq L^0(x', p^1, \lambda^1).$$

Since x' and (p', λ') were arbitrary, we conclude that $(x^0, p^1, \lambda^1) \in \mathcal{S}(u)$.

Let $(x, p, \lambda) \in \mathcal{S}(u)$. As a saddle point, for each $x' \in \mathbb{R}^N$, $L^0(x, p, \lambda) \geq L^0(x', p, \lambda)$. In particular, let $x' := (x_i + \varepsilon, x_{-i})$. Then we have

$$\frac{u_i(x_i+\varepsilon) - u_i(x_i)}{\varepsilon} \leq (p - \lambda_i).$$

Assuming $x_i > 0$, we find the inequality corresponding to $x - \varepsilon$, take limits, and deduce

$$d^+u_i(x_i) \leq (p - \lambda_i) \leq d^-u_i(x_i).$$

If $x_i > 0$, then complementary slackness implies $\lambda_i = 0$ and the lemma is shown. If $x_i = 0$, then $d^-u_i(x_i) = \infty$, and since $\lambda_i \geq 0$, we have $d^+u_i(x_i) \leq p$. □

Corollary 6. *Either* $|\mathcal{P}^*(u)| = 1$ *or* $|\mathcal{X}^*(u)| = 1$.

Proof. Let $x \in \mathcal{X}^*(u)$. Assume that $|\mathcal{P}^*(u)| > 1$. By Lemma 5, $d^+u_i(x_i) < d^-u_i(x_i)$. Since u_i is concave, for each $x_i' < x_i$, $d^+u_i(x_i') \geq d^-u_i(x_i) > d^+u_i(x_i)$ and so $d^+u_i(x_i) \notin Du_i(x_i')$. It follows that $Du_i(x_i') \cap Du_i(x_i)$ is either empty or contains the single point $d^+u_i(x_i') = d^-u_i(x_i)$. Since $|\mathcal{P}^*(u)| > 1$ and $\mathcal{P}^*(u) \subseteq Du_i(x_i)$, it is not possible for $\mathcal{P}^*(u) \subseteq Du_i(x_i')$. Thus no element of $\mathcal{X}^*(u)$ has $x_i' < x_i$. A symmetric argument shows that no element of $\mathcal{X}^*(u)$ has $x_i' > x_i$. Since i is arbitrary, $\mathcal{X}^*(u)$ is a singleton. □

We partition the Pareto set into two subsets, $\mathcal{Z}^*(u) := \{z = (x, m) \in Z : x \in \mathcal{X}^*(u)\}$ and its complement. The second welfare theorem allows us to study the Pareto set via each individual's optimal choice from a Walrasian budget. For each $z \in \mathcal{E}(u)$, there is a price p and a list $(w_i)_{i\in N} \in \mathbb{R}_+^N$ such that each $z_i := (x_i, m_i)$ solves the program

$$\begin{aligned} \max \quad & u_i(x_i) + m_i \\ \text{s.t.} \quad & w_i - px_i - m_i \geq 0 \\ & x_i \geq 0, m_i \geq 0. \end{aligned} \tag{A.2}$$

If $z_i = 0$, then $w_i = 0$ and the problem is trivial. Otherwise, $w_i > 0$ and Slater's constraint qualification is satisfied. We again use the saddle-point method. Let β_i denote the Lagrange multiplier on the budget constraint. Let λ_{ix} denote the multiplier for the nonnegativity constraint of the commodity and let λ_{im} denote the corresponding multiplier for money. Denote the Lagrangian $L(x_i, m_i, \beta_i, \lambda_i)$.

Assume $(x_i, m_i, \beta_i, \lambda_i)$ is a saddle point for the problem with price p and wealth w_i. By studying the expression $L(x_i \pm \varepsilon, m_i, \beta_i, \lambda_i) \leq L(x_i, m_i, \beta_i, \lambda_i)$, we find

$$p\beta_i - \lambda_{ix} \in Du_i(x_i). \tag{A.3}$$

The expression $L(x_i, m_i \pm \varepsilon, \beta_i, \lambda_i) \le L(x_i, m_i, \beta_i, \lambda_i)$ in turn yields that

$$\beta_i = 1 + \lambda_{im}. \tag{A.4}$$

LEMMA 7. *Assume $z := (x, m) \in \mathcal{E}(u)$ and p supports z for u. Then $p \le \max \mathcal{P}^*(u)$.*

PROOF. There is $i \in N$ with $x_i > 0$ and $x_i \ge \min \mathcal{X}_i^*(u)$. Then $\lambda_{ix} = 0$ and by (A.3), we find $p\beta_i \in Du_i(x_i)$. Let $x_i^* \in \mathcal{X}_i^*(u)$ satisfy $x_i^* = \min \mathcal{X}_i^*(u)$. By concavity, $d^- u_i(x_i) \le d^+ u_i(x_i^*)$.[9] By Lemma 5, $\max \mathcal{P}^*(u) \subseteq Du_i(x_i^*)$. Therefore, we use (A.4) and the fact that $\lambda_{im} \ge 0$ to find

$$\max \mathcal{P}^*(u) \ge p\beta_i = p(1 + \lambda_{im}) \ge p. \qquad \square$$

LEMMA 8. *Let $z := (x, m) \in \mathcal{E}(u)$ be supported by price p. Assume that the individual optimization problem (A.2) for each $i \in N$ has a saddle point of the form $(x_i, m_i, 1, (\lambda_{ix}, 0))$. Then $z \in \mathcal{Z}^*(u)$.*

