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Efficiency and endogenous fertility

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This paper explores the properties of the notions of $\mathcal{A}$-efficiency and $\mathcal{P}$-efficiency, which were proposed by Golosov et al. (2007), to evaluate allocations in a general overlapping generations setting in which fertility choices are endogenously selected from a continuum and any two agents of the same generation are identical. First, we show that the properties of $\mathcal{A}$-efficient allocations vary depending on the criterion used to identify potential agents. If one identifies potential agents by their position in their siblings' birth order, as Golosov, Jones, and Tertilt do, then $\mathcal{A}$-efficiency requires that a positive measure of agents use most of their endowment to maximize the utility of the dynasty head, which, in environments with finite-horizon altruism, implies that some agents—the youngest in every family—obtain an arbitrary low income to finance their own consumption and fertility plans. If potential agents are identified by the dates in which they may be born, then $\mathcal{A}$-efficiency reduces to dynastic maximization, which, in environments with finite-horizon altruism, drives the economy to a collapse in finite time. To deal with situations like those arising in economies with finite-horizon altruism, in which $\mathcal{A}$-efficiency may be in conflict with individual rights, we propose to evaluate the efficiency of a given allocation with a particular class of specifications of $\mathcal{P}$-efficiency for which the utility attributed to the unborn depends on the utility obtained by their living siblings. Under certain concavity assumptions on value functions, we also characterize every symmetric, $\mathcal{P}$-efficient allocation as a Millian efficient allocation, that is, as a symmetric allocation that is not $\mathcal{A}$-dominated—with the birth-order criterion—by any other symmetric allocation.

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1. Introduction

The most commonly used optimality notion in normative economic analysis—the notion of Pareto efficiency—relies on the well known Pareto criterion to compare social alternatives. The Pareto criterion allows one to construct a partial ordering on a set of alternatives from the complete preference orderings (defined on this set) of a *fixed* group of agents. An efficient allocation can be described as a maximal element of the partial ordering induced by the Pareto criterion on the set of feasible allocations.

When fertility is endogenous, one can still use the Pareto criterion to rank feasible allocations using the partial orderings of all potential agents, represented by the utility functions of the *living* agents. However, any two allocations with different fertility choices cannot be ranked, since there is no way to know whether an agent who lives in one allocation a but not in another allocation a' is better off in the latter than he is in the former. To avoid this problem and preserve the partial ordering induced by the Pareto criterion, one needs to extend it to compare also allocations with different fertility choices.

Although the issue seems to concern policymakers everywhere, the theoretical foundations of many proposals to alter fertility rates are rather weak. Most of the literature simply identifies optimal allocations with the solutions to alternative social welfare maximization problems, referred to as *Millian* or *Benthamite*, according to whether or not the welfare weight given to a generation in the social welfare function depends on the size of that generation.[1] But this approach does not take into account the fact that the Pareto criterion is not directly applicable to environments in which the set of agents is endogenous. Unlike this literature, Lang (2005) and Michel and Wigniolle (2007) have provided normative principles to evaluate population policies in the context of an overlapping generations framework without altruism. These papers restrict their analysis to *symmetric* allocations, that is, to allocations in which any two agents of the same generation obtain the same consumption bundle, and focus on a notion of efficiency—referred to as *modified Pareto optimality* by Lang and as *representative consumer efficiency* by Michel et al.—based on an extension of the Pareto criterion that ranks any two symmetric allocations of different population size by comparing the welfare obtained by all generations of living consumers.

But the restriction to symmetric allocations in environments without altruism limits the scope of these papers. To fill this gap, Golosov et al. (2007) have proposed two alternative extensions of the Pareto criterion in the context of a general model in which the set of fertility choices is discrete. The first criterion, referred to as the *$\mathcal{A}$-dominance criterion*, ranks any two allocations by comparing the welfare profiles of all agents *alive*

[1] For a discussion on the different notions of optimality arising in settings with endogenous fertility, see, e.g., Razin and Sadka (1995, Chapter 5).

in the two allocations. The second, referred to as the $\mathcal{P}$*-dominance criterion*, is constructed from a preliminary assumption on the utility level obtained by *potential* unborn agents. These two extensions of the Pareto criterion give rise to two notions of efficiency, respectively referred to as $\mathcal{A}$-efficiency and $\mathcal{P}$-efficiency, to evaluate allocations with endogenous populations. Golosov, Jones, and Tertilt (henceforth GJT) provide partial characterizations of the two notions of efficiency as the solutions to welfare maximization problems and prove that under relatively mild assumptions, $\mathcal{A}$-efficient allocations are either $\mathcal{P}$-efficient or are arbitrarily close to allocations that are also $\mathcal{P}$-efficient (see Golosov et al. 2007, Section 4.3, Result 3). Thus, the $\mathcal{P}$-efficiency of $\mathcal{A}$-efficient allocations is robust to different specifications of the utility levels attributed to the unborn. The extension of the Pareto criterion proposed by Lang and Michel and Wigniolle can be seen as the restriction of the partial order induced by the $\mathcal{A}$-dominance criterion to the set of symmetric allocations, as we show in Conde-Ruiz et al. (2010). We also propose to call the notion of efficiency arising with that criterion *Millian efficiency*, since it generalizes the notion of Millian optimality used in the early literature.

A seemingly irrelevant feature of GJT's work deserves some discussion. The authors, for whom fertility choices are represented by the number of children that parents decide to bear during a child-rearing period, identify the agents who may be potentially alive in an economy by the identities of their parents and their position in their siblings' birth order. For example, agent $i = (1, 2)$ identifies the second child of the first child of the dynasty head. There are, however, other possibilities. If people are born at different points in time (as GTJ implicitly assume when they identify potential agents by their position in their siblings' birth order) and parents are able to choose not only the number of children they are willing to rear, but also the specific points in time—within a child-rearing period—at which they want to give birth to these children, one may identify potential agents by their parents' identities and the points in time at which they may be born. Under this criterion, for example, agent $i = (1, 2)$ identifies the child born at $t_2 = 2$ from the descendant of the dynasty head born at $t_1 = 1$. One might think that if none of the agents potentially alive in an economy cares about the specific dates at which they give birth to their children, then the criterion used to distinguish among potential agents should be irrelevant. However, from the points of view of the $\mathcal{A}$- and $\mathcal{P}$-dominance criteria, it is not. As we explain throughout the paper, if one evaluates an increase in the family size using the $\mathcal{A}$-dominance or the $\mathcal{P}$-dominance criterion and potential agents are identified by their positions in the birth order, then the eldest children in a family will be alive in all allocations in which their parents have children and, therefore, their preferences must be taken into account. In contrast, when potential agents are identified by their birth dates, the eldest children in a family might not even be born in many alternative allocations in which their parents have more children. Therefore, their preferences need not be taken into account in welfare comparisons between allocations in which they are alive and allocations in which they are not.

To understand the differences in the set of $\mathcal{A}$-efficient allocations that arise with the two criteria to identify potential agents, it might be useful to think of a couple who would like to have two children, who they wish to call Alex and Robin—irrespective of their gender—in order of birth. They would also like to treat them equally and provide them with the same support and bequests after they pass away. In such a context, an allocation in which only Alex is born, say, in November 2018, and receives resources as if he/she were the family's only heir may be $\mathcal{A}$-efficient if agents are identified by their positions in the birth order. To see this, suppose Alex's parents are desperate to have (at least) one child, so not having any children would make them worse off. However, having more children may not be welfare improving if agents are identified by their positions in the birth order. Why? Because Alex must be alive in any reallocation of resources in which Robin is also born, and providing him/her with fewer resources would make him/her worse off. But even if the agents' order at birth determines their names, it is conceivable to think of an individual as being determined by birth date, because randomness in the environment make the genetic mix and the upbringing of a child different when conception, hence birth date, is anticipated or delayed by just 1 month. Under this perspective, Alex's identity would not be the same if his/her parents decide to anticipate his/her birth date by a month or to postpone it and give birth to Alex after Robin is born. Taking this into account, the allocation in which the Alex born in November 2018 is the only child in his/her family can be easily $\mathcal{A}$-dominated by an allocation in which two children—named Alex and Robin—are born and their parents divide the inheritance equally between the two siblings.

In this paper, we study the properties of different notions of efficiency in a general, overlapping generations setting in which (a) there is a single individual or a continuum of identical individuals, referred to as the *dynasty head*, alive when the economy starts; (b) the number of children within each family is endogenously selected from a bounded interval in the positive real line, the child-rearing period, that also represents the continuum of time instants at which children may be born; (c) the agents are able to decide, possibly, with constraints, not only the number of children they bear, but also the set of specific points in time at which they want to give birth to their children; (d) any two living agents of the same generation have the same labor endowment and the same preferences, which depend on their own consumption of a homogeneous good, on the number of children they decide to bear and on the welfare obtained by their descendants. Thus, our setting generalizes GJT's continuous model of fertility choices in two respects. First, it covers, as particular cases, a wide range of positive models of fertility choice, including not only Barro and Becker's (1989), but also others in which altruism lies between the two polar representations (no altruism and dynastic altruism) mainly considered in the literature. Second, even though, in our general framework, agents choose the dates at which their children are born, potential agents may be identified, for normative purposes, with the two criteria described above. That is, potential agents may be identified by their positions in their siblings' birth order, a criterion referred to as the *birth-order criterion*, or by their birth dates, a criterion referred to as the *birth-date criterion*. The two criteria to identify potential agents give rise to two notions of $\mathcal{A}$-dominance (and, hence, of $\mathcal{A}$-efficiency) and two notions of $\mathcal{P}$-dominance (and, hence, of $\mathcal{P}$-efficiency). To distinguish among them, we refer to the notions of $\mathcal{A}$- and $\mathcal{P}$-dominance that arise with the birth-date criterion as $\mathcal{A}^{\mathcal{D}}$- and $\mathcal{P}^{\mathcal{D}}$-dominance, while the notions of dominance that arise with the birth-order criteria are referred to as $\mathcal{A}^{\mathcal{O}}$- and $\mathcal{P}^{\mathcal{O}}$-dominance. Our results can be gathered in two blocks.

$\mathcal{A}$-efficiency Our general framework allows for a constrained setting in which parents cannot postpone the dates at which they decide to have children. Since, in such a setting, the date at which an agent may be born coincides with the agent's position in his/her siblings order, the birth-date and the birth-order criteria to identify potential agents give rise to the same set of $\mathcal{A}$-efficient (and $\mathcal{P}$-efficient) allocations. When the agents' fertility choices are unconstrained and parents can decide the dates at which their children are born, the two criteria to identify potential agents give rise to a different set of efficient allocations. To be more precise, every $\mathcal{A}^{\mathcal{O}}$-efficient allocation is equivalent to an allocation that arises as $\mathcal{A}^{\mathcal{D}}$-efficient in the constrained setting described above; that is, for the two allocations, every agent who occupies the same position in the siblings' birth order obtains the same consumption bundle and, hence, the same utility. Thus, when fertility choices are unconstrained, an allocation that is $\mathcal{A}^{\mathcal{O}}$-efficient may be $\mathcal{A}^{\mathcal{D}}$-dominated by a fertility choice that was not feasible in the constrained setting. It follows that an $\mathcal{A}^{\mathcal{D}}$-efficient allocation must also be $\mathcal{A}^{\mathcal{O}}$-efficient, although the converse is not, in general, true. Identifying the agents by their position in their siblings' birth order gives rise, therefore, to a larger set of efficient allocations.

(i) Taking this into account, we provide first a necessary condition for $\mathcal{A}$-efficiency that applies to both criteria to identify potential agents. To be more precise, for every $\mathcal{A}^{\mathcal{O}}$-efficient or $\mathcal{A}^{\mathcal{D}}$-efficient allocation, the youngest individuals in every family must devote most of their entire income to maximize not their own utility, but the utility of their parents and, hence, of the dynasty head (Theorem 1). When altruism of the agents is of the finite-horizon type, Theorem 1 implies that *every* $\mathcal{A}$-efficient allocation is characterized by providing a positive measure of agents of every generation born after period $t = 2$ with almost no resources to finance their own consumption (Corollary 1).

Indeed, starting from an allocation a in which none of the agents uses his/her income to maximize the utility of the dynasty head, it is always possible to find another allocation a' that has more individuals that makes all people living under both a and a' better off than they were under a, irrespective of how potential agents are identified. Welfare improvements of this type (in the sense given by the $\mathcal{A}$-dominance criterion) can be achieved by enforcing every newcomer, that is, every individual living under a' who is not born under a, to use his/her endowment to maximize his/her parents' utilities. Newcomers can provide their parents with at least the same utility as those already living in a. Moreover, since marginal agents already living in a were not maximizing their parents' utilities, newcomers require fewer resources so as to achieve this objective. Finally, even though newcomers would rather prefer to obtain the same consumption bundles as those already living in a, this involves no welfare losses from the point of view of the $\mathcal{A}$-dominance criterion.

(ii) When potential agents are identified with the birth-date criterion, fertility choices are unrestricted, and the child-rearing period is long enough, every allocation arising as $\mathcal{A}^{\mathcal{D}}$-efficient can be characterized as a dynastic optimum (Theorem 2), that is, as an allocation that maximizes the utility of the dynasty head among all feasible allocations. This result can be extended to discrete settings and, therefore, contrasts with GJT's examples showing that $\mathcal{A}$-efficiency differs from dynastic maximization. If, in addition, altruism of the agents is of the finite-horizon type, Theorem 2 implies that in *e*very $\mathcal{A}^{\mathcal{D}}$-efficient allocation, the economy collapses in finite time and no economic activity takes place thereafter. The argument behind Theorem 2 is simple: if some of the agents born in a given allocation do not use their endowment to maximize the utility of the dynasty head, she can always replace these agents by a set of agents of equal measure who do behave as she wishes.

As the example described above illustrates, determining which of the two notions of $\mathcal{A}$-efficiency we should opt for depends on whether the ith child in a family should be considered as the same person independently of when he/she is born, a difficult question indeed. In any case, the fact that achieving $\mathcal{A}$-efficiency always requires that some—and perhaps all—of the agents must devote their entire endowment to maximize the utility of the dynasty head shows that there are significant differences between the notion of $\mathcal{A}$-efficiency and the notion of Pareto efficiency. These differences might introduce difficulties in the implementation of $\mathcal{A}$-efficient outcomes, not only because, in some environments, achieving $\mathcal{A}$-efficiency may leave some of the living agents with almost no resources, but also because, even in environments with dynastic altruism, $\mathcal{A}$-efficiency might require that the agents bequeath their debts to their children, which might be incompatible with the agents' rights to use some of their resources, for example, their labor capacities, as they wish. But beyond these differences on the properties of efficient allocations, our analysis of $\mathcal{A}$-efficiency in different environments raises, in our view, some doubts on the very notion of $\mathcal{A}$-dominance as a criterion to aggregate individual preferences: with this criterion, increasing the population size increases social welfare as long as all (already) living agents are better off, irrespective of the living conditions of the newborn.

Millian efficiency as robust $\mathcal{P}$-efficiency In this paper, we also explore whether the results in the preceding statements (i) or (ii) hold for the notions of $\mathcal{P}$-efficiency that arise from the two criteria—birth order or birth date—to identify potential agents. Differently from GJT, for whom the utility of the unborn is a constant value $\overline{u}$, we explore the possibility that the utility attributed to the unborn is a symmetric function of the utility achieved by the agent's living siblings. By making the utility attributed to the unborn depend on the utility obtained by those alive in a given allocation, we introduce, in welfare evaluation, principles such as "An increase in the population size is not welfare improving if the newborns are worse off than any, most, or the average of their siblings already alive." Under certain (concavity) conditions and independently on the criterion chosen to identify potential agents, a symmetric allocation is $\mathcal{P}$-efficient if and only if it is Millian efficient (see Theorem 3). Furthermore, the $\mathcal{P}$-efficiency of Millian efficient allocations holds for a wide range of specifications of the utility attributed to the unborn; that is, for a wide range of principles that determine under what conditions a new life is welfare improving. By assuming that the utility attributed to an unborn agent depends exclusively on the utility obtained by the agent's living siblings, we avoid cardinal assessments in welfare comparisons. Finally, using a weaker notion of efficiency might be the best option available if altering the agents' property rights on their labor capacities is not viable.

The paper is organized as follows. In the following section (Section 2), we anticipate and discuss our main results in the context of a simple, two-period economy. In Section 3, we introduce our general model. In Section 4, we explore the properties of $\mathcal{A}$-efficient allocations in the context of the model described in the previous section. In Section 5, we explore the properties of $\mathcal{P}$-efficiency when the utility obtained by the unborn is a function of the utility obtained by his/her living siblings. In Section 6, we present our main conclusions and discuss several possibilities for further research.

2. Efficient fertility choices in a simple, two-period example

To illustrate the two notions of $\mathcal{A}$- and $\mathcal{P}$-efficiency, we present a simple, two-period economy with two generations of agents: the dynasty head and her potential children. The economy is analogous to those described in Examples 2 and 3 in GJT, the differences being that, in our case, (a) the number of children that the dynasty head decides to raise is a *real* number $n_1 \in [0, \overline{n}]$, where $\overline{n}$ represents the length of the time interval in which the dynasty head may have children and, therefore, the maximum number of children that she can bear, and (b) the dynasty head is able to decide not only the number of descendants n_1 that she wants to bear, but also the (set of) points in time at which she wants to give birth to these descendants, represented by a Borel set $\mathsf{D}_1 \in \mathcal{B}[0, \overline{n}]$ such that

$$n_1 = \int_{\mathsf{D}_1} di.$$

In general, a potential agent is represented by the date $i \in [0, \overline{n}]$ at which he/she may be born; that is, potential agents are represented, therefore, by the birth-date criterion. Yet, the setting includes a particular specification in which birth dates cannot be postponed and the set of feasible fertility choices $\mathcal{D}$ adopts the form $\mathcal{D} = \mathcal{D}^{\mathcal{O}} \equiv \{[0, n] : n \in [0, \overline{n}]\}$. Note that in this constrained case, the agents' birth dates and their positions in their siblings' birth order coincide, and a fertility choice is completely described by the number of individuals. In what follows, we assume that $\mathcal{D}^{\mathcal{O}} \subseteq \mathcal{D}$ is satisfied.

