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A foundation for probabilistic beliefs with or without atoms

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We propose two novel axioms for qualitative probability spaces: (i) *unlikely atoms*, which requires that there is an event containing no atoms that is at least as likely as its complement; and (ii) *third-order atom-swarming*, which requires that for each atom, there is a countable pairwise-disjoint collection of less-likely events that can be partitioned into three groups, each with union at least as likely as the given atom. We prove that under *monotone continuity*, each of these axioms is sufficient to guarantee a unique countably-additive probability measure representation, generalizing work by Villegas to allow atoms. Unlike previous contributions that allow atoms, we impose no cancellation or solvability axiom.

Keywords. Beliefs, qualitative probability, unlikely atoms, atom-swarming.

JEL classification. D81, D83.

1. Introduction

1.1 *Overview*

Much behavior, including much of the economic behavior that we observe and strive to model, is the selection of an action with uncertain consequences. Our standard model is founded on the postulate that when these choices are made by someone who is rational, they can be decomposed into (i) beliefs about the relative likelihood of events, and (ii) tastes among outcomes (Ramsey 1931). This article revisits a classic question: when are such beliefs consistent with standard probability theory?

More precisely, suppose we are given a *qualitative probability space* $(\mathcal{A}, \supseteq, \succsim)$: a triple consisting of (i) a σ-algebra of events $(\mathcal{A}, \supseteq)$, and (ii) a binary relation comparing events $\succsim$, which together satisfy minimal probabilistic requirements (Bernstein 1917, de Finetti 1937, Koopman 1940, Savage 1954). Under what conditions can the comparisons be represented by a σ-measure[1] $\mu : \mathcal{A} \to [0, 1]$?

A necessary condition is *monotone continuity* (Villegas 1964, Arrow 1971): if $B_1 \supseteq B_2 \supseteq \cdots$; and if for each $i \in \mathbb{N}$, $B_i \succsim A$; then $\bigcap B_i \succsim A$. On the appeal of this axiom, Arrow writes, "The assumption of Monotone Continuity seems, I believe correctly, to be the

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[1] In this article, a *measure* is a finitely-additive probability measure, *σ-additivity* is *countable-additivity*, and a *σ-measure* is a σ-additive measure.

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harmless simplification almost inevitable in the formalization of any real-life problem." While this continuity axiom is not sufficient on its own (Kraft, Pratt, and Seidenberg 1959), we obtain sufficient conditions when it is paired with the requirement that there are no *atoms*: non-null events for which each subevent is either equally-likely or null (Villegas 1964).

Though the practice of using σ-measures to represent beliefs with atoms is prevalent throughout economic analysis, atoms in fact create significant technical challenges for guaranteeing that such representations are available. The large literature dedicated to this problem can be classified according to which of the following requirements is imposed:

- there are no atoms;
- the qualitative probability space satisfies a "cancellation" axiom;
- the qualitative probability space satisfies a "solvability" axiom; or
- there are additional primitives beyond $\mathcal{A}$, $\supseteq$, and $\succsim$.

The contribution of this article is to proceed without imposing any of these requirements.

We propose two new axioms, each of which is weaker than the requirement that there are no atoms. First, we propose *unlikely atoms*, which requires that there is an event containing no atoms that is at least as likely as its complement. Second, we propose *third-order atom-swarming* (3-*AS*), which requires that for each atom, there is a countable pairwise-disjoint collection of less-likely events that can be partitioned into

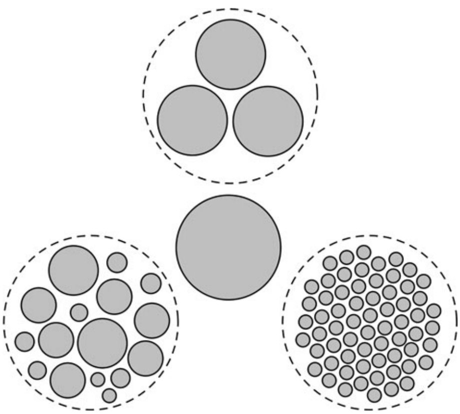

FIGURE 1. Third-order atom-swarming. In the illustration, gray circles and their unions are events, which are compared by area. The center circle is an atom, and the smaller circles together form a countable pairwise-disjoint collection of less-likely events. Each dashed circle shows a subcollection of these less-likely events whose union is at least as likely as the center atom; in this way, the center atom is "sufficiently swarmed" by these less-likely events. The axiom requires that *each* atom is sufficiently swarmed in this manner.

three groups, each with union at least as likely as the given atom (Figure 1). Our main results are that under *monotone continuity*, each of these conditions is necessary and sufficient to guarantee a σ-measure representation in an associated class, and that moreover, this representation is the unique σ-measure representation (Theorem 1 and Theorem 2).

1.2 *Related literature*

Calls for axiomatic foundations of probability can be traced past Hilbert's Sixth Problem (Hilbert 1902) directly to Boole (Boole 1851, 1854); for a detailed history, see Sheynin (2005). While contributors have held many philosophies about how probabilities should be interpreted, the mathematical approaches have not been so diverse, essentially comprising (i) the cardinal approach, definitively associated with Kolmogoroff (1933); and (ii) the ordinal approach, pioneered by Bernstein (1917); see Sheynin (2005) for a translation.

In this article, we are concerned with the problem of identifying axioms that guarantee the ordinal approach is compatible with the cardinal approach, which Machina and Schmeidler (1992) refer to as the first of two lines of inquiry culminating in the modern theory of subjective probability. We categorize the previous literature dedicated to this problem according to which of four assertions is imposed, emphasizing that our contribution is to proceed without any of these assertions. Many of the following contributions assume that events are subsets of a *state space*, so for brevity we use S to denote such a state space throughout this literature review.

1.2.1 *No atoms* The seminal contributions on qualitative probability spaces (Bernstein 1917, de Finetti 1937, Koopman 1940) imposed that S can be partitioned into an arbitrarily large number of equally likely events; this implies that there is a unique measure that "almost represents" beliefs, though it may assign the same probability to two distinguished events (see Kreps 1988). To guarantee representation by an atomless measure, Savage (1954) imposes the stronger *fineness-and-tightness* axiom, together with the additional restriction that $\mathcal{A} = 2^S$. While this additional restriction seriously limits the scope for σ-measure representation,[2] it is in fact unnecessary for the result.[3]

Since Savage, a number of contributions raised arguments suggesting that the beliefs of a rational agent should be represented by a σ-measure; for example, σ-additivity of the representation is required to avoid money pumps (Adams 1962, Seidenfeld and Schervish 1983) and to ensure that choice always respects strict first-order stochastic dominance (Wakker 1993b). Based on these observations and others, Stinchcombe

[2]In particular, if $|S| = |\mathbb{R}|$, then the existence of an atomless σ-measure defined on 2^S is inconsistent with the Continuum Hypothesis (Banach and Kuratowski 1929, Ulam 1930). That said, there are "nice" atomless measures that are not σ-additive; for example, when $S = [0, 1]$, there are measures on 2^S that (i) agree with the Lebesgue measure on those sets where it is defined, and (ii) assign the same number to any pair of congruent sets (Banach 1932).

[3]Savage's axioms imply that there is a measure representation with range $[0, 1]$, which remains true if $\mathcal{A}$ is allowed to be any σ-algebra (Wakker 1981, 1993a). If $\mathcal{A}$ is only required to be an algebra and a weaker version of *fineness-and-tightness* is imposed, then a measure representation is still guaranteed (Wakker 1981), but its range need only be a dense subset of $[0, 1]$ (Marinacci 1993).

(1997) concludes that a measure that is not σ-additive indicates a misspecified state space: "One summary [...] is that countably-infinite constructions require countably additive probabilities." Villegas (1964) identified the appropriate axiom: *monotone continuity* is both necessary and sufficient for a measure representation to be σ-additive (Theorem V1).[4] Moreover, *monotone continuity* and the requirement that there are no atoms are together sufficient to guarantee σ-measure representation (Theorem V2).

While Kopylov (2010) does not explicitly study qualitative probability spaces, it is clear from his analysis that σ-measure representation is guaranteed by *strong monotone continuity* whenever $\mathcal{A}$ is *countably separated* (Mackey 1957); that there are no atoms is then implied. The use of atomless measures to represent preferences also has axiomatic foundations for preferences over slices in classic fair division (Barbanel and Taylor 1995) and for preferences over parcels of land in urban economics (Berliant 1985, 1986).

1.2.2 *Cancellation* When S is finite, there are qualitative probability spaces without measure representations (Kraft, Pratt, and Seidenberg 1959), and the complex *finite cancellation* condition is necessary and sufficient for measure representation (Kraft, Pratt, and Seidenberg 1959; Scott 1964; Fishburn 1970; Krantz, Luce, Suppes, and Tversky 1971); in fact, even without the standard axioms defining qualitative probability spaces, it is alone necessary and sufficient to guarantee an additive representation (Fishburn 1970). Stronger and more complex conditions have been identified that guarantee measure representation while allowing atoms when S is infinite (Domotor 1969, Chateauneuf and Jaffray 1984, Chateauneuf 1985), and in fact Chateauneuf (1985) provides necessary and sufficient conditions using an axiom that implies *finite cancellation*; see Fishburn (1986) for a survey.

1.2.3 *Solvability* To our knowledge, there are two articles that study qualitative probability spaces with atoms that do not impose cancellation (Abdellaoui and Wakker 2005, Chew and Sagi 2006). Both involve versions of *solvability*: for each pair of disjoint events, there is a subevent of one that is as likely as the other.[5] Abdellaoui and Wakker (2005) allow for mosaics instead of σ-algebras and allow for measures that are not convex-ranged, while Chew and Sagi (2006) work with an ordering of events induced from preferences over acts through "exchangeability" and proceed without *monotonicity*. In both cases, there may be atoms, provided any pairwise-disjoint collection of atoms is finite with equally likely members.

1.2.4 *Additional primitives* Many contributions use the entire Savage (1954) model: a state space S, an outcome space X, and preferences over *acts* $f: S \to X$. Implicitly $\mathcal{A} = 2^S$, and under standard assumptions one can uncover an embedded qualitative probability space. While Savage only used this embedded qualitative probability space to guarantee a measure representation of beliefs, many others have used the entire model

[4] In fact, this is the appropriate axiom even in the multiple priors model (Gilboa and Schmeidler 1989), guaranteeing that the set of priors is a relatively weak compact set of σ-measures (Chateauneuf, Maccheroni, Marinacci, and Tallon 2005).

[5] This is not quite the language used in either article. Abdellaoui and Wakker (2005) use "solvability" to refer to a stronger axiom, while Chew and Sagi (2006) use "completeness" to refer to the given axiom and use "solvability" to refer to a related property for measures.

to allow for atoms. Typically this involves imposing, at a minimum, that X has cardinality of at least the continuum and that X has a rich topological structure. For example, X might be a simplex of objective "roulette lotteries" that can be mixed (as in Anscombe and Aumann 1963 and the vast literature that followed), or an interval of dollar amounts (see Wakker 1989, Gul 1992, and references therein), or Euclidean commodity space (as in the literature on intertemporal preferences; for example, Koopmans 1960).

In contrast to the first group, we allow for atoms. In contrast to the second group, we seek simpler sufficient conditions. In contrast to the third group, we allow for distinct atoms that need not be equally likely. In contrast to the fourth group, our analysis is focused on beliefs.

2. Model

It is standard in decision theory to introduce events as members of a *set-σ-algebra* (also known as a *concrete σ-algebra*): there is a state space; each event is a subset of the state space; the state space and the empty set are both events; and the collection of events is closed under complementation, countable unions, and countable intersections. There is, however, a larger class of σ-algebras, and it turns out that the most direct path to applying our results as widely as possible to the standard class involves using this larger class.[6] There are several equivalent definitions; we choose the order-theoretic one:

Definition 1. A partially ordered set $(\mathcal{A}, \supseteq)$ is a *lattice* if and only if for each pair $A, B \in \mathcal{A}$, there is a supremum $A \cup B$ and an infimum $A \cap B$. A lattice $(\mathcal{A}, \supseteq)$ is a *σ-algebra*[7] if and only if

- it is *distributive*: for each triple $A, B, C \in \mathcal{A}$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;[8]
- it is *complemented*: there are $S, \varnothing \in \mathcal{A}$ such that for each $A \in \mathcal{A}$, $S \supseteq A \supseteq \varnothing$; and moreover, for each $A \in \mathcal{A}$, there is a unique $\neg A \in \mathcal{A}$ such that $A \cup \neg A = S$ and $A \cap \neg A = \varnothing$;[9] and
- each countably-infinite collection $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ has both supremum $\bigcup A_i$ and infimum $\bigcap A_i$.

In this case, for each pair $A, B \in \mathcal{A}$, we write $A \setminus B$ to denote $A \cap \neg B$; henceforth, we often write $S \setminus A$ instead of $\neg A$. We refer to each $A \in \mathcal{A}$ as an *event*. We frequently abuse

[6]Others are more general still, allowing $\mathcal{A}$ to be any algebra (Wakker 1981, Marinacci 1993) or even any mosaic (Kopylov 2007, Abdellaoui and Wakker 2005). This last generalization is motivated by the distinction between risk and ambiguity (Knight 1921), as the collection of "subjectively risky events" (Epstein and Zhang 2001) need only be a mosaic.

[7]This is slight abuse of terminology. As an algebra is a set together with some operations, it would be more accurate to refer to $(\mathcal{A}, (\bigcup_i)_{i \in \mathbb{N} \cup \{|\mathbb{N}|\}}, (\bigcap_i)_{i \in \mathbb{N} \cup \{|\mathbb{N}|\}}, \neg)$ as the σ-algebra, where for each $i \in \mathbb{N} \cup \{|\mathbb{N}|\}$, $\bigcup_i$ and $\bigcap_i$ are the associated i-ary operations, and where $\neg$ is the unary operation. This abuse is for brevity.

[8]In fact, for lattices, each of these distribution conditions implies the other.

[9]In fact, for distributive lattices, the uniqueness of each complement is implied.

notation: when we say $(\mathcal{A}, \supseteq)$ is a σ-algebra, we implicitly take all notation defined here as given.

We use set-theoretic notation to emphasize that this larger class of σ-algebras is more familiar than exotic. Many standard arguments involving inclusions, complements, countable unions, and countable intersections for set-σ-algebras generalize; see the Algebra Lemma (Lemma 1) in Appendix A. That said, we must take care to avoid arguments involving *other* set-theoretic concepts for events; for example, we cannot make reference to an event's cardinality. (We can, however, make reference to the cardinality of a collection of events, as we do in the final bulleted entry of the above definition.)

In the context of a given σ-algebra $(\mathcal{A}, \supseteq)$, we refer to a binary relation $\succsim$ on $\mathcal{A}$ as a *likelihood relation*, interpreting $A \succsim B$ to mean that A is at least as likely as B. Both $A \succ B$ and $A \sim B$ have the obvious definitions. In this article, we restrict attention to ordered σ-algebras that satisfy standard probabilistic axioms (Bernstein 1917, de Finetti 1937, Koopman 1940, Savage 1954):

Definition 2. Let $(\mathcal{A}, \supseteq)$ be a σ-algebra and let $\succsim$ be a binary relation on $\mathcal{A}$. We say that $(\mathcal{A}, \supseteq, \succsim)$ satisfies

- *order*[10] if and only if $\succsim$ is complete and transitive;
- *separability*[11] if and only if for each triple $A, B, C \in \mathcal{A}$ such that $A \cap C = B \cap C = \varnothing$,
$$A \succsim B \quad \text{if and only if} \quad A \cup C \succsim B \cup C;$$
- *monotonicity*[12] if and only if for each pair $A, B \in \mathcal{A}$, $A \subseteq B$ implies $B \succsim A$; and
- *nondegeneracy* if and only if there are $A, B \in \mathcal{A}$ such that $A \succ B$.

We say that $(\mathcal{A}, \supseteq, \succsim)$ is a *qualitative probability space* if and only if it satisfies *order*, *separability*, *monotonicity*, and *nondegeneracy*.

We seek conditions under which our ordinal notion of qualitative probability space is compatible with the standard cardinal notion of probability (Kolmogoroff 1933):

Definition 3. Let $(\mathcal{A}, \supseteq)$ be a σ-algebra. A collection $\mathcal{A}' \subseteq \mathcal{A}$ is *pairwise-disjoint* if and only if for each pair $A, B \in \mathcal{A}'$, $A \cap B = \varnothing$. A function $\mu : \mathcal{A} \to [0, 1]$ is a (*probability*)

[10] We remark that violations of *completeness* can be observed and distinguished from indifference (Eliaz and Ok 2006), and in fact have been in a recent experiment (Cettolin and Riedl 2018); see Nehring (2009) and Alon and Lehrer (2014) for recent contributions studying likelihood relations without this assumption.

[11] We remark that the systematic violation of this axiom has been observed in an experiment where certain events feature probability appraisals while others do not (Ellsberg 1961); the favoring of the appraised events is a phenomenon typically ascribed to *ambiguity aversion* (though not always; see Ergin and Gul 2009).

[12] A referee observed that due to *separability*, we need only impose the weaker monotonicity axiom that for each $A \in \mathcal{A}$, $A \succsim \varnothing$. We use this stronger statement simply for convenience in the proofs.

measure (*on* $(\mathcal{A}, \supseteq)$) if and only if (i) $\mu(S) = 1$; and (ii) for each finite, pairwise-disjoint $\{A_i\}_{i\in I} \subseteq \mathcal{A}$,

$$\mu\Big(\bigcup A_i\Big) = \sum \mu(A_i).$$

A measure μ is moreover a σ-*measure* (*on* $(\mathcal{A}, \supseteq)$) if and only if (iii) for each countably-infinite, pairwise-disjoint $\{A_i\}_{i\in I} \subseteq \mathcal{A}$,

$$\mu\Big(\bigcup A_i\Big) = \sum \mu(A_i).$$

Abusing notation, let $\mathbb{M}(\mathcal{A}) \subseteq [0,1]^{\mathcal{A}}$ denote the class of measures and let $\mathbb{M}^\sigma(\mathcal{A}) \subseteq \mathbb{M}(\mathcal{A})$ denote the class of σ-measures.

We say that the ordinal and cardinal notions are compatible when the likelihood comparisons and the probability comparisons perfectly match:

Definition 4. If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space and $\mu \in \mathbb{M}(\mathcal{A})$, then we say μ is a *representation of* $(\mathcal{A}, \supseteq, \succsim)$ if and only if for each pair $A, B \in \mathcal{A}$,

$$A \succsim B \quad \text{if and only if} \quad \mu(A) \geq \mu(B).$$

In this case we say μ *represents* $\succsim$.

3. Results

3.1 *Axioms*

As discussed in Section 1.2, a number of contributions have raised arguments suggesting that the beliefs of a rational agent should be represented by a σ-measure, and the appropriate axiom has been identified:

Definition 5. A qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$ satisfies *monotone continuity* if and only if for each $A \in \mathcal{A}$ and each $(B_i) \in \mathcal{A}^{\mathbb{N}}$ such that (i) $B_1 \supseteq B_2 \supseteq \cdots$; and (ii) for each $i \in \mathbb{N}$, $B_i \succsim A$; we have $\bigcap B_i \succsim A$.

Theorem V1 (Villegas 1964). *If* $(\mathcal{A}, \supseteq, \succsim)$ *is a qualitative probability space with representation* $\mu \in \mathbb{M}(\mathcal{A})$, *then* $(\mathcal{A}, \supseteq, \succsim)$ *satisfies monotone continuity if and only if* $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

That said, this axiom alone is not sufficient for our purposes: there is a qualitative probability space satisfying *monotone continuity*, that moreover has only a finite number of events, with no measure representation (Kraft, Pratt, and Seidenberg 1959). This is in fact due to non-null events that contain no "smaller" non-null events, a concept that we formalize both ordinally and cardinally:

Definition 6. Let $(\mathcal{A}, \supseteq)$ be a σ-algebra. For each binary relation $\succsim$ on $\mathcal{A}$, we say that an event $\alpha \in \mathcal{A}$ is a (*likelihood*) *atom in* $(\mathcal{A}, \supseteq, \succsim)$ if and only if

- $\alpha \succ \varnothing$; and

- for each $B \in \mathcal{A}$ such that $B \subseteq \alpha$, we have $B \sim \alpha$ or $B \sim \varnothing$.

Similarly, for each $\mu \in \mathbb{M}(\mathcal{A})$, we say that an event $\alpha \in \mathcal{A}$ is a *measure-atom in* $(\mathcal{A}, \supseteq, \mu)$ if and only if

- $\mu(\alpha) > 0$; and
- for each $B \in \mathcal{A}$ such that $B \subseteq \alpha$, we have $\mu(B) = \mu(\alpha)$ or $\mu(B) = 0$.

Abusing notation, we let $\mathcal{A}^\bullet$ denote the collection of atoms in $(\mathcal{A}, \supseteq, \succsim)$ and we let $\mathcal{A}^{\bullet|\mu}$ denote the collection of measure-atoms in $(\mathcal{A}, \supseteq, \mu)$. Clearly, if $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space represented by μ, then $\mathcal{A}^\bullet = \mathcal{A}^{\bullet|\mu}$.

To introduce one result from the literature and two new results, we first introduce three conditions that, together with *monotone continuity*, are each sufficient for σ-measure representation. The first is standard, while the second and third are novel:

Definition 7. Let $n \in \mathbb{N}$. A qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$ satisfies

- *no atoms* if and only if $\mathcal{A}^\bullet = \varnothing$;
- *unlikely atoms* if and only if there is $A \in \mathcal{A}$ such that (i) $A \succsim S \setminus A$; and (ii) for each $\alpha \in \mathcal{A}^\bullet$, $\alpha \nsubseteq A$; and
- *nth-order atom-swarming* (n-*AS*) if and only if for each $\alpha \in \mathcal{A}^\bullet$, there are $I \subseteq \mathbb{N}$, pairwise-disjoint $\{B_i\}_{i\in I} \subseteq \mathcal{A}$, and $I_1, I_2, \dots, I_n$ partitioning I such that (i) for each $i \in I$, $\alpha \succ B_i$; and (ii) for each $j \in \{1, 2, \dots, n\}$, $\bigcup_{I_j} B_i \succsim \alpha$.

We associate each of these conditions with a corresponding class of σ-measures.

Definition 8. Let $(\mathcal{A}, \supseteq)$ be a σ-algebra, let $\mu \in \mathbb{M}^\sigma(\mathcal{A})$, and let $n \in \mathbb{N}$. We say that μ belongs to

- $\mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}) \subseteq \mathbb{M}^\sigma(\mathcal{A})$ if and only if for each $A \in \mathcal{A}$ and each $\lambda \in [0, 1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$;
- $\mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A}) \subseteq \mathbb{M}^\sigma(\mathcal{A})$ if and only if there is $A \in \mathcal{A}$ such that (i) $\mu(A) \geq \frac{1}{2}$; and (ii) for each $\alpha \in \mathcal{A}^{\bullet|\mu}$, $\alpha \nsubseteq A$; and
- $\mathbb{M}^\sigma_{n\text{-AS}}(\mathcal{A}) \subseteq \mathbb{M}^\sigma(\mathcal{A})$ if and only if for each $\alpha \in \mathcal{A}^{\bullet|\mu}$, there are $I \subseteq \mathbb{N}$ and pairwise-disjoint $\{B_i\}_{i\in I} \subseteq \mathcal{A}$ such that (i) for each $i \in I$, $\mu(\alpha) > \mu(B_i)$; and (ii) $\mu(\bigcup B_i) \geq n\mu(\alpha)$.

3.2 *Characterizations*

It is already known that *monotone continuity* and *no atoms* together guarantee a representation from the corresponding class of σ-measures:

Theorem V2 (Villegas 1964). *A qualitative probability space* $(\mathcal{A}, \supseteq, \succsim)$ *satisfies monotone continuity and no atoms if and only if it has representation* $\mu \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A})$. *In this case,* $(\mathcal{A}, \supseteq, \succsim)$ *has no other* σ*-measure representation.*

$$\begin{array}{ccc}
A_1 \succsim B_1 & & \mu(A_1) \geq \mu(B_1) \\
\Updownarrow & & \Updownarrow \\
A_2 \succsim B_2 & & \mu(A_2) \geq \mu(B_2) \\
\Updownarrow & & \Updownarrow \\
\vdots & & \vdots \\
\Updownarrow & & \Updownarrow \\
A_\Omega \succsim B_\Omega & \Leftrightarrow & \mu(A_\Omega) \geq \mu(B_\Omega)
\end{array}$$

FIGURE 2. Proof of Theorem 1. More specifically, this is the structure of the central argument in Step 4 of the proof of Proposition 1: we begin with a pair of events (A_1, B_1), then follow an algorithm to iteratively modify pairs until we can terminate with a pair (A_Ω, B_Ω) where it is obvious that the likelihood comparison agrees with the probability comparison; moreover, we do so in a manner that allows us to deduce this for the original pair.

Our first main result generalizes Theorem V2:

THEOREM 1. *A qualitative probability space* $(\mathcal{A}, \supseteq, \succsim)$ *satisfies monotone continuity and unlikely atoms if and only if it has representation* $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$*. In this case,* $(\mathcal{A}, \supseteq, \succsim)$ *has no other* σ*-measure representation.*

The first three appendices are dedicated to the proof. In Appendix A, we prove or provide proof references for basic results about σ-algebras and qualitative probability spaces, including some observations involving continuity that to the best of our knowledge are novel. In Appendix B, we apply a classic σ-algebra homomorphism theorem to formalize the notion that it is without loss of generality to focus our analysis on *idealized spaces*, where there is a unique null event.

In Appendix C, we first focus on idealized spaces. To begin, we use Theorem V2 to assign probabilities to subevents of the largest event that contains no atoms, then associate each atom with an equally-likely event that has been assigned a probability, and finally complete our construction of the σ-measure in the obvious manner. To verify that this is in fact a representation, we introduce an algorithm to begin with an arbitrary pair of events, then iteratively modify the pair until it is obvious that the likelihood comparison and the probability comparison agree, in a manner that allows us to conclude this about the original pair; see Figure 2. Finally, we make some concluding arguments, including that the result extends from all idealized spaces to all spaces.

Our second main result also generalizes Theorem V2:

THEOREM 2. *Let* $n \in \mathbb{N}$ *such that* $n \geq 3$*. A qualitative probability space* $(\mathcal{A}, \supseteq, \succsim)$ *satisfies monotone continuity and* n*-AS if and only if it has representation* $\mu \in \mathbb{M}^\sigma_{n\text{-AS}}(\mathcal{A})$*. In this case,* $(\mathcal{A}, \supseteq, \succsim)$ *has no other* σ*-measure representation.*[13]

The final four appendices are dedicated to the proof, which also relies on the first two appendices. As this result covers qualitative probability spaces for which every

[13]The current statement of this theorem is the result of two considerable generalizations of its original statement, both of which were suggested by the same referee, to whom I am extremely grateful.

non-null event contains an atom, we cannot again rely on Theorem V2 to construct our σ-measure. Instead, we pursue an approach based on the following observation: if two disjoint events are equally likely, then in any σ-measure representation, the measure of both must be half the measure of their union. Informally, if in some sense we were able to iteratively take such halves and then take disjoint unions, then in constructing our σ-measure representation we would necessarily assign numbers of the form $p/2^q$, where $q \in \{0, 1, \ldots\}$ and $p \in \{0, 1, \ldots, 2^q\}$, to certain likelihood equivalence classes. In Appendix D, we propose the notion of *supercabinet* to capture a structured family of equivalence classes with these labels, then prove that any qualitative probability space with a supercabinet has a unique σ-measure representation.

The next two appendices are dedicated to developing the tools to construct a supercabinet. In Appendix E, we propose the notion of *greedy transform* for selecting a subevent from a parent event, given a target event, which under some conditions guarantees that the selection is just as likely as the target. The basic idea is closely related to the observation of Kakeya (Kakeya 1914a,b) that if $(v_i) \in \mathbb{R}_+^{\mathbb{N}}$ is non-increasing with finite sum, then the collection of its subsequence sums $\{\sum_I v_i | I \subseteq \mathbb{N}\}$ is convex if and only if for each $i \in \mathbb{N}$, $v_i \leq \sum_{j>i} v_j$; see also Nitecki (2015). Indeed, for each $v \in [0, \sum v_i]$, a subsequence with sum v can be constructed by including v_1 unless the partial sum will exceed v, then including v_2 unless the partial sum will exceed v, and so on. Loosely, this works because at each stage there is never too much and always enough, and this ordinal argument extends to our setting.

Under 2-*AS*, our greedy transforms allow us to take an arbitrary pair of events and associate it with an equivalent disjoint pair, provided that the complement of the first is at least as likely as the second. In Appendix F, we show that a notion of halves for events is well-defined, then show that under 2-*AS*, we can construct two disjoint halves for any event.

Finally, in Appendix G, we apply the tools developed in the previous two appendices to construct a supercabinet; see Figure 3. To construct the supercabinet, we require only 2-*AS*, but to verify that it is a supercabinet, we require 3-*AS*. The need for 3-*AS* in our proof is most clear in Step 6 and Step 7 of the Supercabinet Construction Lemma (Lemma 18). Loosely, at this stage of the proof, we need to show that if we have a disjoint pair of events from the equivalence classes labeled $p/2^q$ and $p'/2^q$ whose union belongs to the equivalence class labeled $(p + p')/2^q$, then we have a disjoint pair of events from the equivalence classes labeled $p/2^q$ and $(p' + 1)/2^q$ whose union belongs to the equivalence class labeled $(p + p' + 1)/2^q$, and the argument requires us to construct a pairwise-disjoint collection of events from the equivalence classes labeled $p/2^q$, $p'/2^q$, and $1/2^q$, for which we require 3-*AS*. We remark that this succession-based approach to addition is related to Peano (1889).

3.3 *Discussion*

Because Theorem V2, Theorem 1, and Theorem 2 each state that any relevant qualitative probability space has a unique associated σ-measure, we can conveniently explore the logical relationships between these three theorems using σ-measures:

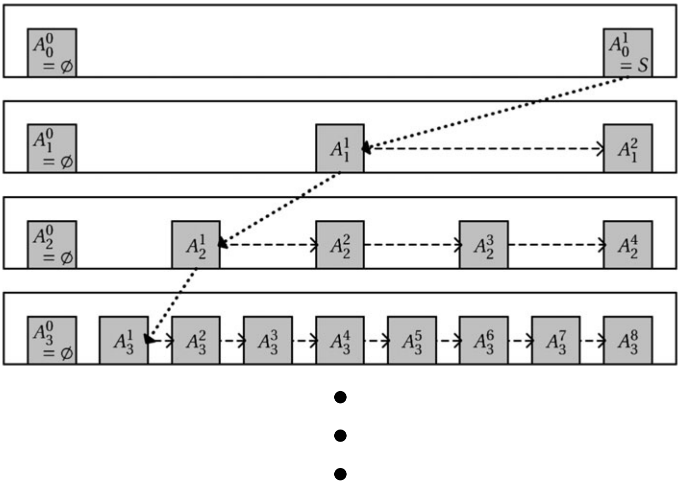

FIGURE 3. Proof of Theorem 2. More specifically, this is the structure of Step 1 and Step 2 of the Supercabinet Construction Lemma (Lemma 18). We construct events A_q^p, where $q \in \{0, 1, \dots\}$ and $p \in \{0, 1, \dots, 2^q\}$, such that we intend to associate each A_q^p with the measure $p/2^q$. For the dotted lines, we use our halving technique from Appendix F. For the dashed lines, we use our disjoint-pair technique from Appendix E. The rest of the proof is dedicated to establishing that the associated likelihood equivalence classes are in fact a supercabinet; this is where we require 3-AS.

EXAMPLE 1 (Hybrid measures). For each $x \in [0, 1]$, define $S_x \equiv [0, x] \cup \mathbb{N}$; define $(\mathcal{A}_x, \supseteq_x)$ to be the collection of Borel sets contained in S_x with ordinary set inclusion, which is indeed a σ-algebra; and define $\mathbb{M}_*^\sigma(\mathcal{A}_x)$ to be the collection of σ-measures that assign each subevent of $[0, x]$ its Borel measure. We consider four cases.

