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School choice under partial fairness

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We generalize the school choice problem by defining a notion of allowable priority violations. In this setting, a weak axiom of stability (partial stability) allows only certain priority violations. We introduce a class of algorithms called the student exchange under partial fairness (SEPF). Each member of this class gives a partially stable matching that is not Pareto dominated by another partially stable matching (i.e., constrained efficient in the class of partially stable matchings). Moreover, any constrained efficient matching that Pareto improves upon a partially stable matching can be obtained via an algorithm within the SEPF class. We characterize the unique algorithm in the SEPF class that satisfies a desirable incentive property. The extension of the model to an environment with weak priorities enables us to provide a characterization result that proves the counterpart of the main result in Erdil and Ergin (2008).

Keywords. School choice, stability, efficiency.

JEL classification. C78, D61, D78, I20.

1. Introduction

School choice has become an important policy tool for school districts to provide students with the opportunity to choose their school. In a school choice problem, students submit their preferences over schools to a central placement authority and the authority decides on an assignment (or a matching) based on schools' capacities, schools' priorities over students, and submitted preferences. The main criteria are improving students' welfare (efficiency) and respecting schools' priorities (fairness). Unfortunately, efficiency and fairness are incompatible in this setting (Balinski and Sönmez 1999). This

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incompatibility has led school districts and scholars to search for a compromise between efficiency and fairness. The current work contributes to this research program.

Our approach is to improve efficiency by allowing priority violations. We generalize the standard school choice problem by including a set of allowable priority violations. The standard fairness notion, i.e., stability, is projected onto this framework in a straightforward way, i.e., partial stability (Section 2.2). We propose a class of algorithms, student exchange under partial fairness (SEPF) such that each algorithm in this class gives a partially stable matching that is not Pareto dominated by another partially stable matching (in other words, is constrained efficient in the class of partially stable matchings). One can begin with any partially stable matching and each algorithm in the SEPF class selects a constrained efficient matching that weakly Pareto dominates the initial matching. Moreover, each such matching can be obtained by an algorithm in the SEPF class (Theorem 1).[1]

The set of allowable priority violations is taken as a premise of the model, and the concept has several interpretations. One interpretation involves the following scenario (Section 4): the school district can ask students' consent for priority violations and then select a matching that takes the students' consent into account (Kesten 2010). The idea is to design school choice rules[2] such that students have incentives to consent for their priorities to be violated: a consenting student is never hurt by her decision to consent. This is an indispensable property that assures that the idea of consent is operational. When priorities are consented to be violated, stability constraints are relaxed and each student's welfare can be weakly improved. A rule motivated by this interpretation is the efficiency adjusted deferred acceptance mechanism (EADAM) (Kesten 2010). The EADAM gives students incentives to consent and finds a constrained efficient matching. Our characterization result implies that the EADAM is in the SEPF class (Proposition 1). We argue that a particular rule, the top priority (TP) rule, stands out in the SEPF class: the TP is the unique constrained efficient rule under which students are not hurt by consenting (Theorem 2). An immediate corollary is the outcome equivalence of the EADAM and the TP rule: they select the same outcome for the same preference and consent profiles. Unfortunately, no rule within the SEPF class is immune to violations through misrepresentation of preferences (Proposition 3). This incompatibility is indeed more general: a constrained efficient rule can never be strategy-proof (Theorem 3).

Another interpretation (Section 5) pertains to the case where certain *types* of priorities can be violated, while other priorities must always be respected. For instance, the Boston Public School System (BPSS) recently removed proximity from the priority structure and started prioritizing students based on sibling status only (Dur et al. 2018). In another example, the Recovery School District in New Orleans replaced an efficient but nonstable matching rule with a stable rule after including the private scholarship schools, whose priorities cannot be violated by law (Pathak 2017). Similarly, some school

[1] In an independent study, Combe et al. (2018) study a similar class of rules. Motivated by the teacher reassignments in France, they propose a class of algorithms. Each algorithm in this class starts with an initial matching and in each iteration, the set of blocking pairs shrinks so as to improve the welfare of the teachers. Combe et al. (2018) prove non-existence of a strategy-proof rule in this class.

[2] A *school choice rule* is a systematic way to match students with schools for each school choice problem.

districts include both exam and regular schools. Exam schools' priorities are determined based on centralized test scores and cannot be violated by law. Regular schools' priorities are exogenously determined and respecting them is plausible but not necessary (Abdulkadiroğlu 2011, Sönmez and Ünver 2011). These examples show that school districts may consider allowing some priorities to be violated, which would render further efficiency gains possible. An intuitive approach to elicit these efficiency gains is to remove these priority classes. Nevertheless, we show (in Section 5.1) that suppressing a priority class may yield a perverse consequence: it may make each student worse off compared to the outcome of the student-proposing deferred acceptance (DA) rule where all existing priority classes are kept. Alternatively, the SEPF class produces matchings that weakly improve on the outcome of the student-proposing DA rule and implicitly "determines" which priorities (among those that are allowed to be violated) to violate so as to obtain these efficiency gains.

Our basic model has strict priorities, but its extension to an environment with weak priorities is straightforward (Section 7). We show that a characterization result analogous to Theorem 1 holds for this domain, which completes the main result of Erdil and Ergin (2008) by proving its converse. This result reveals an important connection between the SEPF and the stable improvement cycles algorithm proposed by Erdil and Ergin (2008).

Related literature

The school choice problem was introduced by Abdulkadiroğlu and Sönmez (2003). A major concern in a school choice problem is *stability*: for each school s, there should not be a student who prefers s to her assigned school and another student assigned to s with a lower priority at s, and there should not also be an unfilled seat at s if a student prefers s to her assigned school. There are rules that always select stable matchings. The student-proposing DA rule is such an example (Gale and Shapley 1962). The student-proposing DA gives the student-optimal stable matching (SOSM), which is the best matching in terms of the students' welfare among all the stable matchings (Gale and Shapley 1962, Balinski and Sönmez 1999). Furthermore, the student-proposing DA is *strategy-proof*: revealing preferences truthfully is a weakly dominant strategy for each student (Dubins and Freedman 1981, Roth 1982). However, there is a serious drawback of the SOSM: it is not *Pareto efficient* (Balinski and Sönmez 1999, Abdulkadiroğlu and Sönmez 2003). Moreover, this inefficiency can be quite severe (Kesten 2010) and there is empirical support for this insight: in New York City (NYC) high school match, possible welfare gains over the SOSM are significant (Abdulkadiroğlu et al. 2009). The inefficiency of the SOSM is actually due to a deeper issue: stability and efficiency are incompatible (Gale and Shapley 1962, Roth 1982, Balinski and Sönmez 1999).

A possible remedy for the SOSM's inefficiency is to relax stability. One alternative in this direction is to focus on efficiency via the top trading cycles (TTC) rule, which always gives an efficient matching (Abdulkadiroğlu and Sönmez 2003, Hakimov and Kesten 2018, Morrill 2013, Morrill 2015). Another alternative is to weaken the stability notion. Such a weakening is *reasonable stability*: a matching is *reasonably stable* if whenever a student i's priority is violated at school s, there does not exist a stable matching in which i is assigned to s.[3] Another weakening is *α-equitability*: a matching with a priority violation is not deemed as unfair if a student's objection to that priority violation is counter-objected by another student (Alcade and Romero-Medina 2017). Alva and Manjunath (2019) define another weakening of stability: a rule is *stable-dominating* if it selects an allocation that Pareto-improves *some* stable allocation at every preference profile. They show that the SOSM is the only stable-dominating and strategy-proof rule. Troyan et al. (2016) and Morrill (2016) focus on the alternative ways to improve efficiency by relaxing the fairness constraint. In particular, under the Morrill (2016) definition of *fairness*, a student i's priority at school s needs to be respected only if there exists a textitlegitimate matching in which i is assigned to s. Morrill (2016) introduces an iterative procedure to find the set of legitimate matchings. He shows that there is a unique legitimate set of matchings and a unique Pareto efficient and legitimate matching which corresponds to EADAM's outcome when all students consent. According to Troyan et al. (2016), a matching is *essentially stable* if any objection of student i to her priority violation at school s initiates a rejection chain that results in her rejection from s. They show that the EADAM produces an essentially stable matching.

