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Optimal structure and dissolution of partnerships

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For a partnership model with general type distributions and interdependent values, we derive the optimal dissolution mechanisms that, for arbitrary initial ownership, maximize any convex combination of revenue and social surplus. The solution involves ironing around typically interior worst-off types, which are endogenously determined. The optimal ownership structures are such that, with identical distributions, equal shares are always optimal. With nonidentical distributions, the optimal shares are typically asymmetric, the identity of the agents with large shares may change with the importance of revenue generation, and even fully concentrated initial ownership and assigning zero shares to the strongest agents can be optimal.

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JEL classification. D23, D61, D82.

1. Introduction

The Coase theorem provides the fundamental insight that the connection between the efficiency of the final allocation and the initial ownership structure depends on the ease with which property rights can be reallocated. The final allocation will be efficient irrespective of initial ownership if transaction costs are negligible and property rights are

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well defined. By now, there is, however, ample evidence that initial misallocations are not always easily and quickly mended through subsequent transactions, indicating in the light of the Coase theorem that transaction costs can be substantive.[1] For example, business deadlocks are notoriously costly. Likewise, land reallocation has proved difficult and time-consuming even in countries like the United States with a well functioning legal system and well defined property rights. It poses major challenges for countries with less well defined property rights such as China.[2] Similarly, the reallocation of spectrum licenses whose initial allocation was deemed inefficient has been slow and costly. This brings to the forefront the question of what, in the presence of transaction costs, are optimal ownership structures.

Partnership dissolution models, initiated by Cramton et al. (1987), provide a flexible framework to analyze this question when one source of transaction costs is the private information held by economic agents and when there may be additional costs associated with reallocating ownership shares such as legal costs, rent extraction by a designer, or other resource costs. Previous literature has analyzed the conditions on the initial allocation of property rights under which ex post efficient reallocation is (im)possible without running a budget deficit. With private values, ex post efficient reallocation is impossible with an extreme ownership structure that concentrates all property rights at a single agent, and ex post efficient reallocation is possible with a sufficiently non-extreme ownership structure. Moreover, when types are identically distributed, symmetric ownership is the only ownership structure that maximizes the revenue that can be generated under ex post efficiency, making it easiest to avoid a deficit. When the agents' type distributions are ranked according to first-order stochastic dominance, the revenue-maximizing ownership structure under ex post efficiency assigns larger shares to stronger agents in terms of stochastic dominance, but also strictly positive shares to the weakest agents.[3]

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[1]For example, Bleakley and Ferrie (2014) show that initial land parcel size after the opening of the frontier in Georgia predicts farm size essentially one-for-one for 50–80 years after land opening, with the effect of initial conditions attenuating gradually and disappearing only after 150 years. Milgrom (2004) makes a similar point in the context of the allocation of radio spectrum licenses, and Che and Cho (2011) describe vividly the inefficiencies associated with the Oklahoma land rush at the turn to the 20th century. Interestingly, Coase's own argument (Coase 1959) favoring the use of auctions to allocate spectrum licenses is consistent with the notion that subsequent market transactions will not easily fix initial misallocations, which is the central premise of the insightful theorem that bears his name (Coase 1960) and that continues to be influential in public policy debates. As a case in point, consider Fowlie and Perloff (2013), whose abstract states, "Standard economic theory predicts that if property rights to pollute are clearly established, equilibrium outcomes in an efficient emissions permit market will be independent of how the emissions permits are initially distributed."

[2]For legal and other costs associated with business deadlock, see, for example, Brooks et al. (2010), Landeo and Spier (2014b), and Landeo and Spier (2014a) and the accounts therein. Bleakley and Ferrie (2014) document the time-consuming nature of reallocating land property that was initially allocated using lotteries. Popular press reports tell of the challenges modern China faces in reallocating farm lots that are deemed inefficiently small; see, for example, https://www.ft.com/content/9d18ee2a-a1a7-11e6-86d5-4e36b35c3550. Milgrom (2004) provides an account of the slow reallocation of spectrum licenses.

[3]See Myerson and Satterthwaite (1983), Cramton et al. (1987), Che (2006), and Figueroa and Skreta (2012).

In this paper, we drop the restriction to ex post efficient reallocation. We study which ownership structures are optimal in partnership dissolution problems where the designer maximizes a convex combination of social surplus and revenue, and, therefore, optimally deviates from ex post efficient reallocation in line with the importance of revenue. If types are identically distributed, symmetric ownership is always optimal, independently of the revenue weight. However, asymmetric but sufficiently equal ownership is also optimal. Interestingly, the set of optimal ownership structures expands as the revenue weight increases, but never includes structures where some shares are zero. In contrast, if the agents' types are drawn from different distributions, optimality of extreme ownership structures is a robust and prevalent phenomenon. In particular, we show extreme ownership structures to be optimal when revenue is sufficiently important and asymmetries are sufficiently pronounced. Moreover, who optimally obtains the largest ownership share depends on how important generating revenue is. As mentioned, under ex post efficiency, the strongest agents in terms of first-order stochastic dominance receive the largest shares. As the importance of generating revenue increases, weaker agents may receive larger shares, even to the point that the strongest agents receive shares of zero.

We also show that these results are, by and large, robust to the introduction of interdependent values, where there may not exist any ownership structure that avoids running a deficit under ex post efficient reallocation (Fieseler et al. 2003).

Exclusively assigning ownership to the weakest agents occurs, for example, in startups that are set up with the goal of being bought up further down the track because the strong agent, that is, the ultimate buyer like Google or a pharmaceutical company, has no share in the partnership at the outset. It also resonates with the ownership structure chosen by the U.S. government in the lead up to the "incentive auction," where the arguably stronger agents such as telecom companies received shares of zero and where revenue generation was an explicit goal of the government.[4,5]

Our optimal ownership results when the agents draw their types from identical distributions imply, for example, that for the case for which Fieseler et al. (2003) establish the impossibility of ex post efficiency with interdependent values, symmetric ownership is an optimal ownership structure under second best. More fundamentally, the general optimality of symmetric ownership with identically distributed types means that, given ex ante symmetric agents, symmetric ownership is robust in the sense that it remains

[4]For examples, see Milgrom and Segal (2015), the *Middle Class Tax Relief and Job Creation Act of 2012*, which specifies in section 6402G(iii)(I) that "$1,750,000,000 of the proceeds from the incentive auction of broadcast television spectrum [...] shall be deposited in the TV Broadcaster Relocation Fund," and Milgrom (2017). Further evidence of the importance and explicit emphasis of revenue generation for the "incentive auction" is provided in footnote 30 in Loertscher et al. (2015). Both commissioners of the Federal Communications Commission and lawmakers emphasized the importance of generating revenue, with one commissioner saying, "Yes, we should focus on maximizing revenue" and one congressman saying that the law required "maximizing the proceeds from the auction."

[5]Of course, a more basic and perhaps simpler motivation for assigning full ownership to current license holders may have been to avoid lengthy and costly battles in court about the somewhat grey area regarding what a licensee was entitled to by holding a license. But there is comfort in knowing that the two goals—avoiding time-consuming legal battles and generating revenue—can be almost perfectly aligned.

optimal even if, at the stage at which the ownership structure is determined, there is uncertainty about the importance of rent extraction or interdependent values at the dissolution stage. Consequently, our analysis also provides an explanation for the widespread organizational structure of fifty–fifty, and more generally, symmetric partnerships.

Of course, to determine the optimal ownership structure, we first need to derive the optimal dissolution (or reallocation) mechanisms for the general partnership model that we study. The initial property rights define the agents' outside option in the dissolution process and, hence, represent the individual rationality constraints in the problem of optimally designing this process. The broad intuition is that an ownership structure is optimal if it relaxes individual rationality constraints as much as possible. With the nature of optimal ownership thus being tightly connected to the properties of optimal dissolution mechanisms, we develop the intuition for the optimal ownership structures after discussing these mechanisms.

Solving for the optimal dissolution mechanisms that maximize any weighted sum of designer's revenue and social surplus is in itself economically relevant and, regardless, technically challenging. At the heart of the challenge lies the fact that shared initial ownership creates countervailing incentives (Lewis and Sappington 1989): A high type of an agent is more likely to buy additional shares and has an incentive to underreport, while a low type is more likely to sell his share and has an incentive to overreport. Types for whom the expected after-dissolution share equals the initial share have no incentive to misreport. Because they thus enjoy no information rent, they are worst off among all types. As they depend on the allocation rule, the worst-off types—which are the types for which individual rationality binds—are endogenous to the design problem.[6]

We overcome the problem of simultaneously determining the optimal allocation rule and the endogenous worst-off types by noticing and exploiting a saddle point property of the problem. Given a critical type for each agent, we define the virtual surplus as the value of the allocation in terms of virtual types. An agent's virtual type equals his virtual cost for types below the critical type and his virtual valuation for types above it, reflecting binding upward and downward incentive constraints. We show that there is an essentially unique combination of critical types and an allocation rule such that, first, the allocation rule maximizes the virtual surplus given the critical types and, second, the critical types are worst-off types under the allocation rule. This is the allocation rule of all optimal dissolution mechanisms. Because virtual costs always exceed virtual valuations, the optimal dissolution mechanisms allocate based on ironed virtual type functions that are flat for types around the critical type. These are all worst-off types. Depending on the ownership structure, ties in terms of ironed virtual types may happen with positive probability. In this case, an appropriately specified tie-breaking rule is an essential ingredient in the optimal allocation rule.

We are now in a position to develop the intuition behind the results on optimal ownership structures. Consider a situation in which the worst-off types of, say, agent

[6]The initial shares in our model represent type-dependent outside options. For the case of a single agent, principal–agent problems with this feature are well understood (see Jullien 2000, for the most general treatment). Much less is known for the case of multiple agents, making our analysis of such a setting a relevant contribution to that literature also.

1, have a higher expected valuation than those of another agent, say, agent 2. Intuitively, a marginal transfer of ownership from the first to the second agent would then relax the first agent's individual rationality constraint by more than it would tighten the second agent's and thus allow for more rent extraction. Under an optimal ownership structure, such beneficial transfers should not be possible. Indeed, we formally show that the optimal ownership structures are fully characterized by the condition that the expected valuation of the critical worst-off types is the same for all agents with nonzero shares and is higher for agents with zero shares. Intuitively, this condition specifically considers the critical among the worst-off types to also account for the marginal changes in the optimal allocation rule in response to a marginal transfer of ownership.

When types are identically distributed, this condition immediately yields optimality of symmetric ownership because this ensures that optimal dissolution mechanisms treat all agents the same, yielding equal critical worst-off types. As long as the designer puts positive weight on revenue, ownership structures that are asymmetric but sufficiently equal require only adjusting the tie-breaking rule of the dissolution mechanism, which does not change the critical worst-off types. Consequently, such ownership structures are also optimal. Moreover, as rent extraction becomes more important and distortions in the allocation hence grow, tie-breaking plays a greater role, which explains why the set of optimal ownership structures expands. However, ownership structures that assign zero shares to some agents can never be optimal with identical distributions since this necessarily results in the lowest types of those agents being worst off, whereas worst-off types of the agents with positive shares are higher.

When types are drawn from different distributions, the expected valuation of each agent's critical worst-off type is typically not equal under symmetric ownership. In this case, unequal initial shares are needed to account for the heterogeneity of agents. As mentioned, even extreme ownership that concentrates all property rights at one agent is optimal under sufficient heterogeneity in type distributions.

To build intuition for this result, consider a bilateral partnership with private values, implying that valuations are equal to types. Under an extreme ownership structure, the two agents are seller and buyer (as in Myerson and Satterthwaite 1983). The optimal allocation rule compares the seller's virtual cost with the buyer's virtual valuation, resulting in the highest (lowest) types of the seller (buyer) being excluded from trade and being worst off. This ownership structure is optimal if the critical worst-off type of the seller is weakly below that of the buyer. Because the critical type of the seller (buyer) is in the interior of the set of worst-off types at the top (bottom) of the support, this condition can indeed hold. Intuitively, it requires that sufficiently many types are excluded and, therefore, are worst off, which happens if high (low) types are sufficiently unlikely for the seller (buyer) and the distortions caused by the revenue weight are sufficiently pronounced. This also explains why the agent with the lowest expected valuation is the sole owner if extreme ownership is optimal. Note that the same forces are at work under interdependent values and when considering more than two agents, which can be thought of as adding replicas of the buyer.

Now suppose we let the weight on revenue go to zero in the buyer–seller example with private values just discussed. As the distortions disappear, the optimal allocation rule becomes ex post efficient, and worst-off types become unique and are at the top of the support for the seller and at the bottom for the buyer. Of course, this makes it impossible that extreme ownership is optimal. Moreover, the expected valuation of the highest type of an agent with a share of one cannot be below that of the lowest possible type of agents with shares of zero even with interdependent values as long as an agent's own type affects his valuation more than the other agents' types, rendering extreme ownership always suboptimal under ex post efficiency.

With a few notable exceptions, which we discuss below, the literature on partnership dissolution has focused on ex post efficient allocation rules and on the question of the conditions on distributions, valuations, and property rights under which ex post efficient reallocation is possible subject to incentive compatibility and individual rationality without running a deficit. For the case in which all agents draw their types from the same distribution, Cramton et al. (1987) and Fieseler et al. (2003) analyze, respectively, models with private values and with interdependent values. Cramton et al. show that with equal ownership, ex post efficiency is always possible. In contrast, Fieseler et al. establish that if interdependence is positive, ex post efficient reallocation may be impossible for any initial ownership structure. Their analysis gives thus additional salience to the question of what are optimal dissolution mechanisms, which is part of our study. Subsequent contributions with interdependent values are made by Kittsteiner (2003), Jehiel and Pauzner (2006), and Chien (2007).

Considering symmetric bilateral partnerships, Kittsteiner (2003) performs a first attack on the problem of having to avoid deficits when valuations are positively interdependent. He shows that adding veto rights restores individual rationality of double auctions, but noticed at the same time that the resulting allocation is suboptimal by providing a superior mechanism for an example with uniformly distributed types, which he (as we show) correctly conjectures to be the second-best mechanism. One contribution of our paper is that it generally derives the optimal mechanisms, thereby providing a benchmark to evaluate specific mechanisms that are or have been proposed to be used in practice, such as Kittsteiner's and those analyzed by Brooks et al. (2010) and Landeo and Spier (2014b). Moreover, for the important special case of identical distributions and equal shares, which was, for example, the focus of Kittsteiner (2003), we introduce a simple two-stage game that implements the optimal dissolution mechanism for a bilateral partnership. In this game, the designer first asks the agents to report BUY, SELL, or HOLD. Trade occurs at posted prices that are contingent on the agents' reports unless both agents report BUY or both report SELL.[7] If both report BUY (SELL), a standard (reverse) auction ensues to allocate the good efficiently.

Focusing on private values, Che (2006) and Figueroa and Skreta (2012), with the latter building on the results of Schweizer (2006), extend the analysis to settings where each agent's type is drawn from a different distribution. When distributions can be ranked by stochastic dominance, Che and Figueroa and Skreta show that the ownership structure that maximizes revenue, given an ex post efficient allocation rule, assigns larger shares

[7] To be precise, if both agents report HOLD, there is no trade.

to stronger agents. Segal and Whinston (2011) provide, among other things, a generalization of the results of Schweizer (2006) to interdependent values. However, their conditions for possibility of ex post efficiency with interdependent values preclude those under which Fieseler et al. (2003) establish impossibility.

To the best of our knowledge, the following papers are the only ones that analyze objectives other than ex post efficiency for partnership models with multilateral private information. Segal and Whinston (2016) study a second-best bargaining problem under a liability rule with two agents and private values. Our work complements theirs. While Segal and Whinston study a richer class of property rights, called liability rules, their analysis in this part of the paper is confined to two agents, private values, and the second-best mechanism, taking as given the initial allocation of property rights. In contrast, we first characterize the efficient frontier for an arbitrary number of agents, allowing for interdependent values and asymmetric distributions, and then derive the optimal ownership structure for any such partnership. Mylovanov and Tröger (2014) solve the informed principal problem one obtains when maximizing one agent's payoff in a bilateral partnership with private values. Our analysis differs from theirs insofar as our designer is not a member of the partnership and his objective attaches the same welfare weight to all agents. Other precursors to our paper are Lu and Robert (2001) and the unpublished paper by Chien (2007). Lu and Robert study the same objective function as we do in the derivation of optimal dissolution mechanisms, but they confine attention to private values and identical type distributions, and they do not address which allocation of initial shares is optimal. Allowing for interdependent values, Chien solves for the second-best mechanism under given initial ownership for the special case of two agents. Our approach is both simpler and more general. Moreover, unless types are identically distributed, the second-best mechanism differs from what Chien's analysis suggests.

The remainder of this paper is organized as follows. Section 2 introduces the setup as well as basic mechanism design results. Section 3 derives and characterizes the optimal dissolution mechanisms and introduces a simple implementation game. Section 4 determines the optimal ownership structures, that is, the initial property rights that maximize the designer's objective. As an extension, Section 5 studies the ownership structures that the partners would choose. Section 6 concludes. The Appendix contains omitted proofs.

2. Model

2.1 *Setup*

There is a set of n risk-neutral agents $\mathcal{N} := \{1, 2, \ldots, n\}$ who jointly own one object. Each agent $i \in \mathcal{N}$ owns share $r_i \in [0, 1]$ in the object, where $\sum_{i\in\mathcal{N}} r_i = 1$. Accordingly, the initial property rights are represented by a point $\mathbf{r} := (r_1, \ldots, r_n)$ in the $(n-1)$-dimensional standard simplex $\Delta^{n-1} := \{\mathbf{r} \in [0, 1]^n : \sum_{i=1}^{n} r_i = 1\}$.

Each agent i privately learns his type x_i, which is a realization of the continuous random variable X_i. Each X_i is independently distributed according to a twice continuously differentiable cumulative distribution function F_i with support $[0,1]$ and density f_i. Agent i's ex post valuation for the object is

$$v_i(\mathbf{x}) := x_i + \sum_{j \neq i} \eta(x_j),$$

where $\mathbf{x} := (x_1, \dots, x_n)$ and where η is a differentiable function with $\eta'(x_j) < 1$ for all x_j. Agent i's status quo utility from owning share r_i is $r_i v_i(\mathbf{x})$.

Dissolving the partnership results in a reallocation of initial property rights $\mathbf{r}$ and monetary transfers. By the revelation principle, it is without loss to focus on incentive compatible direct dissolution mechanisms. A direct dissolution mechanism $(\mathbf{s}, \mathbf{t})$ consists of an allocation rule $\mathbf{s}\colon [0,1]^n \to \Delta^{n-1}$ and a payment rule $\mathbf{t}\colon [0,1]^n \to \mathbb{R}^n$, where $\mathbf{s}(\mathbf{x}) = (s_1(\mathbf{x}), \dots, s_n(\mathbf{x}))$ and $\mathbf{t}(\mathbf{x}) = (t_1(\mathbf{x}), \dots, t_n(\mathbf{x}))$. The agents report their types $\mathbf{x}$ whereupon agent i receives share $s_i(\mathbf{x})$ and pays the amount $t_i(\mathbf{x})$, resulting in ex post payoff $v_i(\mathbf{x}) s_i(\mathbf{x}) - t_i(\mathbf{x})$.[8]

Define $S_i(x_i) := E[s_i(x_i, \mathbf{X}_{-i})]$ and $T_i(x_i) := E[t_i(x_i, \mathbf{X}_{-i})]$ to be the interim expected share and payment of agent i. Moreover, let

$$U_i(x_i) := E\big[v_i(x_i, \mathbf{X}_{-i})\big(s_i(x_i, \mathbf{X}_{-i}) - r_i\big)\big] - T_i(x_i)$$

denote i's interim expected net payoff from taking part in the dissolution. A direct dissolution mechanism is Bayesian incentive compatible if

$$U_i(x_i) \geq E\big[v_i(x_i, \mathbf{X}_{-i})\big(s_i(\tilde{x}_i, \mathbf{X}_{-i}) - r_i\big)\big] - T_i(\tilde{x}_i) \quad \forall x_i, \tilde{x}_i \in [0,1], i \in \mathcal{N}, \tag{IC}$$

and is interim individually rational if

$$U_i(x_i) \geq 0 \quad \forall x_i \in [0,1], i \in \mathcal{N}. \tag{IR}$$

The designer's objective is to maximize a weighted sum of the ex ante expected social surplus $E[\sum_i v_i(\mathbf{X}) s_i(\mathbf{X})]$, which is the value of the final allocation, and the ex ante expected revenue $E[\sum_i t_i(\mathbf{X})]$ subject to the incentive compatibility and individual rationality constraints. Suppose the designer puts weight $\alpha \in [0,1]$ on revenue[9] and let

$$W_\alpha(\mathbf{s}, \mathbf{t}) := (1-\alpha) \sum_{i \in \mathcal{N}} E\big[v_i(\mathbf{X}) s_i(\mathbf{X})\big] + \alpha \sum_{i \in \mathcal{N}} E\big[t_i(\mathbf{X})\big].$$

In Section 3, we take the initial property rights $\mathbf{r}$ as given and study *optimal dissolution mechanisms* that solve

$$\max_{\mathbf{s}, \mathbf{t}} W_\alpha(\mathbf{s}, \mathbf{t}) \quad \text{s.t. (IC) and (IR)}. \tag{1}$$

[8]Note that restricting attention to deterministic allocation rules is without loss of generality since payoffs are linear in the ex post shares. Agent i obtaining share $s_i = \sigma$ can equivalently be interpreted as i becoming the sole owner with probability σ and some other agent becoming the sole owner with probability $1 - \sigma$.

[9]Note that α can also be interpreted as a reduced-form parameter measuring the degree of competition the designer faces, ranging from perfect competition to monopoly.

Note that the initial shares $\mathbf{r}$ enter this problem solely through the constraint (IR). Optimal dissolution mechanisms are denoted by $(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$.

In Section 4, we then turn to analyzing *optimal ownership structures* that solve

$$\max_{\mathbf{r}} W_\alpha\big(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha\big) = \max_{\mathbf{r},\mathbf{s},\mathbf{t}} W_\alpha(\mathbf{s}, \mathbf{t}) \quad \text{s.t. (IC) and (IR).} \tag{2}$$

The set of all optimal ownership structures will be denoted by $R^*(\alpha)$.

2.2 *Incentive compatibility, worst-off types, and virtual surplus*

The standard characterization of Bayesian incentive compatibility applies to our environment (see, e.g., Myerson 1981): (IC) holds if and only if

$$S_i \text{ is nondecreasing} \tag{IC1}$$

$$U_i(x_i) = U_i(\hat{x}_i) + \int_{\hat{x}_i}^{x_i} \big(S_i(y) - r_i\big)\, dy \quad \forall x_i, \hat{x}_i \in [0, 1]. \tag{IC2}$$

For a given monotone allocation rule, payoff equivalence (IC2) pins down interim expected payoffs U_i and payments T_i up to a constant.

Consider a dissolution mechanism $(\mathbf{s}, \mathbf{t})$ that satisfies (IC1) and (IC2). Let the set of *worst-off types* of agent i be denoted by $\Omega_i(\mathbf{s}) := \arg\min_{x_i} U_i(x_i)$. By (IC2), U_i is differentiable almost everywhere and $U_i'(x_i) = S_i(x_i) - r_i$ wherever U_i is differentiable. The monotonicity of S_i implies the following characterization of the set of worst-off types (see also Cramton et al. 1987, Lemma 2). If there is an x_i such that $S_i(x_i) = r_i$, then $\Omega_i(\mathbf{s})$ is a (possibly degenerate) interval and

$$\Omega_i(\mathbf{s}) = \big\{x_i : S_i(x_i) = r_i\big\}.$$

If $S_i(x_i) \neq r_i$ for all $x_i \in [0, 1]$, then $\Omega_i(\mathbf{s})$ is a singleton and

$$\Omega_i(\mathbf{s}) = \big\{x_i : S_i(z) < r_i \; \forall z < x_i \text{ and } S_i(z) > r_i \; \forall z > x_i\big\}.$$

Let $\Omega(\mathbf{s}) := \Omega_1(\mathbf{s}) \times \cdots \times \Omega_n(\mathbf{s})$.

In addition to identifying the set of worst-off types, (IC2) also allows us to eliminate $\mathbf{t}$ from the designer's objective and rewrite it as a function of the the interim payoff of an arbitrarily fixed critical type for each agent and the virtual surplus generated by $\mathbf{s}$ under these critical types. To do so, we first define, for each i, the α-weighted virtual cost and virtual valuation

$$\psi^S_{\alpha,i}(x_i) := x_i - \eta(x_i) + \alpha\frac{F_i(x_i)}{f_i(x_i)} \quad \text{and} \quad \psi^B_{\alpha,i}(x_i) := x_i - \eta(x_i) - \alpha\frac{1 - F_i(x_i)}{f_i(x_i)}.$$

The first part of $\psi^S_{\alpha,i}$ and $\psi^B_{\alpha,i}$, the term $x_i - \eta(x_i)$, represents the effect of i's type on the cost from reducing and gain from increasing, respectively, i's share.[10] The second part,

[10]The change in surplus from moving the object from j to i is $v_i(\mathbf{x}) - v_j(\mathbf{x}) = x_i - \eta(x_i) - x_j + \eta(x_j)$.

with the designer's revenue weight α, accounts for the information rent that has to be granted to i to prevent him from overstating and understating, respectively, his type.