CASE 1. If $x = 1$ and $\mu \in \mathbb{M}_*^\sigma(\mathcal{A}_x)$, then

- $\mu \in \mathbb{M}_{\text{NA}}^\sigma(\mathcal{A}_x)$,
- $\mu \in \mathbb{M}_{\text{UA}}^\sigma(\mathcal{A}_x)$, and
- $\mu \in \mathbb{M}_{\text{3-AS}}^\sigma(\mathcal{A}_x)$.

Moreover, $\mathbb{M}_*^\sigma(\mathcal{A}_x)$ has a unique σ-measure.

CASE 2. If $x \in [\frac{3}{4}, 1)$ and $\mu \in \mathbb{M}_*^\sigma(\mathcal{A}_x)$, then

- $\mu \notin \mathbb{M}_{\text{NA}}^\sigma(\mathcal{A}_x)$,
- $\mu \in \mathbb{M}_{\text{UA}}^\sigma(\mathcal{A}_x)$, and
- $\mu \in \mathbb{M}_{\text{3-AS}}^\sigma(\mathcal{A}_x)$.

We remark that some σ-measures in $\mathbb{M}^{\sigma}_{*}(\mathcal{A}_x)$ have a finite collection of measure-atoms, while others have a countably-infinite collection of measure-atoms.

CASE 3. If $x \in [\frac{1}{2}, \frac{3}{4})$ and $\mu \in \mathbb{M}^{\sigma}_{*}(\mathcal{A}_x)$, then

- $\mu \notin \mathbb{M}^{\sigma}_{\text{NA}}(\mathcal{A}_x)$,
- $\mu \in \mathbb{M}^{\sigma}_{\text{UA}}(\mathcal{A}_x)$, and
- μ may or may not belong to $\mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$.

If there is $s \in \mathbb{N}$ such that $\mu(\{s\}) > \frac{1}{4}$; or if there are $s, s' \in \mathbb{N}$ such that $\mu(\{s\}) = \mu(\{s'\}) > \frac{1}{5}$; then $\mu \notin \mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$; otherwise, $\mu \in \mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$.

CASE 4. If $x \in [0, \frac{1}{2})$ and $\mu \in \mathbb{M}^{\sigma}_{*}(\mathcal{A}_x)$, then

- $\mu \notin \mathbb{M}^{\sigma}_{\text{NA}}(\mathcal{A}_x)$,
- $\mu \notin \mathbb{M}^{\sigma}_{\text{UA}}(\mathcal{A}_x)$, and
- μ may or may not belong to $\mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$.

For example, if there is $s \in \mathbb{N}$ such that $\mu(\{s\}) > \frac{1}{4}$, then $\mu \notin \mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$. If $x > 0$ and if for each $s \in \mathbb{N}$, $\mu(\{s\}) \in [0, x/3]$, then $\mu \in \mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$. If $x = 0$, if $n \in \mathbb{N}$, and if there is $\delta \in [n/(n+1), 1)$ such that for each $t \in \{0, 1, 2, \dots\}$,

$$\mu\big(\{t\}\big) = (1-\delta)\delta^t,$$

then $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A}_x)$. In particular, if $\delta \in [\frac{3}{4}, 1)$, then $\mu \in \mathbb{M}^{\sigma}_{3\text{-AS}}(\mathcal{A}_x)$. $\Diamond$

Case 1 is the only case where Theorem V2 applies, and Case 2 illustrates that both Theorem 1 and Theorem 2 are proper generalizations of Theorem V2. Case 3 establishes that Theorem 1 is not implied by Theorem 2, while Case 4 establishes that Theorem 2 is not implied by Theorem 1.

Our example for Case 4 when $x = 0$ is related to Kochov (2013), who considers a model that covers this setting with an intertemporal interpretation. Kochov proposes *patience*, the requirement that $\{2, 3, \dots\}$ is preferred to $\{1\}$, and establishes that if preferences satisfy *patience* and have a geometric representation, then preferences have no other geometric representation; a corollary is that a geometric representation is unique under 1-*AS*.

4. CONCLUSION

We conclude with an open problem:

CONJECTURE. *A qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$ satisfies monotone continuity and 1-AS if and only if it has representation $\mu \in \mathbb{M}^{\sigma}_{1\text{-AS}}(\mathcal{A})$. In this case, $(\mathcal{A}, \supseteq, \succsim)$ has no other σ-measure representation.*

It is straightforward to show that (i) *unlikely atoms* implies 1-*AS*, and (ii) 3-*AS* implies 1-*AS*; thus if the above conjecture were proven, then both Theorem 1 and Theorem 2 would follow as corollaries.

Appendix A

In this appendix, we state and prove (or provide a proof reference for) two basic lemmas about σ-algebras (the Algebra Lemma and the Measure Lemma) and five basic lemmas about qualitative probabilities (the Complement Lemma, the Domination Lemma, the Continuity Lemma, the Limit-Order Lemma, and the Carving Lemma). These are Lemmas 1–7, respectively.

To establish our first two lemmas, we take a direct approach using elementary arguments. As observed by a referee, an alternative approach is to derive these lemmas as consequences of the powerful Loomis–Sikorski Theorem (Loomis 1947, Sikorski 1960). This alternative approach is more elegant, but involves some digression; we therefore choose to provide details for the direct approach and simply mention this elegant approach for the interested reader.[14]

To begin, there are many algebraic manipulations of the symbols $\supseteq$, $\cup$, $\cap$, and $\setminus$ throughout these proofs that would give the reader well-acquainted with set theory no cause for hesitation if only these symbols had their usual set-theoretic meanings. As each of these symbols has a more general order-theoretic meaning, however, some reassurance is necessary; the Algebra Lemma (Lemma 1) provides this reassurance. Some of these manipulations involve a particular notion of limit, defined below; we remark that the associated notion of convergence, known as *sequential order convergence* (Kantorovich 1935, Birkhoff 1935, 1940), coincides with standard pointwise convergence for set-σ-algebras:

Definition 9. For each event sequence $(A_i) \in \mathcal{A}^{\mathbb{N}}$, we define

(i) the *limit superior of* (A_i), $\limsup A_i \equiv \bigcap_{i\in\mathbb{N}}(\bigcup_{j\geq i} A_j)$; and

(ii) the *limit inferior of* (A_i), $\liminf A_i \equiv \bigcup_{i\in\mathbb{N}}(\bigcap_{j\geq i} A_j)$.

We say that (A_i) is *convergent* if and only if $\limsup A_i = \liminf A_i$, in which case we define the *limit of* (A_i) to be $\lim A_i \equiv \limsup A_i = \liminf A_i$.

Lemma 1 (Algebra Lemma). *If $(\mathcal{A}, \supseteq)$ is a σ-algebra, then for each four $A, B, C, D \in \mathcal{A}$, each pair $I, J \subseteq \mathbb{N}$, each pair $\{A_i\}_{i\in I}, \{A_{ij}\}_{(i,j)\in I\times J} \in 2^{\mathcal{A}}$, each $j \in \mathbb{N}$, each list $I_1, I_2, \ldots, I_j \subseteq \mathbb{N}$, each list of collections $\{B_i\}_{I_1}, \{B_i\}_{I_2}, \ldots, \{B_i\}_{I_j} \in 2^{\mathcal{A}}$, and each convergent pair (C_i), $(D_i) \in \mathcal{A}^{\mathbb{N}}$,*

[14] Using terminology we introduce in Appendix B, the Loomis–Sikorski Theorem states that for each σ-algebra $(\mathcal{A}^*, \supseteq^*)$, there are a set-$\sigma$-algebra $(S, \mathcal{A}, \supseteq)$, a σ-ideal $\mathcal{I} \subseteq \mathcal{A}$, and a bijective σ-homomorphism $h : \mathcal{A}^* \to \mathcal{A}^{\mathcal{I}}$. In this way, each σ-algebra is isomorphic to a quotient of a set-σ-algebra, and the desired properties for a σ-algebra can be derived from the familiar fact that these properties hold for its associated set-σ-algebra.

(i) $S \setminus \varnothing = S$ *and* $S \setminus S = \varnothing$*;*

(ii) $A \cap \varnothing = \varnothing$ *and* $A \cup S = S$*;*

(iii) $A \cap S = A$ *and* $A \cup \varnothing = A$*;*

(iv) $A \cap (S \setminus A) = \varnothing$ *and* $A \cup (S \setminus A) = S$*;*

(v) $S \setminus (S \setminus A) = A$*;*

(vi) $A \cap A = A$ *and* $A \cup A = A$*;*

(vii) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$ *and* $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$*;*

(viii) $A \cap B = B \cap A$ *and* $A \cup B = B \cup A$*;*

(ix) $A \cap (B \cap C) = (A \cap B) \cap C$ *and* $A \cup (B \cup C) = (A \cup B) \cup C$*;*

(x) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*;*

(xi) $A \subseteq C$ *and* $B \subseteq D$ *implies* $A \cap B \subseteq C \cap D$ *and* $A \cup B \subseteq C \cup D$*;*

(xii) $A \subseteq B$ *implies* $(S \setminus A) \supseteq (S \setminus B)$*;*

(xiii) $S \setminus (\bigcup_I A_i) = \bigcap_I (S \setminus A_i)$ *and* $S \setminus (\bigcap_I A_i) = \bigcup_I (S \setminus A_i)$*;*

(xiv) $B \cap (\bigcup_I A_i) = \bigcup_I (B \cap A_i)$ *and* $B \cup (\bigcap A_i) = \bigcap_I (B \cup A_i)$*;*

(xv) $\bigcup_j (\bigcup_{I_j} B_i) = \bigcup_{\bigcup I_j} B_i$ *and* $\bigcap_j (\bigcap_{I_j} B_i) = \bigcap_{\bigcup I_j} B_i$*;*

(xvi) $\bigcup_I (\bigcup_J A_{ij}) = \bigcup_{(i,j) \in I \times J} A_{ij}$ *and* $\bigcap_I (\bigcap_J A_{ij}) = \bigcap_{(i,j) \in I \times J} A_{ij}$*;*

(xvii) $\lim(S \setminus C_i)$ *exists and* $\lim(S \setminus C_i) = S \setminus \lim C_i$*; and*

(xviii) $\lim(C_i \cup D_i)$ *and* $\lim(C_i \cap D_i)$ *exist,* $\lim(C_i \cup D_i) = \lim C_i \cup \lim D_i$*, and* $\lim(C_i \cap D_i) = \lim C_i \cap \lim D_i$*.*

Proof. For (i)–(xv), see Halmos (1963) (Sections 2, 6, and 7); we give a stronger statement of associativity in (xv) for which the proof given by Halmos goes through with trivial modification. We introduce our additional associativity condition in (xvi) for convenience; we omit its straightforward proof, which shares the structure of the proof by Halmos of (xv). For (xvii), there is a standard proof that $S \setminus \liminf A_i = \limsup(S \setminus A_i)$ and $S \setminus \limsup A_i = \liminf(S \setminus A_i)$ using the generalized De Morgan laws in (xiii), and (xvii) is an immediate corollary. We were unsuccessful in finding a proof of (xviii) anywhere, and therefore supply one.[15]

Let $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$. It is straightforward to show that $\limsup(A_i \cup B_i) \supseteq \liminf(A_i \cup B_i)$, $\liminf(A_i \cap B_i) \subseteq \limsup(A_i \cap B_i)$, $\liminf A_i \cup \liminf B_i \subseteq \liminf(A_i \cup B_i)$, and $\limsup A_i \cap \limsup B_i \supseteq \limsup(A_i \cap B_i)$; we omit the simple order-theoretic arguments. Thus it suffices to show that $\limsup(A_i \cup B_i) = \limsup A_i \cup \limsup B_i$ and $\liminf(A_i \cap B_i) =$

[15]Though (xviii) is a standard result for set-σ-algebras, the standard proof does not generalize.

$\liminf A_i \cap \liminf B_i$; indeed, it is then immediate that the desired consequences follow if (A_i) and (B_i) are convergent.

We first show $\limsup(A_i \cup B_i) = \limsup A_i \cup \limsup B_i$. For each $i \in \mathbb{N}$, define $A_i^+ \equiv \bigcup_{j \geq i} A_j$, $B_i^+ \equiv \bigcup_{j \geq i} B_j$, $J_i \equiv \{j \in \mathbb{N} | j \leq i\}$, and $K_i \equiv \{k \in \mathbb{N} | k \leq i\}$. Then $A_1^+ \supseteq A_2^+ \supseteq \cdots$ and $B_1^+ \supseteq B_2^+ \supseteq \cdots$, so for each $i \in \mathbb{N}$, $A_i^+ = \bigcap_{j \in J_i} A_j^+$ and $B_i^+ = \bigcap_{k \in K_i} B_k^+$. Thus

$$\begin{aligned}
\limsup(A_i \cup B_i) &= \bigcap_{i \in \mathbb{N}} \Bigl(\bigcup_{j \geq i} (A_j \cup B_j) \Bigr) \\
&= \bigcap_{i \in \mathbb{N}} \Bigl(\Bigl(\bigcup_{j \geq i} A_j \Bigr) \cup \Bigl(\bigcup_{j \geq i} B_j \Bigr) \Bigr) \\
&= \bigcap_{i \in \mathbb{N}} (A_i^+ \cup B_i^+) \\
&= \bigcap_{i \in \mathbb{N}} \Bigl(\Bigl(\bigcap_{j \in J_i} A_j^+ \Bigr) \cup \Bigl(\bigcap_{k \in K_i} B_k^+ \Bigr) \Bigr) \\
&= \bigcap_{i \in \mathbb{N}} \Bigl(\bigcap_{j \in J_i} \Bigl(A_j^+ \cup \Bigl(\bigcap_{k \in K_i} B_k^+ \Bigr) \Bigr) \Bigr) \\
&= \bigcap_{i \in \mathbb{N}} \Bigl(\bigcap_{j \in J_i} \Bigl(\bigcap_{k \in K_i} (A_j^+ \cup B_k^+) \Bigr) \Bigr) \\
&= \bigcap_{i \in \mathbb{N}} \Bigl(\bigcap_{(j,k) \in J_i \times K_i} (A_j^+ \cup B_k^+) \Bigr) \\
&= \bigcap_{(j,k) \in \bigcup (J_i \times K_i)} (A_j^+ \cup B_k^+) \\
&= \bigcap_{(j,k) \in \mathbb{N} \times \mathbb{N}} (A_j^+ \cup B_k^+) \\
&= \bigcap_{j \in \mathbb{N}} \Bigl(\bigcap_{k \in \mathbb{N}} (A_j^+ \cup B_k^+) \Bigr) \\
&= \bigcap_{j \in \mathbb{N}} \Bigl(A_j^+ \cup \Bigl(\bigcap_{k \in \mathbb{N}} B_k^+ \Bigr) \Bigr) \\
&= \Bigl(\bigcap_{j \in \mathbb{N}} A_j^+ \Bigr) \cup \Bigl(\bigcap_{k \in \mathbb{N}} B_k^+ \Bigr) \\
&= \limsup A_i \cup \limsup B_i.
\end{aligned}$$

We remark that the second equality holds due to (xvi), which can be shown in tedious detail with additional notation; we omit the straightforward argument.

To prove that $\liminf(A_i \cap B_i) = \liminf A_i \cap \liminf B_i$, we need only appeal to duality: in the above paragraph, simply replace each instance of the symbol $\cup$ with $\cap$, replace each instance of $\bigcup$ with $\bigcap$, and replace each instance of $\limsup$ with $\liminf$. □

As the above proof demonstrates, we frequently make arguments that rely on the Algebra Lemma (Lemma 1) without explicit mention. That said, there will occasionally be an explicit mention when we use infinite operations, as the following proof demonstrates.

The Measure Lemma (Lemma 2) states that each σ-measure respects limits, or equivalently, that each σ-measure is a continuous function in the sequential order convergence topology. This is a known result for set-σ-algebras (for example, this is a corollary of observations about $\limsup$ and $\liminf$ in Halmos 1950, Section 9), but we were unable to find a proof for the general result; we therefore supply one:

Lemma 2 (Measure Lemma). *If $(\mathcal{A}, \supseteq)$ is a σ-algebra and $\mu \in \mathbb{M}^\sigma(\mathcal{A})$, then for each convergent $(A_i) \in \mathcal{A}^{\mathbb{N}}$, we have that $\lim \mu(A_i)$ exists with $\lim \mu(A_i) = \mu(\lim A_i)$.*

Proof. We first prove the statement for "vanishing" sequences and then in general.

Step 1: *For each $(V_i) \in \mathcal{A}^{\mathbb{N}}$ such that $V_1 \supseteq V_2 \supseteq \cdots$ and $\lim V_i = \varnothing$, we have that $\lim \mu(V_i)$ exists and $\lim \mu(V_i) = 0$.* Let $(V_i) \in \mathcal{A}^{\mathbb{N}}$ satisfy the hypotheses. For each $i \in \mathbb{N}$, define $A_i \equiv S \setminus V_i$; by the Algebra Lemma (Lemma 1), $\lim A_i$ exists and $\lim A_i = S \setminus \lim V_i = S$. For each $i \in \mathbb{N}$, define $B_i \equiv A_i \setminus (\bigcup_{j<i} A_j)$; as $A_1 \subseteq A_2 \subseteq \cdots$, altogether we have $\bigcup B_i = \bigcup A_i = \lim A_i = S$. As $\{B_i\}$ is pairwise-disjoint, thus $\sum \mu(B_i) = \mu(S) = 1$. For each $i \in \mathbb{N}$, $\mu(A_i) = \sum_{j\leq i} \mu(B_i)$, so $\lim \mu(A_i)$ exists and $\lim \mu(A_i) = \sum \mu(B_i) = 1$; thus $\lim \mu(V_i) = \lim(1 - \mu(A_i))$ exists and $\lim \mu(V_i) = 1 - \lim \mu(A_i) = 0$.

Step 2: *Conclude.* Let $(A_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent and define $A_\infty \equiv \lim A_i$. For each $i \in \mathbb{N}$, define $A_i^+ \equiv \bigcup_{j\geq i} A_j$, define $A_i^- \equiv \bigcap_{j\geq i} A_j$, and define $V_i \equiv A_i^+ \setminus A_i^-$.

We claim that (V_i) is "vanishing." Indeed, for each $i \in \mathbb{N}$, $A_i^+ \supseteq A_{i+1}^+$ and $A_i^- \subseteq A_{i+1}^-$, so $S \setminus A_i^- \supseteq S \setminus A_{i+1}^-$, so $V_i = A_i^+ \cap (S \setminus A_i^-) \supseteq A_{i+1}^+ \cap (S \setminus A_{i+1}^-) = V_{i+1}$. Moreover, $A_\infty = \bigcap A_i^+ = \bigcup A_i^-$, so by the Algebra Lemma (Lemma 1),

$$\begin{aligned}
\lim V_i &= \lim\bigl(A_i^+ \cap \bigl(S \setminus A_i^-\bigr)\bigr) \\
&= \lim A_i^+ \cap \lim\bigl(S \setminus A_i^-\bigr) \\
&= A_\infty \cap \bigl(S \setminus \lim A_i^-\bigr) \\
&= A_\infty \cap (S \setminus A_\infty) \\
&= \varnothing,
\end{aligned}$$

as desired.

We claim that $\lim(A_i^+)$, $\lim(A_i^-)$ exist and $\lim(A_i^+) = \lim(A_i^-)$. Indeed, by Step 1, $\lim \mu(V_i)$ exists and $\lim \mu(V_i) = 0$. For each $i \in \mathbb{N}$, we have $\mu(A_i^+), \mu(A_i^-) \in [0, 1]$, and moreover $\mu(A_1^+) \geq \mu(A_2^+) \geq \cdots$ and $\mu(A_1^-) \leq \mu(A_2^-) \leq \cdots$; thus $(\mu(A_i^+))$ and $(\mu(A_i^-))$

are both bounded and monotonic, so there are $\lim \mu(A_i^+)$ and $\lim \mu(A_i^-)$. For each $i \in \mathbb{N}$, $A_i^+ \supseteq A_i^-$, so $\mu(A_i^+) = \mu(A_i^-) + \mu(A_i^+ \setminus A_i^-) = \mu(A_i^-) + \mu(V_i)$; thus $\lim(A_i^+) = \lim(\mu(A_i^-) + \mu(V_i)) = \lim \mu(A_i^-) + \lim \mu(V_i) = \lim \mu(A_i^-)$, as desired.

To conclude, for each $i \in \mathbb{N}$, $A_i^+ \supseteq A_i \supseteq A_i^-$ and $A_i^+ \supseteq A_\infty \supseteq A_i^-$, so $\mu(A_i^+) \geq \mu(A_i) \geq \mu(A_i^-)$ and $\mu(A_i^+) \geq \mu(A_\infty) \geq \mu(A_i^-)$. By the standard Squeeze Theorem, $\lim \mu(A_i)$ exists, $\lim \mu(A_i) = \lim \mu(A_i^+)$, and $\mu(A_\infty) = \lim \mu(A_i^+)$; thus $\lim \mu(A_i) = \mu(A_\infty) = \mu(\lim A_i)$, as desired. □

With that, we turn from σ-algebras to qualitative probability spaces. The next two lemmas are particularly general in that they apply to each of these spaces. In fact, there is only one other such result in this article (the Half-Equivalence Lemma (Lemma 15) in Appendix F); all other results involve continuity.

The Complement Lemma (Lemma 3) states that likelihood reverses under complements. This is a slight extension of Exercise 3 of Savage (1972, p. 32), and thus the proof is omitted.

Lemma 3 (Complement Lemma, Savage 1954). *If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space, then for each $A \in \mathcal{A}$ and each pair $B, B' \subseteq A$, $B \succsim B'$ if and only if $A \setminus B' \succsim A \setminus B$.*

The Domination Lemma (Lemma 4) states that for any two pairs such that the first is disjoint and dominates the second in likelihood, the union of the first is at least as likely as the union of the second. Moreover, strict domination implies the union of the first is more likely than the union of the second. (As the application of this lemma requires verifying that one pair of events is disjoint, while the application of *separability* requires verifying that two pairs of events are disjoint, it is often more convenient to use the lemma than to use the axiom.) The lemma is a slight extension of Exercise 5a of Savage (1972, p. 32), and thus the proof is omitted.

Lemma 4 (Domination Lemma, Savage 1954). *If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space, then for each four $A, A', B, B' \in \mathcal{A}$, if*

(i) $A \cap A' = \varnothing$,

(ii) $A \succsim B$, and

(iii) $A' \succsim B'$,

then $A \cup A' \succsim B \cup B'$. If moreover $A \succ B$, then $A \cup A' \succ B \cup B'$.

The final three lemmas in this appendix involve *monotone continuity*, and the following closely-related requirement that upper and lower contour sets are closed in the sequential order convergence topology:

Definition 10. A qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$ satisfies *continuity* if and only if for each $A \in \mathcal{A}$ and each convergent $(B_i) \in \mathcal{A}^{\mathbb{N}}$,

(i) if for each $i \in \mathbb{N}$, $B_i \succsim A$, then $\lim B_i \succsim A$; and

(ii) if for each $i \in \mathbb{N}$, $A \succsim B_i$, then $A \succsim \lim B_i$.

Though *continuity* seems stronger than *monotone continuity*, the Continuity Lemma (Lemma 5) states that in fact the two are equivalent for qualitative probabilities, a point which to our knowledge has not been made previously. Based on this equivalence, we are justified in writing *(monotone) continuity* in stating our results while using *continuity* in our proofs:

Lemma 5 (Continuity Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space, then it satisfies monotone continuity if and only if it satisfies continuity.*

Proof. Because for each sequence $(B_i) \in \mathcal{A}^{\mathbb{N}}$ such that $B_1 \supseteq B_2 \supseteq \cdots$, we have $\bigcap B_i = \lim B_i$, thus clearly *continuity* implies *monotone continuity*. Suppose $\succsim$ is a qualitative probability satisfying *monotone continuity*.

To see that upper contour sets are closed, let $A \in \mathcal{A}$ and let $(B_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent such that for each $i \in \mathbb{N}$, $B_i \succsim A$. Define $B_\infty \equiv \lim B_i$; and for each $i \in \mathbb{N}$, define $B_i^+ \equiv \bigcup_{j \geq i} B_j$. By construction, $B_1^+ \supseteq B_2^+ \supseteq \cdots$; and by *monotonicity*, for each $i \in \mathbb{N}$, $B_i^+ \succsim B_i \succsim A$; thus by *monotone continuity*, $\bigcap B_i^+ \succsim A$. By definition, $B_\infty = \bigcap B_i^+$; thus $B_\infty = \bigcap B_i^+ \succsim A$, as desired.

To see that lower contour sets are closed, let $A \in \mathcal{A}$ and let $(B_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent such that for each $i \in \mathbb{N}$, $A \succsim B_i$. Define $B_\infty = \lim B_i$. By the Algebra Lemma (Lemma 1), $(S \setminus B_\infty) = \lim(S \setminus B_i)$. By the Complement Lemma (Lemma 3), for each $i \in \mathbb{N}$, $(S \setminus B_i) \succsim (S \setminus A)$. As upper contour sets are closed, thus $(S \setminus B_\infty) = \lim(S \setminus B_i) \succsim (S \setminus A)$, so by the Complement Lemma (Lemma 3), $A \succsim B_\infty$, as desired. □

The Limit-Order Lemma (Lemma 6) states that for each pair of convergent sequences, if each member of the first sequence is at least the corresponding member of the second, then the limit of the first sequence is at least the limit of the second. Variants of this result appear in Villegas (1964) and Arrow (1971). This particular result does not require $\mathcal{A}$ to be a σ-algebra; any Hausdorff space will do, as can be seen from the proof:

Lemma 6 (Limit-Order Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ satisfies order and continuity, then for each pair of convergent sequences $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $A_j \succsim B_j$, we have* $\lim(A_i) \succsim \lim(B_i)$.

Proof. Let $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ satisfy the hypothesis. Since $\succsim$ is complete, by a standard argument[16] there is $M \subseteq \mathbb{N}$ such that that $(B_i') \equiv (B_i)|_M$ is a $\succsim$-monotonic sequence. Define $(A_i') \equiv (A_i)|_M$. Necessarily $\lim(A_i) = \lim(A_i')$ and $\lim(B_i) = \lim(B_i')$.

Case 1: *(B_i') is non-decreasing.* Then for each pair $j, k \in \mathbb{N}$ with $k \geq j$,

$$\begin{aligned} A_k' &\succsim B_k' \\ &\succsim B_j', \end{aligned}$$

[16]A common proof of the Bolzano–Weierstrass theorem includes a lemma stating that each real sequence has a monotonic subsequence; the standard proof of that lemma suffices here.

so by *continuity*, $\lim(A_i') \succsim B_j'$. Thus by *continuity*, $\lim(A_i') \succsim \lim(B_i')$, so $\lim(A_i) \succsim \lim(B_i)$.

Case 2: (B_i') *is non-increasing*. Then for each pair $j, k \in \mathbb{N}$ with $k \geq j$,

$$\begin{aligned} A_j' &\succsim B_j' \\ &\succsim B_k', \end{aligned}$$

so by *continuity*, $A_j' \succsim \lim(B_i')$. Thus by *continuity*, $\lim(A_i') \succsim \lim(B_i')$, so $\lim(A_i) \succsim \lim(B_i)$. $\square$

Finally, the Carving Lemma (Lemma 7) states that if the union of (i) an event A° that contains no atoms, and (ii) an event B that is disjoint from A°, is at least as likely as some reference event C, which in turn is at least as likely as B alone, then it is possible to "carve" a subevent C° of A° such that the union $C^\circ \cup B$ is precisely as likely as C. When $B = \varnothing$ and C contains no atoms, this is an immediate corollary of Theorem V2; the lemma states that this is true in general:

Lemma 7 (Carving Lemma). *If* $(\mathcal{A}, \supseteq, \succsim)$ *is a qualitative probability space that satisfies (monotone) continuity, then for each triple* $A^\circ, B, C \in \mathcal{A}$ *such that*

(i) $A^\circ \cap B = \varnothing$*;*

(ii) for each $A' \subseteq A^\circ$, $A' \notin \mathcal{A}^\bullet$*; and*

(iii) $A^\circ \cup B \succsim C \succsim B$*;*

there is $C^\circ \in \mathcal{A}$ *such that* $C^\circ \subseteq A^\circ$ *and* $C^\circ \cup B \sim C$.

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$, A°, B, C satisfy the hypotheses. If $A^\circ \sim \varnothing$, then by the Domination Lemma (Lemma 4), $B \succsim A^\circ \cup B \succsim C \succsim B$, so $B \sim C$, so if we define $C^\circ \equiv \varnothing$ we are done; thus let us assume that $A^\circ \succ \varnothing$.

Step 1: *Define* $\mathcal{A}^* \subseteq \mathcal{A}$, $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$, *and* $V^+, V^- \subseteq [0,1]$. Define $\mathcal{A}^* \equiv \{A \in \mathcal{A} | A \subseteq A^\circ\}$; it is straightforward to show that this is a σ-algebra. Define $\supseteq^*, \succsim^*$ on $\mathcal{A}^*$ as follows: for each pair $A, A' \in \mathcal{A}^*$, (i) $A \supseteq^* A'$ if and only if $A \supseteq A'$, and (ii) $A \succsim^* A'$ if and only if $A \succsim A'$. Since $A^\circ \succ \varnothing$, it is straightforward to verify that $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is a qualitative probability space satisfying *monotone continuity* and *no atoms*; thus by Theorem V2, there is a unique $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$ such that for each pair $A, A' \in \mathcal{A}^*$, $\mu^*(A) \geq \mu^*(A')$ if and only if $A \succsim^* A'$ if and only if $A \succsim A'$. To be more concise, we say that μ^* represents comparisons on $\mathcal{A}^*$. Define $V^+, V^- \subseteq [0,1]$ by

$$\begin{aligned} V^+ &\equiv \{v \in [0,1] | A \in \mathcal{A}^* \text{ and } \mu^*(A) = v \text{ implies } A \cup B \succsim C\}, \text{ and} \\ V^- &\equiv \{v \in [0,1] | A \in \mathcal{A}^* \text{ and } \mu^*(A) = v \text{ implies } C \succsim A \cup B\}. \end{aligned}$$

Observe that for each pair $A, A' \in \mathcal{A}^*$ such that $\mu^*(A) = \mu^*(A')$, we have $A \sim A'$, $A \cap B = \varnothing$, and $A' \cap B = \varnothing$, so by *separability*, we have $A \cup B \sim A' \cup B$. For the next step, we refer to this as *the observation*.

Step 2: $\inf V^+ = \sup V^-$. We prove three claims, then conclude.

Claim 1: *$V^+ \neq \varnothing$ and $V^- \neq \varnothing$.* Indeed, since $A^\circ \cup B \succsim C \succsim B = \varnothing \cup B$, thus by the observation, $1 \in V^+$ and $0 \in V^-$.

Claim 2: *For each pair $v', v \in [0,1]$ such that $v' > v$, (i) $v \in V^+$ implies $v' \in V^+ \setminus V^-$, and (ii) $v' \in V^-$ implies $v \in V^- \setminus V^+$.* Indeed, let v', v satisfy the hypotheses. Since $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$, thus there are $A', A \in \mathcal{A}^*$ such that $\mu^*(A') = v'$ and $\mu^*(A) = v$. Since μ^* represents comparisons on $\mathcal{A}^*$, thus $A' \succ A$. Since $A' \cap B = \varnothing$, thus by the Domination Lemma (Lemma 4), $A' \cup B \succ A \cup B$. For (i), if $v \in V^+$, then $A' \cup B \succ A \cup B \succsim C$, so by the observation, $v' \in V^+ \setminus V^-$. For (ii), if $v' \in V^-$, then $C \succsim A' \cup B \succ A \cup B$, so by the observation, $v \in V^- \setminus V^+$.

Claim 3: $V^+ \cup V^- = [0,1]$. Indeed, since $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$, thus for each $v \in [0,1]$, there is $A \in \mathcal{A}^*$ such that $\mu^*(A) = v$. By *order*, $A \cup B \succsim C$ or $C \succsim A \cup B$; thus by the observation, $v \in V^+$ or $v \in V^-$. Altogether, then, $[0,1] \subseteq V^+ \cup V^- \subseteq [0,1]$, so $V^+ \cup V^- = [0,1]$.

To conclude, by Claim 1 and Claim 2, there are $\inf(V^+)$ and $\sup(V^-)$ such that $\inf(V^+) \geq \sup(V^-)$; thus by Claim 3, we have $\inf(V^+) = \sup(V^-)$.