When students consent for the violation of their priorities, at the SOSM, students assigned to certain schools cannot be made better off and the EADAM can be redefined by taking these schools into account (Tang and Yu 2014). Moreover, the outcome of the EADAM is supported as the *strong Nash equilibrium* of the preference revelation game under the DA (Bando 2014). In the affirmative action context, a variant of the EADAM has recently been proposed as a *minimally responsive* rule (that is, a rule such that changing the affirmative action parameter in favor of the minorities never results in a matching that makes each minority student weakly worse off) (Doğan 2016). Regarding the interpretation where certain priorities can be violated, one example is the NYC school match: motivated by the observation that the efficiency loss under the SOSM is significant, school districts have been considering allowing such violations anywhere but exam schools (Abdulkadiroğlu 2011). Another example of this approach can also be seen in Afacan et al. (2017).[4]

The idea that possible welfare gains can be captured by improvement cycles is first proposed by Erdil and Ergin (2008) in the context of coarse priorities of schools. This idea inspired the rules proposed in some other works (Ehlers et al. 2014, Abdulkadiroğlu 2011) and in the current work as well.

2. The model

We first present the standard school choice problem, and then introduce the extended model with allowable priority violations.

[3]This notion is first discussed in the working paper version of Kesten (2010).

[4]They allow certain priority violations based on the information that the parents will not appeal these violations. The authors introduce the *efficiency-corrected deferred acceptance mechanism* (ECDA) algorithm, which finds a *constrained efficient sticky stable* matching. Besides informational issues, our paper differs from theirs in two major aspects. First, we introduce a class of rules selecting *all constrained efficient* matchings and, thus, provide a full characterization result. Second, we show that there exists a *unique* rule in this class that satisfies desired incentive properties.

2.1 *School choice problem*

A *school choice problem* consists of the following elements:

- A finite set of students $I = \{i_1, i_2, \dots, i_n\}$.
- A finite set of schools $S = \{s_1, s_2, \dots, s_m\}$.
- A strict priority profile of schools $\succ = (\succ_s)_{s \in S}$, where $\succ_s$ is the complete priority order of school s over I.[5]
- A capacity vector $q = (q_s)_{s \in S}$, where q_s is the number of available seats at school s.
- A strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i's strict preferences over $S \cup \{\varnothing\}$, where $\varnothing$ stands for the option of being unassigned with $q_\varnothing = |I|$.

Let R_i denote the *at-least-as-good-as* preference relation associated with P_i, that is, $s\, R_i\, s'$ if and only if $s\, P_i\, s'$ or $s = s'$. Let $R = (R_i)_{i \in I}$ denote the weak preference profile of all students.

A *matching* $\mu : I \to S \cup \{\varnothing\}$ is a function such that for each $s \in S$, $|\mu^{-1}(s)| \le q_s$. A matching μ *violates the priority* of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that $\mu(j) = s$, $s\, P_i\, \mu(i)$, and $i \succ_s j$. A matching μ is *fair* if for each $i \in I$ and $s \in S$, it does not violate the priority of student i at school s. A matching μ is *individually rational* if for each $i \in I$, $\mu(i)\, R_i\, \varnothing$. A matching μ is *non-wasteful* if there do not exist a student $i \in I$ and a school $s \in S$ such that $s\, P_i\, \mu(i)$ and $|\mu^{-1}(s)| < q_s$. A matching μ is *stable* if it is fair, individually rational, and non-wasteful.

A matching μ *weakly Pareto dominates* matching μ' if for each $i \in I$, $\mu(i)\, R_i\, \mu'(i)$. A matching μ *Pareto dominates* μ' if μ weakly Pareto dominates μ' and $\mu(j)\, P_j\, \mu'(j)$ for some $j \in I$. A matching μ is *Pareto efficient* if it is not Pareto dominated by another matching μ'. A stable matching μ is *student-optimal stable matching* (SOSM) if it is not Pareto dominated by another stable matching.

2.2 *School choice problem with allowable priority violations*

A *school choice problem with allowable priority violations* (or simply a problem) is a school choice problem where some priority violations are allowed. These violations are given by a correspondence $C : S \rightrightarrows I \times I$, where $(i, j) \in C(s)$ means that the priority of i at s is allowed to be violated by a student j. We require that $(i, j) \in C(s)$ only if $i \succ_s j$.[6] We impose the following restriction on C throughout the paper.

Assumption 1. *If $(i, j) \in C(s)$, then for each $j' \in I$ such that $i \succ_s j' \succ_s j$, $(i, j') \in C(s)$.*

Assumption 1 implies that if the violation of priority of $i \in I$ at $s \in S$ by some students is allowable, then there is a *cutoff student* j with $i \succ_s j$ such that i's priority at s can only

[5] In Section 7, we allow priorities to be weak.

[6] This requirement follows directly from the definition of priorities (see Section 2.1).

be violated by the students whose priorities are weakly higher than j (and by definition, lower than i).[7]

Throughout the paper, we fix I, S, $\succ$, and q. Thus, a problem is defined by (R, C).[8] A *rule* ψ is a systematic procedure that selects a matching for each problem. For problem (R, C), we denote the matching selected by rule ψ with $\psi_{(R,C)}$ and the assignment of student i in $\psi_{(R,C)}$ with $\psi_{(R,C)}(i)$.

There are several interpretations of the correspondence C, such as those that consider C to be the set of priorities that the central authority can sacrifice or those that come from the consenting decisions of students, such as in Kesten (2010). We discuss these interpretations in detail in Sections 4 and 5; yet, for now, we remain agnostic and give the most general treatment.

The addition of correspondence C relaxes the fairness constraint in the following manner: priorities for which violations are allowable (according to C) are not taken into account when considering fairness. By ignoring these priorities, one can define weaker notions of fairness and stability. A matching μ is *partially fair* if for each $i, j \in I$ and $s \in S$, $\mu(j) = s$, $s \, P_i \, \mu(i)$, and $i \succ_s j$ imply $(i, j) \in C(s)$. A matching μ is *partially stable* if it is partially fair, individually rational, and non-wasteful.

A matching μ is *constrained efficient* if it is partially stable and is not Pareto dominated by any other partially stable matching. A rule ψ is *constrained efficient* if it selects a constrained efficient matching for any problem.

The notion of partial stability stands as a compromise between efficiency and stability. Indeed, one can easily see how the interpolation between the two ends works by investigating the extremes. In one extreme, when no priority violation is allowable, the notion of partial stability collapses to that of stability and the only constrained efficient matching is the SOSM. In particular, the SOSM Pareto dominates any other partially stable matching and the set of of partially stable matchings constitutes a lattice. In the other extreme, when any priority violation is allowable, each matching vacuously satisfies *partial fairness* and constrained efficiency is equivalent to Pareto efficiency. For some problems, there is no partially stable matching that Pareto dominates or is Pareto dominated by any other partially stable matching. Thus, we do not observe lattice structure when the set of partially stable matchings is considered.[9]

3. The student exchange under partial fairness

We now present a class of algorithms to characterize the set of constrained efficient matchings that improve the students' welfare upon a partially stable matching.

3.1 *The algorithm*

Given a partially stable matching μ, for each school $s \in S$, we define two sets:

[7]This assumption is necessary for our main results: Theorem 1 would not hold without it. In Appendix B, we give an example where Assumption 1 does not hold and Theorem 1 fails.

