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Competing mechanisms in markets for lemons

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We study directed search equilibria in a decentralized market with adverse selection, where uninformed buyers post general trading mechanisms and informed sellers select one of them. We show that this has differing and significant implications with respect to the traditional approach, based on bilateral contracting between the parties. In equilibrium, all buyers post the same mechanism and low-quality sellers receive priority in any meeting with a buyer. Also, buyers make strictly higher profits with low- than with high-type sellers. When adverse selection is severe, the equilibrium features rationing and is constrained inefficient. Compared to the equilibrium with bilateral contracting, the equilibrium with general mechanisms yields a higher surplus for most, but not all, parameter specifications.

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1. Introduction

Spurred on by recent events in the financial markets, such as market freezes and the rapid growth of decentralized forms of trading, there has been a renewed interest in the study of markets with adverse selection and richer trading processes. The hiring process in the labor market has also been traditionally viewed as occurring in a decentralized fashion and, sometimes, characterized by the presence of adverse selection. The traditional approach to markets with adverse selection relies on bilateral contracting between the parties, where the trading protocol involves one principal and one agent. However, in many situations the process leading to a transaction features multiple agents interacting with the same principal. In decentralized financial markets, buyers may be approached by several sellers at the same time; for instance, a private equity firm negotiates the possibility of making an investment with several entrepreneurs. In

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the labor market, the analysis of the empirical evidence on these issues has been the subject of a number of papers. In particular, the recent work by Davis and Samaniego de la Parra (2017) provides new support in favor of not only the presence of multiple job applications, but also the fact that they are processed at the same time rather than in sequence (see also De Los Santos et al. 2012 for other evidence from consumers' markets in support of this).

This paper studies the properties of competitive markets with adverse selection when the trading protocol allows for the possibility that several agents interact with the same principal. We argue the importance of this in the light of the previous considerations. Moreover, even though multiple interactions have been considered in markets with public information or independent private values, we show that, with adverse selection, allowing for multiple interactions has significant and novel implications for the properties of market outcomes, in particular for their welfare features.

In markets with adverse selection, uninformed principals have an incentive to use the terms of trade they post to separate the different types of agents. Separation can occur in two ways: first, within the offer made by a single principal (when this entails a menu of contracts or a trading mechanism); second, by having different types self-selecting into different offers posted. A common finding in the literature following the traditional approach with bilateral contracting is that the second way always obtains in equilibrium: agents are separated in their choice of contract. Principals' profits are then equalized across the different contracts posted. Hence, the equilibrium exhibits no cross-subsidization, which has important welfare implications with adverse selection. We show that this feature is overturned when we allow for multiple interactions and, as a consequence, the equilibria may now feature cross-subsidies across types. However, competition among principals implies that the equilibrium allocation must satisfy other properties, which make the welfare comparison between the allocations obtained in the two setups far from trivial and quite rich.

More specifically, in the model we consider, uninformed principals post trading mechanisms, and informed agents select a mechanism and one of the principals posting it. We, therefore, follow a directed search approach, where each principal may be selected by several or no agents. We allow principals to exploit this fact by posting general direct mechanisms that specify trading probabilities and transfers for agents, contingent not only on their own reported type, but also on the number and the reported types of other agents meeting the same principal. A key feature of the directed search model is that the probability distribution over the size and composition of these meetings is determined by the ratio between the mass of participants of each type searching for a given mechanism and the mass of principals posting it. The effective probability for an agent to trade within a certain mechanism thus not only depends on the terms of the mechanism itself, but also on how many other agents the mechanism attracts. To illustrate, we can think of a trading mechanism as a form of auction with endogenous participation rates.

The approach we follow appears natural in situations such as decentralized procurement markets, where procurers meet several firms that are privately informed about the quality of their products. In this context, the terms of trade between a procurer and a contractor may not only depend on the contractor's type, but also on the number and composition of the contractor's competitors. More generally, as already mentioned above, this approach is also relevant for other decentralized markets, where trading protocols are less explicit but still involve multiple agents. Examples include financial markets and also the labor market, where firms meet different workers who are privately informed about their productivity. In addition to the empirical support regarding the importance of considering trading mechanisms that involve several agents, we should also mention some theoretical considerations. The existing microfoundations of the directed search equilibrium concept (see, for example, Peters 1997) consider finite games where the principals propose the terms of trade and the agents select one of the principals. Hence, each principal may meet several agents, and it is natural to consider the case where the terms of trade that he posts can exploit this fact.

In an environment such as in Akerlof (1970), where sellers are privately informed about the quality of their good—assumed to be either high or low—we find that there always exists an equilibrium in which all buyers post and sellers select the same mechanism. The equilibrium mechanism specifies that a low-quality object is traded whenever such an object is present in a meeting between a buyer and one or more sellers, meaning that low-type sellers receive priority in every meeting. Remarkably, this property holds no matter how great the gains from trade for the high-quality good are relative to those for the low-quality good. In equilibrium, buyers make strictly higher profits with low- than with high-type sellers. As a consequence, cream-skimming deviations—in particular those attracting only the high types—are not profitable. This is the feature that makes it possible for high- and low-type sellers to participate in the same mechanism. We show that the equilibrium mechanism can be implemented via a sequence of second-price auctions with increasing reserve prices.

While the described properties of the equilibrium allocation hold regardless of the size of the gains from trade for low- and high-quality goods, the latter matters for other features of the equilibrium. We find that in situations where buyers care sufficiently for quality and when competition for high-quality sellers is sufficiently intense—a situation we refer to as adverse selection being severe—buyers make zero profits with high-quality sellers, and meetings where only those sellers are present do not always end with trade. The intuition is that giving priority to low-quality sellers may not be sufficient to satisfy their incentive compatibility constraint, so the trading probability of sellers with a high-quality good needs to be further reduced by rationing them in meetings where they are alone. Also, in such a case, additional equilibria exist where sellers partially separate themselves through their choice of mechanism: buyers post different mechanisms, attracting different ratios of high-type sellers relative to buyers. A sufficiently low trading probability for high-quality sellers can, therefore, be achieved in three ways: via rationing within the mechanisms, through an asymmetric assignment of high-type sellers across the mechanisms posted by buyers, or by a combination of the two. These are new properties due to adverse selection.

The extent to which adverse selection is (or is not) severe also matters for the welfare properties of the equilibrium. We show that whenever adverse selection is not severe and the gains from trade are higher for the low- than the high-quality object, the equilibrium allocation maximizes total surplus. The result follows directly from the facts that (i) the pooling of sellers on the choice of the same mechanism maximizes the number of meetings and (ii) the equilibrium mechanism gives priority to the good for which the gains from trade are larger. Alternatively, with severe adverse selection, total surplus is no longer maximal in equilibrium and—provided the share of high-type sellers in the population is large enough—the equilibrium allocation can be Pareto improved, even subject to the constraints imposed by incentive compatibility and the meeting technology.

Given the features of the equilibrium outcome we obtain, in particular the possibility of cross-subsidization, it is interesting to compare its welfare properties with those of the equilibria that obtain when contracting can only be bilateral. To this end, we consider the case where the mechanisms available to buyers are restricted to the class of menus from which one seller, who is randomly selected among those meeting a buyer, can choose. As should be expected in the light of the results of earlier work (see, for example, Guerrieri et al. 2010), we find that in equilibrium, buyers post one of two prices: the higher one selected by high-type sellers and the lower one selected by low types. Sellers are thus separated by their choice of mechanism at the search stage, rather than by their choice within the mechanism. We find that for many but not all parameter specifications, total surplus at the equilibrium with general mechanisms is strictly higher than at the equilibrium with bilateral contracts. Hence, there are situations in which more surplus is generated when the only available trading mechanisms are bilateral contracts. Even more surprisingly, this does not arise in the case of severe adverse selection, where the equilibrium with general mechanisms features rationing. It occurs instead when gains from trade for the high-quality good are larger than for the low-quality good but competition for high-type sellers is sufficiently weak, so that every meeting still leads to trade.

Related literature

Since the work of Akerlof (1970) and Rothschild and Stiglitz (1976), the properties of market outcomes in the presence of adverse selection have been the subject of intense study, both in Walrasian models and in models where agents act strategically. In the latter models, agents compete among themselves over contracts that specify the transfer of goods and the price, whereas in the former models, available contracts indicate the transfer of goods while prices are taken as given and set so as to clear the markets. Initiated by Gale (1992), Inderst and Müller (1999), and, more recently, Guerrieri et al. (2010), the use of competitive, directed search models to study markets with adverse selection has generated several interesting insights.[1] This framework allows for a richer specification of the terms of the contracts available for trade, which—in contrast to Walrasian models—also include the price to be paid. Both in competitive search models and in Walrasian models (on the latter, see Dubey and Geanakoplos 2002 and Bisin and Got-

[1] For earlier work on directed search in markets without informational asymmetries see, for instance, Moen (1997) and Burdett et al. (2001).

tardi 2006), it is shown that when contracting is bilateral, an equilibrium always exists and is separating, with agents of different types trading different contracts.[2]

Our paper is also closely related to the work on competing mechanisms in independent private value environments. Peters (1997) and Peters and Severinov (1997) consider—as do we—a framework where buyers are not constrained in their ability to meet and contract with multiple sellers. They show that an equilibrium exists where all buyers post the same mechanism, given by a second-price auction with a reserve price equal to their valuation.[3] The finding that a single mechanism is posted in equilibrium is analogous to that which we obtain in the presence of common value uncertainty (though the posted mechanism we find is different from a second-price auction in several important aspects). Furthermore, the argument we use to establish the property that all buyers post the same mechanism is rather different. In independent private value settings, equilibria are always efficient. It is, therefore, possible to use this feature to solve the planner's problem, find a mechanism that decentralizes the solution, and show that no profitable deviation exists.[4] However, this approach cannot be followed in our environment with adverse selection, since we cannot rely on the fact that the equilibrium is efficient. Nevertheless, we show that once the general space of available mechanisms is projected to a simpler, reduced space, the characterization of the equilibrium is still tractable and is obtained by analyzing the solutions of the buyers' optimization problem. Another important difference with respect to the independent private value case is the fact that in our environment, rationing and partial separation of sellers across different mechanisms at the search stage can occur in equilibrium—two features that never arise when buyers' valuations do not depend on sellers' types.

Eeckhout and Kircher (2010) study the role of the properties of the meeting technology for the allocation that obtains at a directed search equilibrium, again in a setting with independent private values.[5] In particular, they show that when meetings can only be bilateral, in equilibrium buyers post different prices that attract different types of sellers, whereas such equilibria do not exist when buyers face no restriction on the number of buyers they can meet. These results are related to our findings on equilibria when mechanisms are restricted to simple menus, as these constitute the available mechanisms when buyers meet one seller at most. Furthermore, the literature on directed search with public information has considered different specifications of the space of available mechanisms and has obtained results similar to ours: if agents can be discrim-

[2]Other papers study adverse selection in economies where trade takes place over a sequence of periods, both in markets with random search, where offers of contracts are made only after meetings occur, and in markets à la Akerlof (1970), where all trades occur at a single price (see, for example, Blouin and Serrano 2001, Janssen and Roy 2002, Camargo and Lester 2014, Fuchs and Skrzypacz 2013, and Moreno and Wooders 2016). It is interesting to note that in these dynamic models, the equilibrium outcome is similar: separation obtains, in this case with sellers of different types trading at different prices and at different points in time.

[3]Note that in this literature, the labels of buyers and sellers are typically reversed.

[4]See, for instance, Cai et al. (2017a) and Lester et al. (2017). Albrecht et al. (2014) consider an environment with free entry of the uninformed party and show that the equilibrium remains efficient.

[5]Their findings were further developed by Lester et al. (2015) and Cai et al. (2017a).

inated by the mechanism through a priority rule, different types will participate in the same mechanism (see Shi 2002 and Shimer 2005), whereas if the set of available mechanisms is restricted to the set of posted prices, types separate according to their type at the search stage (see Shi 2001). In contrast to these papers, however, our focus lies on the welfare implications of such restriction, for which adverse selection plays a critical role.

The paper is organized as follows. Section 2 presents the economy and the set of available trading mechanisms, and defines the notion of directed search equilibrium that is considered. Section 3 establishes the main result, showing the existence and the uniqueness of the equilibrium allocation. The properties of this allocation are also characterized in this section, together with the key steps of the proof. Section 4 provides the welfare analysis of the equilibria we find and compares their properties to those of the equilibrium that obtains when mechanisms are restricted to bilateral contracts. Section 5 concludes. Proofs can be found in the Appendix.

2. Environment

There is a measure b of uninformed buyers and a measure s of informed sellers. Each seller possesses one unit of an indivisible good with uncertain quality and each buyer wants to buy at most one unit. The good's quality is identically and independently distributed across sellers. Quality can be either high or low, and μ denotes the fraction of sellers who possess a high-quality good. Let $\overline{\lambda}^p = \mu s/b$ denote the ratio of high-type sellers to buyers and let $\underline{\lambda}^p = (1-\mu)s/b$ the ratio of low-type sellers to buyers.

The valuation of buyers and sellers for the low-quality good is denoted, respectively, by $\underline{v}$ and $\underline{c}$, and that of the high quality good is denoted by $\overline{v}$ and $\overline{c}$. We assume that both the buyers and the sellers value the high-quality good more than the low-quality good, that is, $\overline{v} \geq \underline{v}$, $\overline{c} > \underline{c}$. For sellers, this preference is assumed to be strict, while we allow buyers to have the same valuation for both types of good, that is, $\overline{v} = \underline{v}$. When $\overline{v}$ is strictly greater than $\underline{v}$, the buyer's valuation depends on the seller's valuation of the object, a situation we refer to as the common value case. This is no longer true when $\overline{v} = \underline{v}$, which we refer to as the private value case. We further assume that there are always positive gains from trade, meaning that for both types of the good, the buyer's valuation strictly exceeds the seller's valuation, that is, $\overline{v} > \overline{c}$, $\underline{v} > \underline{c}$.

Meeting technology

The trading process operates as follows. Buyers simultaneously post mechanisms that specify how trade takes place with the sellers they meet. Sellers observe the posted mechanisms and select one of the mechanisms they like best, as well as one of the buyers posting it. We refer to the collection of buyers posting the same mechanism and the collection of sellers selecting that mechanism as constituting a submarket. We have in mind markets that are decentralized and anonymous, where interactions are relatively infrequent, as for the situations mentioned in the Introduction. In line with the first feature, the matching between buyers and sellers in any submarket does not occur in a centralized way, but is determined by the individual choices of sellers, each of them selecting a buyer. Anonymity is captured by the assumption that the mechanisms posted do not condition on the identity of sellers and that sellers do not condition their choice on the identity of buyers, only on the mechanism they post. Sellers thus pick at random one of the buyers posting the selected mechanism (see, for example, Shimer 2005). Also, since each seller can only select one buyer, contracting is exclusive.

More specifically, we assume that in any submarket, a seller meets one, randomly selected, of the buyers who are present. Buyers have no capacity constraint in their ability to meet sellers; that is, they can meet all arriving sellers, no matter how many there are. As a result, matching occurs according to the urn–ball technology. Under this meeting technology, the number of sellers who meet a particular buyer follows a Poisson distribution with a mean equal to the queue length in the submarket, defined as the ratio of the measure of sellers to the measure of buyers visiting that submarket.[6] Sellers are certain to meet a buyer, while buyers may end up with many sellers or with no seller at all. Moreover, a buyer's probability of meeting a seller of a given type is fully determined by the queue length of that type of seller in the submarket, while it does not depend on the presence of other types of sellers.[7] Both the latter property and the fact that buyers can meet multiple sellers are essential for the following analysis, while most other features of the meeting technology are not. Furthermore, given that we focus on an environment with one-sided heterogeneity, the consideration of meetings between one principal and several agents is interesting only when the heterogeneity is on the agents' side.[8]

Under urn–ball matching, a buyer's probability of meeting k sellers in a market with queue length λ is given by

$$P_k(\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}.$$

Since the presence of high-type sellers does not affect the meeting chances of low-type sellers and vice versa, the probability of a buyer meeting L low-type sellers and H high-type sellers in a market where the queue length of low-type sellers is $\underline{\lambda}$, and the queue length of high-type sellers is $\overline{\lambda}$ is given by

$$P_L(\underline{\lambda})P_H(\overline{\lambda}) = \frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}.$$

Notice that $P_L(\underline{\lambda})P_H(\overline{\lambda})$ is also the probability for a seller to be in a meeting with other L low-type sellers and H high-type sellers.

[6] In principle, we can have a continuum of active submarkets, each with a measure zero of buyers and sellers. In that case, we can use the Radon–Nikodym derivatives to define queue lengths (see also footnote 13).

[7] The class of meeting technologies that have this property, called *invariance* (Lester et al. 2015), includes the urn–ball matching technology as a special case.

[8] See the concluding remarks for further discussion on possible generalizations of the meeting technology.