For an exogenously fixed *critical type* $\hat{x}_i \in [0, 1]$, we define agent i's *virtual type* function given $\hat{x}_i$ as

$$\psi_{\alpha,i}(x_i, \hat{x}_i) := \begin{cases} \psi^S_{\alpha,i}(x_i) & \text{if } x_i < \hat{x}_i \\ \psi^B_{\alpha,i}(x_i) & \text{if } x_i > \hat{x}_i. \end{cases}$$

The *virtual surplus* under allocation rule $\mathbf{s}$ and exogenously fixed critical types $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_n)$ is then given by

$$\widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) := E\left[\sum_{i\in\mathcal{N}} \big(s_i(\mathbf{X}) - r_i\big)\psi_{\alpha,i}(X_i, \hat{x}_i)\right],$$

i.e., the expected gains from trade in terms of virtual types given $\hat{\mathbf{x}}$ when reallocating property rights from $\mathbf{r}$ to $\mathbf{s}$. Using standard techniques, we obtain the following lemma.

Lemma 1. *Suppose the dissolution mechanism* $(\mathbf{s}, \mathbf{t})$ *satisfies (IC1) and (IC2). Then*

$$W_\alpha(\mathbf{s}, \mathbf{t}) = \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) - \alpha \sum_{i\in\mathcal{N}} U_i(\hat{x}_i) + (1-\alpha)\sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})r_i\big] \quad \textit{for all } \hat{\mathbf{x}} \in [0,1]^n. \tag{3}$$

Moreover,

$$\Omega(\mathbf{s}) = \operatorname*{argmin}_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \tag{4}$$

Most proofs are provided in Appendix B.

According to (3), we can write the designer's objective for any exogenously fixed critical types $\hat{\mathbf{x}}$ as the virtual surplus given $\hat{\mathbf{x}}$ minus the α-weighted sum of the interim payoffs of the critical types plus $(1-\alpha)$ times the value of the initial allocation. Since for a given allocation rule $\mathbf{s}$, (3) is constant over all $\hat{\mathbf{x}}$, the critical types with the smallest interim payoffs must also be the critical types that minimize the virtual surplus, implying (4). Identifying the worst-off types as the critical types that minimize the virtual surplus proves useful below.

2.3 *Regularity and virtual type distributions*

We will throughout impose the regularity assumption that each agent's α-weighted virtual cost and valuation is strictly increasing, i.e.,

$$\frac{d}{dx_i}\psi^S_{\alpha,i}(x_i) > 0 \quad \text{and} \quad \frac{d}{dx_i}\psi^B_{\alpha,i}(x_i) > 0 \quad \text{for all } x_i \in [0,1] \text{ and } i \in \mathcal{N}. \tag{5}$$

This represents a joint assumption on α, η, and $F_1, \dots, F_n$. Note that the higher are α and $\eta'(\cdot)$, the more restrictive is (5) for F_i. Assumption (5) holds for all α and η if each f_i is log-concave.

For our analysis below, it is useful to define the cumulative distribution functions $G^S_{\alpha,i}$ and $G^B_{\alpha,i}$ of agent i's virtual cost $\psi^S_{\alpha,i}(X_i)$ and virtual valuation $\psi^B_{\alpha,i}(X_i)$: Under (5), we have

$$G^J_{\alpha,i}(y) := \begin{cases} 0 & \text{if } y < \psi^J_{\alpha,i}(0) \\ F_i\big(\big(\psi^J_{\alpha,i}\big)^{-1}(y)\big) & \text{if } y \in \big[\psi^J_{\alpha,i}(0), \psi^J_{\alpha,i}(1)\big] \\ 1 & \text{if } y > \psi^J_{\alpha,i}(1) \end{cases}$$

for $J \in \{S, B\}$ and $i \in \mathcal{N}$. Observe that for every i and y, $G^S_{\alpha,i}(y) \leq F_i(y) \leq G^B_{\alpha,i}(y)$.

3. Optimal dissolution mechanisms

3.1 *General partnerships*

In this section, we determine the solution to the designer's problem stated in (1). From Section 2.2, it follows that we can replace the constraints (IC) and (IR) with (IC1), (IC2), and $U_i(\omega_i) \geq 0$ for all i and $\omega_i \in \Omega_i(\mathbf{s})$. Define $\mathfrak{S} := \{\mathbf{s} : S_i \text{ is nondecreasing for each } i \in \mathcal{N}\}$. Consequently, (IC1) is equivalent to $\mathbf{s} \in \mathfrak{S}$.

Consider an allocation rule $\mathbf{s} \in \mathfrak{S}$ and some worst-off types $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in \Omega(\mathbf{s})$. Under (IC2), (3) in Lemma 1 implies that we can write the designer's objective as

$$W_\alpha(\mathbf{s}, \mathbf{t}) = \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) - \alpha \sum_{i\in\mathcal{N}} U_i(\omega_i) + (1-\alpha) \sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})r_i\big]. \tag{6}$$

Note that the individual rationality constraint $U_i(\omega_i) \geq 0$ is binding when choosing payments $\mathbf{t}$ that maximize (6) for a given $\mathbf{s}$. Using $U_i(\omega_i) = 0$, (IC2) implies that any optimal $\mathbf{t}$ has to be such that interim expected payments satisfy, for all i,

$$T_i(x_i) = E\big[v_i(x_i, \mathbf{X}_{-i})\big(s_i(x_i, \mathbf{X}_{-i}) - r_i\big)\big] - \int_{\omega_i}^{x_i} \big(S_i(y) - r_i\big)\, dy. \tag{7}$$

It remains to determine the optimal allocation rule. Since the second term in the objective (6) is zero under optimal payments and the third term is independent of the dissolution mechanism, we can restrict attention to maximizing $\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}) = \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$, where the equality follows from (4) in Lemma 1. Consequently, an optimal allocation rule $\mathbf{s}^{\mathbf{r}}_\alpha$ has to satisfy

$$\mathbf{s}^{\mathbf{r}}_\alpha \in \operatorname*{argmax}_{\mathbf{s}\in\mathfrak{S}} \min_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \tag{8}$$

Instead of directly solving the max-min problem (8), we look for a *saddle point* $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ of $\widetilde{W}_\alpha$ that satisfies

$$\mathbf{s}^* \in \operatorname*{argmax}_{\mathbf{s}\in\mathfrak{S}} \widetilde{W}_\alpha\big(\mathbf{s}, \boldsymbol{\omega}^*\big) \tag{9}$$

$$\boldsymbol{\omega}^* \in \operatorname*{argmin}_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha\big(\mathbf{s}^*, \hat{\mathbf{x}}\big). \tag{10}$$

For a saddle point, (9) requires that the allocation rule $\mathbf{s}^*$ maximizes the virtual surplus $\widetilde{W}_\alpha$ under given critical types $\boldsymbol{\omega}^*$, whereas (10) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{s}^*$, i.e., $\boldsymbol{\omega}^* \in \Omega(\mathbf{s}^*)$.

Note that if a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ exists, then $\mathbf{s}^{\mathbf{r}}_\alpha$ solves the problem in (8) if and only if $(\mathbf{s}^{\mathbf{r}}_\alpha, \boldsymbol{\omega}^*)$ is a saddle point.[11] In what follows, we show that a saddle point $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ exists and that $\mathbf{s}^*$ is essentially unique.[12] The characterization of optimal dissolution mechanisms we thereby obtain represents the main result of this section. We proceed by first determining the class of allocation rules that is consistent with (9). Then we argue that an essentially unique member of this class also satisfies (10).

Consider the optimization problem in (9). Pointwise maximization of

$$\widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*) = E\Big[\sum_{i\in\mathcal{N}}(s_i(\mathbf{X}) - r_i)\psi_{\alpha,i}(X_i, \omega_i^*)\Big]$$

would require allocating the object to the agent i with the highest virtual type $\psi_{\alpha,i}(x_i, \omega_i^*)$. Yet, since $\psi^S_{\alpha,i}(x_i) > \psi^B_{\alpha,i}(x_i)$ for all x_i, $\psi_{\alpha,i}(x_i, \omega_i^*)$ is not monotone at ω_i^*, resulting in the violation of the monotonicity constraint $\mathbf{s} \in \mathfrak{S}$. The solution to (9) therefore involves ironing (Myerson 1981): the object is allocated to an agent i with the highest *ironed virtual type*

$$\overline{\psi}_{\alpha,i}(x_i, z_i) := \begin{cases} \psi^S_{\alpha,i}(x_i) & \text{if } \psi^S_{\alpha,i}(x_i) < z_i \\ z_i & \text{if } \psi^B_{\alpha,i}(x_i) \le z_i \le \psi^S_{\alpha,i}(x_i) \\ \psi^B_{\alpha,i}(x_i) & \text{if } z_i < \psi^B_{\alpha,i}(x_i), \end{cases}$$

where the ironing parameter $z_i \in [\psi^B_{\alpha,i}(\omega_i^*), \psi^S_{\alpha,i}(\omega_i^*)]$ is the unique solution to

$$E\big[\psi_{\alpha,i}(X_i, \omega_i^*)\big] = E\big[\overline{\psi}_{\alpha,i}(X_i, z_i)\big]. \tag{11}$$

According to (11), there is a one-to-one relation between the critical type ω_i^* and the corresponding ironing parameter z_i, which can be expressed in closed form as follows. As is easily verified, $E[\psi_{\alpha,i}(X_i, \hat{x}_i)] = \alpha\hat{x}_i + (1-\alpha)E[X_i] - E[\eta(X_i)]$. Hence, (11) yields

$$\omega_i^* = \omega_{\alpha,i}(z_i) := \frac{1}{\alpha}E\big[\overline{\psi}_{\alpha,i}(X_i, z_i) - (1-\alpha)X_i + \eta(X_i)\big]. \tag{12}$$

Note that $\omega_{\alpha,i}(\cdot)$ is a continuous and strictly increasing function.

Figure 1 illustrates the ironed virtual type function. Agent i's ironed virtual type $\overline{\psi}_{\alpha,i}(x_i, z_i)$ is constant and equal to z_i for an interval of types that contains the critical type $\omega_{\alpha,i}(z_i)$, and it is strictly increasing otherwise. Any allocation rule consistent with (9) allocates based on ironed virtual types and, hence, features for each agent bunching around the critical type. Note that (9) does not pin down the allocation when several agents tie for the highest ironed virtual type. To handle this indeterminacy, we next introduce a convenient class of tie-breaking rules.

Let H denote the set of all $n!$ permutations $(h(1), h(2), \dots, h(n))$ of $(1, 2, \dots, n)$. We call each $h \in H$ a hierarchy among the agents in $\mathcal{N}$. A *hierarchical tie-breaking rule*

[11] Suppose $(\mathbf{s}^*, \boldsymbol{\omega}^*)$ satisfies (9) and (10). Then $\min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}^*, \hat{\mathbf{x}}) = \widetilde{W}_\alpha(\mathbf{s}^*, \boldsymbol{\omega}^*) \ge \widetilde{W}_\alpha(\mathbf{s}, \boldsymbol{\omega}^*) \ge \min_{\hat{\mathbf{x}}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}})$ for all $\mathbf{s} \in \mathfrak{S}$ and, hence, $\mathbf{s}^*$ solves the problem in (8). Conversely, the last two inequalities have to hold with equality for $\mathbf{s} = \mathbf{s}^{\mathbf{r}}_\alpha$ if $\mathbf{s}^{\mathbf{r}}_\alpha$ satisfies (8), implying that $(\mathbf{s}^{\mathbf{r}}_\alpha, \boldsymbol{\omega}^*)$ is a saddle point.

[12] The allocation rule $\mathbf{s}^*$ is unique up to the exact specification of a tie-breaking rule.

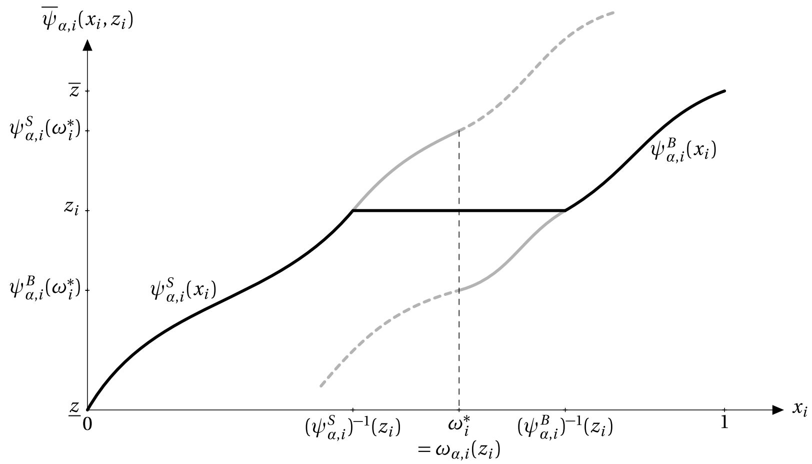


FIGURE 1. Ironed virtual type function.

breaks ties in favor of the agent who is the highest in the hierarchy: If the set of agents $\mathcal{I} \subseteq \mathcal{N}$ tie for the highest ironed virtual type and there is hierarchical tie-breaking according to hierarchy h, the object is assigned to agent $\arg\max_{i \in \mathcal{I}} h(i)$.

Under a *split hierarchical tie-breaking rule* $\mathbf{a}$, ownership of the object is split up into $n!$ shares $\mathbf{a} := (a_1, \dots, a_{n!}) \in \Delta^{n!-1}$, one for each hierarchy in $H = \{h_1, \dots, h_{n!}\}$, and then each a_l is assigned according to hierarchy h_l, i.e., to agent $\arg\max_{i \in \mathcal{I}} h_l(i)$.[13] The outcome in terms of interim expected shares $S_1, \dots, S_n$ of any tie-breaking rule can equivalently be obtained by a split hierarchical tie-breaking rule $\mathbf{a}$.

Define the *ironed virtual type allocation rule* $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ with ironing parameters $\mathbf{z} = (z_1, \dots, z_n)$ and split hierarchical tie-breaking rule $\mathbf{a}$ as, for all $i \in \mathcal{N}$,

$$s_i^{\mathbf{z},\mathbf{a}}(\mathbf{x}) := \begin{cases} 1 & \text{if } \overline{\psi}_{\alpha,i}(x_i, z_i) > \max\limits_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j) \\ \sum\limits_{h \in \hat{H}_i} a_h & \text{if } \overline{\psi}_{\alpha,i}(x_i, z_i) = \max\limits_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j) \\ 0 & \text{if } \overline{\psi}_{\alpha,i}(x_i, z_i) < \max\limits_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j), \end{cases}$$

where $\hat{H}_i := \{h \in H : h(i) > h(k)\ \forall k \in \arg\max_{j \neq i} \overline{\psi}_{\alpha,j}(x_j, z_j)\}$. For a given $\boldsymbol{\omega}^*$, $\mathbf{s}^* = \mathbf{s}^{\mathbf{z},\mathbf{a}}$ solves the problem in (9) for $\mathbf{z} = (\omega^{-1}_{\alpha,1}(\omega^*_1), \dots, \omega^{-1}_{\alpha,n}(\omega^*_n))$ and any tie-breaking rule $\mathbf{a} \in \Delta^{n!-1}$.

Figure 2 presents two examples of ironed virtual type allocation rules $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ for bilateral partnerships. The tie-breaking rule $\mathbf{a} = (a_1, a_2)$ is such that a_1 corresponds to the hierarchy according to which agent 1 beats agent 2. In both examples, the right-hand (left-hand) area represents all type realizations where agent 1's ironed virtual type is strictly greater (smaller) than agent 2's, resulting in full ownership of the object being

[13]An alternative interpretation is that one hierarchy h is randomly selected from H according to the probability distribution $\mathbf{a}$ over H and ties are then broken according to h.

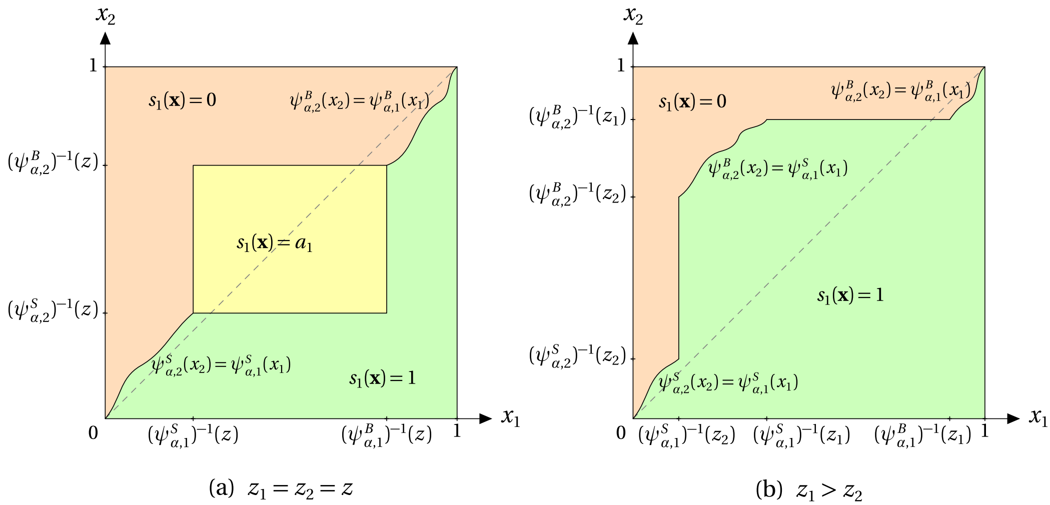


FIGURE 2. Ironed virtual type allocation rules for $n=2$.

allocated to agent 1 (2). In panel (a), ironing parameters are set such that $z_1 = z_2 = z$ for some z. This implies that for all type realizations in the center area, the ironed virtual types are the same for both agents, which happens with strictly positive probability. In this case, share a_1 is allocated to agent 1 and share $a_2 = 1 - a_1$ is allocated to agent 2. The interim expected share of each agent's critical type $S_i(\omega_{\alpha,i}(z))$ therefore depends on the tie-breaking rule. In panel (b), ironing parameters satisfy $z_1 > z_2$, implying that ties have probability 0 and the tie-breaking rule does not affect interim expected shares.

Having established that all allocation rules consistent with (9) are equivalent to ironed virtual type allocation rules $\mathbf{s}^{\mathbf{z},\mathbf{a}}$, we now turn to the second requirement for a saddle point. (10) requires that the critical types $\boldsymbol{\omega}^*$ are worst-off types under allocation rule $\mathbf{s}^*$. A simultaneous solution to (9) and (10) hence corresponds to a vector of ironing parameters $\mathbf{z}$ and a tie-breaking rule $\mathbf{a}$ such that $\omega_{\alpha,i}(z_i) \in \Omega_i(\mathbf{s}^{\mathbf{z},\mathbf{a}})$ for each agent i. Note that because of the bunching property, the interim expected share $S_i^{\mathbf{z},\mathbf{a}}(x_i)$ is constant for an interval of types x_i that contains the critical type $\omega_{\alpha,i}(z_i)$. The characterization of the set of worst-off types in Section 2.2 hence implies that for critical types to be worst off, we must have $S_i^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,i}(z_i)) = r_i$ for all $i \in \mathcal{N}$.

We show that there is typically a unique $\mathbf{z}$ such that $S_i^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,i}(z_i)) = r_i$ for all i for some $\mathbf{a}$, yielding the existence of a saddle point and a characterization of optimal dissolution mechanisms. To prove this result and make its statement precise, the following definitions are useful. Let $\underline{z} := -\eta(0)$ and $\overline{z} := 1 - \eta(1)$, and define the correspondence $\Gamma_n\colon [\underline{z}, \overline{z}]^n \to [0, 1]^n$ such that

$$\Gamma_n(\mathbf{z}) := \left\{\left(S_1^{\mathbf{z},\mathbf{a}}\big(\omega_{\alpha,1}(z_1)\big), \dots, S_n^{\mathbf{z},\mathbf{a}}\big(\omega_{\alpha,n}(z_n)\big)\right) : \mathbf{a} \in \Delta^{n!-1}\right\}.$$

The correspondence $\Gamma_n(\mathbf{z})$ yields the set of all vectors of expected shares for critical types $\omega_{\alpha,1}(z_1), \dots, \omega_{\alpha,n}(z_n)$ that can be obtained with ironing parameters $\mathbf{z}$ and some tie-breaking rule $\mathbf{a}$. If $z_i = z_j$ for two agents i, j, there is a strictly positive probability for a tie and the expected shares depend on tie-breaking. The correspondence $\Gamma_n(\mathbf{z})$ is singleton-valued if and only if $z_i \neq z_j$ for all i and $j \neq i$.

We are now ready to state our main result on optimal dissolution mechanisms.

THEOREM 1. *For each* $\mathbf{r} \in \Delta^{n-1}$*, there exists a unique* $\mathbf{z} \in [\underline{z}, \overline{z}]^n$ *such that* $\mathbf{r} \in \Gamma_n(\mathbf{z})$*. Let* $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$*. All optimal dissolution mechanisms* $(\mathbf{s}_\alpha^\mathbf{r}, \mathbf{t}_\alpha^\mathbf{r})$ *that solve (1) consist of an allocation rule* $\mathbf{s}_\alpha^\mathbf{r}$ *that allocates the object to an agent i with the greatest ironed virtual type* $\overline{\psi}_{\alpha,i}(x_i, z_i^*)$*, where ties are broken such that* $S_{\alpha,i}^\mathbf{r}(\omega_{\alpha,i}(z_i^*)) = r_i$ *for all* $i \in \mathcal{N}$*, and a payment rule* $\mathbf{t}_\alpha^\mathbf{r}$ *such that interim expected payments satisfy*

$$T_{\alpha,i}^\mathbf{r}(x_i) = E\big[v_i(x_i, \mathbf{X}_{-i})\big(s_{\alpha,i}^\mathbf{r}(x_i, \mathbf{X}_{-i}) - r_i\big)\big] - \int_{\omega_{\alpha,i}(z_i^*)}^{x_i} \big(S_{\alpha,i}^\mathbf{r}(y) - r_i\big)\, dy \quad \textit{for all } i \in \mathcal{N}.$$

A split hierarchical tie-breaking rule $\mathbf{a}^*$ *exists such that* $\mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ *is an optimal allocation rule.*

The proof of Theorem 1 is provided in Appendix A. The most challenging part of the proof is to establish the first line of the theorem, which ensures that the inverse correspondence $\Gamma_n^{-1}(\mathbf{r})$ is singleton-valued for all initial shares $\mathbf{r}$. We uncover a recursive structure to Γ_n by partitioning its domain in a suitable way. This allows us to prove that Γ_n has the required properties by induction, using the tractable two-agent case as the base case. The existence of a unique $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$ in turn implies that $\mathbf{s}^* = \mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ and $\boldsymbol{\omega}^* = (\omega_{\alpha,1}(z_1^*), \dots, \omega_{\alpha,n}(z_n^*))$ constitute a saddle point that satisfies (9) and (10) for all split hierarchical tie-breaking rules $\mathbf{a}^*$ such that $(S_1^{\mathbf{z}^*,\mathbf{a}^*}(\omega_{\alpha,1}(z_1^*)), \dots, S_n^{\mathbf{z}^*,\mathbf{a}^*}(\omega_{\alpha,n}(z_n^*))) = \mathbf{r}$. Consequently, $\mathbf{s}_\alpha^\mathbf{r} = \mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ is an optimal allocation rule that solves the max-min problem (8). Any other optimal allocation rule $\mathbf{s}_\alpha^\mathbf{r}$ may differ from $\mathbf{s}^{\mathbf{z}^*,\mathbf{a}^*}$ only with respect to the tie-breaking rule. Finally, interim expected payments are pinned down by payoff equivalence (IC2), as stated in (7).

For all tie-breaking rules $\mathbf{a}$, $(S_1^{\mathbf{z}^*,\mathbf{a}}(\omega_{\alpha,1}(z_1^*)), \dots, S_n^{\mathbf{z}^*,\mathbf{a}}(\omega_{\alpha,n}(z_n^*)))$ is equal to the convex combination with coefficients $\mathbf{a}$ of the $n!$ vectors of the critical types' expected shares under hierarchical tie-breaking. Hence, $\Gamma_n(\mathbf{z}^*)$ is the convex hull of the expected shares under hierarchical tie-breaking. Once the ironing parameters $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$ are determined, finding a corresponding tie-breaking rule $\mathbf{a}^*$ to implement an optimal allocation rule is straightforward: one just has to find a convex combination of the extreme points of the convex hull $\Gamma(\mathbf{z}^*)$ that is equal to $\mathbf{r}$.[14]

To summarize, optimal dissolution mechanisms reallocate shares based on ironed virtual types, where for each ownership structure, the intervals of types where bunching occurs are uniquely determined to ensure, together with the tie-breaking rule, that the corresponding critical type of each agent expects to retain his initial ownership share.