At time $t = 0$, there are $\overline{e}_0$ units of a homogeneous good that can be used to finance the dynasty head's consumption, c_0^m, as well as capital investments, k_1^o, and child bearing activities, represented by a linear cost function $b(n_1) = bn_1$, with $b > 0$. The resource constraint at time 0 is $c_0^m + bn_1 + k_1^o \leq \overline{e}_0$, which, by writing k_1 for capital per worker (that is, $k_1 = k_1^o / n_1$), can be written equivalently as

$$c_0^m + n_1(b + k_1) \leq \overline{e}_0.$$

At time $t = 1$, total output is given by $Y_1 = F(k_1^o, n_1)$. In this simple example, we focus on a linear technology $F(k_1^o, n_1) = Rk_1^o + wn$, with $R > 0$ and $w > 0$ satisfying $Rb > w$. This output is used to provide the consumption good to each of descendants of the dynasty head in case these descendants are alive. Since fertility choices are selected from a continuum, consumption choices made by the descendants of the dynasty head are represented by an (Lebesgue) integrable function $\mathsf{c}_1^m : [0, \overline{n}] \to \mathbb{R}_+$, where $\mathsf{c}_1^m(i)$ represents the amount of the consumption good available to the ith descendant. Denote by

e_1 the average consumption available to the dynasty head's descendants, that is,

$$e_1 = \frac{1}{n_1} \int_{\mathsf{D}_1} \mathsf{c}_1^m(i)\, di. \tag{1}$$

The resource constraint, in per capita terms, that arises at time $t = 1$ adopts the form

$$e_1 \le F(k_1, 1) = f(k_1) = Rk_1 + w.$$

Finally, by letting $k(e_1) = f^{-1}(e_1) = [e_1 - w]/R$, the resource constraint at time $t = 0$ becomes

$$c_0^m + n_1\big[b + k(e_1)\big] \le \overline{e}_0. \tag{2}$$

In the economy described, an allocation $\mathsf{a} = (a, \mathsf{D}_1, \mathsf{c}_1^m)$ is a vector $a = (n_1, c_0^m, e_1) \in [0, \overline{n}] \times \mathbb{R}_+^2$, a set of points in time at which children are born, $\mathsf{D}_1 \in \mathcal{D}$, and a consumption plan $\mathsf{c}_1^m : [0, \overline{n}] \to \mathbb{R}_+$ that satisfy (1) and (2). Write $\mathcal{F}(\mathcal{D})$ for the set of feasible allocations. An allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$ is *ex post symmetric* (or simply symmetric) if it provides every descendant with the same consumption bundle and, hence, satisfies $\mathsf{c}_1^m(i) = e_1$ for every $i \in \mathsf{D}_1$.

While each potential descendant i, if alive (i.e., if $i \in \mathsf{D}_1$), cares monotonically about her own consumption $\mathsf{c}_1^m(i)$, the dynasty head is concerned with her own consumption, the number of descendants she decides to bear, and the amount of the consumption good available to each of her descendants. Her preferences on feasible allocations are represented by a utility function of the form

$$\mathcal{U}_0(\mathsf{a}) = U\left(c_0^m, n_1, \frac{1}{n_1} \int_{\mathsf{D}_1} U^D\big(\mathsf{c}_1^m(i)\big)\, di\right),$$

where U is nondecreasing, continuously differentiable, and concave, and U^D is a nondecreasing, strictly concave function. In this simple example, the preferences of the dynasty head are represented by the altruistic utility function

$$\mathcal{U}_0(\mathsf{a}) = \frac{\alpha}{\sigma}\big[c_0^m\big]^\sigma + \beta \int_{\mathsf{D}_1} \frac{1}{\sigma}\big[\mathsf{c}_1^m(i)\big]^\sigma\, di,$$

which adapts GJT's Examples 1 and 2 to allow for a continuous set of fertility choices. We take the same parametrization as in GJT. Specifically, $\overline{e}_0 = 100$, $b = 24$, $R = 1$, $w = 0$, and $\alpha = \beta = \gamma = \sigma = 1/2$.

$\mathcal{A}$- and $\mathcal{P}$-efficiency Even though, in our setting, the agents' birth dates can be postponed and their order at birth becomes an endogenous variable, it is not clear which of the two criteria to identify potential agents should be taken as the reference to undertake welfare comparisons with the $\mathcal{A}$- and the $\mathcal{P}$-criteria proposed by GJT. To avoid confusion, we use the superscripts $\mathcal{D}$ (for the birth-order criterion) and $\mathcal{O}$ (for the birth-order criterion) to make clear which of the two criteria is taken as the reference.

For expository convenience, the birth-date criterion is considered first. As in GJT, an allocation $\widehat{\mathsf{a}} = (\widehat{a}, \widehat{\mathsf{D}}_1, \widehat{\mathsf{c}}_1^m)$ is said to *$\mathcal{A}^{\mathcal{D}}$-dominate* an allocation $\mathsf{a} = (a, \mathsf{D}_1, \mathsf{c}_1^m)$ if it makes all potential agents—identified by their birth dates—living in both $\widehat{\mathsf{a}}$ and a better off without making any one of them worse off; that is, if (i) $\mathcal{U}_0(\widehat{\mathsf{a}}) \geq \mathcal{U}_0(\mathsf{a})$, (ii) $\widehat{\mathsf{c}}_1^m(i) \geq \mathsf{c}_1^m(i)$ for every $i \in \widehat{\mathsf{D}}_1 \cap \mathsf{D}_1$, and (iii) either $\mathcal{U}_0(\widehat{\mathsf{a}}) > \mathcal{U}_0(\mathsf{a})$ or $\widehat{\mathsf{c}}_1^m(i) > \mathsf{c}_1^m(i)$ is satisfied for a set $\mathsf{D} \subseteq \widehat{\mathsf{D}}_1 \cap \mathsf{D}_1$ of positive measure. As in GTJ, the $\mathcal{A}$ appearing in the notion of $\mathcal{A}^{\mathcal{D}}$-dominance may be justified because it refers to *alive* agents. An allocation $\widehat{\mathsf{a}} \in \mathcal{F}(\mathcal{D})$ is referred to as $\mathcal{A}^{\mathcal{D}}$-efficient if it is not $\mathcal{A}^{\mathcal{D}}$-dominated by any other allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$.

Analogous to the notion of $\mathcal{P}$-dominance proposed by GTJ, the notion of $\mathcal{P}^{\mathcal{D}}$-dominance is associated to a utility function $\mathcal{U}_1^N$, which determines the utility $\mathcal{U}_1^N(\mathsf{a}, i)$ attributed to the ith descendant—identified by the date at which he/she may be born—in the case that this descendant is not born. Complemented by the utility function defined by $\mathcal{U}_1(\mathsf{a}, i) = \mathsf{c}_1^m(\mathsf{i})$, which represents the preferences of every descendant i in those allocations in which i is alive, the preferences of a potential descendant are given by

$$\mathcal{U}_1^P(\mathsf{a}; i) = \begin{cases} \mathsf{c}_1^m(i) & \text{if } i \in \mathsf{D}_1, \\ \mathcal{U}_1^N(\mathsf{a}; i) & \text{otherwise.} \end{cases}$$

Using this notation, an allocation $\widehat{\mathsf{a}}$ $\mathcal{P}^{\mathcal{D}}$-dominates an allocation a if it is unanimously preferred by all potential agents in the economy and strictly preferred by a positive measure of potential agents; that is, if (i) $\mathcal{U}_0(\widehat{\mathsf{a}}) \geq \mathcal{U}_0(\mathsf{a})$, (ii) $\mathcal{U}_1^P(\widehat{\mathsf{a}}; i) \geq \mathcal{U}_1^P(\mathsf{a}; i)$ for every $i \in [0, \overline{n}]$, and (iii) either $\mathcal{U}_0(\widehat{\mathsf{a}}) > \mathcal{U}_0(\mathsf{a})$ or $\mathcal{U}_1^P(\widehat{\mathsf{a}}; i) > \mathcal{U}_1^P(\mathsf{a}; i)$ is satisfied for a set $\mathsf{D} \subseteq [0, \overline{n}]$ of positive measure. Also as in GTJ, the $\mathcal{P}$ appearing in the notion of $\mathcal{P}^{\mathcal{D}}$-efficiency may stand for *population* or *potential*. An allocation $\widehat{\mathsf{a}} \in \mathcal{F}(\mathcal{D})$ is referred to as $\mathcal{P}^{\mathcal{D}}$-efficient if it is not $\mathcal{P}^{\mathcal{D}}$-dominated by any other allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$.

Next, we consider the birth-order criterion. So as to define formally the notion of $\mathcal{A}^{\mathcal{O}}$-efficiency in a setting in which the agents' position in the birth order is endogenous, given a feasible allocation a, for every $i \in [0, n]$, write $\mathsf{o}(i)$ for the position in the siblings' order held by the agent born at date i, that is,

$$\mathsf{o}(i) = \int_{\mathsf{D}_1 \cap [0, i]} ds.$$

Also, given an allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$, for each $i \in [0, \overline{n}]$, write $\mathsf{c}_1^{mO}(i)$ for the amount of the consumption good available to the agent occupying the ith position in the siblings' order, that is,

$$c_1^{mO}(i) = \begin{cases} c_1^m\big(\mathsf{o}^{-1}(i)\big) & \text{if } i \in [0, n_1], \\ 0 & \text{if } i > n_1. \end{cases}$$

With this notation, an allocation a $\mathcal{A}^{\mathcal{O}}$-dominates an allocation $\widehat{\mathsf{a}}$ if $\mathcal{U}_0(\mathsf{a}) \geq \mathcal{U}_0(\widehat{\mathsf{a}})$ and $\mathsf{c}_1^{mO}(i) \geq \widehat{\mathsf{c}}_1^{mO}(i)$ for every $i \leq \min\{\widehat{n}_1, n_1\}$, and either $\mathcal{U}_0(\mathsf{a}) > \mathcal{U}_0(\widehat{\mathsf{a}})$ or $\mathsf{c}_1^{mO}(i) > \widehat{\mathsf{c}}_1^{mO}(i)$ is satisfied for some $i \leq \min\{\widehat{n}_1, n_1\}$. An allocation $\widehat{\mathsf{a}} \in \mathcal{F}(\mathcal{D})$ is $\mathcal{A}^{\mathcal{O}}$-efficient if it is not $\mathcal{A}^{\mathcal{O}}$-dominated by any other allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$.

Given an allocation $\mathsf{a} = (a, \mathsf{c}_1^{mO}, \mathsf{D}_1) \in \mathcal{F}(\mathcal{D})$, we refer to the pair $\mathsf{a}^{\mathcal{O}} = (a, \mathsf{c}_1^{m\mathcal{O}}) \in \mathcal{F}(\mathcal{D}^{\mathcal{O}})$ as the birth-order representation of a. Observe that every $\mathcal{A}^{\mathcal{O}}$-efficient allocation can be equivalently defined as an allocation whose birth-order representation cannot be dominated by any $\mathsf{a} \in \mathcal{F}(\mathcal{D}^{\mathcal{O}})$; that is, for any allocation that is feasible in an economy in which the dynasty head's choices are restricted (i.e., births cannot be postponed) and $\mathcal{D} \equiv \mathcal{D}^{\mathcal{O}}$ is satisfied. The notion of $\mathcal{P}^{\mathcal{O}}$-dominance can be defined analogously, and an allocation is referred to as $\mathcal{P}^{\mathcal{O}}$-efficient if its birth-order representation is not $\mathcal{P}^{\mathcal{O}}$-dominated by any other allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D}^{\mathcal{O}})$.

Note that every $\mathcal{A}^{\mathcal{O}}$-efficient or $\mathcal{P}^{\mathcal{O}}$-efficient allocation is equivalent to an allocation that arises as $\mathcal{A}^{\mathcal{D}}$-efficient or $\mathcal{P}^{\mathcal{D}}$-efficient in the constrained setting in which $\mathcal{D} \equiv \mathcal{D}^{\mathcal{O}}$ holds. Hence, an allocation that is $\mathcal{A}^{\mathcal{O}}$-efficient may be $\mathcal{A}^{\mathcal{D}}$-dominated by an allocation that is not feasible in such a constrained setting. Thus, $\mathcal{A}^{\mathcal{D}}$-efficient (respectively, $\mathcal{P}^{\mathcal{D}}$-efficient) allocations satisfy all the requirements that $\mathcal{A}^{\mathcal{O}}$-efficient (respectively, $\mathcal{P}^{\mathcal{O}}$-efficient) allocations must satisfy, which implies that the set of $\mathcal{A}^{\mathcal{D}}$-efficient (respectively, $\mathcal{P}^{\mathcal{D}}$-efficient) allocations is contained in the set of $\mathcal{A}^{\mathcal{O}}$-efficient (respectively, $\mathcal{P}^{\mathcal{O}}$-efficient) allocations.

Dynastic optima and Millian efficient allocations. So as to explore the properties of $\mathcal{A}^{\mathcal{O}}$-efficient allocations, we assume without loss of generality that $\mathcal{D} \equiv \mathcal{D}^{\mathcal{O}}$ is satisfied. Taking into account that the dynasty head is alive in every feasible allocation, any allocation a^* that maximizes the utility of the dynasty head among all feasible allocations must be both $\mathcal{A}^{\mathcal{O}}$-efficient and $\mathcal{A}^{\mathcal{D}}$-efficient. Moreover, since the function U^D is strictly concave, such a dynastic optimum must be symmetric. The pair (n_1^*, e_1^*) that corresponds to a dynastic optimum a^* must, therefore, solve

$$\max_{(n_1,e_1)\in[0,\overline{n}]\times\mathbb{R}_+} \{U(\overline{e}_0 - n_1[b+k(e_1)], n_1, U^D(e_1))\} \equiv \mathcal{V}_0(\overline{e}_0), \tag{3}$$

which, in our parametric example, yields

$$(c_0^{m*}, n_1^*, e_1^*) = \left(24, \frac{6^{1/2}}{12}, 24\right).$$

Are there other $\mathcal{A}^{\mathcal{O}}$-efficient allocations? A class of seemingly good candidates is the class of Millian efficient allocations; that is, the class of symmetric allocations that cannot be $\mathcal{A}^{\mathcal{O}}$-dominated by any other symmetric allocation. A Millian efficient allocation $\widehat{\mathsf{a}}$ gives the dynasty head the maximum utility that she can obtain with a symmetric allocation if she is restricted to provide *at least* $\widehat{e}_1$ units of resources to *each* of her descendants, i.e., $e_1 \geq \widehat{e}_1$. That is, the pair $(\widehat{n}_1, \widehat{e}_1)$ that corresponds to a Millian efficient allocation solves

$$\max_{(n_1,e_1)\in[0,\overline{n}]\times\mathbb{R}_+} \{U(\overline{e}_0 - n_1[b+k(e_1)], n_1, U^D(e_1)) : e_1 \geq \widehat{e}_1\}. \tag{4}$$

It is also easy to see that, if $\widehat{e}_1 \geq e_1^*$, then the constraint $e_1 \geq \widehat{e}_1$ in (4) must be binding. Therefore, Millian efficient allocations are those for which $\widehat{n}_1$ solves, for some $\widehat{e}_1 \geq e_1^*$,

$$\max_{n_1\in[0,\overline{n}]} \{U(\overline{e}_0 - n_1[b+k(\widehat{e}_1)], n_1, U^D(\widehat{e}_1))\} \equiv W_0(\overline{e}_0, \widehat{e}_1, U^D(\widehat{e}_1)). \tag{5}$$

In our parametric example, for any $\widehat{e}_1 \geq w$, the solution $(\widehat{c}_0^m, \widehat{n}_1, \widehat{e}_1) = (c_0^m(\widehat{e}_1), n_1(\widehat{e}_1), k_1(\widehat{e}_1))$ to the optimization problem in the definition of $W_0(\overline{e}_0, \widehat{e}_1, U^D(\widehat{e}_1))$ is

$$(\widehat{c}_0^m, \widehat{n}_1, \widehat{e}_1) = \left(\frac{[24+\widehat{e}_1]^2}{16\widehat{e}_1}, \frac{100 - \left[\frac{24+\widehat{e}_1}{16\widehat{e}_1}\right]^2}{24+\widehat{e}_1}, \widehat{e}_1 \right),$$

which implies that the fertility rate arising in the dynastic optimum is always higher than that corresponding to any other Millian efficient allocation. Clearly, starting from a symmetric allocation $\widehat{\mathsf{a}}$ for which $(\widehat{n}_1, \widehat{e}_1)$ solves the optimization problem in (4), the only way to increase the utility of the dynasty head with a symmetric allocation is by decreasing e_1, which, taking into account that a lower income $e_1 < \widehat{e}_1$ brings with it a higher fertility level $n_1 \geq \widehat{n}_1$, would make all the descendants who where already living in the original allocation worse off.