Step 3: *Conclude.* Define $v^\circ \equiv \inf V^+ = \sup V^-$. Since $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$, thus there is $C^\circ \in \mathcal{A}^*$ such that $\mu^*(C^\circ) = v^\circ$. We proceed with two claims that together complete the proof.

Claim 1: $C^\circ \cup B \succsim C$. If $v^\circ = 1$, then $C^\circ \sim A^\circ$ and $C^\circ \cap B = \varnothing$, so by the Domination Lemma (Lemma 4) $C^\circ \cup B \succsim A^\circ \cup B \succsim C$ and we are done; thus let us assume $v^\circ < 1$. Then there is decreasing $(v_i) \in [0,1]^{\mathbb{N}}$ such that $v_1 = 1$ and $\lim v_i = v^\circ$. Since $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$, thus there is $(A_i) \in (\mathcal{A}^*)^{\mathbb{N}}$ such that (i) $A^\circ = A_1 \supseteq A_2 \supseteq \cdots$; and (ii) for each $i \in \mathbb{N}$, $\mu^*(A_i) = v_i$. Since $A_1 \supseteq A_2 \supseteq \cdots$, thus there is $\lim A_i$.

We first prove that $C^\circ \cup B \sim (\lim A_i) \cup B$. Indeed, since $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$, thus by the Measure Lemma (Lemma 2),

$$\begin{aligned}
\mu^*(C^\circ) &= v^\circ \\
&= \lim v_i \\
&= \lim \mu^*(A_i) \\
&= \mu^*(\lim A_i),
\end{aligned}$$

so as μ^* represents comparisons on $\mathcal{A}^*$, thus $C^\circ \sim \lim A_i$. By construction, $C^\circ \cap B = \varnothing$. Moreover, for each $i \in \mathbb{N}$, $A_i \cap B = \varnothing$, so by the Algebra Lemma (Lemma 1), there is

$\lim(A_i \cap B)$ and

$$
\begin{aligned}
(\lim A_i) \cap B &= (\lim A_i) \cap (\lim B) \\
&= \lim(A_i \cap B) \\
&= \varnothing.
\end{aligned}
$$

Thus by *separability*, $C^\circ \cup B \sim (\lim A_i) \cup B$, as desired.

Next, we prove that $(\lim A_i) \cup B \succsim C$. Indeed, for each $i \in \mathbb{N}$, $v_i > v^\circ = \sup V^-$, so $v_i \in V^+$, so as $\mu^*(A_i) = v_i$ we have $A_i \cup B \succsim C$. By the Algebra Lemma (Lemma 1), there is $\lim(A_i \cup B)$ and

$$
\begin{aligned}
(\lim A_i) \cup B &= (\lim A_i) \cup (\lim B) \\
&= \lim(A_i \cup B),
\end{aligned}
$$

so by *continuity*, $(\lim A_i) \cup B = \lim(A_i \cup B) \succsim C$, as desired.

Altogether, then, we have $C^\circ \cup B \sim (\lim A_i) \cup B \succsim C$.

Claim 2: $C^\circ \cup B \precsim C$. If $v^\circ = 0$, then $C^\circ \sim \varnothing$ and $\varnothing \cap B = \varnothing$, so by the Domination Lemma (Lemma 4) $C^\circ \cup B \precsim \varnothing \cup B \precsim C$ and we are done; thus let us assume $v^\circ > 0$. Then there is increasing $(v_i) \in [0, 1]^{\mathbb{N}}$ such that $v_1 = 0$ and $\lim v_i = v^\circ$. Since $\mu^* \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^*)$, thus there is $(A'_i) \in (\mathcal{A}^*)^{\mathbb{N}}$ such that (i) $A^\circ = A'_1 \supseteq A'_2 \supseteq \cdots$; and (ii) for each $i \in \mathbb{N}$, $\mu^*(A'_i) = 1 - v_i$. For each $i \in \mathbb{N}$, define $A_i \equiv A^\circ \setminus A'_i$; then (i) $A_1 \subseteq A_2 \subseteq \cdots$; and (ii) for each $i \in \mathbb{N}$, $\mu^*(A_i) = v_i$. Since $A_1 \subseteq A_2 \subseteq \cdots$, thus there is $\lim A_i$.

The remaining arguments for this claim are dual to the corresponding arguments for the previous claim in the following sense: take the first claim's arguments, then replace each instance of $\succsim$ with $\precsim$, each instance of $>$ with $<$, each instance of sup with inf, each instance of V^- with V^+, and each instance of V^+ with V^-; the resulting arguments complete the proof of this claim and thus the proof of this lemma. □

Appendix B

In this appendix, we prove the Null-Quotient Lemma (Lemma 8), which formalizes the idea that it is without loss of generality to focus on the following class of qualitative probability spaces:

Definition 11. A qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$ is an *idealized* (*qualitative probability*) *space* if and only if (i) it satisfies *monotone continuity*, and (ii) $\{A \in \mathcal{A} | A \sim \varnothing\} = \{\varnothing\}$. In this case, by a correction to Lemma 4 of Villegas (1964)[17] and then by Lemma 1 of Villegas (1964), there is $I^\bullet \subseteq \mathbb{N}$ such that (i) $\mathcal{A}^\bullet = \{\alpha_s\}_{s \in I^\bullet}$ is pairwise-disjoint;

[17]Lemma 4 of Villegas (1964) incorrectly states that for each qualitative probability space that satisfies *monotone continuity*, the collection of atoms is at most countably-infinite; as a counterexample, take the qualitative probability space $(2^{[0,1]}, \supseteq, \succsim)$, where $\supseteq$ is ordinary set inclusion and $\{0\} \sim [0, 1]$. That said, the statement is true under the additional condition that the collection of atoms is pairwise-disjoint, which is the case whenever $\varnothing$ is the unique null event, and the original proof of Villegas suffices.

(ii) for each pair $s, s' \in I^\bullet$, $s < s'$ implies $\alpha_s \succsim \alpha_{s'}$; and (iii) for each $s \in I^\bullet$, there is $s^* \in I^\bullet$ such that (a) $s \sim s^*$, and (b) for each $s' \in I^\bullet$ such that $s' > s^*$, $\alpha_s \succ \alpha_{s'}$.

We define $S^\odot \equiv \bigcup \alpha_s$, $\mathcal{A}^\odot \equiv \{A \in \mathcal{A} | A \subseteq S^\odot\}$, $S^\circ \equiv S \setminus S^\odot$, and $\mathcal{A}^\circ \equiv \{A \in \mathcal{A} | A \subseteq S^\circ\}$. It is straightforward to verify that both $\mathcal{A}^\odot$ and $\mathcal{A}^\circ$ are σ-algebras. We frequently abuse language: when we say that $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, we implicitly take a particular choice of index set $I^\bullet$, together with the associated notation defined here, as given.

In an idealized space, $\varnothing$ is the unique null event. As we will see, any qualitative probability space that satisfies *monotone continuity* can be idealized by taking a particular kind of quotient with respect to the σ-ideal of its null events; hence the name.

We begin by introducing the notion of a structure-preserving map from one σ-algebra to another. We choose our definition statement to emphasize that we are using precisely the same notion as Halmos (1963), which is an outstanding reference for all concepts in this appendix:

Definition 12. Let $(\mathcal{A}, \supseteq)$ and $(\mathcal{A}', \supseteq')$ be σ-algebras, with $\cup$, $\cap$, and $\neg$ denoting, respectively, supremum, infimum, and complement for $(\mathcal{A}, \supseteq)$, and with $\cup'$, $\cap'$, and $\neg'$ denoting the respective operations for $(\mathcal{A}', \supseteq')$. A function $h : \mathcal{A} \to \mathcal{A}'$ is a *σ-homomorphism (with respect to $\supseteq$ and $\supseteq'$)* if and only if (i) for each pair $A, B \in \mathcal{A}$, $h(A \cup B) = h(A) \cup' h(B)$ and $h(A \cap B) = h(A) \cap' h(B)$; (ii) for each $A \in \mathcal{A}$, $h(\neg A) = \neg' h(A)$; and (iii) for each $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$, $h(\bigcup A_i) = \bigcup' h(A_i)$.

In other words, a σ-homomorphism preserves complements as well as suprema and infima of countable collections. Note that it need not preserve *all* suprema and infima, and in general will not.[18]

In order to use null events to take quotients of qualitative probability spaces, we first need to be able to use them to take quotients of σ-algebras. There is a standard way to do so, because the collection of null events has the following structure:

Definition 13. Let $(\mathcal{A}, \supseteq)$ be a σ-algebra. A collection $\mathcal{I} \subseteq \mathcal{A}$ is a *σ-ideal of* $\mathcal{A}$ if and only if (i) $\varnothing \in \mathcal{I}$; (ii) for each $A \in \mathcal{I}$ and each $B \subseteq A$, $B \in \mathcal{I}$; and (iii) for each countably-infinite collection $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{I}$, we have $\bigcup A_i \in \mathcal{I}$.

In this case, for each pair $A, B \in \mathcal{A}$, we define $A =^{\mathcal{I}} B$ if and only if $(A \setminus B) \cup (B \setminus A) \in \mathcal{I}$. For each $A \in \mathcal{A}$, we write $[A] \equiv \{B \in \mathcal{A} | A =^{\mathcal{I}} B\}$.

[18]This is a delicate point that bears further comment. A lattice that satisfies all the properties of a σ-algebra except for the requirement that each countably-infinite collection has both a supremum and an infimum is called a (*Boolean*) *algebra*; an algebra whose members are subsets of a state space and whose partial order is set inclusion is called a (*Boolean*) *set-algebra*; and a function from one algebra to another that satisfies only the first two conditions of a σ-homomorphism is called a (*Boolean*) *homomorphism*. Stone's Representation Theorem (Birkhoff 1935, Stone 1936) states that from each algebra (and in particular for our purposes, from each σ-algebra), there is a bijective homomorphism to a particular associated set-algebra. That said, there need not be any such σ-homomorphism: there is no guarantee that this promised homomorphism will preserve suprema and infima of countably-infinite collections, and in fact there are σ-algebras from which there are no bijective σ-homomorphisms to *any* set-σ-algebras. In this sense, the class of σ-algebras is indeed larger than the standard class of set-σ-algebras, even though the classes of algebras and set-algebras coincide; the additional generality is made explicit by the Loomis–Sikorski Theorem (Loomis 1947, Sikorski 1960).

It is straightforward to verify that for each σ-ideal $\mathcal{I}$, the relation $=^{\mathcal{I}}$ is an equivalence relation. We can therefore use a σ-ideal to take a quotient as follows:

Definition 14. If $\mathcal{I}$ is a σ-ideal of $\mathcal{A}$, then the *$\mathcal{I}$-quotient of $\mathcal{A}$*, $(\mathcal{A}^{\mathcal{I}}, \supseteq^{\mathcal{I}})$, is defined by (i) $\mathcal{A}^{\mathcal{I}} \equiv \{[A] | A \in \mathcal{A}\}$; and (ii) for each pair $[A], [B] \in \mathcal{A}^{\mathcal{I}}$, $[A] \supseteq^{\mathcal{I}} [B]$ if and only if there are $A' \in [A]$ and $B' \in [B]$ such that $A' \supseteq B'$.

It is a standard result that such a quotient is always a σ-algebra, and moreover that the ("natural" or "canonical") projection onto it from $\mathcal{A}$ is a σ-homomorphism:

Theorem H (Halmos 1963, Section 13). *If $(\mathcal{A}, \supseteq)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{A}$ is a σ-ideal of $\mathcal{A}$, then $(\mathcal{A}^{\mathcal{I}}, \supseteq^{\mathcal{I}})$ is a σ-algebra. Moreover, the* projection *$h : \mathcal{A} \to \mathcal{A}^{\mathcal{I}}$, defined such that for each $A \in \mathcal{A}$, $h(A) = [A]$, is a σ-homomorphism.*

For our purposes, we take the collection of null events as our σ-ideal; two events are then related to one another if and only if their symmetric difference is null, or equivalently for qualitative probability spaces, if and only if their union is as likely as their intersection. It is straightforward to verify that for any qualitative probability space that satisfies *monotone continuity*, this is indeed a σ-ideal; thus we can define likelihood comparisons for the associated quotient in the natural way:

Definition 15. If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space that satisfies *monotone continuity* and if $\mathcal{I} = \{A \in \mathcal{A} | A \sim \varnothing\}$, then the *null-quotient of* $(\mathcal{A}, \supseteq, \succsim)$ is $(\mathcal{A}^{\mathcal{I}}, \supseteq^{\mathcal{I}}, \succsim^{\mathcal{I}})$, where for each pair $[A], [B] \in \mathcal{A}^{\mathcal{I}}$, $[A] \succsim^{\mathcal{I}} [B]$ if and only if there are $A' \in [A]$ and $B' \in [B]$ such that $A' \succsim B'$.

The Null-Quotient Lemma (Lemma 8) states that the null-quotient of a parent space is an idealized space that inherits all properties we consider in this article, and that moreover σ-measure representations flow from child to parent as well as from parent to child. We can therefore exploit the considerable convenience afforded by focusing on idealized spaces throughout many of our upcoming proofs.

We remark that in order to pursue this approach, it has been necessary to consider σ-algebras that need not be set-σ-algebras precisely because Theorem H guarantees only that a quotient by a σ-ideal is a σ-algebra, not that it is a set-σ-algebra. Indeed, there is no such theorem: if $\mathcal{A}$ is the collection of Borel sets contained in the interval $[0, 1]$ and $\mathcal{I}$ is the σ-ideal consisting of all sets in $\mathcal{A}$ with Borel measure zero, then the quotient is a σ-algebra from which there is no bijective σ-homomorphism to any set-σ-algebra (see Halmos 1963, Section 23 for other examples).

Lemma 8 (Null-Quotient Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space that satisfies monotone continuity and $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is its null quotient, then*

- *the following basic observations hold:*
  - *(i) $[\varnothing]$ is the minimum of $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ and $[S]$ is its maximum;*

(ii) for each pair $A, A' \in \mathcal{A}$ such that $A' \in [A]$, we have $A \sim A'$;

(iii) for each pair $A, B \in \mathcal{A}$, $A \succsim B$ if and only if $[A] \succsim^* [B]$; and

(iv) for each $A \in \mathcal{A}$, $A \in \mathcal{A}^\bullet$ if and only if $[A] \in (\mathcal{A}^*)^\bullet$;

- *$(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space;*
- *if $(\mathcal{A}, \supseteq, \succsim)$ satisfies unlikely atoms, then $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ satisfies unlikely atoms;*
- *for each $n \in \mathbb{N}$, if $(\mathcal{A}, \supseteq, \succsim)$ satisfies n-AS, then $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ satisfies n-AS;*
- *if $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$ represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$, then there is $\mu : \mathcal{A} \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu(A) = \mu^*([A])$, and moreover, $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ represents $(\mathcal{A}, \supseteq, \succsim)$;*
- *if $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ represents $(\mathcal{A}, \supseteq, \succsim)$, then there is $\mu^* : \mathcal{A}^* \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu^*([A]) = \mu(A)$, and moreover, $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$ represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$.*

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ be a qualitative probability space that satisfies *monotone continuity* and let $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ be its null-quotient. It is straightforward to verify that the collection of null events in $\mathcal{A}$ is a σ-ideal; thus by Theorem H, $(\mathcal{A}^*, \supseteq^*)$ is a σ-algebra and the projection from $\mathcal{A}$ to $\mathcal{A}^*$ is a σ-homomorphism. Let $\cup^*$, $\cap^*$, and $\neg^*$ denote, respectively, supremum, infimum, and complement for $(\mathcal{A}^*, \supseteq^*)$. Let $\varnothing^*$ denote its minimum, and for each pair $[A], [B] \in \mathcal{A}^*$, let $[A] \setminus^* [B]$ denote $[A] \cap^* (\neg^*[B])$.

For brevity, we refer to properties (such as *order* and *unlikely atoms*) without explicitly specifying which qualitative probability space the property belongs to, as this will always be clear from the context. The rest of the proof consists of six steps, each establishing the corresponding part of the lemma's statement.

Step 1: *Basic observations, to be used freely throughout the remaining steps.* To see (i), let $A \in \mathcal{A}$. As $\varnothing \subseteq A \subseteq S$, thus $[\varnothing] \subseteq^* [A] \subseteq^* [S]$. Necessarily, then, $[\varnothing]$ is the minimum of $\mathcal{A}^*$ and $[S]$ is the maximum of $\mathcal{A}^*$. Though $[\varnothing] = \varnothing^*$, we will use both $[\varnothing]$ and $\varnothing^*$ to emphasize particular arguments.

To see (ii), let $A, A' \in \mathcal{A}$ such that $A' \in [A]$. Then $(A \setminus A') \cup (A' \setminus A) \sim \varnothing$, which by *monotonicity* and the Domination Lemma (Lemma 4) implies $A \sim A \cap A' \sim A'$.

To see (iii), let $A, B \in \mathcal{A}$. If $A \succsim B$, then by construction $[A] \succsim^* [B]$. If $[A] \succsim^* [B]$, then there are $A' \in [A]$ and $B' \in [B]$ such that $A' \succsim B'$; thus $A \sim A' \succsim B' \sim B$.

To see (iv), first let $A \in \mathcal{A}$ such that $A \in \mathcal{A}^\bullet$. Then $A \succ \varnothing$, so $[A] \succ^* [\varnothing] = \varnothing^*$. For each $[B] \in \mathcal{A}^*$ such that $[B] \subseteq^* [A]$, there are $A' \in [A]$ and $B' \in [B]$ such that $B' \subseteq A'$. Since $(B' \setminus A) \subseteq (A' \setminus A) \subseteq (A \setminus A') \cup (A' \setminus A) \sim \varnothing$, since $(B' \cap A) \subseteq A$, and since $A \in \mathcal{A}^\bullet$, thus by *monotonicity* and the Domination Lemma (Lemma 4), $B' \sim A$ or $B' \sim \varnothing$, so $[B] \sim^* [A]$ or $[B] \sim^* [\varnothing] = \varnothing^*$. Thus $[A] \in (\mathcal{A}^*)^\bullet$.

To complete the proof of (iv), let $A \in \mathcal{A}$ such that $[A] \in (\mathcal{A}^*)^\bullet$. Then $[A] \succ^* \varnothing^* = [\varnothing]$, so $A \succ \varnothing$. For each $B \in \mathcal{A}$ such that $B \subseteq A$, we have $[B] \subseteq^* [A]$, so as $[A] \in (\mathcal{A}^*)^\bullet$, thus $[B] \sim^* [A]$ or $[B] \sim^* \varnothing^* = [\varnothing]$, so $B \sim A$ or $B \sim \varnothing$. Thus $A \in \mathcal{A}^\bullet$.

Step 2: $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ *is an idealized space.* To see that $\varnothing^*$ is the unique null event, let $[A] \in \mathcal{A}^*$ such that $[A] \sim^* \varnothing^*$. Since $\varnothing^* = [\varnothing]$, thus $A \sim \varnothing$, so $(A \setminus \varnothing) \cup (\varnothing \setminus A) = A \sim \varnothing$, so $A \in [\varnothing] = \varnothing^*$, so $[A] = \varnothing^*$.

To see that $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ satisfies *order*, first let $[A], [B] \in \mathcal{A}^*$. Then by *order*, either $A \succsim B$ or $B \succsim A$, so either $[A] \succsim^* [B]$ or $[B] \succsim^* [A]$, as desired. Second, let $[A], [B], [C] \in \mathcal{A}^*$ such that $[A] \succsim^* [B]$ and $[B] \succsim^* [C]$. Then $A \succsim B$ and $B \succsim C$, so by *order*, $A \succsim C$, so $[A] \succsim^* [C]$.

To see that it satisfies *separability*, let $[A], [B], [C] \in \mathcal{A}^*$ such that $[A] \cap^* [C] = [B] \cap^* [C] = \varnothing^*$. Then by Theorem H, $[A \cap C] = [A] \cap^* [C] = \varnothing^* = [\varnothing]$ and $[B \cap C] = [B] \cap^* [C] = \varnothing^* = [\varnothing]$, so $A \cap C \sim \varnothing$ and $B \cap C \sim \varnothing$. Define $A' \equiv A \setminus C$, $B' \equiv B \setminus C$, and $C' \equiv C \setminus (A \cup B)$. It follows from *monotonicity* and the Domination Lemma (Lemma 4) that $A \sim A'$, $B \sim B'$, $C \sim C'$, $A \cup C \sim A' \cup C'$, and $B \cup C \sim B' \cup C'$. Since $A' \cap C' = \varnothing$ and $B' \cap C' = \varnothing$, thus by *separability*, $A' \succsim B'$ if and only if $A' \cup C' \succsim B' \cup C'$; altogether, then, $A \succsim B$ if and only if $A \cup C \succsim B \cup C$. First, if $[A] \succsim^* [B]$, then $A \succsim B$, so $A \cup C \succsim B \cup C$, so $[A \cup C] \succsim^* [B \cup C]$, so by Theorem H, we have $[A] \cup^* [C] \succsim^* [B] \cup^* [C]$. Second, if $[A] \cup^* [C] \succsim^* [B] \cup^* [C]$, then by Theorem H, we have $[A \cup C] \succsim^* [B \cup C]$, so $A \cup C \succsim B \cup C$, so $A \succsim B$, so $[A] \succsim^* [B]$.

To see that it satisfies *monotonicity*, let $[A], [B] \in \mathcal{A}^*$ such that $[A] \subseteq^* [B]$. Then there are $A' \in [A]$ and $B' \in [B]$ such that $A' \subseteq B'$, so by *monotonicity*, we have $B' \succsim A'$, so $[B] \succsim^* [A]$.

To see that it satisfies *nondegeneracy*, by *nondegeneracy* there are $A, B \in \mathcal{A}$ such that $A \succ B$, so $[A] \succ [B]$.

To see that it satisfies *monotone continuity*, let $[A] \in \mathcal{A}^*$ and let $([B_i]) \in (\mathcal{A}^*)^{\mathbb{N}}$ such that (i) $[B_1] \supseteq^* [B_2] \supseteq^* \cdots$; and (ii) for each $i \in \mathbb{N}$, $[B_i] \succsim^* [A]$. Then for each $i \in \mathbb{N}$, there are B_i^i, B_{i+1}^i such that $B_i^i \in [B_i]$, $B_{i+1}^i \in [B_{i+1}]$, and $B_i^i \supseteq B_{i+1}^i$. For each $i \in \mathbb{N}$, define $N_i \equiv (B_{i+1}^{i+1} \setminus B_{i+1}^i) \cup (B_{i+1}^i \setminus B_{i+1}^{i+1})$; as $[B_{i+1}^i] = [B_{i+1}^{i+1}]$, thus $N_i \sim \varnothing$. Define $N \equiv \bigcup N_i$; by *monotonicity*, the Domination Lemma (Lemma 4), and the Limit-Order Lemma (Lemma 6), $N \sim \varnothing$. For each $i \in \mathbb{N}$, define $B_i' \equiv B_i^i \setminus N$; by *monotonicity* and the Domination Lemma (Lemma 4), $(B_i' \setminus B_i^i) \cup (B_i^i \setminus B_i') \sim \varnothing$, so $[B_i'] = [B_i^i] = [B_i]$ and thus $B_i' \sim B_i \succsim A$. Moreover, for each $i \in \mathbb{N}$,

$$\begin{aligned} B_i' &= B_i^i \setminus N \\ &\supseteq B_{i+1}^i \setminus N \\ &= \left(B_{i+1}^{i+1} \cap B_{i+1}^i\right) \setminus N \\ &= B_{i+1}^{i+1} \setminus N \\ &= B_{i+1}'. \end{aligned}$$

Thus $B_1' \supseteq B_2' \supseteq \cdots$, so by *monotone continuity*, we have $\bigcap B_i' \succsim A$, so $[\bigcap B_i'] \succsim^* [A]$. Altogether, by Theorem H, we have $\bigcap^*[B_i] = \bigcap^*[B_i'] = [\bigcap B_i'] \succsim^* [A]$.

Step 3: $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ *inherits unlikely atoms from* $(\mathcal{A}, \supseteq, \succsim)$. Assume $(\mathcal{A}, \supseteq, \succsim)$ satisfies *unlikely atoms*. Then there is $A \in \mathcal{A}$ such that (i) $A \succsim S \setminus A$; and (ii) for each $\alpha \in \mathcal{A}^\bullet$, $\alpha \nsubseteq A$. By Theorem H, $[A] \succsim^* [S \setminus A] = [\neg A] = \neg^*[A]$. Moreover, for each $[B] \in (\mathcal{A}^*)$ such that $[B] \subseteq [A]$, there is $B' \in \mathcal{A}$ such that $B' \subseteq A$, so $B' \notin \mathcal{A}^\bullet$, so $[B] = [B'] \notin (\mathcal{A}^*)^\bullet$.

Step 4: *For each* $n \in \mathbb{N}$, $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ *inherits* n*-AS from* $(\mathcal{A}, \supseteq, \succsim)$. Let $n \in \mathbb{N}$, assume $(\mathcal{A}, \supseteq, \succsim)$ satisfies n*-AS*, and let $[A] \in (\mathcal{A}^*)^\bullet$. Then $A \in \mathcal{A}^\bullet$, so by n*-AS*, there are $I \subseteq \mathbb{N}$, pairwise-disjoint $\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$, and $I_1, I_2, \ldots, I_n$ partitioning I such that (i) for each $i \in I$, $A \succ B_i$; and (ii) for each $j \in \{1, 2, \ldots, n\}$, $\bigcup_{I_j} B_i \succsim A$.

By Theorem H, for each pair $i, j \in I$, $[B_i] \cap^* [B_j] = [B_i \cap B_j] = [\varnothing] = \varnothing^*$, so $\{[B_i]\}_{i\in I}$ is pairwise-disjoint. Moreover, for each $i \in I$, $[A] \succ^* [B_i]$. Finally, by Theorem H, we have $\bigcup^*_{I_j}[B_i] = [\bigcup_{I_j} B_i] \succsim^* [A]$. Altogether, then, I, $\{[B_i]\}_{i\in I}$, and $I_1, I_2, \ldots, I_n$ are as desired.

STEP 5: *Given σ-measure representation μ^*, we can produce σ-measure representation μ as promised.* Assume $(\mathcal{A}^*, \supseteq^* \succsim^*)$ has representation $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$, and define $\mu : \mathcal{A} \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu(A) = \mu^*([A])$. We first prove that $\mu \in \mathbb{M}^\sigma(\mathcal{A})$, then prove that μ represents $(\mathcal{A}, \supseteq, \succsim)$.

To see that $\mu \in \mathbb{M}^\sigma(\mathcal{A})$, first note that by construction $[S]$ is the maximum of $\mathcal{A}^*$, so $\mu^*([S]) = 1$, so $\mu(S) = 1$. Second, let $I \subseteq \mathbb{N}$ and let $\{A_i\}_{i\in I} \subseteq \mathcal{A}$ be pairwise-disjoint. By Theorem H, for each pair $i, j \in I$ such that $i \neq j$, $[A_i] \cap^* [A_j] = [A_i \cap A_j] = [\varnothing] = \varnothing^*$. Then $\{[A_i]\}_{i\in I}$ is pairwise-disjoint, so by Theorem H,

$$
\begin{aligned}
\mu\Bigl(\bigcup A_i\Bigr) &= \mu^*\Bigl(\Bigl[\bigcup A_i\Bigr]\Bigr) \\
&= \mu^*\Bigl(\bigcup^* [A_i]\Bigr) \\
&= \sum \mu^*([A_i]) \\
&= \sum \mu(A_i),
\end{aligned}
$$

as desired.

To see that μ represents $(\mathcal{A}, \supseteq, \succsim)$, let $A, B \in \mathcal{A}$. First, $A \succsim B$ implies $[A] \succsim^* [B]$, which implies $\mu^*([A]) \geq \mu^*([B])$, which implies $\mu(A) \geq \mu(B)$. Second, $\mu(A) \geq \mu(B)$ implies $\mu^*([A]) \geq \mu^*([B])$, which implies $[A] \succsim^* [B]$, which implies $A \succsim B$.

STEP 6: *Given σ-measure representation μ, we can produce σ-measure representation μ^* as promised.* Assume $(A, \supseteq, \succsim)$ has representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$, and define $\mu^* : \mathcal{A} \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu^*([A]) = \mu(A)$; as μ is a representation of $(\mathcal{A}, \supseteq, \succsim)$, this is well-defined. We first prove that $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$, then prove that μ^* represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$.

To see that $\mu^* \in \mathbb{M}^\sigma(\mathcal{A}^*)$, first note that $\mu^*([S]) = \mu(S) = 1$. Second, let $I \subseteq \mathbb{N}$ and let $\{[A_i]\}_{i\in I} \subseteq \mathcal{A}^*$ be pairwise-disjoint. By Theorem H, for each pair $i, j \in I$ such that $i \neq j$, we have $[A_i \cap A_j] = [A_i] \cap^* [A_j] = \varnothing^* = [\varnothing]$, so $A_i \cap A_j \sim \varnothing$. Define $N \in \mathcal{A}$ by

$$
N \equiv \bigcup_{i,j\in I | i\neq j} (A_i \cap A_j).
$$

By *monotonicity*, the Domination Lemma (Lemma 4), and the Limit-Order Lemma (Lemma 6), $N \sim \varnothing$. For each $i \in \mathbb{N}$, define $B_i \equiv A_i \setminus N$; by *monotonicity* and the Domination Lemma (Lemma 4), $(B_i \setminus A_i) \cup (A_i \setminus B_i) \sim \varnothing$, so $[B_i] = [A_i]$ and thus $B_i \sim A_i$. By construction, $\{B_i\}_{i\in I}$ is pairwise-disjoint, so by Theorem H,

$$\begin{aligned}
\mu^*\Big(\bigcup\nolimits^*[A_i]\Big) &= \mu^*\Big(\bigcup\nolimits^*[B_i]\Big) \\
&= \mu^*\Big(\Big[\bigcup B_i\Big]\Big) \\
&= \mu\Big(\bigcup B_i\Big) \\
&= \sum \mu(B_i) \\
&= \sum \mu^*\big([B_i]\big) \\
&= \sum \mu^*\big([A_i]\big),
\end{aligned}$$

as desired.

To see that μ^* represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$, let $[A], [B] \in \mathcal{A}^*$. First, $[A] \succsim^* [B]$ implies $A \succsim B$, which implies $\mu(A) \geq \mu(B)$, which implies $\mu^*([A]) \geq \mu^*([B])$. Second, $\mu^*([A]) \geq \mu^*([B])$ implies $\mu(A) \geq \mu(B)$, which implies $A \succsim B$, which implies $[A] \succsim^* [B]$. □

Appendix C

In this appendix we prove our first main result, which implies that *monotone continuity* and *unlikely atoms* are together sufficient to guarantee σ-measure representation. We first exploit the convenience of idealized spaces to prove the following:

Proposition 1. *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space that satisfies unlikely atoms, then it has representation $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$. In this case, $(\mathcal{A}, \supseteq, \succsim)$ has no other σ-measure representation.*

Proof. By *unlikely atoms* and *monotonicity*, $S^\circ \succsim S \setminus S^\circ = S^\odot$, and by *nondegeneracy* and *monotonicity*, $S = S^\odot \cup S^\circ \succ \varnothing$; thus by *monotonicity* and the Domination Lemma (Lemma 4), $S^\circ \succ \varnothing$.

Step 1: *Define $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, $\mu^\odot : \mathcal{A}^\odot \to [0, 1]$, and $\mu : \mathcal{A} \to [0, 1]$.* Define $\mathcal{A}^\circ \equiv \{A \in \mathcal{A} | A \subseteq S^\circ\}$; it is straightforward to show that this is a σ-algebra. Define $\supseteq^\circ$, $\succsim^\circ$ on $\mathcal{A}^\circ$ as follows: for each pair $A, B \in \mathcal{A}^\circ$, (i) $A \supseteq^\circ B$ if and only if $A \supseteq B$, and (ii) $A \succsim^\circ B$ if and only if $A \succsim B$. Since $S^\circ \succ \varnothing$, it is straightforward to verify that $(\mathcal{A}^\circ, \supseteq^\circ, \succsim^\circ)$ is a qualitative probability space that satisfies *monotone continuity* and *no atoms*; thus by Theorem V2, there is a unique $\mu^\circ \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^\circ)$ such that for each pair $A, B \in \mathcal{A}^\circ$, $\mu^\circ(A) \geq \mu^\circ(B)$ if and only if $A \succsim^\circ B$ if and only if $A \succsim B$. To be more concise, we say that μ° represents comparisons on $\mathcal{A}^\circ$.