As the mechanisms identified by Theorem 1 maximize W_α for any α, there are no mechanisms that yield higher combinations of social surplus and revenue. Importantly, this means that Theorem 1 identifies, at the same time, the second-best dissolution mechanisms that maximize social surplus subject to a revenue constraint (such as budget balance) whenever ex post efficient dissolution is impossible.[15]

[14]One can further show that a tie-breaking rule with these properties that uses at most n hierarchies always exists.

[15]See the beginning of Section 5 for more details.

3.2 *Bilateral partnerships*

To illustrate the working of the optimal dissolution mechanisms, we now specialize the setup to one with two agents. According to Theorem 1, an optimal dissolution mechanism allocates the object to the agent i with the higher ironed virtual type $\overline{\psi}_{\alpha,i}(x_i, z_i^*)$, where $(z_1^*, z_2^*) = \Gamma_2^{-1}(r_1, r_2)$. For bilateral partnerships, characterizing (z_1^*, z_2^*) further is possible at little additional cost.

Suppose $z_1^* > z_2^*$. Then the critical type of agent 1 expects to obtain the object with probability $S_1(\omega_{\alpha,1}(z_1^*)) = G^B_{\alpha,2}(z_1^*)$, whereas the critical type of agent 2 expects to obtain the object with probability $S_2(\omega_{\alpha,2}(z_2^*)) = G^S_{\alpha,1}(z_2^*)$.[16] Moreover, these probabilities are equal to the initial shares r_1 and $1 - r_1$, making the critical types worst off. Consequently, all initial shares that are consistent with $z_1^* > z_2^*$ satisfy $(G^B_{\alpha,2})^{-1}(r_1) > (G^S_{\alpha,1})^{-1}(1 - r_1)$. This is true for all $r_1 \in (\overline{r}_1, 1]$, where $\overline{r}_1$ uniquely solves $(G^B_{\alpha,2})^{-1}(\overline{r}_1) = (G^S_{\alpha,1})^{-1}(1 - \overline{r}_1)$.

Similarly, $z_1^* < z_2^*$ if and only if $r_1 \in [0, \underline{r}_1)$, where $\underline{r}_1$ is the unique solution to $(G^S_{\alpha,2})^{-1}(\underline{r}_1) = (G^B_{\alpha,1})^{-1}(1 - \underline{r}_1)$. Observe that $0 < \underline{r}_1 < \overline{r}_1 < 1$ for all $\alpha > 0$, and that $\underline{r}_1$ is decreasing and $\overline{r}_1$ is increasing in α.

It follows that for $r_1 \in [\underline{r}_1, \overline{r}_1]$ we must have $z_1^* = z_2^*$. In this case, agents tie for the highest ironed virtual type with positive probability. If agent i obtains share a_i in case of a tie, then i's critical type expects to obtain share $S_i(\omega_{\alpha,i}(z_i^*)) = a_i G^B_{\alpha,j}(z_i^*) + (1 - a_i) G^S_{\alpha,j}(z_i^*)$. The optimal allocation rule makes sure that this expected share is equal to r_i. We thus obtain the following corollary to Theorem 1.

Corollary 1. *Suppose $n = 2$. The optimal allocation rule $\mathbf{s}^{\mathbf{r}}_\alpha$ allocates the object to the agent i who has the higher ironed virtual type $\overline{\psi}_{\alpha,i}(x_i, z_i^*)$ and in case of a tie assigns share a_1^* to agent 1 and $1 - a_1^*$ to agent 2.*

(i) If $r_1 \in [0, \underline{r}_1)$, then $z_1^* = (G^S_{\alpha,2})^{-1}(r_1) < (G^B_{\alpha,1})^{-1}(r_2) = z_2^*$ and $a_1^* \in [0, 1]$.

(ii) If $r_1 \in [\underline{r}_1, \overline{r}_1]$, then $z_1^* = z_2^* = z^*$, where z^* and a_1^* are the unique solution to

$$a_1^* G^B_{\alpha,2}(z^*) + (1 - a_1^*) G^S_{\alpha,2}(z^*) = r_1, \qquad a_1^* G^S_{\alpha,1}(z^*) + (1 - a_1^*) G^B_{\alpha,1}(z^*) = r_2.$$

(iii) If $r_1 \in (\overline{r}_1, 1]$, then $z_1^* = (G^B_{\alpha,2})^{-1}(r_1) > (G^S_{\alpha,1})^{-1}(r_2) = z_2^*$ and $a_1^* \in [0, 1]$.

In cases (i) and (iii) of Corollary 1, ties occur with probability 0, which explains why ties can be broken arbitrarily, i.e., why any $a_1^* \in [0, 1]$ is optimal. In contrast, for case (ii) the tie-breaking rule a_1^* of the optimal allocation rule is unique.

The optimal allocation rule described in Corollary 1 is illustrated in Figure 2. Panel (a) depicts case (ii) of Corollary 1 and panel (b) depicts case (iii), which after interchanging the agents' names also applies to case (i). The figures are drawn for a situation where $F_1 \neq F_2$, i.e., where agents draw their types from different distributions. From the figures we can infer how the optimal allocation rule for $\alpha > 0$ differs from the ex post efficient

[16] To see this, note that the cumulative distribution function of agent i's ironed virtual type $Y_i = \overline{\psi}_{\alpha,i}(X_i, z_i^*)$ corresponds to $G^S_{\alpha,i}(y_i)$ for $y_i \le z_i^*$ and $G^B_{\alpha,i}(y_i)$ for $y_i > z_i^*$.

allocation rule that assigns the object to agent 1 (2) if (x_1, x_2) is below (above) the dashed 45-degree line.

Suppose the ownership structure is sufficiently symmetric such that $r_1 \in (\underline{r}_1, \overline{r}_1)$, implying optimal ironing parameters $z_1^* = z_2^* = z^*$ as in panel (a) of Figure 2. Types $x_1 \in [(\psi_{\alpha,1}^S)^{-1}(z^*), (\psi_{\alpha,1}^B)^{-1}(z^*)]$ of agent 1 and types $x_2 \in [(\psi_{\alpha,2}^S)^{-1}(z^*), (\psi_{\alpha,2}^B)^{-1}(z^*)]$ of agent 2 all have the same ironed virtual type z^*. If both type realizations are within these intervals, share $a_1^* \in (0,1)$ of the object is assigned to agent 1, as represented by the center rectangle. This inefficiency of the allocation is reminiscent of the traditional undersupply by a monopolist and of auctions with revenue-maximizing reserve prices. If both agents draw a sufficiently high type, the object is allocated to the agent with the highest virtual valuation, whereas for sufficiently low types, the allocation is based on comparing virtual costs. Thus the object may end up in the hands of the agent who values it less, resulting in a second kind of inefficiency, similar to the optimal auction of Myerson (1981) with asymmetric bidders. Whereas the first kind of inefficiency is always present for $\alpha > 0$, the second kind vanishes if types are identically distributed.

As we increase r_1 within $[\underline{r}_1, \overline{r}_1]$, the share a_1^* increases and z^* may change (it stays constant if $F_1 = F_2$) until we reach $\overline{r}_1$, where $a_1^* = 1$. At this point, we leave the case underlying panel (a) of Figure 2 and switch to the situation depicted in panel (b). As we increase r_1 further, z_1^* increases and z_2^* decreases, eventually reaching $\overline{z}$ and $\underline{z}$, respectively, when $r_1 = 1$.

Now consider $r_1 \in (\overline{r}_1, 1]$, which implies $z_1^* > z_2^*$ as in panel (b) of Figure 2. If types $(x_1, x_2) \in [(\psi_{\alpha,1}^S)^{-1}(z_2^*), 1] \times [0, (\psi_{\alpha,2}^B)^{-1}(z_1^*)]$ realize, the optimal allocation rule assigns the object to agent 1 if his virtual cost $\psi_{\alpha,1}^S(x_1)$ is higher than the virtual valuation $\psi_{\alpha,2}^B(x_2)$ of agent 2. Otherwise, the object is assigned to agent 2. For type realizations within this region, the optimal allocation thus corresponds exactly to the allocation rules derived by Myerson and Satterthwaite (1983), giving rise to the same inefficiency. If $x_1 < (\psi_{\alpha,1}^S)^{-1}(z_2^*)$, the object is allocated on the basis of virtual costs, whereas if $x_2 > (\psi_{\alpha,2}^B)^{-1}(z_1^*)$, the object is assigned to the agent with the higher virtual valuation. In those cases, we again obtain the second kind of inefficiency that disappears if types are drawn from the same distribution. Note that for $r_1 = 1$, where $(\psi_{\alpha,1}^S)^{-1}(z_2^*) = 0$ and $(\psi_{\alpha,2}^B)^{-1}(z_1^*) = 1$, the optimal allocation rule coincides with the solution of Myerson and Satterthwaite (1983) on the entire type space $[0,1]^2$. This is, of course, consistent with the partnership model approaching a bilateral trade setting where agent 1 is the seller and agent 2 is the buyer as r_1 approaches 1.

As α increases while r_1 is kept fixed, the inefficiency of the optimal allocation increases: In panel (a) the center rectangle with tie-breaking becomes larger and in panel (b) the demarcation line where 1's virtual cost coincides with 2's virtual valuation moves upward and to the left. This is because a higher α makes the difference between virtual types and actual net types $x_i - \eta(x_i)$ larger. The comparative static effects of increasing the (positive) interdependence of valuations on the optimal allocation are similar to the effects of increasing α under private values. This is easiest to see for the case with linear interdependence $\eta(x) = ex$ with $e < 1$. In this case, i's virtual type $\psi_{\alpha,i}^K(x_i)$ is larger than j's virtual type $\psi_{\alpha,j}^L(x_j)$ with $K, L \in \{B, S\}$ if and only if for private values (i.e., $\eta'(x) = 0$),

$\psi^K_{\alpha/(1-e),i}(x_i) \geq \psi^L_{\alpha/(1-e),j}(x_j)$. The effect of increasing e in the model with linear interdependence is thus qualitatively the same as that of increasing α in the private values model.

3.3 *Simple mechanisms and implementation*

Dissolution mechanisms used in practice typically have simpler rules than the optimal direct mechanisms we characterize. For example, in the Texas shootout (or buy–sell clause), a commonly used procedure for breaking up bilateral business partnerships, one partner proposes a per-unit price and the other partner decides whether to sell his share or buy out his partner at that price (McAfee 1992, de Frutos and Kittsteiner 2008, Brooks et al. 2010). Under private information, the Texas shootout is typically inefficient, even in settings where ex post efficient dissolution would be feasible. A simple alternative that usually performs better is to run a standard auction among partners, with the highest bidder buying out the others at the price determined in the auction. Under private values, identical distributions, and equal shares, such an auction dissolves efficiently (Cramton et al. 1987). Under nonidentical distributions or unequal shares, while still satisfying individual rationality, the resulting allocation is inefficient, as shown in Wasser (2013) for the bilateral case. For positively interdependent valuations and equal shares, Kittsteiner (2003) showed that partners suffer from both a winner's and a loser's curse, which may result in a violation of individual rationality. Giving veto rights to agents fixes this problem at the cost of losing efficiency.

The optimal dissolution mechanisms of Theorem 1 are an important benchmark for assessing how severe the inefficiencies are of simple mechanisms such as those just discussed. Moreover, our results provide some guidance for improving the design of simple mechanisms. For the important special case of symmetric bilateral partnerships, we propose the following simple mechanism with appealing features for practical use that implements the optimal mechanism.

A simple dissolution procedure Consider a symmetric bilateral partnership, i.e., $n = 2$, $F_1 = F_2$, and $r_1 = r_2 = \frac{1}{2}$. Moreover, suppose $\eta'(x) \in (-1, 1)$ for all x. This setting allows for a particularly simple and intuitive implementation of the optimal dissolution mechanism as a combination of posted prices and standard auctions. The optimal allocation rule of Theorem 1 for this setting is based on ironed virtual types with $z_1^* = z_2^* = z^*$ and leaves ownership equally split in case of a tie. To simplify the notation, let $\omega := \omega_{\alpha,i}(z^*)$ denote each agent's critical worst-off type and let $[\underline{\omega}, \overline{\omega}]$ denote the interval of worst-off types, where $\underline{\omega} := (\psi^S_{\alpha,i})^{-1}(z^*)$ and $\overline{\omega} := (\psi^B_{\alpha,i})^{-1}(z^*)$.

The simple dissolution procedure works as follows. First, the designer announces four per-unit prices: a standard and a modified sell price, denoted respectively p^S and $\hat{p}^S$, as well as a standard and a modified buy price, denoted p^B and $\hat{p}^B$. These prices are

$$p^S := E\big[v_i(\omega, X_j) | \overline{\omega} < X_j\big], \qquad p^B := E\big[v_i(\omega, X_j) | X_j < \underline{\omega}\big]$$

$$\hat{p}^S := E\big[v_i(\underline{\omega}, X_j) | \underline{\omega} \leq X_j \leq \overline{\omega}\big], \qquad \hat{p}^B := E\big[v_i(\overline{\omega}, X_j) | \underline{\omega} \leq X_j \leq \overline{\omega}\big].$$

Then each agent is asked whether he prefers to SELL, HOLD, or BUY. If both agents request something different or if both request HOLD, their requests are called *compatible*. Otherwise, i.e., if both request BUY or both request SELL, their requests are called *incompatible*. If the agents' requests are compatible, shares are traded at fixed prices in the required direction (unless both request HOLD): If agent i requests SELL and agent j requests BUY, i sells his share at p^S to the designer who then sells it to j at p^B, resulting in revenue of $\frac{1}{2}(p^B - p^S)$ for the designer; if agent i requests HOLD while agent j requests SELL (BUY), then agent i buys j's (sells his) share at the standard price p^B (p^S), whereas agent j sells (buys) at the modified price $\hat{p}^S$ ($\hat{p}^B$); and if both agents request HOLD, they keep their shares and no payments are made.

Incompatible requests are resolved using a standard auction. If both agents request BUY, the designer first buys the agents' shares at price p^S and then sells the entire object through an open ascending forward auction, with the price starting at $v_i(\overline{\omega}, \overline{\omega})$. If both agents request SELL, the designer first short sells share $\frac{1}{2}$ to each agent at price p^B and then buys back one unit through an open descending reverse auction, starting at price $v_i(\underline{\omega}, \underline{\omega})$. Figure 3 summarizes the procedure.

The following proposition asserts that the simple dissolution procedure implements an optimal dissolution mechanism of Theorem 1.

PROPOSITION 1. *The simple dissolution procedure has a perfect Bayesian equilibrium in which agent* $i = 1, 2$ *chooses* SELL *if* $x_i < \underline{\omega}$, HOLD *if* $x_i \in [\underline{\omega}, \overline{\omega}]$, *and* BUY *if* $x_i > \overline{\omega}$, *and in case of an auction, agent i drops out when the price reaches* $v_i(x_i, x_i)$. *Moreover, the resulting allocation and payments correspond to those of an optimal dissolution mechanism* $(\mathbf{s}^{\mathbf{r}}_{\alpha}, \mathbf{t}^{\mathbf{r}}_{\alpha})$.

Note that the dissolution procedure is simple in that it often requires only coarse communication and uses a standard auction that preserves the privacy of the winner of the auction when more granular information is required. This simplicity suggests that it may be of practical use for designers of dissolution mechanisms such as courts.

		Agent 2: SELL	Agent 2: HOLD	Agent 2: BUY
Agent 1	SELL	Both buy at p^B & reverse auction	1 sells at $\hat{p}^S$, 2 buys at p^B	1 sells at p^S, 2 buys at p^B
Agent 1	HOLD	1 buys at p^B, 2 sells at $\hat{p}^S$	No trade	1 sells at p^S, 2 buys at $\hat{p}^B$
Agent 1	BUY	1 buys at p^B, 2 sells at p^S	1 buys at $\hat{p}^B$, 2 sells at p^S	Both sell at p^S & forward auction

FIGURE 3. A simple dissolution procedure for symmetric bilateral partnerships.

4. Optimal ownership structures

The optimal dissolution mechanisms $(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$ that solve (1) described in Theorem 1 depend on $\mathbf{r}$ and the designer's preference parameter α. In this section, we study how the designer's value function $W_\alpha(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$ varies with $\mathbf{r}$. Most importantly, we characterize the set of all optimal ownership structures that maximize $W_\alpha(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$, which we denote by

$$R^*(\alpha) := \operatorname*{argmax}_{\mathbf{r}\in\Delta^{n-1}} W_\alpha\big(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}}\big).$$

Hence, the initial shares in $R^*(\alpha)$ are the solutions to the problem stated in (2).

4.1 *The main characterization result*

To develop intuition, suppose, temporarily, that the allocation rule $\mathbf{s}$ is fixed and each interim share S_i is strictly increasing, implying that there is a unique worst-off type ω_i for each agent i. Accordingly, surplus is fixed and revenue extraction is restricted by the fact that each agent i's worst-off type must at least obtain his interim expected status quo utility $r_i E[v_i(\omega_i, X_{-i})]$ to ensure participation. If there are two agents i, j such that $E[v_i(\omega_i, X_{-i})] > E[v_j(\omega_j, X_{-j})]$, then a marginal transfer of ownership from i to j relaxes i's individual rationality constraint by more than it tightens j's and therefore increases the revenue that can be extracted. Since decreasing r_i decreases i's worst-off type ω_i and increasing r_j increases ω_j, the positive effect on revenue gradually diminishes as we transfer ownership further. Revenue is thus concave in the ownership structure and maximized when interim valuations of worst-off types are equal. We now show that this concavity property carries over to any combination of surplus and revenue under an optimal allocation rule that changes with the ownership structure.

Our approach of relating optimal dissolution mechanisms to saddle points of the virtual surplus $\widetilde{W}_\alpha$ is also useful here because it enables us to easily obtain concavity of $W_\alpha(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$ as follows. According to Section 3.1, we have

$$W_\alpha\big(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}}\big) = (1-\alpha)\sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})r_i\big] + \max_{\mathbf{s}\in\mathfrak{S}} \min_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}). \tag{13}$$

The saddle point property of $\widetilde{W}_\alpha$ implies that max-min and min-max are equivalent, i.e.,

$$\begin{aligned}
&\max_{\mathbf{s}\in\mathfrak{S}} \min_{\hat{\mathbf{x}}\in[0,1]^n} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) \\
&\quad = \min_{\hat{\mathbf{x}}\in[0,1]^n} \max_{\mathbf{s}\in\mathfrak{S}} \widetilde{W}_\alpha(\mathbf{s}, \hat{\mathbf{x}}) \\
&\quad = \min_{\hat{\mathbf{x}}\in[0,1]^n} \bigg\{-\sum_{i\in\mathcal{N}} r_i E\big[\psi_{\alpha,i}(X_i, \hat{x}_i)\big] + \max_{\mathbf{s}\in\mathfrak{S}} E\bigg[\sum_{i\in\mathcal{N}} s_i(\mathbf{X})\psi_{\alpha,i}(X_i, \hat{x}_i)\bigg]\bigg\}. \qquad (14)
\end{aligned}$$

Note that (14) is the minimum of a family of affine functions of $\mathbf{r}$ (indexed by $\hat{\mathbf{x}}$). Consequently, (14) is concave in $\mathbf{r}$, and by (13) the same is true for $W_\alpha(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}})$.

Concavity of the objective allows us to study optimal ownership structures using first-order conditions and to obtain an intuitive characterization of optimality that is based on the interim expected valuations of the critical worst-off types. The following theorem is our main result on optimal ownership structures.

THEOREM 2. *(i) The objective* $W_\alpha(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$ *is concave in the ownership structure* $\mathbf{r}$.

(ii) The optimal ownership structures form a convex set $R^*(\alpha)$ *that is a strict subset of* Δ^{n-1}.

(iii) An ownership structure $\mathbf{r}$ *is optimal if and only if, under the corresponding optimal dissolution mechanisms, the interim valuation of the critical worst-off type is the same for all agents with nonzero shares and is higher for all agents with zero shares. That is,* $\mathbf{r} \in R^*(\alpha)$ *if and only if for all* $i \in \mathcal{N}$*, for some* Y*, and for* $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r})$,

$$\begin{aligned} E\big[v_i\big(\omega_{\alpha,i}\big(z_i^*\big), \mathbf{X}_{-i}\big)\big] &= Y \quad \text{if } r_i > 0 \quad \text{and} \\ E\big[v_i\big(\omega_{\alpha,i}\big(z_i^*\big), \mathbf{X}_{-i}\big)\big] &\geq Y \quad \text{if } r_i = 0. \end{aligned} \tag{15}$$

Concavity of the designer's objective, as established in part (i) of Theorem 2, means that if the designer is indifferent between two different ownership structures, then he weakly prefers any convex combination of the two ownership structures. Accordingly, the set of optimal ownership structures is convex. Although multiple ownership structures may be optimal, part (ii) of Theorem 2 implies that it is never the case that all possible ownership structures are optimal. Finally, part (iii) of Theorem 2 provides a clear-cut characterization of optimal ownership structures: exactly those initial shares are optimal that equalize the interim valuations of the induced critical worst-off types across agents with strictly positive initial shares and that lead to higher interim valuations for agents with an initial share of zero.

Recall that under any optimal dissolution mechanism, each agent i's individual rationality constraint is binding for the interval of worst-off types that results from bunching around the critical type $\omega_{\alpha,i}(z_i^*)$. The optimality condition (15) ensures that there is no marginal transfer of ownership from an agent i to an agent j that improves $W_\alpha(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$ via marginally relaxing i's individual rationality constraint by $E[v_i(\omega_{\alpha,i}(z_i^*), \mathbf{X}_{-i})]$ and tightening j's by $E[v_j(\omega_{\alpha,j}(z_j^*), \mathbf{X}_{-j})]$: such marginal changes are either not beneficial (because the interim valuation of i's and j's critical worst-off type are equal) or not feasible (because $r_i = 0$).

As $\alpha \to 0$, every optimal dissolution mechanism approaches a mechanism with the ex post efficient allocation rule and transfers that maximize revenue under this allocation rule. The optimal ownership structure for $\alpha \to 0$ hence yields the initial shares that allow for the highest revenue under ex post efficient allocation (put differently, these shares minimize the subsidy required for efficient trade). As $\alpha \to 0$, $\mathbf{s}^{\mathbf{r}}_\alpha$ approaches the ex post efficient allocation rule and the set of worst-off types $\Omega_i(\mathbf{s}^{\mathbf{r}}_\alpha)$ shrinks to the singleton $\omega_{0,i}$ that solves $\prod_{j\neq i} F_j(\omega_{0,i}) = r_i$, which is thus agent i's critical worst-off type under the ex post efficient allocation rule. As each $\omega_{0,i}$ is strictly increasing in r_i, there is thus a unique ownership structure that meets the optimality condition of Theorem 2, and optimal ownership can then be characterized as follows.[17]

Corollary 2. *The ownership structure* $\{\mathbf{r}^0\} = \lim_{\alpha \to 0} R^*(\alpha)$ *that maximizes revenue under ex post efficiency is unique and is given by*

$$r_i^0 = \prod_{j \neq i} F_j\big(\hat{Y} + E\big[\eta(X_i)\big]\big) \quad \text{for each } i \in \mathcal{N}, \tag{16}$$

where $\hat{Y}$ *solves* $\sum_{i \in \mathcal{N}} \prod_{j \neq i} F_j(\hat{Y} + E[\eta(X_i)]) = 1$. *Moreover, if* $r_i^0 > 0$ *for some* $i \in \mathcal{N}$, *then* $r_j^0 > 0$ *for all* j *such that* $E[\eta(X_j)] \geq E[\eta(X_i)]$.

4.2 *Less inequality is better with identical type distributions*

We explore the implications of Theorem 2, beginning with the case where the types of all agents are drawn from the same distribution F. In this case, the objective $W_\alpha(\mathbf{s}_\alpha^\mathbf{r}, \mathbf{t}_\alpha^\mathbf{r})$ stays constant if we interchange the initial shares of two agents, and concavity of the objective implies that it increases if we assign to both agents the average of their original shares. This suggests that the combination of surplus and revenue that can be achieved through optimal dissolution increases as the ownership structure becomes more equal.