Perhaps surprisingly, the only Millian efficient allocation that is also $\mathcal{A}^{\mathcal{O}}$-efficient is the dynastic optimum. To see why, let $\widehat{a} = (\widehat{c}_0^m, \widehat{n}_1, \widehat{e}_1)$ be Millian efficient allocation that satisfiesy $\widehat{e}_1 > e_1^*$. Now suppose the dynasty head is given the opportunity to choose an asymmetric allocation $\widetilde{\mathsf{a}}$ with more children than those living in $\widehat{a}$ (that is, satisfying $\widetilde{n}_1 \geq \widehat{n}_1$) constructed in such a way that the already living children obtain the same consumption bundle (that is, $\widetilde{c}_1^m(i) = \widehat{e}_1$ for each $i \in [0, \widehat{n}_1]$) while the newborn obtains $\widetilde{e}_1$ (that is, $\widetilde{c}_1^{mO}(i) = \widetilde{e}_1$ for each $i \in (\widehat{n}_1, n_1]$). The dynasty head will choose $\widetilde{c}_0^m$, $\widetilde{n}_1$, $\widetilde{k}_1$, and $\widetilde{e}_1$ to maximize

$$\mathcal{U}_0(\mathsf{a}) = U\left(c_0^m, n_1, \left(\frac{\widehat{n}_1}{n_1}\right) U^D(\widehat{e}_1) + \left(1 - \frac{\widehat{n}_1}{n_1}\right) U^D(e_1)\right)$$

subject to the constraints

$$c_0^m + n_1(b+k_1) \leq \overline{e}_0, \left(\frac{\widehat{n}_1}{n_1}\right)\widehat{e}_1 + \left(1 - \frac{\widehat{n}_1}{n_1}\right) e_1 \leq f(k_1) \quad \text{and} \quad \overline{n}_1 \geq n_1 \geq \widehat{n}_1.$$

In our example, the solution to this problem satisfies $\widetilde{n}_1 = n_1^* - \widehat{n}_1[\widehat{e}_1 - e_1^*]/[Rb - w + e_1^*] > \widehat{n}_1$ and $\widetilde{e}_1 = e_1^* < \widehat{e}_1$. Thus, even though the dynasty head would rather not discriminate among her children, she finds it optimal to do so and increase the number of children if she is constrained to provide each of her already living children with an amount of the consumption good $\widehat{e}_1 > e_1^*$. The possibility to discriminate against the children who are not born in $\widehat{a}$ makes these children cheaper, as their marginal costs—given by $b + k(\widehat{e}_1)$ in the original allocation—jump down to $b + k[(\widehat{n}_1/\widetilde{n}_1)\widehat{e}_1 + (1 - \widehat{n}_1/\widetilde{n}_1)e_1^*]$. Therefore, every Millian efficient allocation $\widehat{a}$ is $\mathcal{A}^{\mathcal{O}}$-dominated by an asymmetric allocation $\widetilde{\mathsf{a}}$ with more individuals, in which all the people born in $\widehat{a}$ obtain at least the same utility as that obtained in $\widetilde{\mathsf{a}}$, while the dynasty head obtains strictly higher utility. Since the dynastic optimum is trivially $\mathcal{A}^{\mathcal{O}}$-efficient, this implies, in turn, that the only symmetric, $\mathcal{A}^{\mathcal{O}}$-efficient allocation is the dynastic optimum.

Other $\mathcal{A}^{\mathcal{O}}$-efficient allocations Observe that, in addition to the dynastic optimum, there are many other $\mathcal{A}^{\mathcal{O}}$-efficient allocations. The asymmetric allocation $\widetilde{\mathsf{a}}$ constructed above (where $\widetilde{e}_1$ is chosen to maximize the dynasty head's utility) is an example. With the allocation $\widetilde{\mathsf{a}}$, the dynasty head cannot obtain higher utility with an allocation with more or with fewer descendants, provided she has to provide the first $\widehat{n}_1$ surviving descendants with $\widehat{e}_1$ units of the consumption good.

In Theorem 1, we show, in the context of a more general model, that in every $\mathcal{A}^{\mathcal{O}}$-efficient allocation, a positive measure of descendants—to be more precise, the youngest individuals in every family—must devote most of their income to maximize the utility of the dynasty head. In the context of the two-period example analyzed here, Theorem 1 implies that in every $\mathcal{A}^{\mathcal{O}}$-efficient allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$, the consumption scheme $\mathsf{c}_1^m(\cdot)$ must satisfy

$$\lim_{\substack{i \to n_1 \\ i < n_1}} \mathsf{c}_1^m(i) = e_1^*. \tag{6}$$

Note that if (6) were not satisfied, the dynasty head might obtain higher utility by altering her fertility choice to provide the youngest descendants with $\mathsf{c}_1^m(i) = e_1^*$ units of the consumption good. This might be problematic if, in our example, the dynasty head is not altruistic and $\beta = 0$ is satisfied. In this case, the dynastic optimum must satisfy $e_1^* = 0$. Moreover, for any other $\mathcal{A}^{\mathcal{O}}$-efficient allocation $\widetilde{\mathsf{a}}$, a positive measure of agents obtain an arbitrarily small amount of resources. Of course, one might argue that non-altruistic preferences are rare, but as we show in Corollary 1, the same result arises in a more general setting with infinite periods of time when the altruism of the agents extends to a *finite* number of generations of their descendants.

$\mathcal{A}^{\mathcal{D}}$-efficiency The fact that most $\mathcal{A}^{\mathcal{O}}$-efficient allocations are asymmetric seems to be at odds with imposing the assumption that all living descendants have identical preferences and capacities. We note, however, that if feasible fertility constraints are unconstrained and the child-bearing period is long enough, none of these asymmetric, $\mathcal{A}^{\mathcal{O}}$-efficient allocations is $\mathcal{A}^{\mathcal{D}}$-efficient. To be more precise, if $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$ and $\overline{n} \geq 2n_1^*$ are satisfied, then the only $\mathcal{A}^{\mathcal{D}}$-efficient allocations are dynastic optima.

To see why, recall that the set of $\mathcal{A}^{\mathcal{O}}$-efficient allocations is isomorphic to the set of $\mathcal{A}^{\mathcal{D}}$-efficient allocations that arise in the restricted setting in which $\mathcal{D} \equiv \mathcal{D}^{\mathcal{O}}$ is satisfied. Thus, in an $\mathcal{A}^{\mathcal{O}}$-efficient allocation a, (i) the youngest children in every family cannot obtain more utility than the utility that he/she would obtain in a dynastic optimum, and (ii) the number of children n_1 cannot be higher than the optimal number n_1^*. Therefore, if $\overline{n} > 2n_1^*$ holds and $c_1^m(i) > e_1^*$ is satisfied for a positive measure of agents alive in a, the dynasty head can always replace these agents by a set of agents of equal measure that obtain $c_1^m(i) = e_1^*$. Clearly, the allocation resulting from such a replacement constitutes an $\mathcal{A}^{\mathcal{D}}$-improvement, since the dynasty head—the only agent living in the two allocations under comparison—will be better off with such replacement. In the absence of altruism, achieving $\mathcal{A}^{\mathcal{D}}$-efficiency requires that $\mathsf{c}_1^m(i) = 0$ is satisfied for every alive descendant $i \in \mathsf{D}_1$. As we show in Section 4, in the absence of altruism or in the presence of finite-horizon altruism, achieving $\mathcal{A}^{\mathcal{D}}$-efficiency may drive an economy to a collapse, as no more individuals are born thereafter.

$\mathcal{A}$-efficiency and individual property rights Even if the dynasty head is altruistic toward her descendants, both $\mathcal{A}^{\mathcal{O}}$- and $\mathcal{A}^{\mathcal{D}}$-efficiency may be incompatible with the existence of property rights on the resources to be allocated in a given economy, as Schoonbroodt and Tertilt (2014) have observed. To see why, suppose we slightly modify the model described above in such a way that the amount of the consumption good k_1^o accumulated as capital by the dynasty head may be negative, in which case $d_1^0 = -k_1^o R$ represents the dynasty head's debt in period 1. Suppose also that in such a model, the dynastic optimum satisfies $e_1^* < w$. Finally, suppose that all descendants of the dynasty head born in any allocation cannot be forced to repay their parent' debts. As none of these agents cares about the welfare enjoyed by the dynasty head, they will all use their entire labor capacity to obtain w units of the consumption good. As $e_1^* < w$ holds, the dynasty head would rather not provide any of her descendants with more resources than the resources they are endowed with. Therefore, the existence of property rights drives the economy to the Millian efficient allocation $\widehat{a}$ for which $\widehat{e}_1 = w > e_1^*$. As explained above, such an allocation is both $\mathcal{A}^{\mathcal{O}}$- and $\mathcal{A}^{\mathcal{D}}$-inefficient. Yet, while such an allocation $\widehat{a}$ can be $\mathcal{A}^{\mathcal{D}}$-dominated by a dynastic optimum, provided all agents living in such dynastic optimum are different than those living in $\widehat{a}$, it can *only* be $\mathcal{A}^{\mathcal{O}}$-dominated by some asymmetric allocations in which only *some* of the agents—the youngest children of the dynasty head—are forced to repay the debt accumulated by their parent to finance her fertility choices.

Discrete choices. Comparison with GJT In the example discussed above, the difference between the fertility choices, $\widetilde{n}_1$ and $\widehat{n}_1$, which correspond to the allocations $\widetilde{\mathsf{a}}$ and $\widehat{a}$, is always strictly positive, and it might become arbitrarily small as $\widehat{e}_1$ approaches e_1^*. That is, ruling out some Millian efficient allocations as being $\mathcal{A}^{\mathcal{O}}$-inefficient may require that the population is increased in arbitrary small amounts. Therefore, the arguments used to characterize the only symmetric, $\mathcal{A}^{\mathcal{O}}$-efficient allocation as a dynastic optimum cannot be applied when the set of fertility choices is discrete, as GJT assume in their examples.

To make this clear, suppose that the the dynasty head is restricted to choose the number of children n_1 she decides to bear out of a discrete set $\{0, 1, 2, 3, \dots, \overline{n}\}$, so that the set $D_1 \subseteq \varnothing \cup \{1, 2, 3, \dots, \overline{n}\}$ of dates at which children of their children are born must satisfy $\mathsf{D}_1 = \varnothing$ if $n_1 = 0$ and $\#\{\mathsf{D}_1\} = n_1$ if $n_1 > 0$. Suppose also that the average income received by the dynasty head's descendants is now given by

$$e_1 = \frac{1}{n_1} \sum_{\mathsf{D}_1} \mathsf{c}_1^m(i),$$

and that the dynasty head's preferences are represented by a utility function of the form

$$\mathcal{U}_0(\mathsf{a}) = U\left(c_0^m, n_1, \frac{1}{n_1} \sum_{\mathsf{D}_1} U^D\left(\mathsf{c}_1^m(i)\right)\right).$$

In such a setting, the dynastic optimum a^* and every Millian efficient allocation $\widehat{a}$ can be characterized, respectively, as the solutions to optimization problems closely analogous to those in the definitions of $\mathcal{V}_0(\overline{e}_0)$ and $W_0(\overline{e}_0, \widehat{e}_1, U^D(\widehat{e}_1))$ (see (3) and (5), respectively), which differ from those in that the optimal choices n_1^* and $\widehat{n}_1$, must now satisfy a constraint of the form $n_1 \in \{0, 1, 2, 3, \ldots, \overline{n}\}$. Thus, as in the model in which fertility choices are drawn from a continuum, the income e_1^* (respectively, the fertility rate n_1^*) that corresponds to a^* provides a lower bound (respectively, an upper bound) for the average consumption (respectively, the fertility rate) that arises with a Millian efficient allocation. Taking this into account, it is straightforward to notice that in the discrete case, many Millian efficient allocations may be $\mathcal{A}^{\mathcal{O}}$-efficient even though they are not dynastic optima. Golosov et al. (2007, Example 2) provide an example in which this occurs.

However, if parents are able to choose any subset of the set of dates $\{1, 2, 3, \ldots, \overline{n}\}$ and $\overline{n} \geq 2n_1^*$ holds, then every $\mathcal{A}^{\mathcal{O}}$-efficient allocation that is $\mathcal{A}^{\mathcal{D}}$-efficient must be a dynastic optimum. As in the continuous case, the fertility rate n_1^* that corresponds to a dynastic optimum provides an upper bound for the fertility rate that arises in an $\mathcal{A}^{\mathcal{O}}$-efficient allocation. Thus, if an allocation a is not a dynastic optimum, the dynasty head is always better off to replace all agents living in a with a new set of agents and provide each of them with the level of consumption e_1^* that corresponds to a dynastic optimum. Such a reallocation of resources is a welfare improvement, since the dynasty head is the only agent alive in the two allocations under comparison, which establishes the result.

$\mathcal{A}^{\mathcal{O}}$-efficiency versus $\mathcal{A}^{\mathcal{D}}$-efficiency. Final remarks In view of the differences between the two criteria to confer the agents their identity, a question arises: Which criterion should we opt for? A possible justification of the birth-order criterion is that the eldest children in a family already are alive when their younger siblings are born. However, in the models presented in this paper, all descendants are already alive by the time parents decide whether they are willing to provide them with any resources. In addition, the objective behind extending the Pareto criterion is to evaluate allocations that result from decisions such as increasing the size of a family *before* potential agents are born, not while potential agents are being born or after they are all already born. Finally, even if, in our model, all children have the same preferences and the dynasty head is willing to treat her children symmetrically, the use of the birth-order criterion delivers a highly asymmetric set of utility possibilities for the agents. To be more precise, while the youngest children in a family cannot obtain, in an $\mathcal{A}^{\mathcal{O}}$-efficient allocation, more utility than the utility they obtain in a dynastic optimum, the oldest may obtain, in the continuous case, arbitrarily high utility. In contrast, the frontier of the set of utility possibilities that correspond to the birth-date criterion is symmetric, although it reduces to a singleton.

The two criteria differ also in many other settings, which, to save on space, we do not analyze formally. For example, consider a setting in which the dynasty head's preferences on her children's utilities depend on their positions in the birth order. With such preferences, dynastic optima, which—for a sufficiently high $\overline{n}$—are the only $\mathcal{A}^{\mathcal{D}}$-efficient allocations, are asymmetric, and those children who occupy higher positions in the birth order obtain higher resources. But this asymmetry is caused by parental preferences, not by the notion of efficiency applied. The birth-order criterion to identify potential agents delivers a much larger set of efficient allocations, in some of which the utility obtained by the eldest child is much higher than that obtained by other siblings. Finally, consider a setting in which parents are concerned about their children's characteristics and these characteristics depend—possibly stochastically—on potential children's birth dates. In such a framework, parent's optimal plans, which may adopt the form "keep having children until you have two kids with a characteristic θ," still characterize $\mathcal{A}^{\mathcal{D}}$-efficiency, but applying the birth-order criterion seems arbitrary. One might argue that applying $\mathcal{A}^{\mathcal{D}}$-efficiency in this type of settings might be in conflict with current laws or with other ethical principles, but it is not clear how applying the birth-order criterion may solve those conflicts.

$\mathcal{P}$-efficient allocations When the function $\mathcal{U}_1^N$ is constant and $\mathcal{U}_1^N(\mathtt{a}, i) = \overline{e}_1$ holds, the properties of $\mathcal{P}$-efficient allocations—with either criterion—are similar to those arising in GJT's setting. If, for example, $\overline{e}_1 < 0$ is satisfied, then the only Millian efficient allocation that is also $\mathcal{P}^{\mathcal{D}}$-efficient or $\mathcal{P}^{\mathcal{O}}$-efficient is the dynastic optimum. There are, however, many other allocations that are $\mathcal{P}^{\mathcal{D}}$-efficient and, hence, $\mathcal{P}^{\mathcal{O}}$-efficient. For any fertility level $\breve{n}_1$ above the fertility level n_1^* that corresponds to the dynastic optimum, an allocation that maximizes the utility of the dynasty head among all allocations for which $n_1 \geq \breve{n}_1$ is satisfied must be $\mathcal{P}$-efficient. Therefore, fertility levels that arise with $\mathcal{P}^{\mathcal{D}}$- or $\mathcal{P}^{\mathcal{O}}$-efficient allocations can be arbitrarily high. As the utility attributed to the unborn $\overline{e}_1$ is higher, a Millian efficient allocation for which $e_1^* \leq \widehat{e}_1 < \overline{e}_1$ is satisfied is also $\mathcal{P}^{\mathcal{D}}$-efficient and $\mathcal{P}^{\mathcal{O}}$-efficient, and other allocations that are not Millian efficient might be $\mathcal{P}^{\mathcal{D}}$-efficient as well.

The two notions of $\mathcal{P}$-efficiency—with constant utility levels attributed to the unborn—bring with them another difficulty: Who sets $\overline{e}$? To avoid this difficulty, Golosov et al. (2007, p. 1054) suggest to take $\overline{e}_1 = e_1^*$, which leaves the dynastic optimum as the only $\mathcal{P}^{\mathcal{D}}$-efficient allocation and the only symmetric, $\mathcal{P}^{\mathcal{O}}$-efficient allocation. In Section 4, we explore the possibility that the utility attributed to the unborn depends on the utility obtained by their living siblings.