Mechanisms and payoffs

We restrict attention to direct mechanisms that do not condition on mechanisms posted by other buyers.[9] A mechanism m is defined by the maps

$$(\underline{X}_m, \overline{X}_m, \underline{T}_m, \overline{T}_m) : \mathbb{N}^2 \to [0,1]^2 \times \mathbb{R}^2,$$

where the first argument is the number of low messages and the second argument is the number of high messages received by a buyer from the sellers he meets. The maps $\underline{X}_m(L,H)$ and $\overline{X}_m(L,H)$ describe the trading probabilities specified by mechanism m for sellers reporting, respectively, to be of low and of high type. The associated transfers (unconditional on whether trade occurs) are given by the maps $\underline{T}_m(L,H)$ and $\overline{T}_m(L,H)$. For instance, $\underline{T}_m(L,H)$ is the transfer to a seller who reports to be of low type when $L-1$ other sellers report to be of low type and H other sellers report to be of high type. We say a mechanism m is feasible if

$$\underline{X}_m(L,H)L + \overline{X}_m(L,H)H \leq 1 \quad \forall (L,H) \in \mathbb{N}^2. \tag{1}$$

Thus, in each meeting the probability that a good is exchanged cannot exceed 1. Let M denote the measurable set of feasible mechanisms.[10]

We assume that when matched with a buyer, a seller does not observe how many other sellers are actually matched with the same buyer or their types.[11] Let $\overline{\lambda}$ denote the expected number of high reports and let $\underline{\lambda}$ denote the expected number of low reports, which under truthful reporting simply correspond to the queue lengths of high- and low-type sellers for mechanism m. The expected trading probabilities for a seller when reporting to be of low and of high type, respectively, are then given by

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \sum_{L=0}^{+\infty} \sum_{H=0}^{+\infty} P_L(\underline{\lambda}) P_H(\overline{\lambda}) \underline{X}(L+1, H)$$

$$\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \sum_{L=0}^{+\infty} \sum_{H=0}^{+\infty} P_L(\underline{\lambda}) P_H(\overline{\lambda}) \overline{X}(L, H+1).$$

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[9] In principle, indirect mechanisms could allow buyers to elicit information about the mechanisms other buyers post. However, given the large market environment we consider, in the equilibria we characterize in Theorem 3.1 below, each buyer knows the realized distribution of mechanisms posted by other buyers and this distribution is not affected by deviations of individual buyers. It then follows that, as noted by Eeckhout and Kircher (2010) in an analogous setup, the buyer does not profit from conditioning on those mechanisms. Hence, the equilibria we characterize are robust to removing the restriction to mechanisms conditioning only on the sellers' reported types, though additional equilibria may exist in that case.

[10] The set M is identified by the collection of sequences of vectors in $\mathbb{R}^4$, indexed by the set $I = \mathbb{N} \times \mathbb{N}$. Letting $\mathcal{M}_i$ denote the Borel σ-algebra of $\mathbb{R}^4$, we can consider the family of measurable spaces $\{(M_i, \mathcal{M}_i), i \in I\}$ and use the Kolmogorov extension theorem to define a probability measure for the set of these sequences.

[11] The assumption that a seller cannot observe the realized number of competitors in a meeting facilitates notation considerably, but is not essential for any of our results. To see this, notice that since traders are assumed to be risk neutral, sellers' payoffs are linear in trading probabilities and transfers. Therefore, if incentive compatibility constraints (2) and (3) below are satisfied, the values of the expected trading probabilities and transfers can always be decomposed in terms of trading probabilities and transfers in each possible meeting in such a way that incentive compatibility is satisfied also when sellers observe the number of competitors in a meeting.

Analogously, we can determine expected transfers $\underline{t}_m(\underline{\lambda}, \overline{\lambda})$ and $\overline{t}_m(\underline{\lambda}, \overline{\lambda})$. The expected payoff for low- and high-type sellers when choosing mechanism m and revealing their type truthfully, net of the utility of their endowment point,[12] is then given by

$$\underline{u}(m|\underline{\lambda}, \overline{\lambda}) = \underline{t}_m(\underline{\lambda}, \overline{\lambda}) - \underline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{c}$$
$$\overline{u}(m|\underline{\lambda}, \overline{\lambda}) = \overline{t}_m(\underline{\lambda}, \overline{\lambda}) - \overline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{c}.$$

Truthful reporting is optimal if the following two inequalities hold:

$$\overline{t}_m(\underline{\lambda}, \overline{\lambda}) - \overline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{c} \leq \underline{t}_m(\underline{\lambda}, \overline{\lambda}) - \underline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{c} \tag{2}$$
$$\underline{t}_m(\underline{\lambda}, \overline{\lambda}) - \underline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{c} \leq \overline{t}_m(\underline{\lambda}, \overline{\lambda}) - \overline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{c}. \tag{3}$$

Note that since incentive compatibility is defined in terms of expected trading probabilities and transfers, whether a given mechanism m is incentive compatible depends on the values of $\underline{\lambda}$ and $\overline{\lambda}$. Let M^{IC} denote the set of tuples $(m, \underline{\lambda}, \overline{\lambda})$ such that $m \in M$ and incentive compatibility with respect to $\underline{\lambda}$ and $\overline{\lambda}$ is satisfied.

The expected payoff for a buyer posting mechanism m, when sellers report truthfully and the expected number of high- and low-type sellers, respectively, is $\overline{\lambda}$ and $\underline{\lambda}$, is then

$$\pi(m|\underline{\lambda}, \overline{\lambda}) = \overline{\lambda}\big(\overline{x}_m(\underline{\lambda}, \overline{\lambda})\overline{v} - \overline{t}_m(\underline{\lambda}, \overline{\lambda})\big) + \underline{\lambda}\big(\underline{x}_m(\underline{\lambda}, \overline{\lambda})\underline{v} - \underline{t}_m(\underline{\lambda}, \overline{\lambda})\big).$$

Equilibrium

An allocation in this setting is defined by a probability measure β on M with support M^β, where $\beta(M')b$ describes the measure of buyers who post mechanisms in $M' \subseteq M$, and two maps $\underline{\lambda}, \overline{\lambda} : M^\beta \to \mathbb{R}^+$ specifying, respectively, the queue lengths of low- and high-type sellers selecting mechanism m. We say that an allocation is *feasible* if[13]

$$\int_{M^\beta} \underline{\lambda}(m)\, d\beta(m) = \underline{\lambda}^p, \qquad \int_{M^\beta} \overline{\lambda}(m)\, d\beta(m) = \overline{\lambda}^p. \tag{4}$$

We call an allocation *incentive compatible* if $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in M^{\mathrm{IC}}$ for all $m \in M^\beta$. We can show that we can restrict our attention to incentive-compatible allocations without loss of generality (w.l.o.g.):[14] for any non-incentive-compatible mechanism there always exists an incentive-compatible mechanism that yields the same payoff for buyers and sellers as the original mechanism.

An equilibrium in the environment considered is given by a feasible and incentive-compatible allocation, such that the values of β, $\underline{\lambda}$, and $\overline{\lambda}$ are consistent with buyers'

[12] For example, the low-type seller has utility $\underline{c}$ at his endowment. His utility gain when participating in mechanism m is then given by $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) + (1 - \underline{x}_m(\underline{\lambda}, \overline{\lambda}))\underline{c} - \underline{c} = \underline{u}(m|\underline{\lambda}, \overline{\lambda})$.

[13] Let $\bar{\sigma}$ and $\underline{\sigma}$ be the probability measures describing, respectively, the high- and low-type sellers' search strategy. The fact that sellers can only search for mechanisms that are posted by some buyers is captured by the assumption that both probability measures are absolutely continuous with respect to β. Condition (4) then says that the queue length $\bar{\lambda}$ is obtained as the product of $\bar{\lambda}^p$ and the Radon–Nikodym derivative of $\bar{\sigma}$ with respect to β. Similarly for $\underline{\lambda}$.

[14] For a formal proof, see the Supplemental Material, available in a supplementary file on the journal website, http://econtheory.org/supp/2921/supplement.pdf.

and sellers' optimal choices. For mechanisms not posted in equilibrium, we extend the maps $\underline{\lambda}$ and $\overline{\lambda}$ to the domain $M \backslash M^\beta$ so as to describe the beliefs of buyers over the queue lengths of low- and high-type sellers that such mechanisms attract. We require these beliefs to be pinned down by a similar consistency condition with sellers' optimal choices out of equilibrium. More specifically, a buyer believes that a deviating mechanism attracts some low-type sellers only if low-type sellers are indifferent between this mechanism and the one they choose in equilibrium, and similarly for high-type sellers:[15]

$$\underline{u}\big(m|\underline{\lambda}(m), \overline{\lambda}(m)\big) \le \max_{m' \in M^\beta} \underline{u}\big(m'|\underline{\lambda}\big(m'\big), \overline{\lambda}\big(m'\big)\big) \quad \text{holding with equality if } \underline{\lambda}(m) > 0 \tag{5}$$

$$\overline{u}\big(m|\underline{\lambda}(m), \overline{\lambda}(m)\big) \le \max_{m' \in M^\beta} \overline{u}\big(m'|\underline{\lambda}\big(m'\big), \overline{\lambda}\big(m'\big)\big) \quad \text{holding with equality if } \overline{\lambda}(m) > 0. \tag{6}$$

Hence, we can say that beliefs are correct also out of equilibrium, should a deviation occur. This specification of the beliefs for mechanisms that are not posted in equilibrium is standard in the literature of directed search, both with and without common value uncertainty (see Guerrieri et al. 2010 and Eeckhout and Kircher 2010, among others).

Formally, we impose the following conditions on out of equilibrium beliefs $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ for $m \notin M^\beta$:

(i) If (5) and (6) admit a unique solution, then $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are given by that solution.

(ii) If (5) and (6) admit no solution, we set $\underline{\lambda}(m)$ and/or $\overline{\lambda}(m)$ equal to $+\infty$ and $\pi(m|\underline{\lambda}(m), \overline{\lambda}(m)) = c$ for some $c \le 0$.[16]

(iii) If (5) and (6) admit multiple solutions, then $\underline{\lambda}(m)$, $\overline{\lambda}(m)$ are given by the solution for which the buyer's payoff $\pi(m|\underline{\lambda}(m), \overline{\lambda}(m))$ is the highest.

Condition (i) says that whenever there is a unique pair $\underline{\lambda}(m)$, $\overline{\lambda}(m)$ satisfying conditions (5) and (6) for an out of equilibrium mechanism $m \notin M^\beta$, buyers' beliefs regarding the queue lengths for such a deviating mechanism are pinned down by these conditions. This includes the solution where both (5) and (6) are satisfied with strict inequality and $\underline{\lambda}(m) = \overline{\lambda}(m) = 0$. In this case, both types of sellers are worse off by choosing mechanism m, whatever the queue length is, so the buyer believes that m does not attract any seller. Since we allow for arbitrary direct mechanisms (which may feature unconditional participation transfers), and since buyers have no constraints in their ability to meet sellers, there are mechanisms for which a solution to (5) and (6) does not exist. That is, for any finite pair $\underline{\lambda}$, $\overline{\lambda}$, at least one type of seller strictly prefers the deviating mechanism over any mechanism in the set M^β. Condition (ii) specifies that in such a case, the queue

[15] Conditions (5) and (6) are perfectly analogous to the sellers' optimality conditions appearing in the equilibrium Definition 1. Hence, conditions (5) and (6) indeed require that queue lengths are consistent with sellers' optimal choices also for out of equilibrium mechanisms, as if all mechanisms were effectively available to sellers. This is analogous to existing refinements in other competitive models with adverse selection, such as Gale (1992) and Dubey and Geanakoplos (2002).

[16] More precisely, if $\underline{u}(m|\underline{\lambda}, \overline{\lambda}) > \max_{m' \in M^\beta} \underline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m'))$ and $\overline{u}(m|\underline{\lambda}, \overline{\lambda}) > \max_{m' \in M^\beta} \overline{u}(m'|\underline{\lambda}(m'), \overline{\lambda}(m'))$ for all $(\underline{\lambda}, \overline{\lambda}) \in \mathbb{R}^2_+$, then $(\underline{\lambda}(m), \overline{\lambda}(m)) = (+\infty, +\infty)$. If only one of the two inequalities is violated, say the first one, then $\underline{\lambda}(m) = +\infty$, while $\overline{\lambda}(m)$ is determined by (6).

lengths $\underline{\lambda}(m)$, $\overline{\lambda}(m)$ are set equal to infinity, while a buyer's associated payoff is nonpositive.[17] Finally, if there are multiple solutions of (5) and (6), we follow McAfee (1993), Eeckhout and Kircher (2010), and others, and assume in condition (iii) that buyers are "optimistic," so that the pair $\underline{\lambda}(m)$, $\overline{\lambda}(m)$ is given by their preferred solution.[18]

We are now ready to define a directed search equilibrium.

Definition 1. A directed search equilibrium is a feasible and incentive-compatible allocation, given by a probability measure β with support M^β and two maps $\underline{\lambda}, \overline{\lambda} : M \to \mathbb{R}^+ \cup +\infty$, such that the following conditions hold:

- Buyers' optimality: For all $m \in M$ such that $(m, \underline{\lambda}(m), \overline{\lambda}(m)) \in M^{\mathrm{IC}}$,

$$\pi\big(m|\underline{\lambda}(m), \overline{\lambda}(m)\big) \leq \max_{m' \in M} \pi\big(m'|\underline{\lambda}\big(m'\big), \overline{\lambda}\big(m'\big)\big) \quad \text{holding with equality if } m \in M^\beta.$$

- Sellers' optimality: For all $m \in M^\beta$,

$$\underline{u}\big(m|\underline{\lambda}(m), \overline{\lambda}(m)\big) \leq \max_{m' \in M^\beta} \underline{u}\big(m'|\underline{\lambda}\big(m'\big), \overline{\lambda}\big(m'\big)\big) \quad \text{holding with equality if } \underline{\lambda}(m) > 0$$

$$\overline{u}\big(m|\underline{\lambda}(m), \overline{\lambda}(m)\big) \leq \max_{m' \in M^\beta} \overline{u}\big(m'|\underline{\lambda}\big(m'\big), \overline{\lambda}\big(m'\big)\big) \quad \text{holding with equality if } \overline{\lambda}(m) > 0.$$

- Beliefs: For all $m \notin M^\beta$, $\underline{\lambda}(m)$ and $\overline{\lambda}(m)$ are determined by conditions (i)–(iii).

3. Directed search equilibrium

We state now our main result, which characterizes the directed search equilibria in the environment described in the previous section.

Theorem 3.1. *There exists a directed search equilibrium with the following properties:*

(i) All buyers post the same mechanism.

(ii) Whenever a low-type seller is present in a match with a buyer, a low-quality good is traded.

The equilibrium is unique in terms of expected payoffs.

Theorem 3.1 states that there always exists an equilibrium in which sellers are pooled at the stage of selecting a mechanism and screened at the mechanism stage. That is, all

[17]The property that mechanism m is strictly preferred by one type of seller for all possible queue lengths requires that m features a strictly positive participation transfer, paid to each seller independently of how many other sellers show up. Such a mechanism attracts infinitely many sellers. Since the possible gains from trade of a buyer are finite, it follows that his payoff associated with such mechanism must be negative.

[18]This is akin to the focus on strongly robust equilibria in the common-agency literature (see Peters 2001 and Han 2007). An equilibrium is said to be strongly robust if the equilibrium payoff for each principal is not less than his payoff when he deviates to another available mechanism, and this is true in any continuation equilibria on and off the equilibrium path (Han 2007). We should point out that in the equilibria that we characterize in the next sections, condition (iii) is not only satisfied for the mechanisms that are not chosen in equilibrium, but also for those that are chosen by buyers.

buyers post identical mechanisms so that everybody trades in a single submarket, and this mechanism specifies different trading probabilities for different types of sellers. In particular, the equilibrium mechanism always gives priority to low-type sellers, meaning that a low-quality good is traded whenever a low-type seller is present in a meeting with a buyer. This not only implies that a low-type seller's probability of trade strictly exceeds a high-type seller's probability of trade,[19] but also that the equilibrium allocation maximizes the trade of low-quality goods in the economy. It is important to point out that this property of the equilibrium holds, regardless of the relative gains from trade and of the fraction of high-type sellers in the population. That is, even when the gains from trade for the low-quality good are arbitrarily small and those for the high-quality good are arbitrarily large, in equilibrium high-type sellers trade only when they are in meetings where there are no low-type sellers. Theorem 3.1 also states that the equilibrium is unique in terms of payoffs. In particular, although we will show that there may also be equilibria where more than one mechanism is posted, all of those equilibria yield the same expected levels of trade and transfers. The remainder of this section is devoted to proving the above result.

3.1 *Buyers' auxiliary optimization problem*

A distinguishing feature of the environment considered here—with asymmetric information and common value uncertainty—is that the equilibrium cannot be found by decentralizing the constrained planner's solution, as noted in the Introduction. To establish the existence and the stated properties of directed search equilibria, we must, therefore, study the solution of the buyers' optimization problem. Given that the space of mechanisms available to buyers is quite large, we first show that we can conveniently restrict our attention to the space of expected trading probabilities and transfers associated with mechanisms in M. To this end, we provide a characterization of the set of expected trading probabilities and transfers associated with the feasible and incentive-compatible mechanisms in our environment.[20] This makes the buyers' problem tractable, and the study of its solution allows us to derive important properties of the equilibrium outcome.

More precisely, the following result provides conditions on the values of expected trading probabilities and transfers that any feasible and incentive-compatible mechanism satisfies, and, vice versa, that are generated by some feasible and incentive-compatible mechanism.

[19]The probability of trade for a low-type seller is strictly larger than his probability of meeting no other low-type seller, while the probability of trade for a high-type seller is strictly smaller than his probability of meeting no low-type seller.

[20]Expected trading probabilities and transfers have been used in the earlier literature on directed search to determine equilibrium payoffs (see, for instance, Shi 2001, 2002 for the case of public information and Peters 1997 for the case of independent private values), but the characterization that we provide of this set is novel.