[8]Since the preference relation is strict, we can define the problem by (P, C) as well.

[9]One can easily see this point by considering an example composed of one school with one seat and two students.

- Let $D_\mu(s) = \{i \in I : s \, P_i \, \mu(i)\}$ (the set of students who prefer s to their assignments under μ).
- Let $X_\mu(s) = \{i \in D_\mu(s) : \forall j \in [D_\mu(s) \setminus (\{k \in I : (k, i) \in C(s)\} \cup \{i\})], i \succ_s j\}$ (the set of students who are eligible for a *partially fair* exchange involving school s).[10]

Let $G = (V, E)$ be a directed graph with the set of vertices V and the set of directed edges E, which is a set of ordered pairs of V.

For each matching μ, $G(\mu) = (I, E(\mu))$ is the *(directed) application graph* associated with μ where the set of directed edges $E(\mu) \subseteq I \times I$ is $ij \in E(\mu)$ (that is, *i points to j*) if and only if $s = \mu(j)$ and $i \in X_\mu(s)$.

A set of edges $\{i_1i_2, i_2i_3, \dots, i_ni_{n+1}\}$ is a *path* if the vertices $i_1, i_2, \dots, i_{n+1}$ are distinct, and is a *cycle* if the vertices $i_1, i_2, \dots, i_n$ are distinct and $i_1 = i_{n+1}$. We say that a *cycle* $\phi = \{i_1i_2, i_2i_3, \dots, i_ni_1\}$ *is solved* when for each $ij \in \phi$, student i is assigned to $\mu(j)$ to obtain a new matching. Formally, we denote the *solution of a cycle* $\phi = \{i_1i_2, i_2i_3, \dots, i_ni_1\}$ by the operation $\circ$; that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_1, i_2, \dots, i_n\}$, $\eta(i') = \mu(i')$. The following algorithm is built on a partially stable matching and is defined by *solving* cycles iteratively:

The Student Exchange under Partial Fairness (SEPF) Algorithm[11]

Step 0. Let μ_0 be a partially stable matching.

Step $k \geq 1$. Given a matching μ_{k-1}, there are two alternatives:

(k.1) If there is no cycle in $G(\mu_{k-1})$, then the algorithm terminates and μ_{k-1} is the matching obtained.

(k.2) Otherwise, solve one of the cycles in $G(\mu_{k-1})$, say ϕ_k, and let $\mu_k = \phi_k \circ \mu_{k-1}$.

The algorithm terminates in a finite number of steps due to a finite number of students and schools and strict preferences. Moreover, the maximum number of possible steps is $0.5|I|(|S| - 1)$.[12] Since the SEPF involves two selections that are not predetermined, it is a class of algorithms: (i) We only require μ_0 to be partially stable, and impose no other restrictions on it, and (ii) at each step of the algorithm, we require one of the cycles to be solved without specifying which one.

[10]Note that this set is always well defined. In particular, whenever $D_\mu(s)$ is nonempty, the student with the highest priority for s among the students in $D_\mu(s)$ is always in $X_\mu(s)$. (See Lemma 3 in Appendix C for a formal treatment.) Moreover, for any $i, j \in D_\mu(s)$ with $i \succ_s j$, if $j \in X_\mu(s)$, then i is also in $X_\mu(s)$ by virtue of Assumption 1.

[11]Example 2 in Appendix J demonstrates how the algorithm works.

[12]This is because there can be at most $|I|(|S| - 1)$ improvements in a problem and at each step a cycle is solved, implying that there are at least two improvements obtained in each step.

One obtains (possibly) different outcomes by choosing different initial partially stable matchings and different cycles at each step of the algorithm.[13] Two questions follow immediately: which initial partially stable matching and which cycle selection procedure should be used? We provide a detailed answer to the second question in Section 4. For the initial partially stable matching, there is a strong argument for choosing the SOSM: many school districts use the student-proposing DA. Thus, the SOSM is the status quo outcome and school districts might be constrained by giving each student at least what she gets under the status quo. For this reason, we focus on algorithms within the SEPF class starting with the SOSM and present the results by using the SOSM as the initial partially stable matching.

3.2 *A characterization result*

Our first result establishes the relationship between constrained efficiency and the SEPF class: For any problem, if one starts with the SOSM, each matching obtained by an algorithm within the SEPF class is constrained efficient and weakly Pareto dominates the SOSM. Moreover, each constrained efficient matching that weakly Pareto dominates the SOSM is attainable through some algorithm within the SEPF that starts with the SOSM.

Let $\Psi(R, C)$ denote the set of all constrained efficient matchings that weakly Pareto dominate the SOSM with respect to the problem (R, C). Let $\Pi(R, C)$ denote the set of all matchings that can be obtained via an algorithm in the SEPF class starting with the SOSM under (R, C).

Theorem 1. *For each problem (R, C), a matching is constrained efficient and weakly Pareto dominates the SOSM if and only if it is obtained by an algorithm within the SEPF class starting with the SOSM; that is, $\Psi(R, C) = \Pi(R, C)$.*

See Appendices C–I[14] for the proofs.

This result states that a group of reasonable normative properties characterize the set of the SEPF outcomes. An important remark is that these properties include a restriction: the characterization concerns only the set of constrained efficient matchings that weakly Pareto dominate the SOSM. In general, there may exist constrained efficient matchings that do not weakly Pareto dominate the SOSM, and the SEPF starting with the SOSM does not find such matchings. Thus, a relevant question is whether this restriction is at the expense of an important property. To answer it, we remove this restriction in Section 6 and seek constrained efficient rules. In Section 6, we also impose incentive compatibility. But it turns out that constrained efficiency and incentive compatibility are incompatible (Theorem 3).

Theorem 1 is a general result without any reference to the nature of allowable priority violations. While there might be different interpretations of allowable priority violations,

[13] The outcome of the SEPF is *a set of* matchings. In Example 2 in Appendix J, there are two matchings obtained by algorithms within the SEPF class.

[14] In Appendix C, we provide the proof for the general case in which μ_0 can be any partially stable matching.

our characterization is a starting point to obtain strong results for the model where priority violations are only possible with students' consent. We analyze this model next.[15]

4. Priority violation via students' consent

An example where the notion of partial fairness is pertinent is the model where priorities are violated as a result of consenting decisions of students, as in Kesten (2010). Before making observations about this interpretation, we emphasize the relationship between the SEPF and the *efficiency adjusted deferred acceptance mechanism* (EADAM) introduced by Kesten (2010).

4.1 *The SEPF and EADAM*

The motivation behind the EADAM is to explore the source of inefficiency of the SOSM due to fairness constraints and improve it on the efficiency dimension. An important observation made by Kesten (2010) is that the priority of student i at school s might not help her to get a better school under the student-proposing DA at all. If this is the case, giving i the lowest priority at s instead of her current priority would not change her assignment and the DA would possibly select a matching that Pareto dominates the SOSM with the original priorities. Motivated by this observation, Kesten (2010) introduces the EADAM in a setting that allows students to *consent* for the violation of their own priorities. This would correspond to student i declaring the violation of her priority at school s by some other student as allowable. That is, there is a clear connection between our setup and that considered in Kesten (2010). Yet, the setup considered in Kesten (2010) is more restrictive than ours: it assumes that a student can either consent for priority violation by each student at each school or not consent for any violation by any student at any school. To capture this difference, we define the following notion.

Definition 1. Correspondence C satisfies *all-or-nothing property* if, for all $i \in I$, only one of the following statements is true:

- $(i, j) \in C(s)$ for each $s \in S$ and $j \in I$ such that $i \succ_s j$
- $(i, j) \notin C(s)$ for each $s \in S$ and $j \in I$.