3. Feasible fertility choices in a general, overlapping generations setting

3.1 *A benchmark framework*

Throughout the remaining of the paper, we focus on a particular class of overlapping generations (OLG) economies with infinite periods of time. Each individual in an economy lives for at most three of these periods, so that individuals living at $t = 0, 1, 2, \ldots$ are referred to as *children*, *middle-aged adults*, or *old adults*, depending on whether it is their first, second, or third period of life. As in GJT, the set of potential agents who are actually alive at any given period is endogenous and depends on the fertility plans selected by the agents. In contrast to these authors, we assume that each middle-aged adult *potentially* alive at $t = 1$ is identified by a nonnegative number $i_1 \in [0, \overline{n}]$ that determines the instant, in the child-rearing period $[0, \overline{n}]$, in which the agent may be born. For $t = 1, 2, 3, \ldots,$ each middle-aged adult potentially alive at t is identified by a vector $i^t = (i^{t-1}, i_t) \in [0, \overline{n}]^t$, where i_t specifies the instant at which the agent may be born and $i^{t-1} = (i_1, \ldots, i_{t-1})$ identifies the agent's parent. To simplify things, all agents belong to the same dynasty, initiated by the only agent who is middle aged at $t=0$—the *dynasty head*, hereafter represented by i^0. Let $\mathcal{B}[0,\overline{n}]^t$ be the class of Borel sets in $[0,\overline{n}]^t$. For every set $\mathsf{D}\in\mathcal{B}[0,\overline{n}]$, the Lebesgue measure of D is denoted by $\mu\{\mathsf{D}\}\equiv\int_D di$, while for every set $\mathsf{D}^t\in\mathcal{B}[0,\overline{n}]^t$ of potential middle-aged agents at t, the Lebesgue measure of D^t is denoted by $\mu^t\{\mathsf{D}^t\}\equiv\int_{\mathsf{D}^t} di^t$. For intervals of the form $\mathsf{D}^t=[a^t,b^t]\in\mathcal{B}[0,\overline{n}]^t$, the Lebesgue integral $\int_{[a^t,b^t]} di^t$ may also be written as $\int_{[a^t,b^t]} di^t=\int_{a^t}^{b^t} di^t$.

The set of feasible fertility choices is represented by a set $\mathcal{D}\subseteq\mathcal{B}[0,\overline{n}]$. Thus, although potential agents are identified, in general, with the *birth-date criterion*, we allow, as a particular specification, that the set of fertility choices adopts the form $\mathcal{D}^{\mathcal{O}}\equiv\{[0,n]:n\in[0,\overline{n}]\}$. Note that in this case, the agents' birth dates and their positions in their siblings' birth order coincide. In what follows, we assume that $\mathcal{D}^{\mathcal{O}}\subseteq\mathcal{D}$ is satisfied.

A fertility plan D is a sequence of mappings $\mathsf{D}=\{\mathsf{D}_{t+1}:[0,\overline{n}]^t\longrightarrow\mathcal{D}\}_{t\geq 0}$. Each mapping $\mathsf{D}_{t+1}:[0,\overline{n}]^t\longrightarrow\mathcal{D}$ represents a schedule that determines, for each $i^t\in[0,\overline{n}]^t$, the set of subperiods $\mathsf{D}_{t+1}(i^t)$ in the child-rearing period in which agent i^t's children are born. We assume that at each point in time, only one child may be born, so that the number of children that agent i^t decides to bear, which we denote by $\mathsf{n}_{t+1}(i^t)$, is given by

$$\mathsf{n}_{t+1}(i^t)\equiv\mu\{\mathsf{D}_{t+1}(i^t)\}=\int_{\mathsf{D}_{t+1}(i^t)} di. \tag{7}$$

For each t and every $i^t=(i^{t-1},i_t)\in[0,\overline{n}]^t$, agent i^t is said to be alive with fertility plan D if agent i^{t-1} is also alive and $i_t\in\mathsf{D}_t(i^{t-1})$ is satisfied. For every individual $i^t\in\mathbb{R}^t_+$ and every $\tau\geq t+1$, the set of descendants of i^t at their middle age at τ is denoted by $\mathsf{D}^\tau(i^t)$. The set of middle-aged adults actually living at t with a fertility plan D is denoted by $\mathsf{D}^t(i^0)$ and its measure is given by

$$\mu^t\{\mathsf{D}^t(i^0)\}=\int_{\mathsf{D}^t(i^0)} di^t=\int_{\mathsf{D}^{t-1}(i^0)}\left(\int_{\mathsf{D}_t(i^{t-1})} di\right) di^{t-1}=\int_{\mathsf{D}^{t-1}(i^0)}\mathsf{n}_t(i^{t-1})\, di^{t-1}.$$

In addition to children, there is only one homogenous good produced at every period $t\geq 1$. This consumption good is produced at each period $t\geq 0$ using, as inputs, a given amount K_t of the same good invested in the previous period $t-1$ as physical capital and a given amount of labor L_t provided by middle-aged adults. That is, $Y_t\leq F_t(K_t,L_t)$, where Y_t is total output and $F_t:\mathbb{R}^2_+\longrightarrow\mathbb{R}_+$ exhibits constant returns to scale, and it is nondecreasing, concave, and continuously differentiable. Rearing children is a production activity that takes place within each household and its costs are represented by a strictly increasing, convex, and continuously differentiable function $b_t:[0,\overline{n}]\longrightarrow\mathbb{R}_+$. Thus, a middle-aged adult who decides to rear n_{t+1} children at period t needs to spend $b_t(n_{t+1})$ units of the consumption good. Fertility and consumption plans of potential agents are represented by a fertility plan D and a sequence of integrable functions $\mathsf{c}=\{(\mathsf{c}^m_t,\mathsf{c}^o_{t+1}):[0,\overline{n}]^t\longrightarrow\mathbb{R}^2_+\}_{t\geq 0}$ that determines, for each $t\geq 0$ and each potential agent $i^t\in\mathbb{R}^t_+$, the consumption vector $(\mathsf{c}^m_t(i^t),\mathsf{c}^o_{t+1}(i^t))$ chosen by agent i^t through her life cycle. Thus, it is assumed that children do not make consumption decisions.

The resource constraint faced by potential agents is described as follows. At time $t=0$, the amount of resources available to finance consumption $(\mathsf{c}^m_0(i^0))$, fertility $(\mathsf{n}_1(i^0))$, and investment decisions ($\mathsf{k}_1^o(i^0)$) of the dynasty head is bounded by an initial endowment $\overline{e}_0$ available for the dynasty head, that is,

$$\mathsf{c}_0^m(i^0) + b_0(\mathsf{n}_1(i^0)) + \mathsf{k}_1^o(i^0) \leq \overline{e}_0. \tag{8}$$

For each period $t \geq 0$, each living agent is endowed with 1 unit of labor time when he/she reaches middle age. Then labor is supplied inelastically, so that labor supply at any given period coincides with the measure of middle-aged agents alive at t, that is, $L_t = \mu^t\{\mathsf{D}^t(i^0)\}$. By writing, for each t and each $i^t \in \mathsf{D}^t(i^0)$, the capital invested per old adult $\mathsf{k}_{t+1}^o(i^t)$ for $\mathsf{k}_{t+1}^o(i^t) = \mathsf{n}_{t+1}(i^t)K_{t+1}/\mu^t\{\mathsf{D}^t(i^0)\}$, the resource constraint at each date $t \geq 1$ is

$$\begin{aligned} &\int_{\mathsf{D}^{t-1}(i^0)} \mathsf{c}_t^o(i^{t-1})\, di^{t-1} + \int_{\mathsf{D}^t(i^0)} \left[\mathsf{c}_t^m(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + \mathsf{k}_{t+1}^o(i^t)\right] di^t \\ &\quad \leq \int_{\mathsf{D}^{t-1}(i^0)} F_t(\mathsf{k}_t^o(i^{t-1}), \mathsf{n}_t(i^{t-1}))\, di^{t-1}. \end{aligned} \tag{9}$$

In what follows, an allocation $\mathsf{a} = (\mathsf{D}, \mathsf{c}, \mathsf{k}^o)$ is a fertility plan D, a consumption plan $\mathsf{c} = \{(\mathsf{c}_t^m, \mathsf{c}_{t+1}^o) : [0, \overline{n}]^t \longrightarrow \mathbb{R}_+^2\}_{t\geq 0}$ that determines consumption choices of every potential agent, and an investment plan $\mathsf{k}^o = \{\mathsf{k}_{t+1}^o : [0, \overline{n}]^t \longrightarrow \mathbb{R}_+\}_{t\geq 0}$ that determines investment decisions in every period. An allocation a is feasible if it satisfies the initial condition in (8), the resource constraint in (9), and condition (7). The set formed by all feasible allocations is denoted by $\mathcal{F}(\mathcal{D})$.

Write $\mathbb{R}^*$ for the set of extended real numbers $\mathbb{R}^* \equiv \{-\infty\} \cup \mathbb{R}$, and write $\mathsf{x}_t(i^t)$ for the consumption–fertility bundle

$$\mathsf{x}_t(i^t) = (\mathsf{c}_t^m(i^t), \mathsf{c}_{t+1}^o(i^t), n_{t+1}(i^t)).$$

Throughout this paper, we assume that preferences of every potential agent of generation t on the set of allocations in which the agent is alive are represented by a utility function $\mathcal{U}_t : \mathcal{F}(\mathcal{D}) \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ that satisfies, for every $\mathsf{a} \in \mathcal{F}(\mathcal{D})$ and $i^t \in \mathsf{D}^t(i^0)$,

$$\mathcal{U}_t(\mathsf{a}; i^t) = U\left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}_{t+1}^D(\mathsf{a}; i^t, i_{t+1})\, di_{t+1}\right),$$

where $U : \mathbb{R}_+^3 \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is nondecreasing, concave, and continuously differentiable, and $\mathcal{U}_{t+1}^D$ is a function that represents the agents' preferences on fertility and consumption choices made by his/her living descendants. Thus, as the functions $\mathcal{U}_t$ and $\mathcal{U}_t^D$ may not coincide, parents' preferences regarding their children's decisions might differ from the preferences of the children themselves.

We assume that for each t, $\mathcal{U}_t^D : \mathcal{F}(\mathcal{D}) \times \mathbb{R}_+^t \longrightarrow \mathbb{R}^*$ is recursively defined, for every $\mathsf{a} \in \mathcal{F}(\mathcal{D})$ and every $i^t \in \mathsf{D}^t(i^0)$, by

$$\mathcal{U}_t^D(\mathsf{a}; i^t) = U^D\left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}_{t+1}^D(\mathsf{a}; i^t, i_{t+1})\, di_{t+1}\right),$$

where $U^D : \mathbb{R}^3_+ \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is also nondecreasing, concave, and continuously differentiable. The fact that each function $\mathcal{U}_t^D$ is defined recursively implies that the preferences of any agent of generation t and those of their children regarding the consumption and fertility choices of any common descendant coincide.

3.2 *The birth-order representation of an allocation*

Even though the agents are identified by their birth dates, one might be interested, for normative purposes, in identifying the agents by the order at which they may be born. Given a feasible allocation a, for every $t \geq 0$ and every $i^t = (i^{t-1}, i_t) \in \mathsf{D}^t(i^0)$, write $\mathsf{o}^t(i^t)$ for the vector that determines the identity of agent i^t according to the birth-order criterion. To be more precise, $\mathsf{o}^t(i^t)$ is recursively defined by $\mathsf{o}^t(i^t) = (\mathsf{o}^{t-1}(i^{t-1}), \mathsf{o}_t(i^{t-1}, i_t))$, with

$$\mathsf{o}_t(i^{t-1}, i_t) = \int_{\mathsf{D}_t(i^{t-1}) \cap [0, i_t]} ds.$$

Also, given an allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$, we refer to the birth-order representation of a to the allocation $\mathsf{a}^{\mathcal{O}} \in \mathcal{F}(\mathcal{D}^{\mathcal{O}})$ as satisfying, for every $t \geq 0$ and every $i^t \in [0, \overline{n}]^t$,

$$(\mathsf{x}_t^{\mathcal{O}}(i^t), \mathsf{k}_{t+1}^{o\mathcal{O}}(i^t)) = \begin{cases} (\mathsf{x}_t((\mathsf{o}^t)^{-1}(i^t)), \mathsf{k}_{t+1}^{o}((\mathsf{o}^t)^{-1}(i^t)) & \text{if } i^t \in \mathsf{D}^t(i^0), \\ 0 & \text{otherwise.} \end{cases}$$

3.3 *On types of altruism*

Our general setting admits a wide range of particular specifications, or environments, frequently studied in the literature of endogenous fertility.

Dynastic altruism. By an environment with dynastic—or *perfect*—altruism, we refer to the class of particular specifications of the model for which $U^D \equiv U$ and U is strictly increasing in c_t^m, n_{t+1}, and u_{t+1}^D. In this particular environment, every agent cares about the utility of his/her immediate descendants, which, proceeding recursively, implies that every agent cares about consumption and fertility decisions of all her descendants. Observe that we are not imposing that $U(\cdot)$ is strictly monotonic in c_{t+1}^o, which allows us to consider, as different specifications of dynastic altruism, (a) models in which the agents live for one period and provide bequests to their immediate descendants, as well as (b) models in which the different generations of agents are truly overlapping and parents provide their immediate descendants with gifts. Examples of the first type of model—although they both restrict their analysis to symmetric allocations—are given in the pioneering work by Razin and Ben-Zion (1975), for whom

$$U(x_t, u_{t+1}^D) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \gamma(n_{t+1}) + \beta u_{t+1}^D,$$

as well as the model developed in Barro and Becker (1989), for whom

$$U(x_t, u_{t+1}^D) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \alpha(n_{t+1}) n_{t+1} u_{t+1}^D,$$

with $\alpha(n_{t+1}) = \alpha n_{t+1}^{-\epsilon}$ being the endogenous discount rate. Finally, an example of a model with dynastic altruism and truly overlapping generations is given by Schoonbroodt and Tertilt (2014), for whom

$$U(x_t, u_{t+1}^D) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \beta v(c_{t+1}^o) + \Psi(n_{t+1}, u_{t+1}^D).$$

No altruism. In many other models that study fertility, the agents are not altruistic at all, and children are viewed as a consumption good.[2] Since we are not imposing that U or U^D must be strictly monotonic in u_{t+1}^D, a setting with no altruism is a particular specification of our general framework, for which $U(x_t, u_{t+1}^D) = u(x_t)$ and $U^D(x_t, u_{t+1}^D) = u(x_t)$.

Non-dynastic altruism. Other possibilities do exist. In the exogenous fertility literature, some authors[3] have studied environments with *limited* (or *non-dynastic*) *altruism* to study the extent to which the positive (for example, Ricardian equivalence) or normative (efficiency) properties of the equilibria that arise with dynastic altruism can be extended to more general settings. Endogenous fertility literature also has abundant specifications of altruism in which the *quality* of children, from which parents derive utility, is not necessarily identified with the children's utilities, and may take the form of goods spent on each child as in Becker and Lewis (1973), income as in Galor and Weil (2000), human capital as in de la Croix and Doepke (2003), or consumption as in Kollmann (1997).

A particular specification of non-dynastic altruism is that for which an agent's altruism extends toward all her future descendants, which corresponds, for example, to the case in which $U^D(x_t, u_{t+1}^D) = U(x_t, \beta u_{t+1}^D)$, with $0 < \beta < 1$ and U, and, hence, U^D, being strictly increasing in u_{t+1}^D. We refer to this type of altruism as *infinite-horizon, non-dynastic altruism*. However, the function U^D need not be strictly increasing in u_{t+1}^D, that is, the agents might be altruistic only toward their immediate descendants. We refer to this type of non-dynastic altruism as *finite-horizon, non-dynastic altruism*, which is represented by utility functions of the form

$$U(x_t, u_{t+1}^D) = v(x_t) + \delta u_{t+1}^D \quad \text{and} \quad U^D(x_t, u_{t+1}^D) = v(x_t),$$

with v being strictly increasing in c_t^m and n_{t+1}, and $\delta \in (0, 1)$.

Although the preferences considered in the present paper allow for non-dynastic altruism, we impose two additional assumptions on preferences that ensure that the agents' preferences and those of their parents are, in a sense that we clarify below, consistent.[4] The first of these assumptions (formalized below) imposes that, keeping fixed the total amount of resources available to any given agent and the decisions taken by the agent's descendants, the agent's preferences on how to distribute these resources among consumption, fertility, and investment coincide with those of her parents.

[2] Examples of this approach, which focus exclusively on symmetric allocations, are Eckstein and Wolpin (1985), Michel and Wigniolle (2007), and Conde-Ruiz et al. (2010).

[3] See, e.g., Bernheim and Ray (1989) and the references therein.

[4] These conditions are needed to obtain the necessary conditions of $\mathcal{A}$-efficiency in Theorem 1 and are also needed to establish Theorem 3, which characterizes symmetric, $\mathcal{P}$-efficient allocations as Millian efficient allocations.

Assumption 1. *For any fixed* $u^D \in \mathbb{R}^*$ *and any two* $(x, \widetilde{x}) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+$, $U^D(x, u^D) \geq U^D(\widetilde{x}, u^D)$ *is satisfied whenever* $U(x, u^D) \geq U(\widetilde{x}, u^D)$ *is satisfied.*

The second assumption imposes that children are not loved by their grandparents more than by their parents. To be more precise, it imposes that the agents discount the utility obtained by their grandchildren at least at the same rate as their parents do, which ensures that whenever an agent is willing to increase the total resources available to any of her grandchildren and, hence, to increase the utility that the agent obtains from consumption decisions of her grandchildren, then the agent's children agree on that decision.

Assumption 2. *For any two* $(x, u^D) \in \mathbb{R}^3_+ \times \mathbb{R}^*$ *and* $(\widetilde{x}, \widetilde{u}^D) \in \mathbb{R}^3_+ \times \mathbb{R}^*$, $U(x, u^D) > U(\widetilde{x}, \widetilde{u}^D)$ *is satisfied whenever* $U^D(x, u^D) > U^D(\widetilde{x}, \widetilde{u}^D)$ *is satisfied.*

Observe that all the examples of the different specifications of preferences given above satisfy Assumptions 1 and 2.