To obtain a precise notion of one ownership structure being more equal than another, we make use of the theory of majorization.[18] Given two vectors $\mathbf{r}$ and $\mathbf{q}$ with n components, we say $\mathbf{r}$ is *majorized* by $\mathbf{q}$, denoted by $\mathbf{r} \prec \mathbf{q}$, if

$$\sum_{i=1}^{k} r_{[i]} \leq \sum_{i=1}^{k} q_{[i]} \quad \text{for } k \in \{1, \dots, n-1\} \quad \text{and} \quad \sum_{i=1}^{n} r_{[i]} = \sum_{i=1}^{n} q_{[i]},$$

where $r_{[1]} \geq \cdots \geq r_{[n]}$ denotes the components of $\mathbf{r} = (r_1, \dots, r_n)$ in decreasing order. Intuitively, $\mathbf{r} \prec \mathbf{q}$ means the components of $\mathbf{r}$ are more equal than the components of $\mathbf{q}$. A real-valued function ϕ is *Schur-concave* if $\mathbf{r} \prec \mathbf{q}$ implies $\phi(\mathbf{r}) \geq \phi(\mathbf{q})$. In particular, a function is Schur-concave if it is symmetric (i.e., $\phi(\mathbf{r}) = \phi(\mathbf{r}')$ if $\mathbf{r}'$ is a permutation of $\mathbf{r}$) and concave (see Marshall et al. 2011, p. 97). Hence, we obtain the following corollary to part (i) of Theorem 2.

Corollary 3. *Suppose* $F_i = F$ *for all* $i \in \mathcal{N}$. *Then* $W_\alpha(\mathbf{s}_\alpha^\mathbf{r}, \mathbf{t}_\alpha^\mathbf{r})$ *is Schur-concave in* $\mathbf{r}$*: the objective increases as the ownership structure becomes more equal.*

Schur-concavity immediately implies that $W_\alpha(\mathbf{s}_\alpha^\mathbf{r}, \mathbf{t}_\alpha^\mathbf{r})$ is minimized when ownership is concentrated at one agent ($r_i = 1$ for one i), which is strictly suboptimal by part (ii) of Theorem 2. The objective is maximized for symmetric ownership ($r_i = 1/n$ for all i).

[17] For the special case of private values where $\eta(x) = 0$, the revenue-maximizing ownership structure under ex post efficiency was obtained by Che (2006) and Figueroa and Skreta (2012). Corollary 2 generalizes their results to interdependent values.

[18] For a comprehensive reference, see Marshall et al. (2011).

Symmetric ownership is optimal because in this case optimal dissolution mechanisms treat all agents the same, which clearly induces the same valuation at the critical worst-off type for each agent. However, this is not the only optimal ownership structure. Any ownership structure that, like symmetric ownership, is associated with optimal dissolution mechanisms where the ironing parameters are all equal (i.e., $z_i^* = z^*$ for all i and some z^*) also results in equal critical worst-off types as required by the optimality condition (15) of Theorem 2. These are ownership structures that are sufficiently equal such that only the tie-breaking rule needs to be adapted to the asymmetry in initial shares while agents are treated the same otherwise. The least equal ownership structures with this property are those that can be accommodated by hierarchical tie-breaking: Using the same ironing parameters and tie-breaking according to hierarchy $h(i) = i$ (i.e., agent i wins ties against all agents $j < i$), for example, yields an optimal dissolution mechanism for the ownership structure

$$\mathbf{r}^\alpha := \left(r_1^\alpha, \dots, r_n^\alpha\right) \quad \text{where } r_i^\alpha := G_\alpha^S\left(z^*\right)^{n-i} G_\alpha^B\left(z^*\right)^{i-1} \text{ for each } i$$

and where z^* is the unique solution to $\sum_{i\in\mathcal{N}} G_\alpha^S(z^*)^{n-i} G_\alpha^B(z^*)^{i-1} = 1$.[19] Moreover, split hierarchical tie-breaking rules yield optimal dissolution mechanisms for any ownership structure that is a convex combination of permutations of $\mathbf{r}^\alpha$, i.e., all ownership structures that are more equal than $\mathbf{r}^\alpha$ in the sense that $\mathbf{r} \prec \mathbf{r}^\alpha$.[20] As all ironing parameters equal z^*, all these ownership structures satisfy the optimality condition (15). All other ownership structures require heterogenous ironing parameters and therefore fail to meet (15).

Note that increasing α strictly increases the difference between virtual costs and valuations, and thus between G_α^S and G_α^B. In turn, the components of $\mathbf{r}^\alpha$ become more spread out, thereby enlarging the set of optimal ownership structures.

Corollary 4. *Suppose $F_i = F$ for all $i \in \mathcal{N}$. Then the optimal ownership structures consist of all initial shares that are more equal than $\mathbf{r}^\alpha$: $R^*(\alpha) = \{\mathbf{r} \in \Delta^{n-1} : \mathbf{r} \prec \mathbf{r}^\alpha\}$. Moreover, the set of optimal ownership structures strictly increases in the revenue weight: for all $\alpha < \alpha'$, $R^*(\alpha) \subset R^*(\alpha')$.*

As $\alpha \to 0$, Corollary 2 implies that symmetric ownership becomes uniquely optimal. In contrast, for $\alpha > 0$, there is always some leeway in specifying an optimal ownership structure, and this leeway is greater the more important is generating revenue.

Most findings of this subsection straightforwardly extend to situations where some but not all agents' types are identically distributed. For any group of agents $\mathcal{I} \subset \mathcal{N}$ with identically distributed types, the designer's objective is Schur-concave in the shares of the agents in $\mathcal{I}$, and it is always optimal to assign equally sized shares to all agents in $\mathcal{I}$.

[19]Under this tie-breaking rule, i's critical type obtains the object if the $n-i$ agents $k > i$ have virtual costs below z^* and the $i-1$ agents $j < i$ have virtual valuations below z^*, resulting in interim expected share $S_i(\omega_\alpha(z^*)) = G_\alpha^S(z^*)^{n-i} G_\alpha^B(z^*)^{i-1}$. Optimal dissolution mechanisms induce $S_i(\omega_\alpha(z^*)) = r_i$, which together with $\sum_{i\in N} r_i = 1$ uniquely pins down z^* since G_α^S and G_α^B are strictly increasing.

[20]The set $\{\mathbf{r} : \mathbf{r} \prec \mathbf{q}\}$ is equal to the convex hull of points obtained by permuting the components of $\mathbf{q}$; see, for example, Marshall et al. (2011, Corollary 2.B.3).

4.3 *(Sub)optimality of extreme ownership structures*

When types are drawn from different distributions, symmetric ownership typically does not result in the same expected valuation for the critical worst-off type of each agent. Optimality hence requires unequal initial shares that account for the ex ante asymmetry among agents. Most importantly, as demonstrated below, even an extreme ownership structure that fully concentrates property rights at one agent can then be optimal. We start with a specific example to illustrate the rich variety of optimal shares that may obtain depending on the interplay of type distributions, value interdependence, and the importance of generating revenue.

EXAMPLE 1. Consider a bilateral partnership ($n=2$) with

$$F_1(x)=x^b, \qquad F_2(x)=1-(1-x)^b, \quad \text{and} \quad \eta(x)=ex, \quad \text{where } b>1 \text{ and } e<1. \tag{17}$$

Under these assumptions, $E[v_1(\mathbf{X})]=(b+e)/(1+b)>(1+eb)/(1+b)=E[v_2(\mathbf{X})]$ and $F_1(x)<F_2(x)$ for all $x\in(0,1)$. Agent 1 is hence the stronger agent in the sense that F_1 first-order stochastically dominates F_2, and the higher is b, the more pronounced is this dominance.

Assume first $b=1.2$, which corresponds to a moderate degree of asymmetry in type distributions. Figure 4 shows the set of optimal shares $R_1^*(\alpha)=\{r_1:(r_1,r_2)\in R^*(\alpha)\}$ for agent 1 as a function of the revenue weight α for $e\in\{-0.35,0,0.35\}$. For private values ($e=0$) and for negative interdependence ($e=-0.35$), the optimal r_1 is unique and decreasing. For positive interdependence ($e=0.35$), however, there is a unique α for which the interval of shares $[\underline{r}_1,\bar{r}_1]$ is optimal as indicated by the vertical line segment in Figure 4.[21] That is, for this value of α, we have $R_1^*(\alpha)=[\underline{r}_1,\bar{r}_1]$, while for all other values of

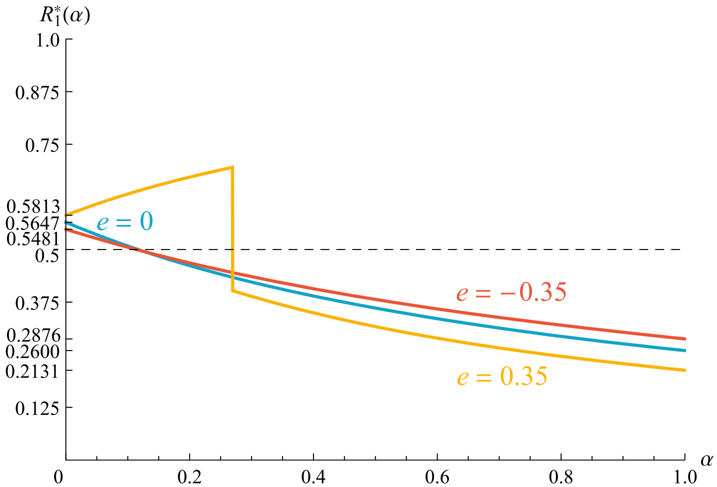


FIGURE 4. Optimal bilateral ownership structures $R_1^*(\alpha)$ under moderate asymmetry ($b=1.2$) for $e=0.35$, $e=0$, and $e=-0.35$.

[21]For bilateral environments where $F_2(x)=1-F_1(1-x)$ for all x and $\eta(x)=ex$ for some $e<1$, as is satisfied here, the virtual type distributions satisfy $G^B_{\alpha,2}(z)=1-G^S_{\alpha,1}(1-e-z)$ and $G^S_{\alpha,2}(z)=1-G^B_{\alpha,1}(1-$

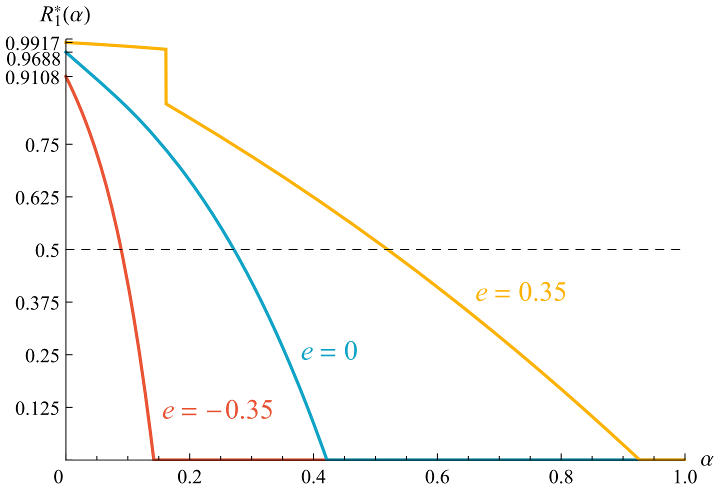


FIGURE 5. Optimal bilateral ownership structures $R_1^*(\alpha)$ under pronounced asymmetry ($b = 5$) for $e = 0.35$, $e = 0$, and $e = -0.35$.

α, the optimal r_1 is unique, increasing in α when α is smaller and decreasing when α is larger. Hence, optimal ownership structures are not necessarily monotone in α. Note also that for small α, $R_1^*(\alpha)$ decreases in e and agent 1 is the majority owner, whereas for large α, it is increasing in e and agent 1 is the minority owner.

Next we consider an increase in the asymmetry in type distributions to $b = 5$, which is illustrated in Figure 5. For all three kinds of interdependence, a change in the revenue weight may now drastically change optimal ownership. Whereas for low values of α, the stronger agent's share is close to but strictly less than 1, it is optimal to assign sole ownership to the weaker agent if generating revenue is important. Moreover, the stronger agent's share is now overall decreasing in α and increasing in the value-interdependence parameter e. $\diamondsuit$

Motivated by Example 1, we proceed by asking under what conditions concentrated ownership is optimal. An extreme ownership structure with $r_i = 1$ for some $i \in \mathcal{N}$ and $r_j = 0$ for all $j \neq i$ represents a trade setting in the spirit of Myerson and Satterthwaite (1983) with one seller and $n - 1$ potential buyers. An optimal allocation rule assigns the object to a buyer with the greatest virtual valuation conditional on this being greater than the seller's virtual cost, which corresponds to optimal dissolution with ironing parameters $z_i = \overline{z}$ and $z_j = \underline{z}$ for $j \neq i$. The intervals of highest seller types $[(\psi^S_{\alpha,i})^{-1}(\overline{z}), 1]$ and of lowest buyer types $[0, (\psi^B_{\alpha,j})^{-1}(\underline{z})]$ are excluded from trade and are thus worst off, with $\omega_{\alpha,i}(\overline{z})$ and $\omega_{\alpha,j}(\underline{z})$, respectively, being the critical among those worst-off types.

$e - z$) for all z. For the optimal dissolution mechanism of Corollary 1, this symmetry property implies that $z_1^* = z_2^* = (1 - e)/2$ for all $r_1 \in [\underline{r}_1, \overline{r}_1]$. Hence, if the optimal ownership structures are such that $z_1^* = z_2^*$, then the entire interval $[\underline{r}_1, \overline{r}_1]$ is optimal.

Condition (15) of Theorem 2 implies that it is optimal to concentrate ownership at agent i if and only if

$$E\big[v_i\big(\omega_{\alpha,i}(\overline{z}),\mathbf{X}_{-i}\big)\big]\leq E\big[v_j\big(\omega_{\alpha,j}(\underline{z}),\mathbf{X}_{-j}\big)\big] \quad \text{for all } j\neq i. \tag{18}$$

That is, the interim expected valuation of the seller's critical excluded type can be no larger than that of each buyer's critical excluded type. Moreover, if (18) holds, then no other ownership structure is optimal (i.e., $R^*(\alpha)$ is a singleton): for any ownership structure with $r_i<1$ and $r_j>0$ for some $j\neq i$, optimal dissolution requires ironing parameters $z_i^*<\overline{z}$ and $z_j^*>\underline{z}$, resulting in a strictly lower (higher) critical worst-off type for i (j), implying that condition (15) cannot hold.

Using the definition of $\omega_{\alpha,i}$ in (12) as well as $\overline{\psi}_{\alpha,i}(X_i,\overline{z})=\min\{\psi^S_{\alpha,i}(X_i),\overline{z}\}$ and $\overline{\psi}_{\alpha,j}(X_j,\underline{z})=\max\{\psi^B_{\alpha,j}(X_j),\underline{z}\}$, one can show that

$$E\big[v_i\big(\omega_{\alpha,i}(\overline{z}),\mathbf{X}_{-i}\big)\big]=E\left[v_i(\mathbf{X})+\min\left\{\frac{F_i(X_i)}{f_i(X_i)},\frac{\overline{z}-X_i+\eta(X_i)}{\alpha}\right\}\right]$$

$$E\big[v_j\big(\omega_{\alpha,j}(\underline{z}),\mathbf{X}_{-j}\big)\big]=E\left[v_j(\mathbf{X})-\min\left\{\frac{1-F_j(X_j)}{f_j(X_j)},\frac{X_j-\eta(X_j)-\underline{z}}{\alpha}\right\}\right].$$

Hence, (18) can be rewritten as (19) below to obtain the following result.

Corollary 5. *The extreme ownership structure with $r_i=1$ is optimal if and only if*

$$E\left[v_i(\mathbf{X})+\min\left\{\frac{F_i(X_i)}{f_i(X_i)},\frac{\overline{z}-X_i+\eta(X_i)}{\alpha}\right\}\right] \leq E\left[v_j(\mathbf{X})-\min\left\{\frac{1-F_j(X_j)}{f_j(X_j)},\frac{X_j-\eta(X_j)-\underline{z}}{\alpha}\right\}\right] \tag{19}$$

for all $j\neq i$. Moreover, if (19) holds, then the optimal ownership structure is unique and assigns sole ownership to the agent with the lowest ex ante expected valuation, i.e., $E[v_i(\mathbf{X})]\leq E[v_j(\mathbf{X})]$ for all $j\neq i$.

Corollary 5 gives a precise condition for extreme ownership to be optimal. In addition, if extreme ownership is optimal, sole ownership is assigned to the agent who is weakest in terms of ex ante expected valuations. Intuitively, the critical among the seller's excluded highest types has to be sufficiently low and the critical among each buyer's excluded lowest types has to be sufficiently high for the former to have a lower valuation than the latter. This requires that sufficiently many types on both sides of the market are excluded: the seller's probability density has to be relatively low for high types (so that the virtual cost is above $\overline{z}$ for many) and each buyer's probability density has to be relatively low for low types (so that the virtual valuation is below $\underline{z}$ for many). Hence, the agent with the lowest ex ante expected valuation must be the seller whenever an extreme ownership structure is optimal.

We now show that optimality of extreme ownership is a robust feature of partnership models insofar as, for any $\alpha>0$ and any valuation interdependence η, there are type distributions such that condition (19) is satisfied.

PROPOSITION 2. *Suppose $\alpha > 0$. For any i, there exist type distributions such that the extreme ownership structure with $r_i = 1$ is optimal. More generally, for any nonempty set of agents $\mathcal{I}$, there exist type distributions such that for all $i \in \mathcal{I}$ and $j \notin \mathcal{I}$, $F_i(x) > F_j(x)$ for all $x \in (0, 1)$ and all optimal ownership structures have $r_i > 0$ and $r_j = 0$.*

The proof makes use of the family of distributions considered in Example 1: if the type of i is drawn from F_2 and the type of each $j \neq i$ is drawn from F_1 specified in (17), then (19) holds if the parameter b is sufficiently large. The second part of Proposition 2 then follows from the insight that if (19) holds and one adds replicas of agent i, the strong agents' optimal shares remain zero. Whenever the revenue weight is positive, there thus are type distributions such that it is optimal to assign zero shares to a group of agents who are strong in the sense of first-order stochastic dominance.

We next consider the role of the revenue weight. Consider an extreme ownership structure and an optimal dissolution mechanism for some given α. Now suppose α increases, which increases the distortions in the optimal allocation rule. In turn, the length of the intervals of worst-off seller types at the top and buyer types at the bottom increases. The critical types move further away from the boundary, which decreases the seller's and increases the buyers' valuations at the critical worst-off types. This is why the left-hand side of (19) is decreasing and the right-hand side is increasing in α. Thus, if an extreme ownership structure is optimal for some value of α, then it remains optimal for any larger value of α, as stated in part (i) of Proposition 3 below. Conversely, if extreme ownership is suboptimal for some α, then it must also be suboptimal for any smaller α.

This implies that the condition for optimality of extreme ownership is most restrictive for $\alpha = 0$. Indeed, under the efficient allocation rule, the seller's unique worst-off type is 1 and the buyers' unique worst-off types are all 0. As long as an agent's own type has a larger effect on his valuation than the other agents' types, it is impossible that the valuation of type 1 of the seller is less than that of type 0 of each buyer. Hence, in this case, extreme ownership is always suboptimal for sufficiently small values of α, as established in part (ii) of Proposition 3. However, if the other agents' types have a large impact, which requires strongly negative interdependence of valuations, then extreme ownership may be optimal even for $\alpha = 0$. Part (iii) of Proposition 3 provides the precise condition under which this happens.

PROPOSITION 3. *(i) If the extreme ownership structure with $r_i = 1$ is optimal for revenue weight $\hat{\alpha}$, then it is also optimal for all revenue weights $\alpha > \hat{\alpha}$.*

(ii) If $\eta'(x) \geq -1$ for all x, then there is an $\tilde{\alpha} > 0$ such that the optimal ownership structures are non-extreme (i.e., $r_i < 1$ for all i) for all revenue weights $\alpha \in [0, \tilde{\alpha})$.

(iii) The extreme ownership structure with $r_i = 1$ is optimal for all $\alpha \in [0, 1]$ if and only if $\eta'(x) < -1$ for some x and $E[\eta(X_i) - \eta(X_j)] \geq 1$ for all $j \neq i$.

For any positive revenue weight, there exist distributions such that extreme ownership is optimal (Proposition 2), and if extreme ownership is optimal, it concentrates all property rights at the agent with the lowest expected valuation (Corollary 5). In contrast, for $\alpha = 0$, extreme ownership is never optimal unless interdependence is strongly negative. Moreover, as the following result shows, if the interdependence is nonnegative and type distributions are ordered by stochastic dominance, the largest share is optimally awarded to the agent with the highest expected valuation.[22]

Proposition 4. *Consider the ownership structure* $\{\mathbf{r}^0\} = \lim_{\alpha \to 0} R^*(\alpha)$ *that maximizes revenue under ex post efficiency and suppose* $\eta'(x) \geq 0$ *for all* x. *If* $F_i(x) > F_j(x)$ *for all* $x \in (0, 1)$ *for two agents* $i, j \in \mathcal{N}$, *then* $0 \leq r_i^0 < r_j^0 < 1$ *or* $r_i^0 = r_j^0 = 0$.

Under ex post efficiency, $F_i > F_j$ implies that i's unique worst-off type is strictly higher than j's when their ownership shares are equal (as for any given type, i is less likely to obtain the object than j). Moreover, under positive interdependence, for the same type x, i's interim valuation $E[v_i(x, \mathbf{X}_{-i})]$ is greater than j's interim valuation $E[v_j(x, \mathbf{X}_{-j})]$. Hence, to equalize interim valuations of worst-off types, a smaller share has to be assigned to agent i. In contrast, under negative interdependence, $E[v_i(x, \mathbf{X}_{-i})] < E[v_j(x, \mathbf{X}_{-j})]$, which may outweigh the asymmetry in worst-off types and induce $r_i^0 > r_j^0$. This happens, for example, in the case of extreme ownership considered in Proposition 3(iii).

5. Extension: Ownership structures chosen by agents

So far we have studied optimal ownership structures and dissolution mechanisms for an exogenously given revenue weight in the designer's objective. We now address the closely related yet distinct question of what initial ownership structures the agents would choose ex ante, that is, before any private information is realized, anticipating the costs associated with subsequent reallocation of property rights due to incentive compatibility and individual rationality constraints and, possibly, additional costs (such as legal costs).

Specifically, we now assume that there are two stages. In the first stage, the agents choose the ownership structure, anticipating that reallocating shares in the second stage will be possible only if the mechanism they use generates revenue that covers the known fixed cost K. We allow for arbitrary K, where $K = 0$ corresponds to requiring budget balance, $K > 0$ can be interpreted as costs caused by the reallocation procedure, and $K < 0$ describes cases where trade is to some extent subsidized.[23] Because there is no private information in the first stage, we assume that agents agree on an ownership structure that maximizes their joint expected surplus. In the second stage, agents privately learn their types and then either reallocate property rights using an individually rational mechanism that generates revenue K or stick to their initial shares.

[22] For the special case of private values where $\eta(x) = 0$, this was observed by Che (2006) and Figueroa and Skreta (2012).

[23] We assume that K does not depend on the ownership structure. That is, if K is the fee charged by the designer of the dissolution mechanism, we assume that this designer cannot price-discriminate across partnerships. While interesting, we leave for future research the analysis of ownership structures chosen by agents who anticipate that a price-discriminating designer wants to maximize revenue.

5.1 *Characterization*

Consider the second stage. Having privately learned their types, agents may now reallocate property rights or stick with the initial shares $\mathbf{r}$. In case of reallocation, the agents optimally use a dissolution mechanism $(\mathbf{s}, \mathbf{t})$ that maximizes surplus subject to raising revenue K. That is, they solve the problem

$$\max_{\mathbf{s},\mathbf{t}} \sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})s_i(\mathbf{X})\big] \quad \text{s.t.} \sum_{i\in\mathcal{N}} E\big[t_i(\mathbf{X})\big] \geq K, \text{(IC), and (IR).} \tag{20}$$

Whether this problem has a solution, which means that reallocation is feasible, depends on the cost K and property rights $\mathbf{r}$ chosen in the first stage.

For any given α and $\mathbf{r}$, let $W_0(\alpha, \mathbf{r}) := \sum_{i\in\mathcal{N}} E[s^{\mathbf{r}}_{\alpha,i}(\mathbf{X})v_i(\mathbf{X})]$ be the surplus and let $W_1(\alpha, \mathbf{r}) := \sum_{i\in\mathcal{N}} E[t^{\mathbf{r}}_{\alpha,i}(\mathbf{X})]$ be the revenue generated by any optimal dissolution mechanism $(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$.[24] Because $\alpha = 1$ means maximizing revenue, $W_1(1, \mathbf{r})$ is the highest revenue that can be generated given ownership structure $\mathbf{r}$.

If $W_1(1, \mathbf{r}) \geq K$, reallocation is thus feasible. It is easy to see that the solutions to problem (20) are then the optimal dissolution mechanisms $(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$ for the revenue weight $\alpha = \alpha^*(K, \mathbf{r})$,[25] where

$$\alpha^*(K, \mathbf{r}) := \min\big\{\alpha \in [0, 1] : W_1(\alpha, \mathbf{r}) \geq K\big\}.$$

If $\alpha^*(K, \mathbf{r}) = 0$, the revenue constraint in (20) is not binding and the resulting allocation is ex post efficient. Otherwise, the revenue constraint is binding and the second-best reallocation is given by Theorem 1 for $\alpha = \alpha^*(K, \mathbf{r})$. In any case, the resulting payoff to the agents is $W_0(\alpha^*(K, \mathbf{r}), \mathbf{r}) - K$. Note that agents always prefer this over sticking to the initial shares $\mathbf{r}$ and avoiding cost K because the reallocation mechanism satisfies (IR). In contrast, if $W_1(1, \mathbf{r}) < K$, reallocating property rights at cost K is not feasible.