If $\mathcal{A}^\bullet = \varnothing$, then define $\mu = \mu^\circ$ and we are done; thus let us assume $\mathcal{A}^\bullet \neq \varnothing$. To define $\mu^\odot : \mathcal{A}^\odot \to [0, 1]$, let $A \in \mathcal{A}^\odot$. By *monotonicity*, $S^\circ \succsim S^\odot \succsim A$; thus by the Carving Lemma (Lemma 7), there is $A^\circ \in \mathcal{A}^\circ$ such that $A \sim A^\circ$. Define $\mu^\odot(A) \equiv \mu^\circ(A^\circ)$; as μ° represents comparisons on $\mathcal{A}^\circ$, this is well-defined. For each pair $A, B \in \mathcal{A}^\odot$, by the previous argument there are $A^\circ, B^\circ \in \mathcal{A}^\circ$ such that $A \sim A^\circ$ and $B \sim B^\circ$, so as μ° represents comparisons on $\mathcal{A}^\circ$, thus $A \succsim B$ if and only if $A^\circ \succsim B^\circ$ if and only if $\mu^\circ(A^\circ) \geq \mu^\circ(B^\circ)$ if and only if $\mu^\odot(A) \geq \mu^\odot(B)$. Thus $\mu^\odot$ represents comparisons on $\mathcal{A}^\odot$.

For each $A \in \mathcal{A}$, define

$$\mu(A) \equiv \frac{\mu^\odot(A \cap S^\odot) + \mu^\circ(A \cap S^\circ)}{\mu^\odot(S^\odot) + \mu^\circ(S^\circ)};$$

as $\mu^\odot(S^\odot) \geq 0$ and $\mu^\circ(S^\circ) = 1$, this is well-defined.

Step 2: *For each disjoint pair $A^\odot, B^\odot \in \mathcal{A}^\odot$, there are disjoint $A^\circ, B^\circ \in \mathcal{A}^\circ$ such that $A^\odot \sim A^\circ$ and $B^\odot \sim B^\circ$.* Let $A^\odot, B^\odot \in \mathcal{A}^\odot$ be disjoint. By *monotonicity*, $S^\circ \succsim S^\odot \succsim A^\odot$; thus by the Carving Lemma (Lemma 7), there is $A^\circ \in \mathcal{A}^\circ$ such that $A^\odot \sim A^\circ$. By *separability*, $A^\odot \cup B^\odot \sim A^\circ \cup B^\odot$.

Define $C^\circ \equiv S^\circ \setminus A^\circ$. By *monotonicity* and the above observation,

$$\begin{aligned} A^\circ \cup C^\circ &= S^\circ \\ &\succsim S^\odot \\ &\succsim A^\odot \cup B^\odot \\ &\sim A^\circ \cup B^\odot, \end{aligned}$$

so by *separability*, $C^\circ \succsim B^\odot$. Thus by the Carving Lemma (Lemma 7), there is $B^\circ \in \mathcal{A}^\circ$ such that $B^\circ \subseteq C^\circ$ and $B^\odot \sim B^\circ$. By construction, A° and B° are disjoint, as desired.

Step 3: $\mu \in \mathbb{M}(\mathcal{A})$. By construction, $\mu(S) = 1$, and for each $A \in \mathcal{A}$, $\mu(A) \in [0, 1]$. Let $A, B \in \mathcal{A}$ be disjoint, and define $A^\odot \equiv A \cap S^\odot$, $A^\circ \equiv A \cap S^\circ$, $B^\odot \equiv B \cap S^\odot$, and $B^\circ \equiv B \cap S^\circ$. By definition,

$$\begin{aligned} \mu(A \cup B) &= \frac{\mu^\odot(A^\odot \cup B^\odot) + \mu^\circ(A^\circ \cup B^\circ)}{\mu^\odot(S^\odot) + \mu^\circ(S^\circ)}, \\ \mu(A) &= \frac{\mu^\odot(A^\odot) + \mu^\circ(A^\circ)}{\mu^\odot(S^\odot) + \mu^\circ(S^\circ)}, \text{ and} \\ \mu(B) &= \frac{\mu^\odot(B^\odot) + \mu^\circ(B^\circ)}{\mu^\odot(S^\odot) + \mu^\circ(S^\circ)}. \end{aligned}$$

Since $A^\odot$ and $B^\odot$ are disjoint, thus by Step 2, there are disjoint $A', B' \in \mathcal{A}^\circ$ such that $A^\odot \sim A'$ and $B^\odot \sim B'$. By two applications of the Domination Lemma (Lemma 4), $A^\odot \cup B^\odot \sim A' \cup B'$. As $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, thus

$$\begin{aligned} \mu^\odot(A^\odot \cup B^\odot) &= \mu^\circ(A' \cup B') \\ &= \mu^\circ(A') + \mu^\circ(B') \\ &= \mu^\odot(A^\odot) + \mu^\odot(B^\odot). \end{aligned}$$

Moreover, since $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, thus $\mu^\circ(A^\circ \cup B^\circ) = \mu^\circ(A^\circ) + \mu^\circ(B^\circ)$. Altogether, then, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Since $A, B \in \mathcal{A}$ were an arbitrary disjoint pair, thus by induction, $\mu \in \mathbb{M}(\mathcal{A})$.

STEP 4: *Conclude.* We first claim that μ represents comparisons on $\mathcal{A}$. Indeed, let $A, B \in \mathcal{A}$. We proceed by constructing a list of event pairs $(A, B) = (A_1, B_1), (A_2, B_2), \dots, (A_\Omega, B_\Omega)$ such that

(i) for each $i \in \{1, 2, \dots, \Omega - 1\}$, $A_i \succsim B_i$ if and only if $A_{i+1} \succsim B_{i+1}$;

(ii) for each $i \in \{1, 2, \dots, \Omega - 1\}$, $\mu(A_i) \geq \mu(B_i)$ if and only if $\mu(A_{i+1}) \geq \mu(B_{i+1})$; and

(iii) $A_\Omega \succsim B_\Omega$ if and only if $\mu(A_\Omega) \geq \mu(B_\Omega)$.

This immediately establishes that $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$; see Figure 2 in Section 3.2.

Let $i \in \mathbb{N}$ and let $(A_i, B_i) \in \mathcal{A} \times \mathcal{A}$. Given another pair $(A^*, B^*) \in \mathcal{A} \times \mathcal{A}$, let us say that we can continue with A^* and B^* as shorthand for the conjunction of (i) $A_i \succsim B_i$ if and only if $A^* \succsim B^*$, and (ii) $\mu(A_i) \geq \mu(B_i)$ if and only if $\mu(A^*) \geq \mu(B^*)$. We describe a procedure that either constructs (A_{i+1}, B_{i+1}) with which we can continue, or else terminates by declaring $\Omega \equiv i$ and establishing $A_\Omega \succsim B_\Omega$ if and only if $\mu(A_\Omega) \geq \mu(B_\Omega)$. We then observe that the iterative application of this procedure to an initial pair (A_1, B_1) must eventually lead to termination.

First, define the four events

$$
\begin{aligned}
A_i^\circ &\equiv A_i \cap S^\circ, \\
A_i^\odot &\equiv A_i \cap S^\odot, \\
B_i^\circ &\equiv B_i \cap S^\circ, \text{ and} \\
B_i^\odot &\equiv B_i \cap S^\odot.
\end{aligned}
$$

Moreover, define $n_i^\circ, n_i^\odot \in \{0, 1, 2\}$ by

$$
\begin{aligned}
n_i^\circ &\equiv \left|\left\{C \in \left\{A_i^\circ, B_i^\circ\right\} | C \neq \varnothing\right\}\right|, \text{ and} \\
n_i^\odot &\equiv \left|\left\{C \in \left\{A_i^\odot, B_i^\odot\right\} | C \neq \varnothing\right\}\right|.
\end{aligned}
$$

Given A_i, B_i, the procedure is to follow the earliest instruction that applies:

I1. If $n_i^\circ = 0$ or $n_i^\odot = 0$, then terminate.

Since $\mu^\odot$ represents comparisons on $\mathcal{A}^\odot$ and μ° represents comparisons on $\mathcal{A}^\circ$, thus by construction, $A_i \succsim B_i$ if and only if $\mu(A_i) \geq \mu(B_i)$.

I2. If $A_i = \varnothing$ or $B_i = \varnothing$, then terminate.

First, consider the case where $A_i = \varnothing$. If $\varnothing = A_i \succsim B_i$, then by *monotonicity*, $\varnothing \sim B_i^\odot$ and $\varnothing \sim B_i^\circ$. Since $\varnothing \in \mathcal{A}^\odot$, where comparisons are represented by $\mu^\odot$, thus $0 = \mu^\odot(\varnothing) = \mu^\odot(B_i^\odot)$. Since $\varnothing \in \mathcal{A}^\circ$, where comparisons are represented by μ°, thus $0 = \mu^\circ(\varnothing) = \mu^\circ(B_i^\circ)$. By construction, $\mu(A_i) = 0 = \mu(B_i)$, as desired. Conversely, if $0 = \mu(A_i) \geq \mu(B_i)$, then by construction $\mu^\odot(\varnothing) = 0 = \mu^\odot(B_i^\odot)$ and $\mu^\circ(\varnothing) = 0 = \mu^\circ(B_i^\circ)$. Since $\varnothing \in \mathcal{A}^\odot$, where comparisons are represented by $\mu^\odot$, thus $\varnothing \sim B_i^{\odot}$. Since $\varnothing \in \mathcal{A}^\circ$, where comparisons are represented by μ°, thus $\varnothing \sim B_i^\circ$. By two applications of the Domination Lemma (Lemma 4), $A_i = \varnothing \sim B_i^{\odot} \cup B_i^\circ = B_i$, as desired.

Second, consider the case where $B_i = \varnothing$. By *monotonicity*, $A_i \succsim B_i$, and by construction, $\mu(A_i) \geq 0 = \mu(B_i)$. Thus we are done.

I3. If $n_i^\circ = 1$ and $n_i^{\odot} = 1$, and neither A_i nor B_i is $\varnothing$, then terminate.

First, consider the case where $A \in \mathcal{A}^{\odot}$ and $B \in \mathcal{A}^\circ$. By *monotonicity*, $S^\circ \succsim S^{\odot} \succsim A$, so by the Carving Lemma (Lemma 7), there is $A^\circ \subseteq S^\circ$ such that $A^\circ \sim A$. If $A \succsim B$, then by construction and by the representation of comparisons on $\mathcal{A}^\circ$ by μ°, $\mu^{\odot}(A) = \mu^\circ(A^\circ) \geq \mu^\circ(B)$, so by construction, $\mu(A) \geq \mu(B)$, as desired. Conversely, if $\mu(A) \geq \mu(B)$, then by construction, $\mu^\circ(A^\circ) = \mu^{\odot}(A) \geq \mu^\circ(B)$, so by the representation of comparisons on $\mathcal{A}^\circ$ by μ°, $A \sim A^\circ \succsim B$, as desired.

Second, consider the case where $A \in \mathcal{A}^\circ$ and $B \in \mathcal{A}^{\odot}$. By *monotonicity*, $S^\circ \succsim S^{\odot} \succsim B$, so by the Carving Lemma (Lemma 7), there is $B^\circ \subseteq S^\circ$ such that $B^\circ \sim B$. If $A \succsim B$, then by construction and by the representation of comparisons on $\mathcal{A}^\circ$ by μ°, $\mu^\circ(A) \geq \mu^\circ(B^\circ) = \mu^{\odot}(B)$, so by construction, $\mu(A) \geq \mu(B)$, as desired. Conversely, if $\mu(A) \geq \mu(B)$, then by construction, $\mu^\circ(A) \geq \mu^{\odot}(B) = \mu^\circ(B^\circ)$, so by the representation of comparisons on $\mathcal{A}^\circ$ by μ°, $A \succsim B^\circ \sim B$, as desired.

I4. If $n_i^\circ = 2$, then there are two cases.

If $A_i^\circ \succsim B_i^\circ$, then by the Carving Lemma (Lemma 7), there is $C^\circ \subseteq A_i^\circ$ such that $C^\circ \sim B_i^\circ$. Define $B_i' \equiv B^{\odot} \cup C^\circ$. By *separability*, by the representation of comparisons on $\mathcal{A}^\circ$ by μ°, and by construction, we can continue with A_i and B_i'. Define $A_{i+1} \equiv A_i \setminus C^\circ$ and $B_{i+1} \equiv B_i' \setminus C^\circ$. By *separability* and additivity of μ, we can continue with A_{i+1} and B_{i+1}.

If $B_i^\circ \succsim A_i^\circ$, then perform the operation analogous to that in the previous case.

In both cases, $n_{i+1}^\circ \in \{0, 1\}$ and $n_{i+1}^{\odot} = n_i^{\odot}$.

I5. If $n_i^\circ = 1$ and $n_i^{\odot} = 2$, then there are two cases.

If $A_i^\circ = \varnothing$, then by *monotonicity*, $S^\circ \succsim S^{\odot} \succsim A_i^{\odot} = A_i$, so by the Carving Lemma (Lemma 7) there is $A^\circ \in \mathcal{A}^\circ$ such that $A^\circ \sim A_i$. Define $A_{i+1} \equiv A^\circ$ and define $B_{i+1} \equiv B_i$. By construction, we can continue with A_{i+1} and B_{i+1}.

If $B_i^\circ = \varnothing$, then perform the operation analogous to that in the previous case.

In both cases, $n_i^\circ = 2$ and $n_{i+1}^{\odot} = 1$.

Define $A_1 \equiv A$ and $B_1 \equiv B$, then repeatedly apply the above procedure. As the associated sequence $(n_i^{\odot}, n_i^\circ)$ decreases lexicographically, thus after a finite number of iterations, the procedure terminates. (In fact, there will be at most four pairs of events in total.) By the argument illustrated in Figure 2, $A \succsim B$ if and only if $\mu(A) \geq \mu(B)$.

As $A, B \in \mathcal{A}$ were arbitrary, thus μ represents comparisons on $\mathcal{A}$. Moreover, by Step 3, $\mu \in \mathbb{M}(\mathcal{A})$, so by Theorem V1, $\mu \in \mathbb{M}^\sigma(\mathcal{A})$. It is straightforward to verify that any such σ-measure must be compatible with the construction in Step 1, and thus that μ is the unique such σ-measure.

To conclude, since $S^\circ \succsim S \setminus S^\circ$, thus $\mu(S^\circ) \geq \frac{1}{2}$. Since μ is a representation, it is straightforward to show that $\mathcal{A}^{\bullet|\mu} = \mathcal{A}^\bullet$; thus S° contains no members of $\mathcal{A}^\bullet = \mathcal{A}^{\bullet|\mu}$. Altogether, then, $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$, as desired. □

To establish our first main result, we use the tools from the previous appendix to prove that focusing on idealized spaces in Proposition 1 was without loss of generality:

Theorem 1 (Repeated). *A qualitative probability space* $(\mathcal{A}, \supseteq, \succsim)$ *satisfies monotone continuity and unlikely atoms if and only if it has representation* $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$. *In this case,* $(\mathcal{A}, \supseteq, \succsim)$ *has no other* σ*-measure representation.*

Proof. It is straightforward to verify that if $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space with representation $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$, then it satisfies *monotone continuity* and *unlikely atoms*; we therefore omit this proof.

Let $(\mathcal{A}, \supseteq, \succsim)$ be a qualitative probability space and let $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ be its null-quotient. Let $\cup^*$, $\cap^*$, and $\neg^*$ denote, respectively, supremum, infimum, and complement for $(\mathcal{A}^*, \supseteq^*)$. Let $\varnothing^*$ denote its minimum, and for each pair $[A], [B] \in \mathcal{A}^*$, let $[A] \setminus^* [B]$ denote $[A] \cap^* (\neg^*[B])$.

Assume $(\mathcal{A}, \supseteq, \succsim)$ satisfies *monotone continuity* and *unlikely atoms*. By the Null-Quotient Lemma (Lemma 8), $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space that satisfies *unlikely atoms*, so by Proposition 1, it has representation $\mu^* \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A}^*)$; it is straightforward to show that $(\mathcal{A}^*)^\bullet = (\mathcal{A}^*)^{\bullet|\mu^*}$. Define $\mu : \mathcal{A} \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu(A) = \mu^*([A])$. By the Null-Quotient Lemma (Lemma 8), $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ and μ represents $(\mathcal{A}, \supseteq, \succsim)$; it is straightforward to show that $\mathcal{A}^\bullet = \mathcal{A}^{\bullet|\mu}$.

We claim that $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$. Indeed, by *unlikely atoms*, there is $A \in \mathcal{A}$ such that (i) $A \succsim S \setminus A$; and (ii) for each $\alpha \in \mathcal{A}^\bullet$, $\alpha \nsubseteq A$; thus (i) $\mu(A) \geq \frac{1}{2}$; and (ii) for each $\alpha \in \mathcal{A}^{\bullet|\mu} = \mathcal{A}^\bullet$, $\alpha \nsubseteq A$; so $\mu \in \mathbb{M}^\sigma_{\mathrm{UA}}(\mathcal{A})$, as claimed.

For uniqueness, let $\mu' \in \mathbb{M}^\sigma$ be a representation of $(\mathcal{A}, \supseteq, \succsim)$, and define $\mu^{*\prime} : \mathcal{A}^* \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu^{*\prime}([A]) = \mu'(A)$; as μ' is a representation of $(\mathcal{A}, \supseteq, \succsim)$, thus by the Null-Quotient Lemma (Lemma 8), this is well-defined. By the Null-Quotient Lemma (Lemma 8), $\mu^{*\prime} \in \mathbb{M}^\sigma(\mathcal{A}^*)$ and $\mu^{*\prime}$ represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$. By Proposition 1, μ^* is the unique such σ-measure, so $\mu^{*\prime} = \mu^*$ and thus $\mu' = \mu$. □

Appendix D

In this appendix, we state and prove the Supercabinet Blueprint Lemma (Lemma 9). More broadly, the next four appendices are dedicated to the proof of our second main result.

While for Theorem 1, we were able to focus on idealized spaces and construct our representation using the one promised by Theorem V2 for $\mathcal{A}^\circ$, we cannot pursue this approach for Theorem 2 as it covers qualitative probability spaces for which $\mathcal{A}^\circ = \varnothing$. As discussed in Section 3.2, we instead pursue an approach based on the following observation: if two disjoint events are equally likely, then in any σ-measure representation, the measure of both must be half the measure of their union. Informally, if in some sense we were able to iteratively take such halves and then take disjoint unions, then in constructing our σ-measure representation we would necessarily assign the following numbers to certain likelihood equivalence classes:

Definition 16. The set of *dyadic rationals in* $[0,1]$, $\mathcal{2} \subseteq [0,1]$, is defined by

$$\mathcal{2} \equiv \{1\} \cup \left\{ \sum_{i \in F} \left(\frac{1}{2}\right)^i \middle| F \subseteq \mathbb{N}, |F| < |\mathbb{N}| \right\}.$$

It may be helpful to think of a dyadic rational as any number that appears on a sequence of progressively finer rulers, where the first measures units, the second measures half units, the third measures quarter units, and so on. It is straightforward to show that $\mathcal{2}$ is a dense subset of $[0,1]$.

In the next two appendices, we indeed develop techniques for iteratively taking halves and then taking disjoint unions in some sense, and we indeed then assign each number in $\mathcal{2}$ to an associated likelihood equivalence class. This alone is not enough to construct a σ-measure representation, however; thus in this appendix, we provide the map of where we are going by formalizing the additional structure such a labeled family of equivalence classes should have:

Definition 17. For each qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$, a collection of likelihood equivalence classes $\{\mathcal{Z}_v\}_{v \in \mathcal{2}} \subseteq [\mathcal{A}/\sim]$ is a *supercabinet of* $(\mathcal{A}, \supseteq, \succsim)$ if and only if

[SC1] $S \in \mathcal{Z}_1$;

[SC2] for each pair $v, v' \in \mathcal{2}$ such that $v + v' \leq 1$, there are disjoint $Z_v \in \mathcal{Z}_v$ and $Z_{v'} \in \mathcal{Z}_{v'}$ such that $Z_v \cup Z_{v'} \in \mathcal{Z}_{v+v'}$;

[SC3] for each non-increasing $(v_i^+) \in \mathcal{2}^{\mathbb{N}}$ and non-decreasing $(v_i^-) \in \mathcal{2}^{\mathbb{N}}$ such that $\lim v_i^+ = \lim v_i^-$,[19] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$; and

(ii) $\lim A_i^+ \sim \lim A_i^-$; and

[SC4] for each monotonic pair $(v_i), (w_i) \in \mathcal{2}^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$, there are convergent $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that for each $i \in \mathbb{N}$, A_i and B_i are disjoint with $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$.

It may be helpful to think of a supercabinet as a collection of drawers, where each drawer contains a collection of equally likely events, and where each drawer is labeled by the dyadic rational that we intend to assign to each of its events. To provide intuition, we choose to highlight [SC2]: for any pair of labels v, v' whose sum is also a la-

[19] Since both sequences are bounded and monotonic, they are indeed convergent.

bel, one should be able to take a first event from the drawer labeled v and a second event from the drawer labeled v'; and moreover, to do so such that these events are disjoint; and moreover, the union of these events should be correctly filed in the drawer labeled $v + v'$. We remark that supercabinets are closely related to a notion from the classic problem of representing a preference relation with a continuous utility function.[20]

The Supercabinet Blueprint Lemma (Lemma 9) states that if we can construct a supercabinet, then we are done. Though we only apply this result to idealized spaces, there is essentially no benefit to focusing on idealized spaces for the proof; we therefore prove it without this restriction.

Lemma 9 (Supercabinet Blueprint Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space that satisfies (monotone) continuity and has a supercabinet, then it has a unique representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.*

Proof. Let $\{\mathcal{Z}_v\}_{v \in \mathbb{2}} \subseteq [\mathcal{A}/\sim]$ be a supercabinet.

Step 1: *For each $k \in \{0, 1, \ldots\}$, there is $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ such that $Z \succ \varnothing$.* We proceed by induction on k. By [SC1], $S \in \mathcal{Z}_1$, and by *nondegeneracy* and *monotonicity*, $S \succ \varnothing$. For the inductive hypothesis, assume $k \in \{0, 1, \ldots\}$ is such that there is $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ such that $Z \succ \varnothing$. By [SC2], there are disjoint $Z', Z'' \in \mathcal{Z}_{(\frac{1}{2})^{k+1}}$ such that $Z' \cup Z'' \sim Z$. Necessarily $Z' \succ \varnothing$, else by the Domination Lemma (Lemma 4) $\varnothing \succsim Z' \cup Z'' \sim Z$, contradicting $Z \succ \varnothing$. By induction we are done.

Step 2: *For each $v \in \mathbb{2}$ such that $v > 0$, there is $Z \in \mathcal{Z}_v$ such that $Z \succ \varnothing$.* Let $v \in \mathbb{2}$ such that $v > 0$. Since $\mathbb{2}$ is dense in $[0, 1]$, there is $k \in \mathbb{2}$ such that $v > (\frac{1}{2})^k$. Since $v - (\frac{1}{2})^k \in \mathbb{2}$, by [SC2] there are disjoint $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ and $Z' \in \mathcal{Z}_{v-(\frac{1}{2})^k}$ such that $Z \cup Z' \in \mathcal{Z}_v$. By Step 1, $Z \succ \varnothing$, and by *monotonicity*, $Z' \succsim \varnothing$, so by the Domination Lemma (Lemma 4), $Z \cup Z' \succ \varnothing$.

Step 3: *For each pair $v^*, v \in \mathbb{2}$ such that $v^* > v$, each $A \in \mathcal{Z}_{v^*}$, and each $B \in \mathcal{Z}_v$, $A \succ B$.* Let $v^*, v \in \mathbb{2}$ such that $v^* > v$, let $A \in \mathcal{Z}_{v^*}$, and let $B \in \mathcal{Z}_v$. Since $v^* - v \in \mathbb{2}$, by [SC2] there are disjoint $Z \in \mathcal{Z}_v$ and $Z' \in \mathcal{Z}_{v^*-v}$ such that $Z \cup Z' \in \mathcal{Z}_{v^*}$. Since $v^* - v > 0$, thus by Step 2

[20] Let $(X, \succsim)$ be a completely pre-ordered set, and let us say that a collection of equivalence classes $\{\mathcal{Z}_i\}_{i \in \mathbb{N}} \subseteq [X/\sim]$ is a *cabinet of* $(X, \succsim)$ if and only if for each pair $x, y \in X$ such that $x \succ y$, there is $z \in \bigcup \mathcal{Z}_i$ such that $x \succsim z \succsim y$. It is a classic result that if $(X \succsim)$ has a cabinet, then for each topology such that all upper contour sets and lower contour sets are closed, there is a continuous representation (Cantor 1895, Debreu 1954, 1964). For our purposes, by the Measure Lemma (Lemma 2), every σ-measure representation is a continuous representation; we therefore seek a continuous representation that moreover respects the algebraic structure of the events. Indeed, though this is not obvious from the definitions, the supercabinet of a qualitative probability space that satisfies *monotone continuity* is a cabinet with additional structure.

there is $Z_{v^*-v} \in \mathcal{Z}_{v^*-v}$ such that $Z' \sim Z_{v^*-v} \succ \varnothing$. By the Domination Lemma (Lemma 4), $A \sim Z \cup Z' \succ Z \sim B$.

Step 4: *We have* $\varnothing \in \mathcal{Z}_0$. By [SC2], there are disjoint $A, B \in \mathcal{Z}_0$ such that $A \cup B \in \mathcal{Z}_0$. If $A \succ \varnothing$, then by the Domination Lemma (Lemma 4) $A \cup B \succ B$, contradicting that $A \cup B \sim B$. Thus by *monotonicity*, $A \sim \varnothing$, so $\varnothing \in \mathcal{Z}_0$.

Step 5: *Define* $\mu : \mathcal{A} \to [0,1]$. For each $v \in \mathcal{Q}$ and each $A \in \mathcal{Z}_v$, define $\mu(A) \equiv v$. Let $A \in \mathcal{A} \setminus (\bigcup \mathcal{Z}_v)$, and define

$$\mathcal{Q}^+ \equiv \{v \in \mathcal{Q} | B \in \mathcal{Z}_v \text{ implies } B \succsim A\}, \text{ and}$$

$$\mathcal{Q}^- \equiv \{v \in \mathcal{Q} | B \in \mathcal{Z}_v \text{ implies } A \succsim B\}.$$

We prove that $\inf(\mathcal{Q}^+) = \sup(\mathcal{Q}^-)$ using three observations.[21] By *monotonicity*, $S \succsim A \succsim \varnothing$, so by [SC1] and Step 4, we have (i) $\mathcal{Q}^+ \neq \varnothing$ and $\mathcal{Q}^- \neq \varnothing$. By [SC2], for each $v \in \mathcal{Q}$, there is $B \in \mathcal{Z}_v$; thus by Step 3, we have (ii) for each pair $v^*, v \in \mathcal{Q}$ such that $v^* > v$, (a) $v \in \mathcal{Q}^+$ implies $v^* \in \mathcal{Q}^+ \setminus \mathcal{Q}^-$, and (b) $v^* \in \mathcal{Q}^-$ implies $v \in \mathcal{Q}^- \setminus \mathcal{Q}^+$; moreover, we have (iii) $\mathcal{Q}^+ \cup \mathcal{Q}^- = \mathcal{Q}$. By (i) and (ii), there are $\inf(\mathcal{Q}^+)$ and $\sup(\mathcal{Q}^-)$ such that $\inf(\mathcal{Q}^+) \geq \sup(\mathcal{Q}^-)$; thus by (iii) and the density of $\mathcal{Q}$ in $[0,1]$, we have $\inf(\mathcal{Q}^+) = \sup(\mathcal{Q}^-)$. Define

$$\begin{aligned} \mu(A) &\equiv \inf(\mathcal{Q}^+) \\ &= \sup(\mathcal{Q}^-). \end{aligned}$$

Step 6: *For each pair* $A, B \in \mathcal{A}$, $\mu(A) > \mu(B)$ *implies* $A \succ B$. Let $A, B \in \mathcal{A}$ such that $\mu(A) > \mu(B)$. Since $\mathcal{Q}$ is dense in $[0,1]$, there is $v' \in \mathcal{Q}$ such that $\mu(A) > v' > \mu(B)$. Let $Z_{v'} \in \mathcal{Z}_{v'}$.

If $A \in \bigcup \mathcal{Z}_v$, then by construction, $A \in \mathcal{Z}_{\mu(A)}$, so by Step 3, $A \succ Z_{v'}$. If $A \notin \bigcup \mathcal{Z}_v$, then $\inf\{v \in \mathcal{Q} | C \in \mathcal{Z}_v \text{ implies } C \succsim A\} = \mu(A) > v'$; thus $A \succ Z_{v'}$.

If $B \in \bigcup \mathcal{Z}_v$, then by construction, $B \in \mathcal{Z}_{\mu(B)}$, so by Step 3, $Z_{v'} \succ B$. If $B \notin \bigcup \mathcal{Z}_v$, then $v' > \mu(B) = \sup\{v \in \mathcal{Q} | C \in \mathcal{Z}_v \text{ implies } B \succsim C\}$; thus $Z_{v'} \succ B$.

Altogether, then, $A \succ Z_{v'} \succ B$.

Step 7: *For each pair* $A, B \in \mathcal{A}$, $\mu(A) = \mu(B)$ *implies* $A \sim B$. Let $A, B \in \mathcal{A}$ such that $\mu(A) = \mu(B)$. We proceed with three cases whose proofs are similar.

Case 1: $\mu(A) = 0$. Since $\mathcal{Q}$ is dense in $[0,1]$, there is decreasing $(v_i^+) \in \mathcal{Q}^{\mathbb{N}}$ such that $\lim v_i^+ = 0$. For each $i \in \mathbb{N}$, define $v_i^- \equiv 0$. Since $\lim v_i^+ = \lim v_i^-$, thus by [SC3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$; and

(ii) $\lim A_i^+ \sim \lim A_i^-$.

[21] These three observations correspond closely to the three claims in Step 2 of the proof of the Carving Lemma (Lemma 7), but the arguments are simpler here.

By Step 4, for each $i \in \mathbb{N}$, $\varnothing \sim A_i^-$; thus by *continuity*, $\varnothing \sim \lim A_i^-$. By construction, for each $i \in \mathbb{N}$, $\mu(A_i^+) > 0 = \mu(A)$, so by Step 6, $A_i^+ \succ A$; thus by *continuity*, $\lim A_i^+ \succsim A$. Altogether, $\varnothing \sim \lim A_i^- \sim \lim A_i^+ \succsim A$, so by *monotonicity*, $A \sim \varnothing$. By the same argument, $B \sim \varnothing$, so $A \sim B$.

Case 2: $\mu(A) = 1$. Since $\mathcal{2}$ is dense in $[0, 1]$, there is increasing $(v_i^-) \in \mathcal{2}^{\mathbb{N}}$ such that $\lim v_i^- = 1$. For each $i \in \mathbb{N}$, define $v_i^+ \equiv 1$. Since $\lim v_i^+ = \lim v_i^-$, thus by [SC3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$; and

(ii) $\lim A_i^+ \sim \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A) = 1 > \mu(A_i^-)$, so by Step 6, $A \succ A_i^-$; thus by *continuity*, $A \succsim \lim A_i^-$. By [SC1], for each $i \in \mathbb{N}$, $A_i^+ \sim S$; thus by *continuity*, $\lim A_i^+ \sim S$. Altogether, $A \succsim \lim A_i^- \sim \lim A_i^+ \sim S$, so by *monotonicity*, $A \sim S$. By the same argument, $B \sim S$, so $A \sim B$.