3.4 *Dynastic optima and value functions*

Since we are assuming that $\mathcal{D}^{\mathcal{O}} \subseteq \mathcal{D}$ is satisfied and the agents care only about the number of children they bear and not about the specific points in time at which their children are born, the maximum utility that the dynasty head can obtain with a feasible allocation is not affected by the constraints in fertility choices that parents may face. To characterize dynastic optima, for every allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$ and every $t \geq 0$ and $i^t \in \mathsf{D}_t(i^0)$, write $\mathsf{e}_t(i^t)$ for the amount of physical resources (income) available to finance agent i^t's consumption, fertility, and investment decisions at period t; that is,

$$\mathsf{e}_t(i^t) := \mathsf{c}^m_t(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + \mathsf{k}^o_{t+1}(i^t).$$

Consider now an arbitrary i^t and an arbitrary e_t, and let $\mathcal{F}(e_t; i^t)$ be the set formed by all sequences

$$\mathsf{a}_t = \{(\mathsf{c}^m_\tau, \mathsf{c}^o_{\tau+1}, \mathsf{k}^o_{\tau+1}, \mathsf{D}_{\tau+1}) : [0, \overline{n}]^\tau \longrightarrow \mathbb{R}^3_+ \times \mathcal{D}\}_{\tau \geq t}$$

that satisfy $\mathsf{e}_t(i^t) \leq e_t$ as well as the feasibility constraints that all potential descendants of agent i^t would face at τ if they were not allowed to obtain resources from other agents in the economy, that is, that satisfy

$$\begin{aligned}\int_{\mathsf{D}^{\tau-1}(i^t)} &\mathsf{c}^o_\tau(i^{\tau-1})\, di^{\tau-1} + \int_{\mathsf{D}^\tau(i^t)} [\mathsf{c}^m_\tau(i^\tau) + b_\tau(\mathsf{n}_{\tau+1}(i^\tau)) + \mathsf{k}^o_{\tau+1}(i^\tau)]\, di^\tau \\ &\leq \int_{\mathsf{D}^{\tau-1}(i^t)} F_\tau(\mathsf{k}^o_\tau(i^{\tau-1}), \mathsf{n}_\tau(i^{\tau-1}))\, di^{\tau-1}\end{aligned}$$

for $\tau \geq t$. Also, for each $t \geq 1$ and $e_t \geq 0$, let $\mathcal{V}^D_t(e_t)$ be defined as the maximum utility that the dynasty heads can obtain from their descendants born at t by endowing any of their immediate descendants with e_t units of resources, that is,[5]

$$\mathcal{V}_t^D(e_t) := \max_{i^t \in [0,\overline{n}]^t} \left\{ \max_{\mathsf{a}_t \in \mathcal{F}(e_t; i^t)} \mathcal{U}_t^D\big(\mathsf{a}_t; i^t\big) \right\}.$$

With this notation, the maximum utility that the dynasty head can obtain with a feasible allocation can be characterized as the solution to

$$\mathcal{V}_0(\overline{e}_0) = \max_{\substack{(k_1^o, x_0) \in \mathbb{R}_+^3 \times [0,\overline{n}] \\ \mathsf{e}_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+}} \left\{ U\left(x_0, \frac{1}{n_1} \int_0^{n_1} \mathcal{V}_1^D\big(\mathsf{e}_1(i)\big)\, di\right) : c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0; \right.$$

$$\left. c_1^o + \int_0^{n_1} \mathsf{e}_1(i)\, di \leq F_1\big(k_1^o, n_1\big) \right\}.$$

Throughout this paper, it is assumed that $\mathcal{V}_0(\overline{e}_0)$ is well defined.

3.5 *Symmetric allocations*

As preferences and labor capacities of any two agents of the same generation are identical, it seems innocuous, both from normative and positive concerns, to restrict attention to ex post symmetric allocations, that is, to allocations for which any two agents of the same generation choose the same consumption and fertility bundles. Formally, a feasible allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$ is said to be *ex post symmetric* (or simply symmetric) if for any t and any two agents $i^t, \tilde{i}^t \in \mathsf{D}^t(i^0)$, one has $\mathsf{x}_t(i^t) = \mathsf{x}_t(\tilde{i}^t) = x_t$ and $\mathsf{k}_{t+1}^o(i^t) = \mathsf{k}_{t+1}^o(\tilde{i}^t) = k_{t+1}^o$. A symmetric allocation is, therefore, represented by a pair of sequences $a \equiv \{(k_{t+1}^o, x_t) \in \mathbb{R}_+^3 \times [0, \overline{n}]\}_{t=0,1,2,\ldots}$ that satisfy the initial condition $c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0$ and, for each $t \geq 1$, the feasibility constraint

$$c_t^o + n_{t+1}\big[c_t^m + b_t(n_{t+1}) + k_{t+1}^o\big] \leq F_t\big(k_t^o, n_t\big).$$

Denote by $\mathcal{S}$ the set that contains all feasible, symmetric allocations. Note that for every t, the utility obtained by the dynasty head from consumption and fertility decisions of the living agents with a symmetric allocation a satisfies $\mathcal{U}_t^D(\mathsf{a}; \tilde{i}^t) = U_t^D(a)$, where $U_t^D : \mathcal{S} \longrightarrow \mathbb{R}^*$ is recursively defined, for each t, by $U_t^D(a) = U^D(x_t, U_{t+1}^D(a))$. The utility obtained by an agent of generation t with a symmetric allocation is $U_t(a) = U(x_t, U_{t+1}^D(a))$. A symmetric allocation $\widehat{a}$ is Millian efficient if there does not exist an alternative symmetric allocation a that makes all generations of agents better off without making any of these generations worse off, that is, if there does not exist $a \in \mathcal{S}$ that satisfies $U_t(a) \geq U_t(\widehat{a})$ for $t \geq 0$ and $U_\tau(a) > U_\tau(\widehat{a})$ for some $\tau \geq 0$.

Some allocations can be regarded as being symmetric in a weaker, ex ante sense. Given a sequence e of income schemes that correspond to a feasible allocation a, for every $t \geq 0$, $i^t \in \mathsf{D}^t(i^0)$, let $E_{t+1}^{\mathsf{e}} : \mathbb{R}^{t+1} \longrightarrow [0, 1]$ be the function that is determined, for each $i^t \in \mathbb{R}^{t+1}$, by

$$E_{t+1}^{\mathsf{e}}\big(e, i^t\big) = \frac{\mu\big\{i \in \mathsf{D}_{t+1}\big(i^t\big) : \mathsf{e}_{t+1}\big(i^t, i\big) \leq e\big\}}{\mu\big\{\mathsf{D}_{t+1}\big(i^t\big)\big\}}.$$

[5]Since the utility received by the dynasty head from consumption of any of her descendants is the same, any choice of i^{t+1} in the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ is optimal.

Thus, $E^{\mathsf{e}}_{t+1}(e, i^t)$ determines the probability that a randomly chosen, immediate descendant of i^t spends on consumption, fertility, and investment decisions at most e units of the homogeneous good at time $t+1$ with the income scheme $\mathsf{e}_{t+1}(i^t, \cdot)$.

With this notation, an allocation a is referred to as ex ante symmetric if the distribution of income among any agent's descendants is determined randomly, so that the income accumulated by, say, the eldest n children of an agent is the same as the average income accumulated by the youngest $[\mathsf{n}_{t+1}(i^t) - n]$ children; that is, if for each $t = 0, 1, 2, \ldots$ each $i^t \in \mathbb{R}^t$, and each $D_{t+1} \subseteq \mathsf{D}_{t+1}(i^t)$, one has

$$\int_{D_{t+1}} \mathsf{e}_{t+1}\left(i^{t+1}\right) di^{t+1} = \mu\{D_{t+1}\} \int_{\mathbb{R}_+} e\, dE^{\mathsf{e}}_{t+1}\left(e, i^t\right).$$

In fact, dynastic maximization may require randomization. To see this, let $\Delta\mathbb{R}_+$ be defined as the set formed by all nondecreasing, measurable (distribution) functions $E : \mathbb{R}_+ \longrightarrow [0, 1]$ that satisfy $\lim_{e\to\infty} E(e) = 1$. Taking this into account, it is straightforward to show that the expenditure function $E^{\mathsf{e}^*}_1(\cdot, i^0)$ that corresponds to a dynastic optimum must solve

$$\mathcal{V}_0(\overline{e}_0) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ W_0\left(\overline{e}_0, \int_{\mathbb{R}_+} e\, dE(e), \int_{\mathbb{R}_+} \mathcal{V}^D_{t+1}(e)\, dE(e) \right) \right\}, \tag{10}$$

where $W_0 : \mathbb{R}^2 \times \mathbb{R}^* \to \mathbb{R}_+$ is defined, for each $(\overline{e}_0, e_1, u^D_1)$, by

$$\begin{aligned} &W_0\left(\overline{e}_0, e_1, u^D_1\right) \\ &\quad = \max_{(k^o_{\tau+1}, x_\tau)\in\mathbb{R}^3_+\times[0,\overline{n}]} \left\{ U\left(x_0, u^D_1\right) : c^m_0 + b_0(n_1) + k^o_1 \le \overline{e}_0;\, c^o_1 + n_1 e_1 \le F_1\left(k^o_1, n_1\right) \right\}. \end{aligned}$$

Also, by writing $\overline{\mathcal{V}}^D_{t+1}(e_{t+1})$ for the maximum utility that the dynasty head obtains by providing a positive measure of her immediate descendants with an average income e_{t+1}, that is,

$$\overline{\mathcal{V}}^D_{t+1}(e_{t+1}) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ \int_{\mathbb{R}_+} \mathcal{V}^D_{t+1}(e)\, dE(e) : \int_{\mathbb{R}_+} e\, dE(e) = e_{t+1} \right\},$$

the value function $\mathcal{V}_0$ can be written as

$$\mathcal{V}_0(\overline{e}_0) = \max_{e_1\ge 0} \left\{ W_0\left(\overline{e}_0, e_1, \overline{\mathcal{V}}^D_1(e_1)\right) \right\}. \tag{11}$$

With this representation, it becomes clear that although dynastic optima may be always symmetric in the weaker, ex ante sense, they might be nonsymmetric in the stronger, ex post sense. Dynastic optima are (ex post) nonsymmetric when the sequence of value functions $\{\mathcal{V}^D_{t+1}\}_{t\ge 0}$ differs from the sequence of value functions $\{V^D_{t+1}\}_{t\ge 0}$ that arise from a dynastic maximization problem in which the dynasty head is restricted to select (ex post) symmetric allocations. To be more precise, let $\widehat{a}$ be any allocation that maximizes the utility of the dynasty head among symmetric allocations and suppose that there exists a period $t \ge 0$ for which $\mathcal{V}^D_{t+1}(\widehat{e}_{t+1}) > V^D_{t+1}(\widehat{e}_{t+1})$. In this case, a dynastic optimum cannot be (ex post) symmetric. It is straightforward to show, using (10) and Jensen's inequality, that a sufficient condition that ensures that dynastic optima are ex post symmetric is that each value function $\mathcal{V}_{t+1}^{D}$ is concave.

Observe, alternatively, that each value function $\overline{\mathcal{V}}_{t+1}^{D}$ must be concave[6] and, hence, absolutely continuous and differentiable almost everywhere.[7] Taking this into account, it follows from (11) that a sufficient condition that ensures the concavity of the value function $\mathcal{V}_0$ is that the indirect utility function W_0 is also concave. Unfortunately, the non-convexities that appear in the feasibility constraints in the definition of W_0 may give rise to nonconcave value functions.[8]

4. $\mathcal{A}$-efficiency

In this section, we explore the properties of $\mathcal{A}$-efficient allocations in the context of the general framework described in Section 3. As in the previous example, we distinguish between $\mathcal{A}^{\mathcal{D}}$- and $\mathcal{A}^{\mathcal{O}}$–-efficiency, depending on whether potential agents are identified with the birth-date or the birth-order criterion. An allocation $\widehat{\mathsf{a}}$ is $\mathcal{A}^{\mathcal{D}}$-efficient if it is not $\mathcal{A}^{\mathcal{D}}$-dominated by an allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$, that is, if there does not exist a feasible allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$ such that (i) for every $t \geq 0$ and every $i^t \in \widehat{\mathsf{D}}^t(i^0) \cap \mathsf{D}^t(i^0)$, one has $\mathcal{U}_t(\mathsf{a}; i^t) \geq \mathcal{U}_t(\widehat{\mathsf{a}}; i^t)$, and (ii) there exists a period τ and a set $\mathsf{D}^\tau \subseteq \widehat{\mathsf{D}}^\tau(i^0) \cap \mathsf{D}^\tau(i^0)$ of positive measure for which $\mathcal{U}_t(\mathsf{a}; i^\tau) > \mathcal{U}_t(\widehat{\mathsf{a}}; i^\tau)$ holds for every $i^\tau \in \mathsf{D}_\tau$. As in Section 2, an allocation $\widehat{\mathsf{a}}$ is $\mathcal{A}^{\mathcal{O}}$-efficient if its reduced form representation $\widehat{\mathsf{a}}^{\mathcal{O}}$ is not $\mathcal{A}^{\mathcal{D}}$-dominated by an allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D}^{\mathcal{O}})$.

In Theorem 1, we provide a necessary condition for $\mathcal{A}$-efficiency that applies to the two criteria to identify potential agents considered in this paper. Given an allocation $\widehat{\mathsf{a}}$, for each t and each $i^t \in \mathsf{D}^t(i^0)$, write $\hat{\imath}_{t+1}^{y}(i^t)$ for the birth date of the youngest descendant of i^t in the allocation $\widehat{\mathsf{a}}$, that is,

$$\hat{\imath}_{t+1}^{y}(i^t) = \sup\{i : i \in \widehat{\mathsf{D}}_{t+1}(i^t)\}.$$

Assume, without loss of generality, that in any feasible allocation a, $\mathsf{e}_t(i^t) = 0$ and $\mathcal{U}_t^{D}(\mathsf{a}; i^t) = 0$ hold for every $i^t \notin \mathsf{D}^t(i^0)$. Taking this into account, the total welfare of the dynasty head's living descendants born at period $t+1$ in allocation a can be equivalently written as

$$\int_{\mathsf{D}_{t+1}(i^t)} \mathcal{U}_{t+1}^{D}(\mathsf{a}; i^t, i)\, di^i = \int_0^{\hat{\imath}_{t+1}^{y}(i^t)} \mathcal{U}_{t+1}^{D}(\mathsf{a}; i^t, i)\, di.$$

[6]Note that if a distribution function E_{t+1} solves the optimization problem in the definition of $\overline{\mathcal{V}}_{t+1}^{D}(e_{t+1})$ and a distribution function E'_{t+1} solves the optimization problem in the definition of $\overline{\mathcal{V}}_{t+1}^{D}(e'_{t+1})$, then for any $\alpha \in (0,1)$, the distribution function $E_{t+1}^{\alpha} \equiv \alpha E_{t+1} + (1-\alpha)E'_{t+1}$ is feasible in the optimization problem in the definition of $\overline{\mathcal{V}}_{t+1}^{D}(\alpha e_{t+1} + (1-\alpha)e'_{t+1})$, which establishes that $\overline{\mathcal{V}}_{t+1}^{D}$ must be concave.

[7]See, e.g., Theorem 10 in Royden (1988, p. 107).

[8]An example of an economy for which value functions are nonconcave is available from the authors upon request.

With this notation, Theorem 1 states that in every $\mathcal{A}$-efficient allocation, some of the agents—specifically, the youngest within each family—must devote most of their entire income to maximize not their own utility, but their parents' utility and, hence, the dynasty head's utility. All the proofs in this paper are relegated to the Appendix.

Theorem 1 (A necessary condition for $\mathcal{A}$-efficiency). *Irrespective of the criterion used to identify potential agents, every $\mathcal{A}$-efficient (i.e., $\mathcal{A}^{\mathcal{D}}$- or $\mathcal{A}^{\mathcal{O}}$-efficient) allocation $\widehat{\mathsf{a}}$ satisfies, for each $t \geq 1$,*

$$\lim_{\substack{i_{t+1} \longrightarrow \hat{\imath}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \left(\frac{\int_{i_{t+1}}^{\hat{\imath}^y_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\widehat{\mathsf{a}}; i^t, i)\, di}{\hat{\imath}^y_{t+1}(i^t) - i_{t+1}} \right) = \lim_{\substack{i_{t+1} \longrightarrow \hat{\imath}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \overline{\mathcal{V}}^D_{t+1} \left(\frac{\int_{i_{t+1}}^{\hat{\imath}^y_{t+1}(i^t)} \mathsf{e}_{t+1}(i^t, i)\, di}{\hat{\imath}^y_{t+1}(i^t) - i_{t+1}} \right), \tag{12}$$

which, in environments in which $\mathcal{V}^D_{t+1}$ is concave, yields

$$\lim_{\substack{i_{t+1} \longrightarrow \hat{\imath}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \mathcal{U}^D_{t+1}(\widehat{\mathsf{a}}; i^t, i_{t+1}) = \lim_{\substack{i_{t+1} \longrightarrow \hat{\imath}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \mathcal{V}^D_{t+1}(\mathsf{e}_{t+1}(i^t, i_{t+1})). \tag{13}$$

The intuition behind Theorem 1 is as follows. If marginal children, that is, the youngest individuals in each family, living in a given allocation a do not use their income to maximize the utility of the dynasty head, she can improve on a by having more children who do use their endowment to maximize her utility. Since these newborn children require a lower income to provide the dynasty head with at least the same utility, they are, in a sense, "cheaper" than those children already living in a.