The all-or-nothing property is used only in Section 4.1 so as to establish the connection between the SEPF class and the EADAM. The property is crucial for the EADAM to be well defined under our setup. Since the all-or-nothing property merely puts a restriction on the correspondence C, the algorithms in the SEPF class would still be well defined. The following result is immediate from Theorem 1 and from Theorem 1 of Tang and Yu (2014).

[15]In Section 5, we discuss another interpretation of our model inspired by a recent change in a school choice policy: we show that our approach can be adopted for the case where a school district is willing to violate certain types of priorities.

Proposition 1. *For each problem (R, C), where C satisfies all-or-nothing property, the matching obtained by the EADAM can also be obtained by an algorithm in the SEPF class starting with the SOSM.*

Each particular algorithm in the SEPF class (starting with the SOSM) defines a constrained efficient rule that Pareto dominates the SOSM (we call these rules the *class of SEPF rules*), and the EADAM is a rule in this class (Proposition 1). It turns out that the EADAM is a special member of the class of SEPF rules: it defines a *unique* rule that satisfies a desirable incentive compatibility property related to the consenting interpretation. Before we state this result, we provide some insights.

4.2 *The concept of underdemanded schools*

At each step of an algorithm in the SEPF class, only the welfare of students in the solved cycle improves. Clearly, if a student's current school is not demanded by other students, then she is not part of any cycle and a welfare improvement is not possible for her. To formalize this idea, we introduce the concept of underdemanded schools.[16]

A school s has no demand at μ if there does not exist a student i who prefers s to $\mu(i)$. A school s is *underdemanded* at μ if it has no demand at μ or if no student on a path to a student in $\mu^{-1}(s)$ is part of a cycle in the graph $G(\mu)$.

Let (R, C) be a problem and let ψ be a rule in the SEPF class. Assume that the underlying SEPF algorithm terminates at Step K and the outcome is the matching μ_{K-1}; that is, $\psi_{(R,C)} = \mu_{K-1}$. Take a step $k \leq K-1$. If student i is not pointed to by another student in the graph $G(\mu_k)$, then school $\mu_k(i)$ has no demand at $\mu_{k'}$ for any $k' \geq k$.[17] Consequently, a student assigned to an underdemanded school in the matching selected at Step k is not part of any cycle at any step $k' \geq k$. Consequently, $\psi_{(R,C)}(i) = \mu_k(i)$.[18] Consistent with this observation, we say that a student is *permanently matched at* μ if she is assigned to an underdemanded school at μ, and a student is *temporarily matched at* μ if she is not permanently matched.

4.3 *Incentives to consent*

A student's decision to consent (or not consent) to priority violations is a strategic one. Consequently, the main issue for a rule based on the idea of consent is whether students have incentives to consent. The mechanism designer would like to give incentives to the students to consent, because each additional consent relaxes the partial fairness constraint and provides the mechanism designer with the opportunity to choose a matching closer to the efficiency frontier. Consequently, one should look for rules that guarantee a student will not be made worse off if she consents.

[16] See also Kesten and Kurino (2013) and Tang and Yu (2014) for a discussion of the same concept. The terminology in Tang and Yu (2014) is different: the authors refer to a school with no demand as a *tier-0 underdemanded* school and refer to an underdemanded school as a *tier-k underdemanded* school for $k > 0$.

[17] This easily follows from Remark 3 in Appendix C.

[18] See Lemma 8 in Appendix E for a formal statement and proof.

Definition 2. A rule ψ *provides incentives to consent* if for any problem, for each $i \in I$ and $s \in S$, student i who consents for s does not get a better assignment by not consenting for s. That is, take any (R, C) and define C' as $C'(s) = C(s) \setminus \bigcup_{j \in I}\{(i, j)\}$ and $C'(s') = C(s')$ for each $s' \in S \setminus \{s\}$. A rule ψ provides incentives to consent if $\psi_{(R,C)}(i) \; R_i \; \psi_{(R,C')}(i)$ for each $i \in I$ and $s \in S$.[19]

Our goal is to search for rules that provide incentives to consent and always yield constrained efficient outcomes that weakly Pareto dominate the SOSM. Our characterization (Theorem 1) reduces this search to the class of SEPF rules. However, a rule within the SEPF class may not provide incentives to consent.[20] Next, we introduce a rule that provides incentives to consent, and it turns out that it is the unique such rule in the SEPF class. We refer to this rule as the *top priority rule*.

4.4 *The top priority rule*

The top priority (TP) rule is a member of the class of SEPF rules; thus, its underlying algorithm corresponds to a specific cycle selection procedure.[21] So as to construct this cycle selection procedure, we introduce some definitions.

For each matching μ, let $G^T(\mu) = (I, E^T(\mu))$ be the *top priority graph* associated with μ, where the set of directed edges $E^T(\mu)$ is defined as follows: $ij \in E^T(\mu)$ if and only if student i has the highest priority for $\mu(j)$ among the students who are temporarily matched at μ and point to j in $G(\mu)$.

The top priority algorithm[22]

Step 0. Let μ_0 be the SOSM.[23]

Step $k \geq 1$. Given a matching μ_{k-1}, there are two alternatives:

(k.1) If there is no cycle in $G(\mu_{k-1})$, then the algorithm terminates and μ_{k-1} is the matching obtained.

(k.2) Otherwise, solve one of the cycles in $G^T(\mu_{k-1})$, say ϕ_k, and let $\mu_k = \phi_k \circ \mu_{k-1}$.

By definition, if i points to j in $G^T(\mu)$, then i points to j in $G(\mu)$ as well. This implies the following remark.

Remark 1. Any cycle that appears in $G^T(\mu)$ also appears in $G(\mu)$.

[19] One can easily extend the definition by constructing $C'(s)$ by only removing a student pair (i, j) from $C(s)$. All our results are robust to this extension.

[20] In Example 2 in Appendix J, student i_2 is better off by not consenting for s_3 under any rule selecting matching μ'.

[21] Note that the class of SEPF rules is defined such that the SOSM is the initial partially stable matching (Section 4.1).

[22] Example 3 in Appendix J demonstrates how the TP algorithm works.

[23] The TP algorithm can be defined with any stable matching as the initial matching. All our results are robust to this extension.

The following lemma states that if there is a cycle in $G(\mu)$, there is a cycle in $G^T(\mu)$ as well.

Lemma 1. *Let μ be a partially stable matching. There is a cycle in the graph $G(\mu)$ if and only if there is a cycle in the graph $G^T(\mu)$.*

Remark 1 and Lemma 1 imply that the TP algorithm is in the SEPF class. Consequently, the TP algorithm is nothing more than a cycle selection procedure within the class of SEPF rules. The only restriction we impose on the cycle selection procedure is that it selects a cycle in $G^T(\mu_k)$. This raises a question: Does it matter which cycle in $G^T(\mu_k)$ is chosen? We demonstrate that the outcome of the TP algorithm does not depend on the order of cycles solved. Thus, the TP algorithm defines a *rule*.

Proposition 2. *The TP algorithm defines a rule in the class of SEPF rules.*

Actually, the TP rule stands out: it is the *unique* SEPF rule that provides incentives to consent.

Theorem 2. *A rule is constrained efficient, provides incentives to consent, and improves the SOSM if and only if it is the TP rule.*

Recall that the EADAM satisfies the following property: for any student, consenting for all schools does not hurt any student (this follows from Proposition 3 of Kesten (2010)). This implies that when C satisfies the all-or-nothing property, the EADAM is outcome-equivalent to the TP rule: they select the same outcome for the same preference and consent profile. The following important implication of this result is immediate: in the Kesten (2010) setup, one cannot do better than the EADAM without sacrificing incentives to consent.

Alternatively, one can interpret incentives to consent as a protection from objections by the students due to priority violation. In particular, if a student objects to a priority violation under a rule that provides incentives to consent, then she cannot be assigned to a better school when her priorities are respected under the same rule. Thus, even if the students are not asked to consent (the case we consider in the next section), providing incentives to consent can be considered as a desirable feature.