Case 3: $\mu(A) \in (0, 1)$. Since $\mathcal{2}$ is dense in $[0, 1]$, there are decreasing $(v_i^+) \in \mathcal{2}^{\mathbb{N}}$ such that $\lim v_i^+ = \mu(A)$ and increasing $(v_i^-) \in \mathcal{2}^{\mathbb{N}}$ such that $\lim v_i^- = \mu(A)$. Since $\lim v_i^+ = \lim v_i^-$, thus by [SC3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$; and

(ii) $\lim A_i^+ \sim \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A_i^+) > \mu(A) > \mu(A_i^-)$, so by Step 6, $A_i^+ \succ A \succ A_i^-$; thus by *continuity*, $\lim A_i^+ \succsim A \succsim \lim A_i^- \sim \lim A_i^+$, so $A \sim \lim A_i^+$. By the same argument, $B \sim \lim A_i^+$, so $A \sim B$.

Step 8: *μ represents $\succsim$.* Let $A, B \in \mathcal{A}$. Then both statements (i) $\mu(A) \geq \mu(B)$ implies $A \succsim B$, and (ii) $A \succsim B$ implies $\mu(A) \geq \mu(B)$, follow immediately from Step 6 and Step 7.

Step 9: *For each disjoint pair $A, B \in \mathcal{A}$, $\mu(A) + \mu(B) \leq 1$.* Let $A, B \in \mathcal{A}$ be disjoint, and assume, by way of contradiction, $\mu(A) + \mu(B) > 1$. Then $\mu(A) > 0$ and $\mu(B) > 0$. Since $\mathcal{2}$ is dense in $[0, 1]$, thus there are $v, v' \in \mathcal{2}$ such that $\mu(A) > v$, $\mu(B) > v'$, and $v + v' > 1$. By [SC2], there are disjoint $Z_v \in \mathcal{Z}_v$ and $Z_{1-v} \in \mathcal{Z}_{1-v}$ such that $Z_v \cup Z_{1-v} \in \mathcal{Z}_1$. Since $\mu(A) > v = \mu(Z_v)$ and $\mu(B) > v' > 1 - v = \mu(Z_{1-v})$, thus by Step 8, $A \succ Z_v$ and $B \succ Z_{1-v}$. But then by the Domination Lemma (Lemma 4) and [SC1], $A \cup B \succ Z_v \cup Z_{1-v} \sim S$, contradicting *monotonicity*.

Step 10: *For each disjoint pair $A, B \in \bigcup \mathcal{Z}_v$, $\mu(A \cup B) = \mu(A) + \mu(B)$.* Let $A, B \in \bigcup \mathcal{Z}_v$ be disjoint. Then $\mu(A), \mu(B) \in \mathcal{2}$, and by Step 9, $\mu(A) + \mu(B) \leq 1$, so $\mu(A) + \mu(B) \in \mathcal{2}$; thus by [SC2] there are disjoint $A' \in \mathcal{Z}_{\mu(A)}$ and $B' \in \mathcal{Z}_{\mu(B)}$ such that $A' \cup B' \in \mathcal{Z}_{\mu(A)+\mu(B)}$. Since

$A \sim A'$ and $B \sim B'$, thus by two applications of the Domination Lemma (Lemma 4), $A \cup B \sim A' \cup B'$, so $A \cup B \in \mathcal{Z}_{\mu(A)+\mu(B)}$, so by construction $\mu(A \cup B) = \mu(A) + \mu(B)$.

Step 11: *For each disjoint pair $A, B \in \mathcal{A}$, $\mu(A \cup B) = \mu(A) + \mu(B)$.* Let $A, B \in \mathcal{A}$ be disjoint. By Step 9, $\mu(A) + \mu(B) \in [0, 1]$. We proceed with two claims whose proofs are similar, though not quite dual.

Claim 1: $\mu(A) + \mu(B) \geq \mu(A \cup B)$. If $\mu(A) + \mu(B) = 1$ we are done, so assume $1 > \mu(A) + \mu(B)$. Then since $\mathbb{2}$ is dense in $[0, 1]$, there are non-increasing $(v_i), (w_i) \in \mathbb{2}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $v_i \geq \mu(A)$, $w_i \geq \mu(B)$, and $1 \geq v_i + w_i$; and

(ii) $\lim v_i = \mu(A)$ and $\lim w_i = \mu(B)$.

By [SC4], there are convergent $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$; and

(ii) for each $i \in \mathbb{N}$, $A_i \cap B_i = \varnothing$.

Define $A_\infty \equiv \lim A_i$ and $B_\infty \equiv \lim B_i$. By construction, for each $i \in \mathbb{N}$, $\mu(A_i) = v_i$ and $\mu(B_i) = w_i$.

Let $i \in \mathbb{N}$. By Step 8, $A_i \succsim A_{i+1} \succsim A_{i+2} \succsim \cdots$, so by *continuity* $A_i \succsim \lim_{j \geq i} A_j = A_\infty$. By the same argument, $B_i \succsim B_\infty$. By the Domination Lemma (Lemma 4), $A_i \cup B_i \succsim A_\infty \cup B_\infty$. By Step 10 and Step 8, $v_i + w_i = \mu(A_i) + \mu(B_i) = \mu(A_i \cup B_i) \geq \mu(A_\infty \cup B_\infty)$.

Since for each $i \in \mathbb{N}$, $v_i + w_i \geq \mu(A_\infty \cup B_\infty)$, thus $\mu(A) + \mu(B) = \lim v_i + \lim w_i = \lim(v_i + w_i) \geq \mu(A_\infty \cup B_\infty)$.

By Step 8, for each $i \in \mathbb{N}$, $A_i \succsim A$ and $B_i \succsim B$; thus by *continuity* $A_\infty \succsim A$ and $B_\infty \succsim B$. By the Algebra Lemma (Lemma 1), $A_\infty \cap B_\infty = \lim(A_i \cap B_i) = \varnothing$, so by the Domination Lemma (Lemma 4), $A_\infty \cup B_\infty \succsim A \cup B$. By Step 8, $\mu(A_\infty \cup B_\infty) \geq \mu(A \cup B)$.

Altogether, then, $\mu(A) + \mu(B) \geq \mu(A_\infty \cup B_\infty) \geq \mu(A \cup B)$, as desired.

Claim 2: $\mu(A) + \mu(B) \leq \mu(A \cup B)$. If $\mu(A) + \mu(B) = 0$ we are done, so assume $0 < \mu(A) + \mu(B)$. Then since $\mathbb{2}$ is dense in $[0, 1]$, there are non-decreasing $(v_i), (w_i) \in \mathbb{2}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $v_i \leq \mu(A)$ and $w_i \leq \mu(B)$; and

(ii) $\lim v_i = \mu(A)$ and $\lim w_i = \mu(B)$.

Then $v_i + w_i \leq \mu(A) + \mu(B) \leq 1$, so by [SC4], there are convergent $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$; and

(ii) for each $i \in \mathbb{N}$, $A_i \cap B_i = \varnothing$.

Define $A_\infty \equiv \lim A_i$ and $B_\infty \equiv \lim B_i$. By construction, for each $i \in \mathbb{N}$, $\mu(A_i) = v_i$ and $\mu(B_i) = w_i$.

Let $i \in \mathbb{N}$. By Step 8, $A_i \precsim A_{i+1} \precsim A_{i+2} \precsim \cdots$, so by *continuity* $A_i \precsim \lim_{j \geq i} A_j = A_\infty$. By the same argument, $B_i \precsim B_\infty$. By the Algebra Lemma (Lemma 1), $A_\infty \cap B_\infty = \lim(A_i \cap B_i) = \varnothing$, so by the Domination Lemma (Lemma 4), $A_i \cup B_i \precsim A_\infty \cup B_\infty$. By Step 10 and Step 8, $v_i + w_i = \mu(A_i) + \mu(B_i) = \mu(A_i \cup B_i) \leq \mu(A_\infty \cup B_\infty)$.

Since for each $i \in \mathbb{N}$, $v_i + w_i \leq \mu(A_\infty \cup B_\infty)$, thus $\mu(A) + \mu(B) = \lim v_i + \lim w_i = \lim(v_i + w_i) \leq \mu(A_\infty \cup B_\infty)$.

By Step 8, for each $i \in \mathbb{N}$, $A_i \precsim A$ and $B_i \precsim B$; thus by *continuity* $A_\infty \precsim A$ and $B_\infty \precsim B$. By the Domination Lemma (Lemma 4), $A_\infty \cup B_\infty \precsim A \cup B$. By Step 8, $\mu(A_\infty \cup B_\infty) \leq \mu(A \cup B)$.

Altogether, then, $\mu(A) + \mu(B) \leq \mu(A_\infty \cup B_\infty) \leq \mu(A \cup B)$, as desired.

Step 12: *We have* $\mu \in \mathbb{M}^\sigma(\mathcal{A})$. Since $\mu(S) = 1$, by Step 11 and induction, $\mu \in \mathbb{M}(\mathcal{A})$. By Step 8, μ represents $\succsim$, so by Theorem V1, $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

Step 13: *If* $\mu' \in \mathbb{M}^\sigma(\mathcal{A})$ *represents* $\succsim$, *then* $\mu' = \mu$. If $\mu' \in \mathbb{M}^\sigma(\mathcal{A})$ represents $\succsim$, then using [SC1] and [SC2], it is straightforward to verify that μ' must be defined as in Step 5. □

We remark that the converse is not true: the qualitative probability space with two events satisfies *monotone continuity* and has a unique σ-measure representation, yet has no supercabinet.

Appendix E

In this appendix, we introduce and establish properties of greedy transforms, our primary technique for constructing supercabinets. In particular, we prove the Erosion Lemma, the Greedy Basics Lemma, the 1-Division Lemma, the Greedy Removal Lemma, and the List-Transform Lemma. These are Lemmas 10–14, respectively. Throughout this appendix, we focus on idealized spaces.

To begin, recall that by the Carving Lemma (Lemma 7), if we take an event that contains no atoms as a parent, and provide an event that is no more likely than the parent as input, then we can receive as output a subevent of the parent that is just as likely as the input. Greedy transforms provide a method for doing this with certain parents that may contain atoms, and moreover for doing so systematically across parents in a desirable manner.

In Section 3.2, when we discussed Kakeya's observation, we demonstrated how under some conditions, one could use a greedy algorithm to take a sequence and construct from it a subsequence whose sum is precisely a given target. This technique suggests well enough how a greedy transform selects atoms from its parent, but for our purposes we must also be systematic about what it does when it has exhausted the parent's atoms yet still falls short. In order to systematically select events that contain no atoms for these situations, we introduce the following notion of a nested family of subevents of S° that passes through all the likelihood equivalence classes in $\mathcal{A}^\circ$:

Definition 18. Let $(\mathcal{A}, \supseteq, \succsim)$ be an idealized space. Then a family $\mathcal{E} = \{E_v\}_{v \in [0,1]} \in (\mathcal{A}^\circ)^{[0,1]}$ is an *erosion of* $(\mathcal{A}, \supseteq, \succsim)$ if and only if

(i) $E_0 = \varnothing$;

(ii) for each pair $v^*, v \in [0,1]$ such that $v^* > v$, $S^\circ \supseteq E_{v^*} \supseteq E_v$;

(iii) for each $A \in \mathcal{A}$ such that $S^\circ \succsim A$, there is $v \in [0,1]$ such that $E_v \sim A$; and

(iv) for each monotonic $(v_i) \in [0,1]^{\mathbb{N}}$, $\lim E_{v_i} = E_{\lim v_i}$.

In this case, the collection of *erodable events* (*given* $\mathcal{E}$) is $\mathcal{A}^{\mathcal{E}} \equiv \{A \in \mathcal{A} | (S^\circ \setminus A) \in \mathcal{E}\}$.

As an example, suppose we first take the σ-algebra of the Borel subsets of the unit circle with events compared according to the Borel measure, then work with the null-quotient. If for each $v \in [0,1]$, E_v is the projection of the circle centered at $(0,0)$ with radius v, then $\{E_v\}_{v\in[0,1]}$ is an erosion. An event is erodable if there is $r \in [0,1]$ such that it is the projection of $\{(x,y) \in \mathbb{R}^2_+ | x^2 + y^2 \in [r,1]\}$—note that modifying the zero-measure boundary of the Borel set we project does not alter the projection—and thus the typical erodable event is the projection of something that looks like a centered donut. It can be helpful to imagine that the unit circle is eroded as it is eaten away from the center out; at any moment, we can freeze this process and observe an erodable event; we can then resume the process, and our erodable event will be further eroded.

In order to systematically use an erosion across all greedy transforms, there must actually be one; fortunately, the Erosion Lemma (Lemma 10) assures us that this is the case:[22]

Lemma 10 (Erosion Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, then it has an erosion $\mathcal{E} = \{E_v\}_{v\in[0,1]}$. If, moreover, $S^\circ \succ \varnothing$, then there is $\mu^\circ \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^\circ)$ such that*

(i) for each pair $A, B \in \mathcal{A}^\circ$, $A \succsim B$ if and only if $\mu^\circ(A) \geq \mu^\circ(B)$, and

(ii) for each $v \in [0,1]$, $\mu^\circ(E_v) = v$.

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ satisfy the hypotheses. If $S^\circ \sim \varnothing$, then for each $v \in [0,1]$, define $E_v \equiv \varnothing$; it is straightforward to verify that $\{E_v\}_{v\in[0,1]}$ is an erosion of $(\mathcal{A}, \supseteq, \succsim)$. Thus let us assume that $S^\circ \succ \varnothing$.

Define $\supseteq^\circ$, $\succsim^\circ$ on $\mathcal{A}^\circ$ as follows: for each pair $A, B \in \mathcal{A}^\circ$, (i) $A \supseteq^\circ B$ if and only if $A \supseteq B$, and (ii) $A \succsim^\circ B$ if and only if $A \succsim B$. Since $S^\circ \succ \varnothing$, it is straightforward to verify that $(\mathcal{A}^\circ, \supseteq^\circ, \succsim^\circ)$ is a qualitative probability space satisfying *monotone continuity* and *no atoms*; thus by Theorem V2, there is $\mu^\circ \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^\circ)$ such that for each pair $A, B \in \mathcal{A}^\circ$, $\mu^\circ(A) \geq \mu^\circ(B)$ if and only if $A \succsim^\circ B$ if and only if $A \succsim B$.

To begin, we construct $\{E_v\}_{v\in\mathbb{2}} \in (\mathcal{A}^\circ)^{\mathbb{2}}$ such that for each pair $v^*, v \in \mathbb{2}$ such that $v^* > v$, we have (i) $\mu^\circ(E_{v^*}) = v^*$, (ii) $\mu^\circ(E_v) = v$, and (iii) $E_{v^*} \supseteq E_v$. We proceed by induction. For the base step, define $E_1 \equiv S^\circ$ and $E_0 \equiv \varnothing$; clearly $\mu^\circ(E_1) = 1$, $\mu^\circ(E_0) = 0$,

[22]This is closely related to a well-known fact mentioned to me by a referee: for each set-σ-algebra with a nonatomic probability measure μ and for each event A with positive measure, there is a collection of events $\{A_v\}_{v\in[0,\mu(A)]}$ such that (i) $A_{\mu(A)} = A$ and $A_0 = \varnothing$; (ii) for each pair $v^*, v \in [0,1]$ such that $v^* > v$, $A_{v^*} \supseteq A_v$; and (iii) for each $v \in [0, \mu(A)]$, $\mu(A_v) = v$. Our lemma provides a version of this result for σ-algebras, which need not be set-σ-algebras, that are equipped with likelihood relations.

and $E_1 \supseteq E_0$. For the inductive step, assume that $q \in \{0\} \cup \mathbb{N}$ is such that for each pair $p^*, p \in \{0, 1, \dots, 2^q\}$ such that $p^* > p$, we have defined $E_{\frac{p^*}{2^q}}, E_{\frac{p}{2^q}} \in \mathcal{A}^\circ$ such that (i) $\mu^\circ(E_{\frac{p^*}{2^q}}) = \frac{p^*}{2^q}$, (ii) $\mu^\circ(E_{\frac{p}{2^q}}) = \frac{p}{2^q}$, and (iii) $E_{\frac{p^*}{2^q}} \supseteq E_{\frac{p}{2^q}}$. For each even $p \leq 2^{q+1}$, $E_{\frac{p}{2^{q+1}}} = E_{\frac{(\frac{p}{2})}{2^q}}$ is already defined, so let $p \leq 2^{q+1}$ be odd. Then $E_{\frac{p+1}{2^{q+1}}}, E_{\frac{p-1}{2^{q+1}}}$ are already defined such that (i) $\mu^\circ(E_{\frac{p+1}{2^{q+1}}}) = \frac{p+1}{2^{q+1}}$, (ii) $\mu^\circ(E_{\frac{p-1}{2^{q+1}}}) = \frac{p-1}{2^{q+1}}$, and (iii) $E_{\frac{p+1}{2^{q+1}}} \supseteq E_{\frac{p-1}{2^{q+1}}}$. As μ° is additive, thus $\mu^\circ(E_{\frac{p+1}{2^{q+1}}} \setminus E_{\frac{p-1}{2^{q+1}}}) = \frac{p+1}{2^{q+1}} - \frac{p-1}{2^{q+1}} = \frac{2}{2^{q+1}}$. Since $\mu^\circ \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^\circ)$, thus there is $A_p \subseteq E_{\frac{p+1}{2^{q+1}}} \setminus E_{\frac{p-1}{2^{q+1}}}$ such that $\mu^\circ(A_p) = \frac{1}{2^{q+1}}$. Define $E_{\frac{p}{2^{q+1}}} \equiv E_{\frac{p-1}{2^{q+1}}} \cup A_p$. By construction, $E_{\frac{p+1}{2^{q+1}}} \supseteq E_{\frac{p}{2^{q+1}}} \supseteq E_{\frac{p-1}{2^{q+1}}}$, and as $E_{\frac{p-1}{2^{q+1}}} \cap A_p = \varnothing$, thus $\mu^\circ(E_{\frac{p}{2^{q+1}}}) = \mu^\circ(E_{\frac{p-1}{2^{q+1}}}) + \mu^\circ(A_p) = \frac{p-1}{2^{q+1}} + \frac{1}{2^{q+1}} = \frac{p}{2^{q+1}}$. As odd $p \leq 2^q$ was arbitrary, altogether we have that for each pair $p^*, p \in \{0, 1, \dots, 2^{q+1}\}$ such that $p^* > p$, we have defined $E_{\frac{p^*}{2^{q+1}}}, E_{\frac{p}{2^{q+1}}} \in \mathcal{A}^\circ$ such that (i) $\mu^\circ(E_{\frac{p^*}{2^{q+1}}}) = \frac{p^*}{2^{q+1}}$, (ii) $\mu^\circ(E_{\frac{p}{2^{q+1}}}) = \frac{p}{2^{q+1}}$, and (iii) $E_{\frac{p^*}{2^{q+1}}} \supseteq E_{\frac{p}{2^{q+1}}}$. By induction, this completes the construction of $\{E_v\}_{v \in \mathbb{2}}$, as desired.

To complete the construction of $\mathcal{E}$, let $v \in [0, 1] \setminus \mathbb{2}$, and define

$$E_v \equiv \bigcup_{\{v' \in \mathbb{2} | v' < v\}} E_{v'}.$$

Since $v > 0$ and $\mathbb{2}$ is countably-infinite, this is well-defined.

Next, we claim that for each $v \in [0, 1]$, $\mu^\circ(E_v) = v$. By construction this is true for each $v \in \mathbb{2}$, so let $v \in [0, 1] \setminus \mathbb{2}$. Then $v > 0$, so there is increasing $(v_i) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i = v$. For each $v' \in \mathbb{2}$ such that $v' < v$, there is $i \in \mathbb{N}$ such that $v' < v_i$ and thus $E_{v'} \subseteq E_{v_i} \subseteq \bigcup E_{v_i}$; thus by construction, $E_v = \bigcup E_{v_i}$. Since $E_{v_1} \subseteq E_{v_2} \subseteq \cdots$, thus $\bigcup E_{v_i} = \lim E_{v_i}$. Since $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, thus by the Measure Lemma (Lemma 2), $\mu^\circ(E_v) = \mu^\circ(\lim E_{v_i}) = \lim \mu^\circ(E_{v_i}) = \lim v_i = v$, as desired.

Finally, we verify that $\mathcal{E}$ is an erosion. For (i), by construction, $E_0 = \varnothing$.

For (ii), let $v^*, v \in [0, 1]$ such that $v^* > v$. If $v \in \mathbb{2}$, then by construction, $S^\circ \supseteq E_{v^*} \supseteq E_v$. If $v \notin \mathbb{2}$, then whether $v^* \in \mathbb{2}$ or $v^* \notin \mathbb{2}$, we have by construction that for each $v' \in \mathbb{2}$ such that $v > v'$, $E_{v^*} \supseteq E_{v'}$; thus $S^\circ \supseteq E_{v^*} \supseteq \bigcup_{\{v' \in \mathbb{2} | v' < v\}} E_{v'} = E_v$.

For (iii), let $A \in \mathcal{A}$ such that $S^\circ \succsim A$. By *monotonicity*, $S^\circ \succsim A \succsim \varnothing$, so by the Carving Lemma (Lemma 7), there is $A^\circ \in \mathcal{A}$ such that $A^\circ \subseteq S^\circ$ and $A^\circ \sim A$. Then $A^\circ \in \mathcal{A}^\circ$ and $\mu^\circ(E_{\mu^\circ(A^\circ)}) = \mu^\circ(A^\circ)$, so $E_{\mu^\circ(A^\circ)} \sim A^\circ \sim A$, as desired.

For (iv), let $(v_i) \in [0, 1]^{\mathbb{N}}$ be monotonic with limit v. If (v_i) is non-increasing, then for each $i \in \mathbb{N}$, $E_{v_i} \supseteq E_v$, so since $E_{v_1} \supseteq E_{v_2} \supseteq \cdots$, thus $\lim E_{v_i} = \bigcap E_{v_i} \supseteq E_v$. Moreover, since $\mu^\circ \in \mathbb{M}^\sigma(\mathcal{A}^\circ)$, thus by the Measure Lemma (Lemma 2), $\mu^\circ(\lim E_{v_i}) = \lim \mu^\circ(E_{v_i}) = \lim v_i = v = \mu^\circ(E_v)$, so $\lim E_{v_i} \sim E_v$. By *monotonicity* and *separability*, $(\lim E_{v_i}) \setminus E_v \sim \varnothing$, so as our space is idealized, thus $(\lim E_{v_i}) \setminus E_v = \varnothing$, so $\lim E_{v_i} = E_v = E_{\lim v_i}$, as desired. It is straightforward to adapt this argument to the case where $(v_i) \in [0, 1]^{\mathbb{N}}$ is non-decreasing, and we omit the argument. □

Due to the Erosion Lemma (Lemma 10), we frequently abuse language: when we say that $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, we implicitly take a particular choice of erosion $\mathcal{E}$, together with the associated notation in the definition of erosion, as given.

We can now formally introduce the greedy transforms:

Definition 19. Let $(\mathcal{A}, \supseteq, \succsim)$ be an idealized space and let $A \in \mathcal{A}^{\mathcal{E}}$. The *greedy transform of* A (*given* $\mathcal{E}$), which abusing notation we write as $\mathcal{G}^A : \mathcal{A} \to \mathcal{A}$, is such that for each $B \in \mathcal{A}$, $\mathcal{G}^A(B) \subseteq A$ is defined as follows:

- Define $\mathcal{G}_0^A(B) \equiv \varnothing$.
- For each $s \in \mathbb{N}$, define

$$\mathcal{G}_s^A(B) \equiv \begin{cases} \mathcal{G}_{s-1}^A(B) \cup \alpha_s, & s \in I^\bullet, \alpha_s \subseteq A, \text{and } B \succsim \mathcal{G}_{s-1}^A(B) \cup \alpha_s; \\ \mathcal{G}_{s-1}^A(B), & \text{else.} \end{cases}$$

- Define $\mathcal{G}_\odot^A(B) \equiv \bigcup_{s\in\mathbb{N}} \mathcal{G}_s^A(B)$.
- By construction and *continuity*, $B \succsim \mathcal{G}_\odot^A(B) = \mathcal{G}_\odot^A(B) \cup (E_0 \cap A)$. Define

$$v^A(B) \equiv \sup\{v \in [0,1] | B \succsim \mathcal{G}_\odot^A(B) \cup (E_v \cap A)\}.$$

- Define $\mathcal{G}^A(B) \equiv \mathcal{G}_\odot^A(B) \cup (E_{v^A(B)} \cap A)$.

We remark that for each list of events $(B_1, B_2, \ldots, B_n)$, we have that S is erodable, and $S \setminus \mathcal{G}^S(B_1)$ is erodable, and $(S \setminus \mathcal{G}^S(B_1)) \setminus (\mathcal{G}^{S\setminus\mathcal{G}^S(B_1)}(B_2))$ is erodable, and so on. In terms of our earlier example where the erosion is a family of projections of concentric circles, the idea is loosely that if a circle is removed, and then a donut is removed, the result is that a larger circle has been removed. Moreover, if a circle is removed, and then a series of donuts are iteratively removed, the result is again that a circle has been removed.

The Greedy Basics Lemma (Lemma 11) provides some basic properties of all greedy transforms, including notably that each greedy transform is idempotent:

Lemma 11 (Greedy Basics Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, then for each $A \in \mathcal{A}^{\mathcal{E}}$ and each pair $B, B' \in \mathcal{A}$, we have*

(i) $(A \setminus \mathcal{G}^A(B)) \in \mathcal{A}^{\mathcal{E}}$,

(ii) $B \succsim \mathcal{G}^A(B)$,

(iii) $\mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(B)$, *and*

(iv) $\mathcal{G}^A(B) \sim \mathcal{G}^A(B')$ *implies* $\mathcal{G}^A(B) = \mathcal{G}^A(B')$.

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ satisfy the hypotheses, let $A \in \mathcal{A}^{\mathcal{E}}$, and let $B, B' \in \mathcal{A}$. Finally, let $\{\mathcal{G}_s^A\}_{s\in\{0\}\cup\mathbb{N}}$, $\mathcal{G}_\odot^A$, and $v^A(B)$ be as in the definition of *greedy transform of A given $\mathcal{E}$*.

Step 1: $(A \setminus \mathcal{G}^A(B)) \in \mathcal{A}^{\mathcal{E}}$. Since $A \in \mathcal{A}^{\mathcal{E}}$, thus $(S^\circ \setminus A) \in \mathcal{E}$. Moreover, $S^\circ \setminus (A \setminus \mathcal{G}^A(B)) = (S^\circ \setminus A) \cup (S^\circ \cap \mathcal{G}^A(B))$. If $(S^\circ \cap \mathcal{G}^A(B)) = \varnothing$, then $S^\circ \setminus (A \setminus \mathcal{G}^A(B)) = (S^\circ \setminus A) \in \mathcal{E}$, so $(A \setminus \mathcal{G}^A(B)) \in \mathcal{A}^{\mathcal{E}}$ and we are done; thus let us assume that $(\mathcal{G}^A(B) \cap S^\circ) \neq \varnothing$.

By construction, $(S^\circ \cap \mathcal{G}^A(B)) = (E_{v^A(B)} \cap A)$. Since $(S^\circ \setminus A) \in \mathcal{E}$, thus there is $v \in [0,1]$ such that $(S^\circ \setminus A) = E_v$. Since $E_{v^A(B)}$ and E_v are nested, and since $(E_{v^A(B)} \cap A) \supsetneq \varnothing =$

$(E_v \cap A)$, thus $E_{v^A(B)} \supseteq E_v$. Then $(E_{v^A(B)} \setminus A) \supseteq (E_v \setminus A) = (S^\circ \setminus A) \setminus A = (S^\circ \setminus A)$, so as $S^\circ \supseteq E_{v^A(B)}$ and thus $(S^\circ \setminus A) \supseteq (E_{v^A(B)} \setminus A)$, altogether we have $(E_{v^A(B)} \setminus A) = (S^\circ \setminus A)$. Thus

$$\begin{aligned} S^\circ \setminus \big(A \setminus \mathcal{G}^A(B)\big) &= \big(S^\circ \setminus A\big) \cup \big(S^\circ \cap \mathcal{G}^A(B)\big) \\ &= (E_{v^A(B)} \setminus A) \cup (E_{v^A(B)} \cap A) \\ &= E_{v^A(B)}, \end{aligned}$$

so $(A \setminus \mathcal{G}^A(B)) \in \mathcal{A}^{\mathcal{E}}$, as desired.

Step 2: $B \succsim \mathcal{G}^A(B)$. By construction, $B \succsim \mathcal{G}^A_\odot(B)$. If $S^\circ \sim \varnothing$, then as our space is idealized, $S^\circ = \varnothing$, so $B \succsim \mathcal{G}^A_\odot(B) = \mathcal{G}^A(B)$. If $v^A(B) = 0$, then $B \succsim \mathcal{G}^A_\odot(B) = \mathcal{G}^A(B)$. Thus let us assume $S^\circ \succ \varnothing$ and $v^A(B) > 0$.

Let $(v_i) \in [0,1]^{\mathbb{N}}$ be an increasing sequence that converges to $v^A(B)$; then $\lim E_{v_i} = E_{\lim v_i} = E_{v^A(B)}$. By construction and *monotonicity*, for each $i \in \mathbb{N}$, $B \succsim \mathcal{G}^A_\odot(B) \cup (E_{v_i} \cap A)$; thus by *continuity* and the Algebra Lemma (Lemma 1),

$$\begin{aligned} B &\succsim \lim\big(\mathcal{G}^A_\odot(B) \cup (E_{v_i} \cap A)\big) \\ &= \big(\lim \mathcal{G}^A_\odot(B)\big) \cup \big(\lim(E_{v_i} \cap A)\big) \\ &= \mathcal{G}^A_\odot(B) \cup \big((\lim E_{v_i}) \cap (\lim A)\big) \\ &= \mathcal{G}^A_\odot(B) \cup (E_{v^A(B)} \cap A) \\ &= \mathcal{G}^A(B), \end{aligned}$$

as desired.

Step 3: *Conclude.* We first prove $\mathcal{G}^A_\odot(\mathcal{G}^A(B)) = \mathcal{G}^A_\odot(B)$ using induction. For the base step, by construction, $\mathcal{G}^A_0(\mathcal{G}^A(B)) = \mathcal{G}^A_0(B) = \varnothing$. For the inductive hypothesis, let $s \in \mathbb{N}$ be such that $\mathcal{G}^A_s(\mathcal{G}^A(B)) = \mathcal{G}^A_s(B)$. If $s+1 \notin I^\bullet$ or $\alpha_{s+1} \nsubseteq A$, then $\mathcal{G}^A_{s+1}(\mathcal{G}^A(B)) = \mathcal{G}^A_s(\mathcal{G}^A(B)) = \mathcal{G}^A_s(B) = \mathcal{G}^A_{s+1}(B)$ and we are done, so assume $s+1 \in I^\bullet$ and $\alpha_{s+1} \subseteq A$.

If $\alpha_{s+1} \subseteq \mathcal{G}^A_{s+1}(B)$, then by *monotonicity* and the inductive hypothesis,

$$\begin{aligned} \mathcal{G}^A(B) &\succsim \mathcal{G}^A_s(B) \cup \alpha_{s+1} \\ &= \mathcal{G}^A_s\big(\mathcal{G}^A(B)\big) \cup \alpha_{s+1}, \end{aligned}$$

so by construction, $\alpha_{s+1} \subseteq \mathcal{G}^A_{s+1}(\mathcal{G}^A(B))$.

If $\alpha_{s+1} \nsubseteq \mathcal{G}^A_{s+1}(B)$, then by the inductive hypothesis, construction, and Step 2,

$$\begin{aligned} \mathcal{G}^A_s\big(\mathcal{G}^A(B)\big) \cup \alpha_{s+1} &= \mathcal{G}^A_s(B) \cup \alpha_{s+1} \\ &\succ B \\ &\succsim \mathcal{G}^A(B), \end{aligned}$$

so by construction, $\alpha_{s+1} \nsubseteq \mathcal{G}^A_{s+1}(\mathcal{G}^A(B))$.