To understand the role played by the function $\overline{\mathcal{V}}^D_{t+1}$ in Theorem 1, a brief remark is in order. For an arbitrary agent i^t, write $e^y_{t+1}(i^t)$ for the limit of the average income available to the youngest children of i^t as i_{t+1} approaches $\hat{\imath}^y_{t+1}(i^t)$, that is,

$$\lim_{\substack{i_{t+1} \longrightarrow \hat{\imath}^y_{t+1}(i^t) \\ i_{t+1} \in \mathsf{D}_{t+1}(i^t)}} \frac{\int_{i_{t+1}}^{\hat{\imath}^y_{t+1}(i^t)} \mathsf{e}_{t+1}(i^t, i)\, di}{\hat{\imath}^y_{t+1}(i^t) - i_{t+1}} = e^y_{t+1}(i^t).$$

Recall that $\overline{\mathcal{V}}^D_{t+1}(e_{t+1})$ represents the maximum utility that the dynasty head can obtain by providing a set D_{t+1}—of positive measure—of her descendants with an average income e_{t+1}. Additionally observe that the function $\overline{\mathcal{V}}^D_{t+1}$ coincides with the function $\mathcal{V}^D_{t+1}$ if the latter is concave, but not in general. Moreover, when they do not coincide at a given point e_{t+1}, achieving $\overline{\mathcal{V}}^D_{t+1}(e_{t+1}) > \mathcal{V}^D_{t+1}(e_{t+1})$ is feasible because the dynasty head can select randomly the income available to each of her descendants in D_{t+1}. By Theorem 1, the average utility obtained from the youngest descendants of i^t must converge to $\overline{\mathcal{V}}^D_{t+1}(e^y_t(i^t))$. Therefore, if $\overline{\mathcal{V}}^D_{t+1}(e^y_t(i^t)) > \mathcal{V}^D_{t+1}(e^y_t(i^t))$ holds, then achieving $\mathcal{A}$-efficiency may require that the income available to the youngest descendants is determined randomly. Thus, non-convexities in the feasible set associated to endogenous fertility may cause the only efficient allocations to be possibly stochastic, a result also present (for Pareto efficiency) in Rogerson (1988).

Another implication of Theorem 1 is that every symmetric (in an ex ante or ex post sense), $\mathcal{A}$-efficient allocation must be a dynastic optimum, which, in turn, implies that all Millian efficient allocations that are not dynastic optima are $\mathcal{A}$-inefficient. To save on space, we do not prove this claim here.

Corollary 1 below states another implication of Theorem 1, which arises in environments without altruism[9] or with finite-horizon altruism. In both environments, each function $\mathcal{V}_t^D$ is concave and the path $\{e_\tau\}_{\tau \geq t+1}$ that solves the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ satisfies $e_\tau = 0$ for $\tau \geq t+1$. Theorem 1 states that for each agent i^t alive at any given time, there must be a positive measure of agents whose income and utility are arbitrarily close to the the income and utility obtained by the agent's marginal child $i_{t+1}^y(i^t)$. Therefore, all descendants of these agents must also receive an income close to that received by the marginal child's descendants, that is, close to zero.

Corollary 1. *Suppose that utility functions adopt the finite-horizon altruistic forms* $U(x_t, u_{t+1}^D) = v(x_t) + \delta u_{t+1}^D$ *and* $U^D(x_t, u_{t+1}^D) = v(x_t)$*, with* $\delta \in [0, 1]$ *and* v *being strictly increasing in* c_t^m *and* n_{t+1}*. In this environment, for every* $\mathcal{A}$*-efficient (i.e.,* $\mathcal{A}^{\mathcal{D}}$*- or* $\mathcal{A}^{\mathcal{O}}$*-efficient) allocation* a*, every* $t \geq 2$ *for which* $\mu^t\{\mathsf{D}^t(i^0)\} > 0$ *holds, and every* $\epsilon_t > 0$*, there exists a set* $\mathsf{D}^t \subseteq \mathsf{D}^t(i^0)$ *of agents alive at* t *of strictly positive measure for whom*

$$0 \leq \mathsf{e}_t(i^t) < \epsilon_t \tag{14}$$

is satisfied for all $i^t \in \mathsf{D}^t$*.*

Two more comments are in order. First of all, observe that Corollary 1 requires that the utility function v is strictly increasing in c_t^m and n_{t+1}. Without this assumption, the set of agents for whom (14) holds might not even be born. With this assumption, parents who exhibit finite-horizon altruistic preferences would like their descendants to have some grandchildren, so that they can use the labor capacities of the latter to obtain resources not for themselves, but for their parents. Thus, condition (14) must hold for a positive measure of *alive* agents born after $t = 2$. Second of all, we note that in our benchmark model, each function $\mathcal{U}_t^D$ is defined recursively and, therefore, the only environment that exhibits finite-horizon altruism in our general setting is that in which agents care only about consumption decisions by their *immediate* descendants. However, Corollary 1 suggests that a similar result characterizes $\mathcal{A}$-efficiency in every model in which the agents care about the welfare of a *finite* number of generations of their descendants.

[9]Apparently, Theorem 1 has no implications in environments with no altruism in which $\mathcal{U}_t(\mathsf{x}, i^t) = u(x(i^t))$ holds for every i^t and the function $\mathcal{U}_t^D$ does not need to be well defined. Yet, it does have implications if we assume that in this type of environments, the function $\mathcal{U}_t^D$ represents the preferences of those born at t and, therefore, it satisfies $\mathcal{U}_t^D(x, i^t) = \mathcal{U}_t(x, i^t) = u(x(i^t))$ for every i^t. Note that in this case, the functions $\overline{\mathcal{V}}_t^D$ and $\mathcal{V}_t^D$ coincide and adopt the form $\overline{\mathcal{V}}_t^D(e_t) = \mathcal{V}_t^D(e_t) = W(e_t, 0)$ for every $t \geq 1$ and every $e_t \geq 0$.

When fertility choices are unrestricted (i.e., $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$), the set of $\mathcal{A}^{\mathcal{D}}$-efficient allocations is strictly smaller than the set of $\mathcal{A}^{\mathcal{O}}$-efficient allocations. In such environments, if the child-rearing period $[0, \overline{n}]$ is long enough to allow the dynasty head to replace all her living children in a dynastic optimum, n_1^*, then $\mathcal{A}^{\mathcal{D}}$-efficiency can be characterized as dynastic maximization, as Theorem 2 below shows.

Theorem 2 ($\mathcal{A}^{\mathcal{D}}$-efficiency as dynastic maximization). *Assume that $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$ and $n_1^* < \overline{n}/2$ hold. In this setting, an allocation a^* is $\mathcal{A}^{\mathcal{D}}$-efficient if and only if it is a dynastic optimum.*

The intuition behind Theorem 2 is simple: the necessary condition in Theorem 1 implies that the number of children that correspond to an $\mathcal{A}^{\mathcal{D}}$-efficient allocation cannot be higher than the number of children of the first generation, n_1^*, that the dynasty head would choose as optimum. Thus, if any dynasty head's children can be "replaced," i.e., $2n_1^* < \overline{n}$ is satisfied, then any allocation a, which is not a dynastic optimum, is always $\mathcal{A}$-dominated by a dynastic optimum with a different set of living agents maximizing the utility of the dynasty head.

In environments with no altruism or finite-horizon altruism, a straightforward implication of Theorem 2 is that every $\mathcal{A}^{\mathcal{D}}$-efficient allocation must collapse in finite time. We state this result formally in Corollary 2 below.

Corollary 2. *Assume that $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$ and $n_1^* < \overline{n}/2$ hold. If, in addition, preferences exhibit no altruism or finite-horizon altruism, then every $\mathcal{A}^{\mathcal{D}}$-efficient allocation a^* must satisfy $\mathsf{e}_t^*(i^t) = 0$ and $\mathsf{x}_t^*(i^t) = 0$ for each $t \geq 2$ and almost all $i^t \in \mathsf{D}^t(i^0)$. That is, in every $\mathcal{A}^{\mathcal{D}}$-efficient allocation, the economy collapses in finite time.*

To summarize the results in this section, the notion of $\mathcal{A}$-efficiency is sensitive to the criterion by which potential lives are distinguished from one another. To be more precise, identifying potential agents by the date at which they may be born, rather than by the agents' position in their siblings' birth order, reduces the set of allocations that can be regarded as $\mathcal{A}$-efficient. Yet, even the weakest form of $\mathcal{A}$-efficiency, that is, $\mathcal{A}^{\mathcal{O}}$-efficiency, might be too demanding in environments with finite-horizon altruism, as it requires that some of the agents obtain an arbitrarily low income.

5. Millian efficiency as robust $\mathcal{P}$-efficiency

As for $\mathcal{A}$-efficiency, we distinguish between two notions of $\mathcal{P}$-efficiency, each of them associated to a criterion to identify potential agents. Both notions are associated to a sequence of functions $\mathcal{U}^N = \{\mathcal{U}_t : \mathcal{F}(\mathcal{D}) \times [0, \overline{n}]^t \longrightarrow \mathbb{R}^*\}$ that determine the utility attributed to the unborn. Then, for any t and any potential agent of generation t, let $\mathcal{U}_t^P : \mathcal{F}(\mathcal{D}) \times [0, \overline{n}]^t \longrightarrow \mathbb{R}^*$ be defined, for all (a, i^t), by

$$\mathcal{U}_t^P(\mathsf{a}; i^t) = \begin{cases} \mathcal{U}_t(\mathsf{a}; i^t) & \text{if } i^t \in \mathsf{D}^t(i^0), \\ \mathcal{U}_t^N(\mathsf{a}; i^t) & \text{otherwise.} \end{cases}$$

With this notation, an allocation $\widehat{\mathsf{a}}$ $\mathcal{P}^{\mathcal{D}}$-dominates an allocation a if for every t and every $i^t \in [0, \overline{n}]^t$, one has $\mathcal{U}_t^P(\widehat{\mathsf{a}}; i^t) \geq \mathcal{U}_t^P(\mathsf{a}; i^t)$, and there exists a period τ and a set of individuals $\mathsf{D}^t \in \mathcal{B}[0, \overline{n}]^t$ of positive measure for which $\mathcal{U}_\tau^P(\widehat{\mathsf{a}}; i^\tau) > \mathcal{U}_\tau^P(\mathsf{a}; i^\tau)$ is satisfied for all $i^\tau \in \mathsf{D}^t$. An allocation $\widehat{\mathsf{a}} \in \mathcal{F}(\mathcal{D})$ is $\mathcal{P}^{\mathcal{D}}$-efficient if it is not $\mathcal{P}^{\mathcal{D}}$-dominated by an allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D})$, and it is $\mathcal{P}^{\mathcal{O}}$-efficient if its birth-order representation is not $\mathcal{P}^{\mathcal{D}}$-dominated by an allocation $\mathsf{a} \in \mathcal{F}(\mathcal{D}^{\mathcal{O}})$.

In their applications, GJT restrict the use of the term $\mathcal{P}$-dominance to a particular specification of the utilities attributed to the unborn for which the utility attributed to the unborn is constant, that is, $\mathcal{U}_t^N(\mathsf{a}; i^t) = \overline{u}$, and their conclusions are analogous to those discussed in the two-period economy discussed in Section 2. The main problem is that determining whether an allocation is optimal (i.e., $\mathcal{P}$-efficient) becomes heavily dependent on the specific value attributed to the unborn—an unknown number.

There are, however, other possibilities. For example, the utility level attributed to an unborn agent might correspond to the lowest utility obtained by the agent's living siblings, that is, the utility function attributed to a potential agent $i^t = (i^{t-1}, i_t)$, if unborn, adopts the form

$$\mathcal{U}_t^N(\mathsf{a}; i^{t-1}, i_t) = \begin{cases} \inf\{\mathcal{U}_t(\mathsf{a}; i^{t-1}, i) : i \in \mathsf{D}_t(i^{t-1})\} & \text{if } \mathsf{n}_t(i^{t-1}) > 0, \\ \inf\{\mathcal{U}_t(\mathsf{a}; i^t) : i^t \in \mathsf{D}^t(i^0)\} & \text{if } \mathsf{n}_t(i^{t-1}) = 0. \end{cases} \tag{15}$$

Alternatively, we might assume that the utility attributed to an unborn agent is the average utility obtained by his/her living siblings, that is,

$$\mathcal{U}_t^N(\mathsf{a}; i^{t-1}, i_t) = \begin{cases} \dfrac{1}{\mathsf{n}_t(i^{t-1})}\left(\displaystyle\int_{\mathsf{D}_t(i^{t-1})} \mathcal{U}_t(\mathsf{a}; i^{t-1}, i)\, di\right) & \text{if } \mathsf{n}_t(i^{t-1}) > 0, \\ \dfrac{1}{\mu^t\{\mathsf{D}^t(i^0)\}}\left(\displaystyle\int_{\mathsf{D}^t(i^0)} \mathcal{U}_t(\mathsf{a}; i^{t-1}, i)\, di\right) & \text{if } \mathsf{n}_t(i^{t-1}) = 0. \end{cases} \tag{16}$$

With such specifications of the utility attributed to the unborn, an allocation a' with more individuals than an allocation a might not $\mathcal{P}^{\mathcal{D}}$-dominate or $\mathcal{P}^{\mathcal{O}}$-dominate a, even though all individuals living in both a and a' are better off in the new allocation a'.

The specifications of the utility for the unborn given in (15) or (16) might not represent the true preferences of potential agents and are, therefore, questionable. But any specification is, in our view, equally questionable. This is the initial stumbling block for any attempt to extend the Pareto criterion to compare allocations with different population size: there is no way to know whether a potential agent is willing to be alive in any given allocation. Thus, rather than as assumptions on the utility obtained by nonborn agents, we regard these specifications as a means to represent normative principles that determine the conditions under which increasing or reducing the population size increases aggregate welfare. For example, according to the principle underlying the notions of $\mathcal{P}$-dominance that arise when each function $\mathcal{U}_t^N$ is defined as in (15), a new life increases aggregate welfare only when the agent is not worse off than any of his/her living siblings with the same tastes and capacities. After all, one might argue that the notion of $\mathcal{A}^{\mathcal{D}}$-dominance also represents a questionable normative principle, according to which a potential life should be worth living if the dynasty head decides that it is.

We should also emphasize that the $\mathcal{P}$-dominance criteria that arise, for example, when each function $\mathcal{U}_t^N$ is defined as in (15), do not state that replacing an allocation a by an allocation a' with more individuals obtaining lower utility than any of their living siblings *decreases* aggregate welfare. If the dynasty head is better off with the allocation a', then a and a' become noncomparable with the two $\mathcal{P}$-dominance criteria. Despite the analogies of the functions defined in (15) or (16) with well known social welfare functions, as an extension of the Pareto criterion, the $\mathcal{P}$-dominance criteria associated to each of these functions say nothing about whether any two living agents should redistribute their resources. Thus, basing the $\mathcal{P}$-dominance criteria on the specification given in (15) does not imply that a $\mathcal{P}$-efficient allocation must be symmetric, and using (16) does not mean that a $\mathcal{P}$-efficient allocation must maximize the average welfare obtained by the living agents. In fact, $\mathcal{A}^{\mathcal{O}}$-efficient allocations, which as we have seen in the previous sections, are not necessarily symmetric, are always $\mathcal{P}^{\mathcal{O}}$-efficient, and there are many other nonsymmetric allocations that may be $\mathcal{P}^{\mathcal{O}}$-efficient but are not $\mathcal{A}^{\mathcal{O}}$-efficient.

It is true that while the Pareto criterion does not introduce distributive concerns at all, when complemented with either (15) or (16), the notions of $\mathcal{P}$-dominance do introduce weak distributive concerns to determine the conditions under which altering the population size is better from the point of view of aggregate welfare. And why not? It would surely be odd to reject a reallocation of resources involving a unanimous welfare gain because some of the agents would envy those most favored by such a reallocation, *when the agents themselves have the opportunity to accept or reject it*. But again, one cannot be sure that an increase in population size that makes the newborn envy all his/her living siblings is a true welfare improvement, as the newborn will never have the opportunity to accept or reject that decision.

In the context of the two-period example in Section 2, it is easy to show that (i) for each of the functions that determine the utility of the unborn defined in (15) or (16), the two $\mathcal{P}$-dominance criteria rule out many allocations as being $\mathcal{P}$-inefficient, (ii) the $\mathcal{P}$-efficiency of a given allocation depends on whether potential agents are identified with the birth-order criterion or with the birth-date criterion, and (iii) the $\mathcal{P}$-efficiency of an allocation depends, in general, on the utility function that determines the utility of the unborn. Yet, Millian efficient allocations arise as $\mathcal{P}$-efficient independently of the criterion used to distinguish among potential agents and also of the function that determines the utility of the unborn. That is, the $\mathcal{P}$-efficiency of Millian efficient allocations holds regardless of the principle that determines whether altering the population size increases aggregate welfare is that captured by (15) or that captured by (16). In fact, in the context of the example, the $\mathcal{P}$-efficiency of Millian efficiency holds if the utility attributed to the unborn coincides with the median, the qth quantile, or the average utility obtained by the agent's *elder* siblings. In summary, to determine the utility attributed to the unborn $\mathcal{U}_t^N$, one might take any *symmetric* function of the utility obtained by the agent's living siblings, as the following property formally states.