5. Violating certain types of priorities

We now discuss another interpretation of C. In particular, we show that our approach can be adopted for the case where a school district allows certain types of priority violations, and we compare our approach to an alternative where certain priorities are removed by school districts.

The school districts are using priority classes when they rank the students. For instance, the Boston Public Schools System (BPSS) until recently used four priority classes: sibling+walk zone, sibling, walk zone, and others. Each school was giving the highest priority to the students in the sibling+walk-zone priority class, the second highest priority to the students in the sibling priority class, and third highest priority to the students in the walk-zone priority class. The ranks of the students within each priority class were determined according to a random draw. Recently, the BPSS decided to suppress walk-zone priorities and now uses only two priority classes: sibling and others (Dur et al. 2018). This demonstrates that although walk-zone priority is important for the BPSS, there is enough justification for the removal of this priority class. The school choice problem with allowable priority violations captures this phenomenon: the acquiescence to violate walk-zone priorities can be modelled as (i, j) being in $C(s)$ if and only if $i \succ_s j$, and either i has walk-zone priority at school s or i has sibling+walk-zone priority and j has sibling priority at s.[24] Thus, instead of removing a priority class, the mechanism designer may consider modelling these priority violations as allowable and using a rule within the SEPF class.

An important clarification needs to be made at this point. Our motivation for proposing the SEPF class is to improve the SOSM in terms of efficiency. However, a school district's motivation in suppressing certain priority classes is *not* necessarily to improve efficiency, and our proposal does not capture other concerns behind suppressing a priority class. Yet, our purpose here is to point out a side effect of this policy: *suppressing priorities may have perverse efficiency implications*. Below, we compare the SEPF to the approach of suppressing some priority classes on the efficiency dimension alone to point out the unintended consequences of the latter approach.

5.1 *Suppressing a priority class versus allowing violation of priorities*

One might think that when a priority class is suppressed, it reduces the number of rejection chains (the source of inefficiency of the SOSM) and leads to efficiency gains. Example 1 indicates this is not always true.

Example 1. There are two schools, $S = \{a, b\}$, and three students, $I = \{i, j, k\}$. Each school has one available seat. The preferences of students are $b\ P_i\ a\ P_i\ \varnothing$, $a\ P_j\ b\ P_j\ \varnothing$, and $a\ P_k\ b\ P_k\ \varnothing$. Each school uses four priority classes: sibling+walk zone, sibling, walk zone, and other. Student i has sibling priority at school a and walk-zone priority at school b. Students j and k belong to the "other" priority class for both schools. Suppose the random draw favors j most and i least. In this problem, the SOSM assigns i to b and j to a. However, when the walk-zone priority is suppressed, the SOSM assigns i to a and j to b without changing the assignment of k. $\diamondsuit$

Example 1 shows that suppressing walk-zone priorities might result in a stable (with respect to relaxed priorities) matching that is *Pareto inferior* to the SOSM (under the original priorities). Moreover, this approach might weaken the fairness aspect of the outcome (with respect to the suppressed priorities) without improving it in terms of efficiency and yield an undesirable allocation (with respect to the balance between efficiency improvements, fairness, and other possible concerns). This balance is already embedded in the class of SEPF rules, since each rule in this class always yields stable

[24] The $C(s)$ obtained in this manner satisfies Assumption 1.

and constrained efficient matchings (with respect to relaxed priorities) which are *Pareto superior* to the SOSM, and walk-zone priorities are violated only if they lead to inefficiencies.

6. Manipulation through preference revelation

We say that a rule is strategy-proof if, given a profile of allowable priority violations C, truthful revelation of preferences is a weakly dominant strategy for each $i \in I$. Unfortunately, there is no strategy-proof rule whose outcome Pareto dominates the SOSM (Kesten and Kurino 2013, Abdulkadiroğlu et al. 2009).[25] Since each rule within the SEPF class *Pareto-improves* over the SOSM, the following is obtained as a corollary of this result.

Proposition 3. *There is no rule within the class of SEPF rules that satisfies strategy-proofness.*

This result, combined with Theorem 1, demonstrates that constrained efficiency, weakly Pareto dominating the SOSM, and *strategy-proofness* are incompatible. Given this incompatibility result, since constrained efficiency is an indispensable property in our model, the only possible way to gain strategy-proofness is to consider all the constrained efficient matchings instead of only those that weakly Pareto dominate the SOSM. The impossibility extends.

Theorem 3. *There is no strategy-proof and constrained efficient rule.*

The notion of strategy-proofness used here requires truthful revelation to be a dominant strategy for every C. In the settings where C is exogenously given, one may wonder whether there is a characterization of C for which strategy-proofness and constrained efficiency are compatible. In Appendix H, we provide sufficient conditions that guarantee the existence of strategy-proof and constrained efficient rules, and we propose such a rule.

7. The SEPF for weak priorities

School districts usually rank students using some predetermined criteria such as proximity and sibling status. Thus, many students end up being grouped under the same priority classes. Since the student-proposing DA is defined only under strict priority orders, school districts use exogenously determined tie-breaking rules to order students within

[25]Both papers consider the environments where schools have indifferences among students and some tie-breaking rule is needed. Nevertheless, our setup can easily be considered as a special case where each indifference class consists of one student only. The Kesten and Kurino (2013) result is more general than that of Abdulkadiroğlu et al. (2009), as it does not require the existence of a null school. Since we allow for the null school, both theorems are applicable in our setup. In a recent paper, Alva and Manjunath (2019) provide a stronger result such that student-proposing DA is the unique strategy-proof rule that (weakly) Pareto improves any stable matching.

priority classes. Unfortunately, exogenous tie-breaking causes efficiency loss (Erdil and Ergin 2008, Abdulkadiroğlu et al. 2009, Kesten 2010). A solution for this efficiency loss problem is the stable improvement cycles (SIC) algorithm (Erdil and Ergin 2008). The SIC algorithm takes the SOSM for a given tie-breaking rule and then improves the assignment by utilizing trade cycles between students. The SIC algorithm defines a class of rules, since at each step there may exist more than one cycle and each cycle selection procedure possibly gives a different outcome. An alternative solution is a modified version of the EADAM (defined in Section V.D. of Kesten (2010)), which we denote as the *EADAM for weak priorities*. It turns out that there is a close connection between the SIC and the EADAM for weak priorities, which becomes apparent through their relationships to the SEPF.

We extend our model by allowing each $s \in S$ to have a weak priority order denoted by $\succsim_s$. We denote the strict priority order of school s by $\succ_s$ and denote the associated indifference relation by $\sim_s$. For this section, we assume that the set of allowable priority violations is not part of the problem (indeed, we construct this object). Thus, a problem is given by a preference profile R. All the other definitions remain the same as they are defined in Section 2.1. Given a problem R and a priority profile $\succsim$, let $f^{\succsim}(R)$ denote the set of student-optimal stable matchings in the problem. If $i \sim_s j$, then, by definition, the assignment of j to s while i prefers s to her assignment does not violate fairness. Consequently, $f^{\succsim}(R)$ may no longer be a singleton.[26] Moreover, the common method of using a single tie-breaker, generating a strict priority profile $\succ'$, and finding the SOSM according to $\succ'$ may not yield a matching in $f^{\succsim}(R)$. That is, letting $\mathcal{T}(\succsim)$ denote the set of strict priority profiles obtained by breaking the ties in $\succsim$,[27] we have $f^{\succsim}(R) \subseteq \bigcup_{\succ' \in \mathcal{T}(\succsim)} f^{\succ'}(R)$, where, as discussed in Erdil and Ergin (2008), the inclusion may be proper. This observation calls for an improvement of the SOSM with single tie-breaking, such that the end matching is in $f^{\succsim}(R)$.