In the first stage, agents anticipate the effect of the initial shares on the second stage. The sum of the agents' payoffs from agreeing on $\mathbf{r}$ initially is

$$P(K, \mathbf{r}) := \begin{cases} W_0\big(\alpha^*(K, \mathbf{r}), \mathbf{r}\big) - K & \text{if } W_1(1, \mathbf{r}) \geq K \\ \sum_{i\in\mathcal{N}} r_i E\big[v_i(\mathbf{X})\big] & \text{if } W_1(1, \mathbf{r}) < K, \end{cases}$$

i.e., the value of the final allocation minus the cost K in case of reallocation. Bargaining efficiently prior to the realization of any private information, the agents will agree on an ownership structure that solves $\max_{\mathbf{r}\in\Delta^{n-1}} P(K, \mathbf{r})$. Let

$$R(K) := \operatorname*{argmax}_{\mathbf{r}\in\Delta^{n-1}} P(K, \mathbf{r})$$

denote the set of all *agent-optimal ownership structures* given reallocation cost K.

[24] Surplus and revenue are the same across all optimal dissolution mechanisms because the interim expected shares $S^{\mathbf{r}}_{\alpha,i}(x_i)$ are uniquely pinned down by Theorem 1 and, e.g., using Lemma 1, one can show that $W_0(\mathbf{s}, \mathbf{t}) = \sum_{i\in\mathcal{N}} E[(S_i(X_i) - r_i)(X_i - \eta(X_i)) + r_i v_i(\mathbf{X})]$.

[25] Denoting by $\lambda \geq 0$ the Lagrange multiplier on the revenue constraint, the Lagrangian for problem (20) is $\sum_{i\in\mathcal{N}} E[v_i(\mathbf{X})s_i(\mathbf{X})] - \lambda(K - \sum_{i\in\mathcal{N}} E[t_i(\mathbf{X})]) = (1 + \lambda)W_{\lambda/(1+\lambda)}(\mathbf{s}, \mathbf{t}) - \lambda K$, so the statement follows by setting $\alpha = \lambda/(1 + \lambda)$.

In the following discussion, we distinguish between three different ranges of K, which we characterize using the optimal ownership structures $R^*(\alpha)$ for fixed α from Section 4. Define $K^1 := W_1(1, \mathbf{r}^1)$ for $\mathbf{r}^1 \in R^*(1)$ to be the highest revenue that can be generated under any ownership structure. Moreover, consider the ownership structure $\{\mathbf{r}^0\} = \lim_{\alpha \to 0} R^*(\alpha)$ that maximizes revenue under ex post efficiency (cf. Corollary 2) and let $K^0 := W_1(0, \mathbf{r}^0)$ denote the associated revenue.

If $K > K^1$, reallocation costs are prohibitively large: even if agents were to choose revenue-maximizing initial shares, reallocation would still not be feasible. Hence, the agents choose an ownership structure that maximizes the expected surplus in the absence of reallocation, resulting in joint payoff $\max_{i \in \mathcal{N}} E[v_i(\mathbf{X})]$.

If $K \leq K^1$, although reallocating shares is feasible provided that the agents choose the initial property rights appropriately, doing so is in the agents' interest if and only if

$$\max_{\mathbf{r} \in \Delta^{n-1}} W_0\big(\alpha^*(K, \mathbf{r}), \mathbf{r}\big) - K \geq \max_{i \in \mathcal{N}} E\big[v_i(\mathbf{X})\big], \tag{21}$$

i.e., if and only if the best ownership structures that render reallocation feasible yield a higher payoff than an ownership structure that is optimal without reallocation. Note that the left-hand side of (21) is strictly decreasing in K since $W_0(\alpha^*(K, \mathbf{r}), \mathbf{r})$ is nonincreasing in K for any fixed $\mathbf{r}$. Consequently, there is a unique critical cost level $\hat{K} \leq K^1$ such that (21) is satisfied if and only if $K \leq \hat{K}$. For any $K > \hat{K}$, the agents abstain from reallocation and $R(K) = \operatorname{argmax}_{\mathbf{r} \in \Delta^{n-1}} \sum_{i \in \mathcal{N}} r_i E[v_i(\mathbf{X})]$.[26]

If $K \leq \hat{K}$ and $K \leq K^0$, generating revenue K and reallocating ex post efficiently is feasible under appropriately chosen initial ownership. As doing so results in the highest possible surplus, agents choose an ownership structure in $\{\mathbf{r} \in \Delta^{n-1} : W_1(0, \mathbf{r}) \geq K\}$ and then use an ex post efficient dissolution mechanism to reallocate.

If $K \leq \hat{K}$ and $K > K^0$, ex post efficient reallocation is not feasible under any initial ownership. In this case, agents choose an ownership structure that maximizes surplus given that they reallocate using an optimal dissolution mechanism with $\alpha > 0$ that generates revenue K. The following proposition provides a summary.

Proposition 5. *There is a critical cost level $\hat{K} \leq K^1$ such that the agent-optimal ownership structures $R(K)$ are as follows:*

(i) If $K \leq \min\{K^0, \hat{K}\}$, then $R(K) = \{\mathbf{r} \in \Delta^{n-1} : W_1(0, \mathbf{r}) \geq K\}$ and agents reallocate using an ex post efficient dissolution mechanism.

(ii) If $K \in (\min\{K^0, \hat{K}\}, \hat{K}]$, then $R(K) = \operatorname{argmax}_{\mathbf{r} \in \Delta^{n-1}} W_0(\alpha^*(K, \mathbf{r}), \mathbf{r})$ and agents reallocate using an optimal dissolution mechanism for $\alpha^*(K, \mathbf{r}_K) > 0$, $\mathbf{r}_K \in R(K)$.

(iii) If $K > \hat{K}$, then $R(K) = \operatorname{argmax}_{\mathbf{r} \in \Delta^{n-1}} \sum_{i \in \mathcal{N}} r_i E[v_i(\mathbf{X})]$ and there is no reallocation.

There are two slightly different reasons why agents do not reallocate in part (iii) of Proposition 5: for $K > K^1$, reallocation is not feasible for any ownership structure, while for $K \in (\hat{K}, K^1]$, there are ownership structures that would permit reallocation but they are dominated by ownership structures that are optimal without reallocation.

[26]Note that this does not require that agents can commit to abstain from reallocating: when (21) is violated, there is no individually rational dissolution mechanism that yields revenue K under $\mathbf{r} \in R(K)$.

5.2 *Illustration and discussion*

We now briefly illustrate Proposition 5 and its implications, beginning with the case of identical type distributions. If $F_i = F$ for all $i \in \mathcal{N}$, the expected surplus without reallocation is independent of the choice of the initial ownership structure. As any individually rational dissolution mechanism guarantees a higher payoff than sticking to the initial shares, agents therefore never gain from choosing an ownership structure that prevents reallocation, implying $\hat{K} = K^1$. From Corollary 4, we know that equal ownership $\mathbf{r}_E := (1/n, \dots, 1/n)$ is both an optimal ownership structure for all α and maximizes revenue under ex post efficiency. Hence, for all K, it is optimal for agents to choose $\mathbf{r}_E$ and reallocate whenever feasible.

While equal ownership is always contained in $R(K)$, $R(K)$ is not a singleton unless $K = K^0$. For $K < K^0$, $R(K)$ contains all sufficiently symmetric ownership structures that allow for ex post efficient reallocation, and the size of $R(K)$ decreases in K as the revenue constraint tightens. In contrast, for $K \in (K^0, K^1]$, we have $R(K) = R^*(\alpha^*(K, \mathbf{r}_E))$ and the size of $R(K)$ increases in K because $R^*(\alpha)$ is increasing in α (Corollary 4): the growing distortions in the allocation again allow for more flexibility in the optimal choice of ownership. At $K = K^1$, there is a discontinuity since $R(K) = \Delta^{n-1}$ for all $K > K^1$.

Figure 6 illustrates how the set of agent-optimal shares for agent 1 $R_1(K) = \{r_1 : (r_1, r_2) \in R(K)\}$, changes with K for a bilateral partnership with uniformly distributed types and $\eta(x) = 0.6$. Note that under this specification with positive interdependence of valuations, $K^0 < 0$, i.e., ex post efficient dissolution is impossible under any initial ownership structure even if $K = 0$.

Under identically distributed types, equal initial ownership is robust in the sense that it is always agent optimal, independently of K. Equal shares are hence also an optimal choice should there be some uncertainty at the ex ante stage regarding the size of K. In contrast, how much flexibility the agents have in choosing the initial ownership, i.e., the size of $R(K)$, depends nonmonotonically on K.

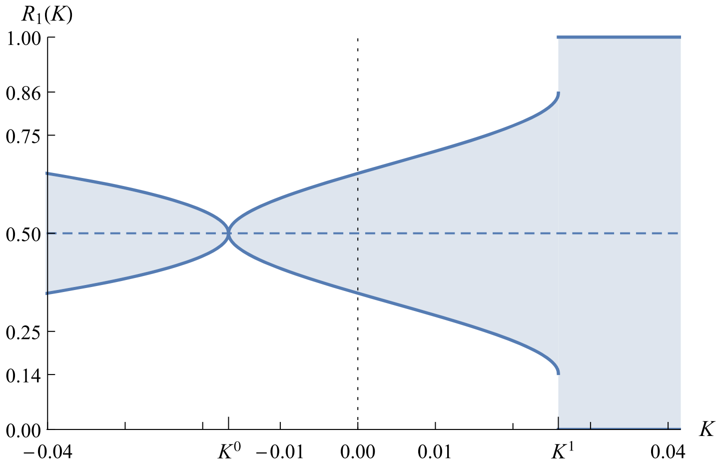


FIGURE 6. Set of agent-optimal r_1 as a function of K for $n = 2$, $\eta(x) = 0.6x$, and with identically and uniformly distributed types, where $K^0 = -\frac{1}{60}$ and $\hat{K} = K^1 = \frac{19}{735}$.

To illustrate the effects of non-identical type distributions, we turn to a specific parametrization.

Example 2. Consider a bilateral partnership with private values ($\eta(x) = 0$), assuming that type distributions are $F_1(x) = x^b$ and $F_2(x) = 1 - (1-x)^b$ for some $b > 1$ as in (17) in Example 1. Numerical computations reveal that $\hat{K}$ decreases in b, satisfying $\hat{K} = K^1$ for $b < 1.116$, $\hat{K} \in (K^0, K^1)$ for $b \in (1.116, 1.579)$, and $\hat{K} \leq K^0$ for $b > 1.579$. Thus, for b sufficiently close to 1, the agents choose initial shares that induce reallocation whenever feasible (as with identically distributed types). In contrast, for b sufficiently large, $\hat{K} \leq K^0$ such that there is either ex post efficient reallocation or no reallocation.

Assume now that $b = 1.2$, resulting in $K^0 = 0.0781 < \hat{K} = 0.1068 < K^1 = 0.1131$. Figure 7 displays the set of agent-optimal initial shares $R_1(K)$ for agent 1. Similar to the case of identically distributed types, for $K < K^0$, $R_1(K)$ is not a singleton and its size is decreasing in K. The revenue-maximizing share under ex post efficiency, which is characterized in Corollary 2 and always contained in $R_1(K)$, is equal to 0.56, favoring the stronger agent. For all $K > K^0$, however, $R_1(K)$ is a singleton: it is first decreasing in K until it reaches 0.34 at $\hat{K}$, favoring the weaker agent, and then jumps to 1 (sole ownership by the stronger agent), even though, for $K \in (\hat{K}, K^1]$, reallocation would still be feasible (the heavy dashed line indicates the optimal share for that purpose). For $K \in (K^0, \hat{K}]$, $R(K)$ coincides with the optimal ownership structure $R^*(\alpha)$ for the unique α that yields revenue K, where $R_1^*(\alpha)$ is depicted in Figure 4 of Section 4. As the revenue constraint tightens, the weaker agent has to be favored to equalize critical worst-off types of the two agents (which is required for optimal reallocation). For $K \in (\hat{K}, K^1]$, however, these distortions are too costly, making reallocation undesirable. ◊

The parameter K can equivalently be interpreted as measuring costly frictions for reallocating property rights due to legal uncertainty, lengthy and costly legal procedures,

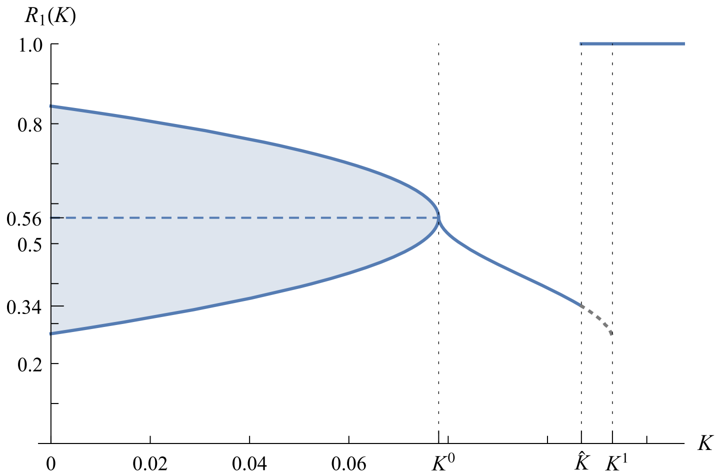


Figure 7. Set of agent-optimal r_1 as a function of K for $b = 1.2$ in Example 2, where $K^0 = 0.0781$, $\hat{K} = 0.1068$, and $K^1 = 0.1131$.

or simply the level of corruption. Our findings suggest that such frictions have a profound impact on the structure of businesses, which in turn lead to indirect and additional costs to society. Moderate to relatively large frictions limit the partners' ability to restructure their business (case (ii) of Proposition 5) and can even prevent them from doing so altogether (case (iii) of Proposition 5). Unless partners are ex ante symmetric, a small increase in frictions K from below to above $\hat{K}$ can also drastically change the initially chosen ownership structure. For the example displayed in Figure 7, the stronger partner's position changes from owning a minority share to sole ownership. Moreover, even for relatively small frictions (case (i) of Proposition 5) that allow for agent-optimal ownership structures that permit efficient restructuring, increasing K has the effect that the set of optimal ownership structures shrinks. This is costly to the extent that it constrains the partners' flexibility in setting up initial ownership structures, which is valuable if incentive reasons as in the work of Grossman and Hart (1986) and Hart and Moore (1990) or other reasons that are outside our model constrain the set of desirable ownership structures.

6. Conclusions

We fully characterize the optimal dissolution mechanisms for a general partnership model. Beyond identifying optimal breakup procedures for business partnerships, our results broadly provide guidance for the design of trading platforms for homogeneous goods with arbitrary initial endowments. To curb information rents, the optimal mechanisms allocate based on uniquely determined ironed virtual type functions, which for each agent are constant for a (typically interior) range of types while corresponding to virtual costs and valuations, respectively, for lower and higher types. Ceteris paribus, the allocation is biased toward agents with larger initial shares through an upward shift of the range where the virtual type is constant or through favorable treatment in case of a tie. Ex ante heterogeneity in type distributions directly translates to asymmetric virtual types, such that, for example, stronger agents in terms of (reverse) hazard rate dominance are discriminated against.

We use the optimal dissolution mechanisms to derive the optimal ownership structures. For identically distributed types, we show that symmetric (extreme) ownership is always (never) optimal and that the set of optimal ownership structures expands as rent extraction by the designer becomes more important. In contrast, with heterogeneously distributed types, optimal ownership structures are typically asymmetric and vary with the weight the designer puts on rent extraction. Even extreme ownership structures are optimal when heterogeneity and the motive of rent extraction are both sufficiently strong. Interestingly, whenever extreme ownership is optimal, sole ownership is assigned to the agent with the lowest ex ante expected valuation.

There are several promising avenues for future research that emerge from this paper. For example, one could relax the assumption that payoffs are linear in types and shares by allowing for complementarities across units an individual agent receives. In a related vein, one could extend the setup to include public goods such as the classic problem of an upstream firm that wants to set up a plant and downstream residents who want clean water, with the residents' property rights representing the probability that they would prevail in court. Finally, one could consider agents' incentives to invest. Assuming an incomplete contracting environment in which an agent's investment improves his type distribution and type realizations are private information, this would permit an incomplete information counterpart to the canonical framework in the property rights literature. While some of these questions have been tackled in the previous literature under ex post efficiency, our methods may prove helpful in solving problems like these more generally.

Appendix A: Proof of Theorem 1

We first prove the second part of the theorem, i.e., the statements after the first line, taking as given that the statement in the first line is true. Then we prove the statement in the first line.

Suppose there exists a unique $\mathbf{z} \in [\underline{z}, \overline{z}]^n$ such that $\mathbf{r} \in \Gamma_n(\mathbf{z})$, as stated in the first line of the theorem. It follows that for this unique $\mathbf{z}$ and some tie-breaking rule $\mathbf{a}$, the ironed virtual type allocation rule $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ and the critical types $(\omega_{\alpha,1}(z_1), \dots, \omega_{\alpha,n}(z_n))$ constitute a saddle point satisfying (9) and (10), making $\mathbf{s}^{\mathbf{z},\mathbf{a}}$ an optimal allocation rule consistent with (8).

Note that by restricting the definition of Γ_n and the statement of Theorem 1 to $z_i \in [\underline{z}, \overline{z}] = [\psi^S_{\alpha,i}(0), \psi^B_{\alpha,i}(1)] \subset [\psi^B_{\alpha,i}(0), \psi^S_{\alpha,i}(1)]$, we have confined attention to critical types $\omega^*_i \in [\omega_{\alpha,i}(\underline{z}), \omega_{\alpha,i}(\overline{z})] \subset [0, 1]$. This restriction is without loss when we are looking for optimal allocation rules. As becomes apparent below, for $\mathbf{z} = \Gamma_n^{-1}(\mathbf{r})$ we have $z_i = \underline{z}$ if and only if $r_i = 0$. Hence for all $\mathbf{r}$, $z_j > \underline{z}$ for at least one j. Accordingly, $S^{\mathbf{z},\mathbf{a}}_i(\omega_{\alpha,i}(z_i)) = 0$ for all $z_i \leq \underline{z}$. If there is a saddle point involving critical type $\omega^*_i = \omega_{\alpha,i}(\underline{z})$, then there is also a saddle point for each $\omega^*_i \in [0, \omega_{\alpha,i}(\underline{z}))$. However, all these saddle points are equivalent in terms of the implied allocation rule $\mathbf{s}^*$ and i's worst-off types $\Omega_i(\mathbf{s}^*) = \{x_i : S^*_i(x_i) = 0\} = [0, (\psi^B_{\alpha,i})^{-1}(\underline{z})]$. A similar line of argument can be invoked for $z_i \geq \overline{z}$, which only occurs if $r_i = 1$.

From the preceding paragraph, we conclude that whereas there can be multiple saddle points satisfying (9) and (10), the corresponding allocation rule $\mathbf{s}^*$ is unique up to the tie-breaking rule and can be defined as allocating to the greatest ironed virtual type for ironing parameters $\mathbf{z} = \Gamma_n^{-1}(\mathbf{r})$. Whereas the exact specification of the tie-breaking rule may differ, all optimal allocation rules result in the same interim expected shares, which in turn pin down interim expected payments, as explained in the main text.

It remains to prove the first line of the theorem. For any $A \subseteq [\underline{z}, \overline{z}]^n$, let $\Gamma_n(A) = \{\mathbf{y} \in [0, 1]^n : \mathbf{y} \in \Gamma_n(\mathbf{z}) \text{ for some } \mathbf{z} \in A\}$ denote the image of A under Γ_n. To prove that for each $\mathbf{r} \in \Delta^{n-1}$, there is a unique $\mathbf{z} \in [\underline{z}, \overline{z}]^n$ such that $\mathbf{r} \in \Gamma_n(\mathbf{z})$, we show that Γ_n has the following two properties.

Property 1. For every $\mathbf{y} \in \Gamma_n([\underline{z}, \overline{z}]^n)$, there is a unique $\mathbf{z}$ such that $\mathbf{y} \in \Gamma_n(\mathbf{z})$.

Property 2. We have $\Delta^{n-1} \subset \Gamma_n([\underline{z}, \overline{z}]^n)$.

Property 1 implies the uniqueness part. It says that every point in the image of Γ_n corresponds to exactly one $\mathbf{z}$. Put differently, the inverse correspondence $\Gamma_n^{-1}(\mathbf{y}) := \{\mathbf{z} \in [\underline{z}, \overline{z}]^n : \mathbf{y} \in \Gamma_n(\mathbf{z})\}$ is singleton-valued for all $\mathbf{y} \in \Gamma_n([\underline{z}, \overline{z}]^n)$. Property 2 implies the existence part. It says that the image of Γ_n contains the standard simplex Δ^{n-1}.

The proof proceeds as follows. After some definitions and preliminary results in Appendix A.1, we show in Appendix A.2 that Property 1 and Property 2 hold for $n = 2$. In Appendix A.3, we first uncover the recursive structure of Γ_n. This then allows us to prove by induction that the two properties hold for all n, using $n = 2$ as the base case.

A.1 *Preliminaries*

Recall the virtual cost distributions $G^S_{\alpha,i}$ and the virtual valuation distributions $G^B_{\alpha,i}$ defined in Section 2. In what follows, we drop the subscript α and write G^S_i and G^B_i instead. Suppose $z_i > z_j$. Then agent i's critical type $\omega_{\alpha,i}(z_i)$ expects that his ironed virtual type $\overline{\psi}_{\alpha,i}(\omega_{\alpha,i}(z_i), z_i) = z_i$ is greater than the ironed virtual type $\overline{\psi}_{\alpha,j}(x_j, z_j)$ of agent j with probability $G^B_j(z_i)$. Similarly, the critical type $\omega_{\alpha,j}(z_j)$ of agent j expects to have a higher ironed virtual type than agent i with probability $G^S_i(z_j)$. Note that G^S_i and G^B_i are strictly increasing, $G^S_i(z_i) < G^B_i(z_i)$ for all $z_i \in [\underline{z}, \overline{z}]$, $G^S_i(\underline{z}) = 0$, and $G^B_i(\overline{z}) = 1$.

Consider agent i and a vector of ironing parameters $\mathbf{z}$. Let the set of agents other than i that have an ironing parameter less than z_i be denoted by $\mathcal{L}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j < z_i\}$. Similarly, let the sets of agents with ironing parameter equal to and greater than z_i be denoted by $\mathcal{E}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j = z_i\}$ and $\mathcal{G}_i(\mathbf{z}) := \{j : j \neq i \text{ and } z_j > z_i\}$, respectively. If $\mathcal{E}_i(\mathbf{z}) \neq \varnothing$ for some i, ties in terms of ironed virtual type have strictly positive probability.

Suppose ties are broken hierarchically according to h. For each agent i, let $\underline{\mathcal{E}}_i(\mathbf{z}, h) := \{j \in \mathcal{E}_i(\mathbf{z}) : h(j) < h(i)\}$ and $\overline{\mathcal{E}}_i(\mathbf{z}, h) := \{j \in \mathcal{E}_i(\mathbf{z}) : h(j) > h(i)\}$ denote the set of other agents with the same ironing parameter against whom agent i wins and loses ties, respectively. Hence, under hierarchy h, the expected share of critical type $\omega_{\alpha,i}(z_i)$ of agent i is

$$S_i\big(\omega_{\alpha,i}(z_i)\big) = p_i(\mathbf{z}, h) := \prod_{j \in \mathcal{L}_i(\mathbf{z}) \cup \underline{\mathcal{E}}_i(\mathbf{z},h)} G^B_j(z_i) \prod_{k \in \mathcal{G}_i(\mathbf{z}) \cup \overline{\mathcal{E}}_i(\mathbf{z},h)} G^S_k(z_i).$$

Let $\mathbf{p}(\mathbf{z}, h) := (p_1(\mathbf{z}, h), \dots, p_n(\mathbf{z}, h))$. The outcome $(S_1^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,1}(z_1)), \dots, S_n^{\mathbf{z},\mathbf{a}}(\omega_{\alpha,n}(z_n)))$ of every split hierarchical tie-breaking rule $\mathbf{a}$ is equal to a convex combination of $\mathbf{p}(\mathbf{z}, h)$ for different hierarchies $h \in H$. Consequently, the set of all possible expected shares given $\mathbf{z}$ is equal to the convex hull of the expected shares under fixed hierarchies, i.e.,

$$\Gamma_n(\mathbf{z}) = \operatorname{Conv}\big(\{\mathbf{p}(\mathbf{z}, h) : h \in H\}\big).$$

Note that, depending on $\mathbf{z}$, we may have $\mathbf{p}(\mathbf{z}, h_1) = \mathbf{p}(\mathbf{z}, h_2)$ for some $h_1 \neq h_2$. In particular, if all n elements of $\mathbf{z}$ are distinct, i.e., $\mathcal{E}_i(\mathbf{z}) = \varnothing$ for all i, then tie-breaking has no bite and all $\mathbf{p}(\mathbf{z}, h)$ coincide. In this case, $\Gamma_n(\mathbf{z})$ is a singleton. By contrast, if $\mathbf{z}$ is such that $z_i = z$ for all i, i.e., $\mathcal{L}_i(\mathbf{z}) = \mathcal{G}_i(\mathbf{z}) = \varnothing$, then all $n!$ points $\mathbf{p}(\mathbf{z}, h)$ are distinct extreme points of the convex hull $\Gamma_n(\mathbf{z})$. In general, if $\mathbf{z}$ is such that its elements take $k \leq n$ distinct values $z^1, \dots, z^k$, then $\Gamma_n(\mathbf{z})$ is equal to the convex hull of $\prod_{l=1}^k m_l!$ distinct extreme points, where m_l denotes the number of agents i with $z_i = z^l$.