Proposition 3.2. *For any* $(\underline{x}, \overline{x}, \underline{t}, \overline{t}) \in [0,1]^2 \times \mathbb{R}^2$ *and* $\overline{\lambda}, \underline{\lambda} \in [0, \infty)$, *there exists a feasible and incentive-compatible mechanism* m, *such that*

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}, \qquad \overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}, \qquad \underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}, \qquad \overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$$

if and only if

$$\overline{t} - \overline{x}\underline{c} \leq \underline{t} - \underline{x}\underline{c} \tag{7}$$

$$\underline{t} - \underline{x}\overline{c} \leq \overline{t} - \overline{x}\overline{c} \tag{8}$$

$$\overline{\lambda}\overline{x} \leq 1 - e^{-\overline{\lambda}} \tag{9}$$

$$\underline{\lambda}\underline{x} \leq 1 - e^{-\underline{\lambda}} \tag{10}$$

$$\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x} \leq 1 - e^{-\overline{\lambda}-\underline{\lambda}}. \tag{11}$$

Please note that most proofs are provided in the Appendix.

Conditions (7) and (8) are analogous to the sellers' incentive compatibility constraints (2) and (3). It is immediate to see that these two conditions imply $\underline{x} \geq \overline{x}$; that is, the expected trading probability is higher for low- than for high-type sellers. The remaining three conditions guarantee that the mechanism m associated with $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ is feasible according to (1), and that meetings take place according to the urn–ball technology. In particular, inequality (9) requires that a buyer's probability of trading with a high-type seller, $\overline{\lambda}\overline{x}$, is weakly smaller than a buyer's probability of meeting at least one high-type seller, $\sum_{k=1}^{+\infty} P_k(\overline{\lambda}) = 1 - e^{-\overline{\lambda}}$. Inequality (10) is the analogous condition for the low-type seller and inequality (11) requires that a buyer's probability of trading with any seller, $\overline{\lambda}\overline{x} + \underline{\lambda}\underline{x}$, is weakly smaller than the probability of meeting at least one seller, $1 - e^{-\overline{\lambda}-\underline{\lambda}}$. While it is easy to see that any feasible and incentive-compatible mechanism must satisfy conditions (7)–(11), the main achievement of the proof of Proposition 3.2 is to show, using a constructive argument, that any trading probabilities and transfers that satisfy these conditions can be generated by some feasible and incentive-compatible mechanism in the original space described in Section 2.

Next, we rewrite the buyers' optimization problem in terms of an auxiliary problem. This problem consists of choosing a mechanism m, with the associated queue lengths $\overline{\lambda}$, $\underline{\lambda}$, so as to maximize a buyer's payoff, taking as given the utility gain attained in the market by low- and high-type sellers relative to their endowment points, denoted by $\underline{U}$ and $\overline{U}$ (in short, their market utilities). More precisely, in this problem a buyer chooses a mechanism and the queue lengths, subject to the constraint that the queue lengths must satisfy the sellers' optimality conditions in Definition 1 for the mechanisms posted in equilibrium and conditions (5) and (6) restricting out of equilibrium beliefs. These constraints can be viewed as a form of participation constraints.[21]

[21] Letting the representative buyer optimize over queue lengths means that in cases where there are multiple solutions to (5) and (6), the buyer picks the preferred pair, which is consistent with condition (iii) pinning down out of equilibrium beliefs. The auxiliary optimization problem does not allow the buyer to choose values of $\underline{\lambda}$, $\overline{\lambda}$ for which the set of inequalities (5) and (6) does not have a solution. This comes without loss of generality because, as we argued in footnote 17, such a mechanism must yield a strictly negative payoff and can, therefore, not be optimal.

The approach based on the study of a constrained optimization problem of the buyer taking market utilities as given has been followed in earlier work in the literature on directed search (for instance, see Shi 2001, 2002 for the case of public information and Eeckhout and Kircher 2010 for the case of independent private values). In these papers, the optimization problem is formulated over the original space of mechanisms being considered. In our setup, alternatively, we rely on Proposition 3.2 to state it simply in terms of values of expected trading probabilities and transfers satisfying the feasibility and incentive constraints (7)–(11). Hence, we have the optimization problem

$$\max_{\underline{x},\overline{x},\underline{t},\overline{t},\underline{\lambda},\overline{\lambda}} \overline{\lambda}(\overline{xv}-\overline{t})+\underline{\lambda}(\underline{xv}-\underline{t}) \tag{P^{aux}}$$

subject to

$$\begin{aligned} \overline{t}-\overline{xc} &\le \overline{U} \quad \text{holding with equality if } \overline{\lambda}>0 \\ \underline{t}-\underline{xc} &\le \underline{U} \quad \text{holding with equality if } \underline{\lambda}>0 \end{aligned} \tag{12}$$

and constraints (7)–(11).

Whenever the values of $\overline{\lambda}$ and $\underline{\lambda}$ are *consistent with the population parameters*, the solutions of the buyer's auxiliary problem give us the expected trading probabilities and transfers in a candidate equilibrium. By consistent we mean that a probability measure β can be assigned to the set of solutions of problem P^{aux}, so that the feasibility conditions in (4) are satisfied.[22] For instance, if the solution to the auxiliary problem is unique, consistency simply requires that the optimal queue lengths $\underline{\lambda}$ and $\overline{\lambda}$ coincide with the population parameters $\underline{\lambda}^p$ and $\overline{\lambda}^p$. In such a case, we have a candidate equilibrium where all buyers post the same mechanism. If the solution is not unique and the optimal queue lengths differ across the different solutions, consistency requires that the average value of the queue lengths equals the population parameters. In this case, at the candidate equilibrium sellers separate through their choice of mechanisms. Having found the expected trading probabilities and transfers of the candidate equilibrium, it is immediate to verify that the associated mechanisms—which exist by Proposition 3.2—are a solution of the buyer's optimization problem in the original space of mechanisms. Hence, the candidate equilibrium is indeed an equilibrium.

Finding an equilibrium thus amounts to finding values of the market utilities $\underline{U}$ and $\overline{U}$, such that the solutions of the buyer's auxiliary problem can be assigned appropriate weights so as to achieve consistency with the population parameters. In what follows, we show that the analysis of the general properties of these solutions allows us to establish two key properties of equilibria: first, equilibrium mechanisms give priority to low-type sellers; second, equilibria are not separating at the search stage.

[22] More precisely, we can restate condition (4) in the reduced space of mechanisms by replacing the domain M^{β} with the set of solutions of the buyer's auxiliary problem P^{aux}. Letting S be the set of solutions and $\underline{\lambda}(s)$, $\overline{\lambda}(s)$ indicate the associated queue lengths for each $s \in S$, these conditions say $\int_S \underline{\lambda}(s)\,d\beta(s) = \underline{\lambda}^p$ and $\int_S \overline{\lambda}(s)\,d\beta(s) = \overline{\lambda}^p$.

Property 1: Priority for low-type sellers To establish the first property, we show that at any solution of P^{aux}, we have $\underline{\lambda}\underline{x} = 1 - e^{-\underline{\lambda}}$. This condition says that a buyer's probability of trading a low-quality good is equal to his probability of meeting a low-type seller, which immediately implies property (ii) of Theorem 3.1: any mechanism posted in equilibrium must give priority to low-type sellers.[23] As already noted, this also implies that $\underline{x} > \overline{x}$. Thus, no pooling mechanism—treating both types equally—is posted in equilibrium.

Lemma 3.3. *At any solution of P^{aux}, the low-type feasibility constraint (10) is satisfied with equality.*

An important step for proving the above result is the observation that incentive compatibility of equilibrium mechanisms requires that $\underline{U} > \overline{U}$. That is, the expected utility gain obtained by trading in the market is strictly higher for low-type sellers than for high-type sellers (see Appendix A.2). Given this, the proof of Lemma 3.3 shows that whenever the posted mechanisms are such that low-type sellers do not receive priority, buyers have a profitable deviation. This deviation is reminiscent of cream skimming. It consists of offering a mechanism that attracts fewer low types and more high types, but gives priority to low-type sellers, so that the overall probability of trading a low- and high-quality good remains unchanged for the buyer. At the deviating mechanism, sellers get the same utility as with the mechanisms posted in the market, but the buyer's profits are higher because the rents paid to sellers are strictly lower. Since high-type sellers obtain a strictly lower market utility than low-type sellers, they are less costly to attract.

The above discussion illustrates how the priority rule proves an effective device for the principal to relax the incentive and participation constraints he faces, in particular of the low types. It also shows that it is possible to do this without changing the probability of trading a high- and low-quality good, something the buyer cares for in an adverse selection environment (in contrast to the private value case).

Property 2: No equilibria with separation at the search stage Next we show that we cannot have fully separating equilibria, where some buyers attract only high-type sellers and others attract only low-type sellers. For this to happen, P^{aux} should have (at least) two solutions, one with $\underline{\lambda} = 0$ and the other with $\overline{\lambda} = 0$, and we prove that this is not possible.

Indeed the following lemma shows that the solution of P^{aux} is always unique with regard to the variables concerning the low-type sellers, including their queue length. This implies that in equilibrium, there can be no posted mechanism that attracts only high types. The uniqueness of the complete solution of P^{aux} then depends on whether market utilities are such that buyers can or cannot make positive profits with high-type sellers when their trading probability is set at the maximal level that is feasible and incentive compatible.[24]

[23] This property has bite only for mechanisms that attract both low and high types. It still leaves open the possibility of a fully separating equilibrium where certain mechanisms attract only high-type sellers.

[24] It can be shown (see Appendix A.2) that the maximal incentive-compatible and feasible trading probability for high-type sellers is $\overline{x}^{\max} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. We thus distinguish the cases where $\underline{U}$ and $\overline{U}$ are such that the associated value of buyers' profits for such trades, $\overline{x}^{\max}(\overline{v} - \overline{c}) - \overline{U}$, is strictly positive or zero.

LEMMA 3.4. *(i) If market utilities are such that buyers can make strictly positive profits with high-type sellers, there is a unique solution of* P^{aux}.

(ii) If market utilities are such that buyers can at most make zero profits with high-type sellers, the solution of P^{aux} *is unique for* $(\underline{\lambda}, \underline{x}, \underline{t})$.

Lemma 3.4 shows that when there are mechanisms that satisfy the constraints of problem P^{aux} and yield the buyer strictly positive profits with high-type sellers, the problem has a unique solution. Hence, in equilibrium, all buyers and sellers trade in a single submarket. To show the uniqueness of the solution of P^{aux} in this case, we proceed in two steps. First, we show that at any solution of P^{aux}, the overall feasibility constraint (11) is satisfied with equality, meaning that every meeting leads to trade. If not, buyers would have a profitable deviation, which consists of attracting additional high-type sellers while keeping the terms of trade of all of the previous sellers unchanged. The rest of the proof of Lemma 3.4 verifies that given this property and the one established in Lemma 3.3, P^{aux} is a strictly convex problem.

When, instead, market utilities are such that buyers make at most zero profits with high-type sellers, buyers are indifferent regarding the number of high-type sellers they attract. As a result, there are typically multiple values of $\overline{\lambda}$ that solve the buyer's auxiliary problem. Part (ii) of the lemma shows that in this case, there is still a unique solution for the variables that describe the terms of trade and the queue length for low-type sellers. The result follows again from the strict convexity of the choice problem with regard to these variables. Hence, buyers could post different mechanisms, which, however, differ only in the terms of trade and the queue length for high-type sellers.

To gain some intuition as to why an equilibrium with full separation at the search stage cannot exist, we observe that in such an equilibrium, high- and low-type sellers would trade in two separate markets, with each market generating the same profits for buyers. The latter property implies that buyers must be able to make strictly positive profits with high-type sellers. Consider then a buyer who posts a mechanism that attracts only low-type sellers. This buyer can profit by deviating to an alternative mechanism that also attracts high-type sellers but grants priority to low types, providing them with the same trading probability and expected transfer as the original mechanism. Since buyers have no capacity constraints in their ability to meet sellers, the additional meetings with high types do not crowd out any meetings with low types. Hence, the alternative mechanism would attract the same number of low-type sellers, and the buyer would obtain the same expected payoff from them; in addition, the buyer can make some positive profits with high-type sellers.[25]

[25] If buyers can offer high-type sellers the same terms of trade as in the market where only high types trade, each additional high type generates the same profit as in this market and the profitability of the deviation follows immediately. Sometimes providing the same terms of trade is not feasible because the priority for low-type sellers imposes constraints on the trading probability of high-type sellers. Hence, transfers must be adjusted to attract high types, but our proof shows that also in this case additional profits can be generated.

In the next section we build on these properties and characterize the conditions under which the solutions of problem P^{aux} satisfy the consistency condition with the population parameters, thus establishing the existence of an equilibrium and some additional properties.

3.2 *Existence of equilibrium and further properties*

3.2.1 *Mild adverse selection* We begin by considering the case where market utilities are such that buyers can make strictly positive profits with high-type sellers. We know from the analysis in the previous section that both the low-type feasibility constraint (10) and the overall feasibility constraint (11) are binding at all solutions of P^{aux}. We thus have

$$\underline{\lambda}\underline{x} = 1 - e^{-\underline{\lambda}} \quad \text{and} \quad \overline{\lambda}\overline{x} = e^{-\underline{\lambda}}\big(1 - e^{-\overline{\lambda}}\big). \tag{13}$$

As shown in Lemma 3.4(i), the solution to problem P^{aux} is unique. Hence, so as to obtain an equilibrium, we need to find values of $\underline{U}$ and $\overline{U}$ such that the population ratios $\underline{\lambda}^p$ and $\overline{\lambda}^p$, together with the associated values of $\underline{x}$, $\overline{x}$ obtained from (13), solve problem P^{aux}, and the profits with high-type sellers are indeed positive. The following proposition identifies the condition on the parameter values of the economy under which such values of $\underline{U}$, $\overline{U}$ exist. The argument of the proof also allows us to determine the equilibrium values of the market utilities, thereby characterizing the equilibrium allocation in closed form.

Proposition 3.5. *There exists a directed search equilibrium in which buyers make strictly positive profits with high-type sellers if and only if* $(1 - e^{-\overline{\lambda}^p})/\overline{\lambda}^p < (\underline{v} - \underline{c})/(\overline{v} - \underline{c})$. *In any such equilibrium, all buyers post the same mechanism with trading probabilities*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\big(1 - e^{-\underline{\lambda}^p}\big), \qquad \overline{x} = e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\big(1 - e^{-\overline{\lambda}^p}\big).$$

Condition $(1 - e^{-\overline{\lambda}^p})/\overline{\lambda}^p < (\underline{v} - \underline{c})/(\overline{v} - \underline{c})$ is always satisfied in the case of independent private values ($\underline{v} = \overline{v}$). More generally, it holds when buyers do not care sufficiently for quality ($\bar{v} - \underline{v}$ is sufficiently small) and when the competition for high-type sellers is not too intense (their ratio to buyers $\overline{\lambda}^p$ is sufficiently large).[26] In this situation, the equilibrium utility gain of high-type sellers relative to that of low-type sellers is sufficiently small, so that buyers make strictly positive profits with high-type sellers and all meetings lead to trade. We refer to this parameter region as characterizing a situation where adverse selection is "mild."

3.2.2 *Severe adverse selection* Next we examine the alternative scenario where market utilities are such that buyers can make, at most, zero profits with high-type sellers. In this case, buyers who attract high-type sellers offer a mechanism whereby the trading

[26] Notice that the function $f(\lambda) = (1 - e^{-\lambda})/\lambda$ is strictly decreasing on $(0, +\infty)$ and its range is $(0, 1)$.

probability of high-type sellers is at the highest possible level that is feasible and incentive compatible.[27] The trading probability of low-type sellers is again determined by the priority rule and thus is specified by (13). Similar to the previous section, we identify conditions on the parameter values under which the solution of P^{aux} yields an equilibrium. In this case, as we know from Lemma 3.4(ii), problem P^{aux} has multiple solutions for $\overline{\lambda}$. We, therefore, have multiple payoff equivalent equilibria, as the following proposition establishes.

Proposition 3.6. *There exists a directed search equilibrium in which buyers make zero profits with high-type sellers if and only if the condition $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p \geq (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$ is satisfied. In any such equilibrium, buyers post mechanisms such that*

$$\underline{x} = \frac{1}{\underline{\lambda}^p}\left(1-e^{-\underline{\lambda}^p}\right), \qquad \overline{x} = e^{-\underline{\lambda}^p}\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}.$$

When $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p \geq (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$ is satisfied with equality, the equilibrium is unique; otherwise there is a continuum of payoff equivalent equilibria. The set of equilibria always includes one where all buyers post the same mechanism.

The condition under which the above equilibria exist, $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p \geq (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$, is the exact complement of that stated in Proposition 3.5. In each of the two parameter regions, the trading probabilities are uniquely determined, and, as we show in the proofs of Propositions 3.5 and 3.6, the same is true for the expected transfers. Hence, the equilibrium is always unique in terms of expected payoffs, as stated in Theorem 3.1. Note that the condition $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p \geq (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$ can only be satisfied if $\overline{v} > \underline{v}$; that is, in a situation of common values. More precisely, it requires that buyers care sufficiently for quality ($\overline{v}-\underline{v}$ sufficiently large) and that there are relatively few high-type sellers for each buyer ($\overline{\lambda}^p$ sufficiently small). We refer to such a situation as one of "severe" adverse selection. The stated properties imply that there is now intense competition for high-type sellers. This leads buyers to make zero profits in equilibrium in trades with high types, who consequently extract all of the gains from trade that are realized. Hence, in contrast to results obtained in the cited literature on competitive equilibria with adverse selection, buyers' profits are not equalized across trades with the two types.

The feature whereby buyers make zero profits with high-type sellers is closely linked to two other important equilibrium properties, which we discuss next.