PROOF. Since, for each $x_i' \in [0, X]$, $L(x_i, m_i, 1, (\lambda_{ix}, 0)) \ge L(x_i', m_i, 1, (\lambda_{ix}, 0))$, we deduce that

$$u(x_i) - px_i + \lambda_{ix} x_i \ge u(x_i') - px_i' + \lambda_{ix} x_i'.$$

Sum over agents and add pX to each side to arrive at

$$\sum_{i \in N} u_i(x_i) + p\Bigl(X - \sum_{i \in N} x_i\Bigr) + \sum_{i \in N} \lambda_{ix} x_i \ge \sum_{i \in N} u_i(x_i') + p\Bigl(X - \sum_{i \in N} x_i'\Bigr) + \sum_{i \in N} \lambda_{ix} x_i'.$$

Letting $\lambda_x := (\lambda_{ix})_{i \in N}$, note that this is precisely $L^0(x, p, \lambda_x) \ge L^0(x', p, \lambda_x)$. Since $z \in \mathcal{E}(u)$ and preferences are increasing, $X - \sum_{i \in N} x_i = 0$. Since for each $i \in N$, complementary slackness in the individual's problem gives $\lambda_{ix} x_i = 0$, we deduce that for $p' \ge 0$ and $\lambda_x' \in \mathbb{R}_+^N$, $L^0(x, p, \lambda_x) \le L^0(x, p', \lambda_x')$. Thus, (x, p, λ_x) is a saddle point of L^0 and, therefore, $x \in \mathcal{X}^*(u)$. $\square$

LEMMA 9. *Let $z := (x, m) \in \mathcal{E}(u) \setminus \mathcal{Z}^*(u)$. For each $x^* \in \mathcal{X}^*(u)$, there is $i \in N$ such that $m_i = 0$ and $x_i < x_i^*$.*

PROOF. Let p support z for u. Let $x^* \in \mathcal{X}^*(u)$. We show the contrapositive, so assume $z := (x, m) \in \mathcal{E}(u)$ and that for each $i \in N$ with $x_i < x_i^*$, $m_i > 0$. Let i have $x_i < x_i^*$. By Lemma 7, we have $\overline{p} := \max \mathcal{P}^*(u) \ge p$. Let $m_i^* \in \mathbb{R}$ be the quantity of money—possibly negative—such that $U_i(z_i) = U_i(z_i^*)$ with $z_i^* = (x_i^*, m_i^*)$. Since z_i solves i's individual problem with price p and wealth $px_i + m$, we have the partial saddle condition

$$u(x_i') + m_i' + \beta_i\bigl(p(x_i - x_i') + m_i - m_i'\bigr) + \lambda_{ix} x' \le u_i(x_i) + m_i,$$

where complementary slackness cleans up the right hand side. Note that z_i^* solves the relaxed problem (nonnegativity of money is ignored) for individual i with price $\overline{p}$ and

[9] For sets A and $B \subseteq \mathbb{R}$, write $A \ge B$ when $\inf A \ge \sup B$.

wealth $\overline{p}x_i^* + m_i^*$. Since $x_i < x_i^*$, it follows that $x_i^* > 0$ and so the saddle condition for the relaxed problem is

$$u(x_i') + m_i' + \tilde{\beta}_i\big(\overline{p}(x_i^* - x_i') + m_i^* - m_i'\big) \le u_i(x_i^*) + m_i^*.$$

Substituting the solution of one problem into the saddle-point condition of the other yields

$$\beta_i\big(p(x_i - x_i^*) + m_i - m_i^*\big) + \lambda_{ix}x^* \le 0,$$
$$\overline{p}(x_i^* - x_i) + m_i^* - m_i \le 0.$$

Recall that by (A.4), since $m_i > 0$, $\beta_i = 1$. Thus, summing the two inequalities gives $\lambda_{ix}x^* \le (\overline{p} - p)(x_i - x_i^*)$. Since $\overline{p} \ge p$ and $x_i < x_i^*$, the right hand side is nonpositive. Since $\lambda_{ix}x^* \ge 0$, conclude that $p = \overline{p}$ and $\lambda_{ix} = 0$.

Since $p \in \mathcal{P}^*(u)$, p also supports z_i^*. It follows by concavity that $d^- u_i(x_i^*) = d^+ u_i(x_i)$ and that for each $x_i' \in [x_i, x_i^*]$, $Du_i(x_i') = \{p\}$. Thus, $px_i^* + m_i^* = w_i := px_i + m_i$. Since $m_i > 0$, there is $z_i' := (x_i', m_i') \in co(\{z_i, z_i^*\})$ with $z_i' > 0$ and $U_i(z_i') = U_i(z_i)$. For each i with $x_i < x_i^*$, calculate z_i'.

Consider $j \in N$ with $x_j > x_j^*$. We can perform the symmetric operation as in the previous paragraph, where in this case $m_j^* > m_j \ge 0$ and concavity yields $d^- u_j(x_j) = d^+ u_j(x_j^*)$. Note that for both types of agents, we are adjusting their consumption in a direction orthogonal to $(p, 1)$. Thus we can make these redistributions in a way that is both small and balanced. We arrive at a feasible allocation z' such that each agent is indifferent between z and z', and each is consuming the same value of goods given price p. At z', either $x_i' = x_i = x_i^*$ or $z_i' > 0$.

For each i with $x_i = x_i^*$, since x_i optimizes the unconstrained problem, z_i optimizes the individual problem with $\lambda_{im} = 0$ and $\beta_i = 1$. For the remaining agents, since $U_i(z_i') = U_i(z_i)$, we substitute this into the saddle-point condition for z':

$$U(z_i'') + (w_i - px_i'' - m_i'') \le U_i(z_i) \le U_i(z_i) + \lambda_{ix}'' x_i' + \lambda_{im}'' m_i'.$$

Then since λ_{ix}'', x_i', λ_{im}'', and m_i' are all nonnegative, we can replace x_i' and m_i' with x_i and m_i on the right hand side to get that z_i also optimizes the individual problem with $\lambda_{im} = 0$ and $\beta_i = 1$. Now invoke Lemma 8 to conclude that $x \in \mathcal{X}^*(u)$. □

A.2 *Proof of Theorem 1*

Our work in the previous section comes to fruition in this result.