Altogether, $\mathcal{G}^A_{s+1}(\mathcal{G}^A(B)) = \mathcal{G}^A_{s+1}(B)$. Since $s \in \mathbb{N}$ was arbitrary, thus by induction, $(\mathcal{G}^A_s(\mathcal{G}^A(B)))_{s\in\{0\}\cup\mathbb{N}} = (\mathcal{G}^A_s(B))_{s\in\{0\}\cup\mathbb{N}}$, so by construction, $\mathcal{G}^A_\odot(\mathcal{G}^A(B)) = \mathcal{G}^A_\odot(B)$. Since

(i) $\mathcal{G}^A(B) = \mathcal{G}^A_\odot(B) \cup (E_{v^A(B)} \cap A) = \mathcal{G}^A_\odot(\mathcal{G}^A(B)) \cup (E_{v^A(B)} \cap A)$; and

(ii) by Step 2, for each $v \in [0,1]$ such that $v > v^A(B)$, $\mathcal{G}^A_\odot(\mathcal{G}^A(B)) \cup (E_v \cap A) = \mathcal{G}^A_\odot(B) \cup (E_v \cap A) \succ B \succsim \mathcal{G}^A(B)$;

thus $v^A(\mathcal{G}^A(B)) = v^A(B)$, so $\mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(B)$, as desired.

To conclude, if $\mathcal{G}^A(B) \sim \mathcal{G}^A(B')$, then $\mathcal{G}^A(B) = \mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(\mathcal{G}^A(B')) = \mathcal{G}^A(B')$, as desired. □

While the basic properties given above hold for all erodable events and their transforms, we are most interested in the greedy transforms of erodable events that satisfy at least one of the following requirements:

Definition 20. Let $(\mathcal{A}, \supseteq, \succsim)$ be a qualitative probability space and let $n \in \mathbb{N}$. An event $A \in \mathcal{A}$ is *n-divisible* if and only if for each $\alpha \in \mathcal{A}^\bullet$ such that $\alpha \subseteq A$, there are $I \subseteq \mathbb{N}$, pairwise-disjoint $\{B_i\}_{i\in I} \subseteq \mathcal{A}$, and $I_1, I_2, \ldots, I_n$ partitioning I such that

(i) for each $i \in I$, $\alpha \succ B_i$;

(ii) for each $j \in \{1, 2, \ldots, n\}$, $\bigcup_{I_j} B_i \succsim \alpha$; and

(iii) for each $i \in I$, $B_i \subseteq A$.

The terminology is justified by the upcoming lemmas, but already we can make two observations. First, a qualitative probability space satisfies *n-AS* if and only if its largest event is n-divisible. Second, 1-divisibility may be viewed as an ordinal analogue to the condition of Kakeya, which we discussed in Section 3.2.

Along the lines of this second observation, the 1-Division Lemma (Lemma 12) may be viewed as an ordinal analogue of Kakeya's observation: if an erodable event is 1-divisible, then applying its greedy transform to an input event that is no more likely yields an output that is as likely as the input:

Lemma 12 (1-Division Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, then for each $A \in \mathcal{A}^\mathcal{E}$ that is 1-divisible and each $B \in \mathcal{A}$ such that $A \succsim B$, we have $B \sim \mathcal{G}^A(B)$.*

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ satisfy the hypotheses, let $A \in \mathcal{A}^\mathcal{E}$, and let $B \in \mathcal{A}$ such that $B \succsim A$. Finally, let $\{\mathcal{G}^A_s\}_{s\in\{0\}\cup\mathbb{N}}$, $\mathcal{G}^A_\odot$ and $v^A(B)$ be as in the definition of *greedy transform of A given $\mathcal{E}$*. By the Greedy Basics Lemma (Lemma 11), $B \succsim \mathcal{G}^A(B)$; it remains to show $\mathcal{G}^A(B) \succsim B$. Define $A^\odot \equiv A \cap S^\odot$ and $A^\circ \equiv A \cap S^\circ$.

Step 1: $\mathcal{G}^A_\odot(B) \cup A^\circ \succsim B$. Since $\mathcal{G}^A_\odot(B) \subseteq A \subseteq S^\odot$, thus there are $I^\bullet_A \subseteq I^\bullet$ and $I^\bullet_G \subseteq I^\bullet_A$ such that $A^\odot = \bigcup_{I^\bullet_A} \alpha_s$ and $\mathcal{G}^A_\odot(B) = \bigcup_{I^\bullet_G} \alpha_s$. We consider three cases:

Case 1: $|I^\bullet_A \setminus I^\bullet_G| = 0$. Then $\mathcal{G}^A_\odot(B) = A^\odot$, so $\mathcal{G}^A_\odot(B) \cup A^\circ = A^\odot \cup A^\circ = A \succsim B$, as desired.

Case 2: $|I^\bullet_A \setminus I^\bullet_G| \in \mathbb{N}$. Define $s^* \equiv \max(I^\bullet_A \setminus I^\bullet_G)$ and define $L \equiv A \setminus (\bigcup_{s\leq s^*} \alpha_s)$. By construction, $\mathcal{G}^A_{s^*-1}(B) \cup \alpha_{s^*} \succ B$. Since A is 1-divisible, thus by *monotonicity* $L \succsim \alpha_{s^*}$, so by *separability*,

$$\begin{aligned}\mathcal{G}^A_{\odot}(B) \cup A^\circ &= \mathcal{G}^A_{s^*-1}(B) \cup L \\ &\succsim \mathcal{G}^A_{s^*-1}(B) \cup \alpha_{s^*} \\ &\succ B,\end{aligned}$$

as desired.

Case 3: $|I^\bullet_A \setminus I^\bullet_G| = |\mathbb{N}|$. By construction, for each $s \in I^\bullet_A \setminus I^\bullet_G$, $\mathcal{G}^A_{s-1}(B) \cup \alpha_s \succ B$. Thus by *continuity*,

$$\begin{aligned}\mathcal{G}^A_{\odot}(B) &= \lim_{s \in I^\bullet_A \setminus I^\bullet_G} \mathcal{G}^A_{s-1}(B) \cup \alpha_s \\ &\succsim B,\end{aligned}$$

so by *monotonicity*, $\mathcal{G}^A_{\odot}(B) \cup A^\circ \succsim B$, as desired.

Step 2: *Conclude*. If $A^\circ \sim \varnothing$, then as our space is idealized $A^\circ = \varnothing$, so $\mathcal{G}^A(B) = \mathcal{G}^A_{\odot}(B) \cup A^\circ \succsim B$ and we are done; thus let us assume $A^\circ \succ \varnothing$. Then by the Erosion Lemma (Lemma 10), there is $\mu^\circ \in \mathbb{M}^\sigma_{\mathrm{NA}}(\mathcal{A}^\circ)$ that represents comparisons on $\mathcal{A}^\circ$ such that for each $v \in [0,1]$, $\mu^\circ(E_v) = v$. Since $\mu^\circ(S^\circ \setminus A) = \mu^\circ(S^\circ \setminus A^\circ) = \mu^\circ(S^\circ) - \mu^\circ(A^\circ) = 1 - \mu^\circ(A^\circ)$, and since $A \in \mathcal{A}^{\mathcal{E}}$, thus $(S^\circ \setminus A) = E_{1-\mu^\circ(A^\circ)}$.

Since $\mathcal{G}^A_{\odot}(B) \cup A^\circ \succsim B \succsim \mathcal{G}^A_{\odot}(B)$, thus by the Carving Lemma (Lemma 7), there is $B^\circ \subseteq A^\circ$ such that $\mathcal{G}^A_{\odot}(B) \cup B^\circ \sim B$. Since $\mu^\circ((S^\circ \setminus A^\circ) \cup B^\circ) = \mu^\circ(S^\circ \setminus A^\circ) + \mu^\circ(B^\circ) = \mu^\circ(S^\circ) - \mu^\circ(A^\circ) + \mu^\circ(B^\circ) = 1 - \mu^\circ(A^\circ) + \mu^\circ(B^\circ)$, thus $\mu^\circ(B^\circ) + (1 - \mu^\circ(A^\circ)) \in [0,1]$; define $v^* \equiv \mu^\circ(B^\circ) + (1 - \mu^\circ(A^\circ))$.

Since $v^* \geq 1 - \mu^\circ(A^\circ)$, thus $E_{v^*} \supseteq E_{1-\mu^\circ(A^\circ)}$, so

$$\begin{aligned}\mu^\circ(E_{v^*} \cap A) &= \mu^\circ\big(E_{v^*} \setminus (S^\circ \setminus A)\big) \\ &= \mu^\circ(E_{v^*} \setminus E_{1-\mu^\circ(A^\circ)}) \\ &= \mu^\circ(E_{v^*}) - \mu^\circ(E_{1-\mu^\circ(A^\circ)}) \\ &= v^* - \big(1 - \mu^\circ(A^\circ)\big) \\ &= \mu^\circ(B^\circ),\end{aligned}$$

so $(E_{v^*} \cap A) \sim B^\circ$. By *separability*, $\mathcal{G}^A_{\odot}(B) \cup (E_{v^*} \cap A) \sim \mathcal{G}^A_{\odot}(B) \cup B^\circ \sim B$, so by *monotonicity*, $\mathcal{G}^A(B) = \mathcal{G}^A_{\odot}(B) \cup (E_{v^A(B)} \cap A) \succsim \mathcal{G}^A_{\odot}(B) \cup (E_{v^*} \cap A) \sim B$. □

The Greedy Removal Lemma (Lemma 13) states that if A is $(n+1)$-divisible, then removing an image of its greedy transform yields a subevent that is n-divisible. This is an ordinal analogue of an observation of Vilmos Komornik, which I received through private communication about a related problem of fair division.

Lemma 13 (Greedy Removal Lemma). *If* $(\mathcal{A}, \supseteq, \succsim)$ *is an idealized space, then for each* $n \in \mathbb{N}$*, each* $A \in \mathcal{A}^{\mathcal{E}}$ *that is* $(n+1)$*-divisible, and each* $B \in \mathcal{A}$*,*

$$A \setminus \mathcal{G}^A(B) \text{ is } n\text{-divisible.}$$

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$, n, A, and B satisfy the hypotheses. Define $G \equiv \mathcal{G}^A(B)$. If $B \succsim A$, then it is straightforward to show that $A \setminus G = S^\circ \setminus E_1 \sim \varnothing$, so as our space is idealized $A \setminus G = \varnothing$ and we are done; thus let us assume $A \succ B$. Since A is 1-divisible, thus by the 1-Division Lemma (Lemma 12), $B \sim G$.

We wish to show $A \setminus G$ is n-divisible. If $A \setminus G$ contains no atoms then we are done; thus let $s^* \in I^\bullet$ such that $\alpha_{s^*} \subseteq A \setminus G$. Since $s^* \in I^\bullet$, thus there is $s^{**} \in I^\bullet$ such that (i) $s^* \sim s^{**}$; and (ii) for each $s' \in I^\bullet$ such that $s' > s^{**}$, $\alpha_{s^*} \succ \alpha_{s'}$. Define $A^{\leftarrow} \equiv (\bigcup_{s' \leq s^{**}} \alpha_{s'}) \cap A$, define $A^{\rightarrow} \equiv A \setminus A^{\leftarrow}$, and define $A^{\leftarrow *} \equiv A^{\leftarrow} \setminus G$. The notation is intended to suggest that the atoms contained in A are arranged from left to right, with higher-index atoms to the right, and that moreover, the largest infinitely divisible event contained in A is even further to the right than all of these atoms.

Define $A_1 \equiv A$, define $(B_1; B_2, B_3, \ldots, B_{n+1}) \equiv (G \cup A^{\leftarrow *}; \alpha_{s^*}, \alpha_{s^*}, \ldots, \alpha_{s^*})$, and for each $i \in \{1, 2, \ldots, n+1\}$, define

- $G_i \equiv \mathcal{G}^{A_i}(B_i)$, and
- $A_{i+1} \equiv A_i \setminus G_i$.

By the Greedy Basics Lemma (Lemma 11), each of the above transforms is associated with an erodable event, so each of these events is well-defined.

Step 1: $G_1 = G \cup A^{\leftarrow *}$. Throughout this step, for each $C \in \mathcal{A}$ and each $s \in \mathbb{N} \cup \{\infty\}$, we write $[C]_s$ to denote $C \cap (\bigcup_{s' \leq s} \alpha_{s'})$. We freely use the fact that as our space is idealized, each atom contains only the empty set and itself; thus for each $C \in \mathcal{A}$ and each $s \in I^\bullet$, $\alpha_s \nsubseteq C$ implies $\alpha_s \subseteq (S \setminus C)$.

We first prove that $[G_1]_\infty = [G \cup A^{\leftarrow *}]_\infty$ by induction. For the base step, $[G_1]_0 = \varnothing = [G \cup A^{\leftarrow *}]_0$. For the inductive step, assume $s \in \mathbb{N}$ is such that $[G_1]_s = [G \cup A^{\leftarrow *}]_s$. If $s+1 \notin I^\bullet$ or $\alpha_{s+1} \nsubseteq A$, then $[G_1]_{s+1} = [G_1]_s = [G \cup A^{\leftarrow *}]_s = [G \cup A^{\leftarrow *}]_{s+1}$ and we are done, so assume $s+1 \in I^\bullet$ and $\alpha_{s+1} \subseteq A$.

If $\alpha_{s+1} \subseteq [G \cup A^{\leftarrow *}]_{s+1}$, then by *monotonicity,*

$$\begin{aligned} G \cup A^{\leftarrow *} &\succsim \left[G \cup A^{\leftarrow *}\right]_{s+1} \\ &= \left[G \cup A^{\leftarrow *}\right]_s \cup \alpha_{s+1} \\ &= [G_1]_s \cup \alpha_{s+1} \\ &= \left[\mathcal{G}^A\left(G \cup A^{\leftarrow *}\right)\right]_s \cup \alpha_{s+1}, \end{aligned}$$

so by construction, $\alpha_{s+1} \subseteq [\mathcal{G}^A(G \cup A^{\leftarrow *})]_{s+1}$ and thus $[G_1]_{s+1} = [G \cup A^{\leftarrow *}]_{s+1}$.

If $\alpha_{s+1} \nsubseteq [G \cup A^{\leftarrow *}]_{s+1} = [G \cup A^{\leftarrow}]_{s+1} = [G]_{s+1} \cup [A^{\leftarrow}]_{s+1}$, then $\alpha_{s+1} \nsubseteq A^{\leftarrow}$; thus $\alpha_{s+1} \subseteq S \setminus A^{\leftarrow}$ and $\alpha_{s+1} \subseteq A$, so $\alpha_{s+1} \subseteq A^{\rightarrow}$, so $s+1 > s^{**}$. Then $[G \cup A^{\leftarrow *}]_s = [G]_s \cup$

$[A^{\leftarrow *}]_s = [G]_s \cup A^{\leftarrow *}$. Moreover, $\alpha_{s+1} \nsubseteq [G]_{s+1}$, so by construction and the Greedy Basics Lemma (Lemma 11),

$$\begin{aligned} [G]_s \cup \alpha_{s+1} &= \left[\mathcal{G}^A(B)\right]_s \cup \alpha_{s+1} \\ &\succ B \\ &\succsim \mathcal{G}^A(B) \\ &= G, \end{aligned}$$

so since (i) $[G]_s \cap A^{\leftarrow *} \subseteq G \cap A^{\leftarrow *} = \varnothing$, and (ii) $\alpha_{s+1} \subseteq S \setminus A^{\leftarrow} \subseteq S \setminus A^{\leftarrow *}$ and thus $\alpha_{s+1} \cap A^{\leftarrow *} = \varnothing$, altogether by *monotonicity* and *separability*, $[G]_s \cup \alpha_{s+1} \cup A^{\leftarrow *} \succ G \cup A^{\leftarrow *}$. Thus

$$\begin{aligned} \left[\mathcal{G}^A\left(G \cup A^{\leftarrow *}\right)\right]_s \cup \alpha_{s+1} &= [G_1]_s \cup \alpha_{s+1} \\ &= \left[G \cup A^{\leftarrow *}\right]_s \cup \alpha_{s+1} \\ &= [G]_s \cup \alpha_{s+1} \cup A^{\leftarrow *} \\ &\succ G \cup A^{\leftarrow *}, \end{aligned}$$

so by construction, $\alpha_{s+1} \nsubseteq [\mathcal{G}^A(G \cup A^{\leftarrow *})]_{s+1}$ and thus $[G_1]_{s+1} = [G \cup A^{\leftarrow *}]_{s+1}$.

As $s \in \mathbb{N}$ was arbitrary, thus by induction, $[G_1]_\infty = [G \cup A^{\leftarrow *}]_\infty$, as desired. If $S^\circ \sim \varnothing$, then as our space is idealized $S^\circ = \varnothing$ and we are done; thus let us assume $S^\circ \succ \varnothing$. Since $B \sim G$, thus by *separability*, for each $v \in [0,1]$, $[\mathcal{G}^A(B)]_\infty \cup (E_v \cap A) \succsim B$ if and only if $[G]_\infty \cup (E_v \cap A) \succsim G$ if and only if

$$\begin{aligned} \left[\mathcal{G}^A\left(G \cup A^{\leftarrow *}\right)\right]_\infty \cup (E_v \cap A) &= [G_1]_\infty \cup (E_v \cap A) \\ &= \left[G \cup A^{\leftarrow *}\right]_\infty \cup (E_v \cap A) \\ &= [G]_\infty \cup \left[A^{\leftarrow *}\right]_\infty \cup (E_v \cap A) \\ &= [G]_\infty \cup (E_v \cap A) \cup A^{\leftarrow *} \\ &\succsim G \cup A^{\leftarrow *}. \end{aligned}$$

Thus by construction, $G_1 = \mathcal{G}^A(G \cup A^{\leftarrow *}) = G \cup A^{\leftarrow *}$, as desired.

Step 2: *For each $i \in \{1, 2, \dots, n+1\}$, A_i is 1-divisible.* Let $i \in \{1, 2, \dots, n+1\}$. If $i = 1$ we are done; thus let us assume $i > 1$ and define $J \equiv \{1, 2, \dots, i-1\}$. Then $A_i = A \setminus (\bigcup_J G_j)$, so we wish to show $A \setminus (\bigcup_J G_j)$ is 1-divisible. If $A \setminus (\bigcup_J G_j)$ contains no atoms then we are done; thus let $s^+ \in I^\bullet$ such that $\alpha_{s^+} \subseteq A \setminus (\bigcup_J G_j)$.

Since $s^+ \in I^\bullet$, thus there is $s^{++} \in I^\bullet$ such that (i) $s^+ \sim s^{++}$; and (ii) for each $s' \in I^\bullet$ such that $s' > s^{++}$, $\alpha_{s^+} \succ \alpha_{s'}$; define $A^{++\rightarrow} \equiv A \setminus (\bigcup_{s' \leq s^{++}} \alpha_{s'})$. Since A is $(n+1)$-divisible and $n + 1 \geq i$, thus there are pairwise-disjoint $D_1, D_2, \dots, D_i \subseteq A^{++\rightarrow}$ such that for each $j \in \{1, 2, \dots, i\}$, $D_j \succsim \alpha_{s^+}$. It is straightforward to show, using construction, the Greedy Basics Lemma (Lemma 11), and *separability*, that for each $j \in J$, $\alpha_{s^+} \succsim$

$G_j \cap A^{++\to}$, which implies $D_j \succsim G_j \cap A^{++\to}$. By repeated application of the Domination Lemma (Lemma 4),

$$\begin{aligned} D_1 &\succsim G_1 \cap A^{++\to}, \\ D_1 \cup D_2 &\succsim (G_1 \cap A^{++\to}) \cup (G_2 \cap A^{++\to}), \\ &\vdots \\ \bigcup_{j=1}^{i-1} D_j &\succsim \bigcup_{j=1}^{i-1} (G_j \cap A^{++\to}). \end{aligned}$$

It cannot be that $\alpha_{s^+} \succ A^{++\to} \setminus (\bigcup_J G_j)$, else $D_i \succ A^{++\to} \setminus (\bigcup_J G_j)$, so by the Domination Lemma (Lemma 4) and the Algebra Lemma (Lemma 1),

$$\begin{aligned} \bigcup_{j=1}^{i} D_j &\succ \left[\bigcup_{j=1}^{i-1} (G_j \cap A^{++\to})\right] \cup \left[A^{++\to} \setminus \left(\bigcup_J G_j\right)\right] \\ &= \left[A^{++\to} \cap \left(\bigcup_J G_j\right)\right] \cup \left[A^{++\to} \setminus \left(\bigcup_J G_j\right)\right] \\ &= A^{++\to}, \end{aligned}$$

contradicting *monotonicity*. Thus $(A \setminus (\bigcup_J G_j)) \setminus (\bigcup_{s' \leq s^{++}} \alpha_{s'}) = A^{++\to} \setminus (\bigcup_J G_j) \succsim \alpha_s$, and it is straightforward to use Theorem V2 and the Carving Lemma (Lemma 7) to write $(A \setminus (\bigcup_J G_j)) \setminus (\bigcup_{s' \leq s^{++}} \alpha_{s'})$ as a countable union of subevents of $A \setminus (\bigcup_J G_j)$ that are each less likely than α_{s^+}, and to do so with the index set contained in $\mathbb{N}$. Since $s^+ \in I^\bullet$ with $\alpha_{s^+} \subseteq A \setminus (\bigcup_J G_j)$ was arbitrary, we are done.

Step 3: *Conclude.* It is straightforward to show, using construction, the Greedy Basics Lemma (Lemma 11), and *separability*, that $\alpha_{s^*} \succsim G \cap A^{\to}$. Since A is $(n+1)$-divisible, there are pairwise-disjoint $D_1, D_2, \ldots, D_{n+1} \subseteq A^{\to}$ such that for each $i \in \{1, 2, \ldots, n+1\}$, $D_i \succsim \alpha_{s^*}$. In particular, we have that $D_1 \succsim G \cap A^{\to}$. Assume, by way of contradiction, there is $i \in \{2, 3, \ldots, n+1\}$ such that $\alpha_{s^*} \succ A_i$. Let i^* be the least such i. Then $D_{i^*} \succsim \alpha_{s^*} \succ A_{i^*}$, and by the Greedy Basics Lemma (Lemma 11), for each $i \in \{2, 3, \ldots, i^* - 1\}$, $D_i \succsim \alpha_{s^*} \succsim G_i$. But then by repeated application of the Domination Lemma (Lemma 4) as in Step 2, and by Step 1,

$$\begin{aligned} \bigcup_{i=1}^{i^*} D_i &\succ [G \cap A^{\to}] \cup \left[\bigcup_{i=2}^{i^*-1} G_i\right] \cup [A_{i^*}] \\ &= \left[(G \cap A^{\to}) \cup (A^{\leftarrow *} \cap A^{\to})\right] \cup \left[\bigcup_{i=2}^{i^*-1} G_i\right] \cup \left[(A \setminus G_1) \setminus \bigcup_{i=2}^{i^*-1} G_i\right] \\ &= \left[(G \cup A^{\leftarrow^*}) \cap A^{\to}\right] \cup [A \setminus G_1] \\ &= [G_1 \cap A^{\to}] \cup [A \setminus (G \cup A^{\leftarrow *})] \end{aligned}$$

$$\begin{aligned}
&= [G_1 \cap A^{\rightarrow}] \cup [A^{\rightarrow} \setminus (G \cup A^{\leftarrow})] \\
&= [G_1 \cap A^{\rightarrow}] \cup [(A^{\rightarrow} \setminus G) \cap (A^{\rightarrow} \setminus A^{\leftarrow})] \\
&= [G_1 \cap A^{\rightarrow}] \cup [(A^{\rightarrow} \setminus G) \cap A^{\rightarrow}] \\
&= A^{\rightarrow},
\end{aligned}$$

contradicting *monotonicity*.

Thus for each $i \in \{2, 3, \dots, n+1\}$, $A_i \succsim \alpha_{s^*}$, and moreover by Step 2, A_i is 1-divisible, so by the 1-Division Lemma (Lemma 12), $G_i \sim \alpha_{s^*}$. Altogether, by Step 1, $G_2, G_3, \dots, G_{n+1}$ are n pairwise-disjoint subevents of $A_2 = A \setminus G_1 = A \setminus (G \cup A^{\leftarrow *}) = (A \setminus G) \setminus (\bigcup_{s' \leq s^{**}} \alpha_{s'})$, each at least as likely as α_{s^*}. It is straightforward to use Theorem V2 and the Carving Lemma (Lemma 7) to write each as a countable union of subevents of $A \setminus G$ that are each less likely than α_{s^*}, and to do so such that the index sets partition their union contained in $\mathbb{N}$. Since $s^* \in I^\bullet$ such that $\alpha_{s^*} \subseteq A \setminus G$ was arbitrary, we are done. □

It follows from the 1-Division Lemma (Lemma 12) and the Greedy Removal Lemma (Lemma 13) that given an n-divisible event A and a list of target events $(B_1, B_2, \dots, B_n)$, we can iteratively apply n greedy transforms to create a pairwise-disjoint output list $(G_1, G_2, \dots, G_n)$, with each output as likely as the corresponding input, so long as the outputs do not collectively "exhaust" A. Indeed, this technique of iteratively applying greedy transforms was used in the above proof.

Because our focus is 3-*AS*, we usually perform this technique for a short list of target events, and therefore it is not particularly convenient to state the above observation formally. However, there is a different observation about this technique for which we do require a formal statement, and it is for this reason that we introduce the technique formally:

Definition 21. For each $n \in \mathbb{N}$,

- an *n-list* is an ordered list $(B_1, B_2, \dots, B_n) \in \mathcal{A}^n$;
- an *n-list sequence* is a sequence $(B_1^i, B_2^i, \dots, B_n^i)_{i \in \mathbb{N}} \in (\mathcal{A}^n)^{\mathbb{N}}$; and
- a *monotonic n-list sequence* is an n-list sequence $(B_1^i, B_2^i, \dots, B_n^i)_{i \in \mathbb{N}} \in (\mathcal{A}^n)^{\mathbb{N}}$ such that for each $j \in \{1, 2, \dots, n\}$, either
  - (i) $B_j^1 \succsim B_j^2 \succsim \cdots$, or
  - (ii) $B_j^1 \precsim B_j^2 \precsim \cdots$

Our technique of iteratively applying greedy transforms takes as input an n-list and outputs an n-list:

Definition 22. Let $(\mathcal{A}, \supseteq, \succsim)$ be an idealized space, let $A \in \mathcal{A}^{\mathcal{E}}$, and let $n \in \mathbb{N}$. The *greedy n-transform of A (given $\mathcal{E}$)*, which abusing notation we write $\mathcal{G}^{n,A} : \mathcal{A}^n \to \mathcal{A}^n$, is defined as follows. For each $(B_1, B_2, \dots, B_n) \in \mathcal{A}^n$, $\mathcal{G}^{n,A}(B_1, B_2, \dots, B_n) \in \mathcal{A}^n$ is defined such that for each $j \in \{1, 2, \dots, n\}$,

$$\mathcal{G}^{n,A}_j(B_1, B_2, \dots, B_n) \equiv \mathcal{G}^{A\setminus[\bigcup_{j'<j} \mathcal{G}^A_j(B_1,B_2,\dots,B_n)]}(B_j).$$

By the Greedy Basics Lemma (Lemma 11), each of the above transforms is associated with an erodable event, so each of these events is well-defined.

The List-Transform Lemma (Lemma 14) states that for each erodable event and each monotonic sequence of input n-lists, the associated sequence of output n-lists converges pointwise:

Lemma 14 (List-Transform Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, then for each $A \in \mathcal{A}^{\mathcal{E}}$, each $n \in \mathbb{N}$, each monotonic n-list sequence $(B^i_1, B^i_2, \dots, B^i_n)_{i\in\mathbb{N}}$, and each $j \in \{1, 2, \dots, n\}$, $(\mathcal{G}^{n,A}_j(B^i_1, B^i_2, \dots, B^i_n))_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ is convergent.*

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$, A, n, and $(B^1_i, B^2_i, \dots, B^n_i)_{i\in\mathbb{N}}$ satisfy the hypotheses. For each $i \in \mathbb{N}$ and each $j \in \{1, 2, \dots, n\}$, define $G^i_j \in \mathcal{A}$ by

$$G^i_j \equiv \mathcal{G}^{n,A}_j\big(B^i_1, B^i_2, \dots, B^i_n\big).$$

We wish to prove $(G^i_1), (G^i_2), \dots, (G^i_n)$ are convergent. We proceed by induction, covering the base step with our inductive hypothesis: assume $j^* \in \{1, 2, \dots, n\}$ is such that for each $j \in \{1, 2, \dots, n\}$ such that $j < j^*$, (G^i_j) is convergent. We claim $(G^i_{j^*})$ is convergent.

For each $i \in \mathbb{N}$, define $G^{i\odot}_{j^*} \equiv G^i_{j^*} \cap S^{\odot}$ and define $G^{i\circ}_{j^*} \equiv G^i_{j^*} \cap S^{\circ}$.

Step 1: *The sequence $(G^{i\odot}_{j^*})$ is convergent.* Throughout this step, for each $C \in \mathcal{A}$ and each $s \in \mathbb{N}$, we write $[C]_s$ to denote $C \cap (\bigcup_{s'\leq s} \alpha_{s'})$. We freely use the fact that as our space is idealized, each atom contains only the empty set and itself; thus for each $j \in \{1, 2, \dots, n\}$, $(G^{i\odot}_j)$ is convergent if and only if for each $s \in I^\bullet$, there is $i^{**} \in \mathbb{N}$ such that for each pair $i, i' \geq i^{**}$, $G^i_j \cap \alpha_s = G^{i'}_j \cap \alpha_s$.

Within the current inductive argument, we make a second inductive argument, again covering the base step with our inductive hypothesis: assume $s \in \mathbb{N}$ is such that for each $s' \in \mathbb{N}$ such that $s' < s$, there is $i^* \in \mathbb{N}$ such that for each pair $i, i' \geq i^*$, $G^i_{j^*} \cap \alpha_{s'} = G^{i'}_{j^*} \cap \alpha_{s'}$. If $s \notin I^\bullet$ or $\alpha_s \nsubseteq A$, then we are done; thus let us assume $s \in I^\bullet$ and $\alpha_s \subseteq A$.

By the inductive hypothesis on j^*, for each $j \in \{1, 2, \dots, n\}$ such that $j < j^*$, there is $i^* \in \mathbb{N}$ such that for each pair $i, i' \geq i^*$, $G^i_j \cap \alpha_s = G^{i'}_j \cap \alpha_s$. Altogether, there is $i^* \in \mathbb{N}$ such that for each pair $i, i' \geq i^*$,

(i) $s' < s$ implies $G^i_{j^*} \cap \alpha_{s'} = G^{i'}_{j^*} \cap \alpha_{s'}$, and

(ii) $j < j^*$ implies $G^i_j \cap \alpha_s = G^{i'}_j \cap \alpha_s$.

Let this final i^* remain fixed.

Assume, by way of contradiction, that for each $i^{**} \in \mathbb{N}$, there are $i_1 \geq i^{**}$ and $i_2 \geq i^{**}$ such that $\alpha_s \subseteq G^{i_1}_{j^*}$ and $\alpha_s \nsubseteq G^{i_2}_{j^*+1}$. Then there are $i_1, i_2, i_3 \in \mathbb{N}$ with $i_3 > i_2 > i_1 > i^*$ such that $\alpha_s \subseteq G^{i_1}_{j^*}$, $\alpha_s \nsubseteq\in G^{i_2}_{j^*}$, and $\alpha_s \subseteq G^{i_3}_{j^*}$.

By definition of i^*, $[G^{i_1}_{j^*}]_{s-1} = [G^{i_2}_{j^*}]_{s-1} = [G^{i_3}_{j^*}]_{s-1}$. By construction,

$$\begin{aligned} B^{i_1}_{j^*} &\succsim [G^{i_1}_{j^*}]_{s-1} \cup \alpha_s \\ &= [G^{i_2}_{j^*}]_{s-1} \cup \alpha_s \\ &\succ B^{i_2}_{j^*}, \end{aligned}$$

so since $(B^i_1, B^i_2, \dots, B^i_n)$ is a monotonic n-list sequence, thus $(B^i_{j^*})$ is non-increasing in likelihood. But by construction,

$$\begin{aligned} B^{i_3}_{j^*} &\succsim [G^{i_3}_{j^*}]_{s-1} \cup \alpha_s \\ &= [G^{i_2}_{j^*}]_{s-1} \cup \alpha_s \\ &\succ B^{i_2}_{j^*}, \end{aligned}$$

contradicting that $(B^i_{j^*})$ is non-increasing in likelihood.