Rationing Suppose the condition $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p \geq (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$ is satisfied with strict inequality. In the equilibrium where all buyers post the same mechanism, the probability that a buyer ends up trading a high-quality object, given by $\overline{\lambda}^p e^{-\underline{\lambda}^p}(\underline{v}-\underline{c})/(\overline{v}-\underline{c})$, is strictly smaller than the probability that a buyer is in a match with high-type sellers only, $e^{-\underline{\lambda}^p}(1-e^{-\overline{\lambda}^p})$. Thus, meetings where only high-type sellers are present do not always end in trade.

To gain some intuition as to why rationing occurs in equilibrium, it is useful to observe that in equilibrium, the payoff of a buyer conditional on meeting a low-type seller

[27]That is, $\bar{x} = \overline{x}^{\max} = (\underline{U}-\overline{U})/(\overline{c}-\underline{c})$ (see footnote 24).

must be strictly higher than his payoff conditional on meeting a high-type seller. That is,

$$\underline{x}(\underline{v}-\underline{c})-\underline{U}>\overline{x}(\overline{v}-\overline{c})-\overline{U}. \tag{14}$$

Note that this property holds for all parameter values. It ensures that in the situation where all buyers post the same mechanism attracting both types, cream-skimming deviations are not profitable. In contrast, if buyers did not make lower profits with high-type sellers, replacing low-type sellers with a larger number of high-type sellers—while keeping their trading probabilities and transfers unchanged—would be both feasible[28] and profitable.

When gains from trade for the high-quality good are large, compared to those for the low-quality good, condition (14) requires that most of these gains are appropriated by the sellers, which means that the high quality object must trade at a high price. Incentive compatibility then requires that the trading probability of high-type sellers is sufficiently small. Giving priority to low-type sellers may not be sufficient to guarantee a small enough trading probability, especially when the ratio of high-type sellers to buyers is very small. Therefore, high-type sellers must be rationed. Crucially, the higher the buyers' valuation of the high-quality good is, the more severe rationing becomes. This can be seen formally by observing that the high-type sellers' equilibrium trading probability decreases in $\overline{v}$ (see Proposition 3.6).

This feature stands in contrast to a monopolistic auction setting or one with random rather than directed search. In a monopolistic auction setting, larger values of $\overline{v}$ favor the posting of pooling mechanisms, whereby both types trade with the same probability, and, thus, lead to a weakly larger trading probability of the high-quality good.[29] Our environment, with competing buyers posting general mechanisms, delivers the opposite comparative statics result. To gain some intuition for this, we should note that in both the monopolistic auction and the random search setting, a buyer faces a set of sellers with given proportions among the different types. In our setup, each buyer faces competition from other buyers, implying that the composition of the sellers he attracts depends on the mechanism he posts. Specifically, if he were to post a pooling price, thereby offering a higher information rent for low-type sellers, he would attract only low types.[30]

Partial sorting When $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p>(\underline{v}-\underline{c})/(\overline{v}-\underline{c})$ holds, Proposition 3.6 shows that we can sustain multiple, payoff equivalent equilibria that differ in their assignment of high-type sellers across buyers. In particular, rather than having each buyer attract the same average number of high-type sellers, we can distribute high-type sellers asymmetrically across submarkets.

[28] This is true since, as shown, sellers with a high-quality good trade with a lower probability than sellers with a low-quality good.

[29] See Manelli and Vincent (1995) for the case of a single buyer who can purchase the good from multiple sellers, and Chiu and Koeppl (2016) and Maurin (2018) for the case of random search. In the latter case, each buyer meets a single seller and optimally posts a price (see Samuelson 1984).

[30] Some further explanation of this property can be found in Section 4, where we discuss the welfare properties of directed search equilibria.

To illustrate this, consider the following specification with two active submarkets, labelled 1 and 2. The respective queue lengths are $\overline{\lambda}_1$, $\underline{\lambda}_1$ and $\overline{\lambda}_2$, $\underline{\lambda}_2$. For simplicity, set $\overline{\lambda}_2 = 0$ so that high-type sellers visit only submarket 1. The trading probabilities for low-type sellers in the two submarkets and for high-type sellers in submarket 1 are then as specified in Proposition 3.6.[31] By Lemma 3.4, both $\underline{\lambda}_1$ and $\underline{\lambda}_2$ must be equal to $\underline{\lambda}^p$. Let γ denote the fraction of buyers posting mechanism 1. Consistency with the population parameters requires that $\overline{\lambda}_1 = (\mu s)/(\gamma b) = \overline{\lambda}^p/\gamma$; that is, the queue length of high-type sellers in the first submarket, $\overline{\lambda}_1$, is equal to the ratio between all high-type sellers in the economy, μs, and those buyers who visit the first submarket, γb. Furthermore, the value of $\overline{\lambda}_1$ must be such that the overall feasibility constraint (11) is satisfied, which, after substituting the values specified above, requires[32]

$$\frac{1}{\frac{1}{\gamma}\overline{\lambda}^p}\left(1 - e^{-\frac{1}{\gamma}\overline{\lambda}^p}\right) \geq \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}}. \tag{15}$$

Under the stated parameter restriction, the above condition is always satisfied for an interval of values of γ sufficiently close to 1; that is, such that sufficiently many buyers and low-type sellers participate in submarket 1, where both types of objects are traded.

In the situation described, two mechanisms coexist in equilibrium. In submarket 2, buyers post a simple mechanism (effectively a price) that only attracts low-type sellers. In submarket 1, buyers post a more complex mechanism (some form of auction), attracting both high- and low-type sellers. Notice that γ can be set so that the overall feasibility constraint (15) is satisfied with equality, in which case there is no rationing in either market.

These two properties of equilibria with severe adverse selection show that the reduction of the equilibrium trading probability of high-type sellers (needed to satisfy incentive compatibility for low-type sellers), can be achieved in different ways: either by rationing high types within the mechanism, by assigning high types asymmetrically across buyers, or by a combination of the two.

Remark. The parameter region with severe adverse selection includes, as a limiting case, the situation where there is free entry of buyers, captured in our environment by letting the measure of buyers tend to $+\infty$. This case is of interest as it corresponds to the analogue of the Rothschild and Stiglitz (1976) game form in the lemons model. As $b \to +\infty$, the ratio of high-type sellers to buyers, $\overline{\lambda}^p$, tends to 0, implying that $(1 - e^{-\overline{\lambda}^p})/\overline{\lambda}^p$ tends to 1. From the proof of Proposition 3.6, it can be seen that as $b \to +\infty$, the transfer to low-type sellers conditional on trading, $\underline{t}/\underline{x}$, converges to $\underline{v}$, while their trading probability converges to 1. Hence, in the limit, buyers also make

[31] In contrast, in submarket 2 where $\overline{\lambda}_2 = 0$, the value of high types' trading probability can be chosen freely as long as the low-type incentive constraint is satisfied. For instance, it could be equal to that of the low types.

[32] Letting $\overline{x}_1$ denote the trading probability of high types in submarket 1, we have $\overline{\lambda}_1\overline{x}_1 = \overline{\lambda}^p/\gamma \cdot e^{-\underline{\lambda}^p}(\underline{v} - \underline{c})/(\overline{v} - \underline{c}) \leq e^{-\underline{\lambda}^p}(1 - e^{-\overline{\lambda}^p/\gamma})$, which yields the expression in the text.

zero profits with low-type sellers. Alternatively, the transfer to high-type sellers conditional on trading is equal to $\overline{v}$, while their trading probability converges to $(\underline{v}-\underline{c})/(\overline{v}-\underline{c})$. These values correspond precisely to those of the separating candidate equilibrium in Rothschild and Stiglitz (1976).[33]

3.3 *Implementation*

Propositions 3.5 and 3.6 characterize the values of the expected trading probabilities for the mechanism buyers post in equilibrium. In their proofs, we also determine the associated market utilities, which yield the expected transfer payments for these mechanisms. What can we say regarding the properties of trading rules that implement these values? As noted in the Introduction, for the case of independent private values, earlier work (see, in particular, Peters 1997) has shown the existence of a directed search equilibrium, where all buyers post the same mechanism and the equilibrium trading probabilities and transfers can be implemented through a second-price auction with a reserve price equal to the buyers' valuation

Our equilibrium characterization shows that in the case of mild adverse selection, trading probabilities—as determined by the priority for low types and the fact that all meetings of a buyer with some sellers end up with trade—are the same as in a second-price auction. The associated transfers, however, can never be implemented through a standard second-price auction with reserve price r, not even if coupled with a participation fee or transfer κ. To see this, notice that the payoffs of low- and high-type sellers in such an auction, denoted by $m_{r,\kappa}$, when the queue lengths are $\underline{\lambda}^p$, $\overline{\lambda}^p$, are given by[34]

$$\underline{u}\big(m_{r,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p\big)=e^{-\underline{\lambda}^p}\big(e^{-\overline{\lambda}^p}(r-\underline{c})+\big(1-e^{-\overline{\lambda}^p}\big)(\overline{c}-\underline{c})\big)+\kappa$$
$$\overline{u}\big(m_{r,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p\big)=e^{-\underline{\lambda}^p-\overline{\lambda}^p}(r-\overline{c})+\kappa.$$

Setting these values equal to the equilibrium market utilities we obtained yields two linearly dependent equations in r and κ. As can be verified, these equations have no solution, except when $\underline{v}=\overline{v}$ (that is, in the case of independent private values).

In contrast, the equilibrium trading probabilities and transfers can be implemented, for instance, via a sequential auction. Buyers could first run a second-price auction with reserve price $r_1\in(\underline{c},\overline{c})$ and if nobody wins the auction, run another auction with reserve price $r_2>\overline{c}$ with probability 1 (when adverse selection is mild) or less than 1 (when it is severe). It is clear that such a sequential auction generates the trading probabilities stated in Propositions 3.5 and 3.6. Furthermore, it can be easily verified that there always exists a pair (r_1,r_2)—possibly coupled with a participation transfer or fee κ—such

[33] Note that in competitive search models, as well as in Walrasian models, the nonexistence issue found by Rothschild and Stiglitz (1976) in a strategic setting does not arise.

[34] In such a second-price auction, a low-type seller earns a positive payoff if and only if he is the only low-type seller in a meeting, an event with probability $e^{-\underline{\lambda}^p}$ (his actual payoff then differs according to whether there are also high-type sellers). For a high-type seller, a positive payoff arises only if he is the only seller in a meeting (which happens with probability $e^{-\underline{\lambda}^p-\overline{\lambda}^p}$).

that the sellers' expected utilities associated with this mechanism are equal to the equilibrium market utilities specified in the proofs of Propositions 3.5 and 3.6.[35]

4. Welfare

In the economy under consideration, the level of total surplus coincides with the realized gains from trade. At an allocation where the trading probabilities for high- and low-type sellers are, respectively, $\overline{x}$ and $\underline{x}$, total surplus is given by

$$b\big(\overline{\lambda}^p\overline{x}(\overline{v}-\overline{c})+\underline{\lambda}^p\underline{x}(\underline{v}-\underline{c})\big).$$

The welfare properties of the equilibria we characterize depend on the values of the parameters of the economy. We first show that when adverse selection is mild and gains from trade are larger for the low-quality object, maximal surplus is attained in equilibrium (hence, the equilibrium is Pareto efficient).

Proposition 4.1. *At the directed search equilibrium allocation, total surplus is maximal, relative to all feasible allocations, if*

$$\underline{v}-\underline{c}\geq\overline{v}-\overline{c}\quad\textit{and}\quad\frac{1}{\overline{\lambda}^p}\big(1-e^{-\overline{\lambda}^p}\big)\leq\frac{\underline{v}-\underline{c}}{\overline{v}-\underline{c}}.$$

If either of the two conditions is violated, there exists a feasible and incentive-compatible allocation that generates strictly more total surplus. Moreover, if μ is sufficiently large, this allocation constitutes a Pareto improvement with respect to the one of the directed search equilibrium.

In the first part of the proposition, we consider the set of allocations that are attainable, subject only to the meeting friction, that is, ignoring incentive compatibility. In this set, total surplus is maximal if (i) the number of meetings is maximized, (ii) every meeting leads to trade, and (iii) sellers who own the good with the larger gains from trade receive priority. It is well known that under the urn–ball meeting technology, the total number of meetings is maximal whenever the queue length of sellers is the same for all the mechanisms traded in equilibrium.[36] The restriction $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p\leq(\underline{v}-\underline{c})/(\overline{v}-\underline{c})$

[35]Letting $m_{r_1,r_2,\kappa,\rho}$ denote the mechanism with sequential reserve prices r_1, r_2, participation transfer κ, and a conditional probability of going to the second round ρ, the low- and high-type seller's payoff are, respectively, given by

$$\underline{u}\big(m_{r_1,r_2,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p\big)=e^{-\underline{\lambda}^p-\overline{\lambda}^p}(r_1-\underline{c})+\kappa$$
$$\overline{u}\big(m_{r_1,r_2,\kappa}|\underline{\lambda}^p,\overline{\lambda}^p\big)=e^{-\underline{\lambda}^p-\overline{\lambda}^p}\rho(r_2-\overline{c})+\kappa.$$

[36]See for example Eeckhout and Kircher (2010). For the private value case, Cai et al. (2017a) in fact demonstrate that merging any two submarkets increases the total surplus under any type distribution if and only if the meeting technology satisfies a property called *joint concavity*. According to this property, the probability of meeting at least one low-type seller is a concave function of the queue length of low-type sellers and the queue length of high-type sellers. For completeness of the argument, in the Supplemental Material, we provide a formal proof of the property that in our environment the number of meetings is maximized when buyers and sellers trade in a single submarket.

assures that in equilibrium, all sellers trade in a single submarket and that high-type sellers are not rationed in equilibrium; thus, the first two properties stated above are indeed satisfied. When $\underline{v} - \underline{c} \geq \overline{v} - \overline{c}$, the sellers with the larger gains from trade are those with a low-quality good. In this case, the property that low-types sellers receive priority in equilibrium implies that the last requirement is also satisfied.

Notice that the two conditions stated in Proposition 4.1 are always satisfied in the case of independent private values, $\underline{v} = \overline{v}$. The result that the directed search equilibrium maximizes total surplus in private value environments is well established in the literature (see, for example, Eeckhout and Kircher 2010 and Cai et al. 2017b). The first part of Proposition 4 extends this result to the case of common values, provided that gains from trade are still larger for the low-quality object and the ratio of high-type sellers to buyers is not too low. In such a situation, incentive compatibility and adverse selection do not constrain attainable welfare.

When, instead, buyers care sufficiently for quality so that the gains from trade for the low-quality object are strictly smaller than those for the high-quality object or when there are sufficiently few high-type sellers so that they are rationed in equilibrium, the welfare properties of equilibria are quite different. The second part of the proposition shows that in this case, the allocation of the directed search equilibrium no longer maximizes total surplus, even subject to the additional constraint imposed by incentive compatibility. Moreover, if the fraction μ of high-type sellers relative to low-type sellers is sufficiently large, there exists an allocation that satisfies the constraints imposed by the meeting friction as well as incentive compatibility and Pareto improves on the allocation of the directed search equilibrium. Hence, the equilibrium is not incentive constrained efficient.

More specifically, the proof of this last part of the claim shows that when either of the two stated conditions is violated, an increase in the trading probability of high-type sellers relative to their equilibrium level—possibly combined with a reduction of the low types' trading probability and a suitable adjustment of the transfers—is both feasible and incentive compatible. This change in the allocation increases total surplus and improves sellers' utility. It also constitutes a Pareto improvement, in the sense that buyers gain as well, provided that the fraction of high- relative to low-type sellers is sufficiently high. In this case, the additional gains made by buyers with high-type sellers more than compensate for the possible losses with low types.

In the equilibrium we characterized, there exists no profitable deviation that allows buyers to capture the additional gains from trade. The reason is that such a deviation would attract too many low-type sellers so as to be profitable. To see this, note that as shown in the proof of the second part of Proposition 4, to increase the trading probability of high-type sellers, a buyer would have to pay an additional information rent to low-type sellers. This implies that all low-type sellers would have strict incentives to switch to the deviating contract, thereby making the deviation nonprofitable.[37]

[37]More precisely, if a buyer posts a mechanism m that yields a higher trading probability for high types, incentive compatibility requires that he offers a payoff for low-type sellers strictly larger than $\underline{U}$ for any pair $\underline{\lambda}$, $\overline{\lambda}$ (see the proof of Proposition 4.1). The buyer's belief is thus pinned down by condition (ii) of Definition 1: $\underline{\lambda}(m) = +\infty$. Given this belief, the deviating contract yields a strictly negative payoff for the buyer.

4.1 *General versus bilateral mechanisms*

The inefficiency of competitive equilibria with adverse selection we found should not come as a surprise. Analogous inefficiency results were obtained in the literature with bilateral contracting, both for competitive equilibria with directed search when meetings are restricted to be bilateral (see Gale 1992 and Guerrieri et al. 2010) and for Walrasian equilibria without search frictions (Dubey and Geanakoplos 2002 and Bisin and Gottardi 2006). In all of these cases, the forces of competition prevent (uninformed) buyers from internalizing the externalities induced by incentive compatibility, linking the trades of different types of (informed) sellers. The properties of the allocations obtained in equilibrium, and, hence, the nature of the inefficiency, are however quite different. As stated in the Introduction, a common feature of bilateral contracting is that in equilibrium, different types of sellers trade different contracts and buyers' profits are equalized across the trades with each type. In that case, the inefficiency of the equilibrium can be entirely imputed to such "no cross-subsidization" property.[38] In contrast, in our setup, all sellers select the same mechanism and profits are not equalized across types. The fact that inefficiency may still obtain suggests that competition among buyers imposes some conditions on the directions of the cross-subsidies, which may be contrary to what is required to achieve efficiency. To better understand this, it is useful to analyze in more detail how the equilibrium allocations in the two setups compare in terms of welfare.