Lemma 10. *Let $u \in \mathcal{D}$ and $\varphi(u) = (\xi(u), \mu(u)) \in \mathcal{E}(u) \setminus \mathcal{Z}^*(u)$. Then $u \in \mathcal{M}^\varphi$.*

Proof. Let $(x, m) = \varphi(u) \in \mathcal{E}(u) \setminus \mathcal{Z}^*(u)$. Since $u \in \mathcal{D}$, there is $(x^*, m^*) \in W(u) \cap \mathcal{Z}^*(u)$, with supporting price $p^* \in \mathcal{P}^*(u)$. By Lemma 9, there is i with $x_i < x_i^*$ and $m_i \le m_i^*$, which means that i is consuming strictly within his Walrasian budget set at p^*.

At profile u, agent i has a profitable deviation from truth-telling: Let i declare a differentiable u'_i with the properties

$$\frac{d}{dx}u'_i(x^*_i) = p^*,$$
$$U'_i(\omega_i) > U'_i(\varphi_i(u)).$$

To see why the inequality is feasible, consider the function $\tilde{u}(x) = p^* x$. Clearly, $\tilde{U}(\omega_i) > \tilde{U}(\varphi_i(u))$. Let u' be a differentiable, strictly concave function that is sufficiently close to $\tilde{u}$.

Denote $u' := (u'_i, u_{-i})$. Clearly, $x^* \in \mathcal{X}^*(u')$. Since $x^*_i > x_i \geq 0$ and since u'_i is differentiable, in fact $\mathcal{X}^*(u') = \{x^*\}$ and $\mathcal{P}^*(u') = \{p^*\}$. Property 1 (voluntary participation) requires $\varphi_i(u') \neq \varphi_i(u)$. Since φ is efficient, by Lemma 7, it is supported by $p' \leq p^*$. Thus, if $\mu_i(u') > 0$, then since u'_i is smooth and strictly concave, $\xi_i(u') > x^*$. If $\mu_i(u') = 0$, then by Property 1, $\xi_i(u') > \xi_i(u)$. In either case, $\varphi_i(u') \gneqq \varphi_i(u)$. Since preferences are increasing, $U_i(\varphi_i(u')) > U_i(\varphi_i(u))$. □

Lemma 11. *Let $u \in \mathcal{U}^N$ and $x \in \mathcal{X}^*(u)$. Let $\varepsilon > 0$. Then there are $u^\varepsilon \in \mathcal{U}^N$, a list $(\alpha_i, \beta_i, \gamma_i)_{i\in N}$, and a neighborhood $\prod_{i\in N} U_i \ni x$ such that the following statements hold:*

- For each $i \in N$ and each $x'_i \in U_i$, $u^\varepsilon_i(x'_i) = \alpha_i \log(x'_i + \beta_i) + \gamma_i$ and $|u^\varepsilon_i(x'_i) - u_i(x'_i)| < \varepsilon$.
- For each $i \in N$ and each $x'_i \notin U_i$, $u^\varepsilon_i(x'_i) = u_i(x'_i)$.
- We have $\mathcal{X}^*(u^\varepsilon) = \{x\}$.

Proof. The proof is constructive. Let $p \in \mathcal{P}^*(u)$ and $i \in N$. Since u_i is concave and increasing, for each $\delta > 0$, there is $x'_i \in \mathbb{R}$ such that $|x'_i - x_i| < \delta$ and u_i is twice differentiable at x'_i. We first calibrate α_i and β_i so that

$$\frac{d}{dx}\big[\alpha_i \log(x_i + \beta_i)\big] = p,$$
$$\frac{d^2}{dx^2}\big[\alpha_i \log(x'_i + \beta_i)\big] > \frac{d^2}{dx^2}u_i(x'_i).$$

The first condition ensures that $\{x\} = \mathcal{X}^*(u^\varepsilon)$ while the second ensures that u_i uniformly dominates u^ε_i. Let $\bar{\gamma}_i := u_i(x_i) - \alpha_i \log(x'_i + \beta_i)$. For each $\gamma \in]0, \bar{\gamma}_i[$, there is a neighborhood $]\underline{x}(\gamma), \bar{x}(\gamma)[$ containing x_i such that for each $x''_i \in]\underline{x}(\gamma), \bar{x}(\gamma)[$, $\alpha_i \log(x''_i + \beta_i) + \gamma < u_i(x''_i)$, and for each $x''_i \in \{\underline{x}(\gamma), \overline{x}(\gamma)\}$, $\alpha_i \log(x''_i + \beta_i) + \gamma = u_i(x''_i)$. For each $\gamma \in \mathbb{R}$, define u^γ_i such that

$$u^\gamma_i(x'') := \begin{cases} \alpha_i \log(x''_i + \beta_i) + \gamma & x''_i \in]\underline{x}(\gamma), \bar{x}(\gamma)[, \\ u_i(x''_i) & \text{otherwise.} \end{cases}$$

For $\gamma^*_i \in]0, \bar{\gamma}_i[$ with $|\bar{\gamma}_i - \gamma^*_i|$ sufficiently small, $(u^{\gamma^*_i}_i)_{i\in N}$ satisfies all the requirements of the lemma except for the inclusion in $\mathcal{U}^N \subseteq \mathcal{C}^k$, but this can clearly be done by smooth pasting at the boundary as

$$\frac{d}{dx}u_i^{\gamma_i^*}\big(\overline{x}(\gamma)\big) > \frac{d}{dx}u_i\big(\overline{x}(\gamma)\big) \quad \text{and} \quad \frac{d}{dx}u_i^{\gamma_i^*}\big(\underline{x}(\gamma)\big) < \frac{d}{dx}u_i\big(\underline{x}(\gamma)\big)$$

by construction. □

We can now give the proof of Theorem 1.