Lemma 2. *The correspondence* Γ_n *has the following properties:*

(i) *For all* $\mathbf{z} \in [\underline{z}, \overline{z}]^n$, $\Gamma_n(\mathbf{z})$ *is nonempty and convex.*

(ii) *The correspondence* Γ_n *is upper hemicontinuous.*

Proof. Part (i) immediately follows from the discussion above. For part (ii), we have to show that for any two sequences $\mathbf{z}^q \to \mathbf{z}$ and $\mathbf{y}^q \to \mathbf{y}$ such that $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$, we have $\mathbf{y} \in \Gamma_n(\mathbf{z})$. Note that if $\mathbf{z}$ is such that all its components are distinct, $\Gamma_n(\mathbf{z})$ is a singleton that is continuous at $\mathbf{z}$. Moreover, if the sequence $\mathbf{z}^q \to \mathbf{z}$ is such that the sets of agents for which ironing parameters coincide stay the same over the whole sequence, $\Gamma_n(\mathbf{z}^q)$ and $\Gamma_n(\mathbf{z})$ are all equal to the convex hull of the same number of extreme points. Since these extreme points are continuous in $\mathbf{z}^q$, $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$ and $\mathbf{y}^q \to \mathbf{y}$ imply $\mathbf{y} \in \Gamma_n(\mathbf{z})$ in this case. Finally, suppose there are some i, j for which $z_i^q > z_j^q$ but $z_i = z_j$. Then, if $\mathbf{y}^q \to \mathbf{y}$ such that $\mathbf{y}^q \in \Gamma_n(\mathbf{z}^q)$, there exists a hierarchical tie-breaking rule for $\mathbf{z}$ where $h(i) > h(j)$ for all i, j with $z_i^q > z_j^q$ and $z_i = z_j$ that induces $\mathbf{y}$. Hence, $\mathbf{y} \in \Gamma_n(\mathbf{z})$. □

Partitioning the domain of Γ_n So as to study properties of the image of Γ_n, it proves useful to consider the following partition of the domain $[\underline{z}, \overline{z}]^n$. Define

$$\xi_n(z) := \{\mathbf{z} \in [z, \overline{z}]^n : z_i = z \text{ for at least one } i \in \mathcal{N}\}.$$

Note that $\xi_n(z) \cap \xi_n(z') = \varnothing$ for all $z \neq z'$. Moreover, $\bigcup_{z\in[\underline{z},\overline{z}]} \xi_n(z) = [\underline{z}, \overline{z}]^n$. Consequently, ξ_n represents a partition of the domain of Γ_n. In addition, define

$$O_n(z) := \Gamma_n(\xi_n(z)).$$

Hence, the image of Γ_n can be written as $\Gamma_n([\underline{z}, \overline{z}]^n) = \bigcup_{z\in[\underline{z},\overline{z}]} O_n(z)$. Below, we determine properties of $O_n(z)$ and their implications for $\Gamma_n([\underline{z}, \overline{z}]^n)$.

A.2 *Proof of Properties 1 and 2 for* $n = 2$

Suppose $n = 2$. There are only two possible hierarchies between two agents, i.e., $H = \{h_1, h_2\}$. Let h_1 (h_2) be the hierarchy where agent 1 (2) wins ties. Define $\boldsymbol{\zeta}_1(z) := (G_2^B(z), G_1^S(z))$ and $\boldsymbol{\zeta}_2(z) := (G_2^S(z), G_1^B(z))$. Hence, $\mathbf{p}(z, z, h_k) = \boldsymbol{\zeta}_k(z)$ for $k = 1, 2$. The general description of Γ_n in the preceding subsection implies

$$\Gamma_2(z_1, z_2) = \begin{cases} \left(G_2^B(z_1), G_1^S(z_2)\right) & \text{if } z_1 > z_2 \\ \mathrm{Conv}\left(\{\boldsymbol{\zeta}_1(z), \boldsymbol{\zeta}_2(z)\}\right) & \text{if } z_1 = z_2 = z \\ \left(G_2^S(z_1), G_1^B(z_2)\right) & \text{if } z_1 < z_2. \end{cases}$$

Suppose $z_1 = z_2 = z$. Geometrically, $\Gamma_2(z, z)$ is equal to all the points on the line segment from $\boldsymbol{\zeta}_1(z)$ to $\boldsymbol{\zeta}_2(z)$, i.e., all points in $\{a\boldsymbol{\zeta}_1(z) + (1 - a)\boldsymbol{\zeta}_2(z) : a \in [0, 1]\}$, where a is the share allocated to agent 1 in case of a tie (i.e., according to hierarchy h_1).

Now consider $O_2(z) = \Gamma_2(\xi_2(z))$ for some $z \in (\underline{z}, \overline{z})$. In Figure 8, $O_2(z)$ is represented by a polygonal chain. Geometrically, $O_2(z)$ consists of the line segment from $\boldsymbol{\zeta}_2(z)$ to

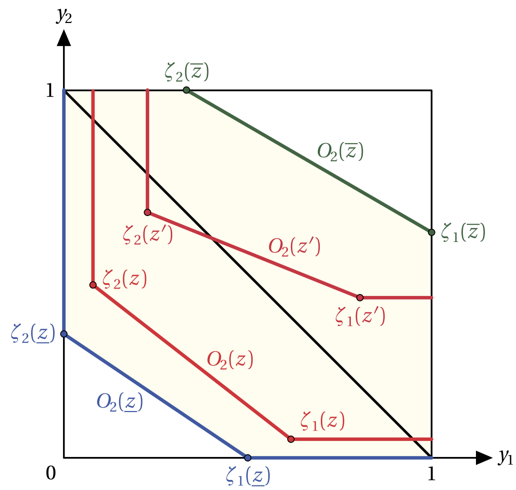


FIGURE 8. The image of Γ_2 and its components $O_2(\underline{z})$, $O_2(z)$, $O_2(z')$, and $O_2(\overline{z})$ for $\underline{z} < z < z' < \overline{z}$.

$\boldsymbol{\zeta}_1(z)$ that represents $\Gamma_2(z, z)$ with two line segments attached to its endpoints: a vertical line segment from $\boldsymbol{\zeta}_2(z)$ to $(G_2^S(z), 1)$ that represents $\Gamma_2(z, z_2)$ for all $z_2 \in (z, \overline{z}]$ and a horizontal line segment from $\boldsymbol{\zeta}_1(z)$ to $(1, G_1^S(z))$ that represents $\Gamma_2(z_1, z)$ for all $z_1 \in (z, \overline{z}]$.

Observe that both coordinates of the vertices $\boldsymbol{\zeta}_1(z)$ and $\boldsymbol{\zeta}_2(z)$ are continuous and strictly increasing in z. Hence, for $z' > z$, $O_2(z') \cap O_2(z) = \varnothing$ and $O_2(z')$ is further away from the origin than $O_2(z)$ (cf. Figure 8). Put differently, O_2 has the following monotonicity property: If $z' > z$, then for all $\mathbf{y}' \in O_2(z')$ and $\mathbf{y} \in O_2(z)$, we have $y_i' > y_i$ for at least one i.

Hence, for every $\mathbf{y} \in \Gamma_2([\underline{z}, \overline{z}]^2)$, there is a unique z such that $\mathbf{y} \in O_2(z)$. Moreover, note that for each $\mathbf{y} \in O_2(z)$, there is a unique point $(z_1, z_2) \in \xi_2(z)$ such that $\mathbf{y} \in \Gamma_2(z_1, z_2)$. Consequently, for every $\mathbf{y} \in \Gamma_2([\underline{z}, \overline{z}]^2)$, there is a unique $\mathbf{z} \in [\underline{z}, \overline{z}]^2$ such that $\mathbf{y} \in \Gamma_2(\mathbf{z})$, i.e., Property 1 holds for $n = 2$.

Consider $O_2(\underline{z})$ and note that $\boldsymbol{\zeta}_1(\underline{z}) = (G_2^B(\underline{z}), 0)$ and $\boldsymbol{\zeta}_2(\underline{z}) = (0, G_1^B(\underline{z}))$. Hence, the points $\mathbf{y} \in \Gamma_2(\underline{z}, \underline{z})$ all lie below the simplex Δ^1, which is represented by the black line segment from $(0, 1)$ to $(1, 0)$ in Figure 8. Moreover, the vertical and horizontal parts of $O_2(\underline{z})$ intersect with the simplex exactly at its boundary since $(G_2^S(\underline{z}), 1) = (0, 1)$ and $(1, G_1^S(\underline{z})) = (1, 0)$, respectively.

Let us increase z. For z small enough, the line segment $\Gamma_2(z, z)$ still lies below the simplex such that the vertical and horizontal parts of $O_2(z)$ intersect with the simplex since the endpoints $(G_2^S(z), 1)$ and $(1, G_1^S(z))$ of $O_2(z)$ are above and to the left of the simplex for all $z > \underline{z}$. As z increases, the two intersection points move inward on the simplex. As z becomes large enough, one of the two vertices $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ crosses the simplex such that one intersection point lies in $\Gamma_2(z, z)$. The two intersection points approach each other until they coincide when the second vertex also crosses the simplex. Finally, for z sufficiently close to $\overline{z}$, both $\boldsymbol{\zeta}_1(z)$ and $\boldsymbol{\zeta}_2(z)$, and therefore the entire polygonal chain $O_2(z)$, lie above the simplex. To see this, note that $\boldsymbol{\zeta}_1(\overline{z}) = (1, G_1^S(\overline{z}))$ and $\boldsymbol{\zeta}_2(\overline{z}) = (G_2^S(\overline{z}), 1)$.

We have just shown that for every $\mathbf{y} \in \Delta^1$, there is a z such that $\mathbf{y} \in O_2(z)$. Consequently, $\Delta^1 \subset \Gamma_2([\underline{z}, \overline{z}]^2) = \bigcup_{z \in [\underline{z}, \overline{z}]} O_2(z)$, i.e., Property 2 holds for $n = 2$. In Figure 8, $\Gamma_2([\underline{z}, \overline{z}]^2)$ is the shaded area between $O_2(\underline{z})$ and $O_2(\overline{z})$, representing a hexagon.

A.3 *Proof of Properties 1 and 2 for $n > 2$*

We now extend the approach of the previous subsection to $n > 2$. Characterizing O_n and Γ_n turns out to be significantly more complex in this case. To handle this complexity, we first uncover the underlying recursive structure of Γ_n: one can construct Γ_n using modified versions of Γ_m for $m < n$. Exploiting this recursive structure, we show that Property 1 and Property 2 hold for n if they hold for all $m < n$. Using $n = 2$ as the base case, the two properties then hold by induction for all n.

Suppose $z_1 = z_2 = \cdots = z_n = z$ and consider $\Gamma_n(z, \dots, z) = \mathrm{Conv}(\{\mathbf{p}(z, \dots, z, h) : h \in H\})$. For each of the $n!$ different hierarchies $h \in H$,

$$\mathbf{p}(z, \dots, z, h) = \Bigl(\overbrace{\prod_{j \in \underline{\mathcal{E}}_1(h)} G_j^B(z) \prod_{k \in \overline{\mathcal{E}}_1(h)} G_k^S(z)}^{=p_1(z,\dots,z,h)}, \dots, \overbrace{\prod_{j \in \underline{\mathcal{E}}_n(h)} G_j^B(z) \prod_{k \in \overline{\mathcal{E}}_n(h)} G_k^S(z)}^{=p_n(z,\dots,z,h)}\Bigr),$$

where we have simplified the notation by writing $\overline{\mathcal{E}}_i(h)$ instead of $\overline{\mathcal{E}}_i(z, \dots, z, h)$. Note that if $z > \underline{z}$, each $h \in H$ yields a distinct $\mathbf{p}(z, \dots, z, h)$. It can be shown that all points $\mathbf{p}(z, \dots, z, h)$ lie in the same $(n-1)$-dimensional hyperplane: For all $h \in H$,

$$\mathbf{p}(z, \dots, z, h) \in \Bigl\{\mathbf{y} \in \mathbb{R}^n : \sum_{i \in \mathcal{N}} \bigl(G_i^B(z) - G_i^S(z)\bigr) y_i = \prod_{j \in \mathcal{N}} G_j^B(z) - \prod_{j \in \mathcal{N}} G_j^S(z)\Bigr\}.$$

Consequently, $\Gamma_n(z, \dots, z)$ is an $(n-1)$-dimensional convex polytope (in the hyperplane defined above) with vertices $\{\mathbf{p}(z, \dots, z, h) : h \in H\}$. Each vertex is connected to $n-1$ other vertices through an edge.

Now consider a nonempty subset of agents $\mathcal{K} \subset \mathcal{N}$ and denote its complement by $\mathcal{K}' := \mathcal{N} \setminus \mathcal{K}$. Define the set of hierarchies $H_{\mathcal{K}} \subset H$ such that for all $h \in H_{\mathcal{K}}$, we have $h(i) > h(j)$ for all $i \in \mathcal{K}$ and $j \in \mathcal{K}'$. If ties are broken based only on hierarchies in $H_{\mathcal{K}}$, agents in $\mathcal{K}$ always win ties against agents in $\mathcal{K}'$. The $(n-2)$-dimensional polytope $\mathrm{Conv}(\{\mathbf{p}(z, \dots, z, h) : h \in H_{\mathcal{K}}\})$ is a facet (i.e., an $(n-2)$-face) of the $(n-1)$-dimensional polytope $\Gamma_n(z, \dots, z)$. The boundary of $\Gamma_n(z, \dots, z)$ consists of $2^n - 2$ such facets, one for each possible nonempty $\mathcal{K} \subset \mathcal{N}$.[27]

Example with three agents In Appendix A.2, we have seen that $\Gamma_2(z, z)$ is a line segment. Assuming $n = 3$, there are six possible hierarchies, i.e., $H = \{h_1, \dots, h_6\}$. Hence, $\Gamma_3(z, z, z)$ is a hexagon (with opposite sides parallel). Let $\boldsymbol{\zeta}_l := \mathbf{p}(z, z, z, h_l)$ and suppose the hierarchies are enumerated in such a way that

$$\boldsymbol{\zeta}_1 = \bigl(G_2^B(z) G_3^B(z), G_1^S(z) G_3^B(z), G_1^S(z) G_2^S(z)\bigr),$$

$$\boldsymbol{\zeta}_2 = \bigl(G_2^B(z) G_3^B(z), G_1^S(z) G_3^S(z), G_1^S(z) G_2^B(z)\bigr),$$

[27] There are $\binom{n}{k}$ facets where $|\mathcal{K}| = k$, each having $k!(n-k)!$ vertices.

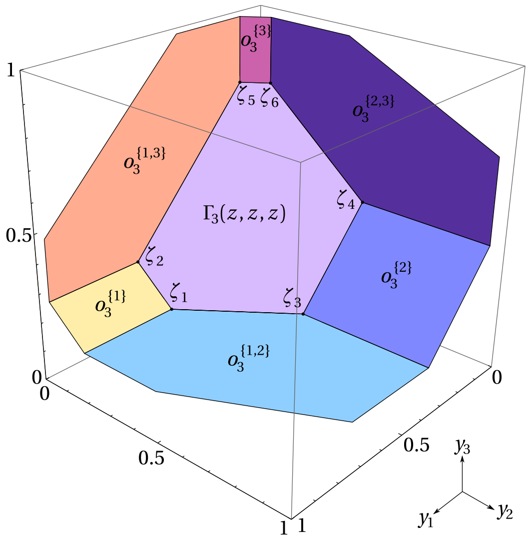


FIGURE 9. The polytopal complex $O_3(z)$ and its components.

$$\boldsymbol{\zeta}_3 = \big(G_2^S(z)G_3^B(z), G_1^B(z)G_3^B(z), G_1^S(z)G_2^S(z)\big),$$
$$\boldsymbol{\zeta}_4 = \big(G_2^S(z)G_3^S(z), G_1^B(z)G_3^B(z), G_1^B(z)G_2^S(z)\big),$$
$$\boldsymbol{\zeta}_5 = \big(G_2^B(z)G_3^S(z), G_1^S(z)G_3^S(z), G_1^B(z)G_2^B(z)\big),$$
$$\boldsymbol{\zeta}_6 = \big(G_2^S(z)G_3^S(z), G_1^B(z)G_3^S(z), G_1^B(z)G_2^B(z)\big).$$

For example, $h_1(1) > h_1(2) > h_1(3)$ and $h_2(1) > h_2(3) > h_2(2)$. As shown in Figure 9, $\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_6$ are the vertices of the hexagon $\Gamma_3(z,z,z)$. The six edges $\overline{\boldsymbol{\zeta}_1\boldsymbol{\zeta}_3}$, $\overline{\boldsymbol{\zeta}_3\boldsymbol{\zeta}_4}$, $\overline{\boldsymbol{\zeta}_4\boldsymbol{\zeta}_6}$, $\overline{\boldsymbol{\zeta}_6\boldsymbol{\zeta}_5}$, $\overline{\boldsymbol{\zeta}_5\boldsymbol{\zeta}_2}$, and $\overline{\boldsymbol{\zeta}_2\boldsymbol{\zeta}_1}$ correspond to tie-breaking using $H_{\{1,2\}}$, $H_{\{2\}}$, $H_{\{2,3\}}$, $H_{\{3\}}$, $H_{\{1,3\}}$, and $H_{\{1\}}$, respectively.[28]

Modified Γ_n correspondences and auxiliary definitions Below, we use the following two modified versions of Γ_n. Let $\mathcal{M} = \{j_1, j_2, \dots, j_m\} \subseteq \mathcal{N}$ be a subset of $m \geq 2$ agents. First, we denote by $\hat{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$ the correspondence Γ_m for a partnership among the m agents in $\mathcal{M}$ with modified virtual type distributions

$$\hat{G}_i^J(z) := G_i^J(z)\Bigg(\prod_{k \in \mathcal{N}\setminus\mathcal{M}} G_k^B(z)\Bigg)^{\frac{1}{m-1}} \quad \text{for } i \in \mathcal{M} \text{ and } J = S, B.$$

Note that all the properties of virtual type distributions G_i^J carry over to modified virtual type distributions $\hat{G}_i^J$. In particular, $\hat{G}_i^B(z) > \hat{G}_i^S(z)$ for all $z \in [\underline{z}, \overline{z}]$, $\hat{G}_i^B(\overline{z}) = 1$, and $\hat{G}_i^S(\underline{z}) = 0$. Hence all results for Γ_m extend to $\hat{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$.

[28]For $n = 4$, $\Gamma_4(z,z,z,z)$ is a truncated octahedron. In general, $\Gamma_n(z,\dots,z)$ is reminiscent of a *permutahedron* (see, e.g., Ziegler 1995), but its facets exhibit less symmetry (unless $F_i = F$ for all i).

Second, we denote by $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$ the correspondence Γ_m for a partnership among the m agents in $\mathcal{M}$ with modified virtual type distributions

$$\check{G}_i^J(z) := G_i^J(z)\left(\prod_{k\in\mathcal{N}\setminus\mathcal{M}} G_k^S(z)\right)^{\frac{1}{m-1}} \quad \text{for } i \in \mathcal{M} \text{ and } J = S, B.$$

Most properties of G_i^J carry over to their modified versions $\check{G}_i^J$, including $\check{G}_i^B(z) > \check{G}_i^S(z)$ for all $z \in (\underline{z}, \overline{z}]$ and $\check{G}_i^S(\underline{z}) = 0$. The only differences are $\check{G}_i^B(\overline{z}) < 1$ and $\check{G}_i^B(\underline{z}) = 0$. Again, all results for Γ_m extend to $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}$, except for those relying on $\check{G}_i^B(\overline{z}) = 1$ or $\check{G}_i^B(\underline{z}) > 0$. In particular, note that $\check{\Gamma}_m^{\mathcal{M}:\mathcal{N}}(z, \dots, z)$ is equivalent to $\Gamma_m(z, \dots, z)$ multiplied by the scalar $\prod_{k\in\mathcal{N}\setminus\mathcal{M}} G_k^S(z)$ (except for the m agents potentially being labeled differently).

We also make use of the following auxiliary definitions for one-agent partnerships where $\mathcal{M}$ is a singleton: $\hat{\Gamma}_1^{\{j\}:\mathcal{N}}(z) := \prod_{i\in\mathcal{N}\setminus j} G_i^B(z)$ and $\check{\Gamma}_1^{\{j\}:\mathcal{N}}(z) := \prod_{i\in\mathcal{N}\setminus j} G_i^S(z)$ for all $z \in [\underline{z}, \overline{z}]$.

Recursive structure of O_n Let us now study $O_n(z) = \Gamma_n(\xi_n(z))$. Define

$$\xi_n^{\mathcal{K}}(z) := \left\{\mathbf{z} \in [z, \overline{z}]^n : z_i > z \text{ for } i \in \mathcal{K} \text{ and } z_j = z \text{ for } j \in \mathcal{K}'\right\} \quad \text{for all } \mathcal{K} \subset \mathcal{N},$$

yielding a partition of $\xi_n(z)$ into $2^n - 1$ sets. Hence, $O_n(z) = \bigcup_{\mathcal{K}\subset\mathcal{N}} \Gamma_n(\xi_n^{\mathcal{K}}(z))$.

Consider a specific $\mathcal{K} \subset \mathcal{N}$ and suppose $z_i > z$ for $i \in \mathcal{K}$ and $z_j = z$ for $j \in \mathcal{K}'$. Then we can treat agents in $\mathcal{K}$ separately from agents in $\mathcal{K}'$. For the former, their critical type's expected share is as in a partnership among $k := |\mathcal{K}|$ agents with modified virtual type distributions $\hat{G}_i^J$ as defined above. For the latter, expected shares are as in $\Gamma_{n-k}(z, \dots, z)$ but multiplied by the scalar $\prod_{i\in\mathcal{K}} G_i^S(z)$, i.e., as in a partnership with $n - k$ agents and modified virtual type distributions $\check{G}_i^J$. Given $\mathbf{y} \in [0, 1]^n$, define $\mathbf{y}_{\mathcal{K}} := (y_{i_1}, y_{i_2}, \dots, y_{i_k})$ for $\mathcal{K} = \{i_1, i_2, \dots, i_k\}$ and $\mathbf{y}_{\mathcal{K}'} := (y_{j_1}, y_{j_2}, \dots, y_{j_{n-k}})$ for $\mathcal{K}' = \{j_1, j_2, \dots, j_{n-k}\}$. Hence, the closure of $\Gamma_n(\xi_n^{\mathcal{K}}(z))$ is

$$o_n^{\mathcal{K}}(z) := \left\{\mathbf{y} \in [0, 1]^n : \mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}\left([z, \overline{z}]^k\right) \text{ and } \mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(z, \dots, z)\right\}.$$

Note that $\Gamma_{n-k}(z, \dots, z)$ is an $(n - k - 1)$-dimensional convex polytope. If, in addition, $\Gamma_m([z, \overline{z}]^m)$ is an m-dimensional convex polytope for all $m < n$ (as we already showed for $m = 2$ above), then $o_n^{\mathcal{K}}(z)$ is an $(n - 1)$-dimensional convex polytope for all $\mathcal{K}$.

With the definition above, $O_n(z) = \bigcup_{\mathcal{K}\subset\mathcal{N}} o_n^{\mathcal{K}}(z)$. Note that $o_n^{\varnothing}(z) = \Gamma_n(z, \dots, z)$. Consequently, $O_n(z)$ is a polytopal complex that consists of $2^n - 1$ polytopes of dimension $(n - 1)$: $\Gamma_n(z, \dots, z)$ with a polytope $o_n^{\mathcal{K}}(z)$ with nonempty $\mathcal{K}$ attached to each of its $2^n - 2$ facets.