Thus there is $i^{**} \in \mathbb{N}$ such that for each pair $i, i' \geq i^{**}$, $G^i_{j^*} \cap \alpha_s = G^{i'}_{j^*} \cap \alpha_s$. By induction on s, $(G^{i\odot}_{j^*})$ is convergent, as desired.

Step 2: *The sequence $(G^{i\circ}_{j^*})$ is convergent.* For each $i \in \mathbb{N}$, define $C^+_i \equiv \bigcup_{j \leq j^*} (G^i_j \cap S^\circ)$ and define $C^-_i \equiv \bigcup_{j<j^*} (G^i_j \cap S^\circ)$. By construction, for each $i \in \mathbb{N}$, $\{(G^i_1 \cap S^\circ), (G^i_2 \cap S^\circ), \dots, (G^i_{j^*} \cap S^\circ)\}$ is pairwise-disjoint; thus $C^+_i \setminus C^-_i = (G^i_{j^*} \cap S^\circ)$.

By construction, for each $i \in \mathbb{N}$, $C^+_i \in \mathcal{E}$ and $C^-_i \in \mathcal{E}$. Since $\mathcal{E}$ is nested, (C^+_i) and (C^-_i) each have subsequences that are monotone with respect to set inclusion, and therefore are convergent. By the Algebra Lemma (Lemma 1), there is $\lim(S \setminus C^-_i)$, and moreover there is $\lim C^+_i \cap \lim(S \setminus C^-_i)$, and moreover

$$\begin{aligned} \lim C^+_i \cap \lim\bigl(S \setminus C^-_i\bigr) &= \lim\bigl(C^+_i \cap \bigl(S \setminus C^-_i\bigr)\bigr) \\ &= \lim\bigl(G^i_{j^*} \cap S^\circ\bigr) \\ &= \lim G^{i\circ}_{j^*}, \end{aligned}$$

as desired.

Step 3: *Conclude.* By Step 1 and Step 2, $(G^{i\odot}_{j^*})$ and $(G^{i\circ}_{j^*})$ are convergent, so by the Algebra Lemma (Lemma 1), there is $\lim(G^{i\odot}_{j^*} \cup G^{i\circ}_{j^*}) = \lim(G^i_{j^*})$. By induction on j^*, $(G^i_1), (G^i_2), \dots, (G^i_n)$ are convergent. $\square$

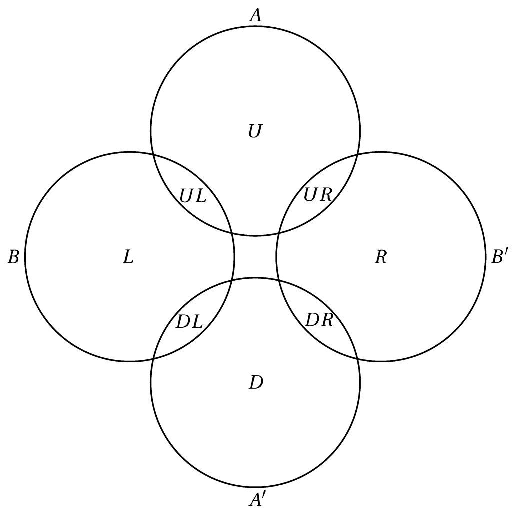

FIGURE 4. Euler diagram for A, B, A', and B'. For example, $UL \equiv A \cap B$ and $D \equiv A' \setminus (B \cup B')$.

APPENDIX F

In this appendix, we prove our primary lemmas about qualitative halves: the Half-Equivalence Lemma (Lemma 15), the First Halving Lemma (Lemma 16), and the Second Halving Lemma (Lemma 17).

As discussed in Appendix D, our broad approach for proving Theorem 2 is based on the observation that if two disjoint events are equally likely, then in any σ-measure representation, the measure of both must be half the measure of their union. In this case, we might say that the disjoint events are each "halves" of any event that is as likely as their union. This appendix is dedicated to pursuing this concept.

To begin, the Half-Equivalence Lemma (Lemma 15) states that this notion of a half is well-defined in the sense that if an event has many halves, then all of them are equally likely. This is a particularly general result in that it applies to all qualitative probability spaces, and it is our last such result in this article:

LEMMA 15 (Half-Equivalence Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is a qualitative probability space, then for each disjoint pair $A, A' \in \mathcal{A}$ and each disjoint pair $B, B' \in \mathcal{A}$, if*

(i) $A \sim A'$,

(ii) $B \sim B'$, *and*

(iii) $A \cup A' \sim B \cup B'$,

then $A \sim B$.

PROOF. Assume, by way of contradiction, $A \nsim B$; without loss of generality, assume $A \succ B$. For convenience, label the components of the Euler diagram for A, A', B, and B' according to Figure 4.

We claim $DL \cup D \succ UR \cup R$. Otherwise, by *separability*,

$$\begin{aligned} B &\sim B' \\ &= DR \cup (UR \cup R) \\ &\succsim DR \cup (DL \cup D) \\ &= A' \\ &\sim A, \end{aligned}$$

contradicting $A \succ B$.

We claim $L \cup R \succsim U \cup D$. Otherwise, by *separability*,

$$\begin{aligned} A \cup A' &= (UL \cup UR \cup DL \cup DR) \cup (U \cup D) \\ &\succ (UL \cup UR \cup DL \cup DR) \cup (L \cup R) \\ &= B \cup B', \end{aligned}$$

contradicting $A \cup A' \sim B \cup B'$. Similarly, $U \cup D \succsim L \cup R$, so $L \cup R \sim U \cup D$.

But then by *separability*,

$$\begin{aligned} (L \cup R) \cup (UL \cup UR) &\sim (U \cup D) \cup (UL \cup UR) \\ &= A \cup D \\ &\succ B \cup D \\ &= (UL \cup L) \cup (DL \cup D) \\ &\succ (UL \cup L) \cup (UR \cup R), \end{aligned}$$

contradicting $L \cup R \cup UL \cup UR \sim L \cup R \cup UL \cup UR$. □

As halves are well-defined, we next move to the problem of producing them. To begin, the First Halving Lemma (Lemma 16) states any 1-divisible event can be partitioned into two halves; we prove this using an approach similar to the greedy transforms of the previous appendix, except that instead of iteratively growing an event so long as it is never more likely than a target event, we iteratively grow an event so long as it is never more likely than its complement:

Lemma 16 (First Halving Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space, then for each $A \in \mathcal{A}$ such that A is 1-divisible and $A \succ \varnothing$, there is $H \subseteq A$ such that $H \sim (A \setminus H)$ and $A \succ H \succ \varnothing$.*

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ and A satisfy the hypotheses. Define $A^{\odot} \equiv A \cap S^{\odot}$; define $A^{\circ} \equiv A \cap S^{\circ}$; define $H_0 \equiv \varnothing$; and for each $s \in \mathbb{N}$, define

$$H_s \equiv \begin{cases} (H_{s-1} \cup \alpha_s), & s \in I^{\bullet}, \alpha_s \subseteq A, \text{and } A \setminus (H_{i-1} \cup \alpha_s) \succsim (H_{i-1} \cup \alpha_s); \\ H_{s-1}, & \text{else.} \end{cases}$$

Finally, define $H^{\odot} \equiv \cup H_i$.

By construction, for each $s \in \mathbb{N}$, $(A \setminus H_s) \succsim H_s$; thus by the Algebra Lemma (Lemma 1) and the Limit-Order Lemma (Lemma 6),

$$\begin{aligned} A \setminus H^{\odot} &= \lim A \cap (S \setminus \lim H_s) \\ &= \lim A \cap \lim(S \setminus H_s) \\ &= \lim(A \setminus H_s) \\ &\succsim \lim H_s \\ &= H^{\odot}. \end{aligned}$$

As $H^{\odot} \subseteq A^{\odot} \subseteq S^{\odot}$, thus there are $I^{\bullet}_A \subseteq I^{\bullet}$ and $I^{\bullet}_H \subseteq I^{\bullet}_A$ such that $H^{\odot} = \bigcup_{I^{\bullet}_H} \alpha_s$ and $A^{\odot} = \bigcup_{I^{\bullet}_A} \alpha_s$. We claim that there is $H \subseteq A$ such that $H \sim (A \setminus H)$, which we establish in two cases:

Case 1: $|I^{\bullet}_A \setminus I^{\bullet}_H| < |\mathbb{N}|$. First, we claim $(H^{\odot} \cup A^{\circ}) \succsim A \setminus (H^{\odot} \cup A^{\circ})$. Indeed, if $|I^{\bullet}_A \setminus I^{\bullet}_H| = 0$, then $H^{\odot} = A^{\odot}$, so

$$\begin{aligned} \left(H^{\odot} \cup A^{\circ}\right) &= A \\ &\succsim \varnothing \\ &= A \setminus A \\ &= A \setminus \left(H^{\odot} \cup A^{\circ}\right). \end{aligned}$$

If $|I^{\bullet}_A \setminus I^{\bullet}_H| \in \mathbb{N}$, then define $s^* \equiv \max(I^{\bullet}_A \setminus I^{\bullet}_H)$, and define $A^{\rightarrow} \equiv A \setminus (\bigcup_{s \leq s^*} \alpha_s)$. Since A is 1-divisible, thus by *monotonicity* $A^{\rightarrow} \succsim \alpha_{s^*}$, so by *separability*, construction, and the Complement Lemma (Lemma 3),

$$\begin{aligned} H^{\odot} \cup A^{\circ} &= H_{s^*-1} \cup A^{\rightarrow} \\ &\succsim H_{s^*-1} \cup \alpha_{s^*} \\ &\succsim A \setminus (H_{s^*-1} \cup \alpha_{s^*}) \\ &\succsim A \setminus \left(H_{s^*-1} \cup A^{\rightarrow}\right) \\ &= A \setminus \left(H^{\odot} \cup A^{\circ}\right). \end{aligned}$$

Thus we have $(H^{\odot} \cup A^{\circ}) \succsim A \setminus (H^{\odot} \cup A^{\circ})$, as claimed. Moreover, by our earlier argument, $A \setminus H^{\odot} \succsim H^{\odot}$. If $A^{\circ} \sim \varnothing$, then as our space is idealized $A^{\circ} = \varnothing$ and we are done; thus let us assume $A^{\circ} \succ \varnothing$.[23]

Define $\mathcal{A}^* \equiv \{A \in \mathcal{A} | A \subseteq A^{\circ}\}$; it is straightforward to show that this is a σ-algebra. Define $\supseteq^*$, $\succsim^*$ on $\mathcal{A}^*$ as follows: for each pair $A, A' \in \mathcal{A}^*$, (i) $A \supseteq^* A'$ if and only if $A \supseteq A'$, and (ii) $A \succsim^* A'$ if and only if $A \succsim A'$. Since $A^{\circ} \succ \varnothing$, it is straightforward to verify that $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space, so by the Erosion Lemma (Lemma 10), it has erosion

[23] In fact, it can be proven that we must have $A^{\circ} \succ \varnothing$ in this case.

$\mathcal{E}^* = \{E^*_v\}_{v\in[0,1]}$. Define

$$V \equiv \big\{v \in [0,1] | A \setminus \big(H^{\odot} \cup E^*_v\big) \succsim H^{\odot} \cup E^*_v\big\}, \text{ and}$$
$$v^* \equiv \sup V.$$

For each $v \in [0,1]$, $v > v^*$ implies $v \notin V$ by definition, and $v < v^*$ implies $v \in V$ by *monotonicity*.

First, we claim $A \setminus (H^{\odot} \cup E^*_{v^*}) \succsim H^{\odot} \cup E^*_{v^*}$. If $v^* = 0$, then $E^*_{v^*} = \varnothing$ and we are done; thus let us assume $v^* > 0$. Let (v_i) be an increasing sequence converging to v^*; then for each $i \in \mathbb{N}$, $A \setminus (H^{\odot} \cup E^*_{v_i}) \succsim H^{\odot} \cup E^*_{v_i}$. By the Algebra Lemma (Lemma 1) and the Limit-Order Lemma (Lemma 6),

$$\begin{aligned}
A \setminus \big(H^{\odot} \cup E^*_{v^*}\big) &= A \cap \big(S \setminus \big(\lim H^{\odot} \cup \lim E^*_{v_i}\big)\big) \\
&= A \cap \big(S \setminus \lim\big(H^{\odot} \cup E^*_{v_i}\big)\big) \\
&= \lim A \cap \lim\big(S \setminus \big(H^{\odot} \cup E^*_{v_i}\big)\big) \\
&= \lim\big(A \setminus \big(H^{\odot} \cup E^*_{v_i}\big)\big) \\
&\succsim \lim\big(H^{\odot} \cup E^*_{v_i}\big) \\
&= \lim H^{\odot} \cup \lim E^*_{v_i} \\
&= H^{\odot} \cup E^*_{v^*},
\end{aligned}$$

as desired.

Second, we claim $A \setminus (H^{\odot} \cup E^*_{v^*}) \precsim H^{\odot} \cup E^*_{v^*}$. If $v^* = 1$, then it is straightforward to show that $A^\circ \setminus E_{v^*} \sim \varnothing$, so as our space is idealized, thus $E_{v^*} = A^\circ$ and we are done; thus let us assume $v^* < 1$. To complete the argument, simply take the corresponding argument in the previous paragraph, then replace each instance of "increasing" with "decreasing" and each instance of $\succsim$ with $\precsim$.

Altogether, then, $A \setminus (H^{\odot} \cup E^*_{v^*}) \sim H^{\odot} \cup E^*_{v^*}$, as desired.

Case 2: $|I^\bullet_A \setminus I^\bullet_H| = |\mathbb{N}|$. By construction, for each $s \in I^\bullet_A \setminus I^\bullet_H$, we have $H_{s-1} \cup \alpha_s \succ A \setminus (H_{s-1} \cup \alpha_s)$. It is then straightforward to show, using the Limit-Order Lemma (Lemma 6) and the Algebra Lemma (Lemma 1), that

$$\begin{aligned}
H^{\odot} &= \lim_{s\in I^\bullet_A \setminus I^\bullet_H} (H_{s-1} \cup \alpha_s) \\
&\succsim \lim_{s\in I^\bullet_A \setminus I^\bullet_H} A \setminus (H_{s-1} \cup \alpha_s) \\
&= A \setminus H^{\odot},
\end{aligned}$$

so as $A \setminus H^{\odot} \succsim H^{\odot}$, thus $H^{\odot} \sim A \setminus H^{\odot}$, as desired.

Thus in both cases, there is $H \subseteq A$ such that $H \sim A \setminus H$. To conclude, since $A \succ \varnothing$, necessarily $A \succ H$, else $H \succsim A$ and $A \setminus H \sim H \succsim A \succ \varnothing$, so by the Domination Lemma

(Lemma 4), $A = H \cup (A \setminus H) \succ A$, contradicting $A \sim A$. Moreover, $H \succ \varnothing$, else $\varnothing \succsim H$ and $\varnothing \succsim H \sim A \setminus H$, so by the Domination Lemma (Lemma 4), $\varnothing \succsim H \cup (A \setminus H) = A \succ \varnothing$, contradicting $\varnothing \sim \varnothing$. Altogether, then, $A \succ H \succ \varnothing$, as desired. □

Finally, the Second Halving Lemma (Lemma 17) states that under 2-*AS*, each event can be associated with two disjoint events analogous to its halves. After proving this result, we will have at last assembled the tools we need to construct our supercabinet, and unsurprisingly this final result proves crucial:

Lemma 17 (Second Halving Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space that satisfies* 2-*AS, then for each $A \in \mathcal{A}$ such that $A \succ \varnothing$, there are disjoint $H(A), H'(A) \in \mathcal{A}$ such that*

(i) $A \sim H(A) \cup H'(A)$, *and*

(ii) $A \succ H(A) \sim H'(A) \succ \varnothing$.

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ and A satisfy the hypotheses. By *monotonicity*, $S \succsim S \setminus A$, so as S is 1-divisible, thus by the 1-Division Lemma (Lemma 12), $\mathcal{G}^S(S \setminus A) \sim S \setminus A$. Since $A \succ \varnothing$, thus by the Complement Lemma (Lemma 3), $S \setminus \mathcal{G}^S(S \setminus A) \sim S \setminus (S \setminus A) = A \succ \varnothing$. Since S is 2-divisible, thus by the Greedy Removal Lemma (Lemma 13), $S \setminus \mathcal{G}^S(S \setminus A)$ is 1-divisible. Altogether, by the First Halving Lemma (Lemma 16), there are disjoint $H(A), H'(A) \subseteq S \setminus \mathcal{G}^S(S \setminus A)$ such that

(i) $A \sim S \setminus \mathcal{G}^S(S \setminus A) = H(A) \cup H'(A)$, and

(ii) $A \sim S \setminus \mathcal{G}^S(S \setminus A) \succ H(A) \sim H'(A) \succ \varnothing$,

as desired. □

Appendix G

In this appendix we prove our second main result, which implies that *monotone continuity* and 3-*AS* are together sufficient to guarantee σ-measure representation.

We first prove the Supercabinet Construction Lemma (Lemma 18), which states that for idealized spaces, 3-*AS* guarantees the existence of a supercabinet. All of the tools developed in the previous two appendices were developed for the purpose of establishing this one result:

Lemma 18 (Supercabinet Construction Lemma). *If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space that satisfies* 3-*AS, then it has a supercabinet.*

Proof. Let $(\mathcal{A}, \supseteq, \succsim)$ satisfy the hypotheses. By 3-*AS*, S is 3-divisible. We proceed through a series of 11 steps; the only notation carried from one step to the following steps is the notation in the step's statement. It is straightforward to verify that each greedy transform we use throughout the proof is indeed associated with an erodable event, and we omit these observations.

In the first two steps, for each $q \in \{0, 1, \dots\}$, we construct events $\{A_q^p\}_{q\in\{0,1,\dots,2^q\}} \subseteq \mathcal{A}$; the notation anticipates that when at last we complete our supercabinet $\{\mathcal{Z}_v\}_{v\in\mathbb{2}}$, the event A_q^p will be the output of the greedy transform $\mathcal{G}^S$ when the input is any event from $\mathcal{Z}_{\frac{p}{2^q}}$. Though we establish that using these events to construct such a collection $\{\mathcal{Z}_v\}_{v\in\mathbb{2}}$ is well-defined as early as Step 4, it is convenient to use the numerator–denominator notation to establish several properties that are useful for verifying that this $\{\mathcal{Z}_v\}_{v\in\mathbb{2}}$ is in fact a supercabinet; we therefore postpone introducing $\{\mathcal{Z}_v\}_{v\in\mathbb{2}}$ until Step 9. The final two steps are dedicated to verifying that it is in fact a supercabinet.

Step 1: *Define* $\{A_q^1\}_{q\in\{0,1,\dots\}}$, $\{H(A_q^1)\}_{q\in\{0,1,\dots\}}$, $\{H'(A_q^1)\}_{q\in\{0,1,\dots\}} \subseteq \mathcal{A}$ *such that for each* $q \in \{0, 1, \dots\}$,

(i) $H(A_q^1) \cap H'(A_q^1) = \varnothing$,

(ii) $A_q^1 \sim H(A_q^1) \cup H'(A_q^1)$, *and*

(iii) $A_q^1 \succ A_{q+1}^1 \sim H(A_q^1) \sim H'(A_q^1) \succ \varnothing$.

We proceed recursively. Define $A_0^1 \equiv \mathcal{G}^S(S)$. By the 1-Division Lemma (Lemma 12), $A_0^1 \sim S$, so by *monotonicity* and *nondegeneracy*, $A_0^1 \succ \varnothing$.

Suppose we have $A_q^1 \in \mathcal{A}$ such that $A_q^1 \succ \varnothing$. By 2-*AS*, we can apply the Second Halving Lemma (Lemma 17); thus there are disjoint $H(A_q^1)$ and $H'(A_q^1)$ such that

(i) $A_q^1 \sim H(A_q^1) \cup H'(A_q^1)$, and

(ii) $A_q^1 \succ H(A_q^1) \sim H'(A_q^1) \succ \varnothing$.

Define $A_{q+1}^1 \equiv \mathcal{G}^S(H(A_q^1))$; by the Half-Equivalence Lemma (Lemma 15) this is well-defined. By *monotonicity*, $S \succsim H(A_q^1)$, so by the 1-Division Lemma (Lemma 12), $A_{q+1}^1 \sim H(A_q^1)$.

Step 2: *For each* $q \in \{0, 1, \dots\}$, *define* $\{A_q^p\}_{q\in\{0,1,\dots,2^q\}} \subseteq \mathcal{A}$ *such that*

(i) *for each* $p \in \{0, 1, \dots, 2^q - 1\}$, $A_q^{p+1} \sim A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1)$; *and*

(ii) *for each* $p \in \{0, 1, \dots, 2^q\}$, $\mathcal{G}^S(A_q^p) = A_q^p$.

Let $q \in \{0, 1, \dots\}$. Define $A_q^0 \equiv \varnothing$, and for each $p \in \{0, 1, \dots, 2^q - 1\}$, define

$$A_q^{p+1} \equiv \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1)\big).$$

By *monotonicity*, $S \succsim A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1)$, so by the 1-Division Lemma (Lemma 12), $A_q^{p+1} \sim A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1)$. (Note that by the Greedy Basics Lemma (Lemma 11), this definition gives the same $\{A_q^1\}_{q\in\{0,1,\dots\}}$ defined before.)

Using the definition of erosion, $\mathcal{G}^S(A_q^0) = \mathcal{G}^S(\varnothing) = \varnothing = A_q^0$. For each $p \in \{0, 1, \ldots, 2^q - 1\}$, by the Greedy Basics Lemma (Lemma 11), $\mathcal{G}^S(A_q^{p+1}) = \mathcal{G}^S[\mathcal{G}^S(A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1))] = \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1)) = A_q^{p+1}$.

Step 3: *For each $q \in \{0, 1, \ldots\}$ and each $p \in \{0, 1, \ldots, 2^q\}$, $S \setminus A_q^p$ is 2-divisible.* Let $q \in \{0, 1, \ldots\}$ and $p \in \{0, 1, \ldots, 2^q\}$. Since S is 3-divisible, thus by Step 2 and the Greedy Removal Lemma (Lemma 13), $S \setminus A_q^p = S \setminus \mathcal{G}^S(A_q^p)$ is 2-divisible.

Step 4: *For each $q \in \{0, 1, \ldots\}$,*

(i) $p \in \{0, 1, \ldots, 2^q - 1\}$ implies $\mathcal{G}^{S\setminus A_q^p}(A_q^1) \sim A_q^1$, and

(ii) $p \in \{0, 1, \ldots, 2^q\}$ implies $A_q^p = A_{q+1}^{2p}$.

We proceed by induction on q. For the base step, let $q = 0$. By the Greedy Basics Lemma (Lemma 11), $\mathcal{G}^{S\setminus A_q^0}(A_q^1) = \mathcal{G}^{S\setminus\varnothing}(A_q^1) = \mathcal{G}^S(A_q^1) = \mathcal{G}^S(\mathcal{G}^S(S)) = \mathcal{G}^S(S) = A_q^1$. For the inductive hypothesis, assume $q \in \{0, 1, \ldots\}$ is such that for each $p \in \{0, 1, \ldots, 2^q - 1\}$, $\mathcal{G}^{S\setminus A_q^p}(A_q^1) \sim A_q^1$.

Within the current inductive argument, we make a second inductive argument on p. For the base step, $A_q^0 = \varnothing = A_{q+1}^0$. For the inductive hypothesis, assume $p \in \{0, 1, \ldots, 2^q - 1\}$ is such that $A_q^p = A_{q+1}^{2p}$. For convenience, define $G_1, G_2, G_2' \in \mathcal{A}$ by

- $G_1 \equiv \mathcal{G}^{S\setminus A_{q+1}^{2p}}(A_{q+1}^1)$,
- $G_2 \equiv \mathcal{G}^{S\setminus A_{q+1}^{2p+1}}(A_{q+1}^1)$, and
- $G_2' \equiv \mathcal{G}^{S\setminus [A_{q+1}^{2p} \cup G_1]}(A_{q+1}^1)$.

We make three claims, which we prove in sequence before concluding:

Claim 1: $G_1 \sim A_{q+1}^1$,

Claim 2: $G_2 \sim A_{q+1}^1$, and

Claim 3: $A_q^{p+1} = A_{q+1}^{2(p+1)}$.

Proof of Claim 1: By the hypothesis on p, $A_q^p = A_{q+1}^{2p}$, so by *monotonicity*, the hypothesis on q, and Step 1,

$$
\begin{aligned}
S \setminus A_{q+1}^{2p} &= S \setminus A_q^p \\
&\succsim \mathcal{G}^{S\setminus A_q^p}(A_q^1) \\
&\sim A_q^1 \\
&\succ A_{q+1}^1.
\end{aligned}
$$

Since by Step 3, $S \setminus A_{q+1}^{2p}$ is 2-divisible, thus by the 1-Division Lemma (Lemma 12), $G_1 = \mathcal{G}^{S\setminus A_{q+1}^{2p}}(A_{q+1}^1) \sim A_{q+1}^1$.

Proof of Claim 2: Since by construction and Claim 1, $H(A_q^1) \sim A_{q+1}^1 \sim G_1$, necessarily $S \setminus [A_{q+1}^{2p} \cup G_1] \succsim H'(A_q^1)$, else by Step 1, the Domination Lemma (Lemma 4), construction, the hypothesis on p, *monotonicity*, and the hypothesis on q,

$$\begin{aligned} A_q^1 &\sim H(A_q^1) \cup H'(A_q^1) \\ &\succ G_1 \cup S \setminus [A_{q+1}^{2p} \cup G_1] \\ &= S \setminus A_{q+1}^{2p} \\ &= S \setminus A_q^p \\ &\succsim \mathcal{G}^{S\setminus A_q^p}(A_q^1) \\ &\sim A_q^1, \end{aligned}$$

contradicting $A_q^1 \sim A_q^1$.

By Step 2, $A_{q+1}^{2p+1} \sim A_{q+1}^{2p} \cup G_1$. By the Complement Lemma (Lemma 3) and the above argument, $S \setminus A_{q+1}^{2p+1} \sim S \setminus [A_{q+1}^{2p} \cup G_1] \succsim H'(A_q^1) \sim A_{q+1}^1$. By Step 3, $S \setminus A_{q+1}^{2p+1}$ is 2-divisible, so by the 1-Division Lemma (Lemma 12), $G_2 = \mathcal{G}^{S\setminus A_{q+1}^{2p+1}}(A_{q+1}^1) \sim A_{q+1}^1$.

Proof of Claim 3: As argued in the proof of Claim 2, $S \setminus [A_{q+1}^{2p} \cup G_1] \succsim H'(A_q^1) \sim A_{q+1}^1$. By Step 3, $S \setminus A_{q+1}^{2p}$ is 2-divisible, so by the Greedy Removal Lemma (Lemma 13), $(S \setminus A_{q+1}^{2p}) \setminus G_1 = S \setminus [A_{q+1}^{2p} \cup G_1]$ is 1-divisible. Thus by the 1-Division Lemma (Lemma 12), $G_2' \sim A_{q+1}^1$.

By Claim 1 and the above argument, $H(A_q^1) \sim G_1$ and $H'(A_q^1) \sim G_2'$, so by Step 1 and two applications of the Domination Lemma (Lemma 4), $A_q^1 \sim H(A_q^1) \cup H'(A_q^1) \sim G_1 \cup G_2'$. By the hypothesis on p and the hypothesis on q, $\mathcal{G}^{S\setminus A_{q+1}^{2p}}(A_q^1) = \mathcal{G}^{S\setminus A_q^p}(A_q^1) \sim A_q^1 \sim G_1 \cup G_2'$. By Step 2, the hypothesis on p, and *separability*,

$$\begin{aligned} A_q^{p+1} &\sim A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1) \\ &= A_{q+1}^{2p} \cup \mathcal{G}^{S\setminus A_{q+1}^{2p}}(A_q^1) \\ &\sim A_{q+1}^{2p} \cup (G_1 \cup G_2') \\ &= [A_{q+1}^{2p} \cup G_1] \cup G_2'. \end{aligned}$$

By Step 2, $[A_{q+1}^{2p} \cup G_1] \sim A_{q+1}^{2p+1}$. By the first paragraph of this claim's proof and Claim 2, $G_2' \sim A_{q+1}^1 \sim G_2$. Thus by two applications of the Domination Lemma (Lemma 4),

$[A_{q+1}^{2p} \cup G_1] \cup G_2' \sim A_{q+1}^{2p+1} \cup G_2$. By Step 2, $A_{q+1}^{2p+1} \cup G_2 \sim A_{q+1}^{2p+2}$. Altogether, $A_q^{p+1} \sim A_{q+1}^{2p+2}$, so $\mathcal{G}^S(A_q^{p+1}) = \mathcal{G}^S(A_{q+1}^{2p+2})$, so by Step 2, $A_q^{p+1} = A_{q+1}^{2p+2}$.

To conclude: by induction on p, if $q \in \{0, 1, \ldots\}$ is such that for each $p \in \{0, 1, \ldots, 2^q - 1\}$, $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim A_q^1$, then

(i) for each $p \in \{0, 1, \ldots, 2^{q+1} - 1\}$, $\mathcal{G}^{S \setminus A_{q+1}^p}(A_{q+1}^1) \sim A_{q+1}^1$; and

(ii) for each $p \in \{0, 1, \ldots, 2^q\}$, $A_q^p = A_{q+1}^{2p}$.

By induction on q, we are done.

STEP 5: *For each* $q \in \{0, 1, \ldots\}$ *and each* $p \in \{0, 1, \ldots, 2^q\}$, $A_q^p \sim S \setminus A_q^{2^q - p}$. Let $q \in \{0, 1, \ldots\}$. For each $p \in \{0, 1, \ldots, 2^q\}$, define $B_q^p \equiv S \setminus A_q^{2^q - p}$. We proceed by induction on p.

For the base step: by Step 4, Step 1, the 1-Division Lemma (Lemma 12), the Complement Lemma (Lemma 3), and Step 2, $B_q^0 = S \setminus A_q^{2^q} = S \setminus A_0^1 = S \setminus \mathcal{G}^S(S) \sim S \setminus S = \varnothing = A_q^0$, so $A_q^0 \sim B_q^0$. For the inductive hypothesis, assume $p \in \{0, 1, \ldots, 2^q - 1\}$ is such that $A_q^p \sim B_q^p$.

Define $A, B, C', B', A' \in \mathcal{A}$ by

$$\begin{aligned} A &\equiv A_q^p, \\ B &\equiv \mathcal{G}^{S \setminus A}\big(A_q^1\big), \\ C' &\equiv A_q^{2^q - (p+1)}, \\ B' &\equiv \mathcal{G}^{S \setminus C'}\big(A_q^1\big), \text{ and} \\ A' &\equiv S \setminus \big(C' \cup B'\big). \end{aligned}$$

By Step 2, $C' \cup B' \sim A_q^{2^q - p}$. By the Complement Lemma (Lemma 3), $A' = S \setminus (C' \cup B') \sim S \setminus A_q^{2^q - p} = B_q^p$. By the hypothesis on p, $A' \sim A_q^p = A$. By Step 4, $B \sim A_q^1 \sim B'$. Thus by two applications of the Domination Lemma (Lemma 4), $A \cup B \sim A' \cup B'$.

By Step 2, $A \cup B \sim A_q^{p+1}$, and by definition, $A' \cup B' = S \setminus A_q^{2^q - (p+1)} = B_q^{p+1}$, so altogether $A_q^{p+1} \sim B_q^{p+1}$. By induction on p, we are done.

STEP 6: *Define the binary operation* $\uplus$. For each $q \in \{0, 1, \ldots\}$ and each pair $p, p' \in \{0, 1, \ldots, 2^q\}$ such that $p + p' \leq 2^q$, define $A_q^p \uplus A_q^{p'} \in \mathcal{A}$ by

$$A_q^p \uplus A_q^{p'} \equiv \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}\big(A_q^{p'}\big)\big).$$

STEP 7: *For each* $q \in \{0, 1, \ldots\}$ *and each pair* $p, p' \in \{0, 1, \ldots, 2^q\}$ *such that* $p + p' \leq 2^q$,

$$A_q^p \uplus A_q^{p'} = A_q^{p+p'}.$$

Let $q \in \{0, 1, \dots\}$ and let $p \in \{0, 1, \dots, 2^q\}$. We proceed by induction on p'. For the base step, let $p' = 0$. Then by Step 2, $A_q^p \uplus A_q^{p'} = \mathcal{G}^S(A_q^p) = A_q^p$.