Proof of Theorem 1. From Lemma 10, we know that $u \in \mathcal{D} \setminus \mathcal{M}^{\varphi}$ implies $\varphi(u) = (\xi(u), \mu(u)) \in \mathcal{Z}^*(u)$. Our proof is by contradiction. Assume $\mathcal{M}^{\varphi}$ is *not* dense in $\mathcal{D}$. Then $\mathcal{D} \setminus \mathcal{M}^{\varphi}$ contains an open set V. Without loss of generality, assume $V = \prod_{i\in N} V_i$. Therefore, $\varphi|_V$ is a strategy-proof rule that implements a selection from the correspondence $\mathcal{Z}^*$. Since $\mathcal{U}$ is smoothly path connected, we may apply the Green–Laffont–Holmström theorem (see Holmström, 1979): $\varphi|_V$ is a VCG mechanism. We then have from Holmström (1977) that $\sum_{i\in N} \mu_i(u) = M$ for each $u \in V$ if and only if there is a list of functions $(f_i)_{i\in N}$, with $f_i : V_{-i} \to \mathbb{R}$, such that for each $u \in V$,

$$\mathbf{V}_N(u) = \sum_{i\in N} f_i(u_{-i}). \tag{A.5}$$

Given $\varepsilon > 0$, let u^{ε} approximate u such that each u_i^{ε} is a logarithm in a neighborhood of $\xi_i(u)$, as in Lemma 11. Consider a one-dimensional subdomain $\mathcal{A} \subset \mathcal{U}$, containing u^{ε}, such that for each $u_i' \in \mathcal{A}$, there is $\alpha_i' \in \mathbb{R}$ with $u_i'(\cdot)|_{V_i} = \alpha_i' \log(\cdot + \beta_i) + \gamma_i$. Identify a profile of preferences $u' \in \mathcal{A}^N$ by its list $\alpha' := (\alpha_i')_{i\in N}$ of parameters. Given this specification, since $\xi \in \mathcal{X}^*$, it has a closed form near x. Letting $\tilde{X} := X + \sum_{i\in N} \beta_i$, it is easy to verify that, for each $\alpha \in \mathcal{A}^N$,

$$\xi_i(\alpha) = \frac{\alpha_i \tilde{X}}{\sum\limits_{i\in N} \alpha_i} - \beta_i.$$

The envelope theorem and further calculation then yield the formula

$$\frac{\partial^k}{\partial\alpha_1 \partial\alpha_2 \cdots \partial\alpha_k} \mathbf{V}_N(\alpha) = (k-2)! \left(\frac{-1}{\sum\limits_{j\in N} \alpha_j} \right)^{k-1}.$$

However, (A.5) implies that for each $\alpha \in \mathcal{A}^N$,

$$\frac{\partial^n \mathbf{V}_N(u)}{\partial\alpha_1 \partial\alpha_2 \cdots \partial\alpha_n} = 0.$$

Therefore, $\mathbf{V}_N(\alpha)$ does not satisfy (A.5). By choosing $\varepsilon > 0$ sufficiently small, we can guarantee that $\mathcal{A}^N \cap V \neq \varnothing$, a contradiction. □

A.3 *Proof of Theorem 2*

A.3.1 *Supporting results* This section develops a few mathematical tools we need for the sequel. Because the content is general, we depart momentarily from the economic environment. For this section, let $K = \{1, \dots, k\} \subseteq \mathbb{N}$ and let $X := \prod_{k \in K} X_k$ be an arbitrary finite product with typical elements x and y.

Baire category in product sets Assume that each X_i is second-countable and that X has the product topology induced by the topologies on the $(X_k)_{k=1}^{K}$. Write $A \sqsubseteq B$ if A is dense in B and $A \trianglelefteq B$ if A is generic in B.

For each $A \subseteq X$ and each $k \in K$, denote by A_k the projection of A onto X_k. Denote by A_{-k} the projection of A onto $\prod_{k' \neq k} X_{k'}$. We study a refinement of various set relations that requires them to hold in a manner that is separable across dimensions. To explain our notion of separability, the following "slicing" correspondence is useful: $A_k(x_{-k}) := \{x_k \in A_k : (x_k, x_{-k}) \in A\}$.

Separations of set relations Given a binary set relation $\prec$, the *separation* of $\prec$, denoted $\prec^{\times}$, is the relation satisfying $A \prec^{\times} B$ if and only if, for each $k \in \{1, \dots, K\}$,

$$\{x_{-k} \in A_{-k} : A_k(x_{-k}) \prec B_k(x_{-k})\} \prec B_{-k}.$$

Note that $\sqsubseteq^{\times}$ is a refinement of $\sqsubseteq$, as $A \sqsubseteq^{\times} B$ implies $A \sqsubseteq B$. An implication of Gruenhage et al. (2007) is that $\sqsubseteq^{\times}$ is a *strict* refinement of $\sqsubseteq$. In fact, Gruenhage et al. (2007) construct dense sets whose slices are nowhere dense.

By the Kuratowski–Ulam theorem, $A \trianglelefteq B$ implies $A \trianglelefteq^{\times} B$; however, we cannot conclude $A \trianglelefteq B$ from $A \trianglelefteq^{\times} B$ unless both A and B have the Baire property. This is analogous to the fact that we cannot say $\int_X \int_Y f(x, y)\, dx\, dy$ is equal to $\int f(x, y)\, dx\, dy$ unless f is measurable on $X \times Y$. However, since $A \trianglelefteq B$ implies $A \sqsubseteq B$, we have the following easy corollary.