Given a weak priority profile $\succsim$, let a strict priority profile $\succ'$ be obtained by breaking the ties in $\succsim$. For each $i \in I$ and $s \in S$, construct the set of allowable priority violations under $\succ'$ as[28]

$$(i, j) \in C(s) \quad \text{if and only if} \quad j \in \left(\left\{j' : i \succ'_s j'\right\} \cap \left\{j' : i \sim_s j'\right\}\right) \quad \text{for each } j \in I. \tag{1}$$

Given the constructed set of allowable priority violations C, we can now start with the SOSM under $\succ'$ and then select a rule within the SEPF class according to C. We refer to this procedure as the *SEPF for weak priorities*. Let $\Omega(R, \succ')$ denote the set of all matchings that can be obtained via the SEPF for weak priorities with a given $\succ'$ according to this procedure.

The SEPF for weak priorities inherits the constrained efficiency of the SEPF class. Consequently, an adaptation of the characterization result in Theorem 1 applies: for a given tie-breaking rule, the set of outcomes obtained by the SEPF for weak priorities is

[26]For a given strict priority profile $\succ$, we define $f^{\succ}(R)$ similarly. It is worth mentioning that $f^{\succ}(R)$ is a singleton and it contains the outcome of the student-proposing DA rule under R and $\succ$.

[27]Formally, $\mathcal{T}(\succsim)$ is the set of priority profiles such that $i \succ_s j$ implies $i \succ'_s j$ for all $i, j \in I$ and $s \in S$.

[28]The set of allowable priority violations constructed in this manner satisfies Assumption 1.

characterized by the set of student-optimal stable matchings that Pareto dominate the SOSM obtained under the given tie-breaking rule.

Proposition 4. *Given R and $\succsim$, let $\succ' \in \mathcal{T}(\succsim)$ and $\mu_0 \in f^{\succ'}(R)$. A matching is student-optimal stable and weakly Pareto dominates μ_0 if and only if it is obtained by the SEPF for weak priorities. That is, $\mu \in f^{\succsim}(R)$ and Pareto dominates μ_0 if and only if $\mu \in \Omega(R, \succ')$.*

It is easy to see that the SEPF for weak priorities and the SIC algorithm of Erdil and Ergin (2008) are outcome equivalent, i.e., for the same problem, the same tie-breaking rule, and the same cycle selection procedure they select the same matching. As a result, Proposition 4 provides a counterpart of Theorem 1 of Erdil and Ergin (2008): not only does the SIC procedure always finds a student-optimal stable matching, but it also finds *all* the student-optimal stable matchings for any given tie-breaking rule.

Another interesting side result of Proposition 4 is the connection between the SIC and EADAM for weak priorities: the EADAM for weak priorities is a member of the SIC algorithms class.

8. Conclusion

This study introduces the school choice problem with allowable priority violations. The main result is a characterization of a class of algorithms, the SEPF, each of which always yields a constrained efficient matching that weakly Pareto dominates the SOSM. A relevant interpretation is the setup where school district officials ask for students' consent for the violation of their priorities (Kesten 2010). Clearly, students having incentives to consent is indispensable in this setting. Our proposal, the TP rule, is the unique rule that satisfies this property within the SEPF class. Indeed, the mechanism designer can also attempt to provide more incentives. One such rule may be the one that favors consenting students in tie-breaking or gives the consenting students higher priorities when they apply to the school system for the higher grades. The characterization of such rules is left for future research.

Our framework also applies to settings where some priority violations are deemed feasible. We demonstrate that the SEPF class under such violations (marked as allowable) has superior welfare properties compared to suppressing these priorities. Another example for allowable priority violations is a setting with centralized placement of students to exam and regular schools. Whereas the priorities to exam schools are legal constraints that cannot be violated, the regular schools are more flexible in terms of their priorities. One can then simply adopt the framework offered in Section 2.2 and specify that priority violations in private schools are allowed for each pair of students. Each cycle selection rule within the SEPF, including the "uniform cycle selection rule," which solves each cycle at each step with equal probabilities, is guaranteed to produce a constrained efficient matching in this case.

Appendix A: The EADAM

Before providing the definition of the EADAM, we first present a notion used in the definition. If student i is tentatively accepted by school s at some step t and is rejected by s

in a later step t' of the student-proposing DA, and there exists another student j who is rejected by s in step $t'' \in \{t, t+1, \dots, t'-1\}$, then i is called an *interrupter* for s and (i, s) is called an *interrupting pair* of step t'.

Under EADAM, each student reports her preferences over the schools and her decision to consent or not. EADAM selects its outcome through the following algorithm:

Round 0. Run the student-proposing DA algorithm.

Round $k > 0$. Find the last step of the student-proposing DA run in Round $k-1$ in which a consenting interrupter is rejected from the school for which she is an interrupter. Identify all the interrupting pairs of that step with consenting interrupters. For each identified interrupting pair (i, s), remove s from the preferences of i without changing the relative order of the other schools. Rerun the student-proposing DA algorithm with the updated preference profile. If there are no more consenting interrupters, then stop.

Appendix B: Necessity of Assumption 1

We demonstrate a problem (R, C) where Assumption 1 does not hold and there are two partially stable matchings μ and $\tilde{\mu}$ such that $\tilde{\mu}$ Pareto dominates μ, but there are no cycles in $G(\mu)$. Consequently, Theorem 1 fails: any algorithm in the SEPF class that starts with μ yields μ, despite μ not being constrained efficient.

Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $q_s = 1$ for each $s \in S$. Take any C such that $(i_1, i_3) \notin C(s_2)$, $(i_1, i_4) \in C(s_2)$, and $(i_3, i_4) \notin C(s_2)$. Let the students' preferences and schools' priorities be

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}		$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
s_2	s_3	s_4	s_2		i_1	i_2	i_3	i_4
s_1	s_2	s_2	s_4		$\vdots$	i_1	i_2	i_3
$\vdots$	$\vdots$	s_3	$\vdots$		$\vdots$	i_3	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	i_4	$\vdots$	$\vdots$

The matchings $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4)\}$ and $\tilde{\mu} = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2)\}$ are partially stable.[29] Also, $\tilde{\mu}$ Pareto dominates μ. But since $X_\mu(s_1) = \varnothing$, $X_\mu(s_2) = \{i_1\}$, $X_\mu(s_3) = \{i_2\}$, and $X_\mu(s_4) = \{i_3\}$, there are no cycles in $G(\mu)$ and the algorithm terminates.

Since $i_3 \in D_\mu(s_2)$, $i_3 \succ_{s_2} i_4$ and $(i_3, i_4) \notin C(s_2)$ (that is, i_3 prevents i_4 from pointing to i_2 in $G(\mu)$), $i_4 \notin X_\mu(s_2)$. Similarly, i_3 is prevented from pointing to i_2 by i_1. This would not occur if Assumption 1 was retained. Since $(i_1, i_3) \notin C(s_2)$ and $i_3 \succ_{s_2} i_4$, Assumption 1 requires that $(i_1, i_4) \notin C(s_2)$, and $\tilde{\mu}$ would not be partially stable.

Appendix C: Proof of Theorem 1

The statement in the theorem is for the case where the initial matching μ_0 is the SOSM. Instead of this special case, we provide the proof of a more general result such that μ_0

[29]The matching μ is the SOSM and *partial stability* of μ follows immediately. For $\tilde{\mu}$, the only student who *envies* another student is i_1. She prefers $s_2 = \tilde{\mu}(i_4)$ to $\tilde{\mu}(i_1)$, and $i_1 \succ_{s_2} i_4$. Nevertheless, $(i_1, i_4) \in C(s_2)$.

can be any partially stable matching. That is, we show that for each problem (R, C) and for each partially stable matching μ_0, a matching is constrained efficient and weakly Pareto dominates μ_0 if and only if it is obtained by an algorithm in the SEPF class starting with μ_0.