For the inductive hypothesis, assume $p' \in \{0, 1, \dots, [2^q - p] - 1\}$ is such that $A_q^p \uplus A_q^{p'} = A_q^{p+p'}$. Define $A, B, C \in \mathcal{A}$ by

$$A \equiv A_q^p,$$
$$B \equiv \mathcal{G}^{S \setminus A}\big(A_q^{p'}\big), \text{ and}$$
$$C \equiv \mathcal{G}^{S \setminus (A \cup B)}\big(A_q^1\big).$$

By hypothesis, $A_q^{p+p'} = A_q^p \uplus A_q^{p'} = \mathcal{G}^S(A \cup B)$. Since S is 1-divisible and, by *monotonicity*, $S \succsim A \cup B$, thus by the 1-Division Lemma (Lemma 12), $\mathcal{G}^S(A \cup B) \sim A \cup B$. Altogether, $A_q^{p+p'} \sim A \cup B$.

By *monotonicity* and Step 4, $S \setminus A_q^{p+p'} \succsim \mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim A_q^1$, so by the Complement Lemma (Lemma 3), $S \setminus (A \cup B) \sim S \setminus A_q^{p+p'} \succsim A_q^1$. By Step 3, $S \setminus A$ is 2-divisible, so by the Greedy Removal Lemma (Lemma 13), $(S \setminus A) \setminus B = S \setminus (A \cup B)$ is 1-divisible. Thus by the 1-Division Lemma (Lemma 12), $A_q^1 \sim C$. By Step 4, $\mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim A_q^1$, so $\mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim C$.

Since $A_q^{p+p'} \sim A \cup B$ and $\mathcal{G}^{S \setminus A_q^{p+p'}}(A_q^1) \sim C$, thus by Step 2 and two applications of the Domination Lemma (Lemma 4),

$$\begin{aligned} A_q^{p+p'+1} &\sim A_q^{p+p'} \cup \mathcal{G}^{S \setminus A_q^{p+p'}}\big(A_q^1\big) \\ &\sim (A \cup B) \cup C. \end{aligned}$$

By Step 5, $S \setminus A_q^p \sim A_q^{2^q - p}$. Since $2^q - p \geq p' + 1$, thus by Step 2 and *monotonicity*, $A_q^{2^q - p} \succsim A_q^{p'+1}$, so $S \setminus A_q^p \succsim A_q^{p'+1}$. By Step 3, $S \setminus A_q^p$ is 2-divisible, so by the 1-Division Lemma (Lemma 12), $\mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1}) \sim A_q^{p'+1}$.

Since $S \setminus A_q^p \succsim A_q^{p'+1}$, by Step 2 and *monotonicity*, $S \setminus A_q^p \succsim A_q^{p'}$. By Step 3, $S \setminus A_q^p$ is 2-divisible, so by the 1-Division Lemma (Lemma 12), $A_q^{p'} \sim B$. By Step 4, $\mathcal{G}^{S \setminus A_q^{p'}}(A_q^1) \sim A_q^1$, and as argued above, $A_q^1 \sim C$, so $\mathcal{G}^{S \setminus A_q^{p'}}(A_q^1) \sim C$. Thus by Step 2 and two applications of the Domination Lemma (Lemma 4), $A_q^{p'+1} \sim A_q^{p'} \cup \mathcal{G}^{S \setminus A_q^{p'}}(A_q^1) \sim B \cup C$.

Altogether, $\mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1}) \sim B \cup C$. Since S is 1-divisible and, by *monotonicity*, $S \succsim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^{p'+1})$, thus by the 1-Division Lemma (Lemma 12) and *separability*,

$$\begin{aligned} A_q^p \uplus A_q^{p'+1} &= \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}\big(A_q^{p'+1}\big)\big) \\ &\sim A_q^p \cup \mathcal{G}^{S \setminus A_q^p}\big(A_q^{p'+1}\big) \\ &\sim A_q^p \cup (B \cup C) \\ &= A \cup (B \cup C) \end{aligned}$$

$$= (A \cup B) \cup C$$
$$\sim A_q^{p+p'+1}.$$

By the Greedy Basics Lemma (Lemma 11) and Step 2, $A_q^p \uplus A_q^{p'+1} = A_q^{p+p'+1}$.

By induction on p', for each $p' \in \{0, 1, \ldots, 2^q - p\}$, $A_q^p \uplus A_q^{p'} = A_q^{p+p'}$. Since $q \in \{0, 1, \ldots\}$ and $p \in \{0, 1, \ldots, 2^q\}$ were arbitrary, we are done.

Step 8: $\lim A_q^1 \sim \varnothing$. For each $q \in \mathbb{N}$, define $B_q \equiv \mathcal{G}_1^{2,S}(A_q^1, A_q^1)$ and define $C_q \equiv \mathcal{G}_2^{2,S}(A_q^1, A_q^1)$. By Step 1, $(A_q^1, A_q^1)_{q\in\mathbb{N}}$ is a monotonic 2-list sequence, so by the List-Transform Lemma (Lemma 14), $(B_q)_{q\in\mathbb{N}}$ and $(C_q)_{q\in\mathbb{N}}$ are convergent. By Step 2 and Step 4, for each $q \in \mathbb{N}$, $B_q = A_q^1$ and $C_q \sim A_q^1$.

By Step 1, for each $q \in \{0, 1, \ldots\}$, $H(A_q^1) \sim B_{q+1}$ and $H'(A_q^1) \sim C_{q+1}$, so by two applications of the Domination Lemma (Lemma 4), $A_q^1 \sim H(A_q^1) \cup H'(A_q^1) \sim B_{q+1} \cup C_{q+1}$. By the Algebra Lemma (Lemma 1) and the Limit-Order Lemma (Lemma 6),

$$\begin{aligned} \lim B_q \cup \lim C_q &= \lim B_{q+1} \cup \lim C_{q+1} \\ &= \lim(B_{q+1} \cup C_{q+1}) \\ &\sim \lim A_q^1 \\ &= \lim B_q. \end{aligned}$$

By the Algebra Lemma (Lemma 1),

$$\begin{aligned} \lim B_q \cap \lim C_q &= \lim(B_q \cap C_q) \\ &= \varnothing. \end{aligned}$$

Thus $\lim C_q \sim \varnothing$, else by *monotonicity* and *separability*, $\lim B_q \cup \lim C_q \succ \lim B_q$, contradicting $\lim B_q \cup \lim C_q \sim \lim B_q$. Thus by the Limit-Order Lemma (Lemma 6), $\lim A_q^1 \sim \lim C_q \sim \varnothing$, as desired.

Step 9: *Define* $\{A_v\}_{v\in\mathbb{2}} \subseteq \mathcal{A}$ *and* $\{\mathcal{Z}_v\}_{v\in\mathbb{2}} \subseteq [\mathcal{A}/{\sim}]$ *such that for each pair* $v, v' \in \mathbb{2}$ *such that* $v' > v$, $A_{v'} \succ A_v$. Let $v \in \mathbb{2}$. Then there are $p, q \in \{0, 1, \ldots\}$ such that $p \le 2^q$ and $v = \frac{p}{2^q}$. Define

$$A_v \equiv A_q^p, \text{ and}$$
$$\mathcal{Z}_v \equiv \{A \in \mathcal{A} | A \sim A_v\}.$$

By Step 4, this is well-defined.

Let $v, v' \in \mathbb{2}$ such that $v' > v$. Then there are $p, p', q \in \{0, 1, \ldots\}$ such that $v = \frac{p}{2^q}$, $v' = \frac{p'}{2^q}$, and $p' > p$. By Step 4 and Step 1, $\mathcal{G}^{S\setminus A_q^p}(A_q^1) \sim A_q^1 \succ \varnothing$, so by Step 2 and *separability*,

$A_q^{p+1} \sim A^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^1) \succ A^p$. Repeating this argument, $A_q^{p'} \succ A_q^p$; thus $A_{v'} = A_q^{p'} \succ A_q^p = A_v$, as desired.

Step 10: *For each convergent pair* $(v_i), (v_i') \in 2^{\mathbb{N}}$ *such that* $\lim v_i = \lim v_i'$, *if* $(A_{v_i}), (A_{v_i'}) \in \mathcal{A}^{\mathbb{N}}$ *are convergent, then* $\lim A_{v_i} \sim \lim A_{v_i'}$. Define $v_\infty \equiv \lim v_i = \lim v_i'$, $A_\infty \equiv \lim A_{v_i}$, and $A'_\infty \equiv \lim A_{v_i'}$. Assume, by way of contradiction, $A_\infty \nsim A'_\infty$; without loss of generality, assume $A_\infty \succ A'_\infty$. Define $G \equiv \mathcal{G}^S(A'_\infty)$. Since S is 1-divisible and, by *monotonicity*, $S \succsim A'_\infty$, thus by the 1-Division Lemma (Lemma 12), $A_\infty \succ A'_\infty \sim G$.

Necessarily $S \setminus G \succ \varnothing$, else by the Complement Lemma (Lemma 3) and *monotonicity*, $G \succsim S \succsim A_\infty$, contradicting $A_\infty \succ G$. Then there is $q^{**} \in \mathbb{N}$ such that $S \setminus G \succ A^1_{q^{**}}$, else by Step 8 and *continuity*, $\varnothing \sim \lim A_q^1 \succsim S \setminus G$, contradicting $S \setminus G \succ \varnothing$.

Since S is 3-divisible, thus by the Greedy Removal Lemma (Lemma 13), $S \setminus G$ is 2-divisible, so by Step 1 and the 1-Division Lemma (Lemma 12), for each $q \in \mathbb{N}$ such that $q \geq q^{**}$, $\mathcal{G}^{S\setminus G}(A_q^1) \sim A_q^1$. Then by Step 1, $(\mathcal{G}^{S\setminus G}(A_q^1))_{q \geq q^{**}} \in \mathcal{A}^{\mathbb{N}}$ is a monotonic 1-list sequence, so by the List-Transform Lemma (Lemma 14), it is convergent.

By the Algebra Lemma (Lemma 1),

$$\begin{aligned} G \cap \lim \mathcal{G}^{S\setminus G}(A_q^1) &= \lim G \cap \lim \mathcal{G}^{S\setminus G}(A_q^1) \\ &= \lim\big(G \cap \mathcal{G}^{S\setminus G}(A_q^1)\big) \\ &= \varnothing. \end{aligned}$$

By the Limit-Order Lemma (Lemma 6) and Step 8, $\lim \mathcal{G}^{S\setminus G}(A_q^1) \sim \lim A_q^1 \sim \varnothing$, so by *separability*, $G \cup \lim \mathcal{G}^{S\setminus G}(A_q^1) \sim G$. By the Algebra Lemma (Lemma 1),

$$\begin{aligned} G \cup \lim \mathcal{G}^{S\setminus G}(A_q^1) &= \lim G \cup \lim \mathcal{G}^{S\setminus G}(A_q^1) \\ &= \lim\big(G \cup \mathcal{G}^{S\setminus G}(A_q^1)\big), \end{aligned}$$

so $\lim(G \cup \mathcal{G}^{S\setminus G}(A_q^1)) \sim G$. Thus there is $q^* \in \mathbb{N}$ such that (i) $q^* \geq q^{**}$ and thus $\mathcal{G}^{S\setminus G}(A^1_{q^*}) \sim A^1_{q^*}$, and (ii) $A_\infty \succ G \cup \mathcal{G}^{S\setminus G}(A^1_{q^*})$, else by *continuity* $G \succsim A_\infty$, contradicting $A_\infty \succ G$.

Before proceeding, we make two observations:

- For each $v \in 2$, $v > v_\infty$ implies $A_v \succ A_\infty$ and $A_v \succ A'_\infty$. Indeed, let $v \in 2$ such that $v > v_\infty$. Then there is $v' \in 2$ such that $v > v' > v_\infty$. Since $\lim v_i = v_\infty$, there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $v' > v_i$. By Step 9, for each $i \geq i^*$, $A_v \succ A_{v'} \succ A_{v_i}$, so by *continuity*, $A_v \succ A_{v'} \succsim A_\infty$. By the same argument, $v > v_\infty$ implies $A_v \succ A'_\infty$.
- For each $v \in 2$, $v_\infty > v$ implies $A_\infty \succ A_v$ and $A'_\infty \succ A_v$. The proof of this observation is analogous to that of the above observation.

We proceed by reaching a contradiction in the following three cases:

Case 1: $v_\infty = 0$. Then $\frac{1}{2^{q^*}} > v_\infty$, so by our first observation and *monotonicity*, $A^1_{q^*} \succ A_\infty \succ G \cup \mathcal{G}^{S\setminus G}(A^1_{q^*}) \succsim \mathcal{G}^{S\setminus G}(A^1_{q^*}) \sim A^1_{q^*}$, contradicting $A^1_{q^*} \sim A^1_{q^*}$.

Case 2: $v_\infty = 1$. Then $v_\infty > \frac{2^{q^*}-1}{2^{q^*}}$, so by our second observation, $G \sim A'_\infty \succ A^{2^{q^*}-1}_{q^*}$. By the Greedy Basics Lemma (Lemma 11), $\mathcal{G}^{S\setminus G}(A^1_{q^*}) \sim A^1_{q^*} \succsim \mathcal{G}^{S\setminus A^{2^{q^*}-1}_{q^*}}(A^1_{q^*})$, so $A_\infty \succ G \cup \mathcal{G}^{S\setminus G}(A^1_{q^*}) \succ A^{2^{q^*}-1}_{q^*} \cup \mathcal{G}^{S\setminus A^{2^{q^*}-1}_{q^*}}(A^1_{q^*})$.

Since S is 1-divisible and, by *monotonicity*, $S \succsim A^{2^{q^*}-1}_{q^*} \cup \mathcal{G}^{S\setminus A^{2^{q^*}-1}_{q^*}}(A^1_{q^*})$, thus by the 1-Division Lemma (Lemma 12), $A^{2^{q^*}-1}_{q^*} \cup \mathcal{G}^{S\setminus A^{2^{q^*}-1}_{q^*}}(A^1_{q^*}) \sim A^{2^{q^*}-1}_{q^*} \uplus A^1_{q^*}$. But then by Step 7, Step 4, Step 1, the 1-Division Lemma (Lemma 12), and *monotonicity*, $A_\infty \succ A^{2^{q^*}-1}_{q^*} \uplus A^1_{q^*} = A^{2^{q^*}}_{q^*} = A^1_0 = \mathcal{G}^S(S) \sim S \succsim A_\infty$, contradicting $A_\infty \sim A_\infty$.

Case 3: $v_\infty \in (0,1)$. Define $\epsilon^* \equiv \frac{1}{2^{q^*}}$. Since $\mathcal{2}$ is dense in $[0,1]$, there is $v^* \in \mathcal{2}$ such that $1 \geq v^* + \epsilon^* > v_\infty > v^*$. By our two observations, $A_{v^*+\epsilon^*} \succ A_\infty \succ G \sim A'_\infty \succ A_{v^*}$.

Since $1 - v^* \geq \epsilon^*$, thus by Step 5 and Step 9, $S \setminus A_{v^*} \sim A_{1-v^*} \succsim A_{\epsilon^*} = A^1_{q^*}$. Then by Step 3 and the 1-Division Lemma (Lemma 12), $\mathcal{G}^{S\setminus G}(A^1_{q^*}) \sim A^1_{q^*} \sim \mathcal{G}^{S\setminus A_{v^*}}(A^1_{q^*})$. Since $G \succ A_{v^*}$, thus by the Domination Lemma (Lemma 4), $A_\infty \succ G \cup \mathcal{G}^{S\setminus G}(A^1_{q^*}) \succ A_{v^*} \cup \mathcal{G}^{S\setminus A_{v^*}}(A^1_{q^*})$.

Since S is 1-divisible and, by *monotonicity*, $S \succsim A_{v^*} \cup \mathcal{G}^{S\setminus A_{v^*}}(A^1_{q^*})$, thus by the 1-Division Lemma (Lemma 12), $A_{v^*} \cup \mathcal{G}^{S\setminus A_{v^*}}(A^1_{q^*}) \sim A_{v^*} \uplus A_{\epsilon^*}$. But then by Step 7, $A_\infty \succ A_{v^*} \uplus A_{\epsilon^*} = A_{v^*+\epsilon^*}$, contradicting $A_{v^*+\epsilon^*} \succ A_\infty$.

Step 11: *Conclude.* We verify that $\{\mathcal{Z}_v\}_{v\in\mathcal{2}}$ satisfies [SC1], [SC2], [SC3], and [SC4].

[SC1]: By Step 1 and the 1-Division Lemma (Lemma 12), $A_1 = A^1_0 = \mathcal{G}^S(S) \sim S$, so $S \in \mathcal{Z}_1$.

[SC2]: Let $v, v' \in \mathcal{2}$ such that $v + v' \leq 1$. Then there are $p, p', q \in \{0,1,\ldots\}$ such that $v = \frac{p}{2^q}$, $v' = \frac{p'}{2^q}$, and $p + p' \leq 2^q$.

By Step 9, $A^p_q \in \mathcal{Z}_v$, $A^{p'}_q \in \mathcal{Z}_{v'}$, and $A^{p+p'}_q \in \mathcal{Z}_{v+v'}$. Since $p' \leq 2^q - p$, thus by Step 5, Step 2, and *monotonicity*, $S \setminus A^p_q \sim A^{2^q-p}_q \succsim A^{p'}_q$. By Step 3, $S \setminus A^p_q$ is 2-divisible, so by the 1-Division Lemma (Lemma 12),

$$\mathcal{G}^{S\setminus A^p_q}\big(A^{p'}_q\big) \sim A^{p'}_q.$$

Thus we have disjoint $A^p_q \in \mathcal{Z}_v$ and $\mathcal{G}^{S\setminus A^p_q}(A^{p'}_q) \in \mathcal{Z}_{v'}$.

Since S is 1-divisible and, by *monotonicity*, $S \succsim A^p_q \cup \mathcal{G}^{S\setminus A^p_q}(A^{p'}_q)$, thus by the 1-Division Lemma (Lemma 12), $\mathcal{G}^S(A^p_q \cup \mathcal{G}^{S\setminus A^p_q}(A^{p'}_q)) \sim A^p_q \cup \mathcal{G}^{S\setminus A^p_q}(A^{p'}_q)$. Thus by Step 7,

$$A^{p+p'}_q = A^p_q \uplus A^{p'}_q$$

$$= \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S\setminus A_q^p}\big(A_q^{p'}\big)\big)$$

$$\sim A_q^p \cup \mathcal{G}^{S\setminus A_q^p}\big(A_q^{p'}\big),$$

so $A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^{p'}) \in \mathcal{Z}_{v+v'}$, as desired.

[SC3]: Let $(v_i^+) \in 2^{\mathbb{N}}$ be non-increasing and let $(v_i^-) \in 2^{\mathbb{N}}$ be non-decreasing such that $\lim v_i^+ = \lim v_i^-$. For each $i \in \mathbb{N}$, define $A_i^+ \equiv A_{v_i^+}$ and $A_i^- \equiv A_{v_i^-}$. By Step 9, for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$. Moreover, by Step 9, (A_i^+) and (A_i^-) are both monotonic 1-list sequences, so by the List-Transform Lemma (Lemma 14), (A_i^+) and (A_i^-) are both convergent; thus by Step 10, $\lim A_i^+ \sim \lim A_i^-$.

[SC4]: Let $(v_i), (w_i) \in 2^{\mathbb{N}}$ be monotonic such that for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$. For each $i \in \mathbb{N}$, define $A_i \equiv A_{v_i}$ and $B_i \equiv \mathcal{G}^{S\setminus A_i}(A_{w_i})$.

By Step 5, for each $i \in \mathbb{N}$, $S \setminus A_i \sim A_{1-v_i}$, so since $1 - v_i \geq w_i$, thus by Step 9, $S \setminus A_{v_i} \succsim A_{w_i}$. Thus by Step 3 and the 1-Division Lemma (Lemma 12), for each $i \in \mathbb{N}$, $B_i \sim A_{w_i}$. Altogether, for each $i \in \mathbb{N}$, A_i and B_i are disjoint with $A_i \in \mathcal{Z}_{v_i}$ and $B_i \in \mathcal{Z}_{w_i}$.

By Step 9, (A_i, B_i) is a monotonic 2-list sequence; thus by the List-Transform Lemma (Lemma 14), $(\mathcal{G}_1^{2,S}(A_i, B_i))$ and $(\mathcal{G}_2^{2,S}(A_i, B_i))$ are convergent. By construction and the Greedy Basics Lemma (Lemma 11), these are, respectively, (A_i) and (B_i). □

Before proceeding, we make one remark about the above proof: constructing $\{A_q^p\}$, and therefore using $\{A_0^0\} \cup \{A_q^p \mid p \text{ is odd}\}$ to construct the very collection $\{\mathcal{Z}_v\}_{v\in 2}$ that we ultimately prove is a supercabinet, requires only 2-*AS*. Though the above proof requires 3-*AS* to verify that this is a supercabinet, we have no example of a qualitative probability space that satisfies 2-*AS* but not 3-*AS* for which $\{\mathcal{Z}_v\}_{v\in 2}$ is not a supercabinet; in other words, we do not know if 3-*AS* is necessary for the result. If $\{\mathcal{Z}_v\}_{v\in 2}$ were verified to be a supercabinet using only 2-*AS*, then it would be straightforward to modify the following two results to prove a new theorem (for 2-*AS*) that is more general than Theorem 2 (for 3-*AS*) but less general than the conjecture discussed in Section 4 (for 1-*AS*).

We are almost done. From here, we proceed as we did with our first main result, first exploiting the convenience of idealized spaces to prove Proposition 2. This is almost an immediate consequence of our results thus far:

Proposition 2. *Let $n \in \mathbb{N}$ such that $n \geq 3$. If $(\mathcal{A}, \supseteq, \succsim)$ is an idealized space that satisfies n-AS, then it has representation $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A})$. In this case, $(\mathcal{A}, \supseteq, \succsim)$ has no other σ-measure representation.*

Proof. Let $n \in \mathbb{N}$ such that $n \geq 3$ and let $(\mathcal{A}, \supseteq, \succsim)$ be an idealized space that satisfies n-*AS*. Then it satisfies 3-*AS*, so by the Supercabinet Construction Lemma (Lemma 18), it has a supercabinet, so by the Supercabinet Blueprint Lemma (Lemma 9), it has a unique representation $\mu \in \mathbb{M}^{\sigma}(\mathcal{A})$. It remains to show that $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A})$.

It is straightforward to show $\mathcal{A}^{\bullet} = \mathcal{A}^{\bullet|\mu}$. Let $\alpha \in \mathcal{A}^{\bullet|\mu} = \mathcal{A}^{\bullet}$. By n-*AS*, there are $I \subseteq \mathbb{N}$, pairwise-disjoint $\{B_i\}_{i\in I} \subseteq \mathcal{A}$, and $I_1, I_2, \ldots, I_n$ partitioning I such that

(i) for each $i \in I$, $\alpha \succ B_i$; and

(ii) for each $j \in \{1, 2, \dots, n\}$, $\bigcup_{I_j} B_j \succsim \alpha$.

Since μ is a representation, thus for each $j \in \{1, 2, \dots, n\}$, $\mu(\bigcup_{I_j} B_j) \geq \mu(\alpha)$, and moreover,

(i) for each $i \in I$, $\mu(\alpha) > \mu(B_i)$; and

(ii) $\mu(\bigcup B_i) = \mu(\bigcup_{I_1} B_i) + \mu(\bigcup_{I_2} B_i) + \cdots + \mu(\bigcup_{I_n} B_i) \geq n\mu(\alpha)$.

Since $\alpha \in \mathcal{A}^{\bullet|\mu}$ was arbitrary, we are done. □

To conclude, we prove Theorem 2 using the Null-Quotient Lemma (Lemma 8) and Proposition 2. The second half of the proof is written with arguments that parallel those in the proof of Theorem 1 in order to emphasize the generality of their common null-quotient technique:

Theorem 2 (Repeated). *Let $n \in \mathbb{N}$ such that $n \geq 3$. A qualitative probability space $(\mathcal{A}, \supseteq, \succsim)$ satisfies monotone continuity and n-AS if and only if it has representation $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A})$. In this case, $(\mathcal{A}, \supseteq, \succsim)$ has no other σ-measure representation.*

Proof. Let $n \in \mathbb{N}$ such that $n \geq 3$. Let $(\mathcal{A}, \supseteq, \succsim)$ be a qualitative probability space and let $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ be its null-quotient. Let $\cup^*$, $\cap^*$, and $\neg^*$ denote, respectively, supremum, infimum, and complement for $(\mathcal{A}^*, \supseteq^*)$. Let $\varnothing^*$ denote its minimum, and for each pair $[A], [B] \in \mathcal{A}^*$, let $[A] \setminus^* [B]$ denote $[A] \cap^* (\neg^*[B])$.

($\Leftarrow$) Assume $(\mathcal{A}, \supseteq, \succsim)$ has representation $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A})$. By Theorem V1, we have that $(\mathcal{A}, \supseteq, \succsim)$ satisfies *monotone continuity*, so by the Null-Quotient Lemma (Lemma 8), $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space.

Define $\mu^* : \mathcal{A}^* \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu^*([A]) = \mu(A)$; as μ is a representation of $(\mathcal{A}, \supseteq, \succsim)$, thus by the Null-Quotient Lemma (Lemma 8), this is well-defined. By the Null-Quotient Lemma (Lemma 8), $\mu^* \in \mathbb{M}^{\sigma}(\mathcal{A}^*)$ and μ^* represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$. It is straightforward to show that $\mathcal{A}^{\bullet} = \mathcal{A}^{\bullet|\mu}$ and $(\mathcal{A}^*)^{\bullet} = (\mathcal{A}^*)^{\bullet|\mu^*}$.

We claim that $\mu^* \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A}^*)$. Indeed, let $[\alpha] \in (\mathcal{A}^*)^{\bullet|\mu^*} = (\mathcal{A}^*)^{\bullet}$. By the Null-Quotient Lemma (Lemma 8), $\alpha \in \mathcal{A}^{\bullet} = \mathcal{A}^{\bullet|\mu}$. Since $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A})$, thus by Theorem H, there are $I \subseteq \mathbb{N}$ and pairwise-disjoint $\{B_i\}_{i \in I} \subseteq \mathcal{A}$ such that

(i) for each $i \in I$, $\mu^*([\alpha]) = \mu(\alpha) > \mu(B_i) = \mu^*([B_i])$; and

(ii) $\mu^*(\bigcup^*[B_i]) = \mu^*([\bigcup B_i]) = \mu(\bigcup B_i) \geq n\mu(\alpha) = n\mu^*([\alpha])$.

By Theorem H, for each pair $i, j \in I$, $[B_i] \cap^* [B_j] = [B_i \cap B_j] = [\varnothing] = \varnothing^*$, so $\{[B_i]\}_{i \in I}$ is pairwise-disjoint. Since $[\alpha] \in (\mathcal{A}^*)^{\bullet|\mu^*}$ was arbitrary, thus $\mu^* \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A}^*)$, as desired.

To conclude, let $\alpha \in \mathcal{A}^{\bullet}$. By the Null-Quotient Lemma (Lemma 8), $[\alpha] \in (\mathcal{A}^*)^{\bullet} = (\mathcal{A}^*)^{\bullet|\mu^*}$. Define

$$[A^*] \equiv [S] \setminus^* \left(\bigcup\nolimits^* \{[\alpha'] \in (\mathcal{A}^*)^{\bullet|\mu^*} | \mu^*([\alpha']) \geq \mu^*([\alpha])\}\right).$$

Since $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space, this is well-defined. Since $\mu^* \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A}^*)$, thus $\mu^*([A^*]) \geq n\mu^*([\alpha])$. Since $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space, it is straightforward to develop cardinal analogues of the greedy transforms, the 1-Division Lemma (Lemma 12), and the Greedy Removal Lemma (Lemma 13); we omit the formal statements and their proofs, which are simpler than the included proofs for their ordinal analogues. Since $\mu^* \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A}^*)$, if we begin with $[A^*]$ as parent and iteratively apply these cardinal greedy transforms to target $\mu^*([\alpha])$, we construct pairwise-disjoint $\{[G_1], [G_2], \dots, [G_n]\}$ such that for each $j \in \{1, 2, \dots, n\}$, $[G_j] \subseteq [A^*]$ and $\mu^*([G_j]) = \mu^*([\alpha])$. It is straightforward to use Theorem V2 to write each G_j as a countable union $\bigcup^*_{I_j} [B_i]$ of subevents of $[A^*]$ that each have smaller measure than $[\alpha]$, and to do so such that $I_1, I_2, \dots, I_n$ partition their union $I \subseteq \mathbb{N}$. By Theorem H,

(i) for each $i \in I$, $\mu(\alpha) = \mu^*([\alpha]) > \mu^*([B_i]) = \mu(B_i)$; and

(ii) for each $j \in \{1, 2, \dots, n\}$, $\mu(\bigcup_{I_j} B_i) = \mu^*([\bigcup_{I_j} B_i]) = \mu^*(\bigcup^*_{I_j} [B_i]) = \mu^*([G_j]) = \mu^*([\alpha]) = \mu(\alpha)$.

By Theorem H, for each pair $i, j \in I$, $[B_i \cap B_j] = [B_i] \cap^* [B_j] = \varnothing^* = [\varnothing]$, so $\mu(B_i \cap B_j) = 0$. Define $N \in \mathcal{A}$ by

$$N \equiv \bigcup_{i,j \in I | i \neq j} (B_i \cap B_j).$$

As I is countable, thus $\mu(N) = 0$. For each $i \in \mathbb{N}$, define $B'_i \equiv B_i \setminus N$; then $\mu(B'_i) = \mu(B_i)$. By construction, $\{B'_i\}_{i \in I}$ is pairwise-disjoint. Since μ is a representation, thus

(i) for each $i \in I$, $\alpha \succ B_i \sim B'_i$; and

(ii) for each $j \in \{1, 2, \dots, n\}$, $\bigcup_{I_j} B'_i \sim \bigcup_{I_j} B_i \sim \alpha$.

As $\alpha \in \mathcal{A}^\bullet$ was arbitrary, thus $(\mathcal{A}, \supseteq, \succsim)$ satisfies *n-AS*, as desired.

($\Rightarrow$) Assume $(\mathcal{A}, \supseteq, \succsim)$ satisfies *monotone continuity* and *n-AS*. By the Null-Quotient Lemma (Lemma 8), $(\mathcal{A}^*, \supseteq^*, \succsim^*)$ is an idealized space satisfying *n-AS*, so by Proposition 2, it has representation $\mu^* \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A}^*)$; it is straightforward to show that $(\mathcal{A}^*)^\bullet = (\mathcal{A}^*)^{\bullet|\mu^*}$. Define $\mu : \mathcal{A} \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu(A) = \mu^*([A])$. By the Null-Quotient Lemma (Lemma 8), $\mu \in \mathbb{M}^{\sigma}(\mathcal{A})$ and μ represents $(\mathcal{A}, \supseteq, \succsim)$; it is straightforward to show that $\mathcal{A}^\bullet = \mathcal{A}^{\bullet|\mu}$.

We claim that $\mu \in \mathbb{M}^{\sigma}_{n\text{-AS}}(\mathcal{A})$. Indeed, the earlier argument from the proof of Proposition 2 suffices.

For uniqueness, let $\mu' \in \mathbb{M}^{\sigma}$ be a representation of $(\mathcal{A}, \supseteq, \succsim)$, and define $\mu^{*\prime} : \mathcal{A}^* \to [0, 1]$ such that for each $A \in \mathcal{A}$, $\mu^{*\prime}([A]) = \mu'(A)$; as μ' is a representation of $(\mathcal{A}, \supseteq, \succsim)$, thus by the Null-Quotient Lemma (Lemma 8), this is well-defined. By the Null-Quotient Lemma (Lemma 8), $\mu^{*\prime} \in \mathbb{M}^{\sigma}(\mathcal{A}^*)$ and $\mu^{*\prime}$ represents $(\mathcal{A}^*, \supseteq^*, \succsim^*)$. By Proposition 2, μ^* is the unique such σ-measure, so $\mu^{*\prime} = \mu^*$ and thus $\mu' = \mu$. □

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