Corollary 12. *A generic set is separately dense, and it follows that the complement of a nowhere separably dense set is non-meager.*

An example in Oxtoby (2013, p. 57) for $\mathbb{R}^2$ can be extended to any Polish space and implies that there are non-meager, nowhere separably dense sets. Such sets fail to have the Baire property and, therefore, fail to be Borel.

Separable functions Assume for this section that X is a vector space. For functions $f : X \to \mathbb{R}$, define a difference operator inductively as $\Delta_y^{\varnothing} f(x) = f(x)$, and for $S = S' \cup \{i\}$, $i \notin S'$, $\Delta_y^S f(x) = \Delta_y^{S'} f(y_i, x_{-i}) - \Delta_y^{S'} f(x)$. By induction,

$$\Delta_y^S f(x) = \sum_{S' \subseteq S} \sigma(S') f(y_{S'}, x_{N \setminus S'}),$$

where

$$\sigma(S') := \begin{cases} 1 & \text{parity}\,(|S'|) = \text{parity}\,(|S|), \\ -1 & \text{otherwise.} \end{cases}$$

We say that function f is $n-1$-separable if there is a list of functions $f_i : X^{K\setminus\{k\}} \to \mathbb{R}$ such that $f = \sum f_i$. Clearly if f is $n-1$-separable, then $\Delta_y^K f(x) \equiv 0$.

Lemma 13. *If for each $\{x, y\} \subseteq X^K$, $\Delta_y^K f(x) = 0$, then f is $n-1$-separable.*

Proof. Fix arbitrary $x \in X^K$. For each $k \in K$, define

$$\mathcal{S}_k := \{S \subseteq K : k \in S \text{ and } \forall j < k, S \notin \mathcal{S}_j\}.$$

Then let

$$f_k(y) = -\sum_{S\in\mathcal{S}_k} \sigma(S^{\complement}) f(y_{S^{\complement}}, x_{N\setminus S^{\complement}}).$$

Since $\Delta_y^K f(x) = 0$,

$$f(y) = -\sum_{S\subsetneq N} \sigma(S) f(y_S, x_{N\setminus S}),$$

so we need only to show that each term on the right is assigned to some f_k and that each f_k is well defined. To be well defined, note that each $\mathcal{S}_k \ni \{k\}$, so each f_k has at least one term. For exhaustion of the terms, since $\bigcup \mathcal{S}_k$ contains all nonempty sets, so $\bigcup\{S^{\complement} : S \in \bigcup \mathcal{S}_k\}$ contains all sets except N, which is as desired since the term corresponding to N is precisely $f(y)$. □

A.3.2 *The category of the manipulable set*

Lemma 14. *Let $V \subseteq \mathcal{R}^N$ be open, and let $U \sqsubseteq V$. Let $\psi = (\zeta, \nu) : U \to Z$ be an individually rational, budget-balanced VCG rule. Then there is $\psi^* = (\zeta, \nu) : V \to Z$, also an individually rational, budget-balanced VCG rule, with $\psi^*|_U = \psi$.*

Proof. By Holmström (1977), a VCG rule is budget balanced if and only if the value function is $n-1$-separable. Therefore, $\Delta_{u'}^N \mathbf{V}_N(u) = 0$ for each pair $\{u, u'\} \subseteq U$. Berge's maximum theorem implies that $\Delta_{u'}^N \mathbf{V}_N(u)$ is a continuous function on V, which is zero on a dense set and therefore zero on all of V. Lemma 13 thus implies that the value function is $n-1$-separable on V, with components $(f_i)_{i\in N}$. As Holmström (1977) found that each $h_i = \nu_i - f_i$, we conclude that ψ^* extends ψ. □

Theorem 15. *The set $\mathcal{D} \setminus \mathcal{M}^{\varphi}$ is nowhere separately dense in $\mathcal{D}$.*

Proof. Suppose, to the contrary, that there is an open product set $V \subseteq \mathcal{D}$ and a set $U \subseteq \mathcal{D} \setminus \mathcal{M}^{\varphi}$ such that $U \sqsubseteq^{\times} V$. By Lemma 10, $\varphi|_U$ implements $\mathcal{Z}^*|_U$. Let $i \in N$ and let U_{-i}^* be the set designated in the definition of separate denseness. Let $u_{-i} \in U_{-i}^*$ and, recalling the slicing operator of the previous section, our hypothesis implies $U_i(u_{-i}) \sqsubseteq V_i$ and $U_{-i}^* \sqsubseteq V_{-i}$.

Since $U \subseteq \mathcal{D} \setminus \mathcal{M}^{\varphi}$, for each pair $\{u_i, u_i'\} \subseteq U_i(u_{-i})$,

$$u_i\big(\xi_i(u_i, u_{-i})\big) + \mu_i(u_i, u_{-i}) \geq u_i\big(\xi_i(u_i', u_{-i})\big) + \mu_i(u_i', u_{-i}). \tag{A.6}$$

Denoting by p^{**} the Vickrey pivot price function, let $h_i(u_{-i}) := \mu_i(u_i, u_{-i}) - p^{**}(u)$.

Claim. *For each* $u_i' \in U_i$, $\mu_i(u_i', u_{-i}) = p^{**}(u) + h_i(u_{-i})$.