To conduct such a comparison within our framework, we consider the case in which the space of available mechanisms is restricted to the class of bilateral menus, while keeping the meeting technology unchanged. Such restriction entails that a buyer picks at random one of the arriving sellers, who can then choose his preferred item from the menu that is proposed. Conditional on being chosen, a seller's trading probabilities and transfers thus depend on his own report but not on the report of others.

The next proposition shows that given the restriction to bilateral menus, the equilibrium outcome features different mechanisms posted in equilibrium. These mechanisms treat all types the same ($\overline{x} = \underline{x}$) and do not ration any sellers. Hence, rather than posting menus, buyers offer single contracts, which take the form of posted prices, and never find it optimal to attract both types of sellers. Sellers separate themselves according to their type across the different prices posted and buyers' profits are equalized across submarkets, in line with the results in the literature on bilateral contracting discussed above.

Proposition 4.2. *If the set of available mechanisms is restricted to bilateral menus, a directed search equilibrium exists and has the following properties:*

- *A fraction* $\gamma \in [0, 1)$ *of buyers post a price* p_h *and attract only high-type sellers.*
- *The remaining fraction* $1 - \gamma \in (0, 1]$ *of buyers post a price* $p_l < p_h$ *and attract only low-type sellers.*

[38] As Davoodalhosseini (2018) shows, if adverse selection is sufficiently severe, a planner can improve on the competitive search equilibrium allocation by taxing the high-quality market and subsidizing the low-quality market. The cross-subsidization relaxes the incentive compatibility constraint of low-type sellers and thereby allows more trades of the high-quality good.

See the Supplemental Material for the proof.

We can now compare the equilibrium allocation with bilateral contracts to that with general mechanisms, in terms of welfare. Recall first that if the gains from trade for the low-quality good exceed those for the high-quality good and the ratio of high-type sellers to buyers is not too small, the equilibrium with general mechanisms maximizes total surplus (see Proposition 4.1). Hence, it always dominates the one with bilateral contracts in terms of total surplus. The interesting case to consider is that whereby the gains from trade are larger for the high-quality good, as the equilibrium with general mechanisms may not be incentive constrained efficient. As we discussed, the inefficiency is due to the fact that high types trade too little. Their level of trade is particularly low when adverse selection is severe, that is, when equilibrium mechanisms not only give priority to low-quality sellers but also ration high-quality sellers. It could be conjectured that the equilibrium with bilateral contracts, characterized by no cross-subsidization, may do better in such situations. However, we find that this is typically not the case, as the next numerical example illustrates.[39]

Example 1. Let $\underline{\lambda}^p = \overline{\lambda}^p = 1$, and $\underline{c} = 0$, $\overline{c} = 1$, $\underline{v} = 1$, and $\overline{v} = 3$. Under this specification, adverse selection is severe and the equilibrium with general mechanisms is as characterized in Proposition 3.6. A buyer's probability to trade, respectively, a high- and low-quality good is given by

$$\overline{\lambda}^p \overline{x} \approx 0.123, \qquad \underline{\lambda}^p \underline{x} \approx 0.632,$$

while a buyer's probability of meeting some high-type seller without meeting a low-type seller is ≈ 0.233. Hence, in meetings without low-type sellers, trade occurs only slightly more than half of the time.

In the equilibrium where mechanisms are restricted to bilateral menus, a fraction $\gamma \approx 0.120$ of buyers post the high price, while the rest post a low price. A buyer's probability of trading a high- and low-quality good is now

$$\overline{\lambda}^p \overline{x} \approx 0.119, \qquad \underline{\lambda}^p \underline{x} \approx 0.598.$$

In the above example, we see that both the probability that a buyer meets a seller and that he trades a low-quality good are lower in the equilibrium with bilateral contracts. The two properties are intuitive: distributing buyers and sellers over two submarkets with different seller–buyer ratios entails a higher possibility that sellers are misallocated across buyers. Moreover, low-type sellers distribute themselves only across a fraction of buyers, rather than across all of them. What is more surprising is that also the probability of trading a high-quality good is slightly lower. To gain some understanding as to why this happens, notice that for low-type sellers to choose the lower price p_l in equilibrium, the trading probability in the market where p_h is posted must be sufficiently low. The higher are the gains from trade for the high-quality good, the lower is the trading probability for high-type sellers, similar to the case where buyers post general mechanisms.

[39]We considered several other specifications of the parameters in the region with severe adverse selection. In all cases we found that the total surplus in the equilibrium with general mechanisms exceeds that in the equilibrium with bilateral contracts.

Nevertheless, total surplus is not always higher in the equilibrium with general mechanisms. Notably, this reversal arises when parameters fall in the region of mild adverse selection, where such equilibrium features no rationing. The next proposition demonstrates that, provided the gains from trade for the high-quality good exceed those for the low-quality good, this happens when the measure of high-type sellers is sufficiently large.

Proposition 4.3. *Assume $\underline{v} - \underline{c} < \overline{v} - \overline{c}$. If the measure of high-type sellers is sufficiently large, total surplus is strictly greater when buyers are restricted to bilateral menus compared to when they can post general mechanisms.*

The result is established by considering the properties of the equilibrium allocations as $\overline{\lambda}^p \to +\infty$ and can be explained as follows. The property that low-type sellers are given priority in every meeting implies that buyers trade the low-quality good with probability $1 - e^{-\underline{\lambda}^p}$, the probability with which they meet at least one low-type seller. As $\overline{\lambda}^p \to +\infty$, a buyer's probability of meeting some high-type seller tends to 1. Since there is no rationing, this implies that in the limit, buyers trade a high-quality good with the residual probability, $e^{-\underline{\lambda}^p}$. Total surplus thus approaches $b(e^{-\underline{\lambda}^p}(\overline{v} - \overline{c}) + (1 - e^{-\underline{\lambda}^p})(\underline{v} - \underline{c}))$.

In the equilibrium with bilateral contracts, the trading probability of sellers converges to 0 as $\overline{\lambda}^p \to +\infty$, both in the submarket where p_h is posted and in the submarket where p_l is posted. As a consequence, the probability that a buyer trades tends to 1 in both submarkets and the measure of buyers posting the high price tends to b. In the limit, the total surplus in the equilibrium with bilateral contracts is thus given by $b(\overline{v}-\overline{c})$, which is equal to the first best level (all buyers purchase the high-quality object). This value strictly exceeds total surplus in the equilibrium with general mechanisms where, as demonstrated, a positive fraction of buyers end up purchasing the low-quality object, even when $\overline{\lambda}^p \to +\infty$.

It is useful to illustrate these features in the environment of Example 1. When $\overline{\lambda}^p = 10$ instead of $\overline{\lambda}^p = 1$, the equilibrium with general mechanisms features no rationing. Due to the high ratio of sellers to buyers, almost all buyers are matched, but given the feature that low-type sellers receive priority in any match, only 37% of them end up purchasing a high-quality good. Conversely, in the equilibrium with bilateral contracts, the probability that a buyer trades is slightly lower (94% instead of 99%), but the probability of trading a high-quality good is considerably higher (52%). Since gains from trade are higher for the high-quality object, the effect of the increased probability of trading a high-quality good outweighs the effect of the reduced overall probability of trade, and surplus is larger in the equilibrium with bilateral contracts.

To summarize, the analysis in this section shows that the features of the trading mechanisms available to buyers matter considerably. While for most parameter specifications the equilibrium with general mechanisms yields a higher level of total surplus than the equilibrium where mechanisms are restricted to bilateral contracts, this is not always the case when there is adverse selection.[40] When gains from trade are higher for the high-quality good than for the low-quality good, competition among buyers generates inefficiencies in the equilibrium outcome, caused by either the priority given to low-type sellers or the absence of cross-subsidization among sellers' types. For different parameter specifications, one or the other kind of inefficiency proves to be more severe.

5. Conclusion

This paper shows that allowing uninformed buyers to post general mechanisms instead of bilateral contracts in markets with adverse selection has important implications for the equilibrium allocation and its welfare properties. In particular, we find that with general mechanisms there always exists a directed search equilibrium, where all buyers post the same mechanism that gives priority to sellers with a low-quality good. Hence, separation of the different types of sellers occurs within the same mechanism that is posted. According to this mechanism, high-type sellers can trade only in meetings with a buyer where no low-type seller is present, no matter how large are the gains from trade of the good that they own. As discussed, this property of the equilibrium can give rise to severe inefficiencies.

Our analysis is based on the consideration of a specific trading protocol and meeting technology that, even though standard, has some important implications worth highlighting. With regard to the trading protocol, the feature that the uninformed party posts the mechanisms avoids issues of signaling and the associated multiplicity of equilibria.[41] Furthermore, in an environment with one-sided heterogeneity as the one considered, the study of general mechanisms—as opposed to bilateral contracts—is only interesting when homogeneous principals post and heterogeneous agents search, as discussed in Section 2. Extending the analysis to markets with two-sided heterogeneity is an interesting task for future research (for more on this, see Albrecht et al. 2016).

With regard to the meeting technology, we assumed that buyers are not constrained in their ability to meet sellers and that sellers can at most meet one buyer. This constitutes a natural benchmark. In ongoing work, we study the case of more general meeting technologies, where buyers face capacity constraints in their ability to meet sellers.[42] In the presence of these constraints, the deviations that prevented the existence of equilibria with separation at the search stage entail a cost (attracting high types requires attracting fewer low types) and, therefore, may no longer be profitable. As a consequence, it may occur that in all equilibria, buyers post different mechanisms that attract different compositions of seller types.

Some interesting issues also arise when sellers are allowed to meet more than one buyer. This amounts to dropping the maintained assumption of exclusivity at the meeting stage and implies that sellers may have more than a single opportunity to trade at

[40] Lester et al. (2019) study a related issue in an environment with random search and imperfect competition. In particular, they examine how the features of the meeting technology affect traders' market power and, hence, the consequences for the welfare properties of equilibria in the presence of adverse selection.

[41] Delacroix and Shi (2013) take steps in that direction by exploring the consequences of signaling in directed search models, though in environments rather different from ours.

[42] This line of research has been pursued for the case of independent private values by Cai et al. (2017a, 2017b).

the same time, possibly with different terms. In this case, some of the offers made by buyers may end up being rejected. Following Galenianos and Kircher (2009), Kircher (2009), and Wolthoff (2018), we plan to pursue this issue in future research. We conjecture that in an environment where sellers can meet multiple buyers, it is no longer possible to separate sellers at the search or at the mechanism stage. The equilibrium may thus exhibit pooling, a feature that is reminiscent of results obtained in the literature on markets with nonexclusive contracting (see Attar et al. 2011, 2017).

Appendix

A.1 *Proof of Proposition 3.2*

"Only if." We first show that for any feasible and incentive-compatible mechanism m, expected trading probabilities and prices satisfy conditions (7)–(11). Let $\underline{x} = \underline{x}_m(\underline{\lambda}, \overline{\lambda})$, $\overline{x} = \overline{x}_m(\underline{\lambda}, \overline{\lambda})$ and $\underline{t} = \underline{t}_m(\underline{\lambda}, \overline{\lambda})$, $\overline{t} = \overline{t}_m(\underline{\lambda}, \overline{\lambda})$. Incentive compatibility of m then trivially implies (7) and (8). Feasibility implies the remaining conditions. To see this, note first that $\underline{X}_m(L,H)L + \overline{X}_m(L,H)H \le 1\ \forall L, H$ requires $\underline{X}_m(L,H) \le 1/L$, which in turn implies

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\underline{X}_m(L+1,H) \le \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\frac{1}{L+1} = \frac{1}{\underline{\lambda}}\big(1-e^{-\underline{\lambda}}\big).$$

Analogously it can be shown that $\overline{X}_m(L,H) \le 1/H$ implies $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) \le (1-e^{-\overline{\lambda}})/\overline{\lambda}$. From the perspective of a buyer, the probability of trading a low-quality good is given by

$$\sum_{L=1}^{+\infty}\sum_{H=0}^{+\infty} \frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L,H)L = \underline{\lambda}\sum_{L=0}^{+\infty}\sum_{H=0}^{+\infty} \frac{\underline{\lambda}^L}{L!}e^{-\underline{\lambda}}\frac{\overline{\lambda}^H}{H!}e^{-\overline{\lambda}}\underline{X}_m(L+1,H) = \underline{\lambda}\underline{x}_m(\underline{\lambda}, \overline{\lambda}).$$

Similarly, the probability for a buyer to trade a high-quality good can be shown to equal $\overline{\lambda}\overline{x}_m(\underline{\lambda}, \overline{\lambda})$. Feasibility then implies

$$\begin{aligned}
\overline{\lambda}\overline{x}_m(\underline{\lambda}, \overline{\lambda}) + \underline{\lambda}\underline{x}_m(\underline{\lambda}, \overline{\lambda}) &= \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda})\big(\underline{X}_m(L,H)L + \overline{X}_m(L,H)H\big), \\
&\le \sum_{H=0}^{+\infty}\sum_{L=0}^{+\infty} P_H(\overline{\lambda})P_L(\underline{\lambda}) \cdot 1 - P_0(\overline{\lambda})P_0(\underline{\lambda}), \\
&= 1 - e^{-\underline{\lambda}-\overline{\lambda}}.
\end{aligned}$$

"If." We now show that for any vector $(\underline{x}, \overline{x}, \underline{t}, \overline{t})$ satisfying conditions (7)–(11), there exists a feasible and incentive-compatible mechanism m such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$, $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}$, $\overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. Consider the mechanism

$$\begin{aligned}
\underline{X}_m(L,H) &= \underline{\rho}\frac{1}{L+\alpha H} \qquad \underline{T}_m(L,H) = \underline{t}, \quad L \ge 1 \\
\overline{X}_m(L,H) &= \overline{\rho}\frac{\alpha}{L+\alpha H} \qquad \overline{T}_m(L,H) = \overline{t}, \quad H \ge 1,
\end{aligned}$$

for some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$. For the case $\alpha = 0$, let $\overline{X}_m(0,H) = \overline{\rho}/H$.

This mechanism trivially satisfies $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}$ and $\overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. We thus want to show that there always exists some tuple $(\alpha, \underline{\rho}, \overline{\rho})$ such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$. Ex ante trading probabilities are given by

$$\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{\rho} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L+1+\alpha H}$$

$$\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{\rho} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{\alpha}{L+\alpha(H+1)}.$$

Define the function

$$f(\alpha) = \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L+1+\alpha H}.$$

Note that $f'(\alpha) < 0$. The function's range is given by $[(1 - e^{-\underline{\lambda}-\overline{\lambda}})/(\underline{\lambda} + \overline{\lambda}), (1 - e^{-\underline{\lambda}})/\underline{\lambda}]$. To see this, consider first the case $\alpha = 0$:

$$f(0) = \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L+1} = \sum_{L=0}^{+\infty} P_L(\underline{\lambda}) \frac{1}{L+1} \underbrace{\sum_{H=0}^{+\infty} P_H(\overline{\lambda})}_{=1} = \frac{1}{\underline{\lambda}} \sum_{L=0}^{+\infty} \frac{\underline{\lambda}^{L+1}}{(L+1)!} e^{-\underline{\lambda}}$$

$$= \frac{1}{\underline{\lambda}}\big(1 - e^{-\underline{\lambda}}\big).$$

Consider next the case $\alpha = 1$:

$$f(1) = \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L+1+H} = \sum_{N=0}^{+\infty} P_N(\underline{\lambda} + \overline{\lambda}) \frac{1}{N+1} = \frac{1}{\underline{\lambda} + \overline{\lambda}}\big(1 - e^{-\underline{\lambda}-\overline{\lambda}}\big).$$

Define the function

$$g(\alpha) = \begin{cases} \displaystyle\sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{\alpha}{L+\alpha(H+1)} & \text{if } \alpha > 0, \\ \displaystyle\sum_{H=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{H+1} & \text{if } \alpha = 0. \end{cases}$$

Note that $g'(\alpha) > 0$ and that g is continuous at $\alpha = 0$, i.e., $\lim_{\alpha \to 0} g(\alpha) = g(0)$. At $\alpha = 1$, g is equal to f. At $\alpha = 0$, we have

$$g(0) = \sum_{H=0}^{+\infty} P_H(\overline{\lambda}) P_0(\underline{\lambda}) \frac{1}{H+1} = e^{-\underline{\lambda}} \sum_{H=0}^{+\infty} \frac{\overline{\lambda}^H}{H!} e^{-\overline{\lambda}} \frac{1}{H+1} = \frac{e^{-\underline{\lambda}}}{\overline{\lambda}} \sum_{H=0}^{+\infty} \frac{\overline{\lambda}^{H+1}}{(H+1)!} e^{-\overline{\lambda}}$$

$$= e^{-\underline{\lambda}} \frac{1}{\overline{\lambda}}\big(1 - e^{-\overline{\lambda}}\big).$$

The range of g is consequently $[e^{-\underline{\lambda}}(1 - e^{-\overline{\lambda}})/\overline{\lambda}, (1 - e^{-\underline{\lambda}-\overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})]$.