Example with three agents (continued) The polytopal complex $O_3(z)$ consists of the hexagon $\Gamma_3(z, z, z)$ with one polygon attached to each of its six edges, as shown in Figure 9. Those six polygons can be divided into two groups: $o_3^{\{1\}}(z)$, $o_3^{\{2\}}(z)$, and $o_3^{\{3\}}(z)$ are convex quadrilaterals and $o_3^{\{1,2\}}(z)$, $o_3^{\{1,3\}}(z)$, and $o_3^{\{2,3\}}(z)$ are hexagons. For example,

$$o_3^{\{1\}}(z) = \left\{\mathbf{y} \in [0, 1]^3 : y_1 \in \hat{\Gamma}_1^{\{1\}:\mathcal{N}}\left([z, \overline{z}]\right) \text{ and } (y_2, y_3) \in \check{\Gamma}_2^{\{2,3\}:\mathcal{N}}(z, z)\right\}.$$

Since both $\hat{\Gamma}_1^{\{1\}:\mathcal{N}}([z,\overline{z}])$ and $\check{\Gamma}_2^{\{2,3\}:\mathcal{N}}(z,z)$ are line segments, $o_3^{\{1\}}(z)$ is a convex quadrilateral, sharing the edge $\overline{\boldsymbol{\zeta}_2\boldsymbol{\zeta}_1}$ with the hexagon $\Gamma_3(z,z,z)$. Moreover,

$$o_3^{\{1,2\}}(z)=\{\mathbf{y}\in[0,1]^3:(y_1,y_2)\in\hat{\Gamma}_2^{\{1,2\}:\mathcal{N}}([z,\overline{z}]^2)\text{ and }y_3=\check{\Gamma}_1^{\{3\}:\mathcal{N}}(z)\}.$$

Note that y_3 is constant, whereas $\hat{\Gamma}_2^{\{1,2\}:\mathcal{N}}([z,\overline{z}]^2)$ is a hexagon, which follows from Appendix A.2 (cf. Figure 8). Hence, $o_3^{\{1,2\}}(z)$ is also a hexagon, sharing the edge $\overline{\boldsymbol{\zeta}_1\boldsymbol{\zeta}_3}$ with the hexagon $\Gamma_3(z,z,z)$.

Monotonicity of O_n Observe that all coordinates of each $\mathbf{p}(z,\dots,z,h)$ are continuous and strictly increasing in z. Hence, if $\hat{z}>z$, then for all $\hat{\mathbf{y}}\in\Gamma_n(\hat{z},\dots,\hat{z})$ and $\mathbf{y}\in\Gamma_n(z,\dots,z)$, we have $\hat{y}_i>y_i$ for at least one i. The following lemma shows that the monotonicity property of $\Gamma_n(z,\dots,z)$ extends to $O_n(z)$.

Lemma 3. *If $\hat{z}>z$, then for all $\hat{\mathbf{y}}\in O_n(\hat{z})$ and $\mathbf{y}\in O_n(z)$, $\hat{y}_i>y_i$ for at least one i.*

Proof. We show that $\hat{y}_i>y_i$ for at least one i for all $\mathcal{K},\mathcal{M}\subset\mathcal{N}$ and $\hat{\mathbf{y}}\in o_n^{\mathcal{M}}(\hat{z})$, $\mathbf{y}\in o_n^{\mathcal{K}}(z)$.

Note that each $\hat{\mathbf{y}}\in o_n^{\mathcal{M}}(\hat{z})$ corresponds to a $\hat{\mathbf{z}}\in[\hat{z},\overline{z}]^n$ and a tie-breaking rule. Now, consider the $\tilde{\mathbf{y}}_{\mathcal{K}'}\in\check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z},\dots,\hat{z})$ that is obtained when breaking ties among agents in $\mathcal{K}'$ in such a way that the same rule as for $\hat{\mathbf{y}}$ is applied for all $j,l\in\mathcal{K}'$ where $\hat{z}_j=\hat{z}_l$, whereas j wins against l for all $j,l\in\mathcal{K}'$ where $\hat{z}_j>\hat{z}_l$. This tie-breaking implies $\hat{y}_j>\tilde{y}_j$ for all $j\in\mathcal{K}'\cap\mathcal{M}$ since $p_j(\hat{\mathbf{z}},h)>p_j(\hat{z},\dots,\hat{z},h)$ for all relevant hierarchies h. Moreover $\hat{y}_l\geq\tilde{y}_l$ for all $l\in\mathcal{K}'\cap\mathcal{M}'$. Hence, we conclude that for all $\hat{\mathbf{y}}\in o_n^{\mathcal{M}}(\hat{z})$, there is a $\tilde{\mathbf{y}}_{\mathcal{K}'}\in\check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z},\dots,\hat{z})$ such that $\hat{y}_i\geq\tilde{y}_i$ for all $i\in\mathcal{K}'$.

Since $\hat{z}>z$, there is for all $\tilde{\mathbf{y}}_{\mathcal{K}'}\in\check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(\hat{z},\dots,\hat{z})$ and $\mathbf{y}_{\mathcal{K}'}\in\check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}(z,\dots,z)$ at least one $i\in\mathcal{K}'$ such that $\tilde{y}_i>y_i$. Combining this with the conclusion of the preceding paragraph implies that for all $\hat{\mathbf{y}}\in o_n^{\mathcal{M}}(\hat{z})$ and $\mathbf{y}\in o_n^{\mathcal{K}}(z)$, there is at least one $i\in\mathcal{K}'$ such that $\hat{y}_i\geq\tilde{y}_i>y_i$. □

For the three-agent example displayed in Figure 9, Lemma 3 implies that $O_3(z)$ moves toward the observer as we increase z. See also Figure 10 that depicts $O_3(z)$ for four different values for z.

Induction step for Property 1 Monotonicity of O_n implies that for each $\mathbf{y}\in\Gamma_n([\underline{z},\overline{z}]^n)=\bigcup_{z\in[\underline{z},\overline{z}]}O_n(z)$, there is a unique z such that $\mathbf{y}\in O_n(z)$.

Lemma 4. *If Property 1 holds for all Γ_m with $m<n$, then Property 1 holds for Γ_n.*

Proof. Lemma 3 implies that for every $\mathbf{y}\in\Gamma_n([\underline{z},\overline{z}]^n)$, there is a unique z such that $\mathbf{y}\in O_n(z)$.

We next show that for every $\mathbf{y}\in O_n(z)$, there is a unique $\mathcal{K}\subset\mathcal{N}$ such that $\mathbf{y}\in\Gamma_n(\xi_n^{\mathcal{K}}(z))$. Consider $\mathcal{K},\mathcal{M}\subset\mathcal{N}$ such that $\mathcal{K}\neq\mathcal{M}$. Without loss of generality, suppose $\mathcal{K}\cap\mathcal{M}'\neq\varnothing$. Then, for all $\mathbf{y}\in\Gamma_n(\xi_n^{\mathcal{K}}(z))$ and $\tilde{\mathbf{y}}\in\Gamma_n(\xi_n^{\mathcal{M}}(z))$, $y_i>\tilde{y}_i$ for at least one $i\in\mathcal{K}\cap\mathcal{M}'$. To see this, consider the corresponding $\mathbf{z}\in\xi_n^{\mathcal{K}}(z)$ and $\tilde{\mathbf{z}}\in\xi_n^{\mathcal{M}}(z)$. For $i\in\mathcal{K}\cap\mathcal{M}'$ and $j\in\mathcal{K}'$, we have $z_i>z_j=z$ but $\tilde{z}_i=z\leq\tilde{z}_j$. Hence, in the first case, the critical type of agent i has a strictly higher winning probability against agents in $\mathcal{K}'$ than in the second case. The same is true for $j \in \mathcal{K} \cap \mathcal{M}$, since $z_i > z$ whereas $\tilde{z}_i = z < \tilde{z}_j$. Finally, the winning probability of agent i's critical type against other agents in $\mathcal{K} \cap \mathcal{M}'$ cannot be lower for all $i \in \mathcal{K} \cap \mathcal{M}'$ when considering $\mathbf{z} \in \xi_n^{\mathcal{K}}(z)$ than when considering $\tilde{\mathbf{z}} \in \xi_n^{\mathcal{M}}(z)$. Consequently, $y_i > \tilde{y}_i$ for at least one i.

So far we have shown that for every $\mathbf{y} \in \Gamma_n([\underline{z}, \overline{z}]^n)$, there are unique z, $\mathcal{K}$ such that $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$. This already partially pins down $\mathbf{z}$: for all $i \in \mathcal{K}'$, we have $z_i = z$. Moreover, $\mathbf{y} \in \Gamma_n(\xi_n^{\mathcal{K}}(z))$ implies $\mathbf{y} \in o_n^{\mathcal{K}}(z)$ and, therefore, $\mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}([z, \overline{z}]^k)$. If Property 1 holds for $k < n$, there is a unique $\mathbf{z}_{\mathcal{K}}$ such that $\mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}(\mathbf{z}_{\mathcal{K}})$. This pins down z_i also for $i \in \mathcal{K}$. □

Convexity of $\Gamma_n([z, \overline{z}]^n)$ Suppose $\Gamma_m([z, \overline{z}]^m)$ is a convex polytope for all $m < n$. As observed above, this implies that $O_n(z)$ is a polytopal complex consisting of $2^n - 1$ convex polytopes $o_n^{\mathcal{K}}$ of dimension $n-1$, one for each $\mathcal{K} \subset \mathcal{N}$. If $\mathcal{K} \cap \mathcal{M} \neq \varnothing$, then the two polytopes $o_n^{\mathcal{K}}$ and $o_n^{\mathcal{M}}$ are adjacent, i.e., they share a facet (of dimension $n-2$). Let the boundary of the polytopal complex $O_n(z)$ be defined as all the facets of each polytope $o_n^{\mathcal{K}}$ that are not shared with some other polytope $o_n^{\mathcal{M}}$, where $\mathcal{K} \neq \mathcal{M}$. Each point $\mathbf{y} \in \Gamma_n(\mathbf{z})$ on the boundary of $O_n(z)$ corresponds to a $\mathbf{z}$ where, for some $\mathcal{K} \subset \mathcal{N}$, $z_i = \overline{z}$ for $i \in \mathcal{K}$, whereas $z_j = z$ for $j \in \mathcal{K}'$.

In a manner similar to the construction of O_n above, define

$$Q_n(z) := \Gamma_n\big(\{\mathbf{z} \in [z, \overline{z}]^n : z_i = \overline{z} \text{ for at least one } i \in \mathcal{N}\}\big) = \bigcup_{\mathcal{K} \subset \mathcal{N}} q_n^{\mathcal{K}}(z),$$

where

$$q_n^{\mathcal{K}}(z) := \big\{\mathbf{y} \in [0,1]^n : \mathbf{y}_{\mathcal{K}} \in \hat{\Gamma}_k^{\mathcal{K}:\mathcal{N}}(\overline{z}, \dots, \overline{z}) \text{ and } \mathbf{y}_{\mathcal{K}'} \in \check{\Gamma}_{n-k}^{\mathcal{K}':\mathcal{N}}\big([z, \overline{z}]^{n-k}\big)\big\}.$$

The term $Q_n(z)$ represents the image under Γ_n of the set of all $\mathbf{z}$ where $z_i \geq z$ for all i and $z_i = \overline{z}$ for at least one i. Observe that $Q_n(z)$ contains all the boundary points of $O_n(\tilde{z})$ for each $\tilde{z} \in [z, \overline{z}]$. Moreover, $Q_n(\overline{z}) = O_n(\overline{z})$.

Lemma 5. *The image* $\Gamma_n([z, \overline{z}]^n)$ *is an n-dimensional convex polytope for all* $z < \overline{z}$. *The boundary of this polytope is* $O_n(z) \cup Q_n(z)$.

Proof. From Appendix A.2 we know that $\Gamma_2([z, \overline{z}]^2)$ is a hexagon. We now show that if $\Gamma_m([z, \overline{z}]^m)$ is a convex polytope for all $m < n$, then $\Gamma_n([z, \overline{z}]^n)$ is a convex polytope. Consequently, the first statement in the lemma follows by induction.

Suppose $\Gamma_m([z, \overline{z}]^m)$ is a convex polytope for all $m < n$ and recall that $\Gamma_n([z, \overline{z}]^n) = \bigcup_{\tilde{z} \in [z, \overline{z}]} O_n(\tilde{z})$. As derived above, $O_n(z)$ is a polytopal complex. As all coordinates of the extreme points of $\Gamma_n(z, \dots, z)$ are continuous and strictly increasing in z, Lemma 3 implies that $O_n(z)$ continuously moves further away from the origin as z increases. Hence, $O_n(z)$ is part of the boundary of $\Gamma_n([z, \overline{z}]^n)$.

In addition to $O_n(z)$, all boundary points of $O_n(\tilde{z})$ for each $\tilde{z} \in (z, \overline{z})$ are also part of the boundary of $\Gamma_n([z, \overline{z}]^n)$, whereas all interior points of $O_n(\tilde{z})$ are in the interior of $\Gamma_n([z, \overline{z}]^n)$. Last, note that $O_n(\overline{z})$ consists of only one convex polytope (namely $\Gamma_n(\overline{z}, \dots, \overline{z})$) and that all its points are part of the boundary of $\Gamma_n([z, \overline{z}]^n)$.

The term $Q_n(z)$ represents all points on the boundary of $\Gamma_n([z,\overline{z}]^n)$ described in the preceding paragraph, i.e., boundary points that are not in $O_n(z)$. Consequently, $O_n(z) \cup Q_n(z)$ represents the entire boundary of $\Gamma_n([z,\overline{z}]^n)$. Like $O_n(z)$, $Q_n(z)$ is also a polytopal complex that consists of $2^n - 1$ convex polytopes of dimension $n-1$. The boundary of $\Gamma_n([z,\overline{z}]^n)$ therefore consists of $2^{n+1}-2$ convex polytopes ($o_n^{\mathcal{K}}(z)$ and $q_n^{\mathcal{K}}(z)$ for all $\mathcal{K} \subset \mathcal{N}$), making $\Gamma_n([z,\overline{z}]^n)$ an n-dimensional polytope with $2^{n+1}-2$ facets.

Recall that for all $z < \overline{z}$, $O_n(z)$ consists of $\Gamma_n(z,\dots,z)$ with an $o_n^{\mathcal{K}}(z)$ attached to each facet. The points in each $o_n^{\mathcal{K}}(z)$ are further away from the origin than the points on the corresponding facet of $\Gamma_n(z,\dots,z)$. Because of the monotonicity and continuity properties of $O_n(z)$, for all $\mathbf{y} \in \text{Conv}(O_n(z))$ such that $\mathbf{y} \notin O_n(z)$, there is a $\tilde{z} \in (z,\overline{z}]$ such that $\mathbf{y} \in O_n(\tilde{z})$. Hence, the polytope $\Gamma_n([z,\overline{z}]^n) = \bigcup_{\tilde{z}\in[z,\overline{z}]} O_n(\tilde{z})$ is convex. □

Induction step for Property 2 Consider $O_n(\underline{z})$. This represents a special case since $\Gamma_n(\underline{z},\dots,\underline{z})$ is a general $(n-1)$-simplex with only n vertices rather than a polytope with $n!$ vertices. In particular, note that for each vertex $\mathbf{p}(\underline{z},\dots,\underline{z},h) = (p_1,\dots,p_n)$, $p_i \in (0,1)$ for one i, whereas $p_j = 0$ for all $j \neq i$, resulting in only n distinct vertices. Since $\sum_{i=1}^n p_i < 1$, the general simplex $\Gamma_n(\underline{z},\dots,\underline{z})$ does not intersect with standard simplex Δ^{n-1}: the former lies closer to the origin than the latter.[29]

It follows that $O_n(\underline{z})$ consists of only $n+1$ polytopes of dimension $(n-1)$: the general simplex $\Gamma_n(\underline{z},\dots,\underline{z})$ with a polytope o_n^i attached to each of its n facets (each corresponding to a general $(n-2)$-simplex), where, for each $i \in \mathcal{N}$,

$$o_n^i := \{\mathbf{y} \in [0,1]^n : \mathbf{y}_{\mathcal{N}\setminus i} \in \hat{\Gamma}_{n-1}^{\mathcal{N}\setminus i:\mathcal{N}}([\underline{z},\overline{z}]^{n-1}) \text{ and } y_i = 0\}.$$

Lemma 6. *If Property 2 holds for all Γ_m with $m < n$, then $O_n(\underline{z})$ contains the entire boundary (all n facets) of Δ^{n-1}.*

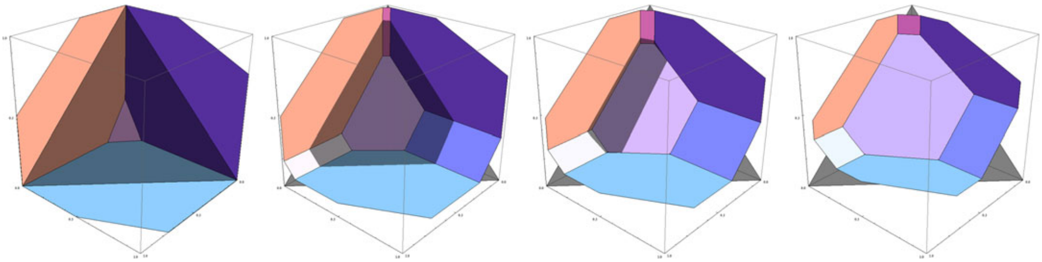
Proof. Recall that $O_n(\underline{z})$ is the union of $\Gamma_n(\underline{z},\dots,\underline{z})$ and n polytopes o_n^i as defined above. Property 2 for $m < n$ implies, in particular, $\Delta^{n-2} \subset \Gamma_{n-1}([\underline{z},\overline{z}]^{n-1})$ and, therefore, $\Delta^{n-2} \subset \hat{\Gamma}_{n-1}^{\mathcal{N}\setminus i:\mathcal{N}}([\underline{z},\overline{z}]^{n-1})$. Moreover, the n facets of Δ^{n-1} all correspond to one coordinate being set to zero, i.e., $\mathbf{y}_{\mathcal{N}\setminus i} \in \Delta^{n-2}$ and $y_i = 0$. □

Panel (a) of Figure 10 illustrates Lemma 6 in the three-agent example. It shows how $O_3(\underline{z})$ intersects with the boundary of the semitransparent black triangle that represents the simplex Δ^2. Figure 10 also conveys that as we increase z, the intersection of $O_3(z)$ with Δ^2 moves inward (panels (b) and (c)) until the entire simplex has been covered and for all higher z, $O_3(z)$ does not intersect with Δ^2 (panel (d)).[30] Hence, Property 2 holds for Γ_3.

Using the convexity of $\Gamma_n([\underline{z},\overline{z}]^n)$, it is now straightforward to obtain the following lemma.

[29] In the three-agent example above, we obtain, for $z = \underline{z}$, $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_2 = (G_2^B(\underline{z})G_3^B(\underline{z}),0,0)$, $\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_4 = (0,G_1^B(\underline{z})G_3^B(\underline{z}),0)$, and $\boldsymbol{\zeta}_5 = \boldsymbol{\zeta}_6 = (0,0,G_1^B(\underline{z})G_2^B(\underline{z}))$.

[30] Let $\hat{z}$ be the smallest z such that $\sum_{i\in\mathcal{N}} p_i(z,\dots,z,h) \geq 1$ for all $h \in H$. Similarly, let $\check{z}$ be the greatest z such that $\sum_{i\in\mathcal{N}} p_i(z,\dots,z,h) \leq 1$ for all $h \in H$. Observe that $\underline{z} < \check{z} \leq \hat{z} < \overline{z}$ (with $\check{z} = \hat{z}$ if $F_i = F$ for all i). The polytopal complex $O_n(z)$ intersects with Δ^{n-1} if and only if $z \leq \hat{z}$, whereas $\Gamma_n(z,\dots,z)$ intersects with Δ^{n-1} if and only if $z \in [\check{z},\hat{z}]$. Panels (b), (c), and (d) of Figure 10 correspond to $z < \check{z} < z' < \hat{z} < z''$.



(a) $O_3(\underline{z})$ (b) $O_3(z)$ (c) $O_3(z')$ (d) $O_3(z'')$

FIGURE 10. Increasing z in the three-agent example: O_3 for some $\underline{z} < z < z' < z'' < \overline{z}$ and the simplex Δ^2 (semitransparent black triangle).

LEMMA 7. *If Property 2 holds for all Γ_m with $m < n$, then Property 2 holds for Γ_n.*

PROOF. If Property 2 holds for all Γ_m with $m < n$, then, according to Lemma 6, $O_n(\underline{z})$ contains the entire boundary of Δ^{n-1}. By Lemma 5, $\Gamma_n([\underline{z}, \overline{z}]^n)$ is convex and $O_n(\underline{z})$ is part of the boundary of $\Gamma_n([\underline{z}, \overline{z}]^n)$. Consequently, the boundary of Δ^{n-1} being contained in the boundary of $\Gamma_n([\underline{z}, \overline{z}]^n)$ implies Property 2 for Γ_n. □

Final step As shown in Appendix A.2, Property 1 and Property 2 hold for $n = 2$. By induction, using Lemmata 4 and 7, Property 1 and Property 2 hold for all n.

APPENDIX B: OTHER PROOFS

B.1 *Proof of Lemma 1*

The definition of U_i implies

$$W_\alpha(\mathbf{s}, \mathbf{t}) = \sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})\big(s_i(\mathbf{X}) - r_i\big)\big] - \alpha \sum_{i\in\mathcal{N}} E\big[U_i(X_i)\big] + (1-\alpha)\sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})r_i\big]. \tag{22}$$

Using the fact that $\sum_{i\in\mathcal{N}}(s_i(\mathbf{X}) - r_i) = 0$, we get

$$\begin{aligned}\sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})\big(s_i(\mathbf{X}) - r_i\big)\big] &= \sum_{i\in\mathcal{N}} E\Bigg[\bigg(X_i - \eta(X_i) + \sum_{j\in\mathcal{N}}\eta(X_j)\bigg)\big(s_i(\mathbf{X}) - r_i\big)\Bigg] \\ &= \sum_{i\in\mathcal{N}} E\big[\big(X_i - \eta(X_i)\big)\big(S_i(X_i) - r_i\big)\big].\end{aligned} \tag{23}$$

Integrating (IC2) by parts, we obtain, for all $\hat{x}_i \in [0, 1]$,

$$\begin{aligned}E\big[U_i(X_i)\big] &= U_i(\hat{x}_i) + \int_0^1 \int_{\hat{x}_i}^{x_i} \big(S_i(y) - r_i\big)\, dy f_i(x_i)\, dx_i \\ &= U_i(\hat{x}_i) - \int_0^{\hat{x}_i} F_i(y)\big(S_i(y) - r_i\big)\, dy + \int_{\hat{x}_i}^1 \big(1 - F_i(y)\big)\big(S_i(y) - r_i\big)\, dy.\end{aligned} \tag{24}$$

Substituting (23) and (24) into (22) yields

$$W_\alpha(\mathbf{s},\mathbf{t}) = \sum_{i\in\mathcal{N}}\left(\int_0^{\hat{x}_i} \psi^S_{\alpha,i}(y)\big(S_i(y)-r_i\big)f_i(y)\,dy + \int_{\hat{x}_i}^1 \psi^B_{\alpha,i}(y)\big(S_i(y)-r_i\big)f_i(y)\,dy\right)$$
$$-\alpha\sum_{i\in\mathcal{N}} U_i(\hat{x}_i) + (1-\alpha)\sum_{i\in\mathcal{N}} E\big[v_i(\mathbf{X})r_i\big],$$

which, by the definitions of $\psi_{\alpha,i}(x_i,\hat{x}_i)$ and $\widetilde{W}_\alpha(\mathbf{s},\hat{\mathbf{x}})$, is equivalent to (3).

Consider $\hat{\mathbf{x}}, \boldsymbol{\omega} \in [0,1]^n$. From (3), we obtain

$$\widetilde{W}_\alpha(\mathbf{s},\hat{\mathbf{x}}) - \widetilde{W}_\alpha(\mathbf{s},\boldsymbol{\omega}) = \alpha\sum_{i\in\mathcal{N}}\big(U_i(\hat{x}_i) - U_i(\omega_i)\big).$$

For all $\boldsymbol{\omega}\in\Omega(\mathbf{s})$ and $\hat{\mathbf{x}}\notin\Omega(\mathbf{s})$, we have $U_i(\hat{x}_i)\geq U_i(\omega_i)$ for all i, where the inequality is strict for at least one i, and, therefore, $\widetilde{W}_\alpha(\mathbf{s},\hat{\mathbf{x}}) > \widetilde{W}_\alpha(\mathbf{s},\boldsymbol{\omega})$. Consequently, $\Omega(\mathbf{s}) = \arg\min_{\hat{\mathbf{x}}}\widetilde{W}_\alpha(\mathbf{s},\hat{\mathbf{x}})$.

B.2 *Proof of Proposition 1*

Suppose agent 2 plays the candidate equilibrium strategy given in the proposition. We show that it is a best response for agent 1 to also play this strategy.