Proof. We construct a strategy-proof function $\psi = (\zeta, \nu) : V_i \to \mathbb{R}^2$ that satisfies the claim. Let $A := \varphi_i(U_i(u_{-i}), u_{-i})$. Inequality (A.6) implies that, for each $u_i' \in U_i(u_{-i})$, $\varphi_i(u_i', u_{-i}) \in \arg\max_{(x,m)\in A} u_i'(x) + m$. Let $\psi_i|_{U_i(u_{-i})} = \varphi_i|_{U_i(u_i-i)}$. For each $u_i' \in V_i \setminus U_i(u_{-i})$, let $\psi_i(u_i', u_{-i}) \in \arg\max_{(x,m)\in \bar{A}} u_i'(x) + m$. Since $\mathcal{X}^*$ is upper hemi-continuous and $U_i(u_{-i}) \sqsubseteq V_i$, then $\zeta_i \in \mathcal{X}^*|_{V_i \times \{u_{-i}\}}$. By the Green–Laffont–Holmström theorem, ψ_i has the required form. The claim follows. $\square$

We have found a list of functions $(h_i)_{i\in N}$ such that φ has the VCG form whenever $u \in U$. Thus $\varphi|_U$ is an individually rational, budget-balanced VCG rule. Lemma 14 then implies that it can be extended to an individually rational, budget-balanced VCG rule on V, contradicting Theorem 1. $\square$

This allows us to give a simple proof of Theorem 2.

Proof of Theorem 2. If $\mathcal{M}^\varphi$ were meager in some open set $U = \prod_{i\in N} \mathcal{D}$, then $U \setminus \mathcal{M}^\varphi \trianglelefteq U$. By Corollary 12, this implies $U \setminus \mathcal{M}^\varphi \sqsubseteq^\times U$, which contradicts Theorem 15. $\square$

Appendix B: Proof of Theorem 4

The proof is by construction. Fix $u^* \in \mathcal{U}^N$ and $z^* = (x^*, m^*) \in W^*(u)$ that satisfy the conditions of Theorem 4. Let p^* be a supporting price. We first construct a family of rules φ^p, $p \in \mathbb{R}$, so that when we plug in p^* as the parameter, rule φ^{p^*} is min-stable and strategy-proof in a neighborhood $V = \prod_{i\in N} V_i$ of u^*. We must then extend this local rule to Φ defined on the set of economies $V^+ \supset V$ with the property that for each $u \in V^+$, there are $\iota(u) \in N$ and $u'_{\iota(u)} \in \mathcal{U}$ such that $(u'_{\iota(u)}, u_{-\iota(u)}) \in V$. If we spell out our incentive constraints, we find that Φ is all we need. For each pair of economies u and u', if $\{u, u'\} \subseteq V$, then we must have

$$\forall i \in N, \quad U_i\big(\varphi_i^{p^*}(u)\big) \geq U_i\big(\varphi_i^{p^*}(u_i', u_{-i})\big) \quad \text{and}$$
$$U_i'\big(\varphi_i^{p^*}(u')\big) \geq U_i'\big(\varphi_i^{p^*}(u_i, u_{-i}')\big).$$

If $u \in V^+ \setminus V$ and $u' := (u'_{\iota(u)}, u_{-\iota(u)}) \in V$,

$$U_{\iota(u)}\big(\varphi_{\iota(u)}^{p^*}(u')\big) \geq U_{\iota(u)}\big(\Phi_{\iota(u)}(u)\big).$$

For every other pair of economies, there are no incentive constraints. Thus, once we have constructed $\Phi : V^+ \to Z$, we can simply use the Walrasian rule for each $u \in \mathcal{U}^N \setminus V^+$.

Once Φ is given, we analyze the conditions under which it is min-stable and non-manipulable in V. We find that these conditions can be guaranteed in a replica economy.

The family φ^p The idea is this: interior traders, $N \setminus N'$, choose their optimal quantity of money and commodity given endowment z_i^* and price p. Their consumption of money will be adjusted to offset the aggregate excess or deficit generated by the *other* interior traders, and as such it will not effect their incentives. The boundary traders, N', will then absorb the excess or deficit in *commodity* generated by the interior traders. In both cases, the deficit is with reference to z.

Given preference $u_i \in \mathcal{U}$, price $p' \in \mathbb{R}_{++}$, and endowment $e'_i \in \mathbb{R}^2$, denote by $D(u_i, p', e'_i) \in \mathbb{R}^2$ agent i's demanded consumption. If $e'_i := z_i^*$, the notation for endowment is suppressed. For each $i \in N$ and each $p' \in \mathbb{R}_{++}$, define functions

$$\Delta^m(i, u; p) := \frac{\sum\limits_{j \in N \setminus (N' \cup \{i\})} D_m\big(u_j^*, p^*\big) - D_m(u_j, p)}{\big|N \setminus N'\big| - 1}$$

and

$$\Delta^x(i, u; p) := \frac{\sum\limits_{j \in N \setminus N'} D_x\big(u_j^*, p^*\big) - D_x(u_j, p)}{\big|N'\big|} + \frac{\sum\limits_{j \in N' \setminus i} D_x\big(u_j^*, p^*\big)}{\big|N'\big| - 1}.$$

Note that Δ^m is the per capita money deficit and Δ^x is the per capita commodity deficit mentioned above. It is more convenient to define the rule in terms of demands, so instead of distributing these deficits, we make them adjustments to endowment. Quasilinearity of preferences helps us to reconcile this approach. Thus we define the adjusted endowment as

$$\omega_i(u; p) := \begin{cases} \big(x_i^*, m_i^* + \Delta^m(i, u; p)\big) & i \in N \setminus N', \\ \big(\Delta^x(i, u; p), 0\big) & i \in N'. \end{cases}$$

Finally, $\varphi_i^p(u) := D(u_i, p, \omega_i(u; p))$. By construction, agents can influence neither their own adjusted endowment nor the price. It follows that the function $\varphi^p|_V$ is strategy-proof. If V is sufficiently small, it is also the case that $\varphi^{p^*}|_V$ is min-stable: Property 2 (efficiency) results when V is small enough that, for each $u \in V$, at $\varphi^{p^*}(u)$, it remains the case that p^* is a subgradient for the consumption of the boundary traders. Similarly, Property 1 (voluntary participation) holds when $\omega_i(u; p^*)$ and z_i^* do not differ by much (since $U_i(z_i^*) > U_i(e_i)$ by assumption).