With this we can show that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)–(11), we can find some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$ such that $\underline{\rho} f(\alpha) = \underline{x}$ and $\overline{\rho} g(\alpha) = \overline{x}$. Given that $\underline{x}, \overline{x} \geq 0$ and $\underline{\rho}, \overline{\rho} \in [0,1]$, this can be satisfied if there exists an α such that $f(\alpha) \geq \underline{x}$ and $g(\alpha) \geq \overline{x}$. The first inequality requires that α is not too large, while the second requires that α is not too small. Consider first the case in which $\underline{x} \leq (1 - e^{-\underline{\lambda}-\overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$. Here $f(\alpha) \geq \underline{x}$ is satisfied for all $\alpha \in [0,1]$. Conditions (7), (8), and (11) together imply $\overline{x} \leq (1 - e^{-\underline{\lambda}-\overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$, from which it follows that $g(\alpha) \geq \overline{x}$ can be satisfied (e.g., $\alpha = 1$). Consider now the case $\underline{x} \geq (1 - e^{-\underline{\lambda}-\overline{\lambda}})/(\underline{\lambda} + \overline{\lambda})$ and let $\tilde{\alpha}$ be such that $f(\tilde{\alpha}) = \underline{x}$. We can show

$$
\begin{aligned}
&\overline{\lambda} g(\tilde{\alpha}) + \underline{\lambda} f(\tilde{\alpha}) \\
&\quad = \overline{\lambda} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{\tilde{\alpha}}{L + \tilde{\alpha}(H+1)} + \underline{\lambda} \sum_{H=0}^{+\infty} \sum_{L=0}^{+\infty} P_H(\overline{\lambda}) P_L(\underline{\lambda}) \frac{1}{L + 1 + \tilde{\alpha} H} \\
&\quad = \overline{\lambda} \sum_{H=1}^{+\infty} \sum_{L=0}^{+\infty} \frac{\overline{\lambda}^{H-1}}{(H-1)!} \frac{\underline{\lambda}^L}{L!} e^{-\underline{\lambda}-\overline{\lambda}} \frac{\tilde{\alpha}}{L + \tilde{\alpha} H} \frac{H}{H} + \underline{\lambda} \sum_{H=0}^{+\infty} \sum_{L=1}^{+\infty} \frac{\overline{\lambda}^{H}}{H!} \frac{\underline{\lambda}^{L-1}}{(L-1)!} \frac{1}{L + \tilde{\alpha} H} \frac{L}{L} \\
&\quad = \sum_{H=1}^{+\infty} \sum_{L=1}^{+\infty} \frac{\overline{\lambda}^{H}}{H!} \frac{\underline{\lambda}^{L}}{L!} e^{-\underline{\lambda}-\overline{\lambda}} \underbrace{\left(\frac{\tilde{\alpha} H}{L + \tilde{\alpha} H} + \frac{L}{L + \tilde{\alpha} H} \right)}_{=1} + \sum_{H=1}^{+\infty} \frac{\overline{\lambda}^{H}}{H!} e^{-\underline{\lambda}-\overline{\lambda}} + \sum_{L=1}^{+\infty} \frac{\underline{\lambda}^{L}}{L!} e^{-\underline{\lambda}-\overline{\lambda}} \\
&\quad = \left(1 - e^{-\overline{\lambda}} - e^{-\underline{\lambda}} + e^{-\underline{\lambda}-\overline{\lambda}}\right) + \left(e^{-\underline{\lambda}} - e^{-\underline{\lambda}-\overline{\lambda}}\right) + \left(e^{-\overline{\lambda}} - e^{-\underline{\lambda}-\overline{\lambda}}\right) \\
&\quad = 1 - e^{-\underline{\lambda}-\overline{\lambda}}.
\end{aligned}
$$

With this,

$$
g(\tilde{\alpha}) = \frac{1}{\overline{\lambda}} \left(1 - e^{-\underline{\lambda}-\overline{\lambda}} - \underline{\lambda}, f(\tilde{\alpha})\right) = \frac{1}{\overline{\lambda}} \left(1 - e^{-\underline{\lambda}-\overline{\lambda}} - \underline{\lambda x}\right) \geq \overline{x},
$$

where the last inequality follows from condition (11). Thus, there exists some $\overline{\rho} \in [0,1]$ such that $\overline{\rho} g(\tilde{\alpha}) = \overline{x}$. Together this implies that for any $\underline{x}$ and $\overline{x}$ satisfying conditions (9)–(11), there exists some $\alpha, \underline{\rho}, \overline{\rho} \in [0,1]$ such that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$ and $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$.

Finally we need to check feasibility and incentive compatibility of the proposed mechanism. Feasibility follows from

$$
\underline{x}(L,H)L + \overline{x}(L,H)H = \underline{\rho} \frac{1}{L + \alpha H} L + \overline{\rho} \frac{\alpha}{L + \alpha H} H \leq \frac{1}{L + \alpha H} L + \frac{\alpha}{L + \alpha H} H = 1.
$$

Incentive compatibility is trivially satisfied given that $\underline{x}_m(\underline{\lambda}, \overline{\lambda}) = \underline{x}$, $\overline{x}_m(\underline{\lambda}, \overline{\lambda}) = \overline{x}$ and $\underline{t}_m(\underline{\lambda}, \overline{\lambda}) = \underline{t}$, $\overline{t}_m(\underline{\lambda}, \overline{\lambda}) = \overline{t}$. □

A.2 *Solving P^{aux}: Preliminaries*

We start by deriving some properties of the market utilities that need to be satisfied in equilibrium.

Lemma A.1. *At a directed search equilibrium, we have*

(i) $\underline{U} > \overline{U}$ *and* $\underline{U} - \overline{U} < \overline{c} - \underline{c}$

(ii) $\underline{U}, \overline{U} > 0$

(iii) $\underline{U} < \underline{v} - \underline{c}$ *and* $\overline{U} \leq (\overline{v} - \overline{c})/(\overline{v} - \underline{c})\underline{U}$.

Proof. Let $(\underline{x}, \underline{t})$ and $(\overline{x}, \overline{t})$ be pairs of expected trading probabilities and transfers associated to (possibly different) mechanisms chosen by low- and high-type sellers in a given equilibrium. These values must then also be part of a solution of P^{aux}. Market utilities are, therefore, $\underline{U} = \underline{t} - \underline{x}\underline{c}$ and $\overline{U} = \overline{t} - \overline{x}\overline{c}$. The following properties must hold:

(i)(a) $\underline{U} > \overline{U}$: The low-type incentive constraint (16) requires $\underline{t} - \underline{x}\underline{c} \geq \overline{t} - \overline{x}\underline{c}$, which can be rewritten as $\overline{x}(\overline{c} - \underline{c}) \leq \underline{U} - \overline{U}$. Since $\overline{x} \geq 0$, this inequality can only be satisfied if $\underline{U} \geq \overline{U}$. Suppose now that $\underline{U} = \overline{U}$ so that $\overline{x} = 0$. Since, under any solution of P^{aux}, buyers must make weakly positive profits with both types of seller,[43] we must have $\overline{t} = 0$ and, hence, $\underline{U} = \overline{U} = 0$. A buyer's expected profit with low-type sellers when $\underline{U} = \underline{t} - \underline{x}\underline{c} = 0$ is $\underline{\lambda}\underline{x}(\underline{v} - \underline{c})$. The maximal value of this last expression at an admissible solution of P^{aux} is attained when $\underline{\lambda}\underline{x} = (1 - e^{-\underline{\lambda}})$; that is, if the buyer trades with low-type sellers whenever possible. However, since $1 - e^{-\underline{\lambda}}$ is strictly increasing in $\underline{\lambda}$, no finite value of $\underline{\lambda}$ can solve P^{aux}, implying that $\underline{U} = \overline{U} = 0$ cannot be admissible equilibrium values.

(b) $\underline{U} - \overline{U} < \overline{c} - \underline{c}$: The high-type incentive constraint (8) requires $\overline{t} - \overline{x}\overline{c} \geq \underline{t} - \underline{x}\overline{c}$ or $\underline{x}(\overline{c} - \underline{c}) \geq \underline{U} - \overline{U}$. Since by condition (10), we have $\underline{x} \leq (1 - e^{-\underline{\lambda}})/\underline{\lambda} < 1$,[44] the inequality $\underline{x}(\overline{c} - \underline{c}) \geq \underline{U} - \overline{U}$ can only be satisfied if $\underline{U} - \overline{U} < \overline{c} - \underline{c}$.

(ii) $\underline{U}, \overline{U} > 0$: The inequality $\underline{U} > 0$ follows directly from $\underline{U} > \overline{U}$ and $\overline{U} \geq 0$. It thus remains to show that $\overline{U} > 0$. Toward a contradiction, suppose $\overline{U} = 0$. In this case, by an argument symmetric to that in (i)(a) above, the expected profit a buyer makes trading with high-type sellers is $\overline{\lambda}\overline{x}(\overline{v} - \overline{c})$. Since $\underline{U} > \overline{U}$, there always exists a strictly positive value of $\overline{x}$ that satisfies the low-type incentive constraint (7) and the overall feasibility constraint (11). Given that $\overline{\lambda}$ can be set equal to 0, a strictly positive value of $\overline{x}$ is also weakly optimal. Thus, w.l.o.g. suppose $\overline{x} > 0$ and consider an increase of $\overline{\lambda}$ together with a decrease of $\overline{x}$ so as to keep $\overline{\lambda}\overline{x}$ unchanged. Adjusting $\overline{t}$ so as to keep the utility of high-type sellers constant while keeping the remaining contracting parameters unchanged, we obtain a tuple $(\underline{\lambda}, \overline{\lambda}', \underline{x}, \overline{x}', \underline{t}, \overline{t})$ that is incentive compatible and satisfies the overall feasibility constraint (11) with strict inequality, i.e., $\underline{\lambda}\underline{x} + \overline{\lambda}'\overline{x}' < 1 - e^{-\underline{\lambda} - \overline{\lambda}'}$. The buyer can then increase his payoff by deviating to $(\underline{\lambda}, \overline{\lambda}' + \varepsilon, \underline{x}, \overline{x}', \underline{t}, \overline{t}')$, with $\varepsilon > 0$, while still satisfying all constraints of P^{aux}, thus, a contradiction.

(iii)(a) $\underline{U} < \underline{v} - \underline{c}$: Suppose not, $\underline{U} \geq \underline{v} - \underline{c}$. Since, as shown in (i)(b) above, $\underline{x} < 1$ whenever $\underline{\lambda} > 0$, this implies that $\underline{x}(\underline{v} - \underline{c}) - \overline{U} < 0$; that is, a buyer's payoff with each low-type seller is strictly negative. As a consequence, at any solution of P^{aux} we have $\underline{\lambda} = 0$. This in turn implies that the low types' market utility $\underline{U}$ must equal 0 and, therefore, $\underline{U} < \underline{v} - \underline{c}$, a contradiction.

[43] If buyers make losses with one type of seller, they can always set the respective ratio, $\underline{\lambda}$ or $\overline{\lambda}$, equal to 0.

[44] The inequality $(1 - e^{-\underline{\lambda}})/\underline{\lambda} < 1$ is a general property for all $\underline{\lambda} \in (0, +\infty)$.

(b) $\overline{U} \leq (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$: Suppose not, $\overline{U} > (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$. As argued above, the low-type incentive constraint (7) can be written as $\overline{x} \leq (\underline{U}-\overline{U})/(\overline{c}-\underline{c})$. This implies that the payoff of a buyer with each high-type seller is negative:

$$\overline{x}(\overline{v}-\overline{c})-\overline{U} \leq \frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c})-\overline{U} = \frac{(\overline{v}-\overline{c})\underline{U}-(\overline{v}-\underline{c})\overline{U}}{\overline{c}-\underline{c}} < 0.$$

All solutions of P^{aux} must, therefore, satisfy $\overline{\lambda}=0$, which in turn implies $\overline{U}=0$ and, therefore, $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$, a contradiction. □

Next, we can show that w.l.o.g. the participation constraints of the two types of sellers in P^{aux} can be set to hold as equalities. Consider first the possibility that at a solution of P^{aux} the buyer only wants to attract high-type sellers, i.e., $\underline{\lambda}=0$. Since $\underline{U}-\overline{U} < \overline{c}-\underline{c}$, there exists a pair $\underline{t}$, $\underline{x}$ so as to satisfy $\underline{t}-\underline{x}\underline{c}=\underline{U}$ and $\underline{t}-\underline{x}\overline{c} \leq \overline{U}$. In particular, any combination of $\underline{x}$ and $\underline{t}$ that satisfies $\underline{x} \geq (\underline{U}-\overline{U})/(\overline{c}-\underline{c})$ and $\underline{t}=\underline{U}+\underline{x}\underline{c}$ makes low-type sellers indifferent and satisfies the incentive constraint of high-type sellers. The actual choice of $\underline{x}$, $\underline{t}$ does not affect the buyer's payoff, as $\underline{x}$ and $\underline{t}$ are multiplied by $\underline{\lambda}=0$ both in the objective and in the remaining constraints. A symmetric argument can be made for the case of $\overline{\lambda}=0$.

Solving the participation constraints for $\underline{t}$, $\overline{t}$ and substituting into the objective function and the remaining constraints, the buyer's optimization problem can be rewritten in the simpler form

$$\max_{\underline{x},\overline{x},\underline{\lambda},\overline{\lambda}} \overline{\lambda}\big[\overline{x}(\overline{v}-\overline{c})-\overline{U}\big]+\underline{\lambda}\big[\underline{x}(\underline{v}-\underline{c})-\underline{U}\big], \qquad (P^{\mathrm{aux}'})$$

subject to

$$\overline{x}(\overline{c}-\underline{c}) \leq \underline{U}-\overline{U} \qquad (16)$$

$$\underline{x}(\overline{c}-\underline{c}) \geq \underline{U}-\overline{U} \qquad (17)$$

$$\underline{\lambda}\underline{x} \leq 1-e^{-\underline{\lambda}} \qquad (18)$$

$$\overline{\lambda}\overline{x}+\underline{\lambda}\underline{x} \leq 1-e^{-\overline{\lambda}-\underline{\lambda}} \qquad (19)$$

$$\overline{\lambda}, \underline{\lambda} \geq 0.$$

All properties of P^{aux} will be proven by considering the simplified problem $P^{\mathrm{aux}'}$.

A.3 *Proof of Lemma 3.3*

Suppose the claim is not valid; that is, suppose there exists a solution $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ of $P^{\mathrm{aux}'}$ with $\underline{x}\underline{\lambda} < 1-e^{-\underline{\lambda}}$. Note that the strict inequality requires $\underline{\lambda} > 0$. Consider then an alternative tuple $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ with $\underline{\lambda}' < \underline{\lambda}$, $\overline{\lambda}' = \overline{\lambda}+(\underline{\lambda}-\underline{\lambda}')$, $\underline{x}' = \underline{\lambda}/\underline{\lambda}'\underline{x}$, and $\overline{x}' = \overline{\lambda}/\overline{\lambda}'\overline{x}$. If $\underline{\lambda}'$ is sufficiently close to $\underline{\lambda}$, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ satisfies constraint (18). Since $\underline{x}'\underline{\lambda}' = \underline{x}\underline{\lambda}$, $\overline{x}'\overline{\lambda}' = \overline{x}\overline{\lambda}$, and $\underline{\lambda}'+\overline{\lambda}' = \underline{\lambda}+\overline{\lambda}$, it also satisfies the overall feasibility constraint (19). Last, since $\underline{x}' > \underline{x}$

and $\overline{x}' < \overline{x}$, the mechanism associated to this alternative tuple always satisfies the incentive compatibility constraints (16) and (17). Consider now the buyer's payoff associated to $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$:

$$\begin{aligned}
&\underline{\lambda}'\underline{x}'(\underline{v}-\underline{c}) + \overline{\lambda}'\overline{x}'(\overline{v}-\overline{c}) - \underline{\lambda}'\underline{U} - \overline{\lambda}'\overline{U} \\
&\quad = \underline{\lambda}\underline{x}(\underline{v}-\underline{c}) + \overline{\lambda}\overline{x}(\overline{v}-\overline{c}) - \underline{\lambda}'\underline{U} - \big(\overline{\lambda} + \big(\underline{\lambda} - \underline{\lambda}'\big)\big)\overline{U} \\
&\quad = \underline{\lambda}\underline{x}(\underline{v}-\underline{c}) + \overline{\lambda}\overline{x}(\overline{v}-\overline{c}) - \underline{\lambda}\underline{U} - \overline{\lambda}\overline{U} + \big(\underline{\lambda} - \underline{\lambda}'\big)(\underline{U} - \overline{U}).
\end{aligned}$$

Since the last term of the last line is strictly positive, $(\underline{x}', \overline{x}', \underline{\lambda}', \overline{\lambda}')$ yields a strictly higher payoff than $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$, a contradiction. □

A.4 *Proof of Lemma 3.4*

Given market utilities, a buyer's profit per high-type seller he attracts is $\overline{x}(\overline{v}-\overline{c}) - \overline{U}$. The maximal incentive feasible trading probability for high-type sellers is $\overline{x} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$ (see constraint (16) in $P^{\text{aux}'}$). Substituting this value of the trading probability into the profits a buyer obtains with each high-type seller, we find that these profits are nonnegative if $\overline{U} \leq (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$.