Property S. *For every $t \geq 1$, every $i^t \in [0, \overline{n}]$, and every ex post symmetric allocation a such that $\mathsf{x}_t(\tilde{\imath}^t) = x_t$ and $\mathsf{n}_{t+1}(\tilde{\imath}^t) = n_{t+1}$ for every $\tilde{\imath}^t \in \mathsf{D}^t(i^0)$ one has*

$$\mathcal{U}_t^N(\mathsf{a}; i^t) = \mathcal{U}_t(\mathsf{a}; i^t) \equiv U_t(a).$$

This property turns out to be a sufficient condition to ensure the Millian efficiency of symmetric $\mathcal{P}$-efficient allocations (Theorem 3(i)). To establish the equivalence between Millian efficiency and symmetric $\mathcal{P}$-efficiency in the context of the general setting described in Section 3 (Theorem 3(ii)) requires an additional weak (concavity) condition. Given a sequence $\widehat{e} \equiv \{\widehat{e}_t\}_{t=0}^{\infty}$, for an arbitrary $t \geq 0$ and each e_t, let the restricted value function $\mathcal{V}^D_{\widehat{e},t} : [\widehat{e}_t, +\infty) \longrightarrow \mathbb{R}$ be defined, for every $e_t \in [\widehat{e}_t, +\infty)$, by

$$\mathcal{V}^D_{\widehat{e},t}(e_t) := \max_{i^t \in [0,\overline{n}]^t} \left\{ \max_{\mathsf{a}_t \in \mathcal{F}(e_t; i^t)} \left\{ \mathcal{U}^D_t\left(\mathsf{a}_t; i^t\right) : \mathsf{e}_\tau\left(i^\tau\right) \geq \widehat{e}_\tau \text{ for all } \tau \geq t+1 \right\} \right\}.$$

Note that in contrast to the unconstrained value function $\mathcal{V}^D_t$, the constrained value function $\mathcal{V}^D_{\widehat{e},t}$ determines the maximum utility that agents born before t can obtain from the consumption decisions of their descendants, *provided* each of these descendants is endowed with at least $\widehat{e}_\tau$ units of resources.

Theorem 3 (A characterization of symmetric, $\mathcal{P}$-efficient allocations as Millian efficient allocations). *Assume Property S holds.*

(i) If $\widehat{a}$ is a symmetric, $\mathcal{P}$-efficient (i.e., $\mathcal{P}^{\mathcal{D}}$- or $\mathcal{P}^{\mathcal{O}}$-efficient) allocation, with either criteria to identify potential agents, then $\widehat{a}$ is Millian efficient.

(ii) If $\widehat{a}$ is a Millian efficient allocation and, for each t, the function $\mathcal{V}^D_{\widehat{e},t}$ is concave on $[\widehat{e}_t, +\infty)$, then $\widehat{a}$ is $\mathcal{P}$-efficient with either criterion to identify potential agents, i.e., $\mathcal{P}^{\mathcal{D}}$- or $\mathcal{P}^{\mathcal{O}}$-efficient.

Observe that a characterization of *any* symmetric and nonsymmetric $\mathcal{P}$-efficient allocation seems subtler, since determining whether a nonsymmetric allocation is $\mathcal{P}$-efficient depends on the specific functional form given to each function $\mathcal{U}^N_t$. Yet, Theorem 3 suffices to rule out any symmetric allocation that is not $\mathcal{M}$-efficient (for example, Benthamite optima) as being $\mathcal{P}$-inefficient.

Theorem 3(ii) establishes that in regular settings in which value functions are concave on a certain range, Millian efficient allocations are $\mathcal{P}$-efficient as long as each function $\mathcal{U}^N_t$ belongs to the class of functions that satisfy Property S. Thus, just as an $\mathcal{A}$-efficient allocation can be described as a $\mathcal{P}$-efficient allocation for which $\mathcal{P}$-efficiency holds irrespective of the utility attributed to the unborn, Millian efficient allocations can be described as $\mathcal{P}$-efficient allocations for which $\mathcal{P}$-efficiency holds for a wide range of specifications of the utility attributed to the unborn and, therefore, for a wide range of principles to compare allocations with different population size.

$\mathcal{P}$-efficiency and property rights In Section 2, we express our concerns regarding the possibility that $\mathcal{A}$-efficiency might be incompatible with individual rights. In the two-period economy, this occurs when the amount of resources that children can obtain with the labor capacity they are endowed with, w, is higher than average income received by children in a dynastic optimum, e^*_1. Also, the allocation that arises with said distribution of rights is Millian efficient and, in view of Theorem 3, is $\mathcal{P}$-efficient. Therefore, as in those environments in which achieving Pareto efficiency is incompatible with

a distribution of rights and achieving Pareto efficiency from an initial status quo is not Pareto improving (as occurs, for example, in environments with asymmetric information or in overlapping generations economies facing macroeconomic risks), it might be convenient to adopt weaker notions of efficiency such as $\mathcal{P}$-efficiency, just as it is useful to adopt *interim* (incentive-constrained) efficiency[10] in the presence of asymmetric information or *interim* (conditional) efficiency in stochastic, overlapping generations economies (see, e.g., Chattopaday and Gotardi 1999). In the latter setting, achieving Pareto efficiency (ex ante efficiency in this case) from an initial status quo requires policies (e.g., social security) that make a generation of agents worse off than they would be in the absence of such policies. In our example, an analogous problem may occur, since achieving $\mathcal{A}$-efficiency (or, with the birth-date criterion, dynastic maximization) requires that the dynasty head accumulates debts that his/her descendants may not be willing to repay.

6. Conclusions

In this paper, we explored the properties of the notions of $\mathcal{A}$-efficiency and $\mathcal{P}$-efficiency, proposed by Golosov et al. (2007), to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism. We first argued that achieving $\mathcal{A}$-efficiency may have different implications depending on the criterion we use to distinguish potential agents from one another. If we identify potential siblings by their birth order, the set of $\mathcal{A}$-efficient allocations, which we refer to as $\mathcal{A}^{\mathcal{O}}$-efficient allocations, is large, although the youngest siblings in every family must devote most of their income to maximize the utility of their parents. Therefore, $\mathcal{A}$-efficiency might conflict with the individuals' rights to use their labor capacity as they wish, and in environments with finite horizon altruism, it implies that the youngest children in almost every family obtain almost zero resources to finance their own consumption and fertility plans. Things might get worse—for everyone except the dynasty head—if we identify potential agents by their birth dates. In this case, achieving $\mathcal{A}$-efficiency (which we refer to as $\mathcal{A}^{\mathcal{D}}$-efficiency) requires that all children, who in our setting are equal in their tastes and capacities, are treated symmetrically and, therefore, it requires that all agents in the economy must devote their entire income to maximize the utility of their parents. Since, in our setting, the preferences of any agent on consumption decisions of his/her descendants coincide with those of the dynasty head, $\mathcal{A}^{\mathcal{D}}$-efficiency reduces, therefore, to dynastic maximization. In environments with finite-horizon altruism, for example, $\mathcal{A}^{\mathcal{D}}$-efficiency is characterized by a collapse.

In this paper, we also showed that these properties might not hold for $\mathcal{P}$-efficiency—at least if the welfare attributed to the unborn (which we regard as a device that represents different principles that determine the circumstances under which new lives increase welfare) depends on the welfare enjoyed by those living in a given allocation. More specifically, if the welfare attributed to the unborn is a symmetric function of the welfare obtained by their living siblings, then every Millian efficient allocation, that is,

[10] See Holmström and Myerson (1983).

every symmetric allocation that is not $\mathcal{A}$-dominated by any other symmetric allocation, is $\mathcal{P}$-efficient independently of the criterion to identify potential agents.

There are several directions that might be worth exploring. A first direction would be to study the efficiency properties of equilibria. Schoonbroodt and Tertilt (2014) have shown, in an environment with dynastic altruism, that an equilibrium in which the non-negativity constraints on transfers are binding cannot be $\mathcal{A}$-efficient or, using our distinction, $\mathcal{A}^{\mathcal{O}}$-efficient. Yet, in their setting, the equilibrium that arises from the interaction of markets and families is symmetric. Thus, our results suggest that such equilibrium might be Millian efficient and, hence, both $\mathcal{P}^{\mathcal{D}}$- and $\mathcal{P}^{\mathcal{O}}$-efficient. Accordingly, an important qualitative conclusion of Golosov, Jones and Tertilt may prevail: in the absence of non-convexities, externalities, missing markets, dynamic efficiency problems, etc., the fact that fertility decisions are endogenous does not mean that markets fail to deliver efficient allocations.

A second direction would be to extend the results to environments in which the agents are heterogeneous. While an environment populated by agents with identical preferences and exhibiting no altruism or finite-horizon altruism is probably extremely rare, an environment populated by heterogeneous agents in which some of the agents' preferences have these properties are probably less rare. Thus, Corollary 1 may become more relevant in models with heterogeneous dynasties. We should also point out that the symmetry restriction that underlies the Millian notion of efficiency requires that every two agents with the same characteristics be treated equally, but this does not mean that agents with different characteristics are treated equally. Thus, in models in which agents are heterogeneous in their characteristics (preferences, endowments, preferences and endowments of their ancestors, and, finally, the agents' birth dates or the agents' order of birth with respect to their siblings), the Millian notion of efficiency may be still applicable if we regard the symmetry restriction as requiring that any two agents of the same generation with the same preferences and endowments—and for whom the preferences and endowments of all their ancestors are also equal—must be treated symmetrically. If the utility attributed to an unborn agent depends only on the utility attributed to those among the agent's siblings who have the same characteristics, then the equivalence between Millian efficiency and symmetric $\mathcal{P}$-efficiency will prevail. As in the setting studied here (in which many asymmetric allocations might be regarded as $\mathcal{P}^{\mathcal{D}}$-efficient even when they are not $\mathcal{A}$-efficient), $\mathcal{P}^{\mathcal{D}}$-efficiency is consistent with symmetry but it does not impose symmetry.

Finally, as a third direction, it might be worthwhile to explore the consequences of different fertility policies in environments in which other potential market failures arise, such as pollution problems and missing markets.

Appendix: Proofs

Proof of Theorem 1. Let $\widehat{\mathsf{a}}$ be an $\mathcal{A}$-efficient allocation. To show that (12) must be satisfied, assume first that for the allocation $\widehat{\mathsf{a}}$, all fertility choices chosen at t belong to the set D^0, so that $\hat{\imath}^y_{t+1}(i^t) = \widehat{n}_{t+1}(i^t)$ is satisfied. Note that this assumption is without loss of generality: since the agents do not care about the specific points in time at which their children are born, for every $\mathcal{A}$-efficient allocation, there must be an allocation whose fertility choices satisfy this property and that is also $\mathcal{A}$-efficient.

Write now $\widehat{x}_t$ and $\widehat{k}_{t+1}$, respectively, for $\widehat{x}_t = \widehat{\mathsf{x}}_t(i^t)$ and $\widehat{k}_{t+1} = \widehat{\mathsf{k}}^o_{t+1}(i^t)$. Also, for each $n \in [0, \overline{n}]$, write $e^{\widehat{\mathsf{a}}}_{t+1}(n)$ and $D^{\widehat{\mathsf{a}}}_{t+1}(n)$, respectively, for $e^{\widehat{\mathsf{a}}}_{t+1}(n) = \int_0^n \widehat{\mathsf{e}}_{t+1}(i^t, i)\, di$ and $D^{\widehat{\mathsf{a}}}_{t+1}(n) = \int_0^n \mathcal{U}^D_{t+1}(\widehat{\mathsf{a}}; i^t, i)\, di$. With this notation, observe that the welfare obtained by the dynasty head by choosing a feasible allocation with $n \le \widehat{n}_{t+1}$ individuals born at period $t+1$, if all descendants in the interval $[0, n]$ take the same decisions as those corresponding to the allocation $\widehat{\mathsf{a}}$, can also be written as

$$U^{D\widehat{\mathsf{a}}}_t(n) = U^D\left(\overline{e}_0 - b_t(n) - \widehat{k}^o_1, F_{t+1}\left(\widehat{k}^o_{t+1}, n\right) - e^{\widehat{\mathsf{a}}}_{t+1}(n), n, \frac{D^{\widehat{\mathsf{a}}}_{t+1}(n)}{n}\right) = \frac{D^{\widehat{\mathsf{a}}}_t(n)}{n}.$$

From the definition of $\mathcal{A}$-efficiency, any allocation differing from $\widehat{\mathsf{a}}$ at a single point or, in general, on a set of measure zero must be also $\mathcal{A}$-efficient. Therefore, since both $e^{\widehat{\mathsf{a}}}_{t+1}$ and $D^{D\widehat{\mathsf{a}}}_{t+1}$ are integrable and the Lebesgue integrals $e^{\widehat{\mathsf{a}}}_{t+1}$ and $D^{D\widehat{\mathsf{a}}}_{t+1}$ are differentiable almost everywhere,[11] there is no loss of generality in assuming that both functions are differentiable from the left—and, hence, continuous from the left—at $\widehat{n}_{t+1}$; that is,

$$\frac{d^- e^{\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})}{dn} = \lim_{\substack{n \to \widehat{n}_{t+1} \\ n < \widehat{n}_{t+1}}} \frac{\int_n^{\widehat{n}_{t+1}} \widehat{\mathsf{e}}_{t+1}\left(i^t, i\right) di}{\widehat{n}_{t+1} - n} = \widehat{\mathsf{e}}_{t+1}\left(i^t, \widehat{n}_{t+1}\right),$$

$$\frac{d^- D^{\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})}{dn} = \lim_{\substack{n \to \widehat{n}_{t+1} \\ n < \widehat{n}_{t+1}}} \frac{\int_n^{\widehat{n}_{t+1}} \mathcal{U}^D_{t+1}\left(\widehat{\mathsf{a}}; i^t, i\right) di}{\widehat{n}_{t+1} - n} = \mathcal{U}^D_{t+1}\left(\widehat{\mathsf{a}}, i^t, \widehat{n}_{t+1}\right).$$

Moreover, since $\widehat{\mathsf{a}}$ is $\mathcal{A}$-efficient and the dynasty head cannot obtain higher utility by reducing the population size, the left-hand side derivative of $U^{\widehat{\mathsf{a}}}_0$ at $\widehat{n}_{t+1}$ satisfies

$$\begin{aligned}\frac{d^- U^{D\widehat{\mathsf{a}}}_t(\widehat{n}_{t+1})}{dn_{t+1}} &= -b'_t(\widehat{n}_{t+1}) D_1 U^D\left(\widehat{x}_t, U^{D\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})\right) \\ &\quad + \left[D_2 F_{t+1}\left(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}\right) - \widehat{\mathsf{e}}_{t+1}\left(i^t, \widehat{n}_{t+1}\right)\right] D_2 U^D\left(\widehat{x}_t, U^{D\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})\right) \\ &\quad + D_3 U^D\left(\widehat{x}, U^{D\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})\right) \\ &\quad + \frac{1}{\widehat{n}_{t+1}}\left[\mathcal{U}^D_{t+1}\left(\widehat{\mathsf{a}}, i^t, \widehat{n}_{t+1}\right) - U^{D\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})\right] D_4 U^D\left(\widehat{x}, U^{D\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})\right) \\ &\ge 0.\end{aligned}$$

With this observation in mind, we now show that condition (12), that is,

$$\lim_{\substack{n \to \widehat{n}_{t+1} \\ n < \widehat{n}_{t+1}}} \left(\frac{\int_n^{\widehat{n}_{t+1}} \mathcal{U}^D_{t+1}\left(\widehat{\mathsf{a}}; i^t, i\right) di}{\widehat{n}_{t+1} - n}\right)$$

[11] See, e.g., Theorem 10 in Royden (1988, p. 107).

$$= \overline{\mathcal{V}}^D_{t+1}\left(\lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \frac{\int_n^{\widehat{n}_{t+1}} \widehat{\mathsf{e}}_{t+1}(i^t, i)\,di}{\widehat{n}_{t+1}-n}\right) = \overline{\mathcal{V}}^D_{t+1}\big(\widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1})\big), \tag{17}$$

must be satisfied. To prove this statement, suppose it is false. Then select $\widetilde{e}_{t+1} < \widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1})$ in such a way that

$$\lim_{\substack{n\to\widehat{n}_{t+1}\\ n<\widehat{n}_{t+1}}} \left(\frac{\int_n^{\widehat{n}_{t+1}} \mathcal{U}^D_{t+1}(\widehat{\mathsf{a}}; i^t, i)\,di}{\widehat{n}_{t+1}-n}\right) = \overline{\mathcal{V}}^D_{t+1}(\widetilde{e}_{t+1})$$

is satisfied, and consider an allocation a such that (i) at time t, agent i^t chooses $\widetilde{n}_{t+1} > \widehat{n}_{t+1}$, (ii) those individuals who were already living in $\widehat{\mathsf{a}}$ receive exactly the same bundle, and (iii) those individuals born at t who were not living under $\widehat{\mathsf{a}}$ receive an endowment $e_{t+1}(i^t, i_t) = \widetilde{e}_{t+1}$ and use this endowment to maximize the utility of the dynasty head, so that $e^{\widehat{\mathsf{a}}}_{t+1}(n) = \int_0^n \widehat{\mathsf{e}}_{t+1}(i^t, i)\,di + [\widetilde{n}_{t+1} - n]\widetilde{e}_{t+1}$ is satisfied. Since the number of individuals alive in the new allocation a is higher than the number of individuals living in $\widehat{\mathsf{a}}$, the left-hand side derivative of $U^{D,\mathsf{a}}_t$ at $n = \widehat{n}_{t+1}$ coincides with that of $U^{D\widehat{\mathsf{a}}}_{t+1}$. Also, the right-hand side derivative of $U^{D,\mathsf{a}}_{t+1}$ at $n = \widehat{n}_{t+1}$ is given by

$$\frac{d^+ U^{D\widehat{\mathsf{a}}}_t(\widehat{n}_{t+1})}{dn_{t+1}} = \frac{d^- U^{D\widehat{\mathsf{a}}}_t(\widehat{n}_{t+1})}{dn_{t+1}} + D_2 U^D\big(\widehat{x}, U^{D\widehat{\mathsf{a}}}_{t+1}(\widehat{n}_{t+1})\big)\big[\widehat{\mathsf{e}}_{t+1}(i^t, \widehat{n}_{t+1}) - \widetilde{e}_{t+1}\big] > 0,$$

which implies that there exists $\widetilde{n}_{t+1} > \widehat{n}_{t+1}$ for which the dynasty head—and, under Assumption 2, agent i^t—obtains more utility with the allocation a than the utility she obtains with $\widehat{\mathsf{a}}$. Thus, some of the agents living in both a and $\widehat{\mathsf{a}}$ are better off with the former allocation than they are with the latter, and no agent living in the two allocations is worse off. This contradicts the assumption that imposes that $\widehat{\mathsf{a}}$ is $\mathcal{A}$-efficient, a contradiction that establishes that (17) and, hence, the equivalent condition in (12), must be satisfied. Note that if the value function $\mathcal{V}^D_{t+1}$ is concave and, hence, satisfies $\mathcal{V}^D_{t+1} = \overline{\mathcal{V}}^D_{t+1}$, then (13) follows straightforwardly from (12), which completes the proof of Theorem 1. □