Extending to Φ Recall that our task now is to incentivize agents not to manipulate a $u \in V$ economy toward a $u' \in V^+$ economy. That is to say, the incentive constraints of agent $\iota(u)$ must be satisfied. To do this, we simply offer each agent a polytope of choices, with their endowment an extreme point, such that the faces are given by the most extreme prices the agent can induce under Walras. As above, the excess or deficit generated by ι is absorbed by the other agents.

Extend the definition of ι so that $\iota(u) = 0$ whenever $u \in V$. Formally, make 0 a dummy agent so that any function $f : \mathcal{U}^{N \cup \{0\}} \to Z$ subsequently defined always has $f_0(u) = (0, 0)$.

Let $\underline{p}_i := \min\{p^*(u_i, u^*_{-i}) : i \in N, u_i \in \mathcal{U}\}$ and $\overline{p}_i := \max\{p^*(u_i, u^*_{-i}) : i \in N, u_i \in \mathcal{U}\}$. Let $A_i := \{y \in \mathbb{R}^2 : (\underline{p}, 1)(y - e_i) \le 0, (\bar{p}, 1)(y - e_i) \le 0\}$. Set

$$\Phi_i(u) := \arg \max_{(x,m)\in A\cup\{D(u_i, p^*, \omega_i(u; p^*))\}} u_i(x) + m.$$

To see that Φ is an extension of φ^{p^*}, recall that we assumed $z^*_i \neq e_i$. Since $p^* \in (\underline{p}, \overline{p})$, A_i passes below agent i's reference Walrasian bundle (x_i, m_i) (see Figure 1). Therefore, for u close enough to u^*, $\Phi_i(u) = D(u_i, p^*, \omega_i(u; p^*))$.

Note that for each $u \in V^+ \setminus V$, $\pi(u) := \text{med}[Du(\Phi_{\iota(u)X}(u))]$ is well defined and $\pi(u) \in (\underline{p}_{\iota(u)}, \overline{p}_{\iota(u)})$. Let $\Delta_\iota(u) := D(u_{\iota(u)}, \pi(u), \omega_{\iota(u)}(u; \pi(u))) - \Phi_{\iota(u)}(u)$ and define

$$\Delta_i(u) := \begin{cases} \left|N \setminus \left(N' \cup \{\iota(u)\}\right)\right|^{-1}\left(0, \Delta_{\iota,m}(u)\right) & i \in N \setminus N', \\ \left|N' \setminus \{\iota(u)\}\right|^{-1}\left(\Delta_{\iota,x}(u), 0\right) & i \in N', \\ (0, 0) & \text{otherwise.} \end{cases}$$

Thus we define, for each $i \neq \iota(u)$, $\Phi_i(u) := D(u_i, \pi(u), \omega_i(u; \pi(u))) + \Delta_i(u)$.

Showing Φ *satisfies our requirements* By construction, Φ is non-manipulable in V; restricted to V, it is φ^{p^*}, which is strategy-proof. An agent contemplating a deviation away from V has a fixed polytope of bundles which to choose. Thus, we must only ensure that the rule is min-stable. This was already assured on V, so we must simply ensure that the agents $i \neq \iota(u)$ can collectively absorb the deficit induced by $\iota(u)$. We do this case by case.

Case 1. Excess Commodity. The boundary traders must absorb the excess or deficit of commodity. Letting $j \in N$ have $x_j = \max_N x_i$, they have to share at most this quantity, in which case the supporting price is $\underline{p}_j \le p^*$. Thus, if for each $i \in N'$,

$$p^* \le u_i\left(x_i + \frac{x_j}{|N'| - 1}\right),$$

$\underline{p}_j$ remains a supporting price and the economy is efficient. Note that this is conservative, since $\underline{p}_{\iota(u)} \le p^*$ implies that some of the interior traders might also absorb this excess. In any case, replicating the economy causes $|N'|$ to multiply without changing any other factors, and thus it is achievable, since $p^* < u_i(x_i)$ by assumption.

Case 2. Deficit of Commodity. Now let $j \in N$ have $x_j = \min_N x_i$. By symmetric reasoning, we must ensure that each $i \in N'$ has

$$\overline{p}_j \le u_i\left(x_i - \frac{x_j}{|N'|}\right).$$

This can also be guaranteed by replication, which reduces the influence one agent can have on the Walrasian price, making $\overline{p}_j$ as close to p^* as we like. For voluntary participation, we must also ensure that

$$u_i(\overline{x}_i) + \overline{m}_i \le u_i\left(x_i - \frac{x_j}{|N'|}\right),$$

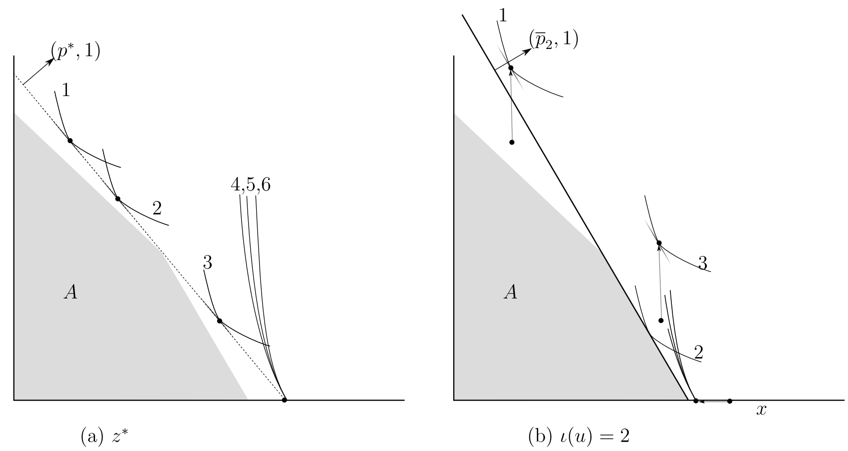

Figure 1. Rule Φ.

which can also be achieved by replication since $U_i(z^*) > U_i(e_i)$.