(i) We start by considering the case where $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$, so that buyers can make strictly positive profits with high-type seller. We show next that at all solutions of $P^{\text{aux}'}$, the overall feasibility condition (19) is satisfied as equality. Let $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ be a tuple that solves $P^{\text{aux}'}$. If $\overline{\lambda} = 0$, constraint (19) is satisfied as equality by Lemma 3.3. Consider now the case $\overline{\lambda} > 0$. Toward a contradiction, suppose that (19) is satisfied as a strict inequality at a solution of $P^{\text{aux}'}$. If this is the case, when the buyer's profit with each high-type seller, $\overline{x}(\overline{v}-\overline{c}) - \overline{U}$, is strictly positive, the buyer can strictly increase his payoff by increasing $\overline{\lambda}$. For this not to be a profitable deviation, we must therefore have $\overline{x}(\overline{v}-\overline{c}) \leq \overline{U}$. Noting that the stated property $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$ is equivalent to $\overline{U}/(\overline{v}-\overline{c}) < (\underline{U}-\overline{U})/(\overline{c}-\underline{c})$, the inequality $\overline{x}(\overline{v}-\overline{c}) \leq \overline{U}$ implies $\overline{x} < (\underline{U}-\overline{U})/(\overline{c}-\underline{c})$. That is, the low-type incentive constraint (16) is satisfied as a strict inequality. By slackness of the constraints (16) and (19), an increase in $\overline{x}$ is then both feasible and incentive compatible. Since increasing $\overline{x}$ strictly increases the buyer's payoff with each high-type seller and since $\overline{\lambda} > 0$, the buyer has a profitable deviation, thus a contradiction.

Having shown that when $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$, feasibility constraint (19) is satisfied with equality and recalling that the same is true for (18) by Lemma 3.3, the values of $\underline{\lambda}$, $\overline{\lambda}$ belonging to a solution of $P^{\text{aux}'}$ maximize

$$\hat{\pi}(\underline{\lambda}, \overline{\lambda}) := \big(1 - e^{-\underline{\lambda}}\big)(\underline{v}-\underline{c}) + e^{-\underline{\lambda}}\big(1 - e^{-\overline{\lambda}}\big)(\overline{v}-\overline{c}) - \underline{\lambda}\underline{U} - \overline{\lambda}\overline{U}, \tag{20}$$

subject to incentive compatibility constraints (16) and (17). The two latter constraints can, respectively, be rewritten as

$$e^{-\underline{\lambda}}\big(1 - e^{-\overline{\lambda}}\big) \leq \overline{\lambda}\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} \tag{21}$$

$$\big(1 - e^{-\underline{\lambda}}\big) \geq \underline{\lambda}\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}}. \tag{22}$$

(a) Consider first the case $\underline{v}-\underline{c}>\overline{v}-\overline{c}$. Under this condition, $\hat{\pi}(\underline{\lambda},\overline{\lambda})$ is strictly concave. To see this, note that $\partial^2\hat{\pi}/\partial\overline{\lambda}^2=-e^{-\underline{\lambda}}e^{-\overline{\lambda}}(\overline{v}-\overline{c})<0$ and that the determinant of the Hessian is given by $e^{-2\underline{\lambda}}e^{-\overline{\lambda}}(\overline{v}-\overline{c})[(\underline{v}-\underline{c})-(\overline{v}-\overline{c})]>0$. The Hessian is thus negative definite.

We can show that, as a consequence, there cannot exist two solutions $(\underline{x}_1,\overline{x}_1,\underline{\lambda}_1,\overline{\lambda}_1)$ and $(\underline{x}_2,\overline{x}_2,\underline{\lambda}_2,\overline{\lambda}_2)$. Strict concavity of $\hat{\pi}(\underline{\lambda},\overline{\lambda})$ would in fact imply that a strictly larger payoff is attained at $\underline{\lambda}=\alpha\underline{\lambda}_1+(1-\alpha)\underline{\lambda}_2$, $\overline{\lambda}=\alpha\overline{\lambda}_1+(1-\alpha)\overline{\lambda}_2$, $\alpha\in(0,1)$. The pair $(\underline{\lambda},\overline{\lambda})$ satisfies the incentive constraints since $(1-e^{-\underline{\lambda}})/\underline{\lambda}\geq \text{Min}\{(1-e^{-\underline{\lambda}_1})/\underline{\lambda}_1,(1-e^{-\underline{\lambda}_2})/\underline{\lambda}_2\}$ and $e^{-\underline{\lambda}}(1-e^{-\overline{\lambda}})/\overline{\lambda}\leq \text{Max}\{e^{-\underline{\lambda}_1}(1-e^{-\overline{\lambda}_1})/\overline{\lambda}_1,e^{-\underline{\lambda}_2}(1-e^{-\overline{\lambda}_2})/\overline{\lambda}_2\}$.[45] Hence $(\underline{\lambda},\overline{\lambda},\underline{x},\overline{x})$ is an admissible solution and yields a higher payoff, a contradiction.

(b) Consider next the case $\underline{v}-\underline{c}\leq\overline{v}-\overline{c}$. We first establish that the incentive constraint of the low-type sellers must be binding at a solution of $P^{\text{aux}'}$.

Lemma A.2. *If $\overline{U}<(\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$ and $\underline{v}-\underline{c}\leq\overline{v}-\overline{c}$, at any solution of $P^{\text{aux}'}$ the low-type incentive compatibility constraint (16) is satisfied with equality.*

Proof. We argue by contradiction. Assume $(\underline{x},\overline{x},\underline{\lambda},\overline{\lambda})$ is a solution of $P^{\text{aux}'}$ (with $\overline{\lambda}>0$) and (16) is satisfied as inequality: $\overline{x}<(\underline{U}-\overline{U})/(\overline{c}-\underline{c})$. Suppose first that $\underline{\lambda}>0$ and consider an alternative tuple $(\underline{x}',\overline{x}',\underline{\lambda}',\overline{\lambda}')$ with $\underline{\lambda}'=\underline{\lambda}-\Delta$, $\Delta>0$, $\overline{\lambda}'=\overline{\lambda}+\Delta$, $\underline{x}'=(1-e^{-\underline{\lambda}'})/\underline{\lambda}'$, and $\overline{x}'=e^{-\underline{\lambda}'}(1-e^{-\overline{\lambda}'})/\overline{\lambda}'$. In the alternative tuple some low types are replaced with high types, while the ratio between buyers and all types of sellers is kept unchanged, and the feasibility constraints (18) and (19) still hold as equality. As a consequence, we have $\underline{x}'>\underline{x}$ and $\overline{x}'>\overline{x}$, implying that $(\underline{x}',\overline{x}',\underline{\lambda}',\overline{\lambda}')$ satisfies all constraints of problem $P^{\text{aux}'}$ as long as Δ is sufficiently small. The difference between the buyer's payoff associated to $(\underline{x}',\overline{x}',\underline{\lambda}',\overline{\lambda}')$ and that associated to $(\underline{x},\overline{x},\underline{\lambda},\overline{\lambda})$ is then

$$\hat{\pi}(\underline{\lambda}',\overline{\lambda}')-\hat{\pi}(\underline{\lambda},\overline{\lambda})=\left(e^{-\underline{\lambda}+\Delta}-e^{-\underline{\lambda}}\right)\left[(\overline{v}-\overline{c})-(\underline{v}-\underline{c})\right]+\Delta(\underline{U}-\overline{U})>0.$$

The first term is the difference in the probability of meeting no low-type seller (in which case a high-quality good is traded) multiplied by the difference between the gains from trade of the high- and low-quality good, while the second term is the difference in rent paid to sellers: since the alternative mechanism on average replaces Δ low-type sellers with high-type sellers, the reduction in rent is $\Delta(\underline{U}-\overline{U})$. Under the assumption $\underline{v}-\underline{c}\leq\overline{v}-\overline{c}$, the sum of the two terms is strictly positive. Hence, $(\underline{x}',\overline{x}',\underline{\lambda}',\overline{\lambda}')$ yields a strictly larger payoff than $(\underline{x},\overline{x},\underline{\lambda},\overline{\lambda})$, a contradiction.

For the case $\underline{\lambda}=0$, notice that $\hat{\pi}(\underline{\lambda},\overline{\lambda})$ as well as the correspondence that determines the set of feasible values of $\overline{\lambda}$, defined by the constraint $e^{-\underline{\lambda}}(1-e^{-\overline{\lambda}})/\overline{\lambda}\leq(\underline{U}-\overline{U})/(\overline{c}-\underline{c})$, are continuous in $\underline{\lambda}$ at $\underline{\lambda}=0$. Consider the constrained optimization problem where we require $\underline{\lambda}\geq\varepsilon>0$. By the previous argument, incentive constraint (16) holds as equality at a solution of this constrained problem for all $\varepsilon>0$, and by the continuity property, the solution of the constrained problem converges to the solution of the original problem where we require only $\underline{\lambda}\geq 0$. Since (16) holds as equality along all points in the sequence, it does so in the limit. □

[45] It can be verified that $e^{-\underline{\lambda}}(1-e^{-\overline{\lambda}})/\overline{\lambda}$ is convex in $\underline{\lambda}$ and $\overline{\lambda}$.

Having shown that the low-type incentive constraint (16) is binding for all $\overline{\lambda} > 0$, we can solve this constraint for $\underline{\lambda}$, yielding

$$\underline{\lambda} = \ln\left(\frac{1-e^{-\overline{\lambda}}}{\overline{\lambda}} \frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right).$$

The condition $\underline{\lambda} \geq 0$ requires that $\overline{\lambda} \leq L$, where L is defined by $(1-e^{-L})/L = (\underline{U} - \overline{U})/(\overline{c}-\underline{c})$. For $\overline{\lambda} \in (0, L]$, the buyer's payoff can then be written as a function of $\overline{\lambda}$ only:

$$\tilde{\pi}(\overline{\lambda}) = \overline{\lambda}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}(\overline{v}-\overline{c}) + \left(1 - \frac{\overline{\lambda}}{1-e^{-\overline{\lambda}}}\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}}\right)(\underline{v}-\underline{c}) - \overline{\lambda}\overline{U} - \ln\left(\frac{1-e^{-\overline{\lambda}}}{\overline{\lambda}}\frac{\overline{c}-\underline{c}}{\underline{U}-\overline{U}}\right)\underline{U}. \quad (23)$$

Since $\underline{x} > \overline{x} =$ for $\underline{\lambda}, \overline{\lambda} > 0$ (by Lemma 3.3), the fact that the low-type incentive constraint (16) holds as equality immediately implies that the high-type incentive constraint (17) is slack. At a solution of $P^{\text{aux}'}$ with $\overline{\lambda} > 0$, $\overline{\lambda}$ is thus the unconstrained maximizer of $\tilde{\pi}$ on $(0, L]$. We show in the Supplemental Material that the function $\tilde{\pi}$ is strictly concave in $\overline{\lambda}$ and that there cannot be a solution of problem $P^{\text{aux}'}$ with $\overline{\lambda} = 0$. Hence, the solution to this problem is unique and the optimal value of $\overline{\lambda}$ is strictly positive.

(ii) We next consider the case where $\overline{U} = (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$ so that buyers make at most zero profits with high-type sellers. In this case, we have $\overline{\lambda}[\overline{x}(\overline{v}-\overline{c}) - \overline{U}] = 0$ for all solutions of $P^{\text{aux}'}$ and $\overline{x} = (\underline{U} - \overline{U})/(\overline{c}-\underline{c})$ for all solutions of $P^{\text{aux}'}$ with $\overline{\lambda} > 0$. We can then show that $\underline{\lambda}$ is the same at all solutions of $P^{\text{aux}'}$. Suppose not. By Lemma 3.3, the buyer's payoff from low-type sellers is given by $(1-e^{-\underline{\lambda}})(\underline{v}-\underline{c}) - \underline{\lambda}\underline{U}$. This term is strictly concave in $\underline{\lambda}$ and attains its maximum at $\underline{\lambda} = \ln((\underline{v}-\underline{c})/\underline{U})$. This implies that any solution of $P^{\text{aux}'}$ must satisfy $\underline{\lambda} \leq \ln((\underline{v}-\underline{c})/\underline{U})$: if in fact the buyer proposes a mechanism with $\underline{\lambda} > \ln((\underline{v}-\underline{c})/\underline{U})$, decreasing $\underline{\lambda}$ and adjusting $\underline{x}$ so that (18) still holds as equality strictly increases the buyer's payoff and weakly relaxes all constraints. Now suppose there exist two values $\underline{\lambda}_1$ and $\underline{\lambda}_2$ with $\underline{\lambda}_1 \neq \underline{\lambda}_2$ that belong to some solution of $P^{\text{aux}'}$. Without loss of generality, assume $\underline{\lambda}_1 < \underline{\lambda}_2$, which implies $(1-e^{-\underline{\lambda}_1})(\underline{v}-\underline{c}) - \underline{\lambda}_1\underline{U} < (1-e^{-\underline{\lambda}_2})(\underline{v}-\underline{c}) - \underline{\lambda}_2\underline{U}$. Since both solutions must yield the same payoff for the buyer, we must then have $\overline{\lambda}_1[\overline{x}_1(\overline{v}-\overline{c}) - \overline{U}] > \overline{\lambda}_2[\overline{x}_2(\overline{v}-\overline{c}) - \overline{U}]$. However, we know that $\overline{\lambda}[\overline{x}(\overline{v}-\overline{c}) - \overline{U}] = 0$ for all solutions of $P^{\text{aux}'}$ and, therefore, we have a contradiction.

Finally, the property that $\underline{\lambda}$ is the same for all solutions of $P^{\text{aux}'}$, together with the result in Lemma 3.3, immediately implies that $\underline{x}$ must have the same value for all solutions of $P^{\text{aux}'}$. Through the low-type participation constraint (12), this in turn implies that also the transfer $\underline{t}$ is the same. □

A.5 *Proof of Proposition 3.5*

Assume buyers make positive profits with high-type sellers in equilibrium. In this case, market utilities must satisfy $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$ (see Appendix A.4, first paragraph). The proof of Lemma 3.4 showed that in this case the overall feasibility constraint is binding; hence, we can use (13) to substitute for $\underline{x}$ and $\overline{x}$, and write $P^{\text{aux}'}$ solely in terms of the variables $\underline{\lambda}$, $\overline{\lambda}$, as already done in the proof of Lemma 3.4. At a solution of $P^{\text{aux}'}$, the values of $\underline{\lambda}$, $\overline{\lambda}$ must then maximize $\hat{\pi}(\underline{\lambda}, \overline{\lambda})$, specified in (20), subject to the incentive constraints, as rewritten in (21) and (22).

(a) Suppose first that the incentive constraints are not binding. The first-order conditions for an interior solution with respect to $\underline{\lambda}$ and $\overline{\lambda}$ are

$$e^{-\underline{\lambda}}(\underline{v}-\underline{c}) - e^{-\underline{\lambda}}\big(1-e^{-\overline{\lambda}}\big)(\overline{v}-\overline{c}) - \underline{U} = 0$$
$$e^{-\underline{\lambda}-\overline{\lambda}}(\overline{v}-\overline{c}) - \overline{U} = 0.$$

The solution of this system of equations is equal to $\underline{\lambda}^p$, $\overline{\lambda}^p$ if and only if market utilities $\underline{U}$ and $\overline{U}$ are given by

$$\underline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c}) + e^{-\underline{\lambda}^p}\big[(\underline{v}-\underline{c}) - (\overline{v}-\overline{c})\big] \tag{24}$$
$$\overline{U} = e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c}). \tag{25}$$

Consider then the buyers' payoff with high-type sellers:

$$\overline{x}(\overline{v}-\overline{c}) - \overline{U} = e^{-\overline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\big(1-e^{-\overline{\lambda}^p} - \overline{\lambda}^p e^{-\overline{\lambda}^p}\big)(\overline{v}-\overline{c}).$$

We have $1-e^{-\overline{\lambda}^p} > \overline{\lambda}^p e^{-\overline{\lambda}^p}$ because $1-e^{-\overline{\lambda}^p}$ equals the probability of at least one arrival when the queue length is $\overline{\lambda}^p$, and this strictly exceeds the probability of exactly one arrival, $\overline{\lambda}^p e^{-\overline{\lambda}^p}$. The buyers' payoff with high-type sellers at the equilibrium mechanism is therefore strictly positive, which implies that market utilities as given in (24) and (25) satisfy the assumed condition $\overline{U} < (\overline{v}-\overline{c})/(\overline{v}-\underline{c})\underline{U}$.

Finally, we need to check under which conditions incentive compatibility constraints (21) and (22) hold at $\underline{\lambda}^p$, $\overline{\lambda}^p$. Substituting the values of $\underline{U}$ and $\overline{U}$ as in (24) and (25) into these constraints, we obtain the inequalities

$$\frac{1}{\overline{\lambda}^p}\big(1-e^{-\overline{\lambda}^p}\big) \le 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}} \tag{26}$$
$$\frac{1-e^{-\underline{\lambda}^p}}{\underline{\lambda}^p e^{-\underline{\lambda}^p}} \ge 1 - \frac{\overline{v}-\underline{v}}{\overline{c}-\underline{c}}. \tag{27}$$

Since $1-e^{-\underline{\lambda}^p} > \underline{\lambda}^p e^{-\underline{\lambda}^p}$, as we just argued, inequality (27) is always satisfied. The described equilibrium thus exists if and only if (26) is satisfied.