First, consider the continuation game after both agents have chosen BUY. Agent 1 infers that $x_2 > \overline{\omega}$ and has to choose his strategy in the open ascending forward auction. Given that agent 2 follows the strategy to stay in the auction until the price reaches $v_2(x_2,x_2)$, agent 1's payoff from winning the auction is $v_1(x_1,x_2) - v_2(x_2,x_2)$, which is positive for $x_2 \leq x_1$. Hence, it is optimal for agent 1 to drop out at price $v_1(x_1,x_1)$. (If $x_1 < \overline{\omega}$, which is off the equilibrium path, agent 1 drops out immediately.) Essentially the same argument applies to the continuation game after both agents have chosen SELL: agent 1 chooses to drop out at price $v_1(x_1,x_1)$ in the open descending reverse auction.

Now consider the first stage where agent 1 chooses between BUY, HOLD, and SELL:

- Conditional on $X_2 < \underline{\omega}$, i.e., agent 2 choosing SELL, agent 1's expected payoff is

$$\frac{1}{2}\big(E\big[v_1(x_1,X_2)|X_2<\underline{\omega}\big] - p^B\big) + E\big[\max\big\{v_2(X_2,X_2) - v_1(x_1,X_2), 0\big\}|X_2<\underline{\omega}\big]$$

  if he chooses SELL and $\frac{1}{2}(E[v_1(x_1,X_2)|X_2<\underline{\omega}] - p^B)$ if he chooses HOLD or BUY.

- Conditional on $X_2\in[\underline{\omega},\overline{\omega}]$, i.e., agent 2 choosing HOLD, agent 1's expected payoff is $\frac{1}{2}(\hat{p}^S - E[v_1(x_1,X_2)|\underline{\omega}\leq X_2\leq\overline{\omega}])$ if he chooses SELL, 0 if he chooses HOLD, and $\frac{1}{2}(E[v_1(x_1,X_2)|\underline{\omega}\leq X_2\leq\overline{\omega}] - \hat{p}^B)$ if he chooses BUY.

- Conditional on $X_2 > \overline{\omega}$, i.e., agent 2 choosing BUY, agent 1's expected payoff is $\frac{1}{2}(p^S - E[v_1(x_1,X_2)|X_2>\overline{\omega}])$ if he chooses SELL or HOLD and

$$\frac{1}{2}\big(p^S - E\big[v_1(x_1,X_2)|X_2>\overline{\omega}\big]\big) + E\big[\max\big\{v_1(x_1,X_2) - v_2(X_2,X_2), 0\big\}|X_2>\overline{\omega}\big]$$

  if he chooses BUY.

Consequently, in all three cases, agent 1 finds it optimal to choose SELL if $x_1 < \underline{\omega}$, HOLD if $\underline{\omega} \le x_1 \le \overline{\omega}$, and BUY otherwise.

Note that the allocation induced in this equilibrium is equal to the optimal allocation of Theorem 1. Hence, by incentive compatibility, the induced payments are also pinned down up to a constant. Now note that according to Theorem 1, the interim payment of types $x_1 \in [\underline{\omega}, \overline{\omega}]$ in an optimal dissolution mechanism is

$$\begin{aligned} T_1(x_1) &= E\left[v_1(x_1, X_2)\left(s_1(x_1, X_2) - \frac{1}{2}\right)\right] \\ &= F(\underline{\omega})E\left[v_1(x_1, X_2)\left(1 - \frac{1}{2}\right)\middle| X_2 < \underline{\omega}\right] \\ &\quad + \left(1 - F(\overline{\omega})\right)E\left[v_1(x_1, X_2)\left(-\frac{1}{2}\right)\middle| X_2 > \overline{\omega}\right] \\ &= F(\underline{\omega})\frac{1}{2}p^B - \left(1 - F(\overline{\omega})\right)\frac{1}{2}p^S, \end{aligned}$$

which coincides with the expected payment of those types in the indirect mechanism.

B.3 *Proof of Theorem 2*

Part (i) was shown in the main text. Concavity of the objective implies that the set $R^*(\alpha)$ of maximizers is convex. To complete the proof of part (ii), we show that at most one of the two ownership structures $\mathbf{r}' = (1, 0, \dots, 0)$ and $\mathbf{r}'' = (0, 1, 0 \dots, 0)$ can be optimal, implying that $R^*(\alpha)$ is a strict subset of Δ^{n-1}. The ironing parameters of optimal dissolution mechanisms for $\mathbf{r}'$ and $\mathbf{r}''$ satisfy $z_1' = \overline{z}$, $z_2' = \underline{z}$ and $z_1'' = \underline{z}$, $z_2'' = \overline{z}$, respectively (cf. the proof of Theorem 1 in Appendix A). For both $\mathbf{r}'$ and $\mathbf{r}''$ to be optimal, condition (15) in part (iii) would require that $E[v_1(\omega_{\alpha,1}(\overline{z}), \mathbf{X}_{-1})] \le E[v_2(\omega_{\alpha,2}(\underline{z}), \mathbf{X}_{-2})]$ and $E[v_1(\omega_{\alpha,1}(\underline{z}), \mathbf{X}_{-1})] \ge E[v_2(\omega_{\alpha,2}(\overline{z}), \mathbf{X}_{-2})]$. But these inequalities cannot hold simultaneously because each $\omega_{\alpha,i}(z_i)$ is strictly increasing in z_i.

It remains to prove part (iii). First, note that

$$E\left[\psi_{\alpha,i}(X_i, \hat{x}_i)\right] = (1 - \alpha)E\left[v_i(\mathbf{X})\right] + \alpha E\left[v_i(\hat{x}_i, \mathbf{X}_{-i})\right] - \sum_{j \in \mathcal{N}} E\left[\eta(X_j)\right].$$

Combining this with (13) and (14), we obtain

$$\begin{aligned} W_\alpha\left(\mathbf{s}_\alpha^{\mathbf{r}}, \mathbf{t}_\alpha^{\mathbf{r}}\right) = \min_{\hat{\mathbf{x}} \in [0,1]^n} \Bigg\{ &-\alpha \sum_{i \in \mathcal{N}} r_i E\left[v_i(\hat{x}_i, \mathbf{X}_{-i})\right] \\ &+ \sum_{i \in \mathcal{N}} E\left[\eta(X_i)\right] + \max_{\mathbf{s} \in \mathfrak{S}} E\left[\sum_{i \in \mathcal{N}} s_i(\mathbf{X})\psi_{\alpha,i}(X_i, \hat{x}_i)\right]\Bigg\}. \end{aligned} \tag{25}$$

For the remainder of this proof, it is more convenient to represent the standard simplex by $\widehat{\Delta}^{n-1} := \{\mathbf{r} \in [0, 1]^{n-1} : \sum_{i=1}^{n-1} r_i \le 1\}$. Note that using this definition, $(r_1, \dots, r_{n-1}) \in \widehat{\Delta}^{n-1}$ is equivalent to $(r_1, \dots, r_{n-1}, 1 - \sum_{i=1}^{n-1} r_i) \in \Delta^{n-1}$. Define the value function

$V_\alpha\colon \widehat{\Delta}^{n-1} \to \mathbb{R}$ such that $V_\alpha(\hat{r}_1, \dots, \hat{r}_{n-1}) = W_\alpha(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$ for $\mathbf{r} = (\hat{r}_1, \dots, \hat{r}_{n-1}, 1 - \sum_{i=1}^{n-1} \hat{r}_i)$. Hence, (25) yields, for each $\mathbf{r} \in \widehat{\Delta}^{n-1}$,

$$V_\alpha(\mathbf{r}) = \min_{\hat{\mathbf{x}} \in [0,1]^n} \left\{ \alpha \sum_{i=1}^{n-1} r_i \big(E\big[v_n(\hat{x}_n, \mathbf{X}_{-n})\big] - E\big[v_i(\hat{x}_i, \mathbf{X}_{-i})\big]\big) - \alpha E\big[v_n(\hat{x}_n, \mathbf{X}_{-n})\big] \right.$$
$$\left. + \sum_{i \in \mathcal{N}} E\big[\eta(X_i)\big] + \max_{\mathbf{s} \in \mathfrak{S}} E\left[\sum_{i \in \mathcal{N}} s_i(\mathbf{X}) \psi_{\alpha,i}(X_i, \hat{x}_i)\right] \right\}.$$

Because $V_\alpha(\mathbf{r})$—like $W_\alpha(\mathbf{s}^{\mathbf{r}}_\alpha, \mathbf{t}^{\mathbf{r}}_\alpha)$—is the minimum of a family of affine functions of $\mathbf{r}$, it is concave and differentiable almost everywhere. By the envelope theorem,

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} = \alpha \big(E\big[v_n\big(\omega^*_n, \mathbf{X}_{-n}\big)\big] - E\big[v_i\big(\omega^*_i, \mathbf{X}_{-i}\big)\big]\big), \tag{26}$$

where $\omega^*_i = \omega_{\alpha,i}(z^*_i)$ for $i \in \mathcal{N}$ and $\mathbf{z}^* = \Gamma_n^{-1}(\mathbf{r}, 1 - \sum_{i=1}^{n-1} r_i)$. Note that since each $\omega_{\alpha,i}$ and Γ_n^{-1} is a continuous function, these partial derivatives are continuous. Therefore, V_α is differentiable on $\widehat{\Delta}^{n-1}$.

Consider the problem of maximizing $V_\alpha(r_1, \dots, r_{n-1})$ subject to $(r_1, \dots, r_{n-1}) \in \widehat{\Delta}^{n-1}$. As we maximize a concave and differentiable function over a convex set, a solution exists and can be identified using Kuhn–Tucker conditions. We represent the requirement $(r_1, \dots, r_{n-1}) \in \widehat{\Delta}^{n-1}$ by the following n inequality constraints: For all $i \in \{1, \dots, n-1\}$, let λ_i denote the Lagrange multiplier on the constraint $r_i \geq 0$ and let λ_n denote the Lagrange multiplier on the constraint $1 - r_n = \sum_{i=1}^{n-1} r_i \leq 1$. Any solution corresponds to shares and nonnegative multipliers that satisfy

$$\frac{\partial V_\alpha(\mathbf{r})}{\partial r_i} + \lambda_i - \lambda_n = 0 \quad \text{and} \quad \lambda_i r_i = 0 \quad \text{for all } i \in \{1, \dots, n-1\}$$

as well as $(\sum_{i=1}^{n-1} r_i - 1)\lambda_n = 0$. Using (26), this implies that an ownership structure $\mathbf{r}$ is optimal if and only if it satisfies (15) in part (iii) of the theorem.

B.4 *Proof of Corollary 2*

Using the definition of v_i, and setting $\omega_{\alpha,i}(z^*_i) = \omega_{0,i}$ and $\hat{Y} = Y - \sum_{i \in \mathcal{N}} E[\eta(X_i)]$, condition (15) can be written as

$$\omega_{0,i} = \hat{Y} + E\big[\eta(X_i)\big] \quad \text{if } r_i > 0 \quad \text{and} \quad 0 = \omega_{0,i} \geq \hat{Y} + E\big[\eta(X_i)\big] \quad \text{if } r_i = 0.$$

Because $r_i = \prod_{j \neq i} F_j(\omega_{0,i})$, the equality and the inequality are both equivalent to

$$r_i = \prod_{j \neq i} F_j(\omega_{0,i}) = \prod_{j \neq i} F_j\big(\hat{Y} + E\big[\eta(X_i)\big]\big),$$

which yields (16), and $\hat{Y}$ is pinned down by the requirement that $\sum_{i \in \mathcal{N}} r_i = 1$.

Moreover, by (16), $r^0_i > 0$ if and only if $\hat{Y} + E[\eta(X_i)] > 0$. Hence, $r^0_i > 0$ and $E[\eta(X_j)] \geq E[\eta(X_i)]$ implies $r^0_j > 0$.

B.5 *Proof of Proposition 2*

Suppose $F_i(x) = 1 - (1-x)^b$ and $F_j(x) = x^b$ for all $j \neq i$, where $b > 1$. We show that (19) holds if b is large enough. Inequality (19) is equivalent to

$$E\left[\min\left\{\frac{F_i(X_i)}{f_i(X_i)}, \frac{\overline{z} - X_i + \eta(X_i)}{\alpha}\right\} + \min\left\{\frac{1 - F_j(X_j)}{f_j(X_j)}, \frac{X_j - \eta(X_j) - \underline{z}}{\alpha}\right\}\right] \\ \leq E\left[v_j(\mathbf{X}) - v_i(\mathbf{X})\right] \tag{27}$$

for all $j \neq i$. Define $\overline{\theta} := \max_{x\in[0,1]} \eta'(x)$ and $\underline{\theta} := \min_{x\in[0,1]} \eta'(x)$. Noting that $F_i(x) \geq F_j(x)$ for all x and using integration by parts, we then have

$$E\left[\eta(X_j) - \eta(X_i)\right] = \int_0^1 \eta'(y)\big(F_i(y) - F_j(y)\big)\,dy \\ \leq \overline{\theta}\int_0^1 \big(F_i(y) - F_j(y)\big)\,dy = \overline{\theta}E[X_j - X_i].$$

Accordingly, we obtain a lower bound for the right-hand side of (27):

$$E\left[v_j(\mathbf{X}) - v_i(\mathbf{X})\right] = E\left[X_j + \eta(X_i) - X_i - \eta(X_j)\right] \geq (1-\overline{\theta})E[X_j - X_i].$$

Moreover, we obtain upper bounds for the following two terms on the left-hand side of (27):

$$\overline{z} - X_i + \eta(X_i) = 1 - X_i - \int_{X_i}^1 \eta'(y)\,dy \leq (1-\underline{\theta})(1-X_i),$$

$$X_j - \eta(X_j) - \underline{z} = X_j - \int_0^{X_j} \eta'(y)\,dy \leq (1-\underline{\theta})X_j.$$

Taken together, (27) hence holds if

$$E\left[\min\left\{\frac{F_i(X_i)}{f_i(X_i)}, \frac{(1-\underline{\theta})(1-X_i)}{\alpha}\right\} + \min\left\{\frac{1-F_j(X_j)}{f_j(X_j)}, \frac{(1-\underline{\theta})X_j}{\alpha}\right\}\right] \\ \leq (1-\overline{\theta})E[X_j - X_i]. \tag{28}$$

Using the explicit assumptions on F_i and F_j, (28) simplifies to

$$2\int_0^1 \min\left\{1 - x^b, \frac{(1-\underline{\theta})b}{\alpha}x^b\right\}dx \leq (1-\overline{\theta})\frac{b-1}{b+1}. \tag{29}$$

With $\tilde{x} := (\alpha/((1-\underline{\theta})b + \alpha))^{1/b}$, carrying out the integral gives

$$\int_0^1 \min\left\{1 - x^b, \frac{(1-\underline{\theta})b}{\alpha}x^b\right\}dx = \int_0^{\tilde{x}} \frac{(1-\underline{\theta})b}{\alpha}x^b\,dx + \int_{\tilde{x}}^1 1 - x^b\,dx = \frac{b}{b+1}(1-\tilde{x}).$$

Hence, (29) is equivalent to

$$1 - \frac{(1-\overline{\theta})(b-1)}{2b} \leq \left(\frac{\alpha}{(1-\underline{\theta})b + \alpha}\right)^{\frac{1}{b}}. \tag{30}$$

Because $\overline{\theta} < 1$, the left-hand side of (30) is strictly less than 1 for all $b > 1$. By contrast,

$$\lim_{b\to\infty}\left(\frac{\alpha}{(1-\underline{\theta})b+\alpha}\right)^{\frac{1}{b}} = \lim_{b\to\infty}\exp\left(\frac{\ln(\alpha)}{b} - \frac{\ln\big((1-\underline{\theta})b+\alpha\big)}{b}\right) = 1.$$

Consequently, (30), and therefore (27), holds for b sufficiently large, which establishes the first part of the proposition.

Now consider a partnership where for a nonempty set of agents $\mathcal{I}$, $F_i(x) = 1-(1-x)^b$ for all $i \in \mathcal{I}$ and $F_j(x) = x^b$ for all $j \notin \mathcal{I}$. Let $b > 1$ be such that (27) holds. Hence,

$$E\big[v_i\big(\omega_{\alpha,i}(\overline{z}), \mathbf{X}_{-i}\big)\big] \le E\big[v_j\big(\omega_{\alpha,j}(\underline{z}), \mathbf{X}_{-j}\big)\big] \quad \text{for all } i \in \mathcal{I} \text{ and } j \notin \mathcal{I}.$$

Since $\omega_{\alpha,i}(z_i^*)$ is increasing in z_i^*, all ownership structures that satisfy the optimality condition (15) of Theorem 2 assign $r_j = 0$ to each $j \notin \mathcal{I}$ and sufficiently equal nonzero shares to $i \in \mathcal{I}$ such that $z_i^* = z^* > \underline{z} = z_j^*$ for all $i \in \mathcal{I}$ and $j \notin \mathcal{I}$ for the associated optimal dissolution mechanisms. This establishes the second part of the proposition.

B.6 *Proof of Proposition 3*

Both sides of (19) are continuous in α. Moreover, the left-hand side of (19) is strictly decreasing and the right-hand side of (19) is strictly increasing in α. This immediately implies (i).

Taking the limit as $\alpha \to 0$, (19) becomes

$$E\left[v_i(\mathbf{X}) + \frac{F_i(X_i)}{f_i(X_i)}\right] \le E\left[v_j(\mathbf{X}) - \frac{1-F_j(X_j)}{f_j(X_j)}\right].$$

This is equivalent to $E[\eta(X_i) - \eta(X_j)] \ge 1$. However, if $\eta'(x) \in [-1, 1)$, then

$$E\big[\eta(X_i) - \eta(X_j)\big] = \int_0^1 \eta'(y)\big(F_j(y) - F_i(y)\big)\, dy < 1,$$

which means that (19) is violated for $\alpha \to 0$. Hence, either there exists a revenue weight $\hat{\alpha} > 0$ such that (19) holds with equality or (19) is violated for all α. In both cases, (ii) is true. Conversely, if $E[\eta(X_i) - \eta(X_j)] \ge 1$ for all $j \ne i$, which requires $\eta'(x) < -1$ for some x, then (19) holds for $\alpha = 0$. By (i), (19) then holds for all α, resulting in (iii).

B.7 *Proof of Proposition 4*

Note that $\eta'(x) \ge 0$ and $F_i(x) < F_j(x)$ for all $x \in (0, 1)$ implies $E[\eta(X_i)] \ge E[\eta(X_j)]$. First, suppose $r_i^0 > 0$. With $F_i(x) < F_j(x)$ and $E[\eta(X_i)] \ge E[\eta(X_j)]$, (16) in Corollary 2 yields

$$r_i^0 = \prod_{k\ne i} F_k\big(\hat{Y} + E\big[\eta(X_i)\big]\big) > \prod_{k\ne j} F_k\big(\hat{Y} + E\big[\eta(X_j)\big]\big) = r_j^0.$$

Now, suppose $r_i^0 = 0$. In this case, $r_j^0 = 0$ is immediate from the last statement in Corollary 2.

References

Bleakley, Hoyt and Joseph Ferrie (2014), "Land openings on the Georgia frontier and the coase theorem in the short- and long-run." Unpublished paper, University of Chicago. [1064]

Brooks, Richard R. W., Claudia M. Landeo, and Kathryn E. Spier (2010), "Trigger happy or gun shy? Dissolving common-value partnerships with Texas shootouts." *RAND Journal of Economics*, 41, 649–673. [1064, 1068, 1080]

Che, Yeon-Koo (2006), "Beyond the Coasian irrelevance: Asymmetric information." Unpublished lecture notes. [1064, 1068, 1084, 1090]

Che, Yeon-Koo and Erin Cho (2011), "Brave new world of market design." *Korean Economic Review*, 27, 33–56. [1064]

Chien, Hung-Ken (2007), "Incentive efficient mechanisms for partnership dissolution." Working Paper, University of Bonn. [1068, 1069]

Coase, Ronald H. (1959), "The federal communications commission." *Journal of Law and Economics*, 2, 1–40. [1064]

Coase, Ronald H. (1960), "The problem of social cost." *Journal of Law and Economics*, 3, 1–44. [1064]

Cramton, Peter, Robert Gibbons, and Paul Klemperer (1987), "Dissolving a partnership efficiently." *Econometrica*, 55, 615–632. [1064, 1068, 1071, 1080]

de Frutos, María-Angeles and Thomas Kittsteiner (2008), "Efficient partnership dissolution under buy-sell clauses." *RAND Journal of Economics*, 39, 184–198. [1080]

Fieseler, Karsten, Thomas Kittsteiner, and Benny Moldovanu (2003), "Partnerships, lemons, and efficient trade." *Journal of Economic Theory*, 113, 223–234. [1065, 1068, 1069]

Figueroa, Nicolás and Vasiliki Skreta (2012), "Asymmetric partnerships." *Economics Letters*, 115, 268–271. [1064, 1068, 1084, 1090]

Fowlie, Meredith and Jeffrey M. Perloff (2013), "Distributing pollution rights in cap-and-trade programs: Are outcomes independent of allocation?" *Review of Economics and Statistics*, 95, 1640–1652. [1064]

Grossman, Sanford J. and Oliver D. Hart (1986), "The costs and benefits of ownership: A theory of vertical and lateral integration." *Journal of Political Economy*, 94, 691–719. [1095]

Hart, Oliver and John Moore (1990), "Property rights and the nature of the firm." *Journal of Political Economy*, 98, 1119–1158. [1095]

Jehiel, Philippe and Ady Pauzner (2006), "Partnership dissolution with interdependent values." *RAND Journal of Economics*, 37, 1–22. [1068]

Jullien, Bruno (2000), “Participation constraints in adverse selection models.” *Journal of Economic Theory*, 93, 1–47. [1066]

Kittsteiner, T. (2003), “Partnerships and double auctions with interdependent valuations.” *Games and Economic Behavior*, 44, 54–76. [1068, 1080]

Landeo, Claudia M. and Kathryn E. Spier (2014a), “Irreconcilable differences: Judicial resolution of business deadlock.” *University of Chicago Law Review*, 81, 203–227. [1064]

Landeo, Claudia M. and Kathryn E. Spier (2014b), “Shotguns and deadlocks.” *Yale Journal on Regulation*, 31, 143–187. [1064, 1068]

Lewis, Tracy R. and David E. M. Sappington (1989), “Countervailing incentives in agency problems.” *Journal of Economic Theory*, 49, 294–313. [1066]

Loertscher, Simon, Leslie Marx, and Tom Wilkening (2015), “A long way coming: Designing centralized markets with privately informed buyers and sellers.” *Journal of Economic Literature*, 53, 857–897. [1065]

Lu, Hu and Jacques Robert (2001), “Optimal trading mechanisms with ex ante unidentified traders.” *Journal of Economic Theory*, 97, 50–80. [1069]

Marshall, Albert W., Ingram Olkin, and Barry C. Arnold (2011), *Inequalities: Theory of Majorization and Its Applications*, second edition. Springer Series in Statistics. Springer-Verlag, New York, New York. [1084, 1085]

McAfee, R. Preston (1992), “Amicable divorce: Dissolving a partnership with simple mechanisms.” *Journal of Economic Theory*, 56, 266–293. [1080]

Milgrom, Paul (2004), *Putting Auction Theory to Work*. Cambridge University Press. [1064]

Milgrom, Paul (2017), *Discovering Prices: Auction Design in Markets With Complex Constraints*. Columbia University Press, New York. [1065]

Milgrom, Paul and Ilya Segal (2015), “Deferred-acceptance auctions and radio spectrum reallocation.” Unpublished Paper, Stanford University. [1065]

Myerson, Roger B. (1981), “Optimal auction design.” *Mathematics of Operations Research*, 6, 58–73. [1071, 1074, 1079]

Myerson, Roger B. and Mark A. Satterthwaite (1983), “Efficient mechanisms for bilateral trading.” *Journal of Economic Theory*, 29, 265–281. [1064, 1067, 1079, 1087]

Mylovanov, Tymofiy and Thomas Tröger (2014), “Mechanism design by an informed principal: Private values with transferable utility.” *Review of Economic Studies*, 81, 1668–1707. [1069]

Schweizer, Urs (2006), “Universal possibility and impossibility results.” *Games and Economic Behavior*, 57, 73–85. [1068, 1069]

Segal, Ilya and Michael D. Whinston (2011), “A simple status quo that ensures participation (with application to efficient bargaining).” *Theoretical Economics*, 6, 109–125. [1069]

Segal, Ilya and Michael D. Whinston (2016), "Property rights and the efficiency of bargaining." *Journal of the European Economic Association*, 14, 1287–1328. [1069]

Wasser, Cédric (2013), "Bilateral $k + 1$-price auctions with asymmetric shares and values." *Games and Economic Behavior*, 82, 350–368. [1080]

Ziegler, Günter M. (1995), *Lectures on Polytopes*, volume 152 of Graduate Texts in Mathematics. Springer-Verlag, New York, New York. [1101]

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