Case 3. Deficit of Money. Reasoning identical to the previous case and the fact that $U_i(z^*) > U_i(e_i)$ for all $i \in N$ ensure that the interior agents can absorb the deficit of money.

It follows that for large enough economies, no replication is necessary.

Figure 1 illustrates the idea of the rule Φ. Panel (a) shows the allocation at the target economy and panel (b) shows how the other agents absorb the drastic movement of agent 2 away from the initial economy.

B.1 *Feasibility of* Φ

Lemma 16. *We have* $\sum_{i \in N} \Phi_i \equiv E$.

Proof. The proof is just an accounting exercise. Assume $u \in V^* \setminus V$. The case $u \in V$ is nested in this proof. Separating out $\iota(u)$, we have

$$\sum_{i \in N} \Phi_i(u) = \Phi_{\iota(u)}(u) + \sum_{i \in N \setminus \{\iota(u)\}} \big(D\big(u_i, \pi(u), \omega_i\big(u; \pi(u)\big)\big) + \Delta_i(u)\big). \tag{B.1}$$

For simpler notation, let $\iota := \iota(u)$ and $\pi := \pi(u)$:

$$\begin{aligned}\sum_{i \in N \setminus \{\iota\}} \Delta_i(u) &= \sum_{i \in N \setminus (N' \cup \{\iota\})} \frac{\big(0, \Delta_{\iota,m}(u)\big)}{\big|N \setminus \big(N' \cup \{\iota\}\big)\big|} + \sum_{i \in N' \setminus \{\iota\}} \frac{\big(\Delta_{\iota,x}(u), 0\big)}{\big|N' \setminus \{\iota\}\big|} \\ &= \big(0, \Delta_{\iota,m}(u)\big) + \big(\Delta_{\iota,m}(u), 0\big) = \Delta_\iota(u) \\ &= D\big(u_\iota, \pi, \omega_\iota(u; \pi)\big) - \Phi_\iota(u).\end{aligned}$$

Thus (B.1) simplifies to

$$\sum_{i \in N} \Phi_i(u) = D\big(u_\iota, \pi, \omega_\iota(u; \pi)\big) + \sum_{i \in N \setminus \{\iota\}} D\big(u_i, \pi, \omega_i(u; \pi)\big) = \sum_{i \in N} D\big(u_i, \pi, \omega_i(u; \pi)\big).$$

By making enough copies of the economy, $\bar{p} - \underline{p}$ can be made arbitrarily small. Thus if V is sufficiently small, the demand of each $i \in N \setminus N'$ given price π and adjusted endowment $\omega_i(u; \pi)$ is interior (recalling that $\pi \in [\underline{p}, \bar{p}]$). Thus, by quasi-linearity $D(u_i, \pi, \omega_i(u; \pi)) = D(u_i, \pi) + (0, \omega_i(u; \pi))$. Similarly, each $i \in N'$ continues to demand a boundary bundle and, therefore, $D(u_i, \pi, \omega_i(u; \pi)) = (\omega_{ix}(u; \pi), 0)$. Recall that

$$\omega_{ix}(u; \pi) = \frac{\sum_{j \in N \setminus N'} D_x(u_j^*, p^*) - D_x(u_j, \pi)}{|N'|} + \frac{\sum_{j \in N' \setminus i} D_x(u_j^*, p^*)}{|N'| - 1}.$$

Therefore,

$$\begin{aligned}
\sum_{i \in N'} \omega_{ix}(\hat{u}; \pi) &= \sum_{i \in N'} \left(\frac{\sum_{j \in N \setminus N'} D_x(u_j^*, p^*) - D_x(u_j, \pi)}{|N'|} + \frac{\sum_{j \in N' \setminus i} D_x(u_j^*, p^*)}{|N'| - 1} \right) \\
&= \sum_{j \in N \setminus N'} D_x(u_j^*, p^*) - D_x(u_j, \pi) + \sum_{j \in N'} D_x(u_j^*, p^*) \\
&= \sum_{j \in N} D_x(u_j^*, p^*) - \sum_{j \in N \setminus N'} D_x(u_j, \pi).
\end{aligned}$$

Thus, for commodity,

$$\begin{aligned}
\sum_{i \in N} D_x(u_i, \pi, \omega_i(u; \pi)) &= \sum_{i \in N \setminus N'} D_x(u_i, \pi) + \sum_{i \in N'} \omega_{ix}(u; \pi) \\
&= \sum_{i \in N \setminus N'} D_x(u_i, \pi) + \sum_{i \in N} D_x(u_i^*, p^*) - \sum_{i \in N \setminus N'} D_x(u_i, \pi) \\
&= \sum_{i \in N} D_x(u_i^*, p^*) = X,
\end{aligned}$$

and for money,

$$\begin{aligned}
\sum_{i \in N \setminus N'} D_m(u_i, \pi, \omega_i(u; \pi)) &= \sum_{i \in N \setminus N'} \left[D_m(u_i, \pi) + \omega_{im}(u; \pi) \right] \\
&= \sum_{i \in N \setminus N'} D_m(u_i, \pi) \\
&\quad + \sum_{i \in N \setminus N'} \left[\frac{\sum_{j \in N \setminus (N' \cup \{i\})} D_m(u_j^*, p^*) - D_m(u_j, \pi)}{|N \setminus N'| - 1} \right] \\
&= \sum_{i \in N \setminus N'} D_m(u_i, \pi) + \sum_{i \in N \setminus N'} D_m(u_j^*, p^*) - D_m(u_j, \pi) \\
&= \sum_{i \in N \setminus N'} D_m(u_j^*, p^*) = M.
\end{aligned}$$

□

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