(b) Suppose next at least one of the incentive constraints (21) and (22) is binding. Letting γ_l and γ_h denote the respective Lagrange multipliers of these constraints, the population parameters $\underline{\lambda}^p$, $\overline{\lambda}^p$ are optimal if they solve the first-order conditions

$$e^{-\underline{\lambda}^p}(\underline{v}-\underline{c}) - e^{-\underline{\lambda}^p}\big(1-e^{-\overline{\lambda}^p}\big)(\overline{v}-\overline{c}) - \underline{U} + \gamma_l e^{-\underline{\lambda}^p}\big(1-e^{-\overline{\lambda}^p}\big) - \gamma_h\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}^p}\right) = 0$$
$$e^{-\underline{\lambda}^p-\overline{\lambda}^p}(\overline{v}-\overline{c}) - \overline{U} + \gamma_l\left(\frac{\underline{U}-\overline{U}}{\overline{c}-\underline{c}} - e^{-\underline{\lambda}^p-\overline{\lambda}^p}\right) = 0.$$

Consider first the possibility that the high-type incentive constraint (22) is binding, i.e., $\gamma_h > 0$, $\gamma_l = 0$. In this case, $\overline{U}$ has the same value as in case (a), given by (25), and $(1 - e^{-\underline{\lambda}^p})/\underline{\lambda}^p = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. Since $e^{-\underline{\lambda}^p} < (1 - e^{-\underline{\lambda}^p})/\underline{\lambda}^p = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$, the term multiplying γ_h is positive. For γ_h to be strictly positive, the following condition must then hold:

$$e^{-\underline{\lambda}^p}(\underline{v} - \underline{c}) - e^{-\underline{\lambda}^p}\big(1 - e^{-\overline{\lambda}^p}\big)(\overline{v} - \overline{c}) - \underline{U} > 0.$$

With $\overline{U}$ determined by (25) and $\underline{U}$ such that $(1 - e^{-\underline{\lambda}^p})/\underline{\lambda}^p = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$, this inequality can be rewritten as

$$\frac{1 - e^{-\underline{\lambda}^p}}{\underline{\lambda}^p e^{-\underline{\lambda}^p}} < 1 - \frac{\overline{v} - \underline{v}}{\overline{c} - \underline{c}}.$$

This inequality is the complement of (27). Following the argument above, it is always violated and, hence, we must have $\gamma_h = 0$.

Consider next the case where the low-type incentive constraint (21) binds: $e^{-\underline{\lambda}}(1 - e^{-\overline{\lambda}})/\overline{\lambda} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$. As shown in the proof of part (i)(b) of Lemma 3.4, the solution of the buyer's auxiliary problem is obtained from the first-order condition $\tilde{\pi}'(\overline{\lambda}) = 0$, with $\tilde{\pi}(\overline{\lambda})$ as in (23). Substituting the population parameter $\overline{\lambda}^p$ into this equation, we get the condition

$$(\overline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} - e^{\overline{\lambda}^p}\frac{e^{\overline{\lambda}^p} - \overline{\lambda}^p - 1}{\big(e^{\overline{\lambda}^p} - 1\big)^2}(\underline{v} - \underline{c})\frac{\underline{U} - \overline{U}}{\overline{c} - \underline{c}} + \frac{e^{\overline{\lambda}^p} - \overline{\lambda}^p e^{\overline{\lambda}^p} - 1}{\overline{\lambda}^p\big(e^{\overline{\lambda}^p} - 1\big)}\underline{U} = 0.$$

Solving this equation, together with $e^{-\underline{\lambda}}(1 - e^{-\overline{\lambda}})/\overline{\lambda} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$, for $\underline{U}$ and $\overline{U}$ yields

$$\underline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\big(1 - e^{-\overline{\lambda}^p}\big)\frac{1 - e^{-\overline{\lambda}^p}}{1 - \frac{1}{\overline{\lambda}^p}\big(1 - e^{-\overline{\lambda}^p}\big)}(\overline{v} - \underline{v})$$

$$\overline{U} = e^{-\underline{\lambda}^p - \overline{\lambda}^p}(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}\frac{1}{\overline{\lambda}^p}\big(1 - e^{-\overline{\lambda}^p}\big)\left[\frac{1 - e^{-\overline{\lambda}^p}}{1 - \frac{1}{\overline{\lambda}^p}\big(1 - e^{-\overline{\lambda}^p}\big)}(\overline{v} - \underline{v}) - (\overline{c} - \underline{c})\right].$$

It is immediate to verify that these values satisfy the condition $\overline{U} < (\overline{v} - \overline{c})/(\overline{v} - \underline{c})\underline{U}$ if and only if $(1 - e^{-\overline{\lambda}^p})/\overline{\lambda}^p < (\underline{v} - \underline{c})/(\overline{v} - \underline{c})$. This condition implies that at the maximal incentive feasible trading probability for high-type sellers, buyers make strictly positive profits with high-type sellers. Since the low-type incentive constraint is binding, this is indeed the case. □

A.6 *Proof of Proposition 3.6*

Assume buyers make zero profits with high-type sellers in equilibrium. We first show that market utilities must then satisfy $\overline{U} = (\overline{v} - \overline{c})/(\overline{v} - \underline{c})\underline{U}$. Suppose instead that there is an equilibrium with buyers making zero profits with high types and $\overline{U} < (\overline{v} - \overline{c})/(\overline{v} - \underline{c})\underline{U}$. In this case, $P^{\text{aux}'}$ must have a solution $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ with $\overline{\lambda} > 0$ and $\overline{x} < (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$.

The latter inequality implies that the low-type incentive constraint (7) is slack at the solution. There exists an alternative tuple $(\underline{x}, \overline{x}', \underline{\lambda}, \overline{\lambda}')$ with $\overline{x} < \overline{x}' \leq (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$, $\overline{\lambda}' > 0$ and $\underline{\lambda x} + \overline{\lambda}'\overline{x}'^{-\underline{\lambda}-\overline{\lambda}'}$. Since the trading probability and queue length of low-type sellers remain unchanged, this tuple satisfies all constraints of $P^{\text{aux}\prime}$ and generates the same profits with low types as $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$. Moreover, it generates strictly positive profits with high types. Hence, $(\underline{x}, \overline{x}, \underline{\lambda}, \overline{\lambda})$ cannot be a solution of $P^{\text{aux}\prime}$ and equilibrium market utilities must satisfy $\overline{U} = (\overline{v} - \overline{c})/(\overline{v} - \underline{c})\underline{U}$.

Given this condition on the market utilities, we have $\overline{x} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$ for all solutions with $\overline{\lambda} > 0$. Furthermore, $\underline{x}$ is determined by (13) and problem $P^{\text{aux}'}$ amounts to maximizing $(1 - e^{-\underline{\lambda}})(\underline{v} - \underline{c}) - \underline{\lambda U}$ over $\underline{\lambda}$ subject to the incentive constraint of high-type sellers, as rewritten in (22). This constraint is not binding, given the above value of $\bar{x}$ and the property $\underline{x} > \overline{x}$ we established. With the high-type incentive constraint (17) not binding, the optimal value of $\underline{\lambda}$ is uniquely characterized by the first-order condition

$$e^{-\underline{\lambda}}(\underline{v} - \underline{c}) - \underline{U} = 0.$$

Since all buyers find it optimal to attract the same queue length of low-type seller, consistency with the population parameters requires that this solution is given by $\underline{\lambda}^p$, which requires

$$\underline{U} = e^{-\underline{\lambda}^p}(\underline{v} - \underline{c}). \tag{28}$$

The value of $\overline{U}$ in the candidate equilibrium is then pinned down by the assumed condition $\overline{U} = (\overline{v} - \overline{c})/(\overline{v} - \underline{c})\underline{U}$ and is, therefore, given by

$$\overline{U} = e^{-\underline{\lambda}^p} \frac{(\underline{v} - \underline{c})(\overline{v} - \overline{c})}{\overline{v} - \underline{c}}. \tag{29}$$

With $\overline{x} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$, $P^{\text{aux}'}$ is solved by all values of $\overline{\lambda}$ that satisfy the overall feasibility constraint (19). Substituting $\overline{x} = (\underline{U} - \overline{U})/(\overline{c} - \underline{c})$ evaluated at the market utilities in (28) and (29), and substituting $\underline{x}$ as determined by (13) and $\underline{\lambda}^p$, this constraint reduces to

$$\frac{1}{\overline{\lambda}}\left(1 - e^{-\overline{\lambda}}\right) \geq \frac{\underline{v} - \underline{c}}{\overline{v} - \underline{c}}. \tag{30}$$

An equilibrium is then given by any distribution of ratios between high-type sellers and buyers G with support Λ such that $\int_\Lambda \overline{\lambda}\, dG(\overline{\lambda}) = \overline{\lambda}^p$ and feasibility constraint (30) is satisfied for all values of $\overline{\lambda}$ in Λ. Since $(1 - e^{-\overline{\lambda}})/\overline{\lambda}$ is decreasing in $\overline{\lambda}$ and since $\max \Lambda \geq \overline{\lambda}^p$, such distribution exists if and only if (30) is satisfied at $\overline{\lambda} = \overline{\lambda}^p$, i.e., at the degenerate distribution localized at $\overline{\lambda}^p$. When (30) holds as an equality, this is the only distribution; otherwise there is a continuum of distributions. □

A.7 *Proof of Proposition 4.1*

The statement that the directed search equilibrium allocation maximizes total surplus if $(1 - e^{-\overline{\lambda}^p})/\overline{\lambda}^p \leq (\underline{v} - \underline{c})/(\overline{v} - \underline{c})$ and $\underline{v} - \underline{c} \geq \overline{v} - \overline{c}$ follows from the argument in the text together with the property that under urn–ball matching, having buyers and sellers trade in a single submarket maximizes the number of meetings (formally shown in the Supplemental Material). We now demonstrate that both conditions are necessary for total surplus to be maximized and that, if they are not satisfied, the allocation is constrained inefficient whenever μ is sufficiently large.

- Consider first the case $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p > (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$. Proposition 3.6 shows that under this condition, at a directed search equilibrium, the overall feasibility constraint (11) is slack. Consider an increase in the trading probability of the high-type seller $\Delta\overline{x}$, small enough so that (11) is not violated. Modify then the expected transfer to the high-type seller so that his utility is kept constant:

$$\Delta\overline{t} = \Delta\overline{x}\overline{c}.$$

The trading probability of the low-type seller is kept unchanged and the expected transfer to the low-type seller is adjusted to ensure that his incentive compatibility constraint is satisfied:

$$\Delta\underline{t} - \underbrace{\Delta\underline{x}\underline{c}}_{=0} = \Delta\overline{t} - \Delta\overline{x}\underline{c} \quad \Leftrightarrow \quad \Delta\underline{t} = (\overline{c}-\underline{c})\Delta\overline{x}.$$

These changes make the high-type sellers indifferent, strictly improve the low-type sellers (since $\Delta\underline{t} > 0$), and increase the total surplus that is generated. They also make buyers weakly better off and thus constitute a Pareto improvement if

$$\overline{\lambda}^p[\Delta\overline{x}\overline{v} - \Delta\overline{t}] + \underline{\lambda}^p[\Delta\underline{x}\underline{v} - \Delta\underline{t}] \geq 0, \quad \Leftrightarrow \quad \frac{\underline{\lambda}^p}{1-\mu}\big[\mu(\overline{v}-\underline{c}) - (\overline{c}-\underline{c})\big]\Delta\overline{x} \geq 0,$$

which is satisfied whenever $\mu \geq (\overline{c}-\underline{c})/(\overline{v}-\underline{c}) \in (0,1)$. □

- Next, consider the case $(1-e^{-\overline{\lambda}^p})/\overline{\lambda}^p \leq (\underline{v}-\underline{c})/(\overline{v}-\underline{c})$ together with $\underline{v}-\underline{c} < \overline{v}-\overline{c}$. Under this specification, the overall feasibility constraint (11) is binding (see Propositions 3.5 and 3.6). Consider an increase in the trading probability of the high-type seller by $\Delta\overline{x}$, while adjusting the trading probability of the low-type seller so that the feasibility constraint is still satisfied as equality:

$$\overline{\lambda}^p\Delta\overline{x} + \underline{\lambda}^p\Delta, \underline{x} = 0 \quad \Leftrightarrow \quad \Delta\underline{x} = -\frac{\mu}{1-\mu}\Delta\overline{x}.$$

Let us again modify the expected transfer to the high-type seller so that his utility remains unchanged,

$$\Delta\overline{t} = \Delta\overline{x}\overline{c},$$

and the expected transfer to the low-type seller so that his incentive compatibility constraint is satisfied with equality:

$$\Delta\underline{t} - \Delta\underline{x}\underline{c} = \Delta\overline{t} - \Delta\overline{x}\underline{c}.$$

Substituting the previous values of $\Delta\underline{x}$ and $\Delta\bar{t}$ into the above equation yields

$$\Delta\underline{t} = \frac{1}{1-\mu}\big((1-\mu)\bar{c} - \underline{c}\big)\Delta\bar{x}.$$

As before, these changes make high-type sellers indifferent and strictly improve the utility of low-type sellers:

$$\Delta\underline{t} - \Delta\underline{x}\underline{c} = \left[\frac{1}{1-\mu}\big((1-\mu)\bar{c} - \underline{c}\big) + \frac{\mu}{1-\mu}\underline{c}\right]\Delta\bar{x} = \bar{c} - \underline{c} > 0.$$

They also increase total surplus (while satisfying incentive compatibility and the feasibility constraints imposed by the matching technology) because trades of the good with the lower gains are substituted by trades of the good with the higher gains. Finally, they make buyers weakly better off and, therefore, constitute a Pareto improvement if

$$\begin{aligned}
&\bar{\lambda}^p[\Delta\bar{x}\bar{v} - \Delta\bar{t}] + \underline{\lambda}^p[\Delta\underline{x}\underline{v} - \Delta\underline{t}] \geq 0 \\
\Leftrightarrow \quad &\frac{\underline{\lambda}^p}{1-\mu}\big[\mu(\bar{v} - \underline{v}) - (\bar{c} - \underline{c})\big]\Delta\bar{x} \geq 0.
\end{aligned}$$

The above inequality is satisfied whenever $\mu \geq (\bar{c} - \underline{c})/(\bar{v} - \underline{v}) \in (0, 1)$.

A.8 *Proof of Proposition 4.3*

Let W^{GM} denote the total surplus in the equilibrium under general mechanisms and W^{BC} the total surplus in the equilibrium under bilateral contracts. We are interested in the limiting case of $\mu s \to +\infty$, while b and $(1-\mu)s$ are kept finite, implying that $\bar{\lambda}^p$ tends to $+\infty$ and $\underline{\lambda}^p$ is finite.

Consider first the case of general mechanisms. Given $\lim_{\bar{\lambda}^p \to +\infty}(1 - e^{-\bar{\lambda}^p})/\bar{\lambda}^p = 0$, the condition $(1-e^{-\bar{\lambda}^p})/\bar{\lambda}^p < (\underline{v}-\underline{c})/(\bar{v}-\underline{c})$ is always satisfied, meaning that the limiting case falls into the parameter region of Proposition 3.5. The limit of total surplus is thus given by

$$\begin{aligned}
\lim_{\bar{\lambda}^p \to +\infty} W^{\mathrm{GM}} &= \lim_{\bar{\lambda}^p \to +\infty} b\big[\big(1 - e^{-\underline{\lambda}^p}\big)(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}\big(1 - e^{-\bar{\lambda}^p}\big)(\bar{v} - \bar{c})\big], \\
&= b\big[\big(1 - e^{-\underline{\lambda}^p}\big)(\underline{v} - \underline{c}) + e^{-\underline{\lambda}^p}(\bar{v} - \bar{c})\big].
\end{aligned}$$

Consider next the case of bilateral contracts. We can first show that as $\bar{\lambda}^p \to +\infty$, the equilibrium fraction of buyers going to the high-quality market, γ, tends to 1. As shown in the proof of Proposition 4.2 in the Supplemental Material, a buyer's profit in the low- and high-quality market, respectively, is given by

$$\left(1 - e^{-\frac{\underline{\lambda}^p}{1-\gamma}} - \frac{\underline{\lambda}^p}{1-\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}\right)(\underline{v} - \underline{c})$$

$$\big(1 - e^{-\frac{\bar{\lambda}^p}{1-\gamma}}\big)(\bar{v} - \underline{c}) - \frac{\bar{\lambda}^p}{\gamma}e^{-\frac{\underline{\lambda}^p}{1-\gamma}}(\underline{v} - \underline{c}).$$

Suppose γ does not tend to 1. Then

$$\lim_{\overline{\lambda}^p\to+\infty}\left(\left(1-e^{-\frac{\overline{\lambda}^p}{1-\gamma}}\right)(\overline{v}-\underline{c})-\frac{\overline{\lambda}^p}{\gamma}e^{-\frac{\lambda^p}{1-\gamma}}(\underline{v}-\underline{c})\right)=-\infty,$$

implying that the indifference condition for buyers cannot be satisfied. Instead we need γ to be a function of $\overline{\lambda}^p$ such that

$$\lim_{\overline{\lambda}^p\to+\infty}\left(\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}e^{-\frac{\lambda^p}{1-\gamma(\overline{\lambda}^p)}}\right)=l\in\mathbb{R},$$

and we need l such that buyers are indifferent between both markets. Since $\lim_{\overline{\lambda}^p\to+\infty}\gamma(\overline{\lambda}^p)=1$, the limit of total surplus in the equilibrium with bilateral contracts is then given by

$$\begin{aligned}\lim_{\overline{\lambda}^p\to+\infty}W^{\mathrm{BC}}&=\lim_{\overline{\lambda}^p\to+\infty}b\left[\left(1-\gamma(\overline{\lambda}^p)\right)\left(1-e^{-\frac{\lambda^p}{1-\gamma(\overline{\lambda}^p)}}\right)(\underline{v}-\underline{c})+\gamma(\overline{\lambda}^p)\left(1-e^{-\frac{\overline{\lambda}^p}{\gamma(\overline{\lambda}^p)}}\right)(\overline{v}-\overline{c})\right]\\&=b(\overline{v}-\overline{c})\\&>\lim_{\overline{\lambda}^p\to+\infty}W^{\mathrm{GM}}.\end{aligned}$$

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