Proof of Corollary 1. Let a be an $\mathcal{A}$-efficient allocation that arises in an environment with no altruism or with finite-horizon altruism, and let $t \geq 1$ and i^t be arbitrary. As in the proof of Theorem 1, we assume, without loss of generality, that $\mathcal{D} \equiv \mathcal{D}^{\mathcal{O}}$ holds. As in this type of environment, $U^D_t(\mathsf{a}, i^t) = u(x_t(i^t))$ is satisfied, it is straightforward to show that the function $\mathcal{V}^D_t$ can be equivalently defined, for each $e_t \geq 0$, by

$$\mathcal{V}^D_t(e_t) = \max\{W_t(e_t, e_{t+1}) : e_{t+1} \geq 0\},$$

where, in turn, $W_t : \mathbb{R}^2_+ \to \mathbb{R}$ is defined, for each $(e_t, e_{t+1}) \in \mathbb{R}^2_+$, by

$$W_t(e_t, e_{t+1}) = \max_{(k^o_{t+1}, x_{t+1}) \in \mathbb{R}^3_+ \times [0, \overline{n}]} \big\{u(x_t) : c^m_t + b_0(n_1) + k^o_1 \leq \overline{e}_t; c^o_1 + n_{t+1}e_{t+1} \leq F_1(k^o_1, n_1)\big\}.$$

It is straightforward to show that W_t is strictly increasing in e_t and strictly decreasing in e_{t+1} , so that $\mathcal{V}_t^D$ is a concave function that satisfies $\mathcal{V}_t^D(e_t) = W_t(e_t, 0)$.

Alternatively, as the agents care only about consumption–fertility decisions of their immediate descendants, it is also straightforward to show that, in every $\mathcal{A}$-efficient allocation, for each $t \geq 1$ and each i^t alive at t, i^t (and, by Assumption 1, the dynasty head as well) must choose $(\mathsf{x}_t(i^t), \mathsf{k}^o_{t+1}(i^t))$ to solve the optimization problem in the definition of $W_t(e_t, \int_{\mathbb{R}_+} e\, dE^{\mathsf{e}}_{t+1}(e, i^t))$, which yields

$$\mathcal{U}_t^D(\mathsf{a}, i^t) = W_t\left(e_t, \int_{\mathbb{R}_+} e\, dE^{\mathsf{e}}_{t+1}(e, i^t)\right).$$

Therefore, Theorem 1 implies that

$$\begin{aligned} &\lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} W_{t+1}\left(e_{t+1}(i^t, i_{t+1}), \int_{\mathbb{R}_+} e\, dE^e_{t+1}(e, i^t, i_{t+1})\right) \\ &\quad = \lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} W_{t+1}\big(e_{t+1}(i^t, i_{t+1}), 0\big) \end{aligned} \tag{18}$$

must be satisfied. Observe that the fact that W_{t+1} is continuous, together with the fact the two limits in (18) exist imply that the limit

$$\lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} e_{t+1}(i^t, i_{t+1})$$

is also well defined. Taking this into account, to complete the proof of Corollary 1, suppose it does not hold for, say, period $\tau = t+2$; that is, there exists $\epsilon_{t+2} > 0$ such that, for (almost) every agent of generation $\tau = t+2$ alive in a, one has $e_{t+1}(i^{t+1}, i_{t+2}) \geq \epsilon_{t+2}$. Observe that this implies that $\int_{\mathbb{R}_+} e\, dE^e_{t+1}(e, i^{t+1}) \geq \epsilon_{t+2}$ must be satisfied for (almost) every agent i^t alive at $\tau = t+1$. But the fact that W_{t+1} is strictly decreasing in e_{t+2} implies that

$$\lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} W_{t+1}\left(e_{t+1}(i^t, i_{t+1}), \int_{\mathbb{R}_+} e\, dE^e_{t+1}(e, i^t, i_{t+1})\right) < \lim_{\substack{i_{t+1} \to \mathsf{n}_{t+1}(i_t) \\ i_{t+1} < \mathsf{n}_{t+1}(i_t)}} W_{t+1}\big(e_{t+1}(i^t, i_{t+1}), 0\big)$$

must hold, which contradicts Theorem 1, a contradiction that completes the proof of Corollary 1. □

Proof of Theorem 2. Let $\widehat{\mathsf{a}}$ be an $\mathcal{A}^{\mathcal{D}}$-efficient allocation in an economy for which $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$ holds, and assume $2n_1^* < \overline{n}$ is satisfied. To prove Theorem 2, we proceed by steps. In the first step, we show that in the allocation $\widehat{\mathsf{a}}$, the number of individuals who obtain an income above the average income e_1^* that corresponds to a dynastic optimum is lower than the number of individuals n_1^* that correspond to such a dynastic optimum. In the second step, we show that under the qualifying condition $2n_1^* < \overline{n}$, we must have $\widehat{\mathsf{n}}_1(i^0) = n_1^*$, which, in turn, implies that $\widehat{\mathsf{a}}$ must be a dynastic optimum.

Step 1. To prove Step 1, for each $t \geq 1$, each $i^t \in \widehat{\mathsf{D}}_t(i^0)$, and each $e \geq 0$, let $\mathcal{V}_t^{D,\widehat{\mathsf{a}}}(e_t, i^t)$ be defined as the maximal utility that agent i^t's parent—or, in models with infinite-horizon altruism, the dynasty head—can obtain from i^t's descendants by endowing i^t with e_t units of resources, provided each descendant i^τ of i^t has to be provided with at least the same resources as the resources she receives in $\widehat{\mathsf{a}}$; that is,

$$\mathcal{V}_t^{D,\widehat{\mathsf{a}}}(e_t, i^t) := \max_{\mathsf{a}_t \in \mathcal{F}(e_t; i^t)} \{\mathcal{U}_t^D(\mathsf{a}; i^t) : \mathsf{e}_\tau(i^\tau) \geq \widehat{\mathsf{e}}_\tau(i^\tau) : \tau > t; i^\tau \in \mathsf{D}^\tau(i^0)\}.$$

With this notation, it is straightforward to show that since $\widehat{\mathsf{a}}$ is $\mathcal{A}^\mathcal{D}$-efficient, one must have, for every $t \geq 1$ and every $i^t \in \widehat{\mathsf{D}}^t(i^0)$,

$$\mathcal{U}_t^D(\widehat{\mathsf{a}}; i^t) = \mathcal{V}_t^{D,\widehat{\mathsf{a}}}(\widehat{\mathsf{e}}_t(i^t); i^t).$$

With this observation in mind, note that for any allocation a that arises as $\mathcal{A}^\mathcal{D}$-efficient in this unrestricted setting, there is an allocation $\widehat{\mathsf{a}}$ that satisfies $\widehat{\mathsf{D}}_{t+1}(i^t) = [0, \widehat{\mathsf{n}}_{t+1}(i^t)]$ for every $i^t \in \mathsf{D}^t(i^0)$ that provides the dynasty head with the same utility as the utility that she obtains with a, that, consequently, is also $\mathcal{A}^\mathcal{D}$-efficient. Thus, we can assume, without loss of generality, that $\widehat{\mathsf{D}}_1(i^0)$ adopts the form $[0, \widehat{\mathsf{n}}_1(i^0)]$. Taking this is this into account, the fact that $\widehat{\mathsf{a}}$ is $\mathcal{A}^\mathcal{D}$-efficient implies that the utility obtained with $\widehat{\mathsf{a}}$ by the dynasty head can be written as

$$\begin{aligned}\mathcal{U}_0(\widehat{\mathsf{a}}; i^0) = \max_{(e_1^y, k_1^o, x_0) \in \mathbb{R}_+^4 \times [0,\overline{n}]} \bigg\{ & U\bigg(x_0, \frac{1}{n_1}\int_0^{\underline{n}_1} \mathcal{V}_1^{D,\widehat{\mathsf{a}}}(\widehat{\mathsf{e}}_1(i), i)\, di + (n_1 - \underline{n}_1)\overline{\mathcal{V}}_1^D(e_1^y)\bigg) : \\ & c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0; c_1^o + \int_0^{\underline{n}_1} \mathsf{e}_1(i)\, di + (n_1 - \underline{n}_1) e_1^y \leq F_1(k_1^o, n_1)\bigg\}\end{aligned}$$

for some $\underline{n}_1 \in [0, \widehat{\mathsf{n}}_1(i^0)]$. Note that if $\underline{n}_1 = 0$, the allocation $\widehat{\mathsf{a}}$ is a dynastic optimum. Therefore, for the remainder of the proof, we assume that $\underline{n}_1 > 0$.

Write now $\widehat{e}_1^o$ and $\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}(e_1^o)$, respectively, for $\widehat{e}_1^o = [1/\underline{n}_1]\int_0^{\underline{n}_1} \widehat{\mathsf{e}}_1(i)\, di$ and

$$\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}(e_1^o) = \max_{e_1:[0,\underline{n}_1]\to R_+} \left\{\frac{1}{\underline{n}_1}\int_0^{\underline{n}_1} \mathcal{V}_1^{D,\widehat{\mathsf{a}}}(\mathsf{e}_1(i), i)\, di : \frac{1}{\underline{n}_1}\int_0^{\underline{n}_1} \mathsf{e}_1(i)\, di = e_1^o\right\}.$$

Using this notation, it can be shown that the utility obtained with $\widehat{\mathsf{a}}$ by the dynasty head can be written as

$$\begin{aligned}\mathcal{U}_0(\widehat{\mathsf{a}}; i^0) = \max_{(e_1^o, e_1^y, k_1^o, x_0) \in \mathbb{R}_+^5 \times [0,\overline{n}]} \bigg\{ & U\bigg(x_0, \bigg(\frac{\underline{n}_1}{n_1}\bigg)\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}(e_1) + \bigg(1 - \frac{\underline{n}_1}{n_1}\bigg)\overline{\mathcal{V}}_1^D(e_1^y)\bigg) : \\ & e_1 \geq \widehat{e}_1^o; c_0^m + b_0(n_1) + k_1^o \leq \overline{e}_0; c_1^o + \bigg(\frac{\underline{n}_1}{n_1}\bigg)e_1 + \bigg(1 - \frac{\underline{n}_1}{n_1}\bigg)e_1^y \leq F_1(k_1^o, n_1)\bigg\}.\end{aligned}$$

Using the first-order conditions that characterize a dynastic optimum, together with those that characterize a solution $(\widehat{x}_0, \widehat{k}_1^o, \widehat{e}_1^o, \widehat{e}_1^y)$ to the optimization problem in (19) and take into account that both $\overline{\mathcal{V}}_1^D$ and $\overline{\mathcal{V}}_1^{D,\widehat{\mathsf{a}},\underline{n}_1}$ are concave—even if $\mathcal{V}_1^{D,\widehat{\mathsf{a}}}$ and $\mathcal{V}_1^D$ are not concave—it can be shown that $\widehat{e}_1^o \geq \widehat{e}_1^y$ and $\widehat{e}_1^o \geq e_1^*$ must be satisfied.

It can be shown, using the first-order conditions of (19) and those that characterize a dynastic optimum, that if $\widehat{e}_1^o \geq \widehat{e}_1^y > e_1^*$, then $\widehat{n}_1 < n_1^*$ must be satisfied. Also, if $\widehat{e}_1^o \geq e_1^* > \widehat{e}_1^y$ is satisfied, then $\underline{n}_1 < n_1^*$ must be satisfied. Therefore, in the allocation $\widehat{\mathsf{a}}$, the number of individuals who obtain an income above the average income e_1^* that corresponds to a dynastic optimum is lower than the number of individuals n_1^* that correspond to such a dynastic optimum, which completes the proof of Step 1.

Step 2: $\widehat{\mathsf{n}}_1(i^0) = n_1^*$. Taking Step 1 into account, it is easy to see that $\widehat{\mathsf{a}}$ cannot be $\mathcal{A}^{\mathcal{D}}$-efficient unless it is a dynastic optimum. If it was $\mathcal{A}^{\mathcal{D}}$-efficient but was not a dynastic optimum, it would be easy to replace $\widehat{\mathsf{a}}$ by a dynastic optimum a^* such that most of the agents living in a^*, except the dynasty head and all those who obtain $\widehat{e}_1^y < e_1^*$, are not alive in $\widehat{\mathsf{a}}$. Such allocation a^* would trivially provide all agents living in both a^* and $\widehat{\mathsf{a}}$ with more utility than the utility that they obtain with $\widehat{\mathsf{a}}$, which contradicts the hypothesis that states that $\widehat{\mathsf{a}}$ is $\mathcal{A}^{\mathcal{D}}$-efficient but is not a dynastic optimum.

Thus, every $\mathcal{A}^{\mathcal{D}}$-efficient allocation must be a dynastic optimum. Since dynastic optima are trivially $\mathcal{A}^{\mathcal{D}}$-efficient, it follows that in a setting in which $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$ holds, an allocation is $\mathcal{A}^{\mathcal{D}}$-efficient if and only if it is a dynastic optimum, which completes the proof of Theorem 2. □

Proof of Corollary 2. Observe, that under the qualifying conditions, $\mathcal{D} \equiv \mathcal{B}[0, \overline{n}]$ and $n_1^* < \overline{n}/2$, every $\mathcal{A}^{\mathcal{D}}$-efficient allocation a^* must be a dynastic optimum by Theorem 2. Also, proceeding as in the proof of Corollary 1, one must have $\mathcal{V}_t^D(e_t) = W_t(e_t, 0)$ for each t. It follows that $e_t = 0$ must be satisfied for $t \geq 2$, which completes the proof. □

Proof of Theorem 3. To prove (i), assume Property S holds and let $\widehat{\mathsf{a}}$ be an ex post symmetric, $\mathcal{P}^{\mathcal{D}}$-efficient and, hence $\mathcal{P}^{\mathcal{O}}$-efficient, allocation that satisfies $\widehat{\mathsf{x}}_t(i^t) = \widehat{x}_t$ for each t and each $i^t \in \mathsf{D}_t(i^0)$. To show that $\widehat{a}$ must be Millian efficient, suppose it is not; that is, suppose there exists an alternative symmetric allocation a that provides all generations of agents living in $\widehat{\mathsf{a}}$ with higher utility. Since Property S holds, choosing a instead of $\widehat{a}$ involves a welfare improvement from the point of view of the $\mathcal{P}^{\mathcal{D}}$-dominance criterion, which contradicts the assumption that imposes that $\widehat{a}$ is $\mathcal{P}^{\mathcal{D}}$-efficient and, hence, completes the proof of statement (i) in Theorem 3.

To prove (ii), assume Property S holds and let $\widehat{a}$ be a Millian efficient allocation such that each function $\mathcal{V}_{\widehat{e},t}^D$ is concave on $[\widehat{e}_t, +\infty)$. To show that $\widehat{a}$ is $\mathcal{P}^{\mathcal{D}}$-efficient, let t be arbitrary and write $\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)$ as

$$\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t) = \max_{E:[\widehat{e}_{t+1},\infty] \to [0,1] \in \Delta\mathbb{R}_+} W_t^D\left(\widehat{e}_t, \int dE(e), \int \mathcal{V}_{\widehat{e},t+1}^D(e)\, dE(e)\right),$$

which, taking into account that $\mathcal{V}_{\widehat{e},t+1}^D$ is concave on $[\widehat{e}_{t+1}, +\infty)$, implies that $\widehat{e}_{t+1}$ solves the sequence of optimization problems in the definition of $\{\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)\}_{t\geq 1}$. Therefore, $\mathcal{U}_t^D(\widehat{\mathsf{a}}, i^t) = \mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)$ must be satisfied for each t and each $i^t \in \mathsf{D}^t(i^0)$. Analogously, $\mathcal{U}_0(\widehat{a}; i^0) = \mathcal{V}_{\widehat{e},0}(\overline{e}_0)$ must hold for $t = 0$. Taking this into account, observe that if $\widehat{a}$ is not $\mathcal{P}^{\mathcal{D}}$-efficient, then it should be $\mathcal{P}^{\mathcal{D}}$-dominated by the allocation that solves the sequence of optimization problems in the definition of $\{\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)\}_{t\geq 1}$, a contradiction that establishes that $\widehat{a}$ is $\mathcal{P}^{\mathcal{D}}$-efficient (and, hence, $\mathcal{P}^{\mathcal{O}}$-efficient) and completes the proof of Theorem 3. □

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