Remark 2. In the graph $G(\mu)$, if i points to j, then i points to each student in $\mu^{-1}(\mu(j))$.

Remark 2 follows from a simple observation: i points to j if and only if $i \in X_\mu(\mu(j))$, and, thus, i also points to student i' if $\mu(i') = \mu(j)$.

For a given problem (R, C) and a partially stable initial matching μ_0, consider an algorithm in the SEPF class. Let K be the last step of the algorithm and let μ_k be the matching selected by the algorithm at step $k \in \{1, \dots, K-1\}$. A cycle is solved at each step of the algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving the cycle. Thus, the matching at each step Pareto dominates the matching in the previous step. This implies that for a student i, if a school s is better than $\mu_k(i)$, then it is also better than $\mu_{k-1}(i)$.

Remark 3. For each $k \geq 1$ and each $s \in S$, $D_{\mu_k}(s) \subseteq D_{\mu_{k-1}}(s)$.

A consequence of Remark 3 is that if i points to j in the graph $G(\mu_{k-1})$ and i is not better off at Step k, then in the graph $G(\mu_k)$, i points to the students who are assigned to school $\mu_{k-1}(j)$ at μ_k. In particular, if i points to j and both are not better off at a step of the algorithm, then i points to j in the next step as well. To see this, let cycle $\phi_k = \{i_1 i_2, i_2 i_3, \dots, i_n i_1\}$ be solved in the graph $G(\mu_{k-1})$ such that $\mu_k = \phi_k \circ \mu_{k-1}$. Suppose i points to j in the graph $G(\mu_{k-1})$ and $i, j \notin \{i_1, i_2, \dots, i_n\}$. By definition of the graph $G(\mu_{k-1})$, $i \in X_{\mu_{k-1}}(s)$, where $s = \mu_{k-1}(j)$. Since $\mu_k(i) = \mu_{k-1}(i)$, $i \in D_{\mu_k}(s)$. Let $i' \in D_{\mu_k}(s)$ be such that $i' \succ_s i$. By Remark 3, $i' \in D_{\mu_{k-1}}(s)$. Thus, since $i \in X_{\mu_{k-1}}(s)$ and $i' \succ_s i$, we have $(i', i) \in C(s)$. We conclude that for each student $\tilde{j} \in D_{\mu_k}(s)$ with a higher priority than student i at school s, $(\tilde{j}, i) \in C(s)$. Consequently, $i \in X_{\mu_k}(s)$. Since $s = \mu_k(j) = \mu_{k-1}(j)$, i points to j in the graph $G(\mu_k)$.

Remark 4. If i points to j in $G(\mu_{k-1})$ and both students' assignments do not change at Step k, then i points to j in $G(\mu_k)$.

C.1 *Proof of the "if" part*

Lemma 2. *Each matching obtained by an algorithm within the SEPF class is partially stable.*

Proof. (i) *Partial fairness.* Let μ_k be the matching at step $k \in \{0, 1, \dots, K-1\}$ of an algorithm in the SEPF class. We prove this statement by induction on k. The initial matching, μ_0, is *partially fair*.

As an inductive hypothesis, suppose μ_{k-1} is partially fair. Take any student–school pair (i, s) such that $s \, P_i \, \mu_k(i)$. At each step of the algorithm, each student is either better off (she is in the solved cycle) or she is assigned to the same school as in the previous step. Thus, for each $\ell \in I$, $\mu_k(\ell)\ R_\ell\ \mu_{k-1}(\ell)$. Since $s\ P_i\ \mu_k(i)$, this implies that $s\ P_i\ \mu_{k-1}(i)$ and $i \in D_{\mu_{k-1}}(s)$. Take any $j \in \mu_k^{-1}(s)$ with $(i, j) \notin C(s)$. If $j \in \mu_{k-1}^{-1}(s)$, then by partial fairness of μ_{k-1}, $j \succ_s i$. Alternatively, suppose $j \notin \mu_{k-1}^{-1}(s)$. Since $j \in \mu_k^{-1}(s)$, j is in the cycle solved at Step k. Thus, $j \in X_{\mu_{k-1}}(s)$. By assumption, $(i, j) \notin C(s)$; thus, $i \in D_{\mu_{k-1}}(s) \setminus \{i' \in I : (i', j) \in C(s)\}$. Since $j \in X_{\mu_{k-1}}(s)$, we have $j \succ_s i$. Thus, μ_k does not violate the priority of i at s and it is partially fair. The induction follows.

(ii) *Individual rationality.* Since μ_0 is individually rational and each student is weakly better off at each step of the SEPF algorithm, its outcome is individually rational.

(iii) *Non-wastefulness.* The initial matching μ_0 is non-wasteful and at each step, students are assigned to (weakly) better schools. By the definition of the SEPF algorithm, for each school s, the number of students assigned to s at each step remains the same as it is under μ_0. Thus, if $|\mu_0^{-1}(s)| = q_s$, then each matching obtained by the SEPF assigns q_s students to s. Suppose $|\mu_0^{-1}(s)| < q_s$. Since μ_0 is non-wasteful, the set $D_{\mu_0}(s)$ is empty. By Remark 3, at each step k, $D_{\mu_k}(s)$ is empty. Thus, each matching obtained by the SEPF satisfies non-wastefulness. □

Lemma 3. *For each partially stable matching μ and $s \in S$, $X_\mu(s) = \varnothing$ if and only if $D_\mu(s) = \varnothing$.*

Proof. *Only if.* Let $X_\mu(s) = \varnothing$. Suppose $D_\mu(s) \neq \varnothing$. By strict priorities, there exists i who has higher priority for s than any other student in $D_\mu(s)$. By definition, $i \in X_\mu(s)$; a contradiction. *If.* The "if" part follows directly from the definition of the set $X_\mu(s)$. □

Lemma 4. *Let μ and η be partially stable matchings such that η Pareto dominates μ. For each $s \in S$, $|\eta^{-1}(s)| = |\mu^{-1}(s)|$.*

Proof. Take a school $s \in S$. First, we show that $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$. Suppose $|\eta^{-1}(s)| > |\mu^{-1}(s)|$ and let $i \in \eta^{-1}(s) \setminus \mu^{-1}(s)$. Since η Pareto dominates μ and preferences are strict, this implies that $s\ P_i\ \mu(i)$. But since $|\mu^{-1}(s)| < q_s$, this violates non-wastefulness of μ. This implies that for each $s \in S$, $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$.

Second, we show that $|\eta^{-1}(s)| \geq |\mu^{-1}(s)|$. Suppose $|\eta^{-1}(s)| < |\mu^{-1}(s)|$. Adding together all schools and using the previous finding, we have $\sum_{s\in S} |\eta^{-1}(s)| < \sum_{s\in S} |\mu^{-1}(s)|$. But since η Pareto dominates μ and both matchings are partially stable, if a student is assigned to a school under μ, then she is also assigned to a school under η. This implies that $\sum_{s\in S} |\eta^{-1}(s)| \geq \sum_{s\in S} |\mu^{-1}(s)|$, a contradiction. □

Lemma 5. *For a given problem (R, C), a matching obtained by an algorithm within the SEPF class is constrained efficient.*

We use the following concept in the proof: given $G = (V, E)$, a path $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a *chain* if for each $j \in V$, $j i_1 \notin E$ and $i_{n+1} j \notin E$, and vertex i_1 is called the *tail* of this chain.

Proof of Lemma 5. Let μ be a matching obtained by an algorithm within the SEPF class. By Lemma 2, μ is partially stable. We show that a partially stable matching that

Pareto dominates μ does not exist. On the contrary, suppose a partially stable matching $\widetilde{\mu}$ Pareto dominates μ. Note that, by definition of the SEPF class, there is no cycle in the graph $G(\mu)$. There are two possible cases.

Case 1: *There are no chains in* $G(\mu)$. Then, for each $s \in S$, $X_\mu(s) = \varnothing$. By Lemma 3, this implies that $D_\mu(s) = \varnothing$. Thus, at μ, each student is assigned to her best school and μ is Pareto efficient.

Case 2: *There is a chain in* $G(\mu)$. Let I_1 be the set of students who are not pointed to by any other student.[30] Let $i_1 \in I_1$ with $\mu(i_1) = s_1$. Since i_1 is not pointed to by any student, $X_\mu(s_1) = \varnothing$. Then, by Lemma 3, $D_\mu(s_1) = \varnothing$. Since $\widetilde{\mu}$ Pareto dominates μ, there does not exist $i \in I$ such that $\mu(i) \neq s_1$ but $\widetilde{\mu}(i) = s_1$. Thus, $\widetilde{\mu}^{-1}(s_1) \subseteq \mu^{-1}(s_1)$. By Lemma 4, $|\widetilde{\mu}^{-1}(s_1)| = |\mu^{-1}(s_1)|$. Thus, $\mu^{-1}(s_1) = \widetilde{\mu}^{-1}(s_1)$. Since i_1 is chosen arbitrarily, this holds for each $s \in S$ such that $\mu^{-1}(s) \subseteq I_1$. Let S_1 denote the set of these schools. That is, for each $s \in S_1$, $\mu^{-1}(s) = \widetilde{\mu}^{-1}(s)$.

There exists at least one student in $I \setminus I_1$ such that she is pointed to only by students in I_1. Otherwise, there is a cycle in $G(\mu)$ or there are no chains in $G(\mu)$, a contradiction. Let I_2 be the set of such students and take some $i_2 \in I_2$, $s_2 = \mu(i_2)$. We first show that there does not exist $j \in I$ such that $\mu(j) \neq s_2$ but $\widetilde{\mu}(j) = s_2$. To see why, assume to the contrary. Since $\widetilde{\mu}$ Pareto dominates μ, this implies that $s_2 \, P_j \, \mu(j)$ and, thus, $j \in D_\mu(s_2)$. Nevertheless, we must have $j \notin X_\mu(s_2)$. This is because otherwise $j \in I_1$ (recall that i_2 is pointed to only by students in I_1). Thus, by the previous paragraph, we must have $\mu(j) = \widetilde{\mu}(j)$, a contradiction. We conclude that $j \notin X_\mu(s_2)$ and it implies that for some $j' \in D_\mu(s_2)$, $(j', j) \notin C(s_2)$ and $j' \succ_{s_2} j$. Let i be the student with highest priority for s_2 among such j'. Note that $i \in X_\mu(s_2)$,[31] and since i_2 is pointed to only by students in I_1, $i \in I_1$. Then, since i is assigned to the same school both under μ and $\widetilde{\mu}$, the assignment of j to s_2 under matching $\widetilde{\mu}$ violates the priority of i at s_2, which contradicts the partial fairness of $\widetilde{\mu}$. Thus, there does not exist a student j such that $\mu(j) \neq s_2$ but $\widetilde{\mu}(j) = s_2$. Once again, by Lemma 4, $\mu^{-1}(s_2) = \widetilde{\mu}^{-1}(s_2)$. Let S_2 denote the set of the schools such that for each $s \in S_2$, $\mu^{-1}(s) \subseteq I_2$.

Now we can continue in the same manner. If there is a student in $I \setminus (I_1 \cup I_2)$ who is pointed to by a student in $G(\mu)$, then at least one of them, say i_3, is pointed to only by students in $I_1 \cup I_2$. By the same argument, the same students are assigned to $\mu(i_3)$ under both μ and $\widetilde{\mu}$.

Repeating the argument once more, we conclude that all students in a chain in $G(\mu)$ have the same assignment under μ and $\widetilde{\mu}$. The students who are not in a chain in $G(\mu)$, i.e., those who are not pointed to by a student and who are not pointing to a student, are contained in I_1, and, as argued above, they have the same assignment under μ and $\widetilde{\mu}$ as well. Thus, the matchings μ and $\widetilde{\mu}$ coincide for all students. We conclude that $\mu = \widetilde{\mu}$ and $\widetilde{\mu}$ cannot Pareto dominate μ. □

[30] Note that, by definition, I_1 contains all the students who are at the tail of some chain in $G(\mu)$.

[31] Otherwise, there must be another student $i' \in D_\mu(s_2)$, $(i', i) \notin C(s_2)$ and $i' \succ_{s_2} i$. But by Assumption 1, $(i', i) \notin C(s_2)$ and $i \succ_{s_2} j$ implies that $(i', j) \notin C(s_2)$. This would be a contradiction to i being the student with highest priority for s_2 among $j' \in D_\mu(s_2)$ with $(j', j) \notin C(s_2)$ and $j' \succ_{s_2} j$.

Since the matching obtained at each step of an algorithm within the SEPF class improves the matching in the previous step, it improves over μ_0. This completes the proof of the "if" part of the theorem.

C.2 *Proof of the "only if" part*

Let μ_0 be a partially stable matching. We prove that each constrained efficient matching that Pareto dominates μ_0 can be obtained by an algorithm within the SEPF class. We first introduce an auxiliary notion called improvement cycle.

Definition 3. An *improvement cycle* ϕ over a matching μ is a cycle such that for each $ij \in \phi$, $\mu(j)\ P_i\ \mu(i)$.

The following lemma is the crucial first step in the construction of a cycle that appears in the graph used by the algorithms in the SEPF class.

Lemma 6. *Let μ and η be partially stable matchings such that η Pareto dominates μ. Then there exists a set of disjoint improvement cycles $\Phi = \{\phi_1, \dots, \phi_m\}$ such that $\eta = \phi_m \circ \cdots \circ \phi_1 \circ \mu$.*

Proof. Let $N \subseteq I$ be the set of students who are strictly better off under η. Let $\widetilde{G}(\mu, \eta) = (N, E)$ be a directed graph where the edges $E \subseteq N \times N$ are such that each $i \in N$ points to a unique student in $N \cap \mu^{-1}(\eta(i))$ and each student in N is pointed to by a unique student.

We claim that such a graph $\widetilde{G}(\mu, \eta)$ can be constructed. By Lemma 4, for each s, $|\mu^{-1}(s)| = |\eta^{-1}(s)|$. Thus, $\mu^{-1}(s) \neq \eta^{-1}(s)$ implies $|\mu^{-1}(s) \setminus \eta^{-1}(s)| = |\eta^{-1}(s) \setminus \mu^{-1}(s)|$. Moreover, each $i \in \mu^{-1}(s) \setminus \eta^{-1}(s)$ is pointed to by one of the students in $\eta^{-1}(s) \setminus \mu^{-1}(s)$. Thus, one can construct the graph $\widetilde{G}(\mu, \eta)$.[32]

Since each student in N is pointed to by a unique student and points to a unique student in N, each student is in a cycle and no two cycles intersect. By construction, each of these disjoint cycles is an improvement cycle over μ, and the matching η is obtained by solving these cycles in any order. $\square$

An improvement cycle does not necessarily appear in the graph constructed by the SEPF algorithm. To complete our proof, we prove a result similar to Lemma 6 for the cycles that appear in the graph constructed by the SEPF at each step.

Lemma 7. *Let μ and η be partially stable matchings such that η Pareto dominates μ. Then there exists a sequence of cycles $(\gamma_1, \dots, \gamma_n)$ such that*

- *γ_1 appears in $G(\mu)$*
- *for each $k \in \{2, \dots, n\}$, γ_k appears in $G(\gamma_{k-1} \circ \cdots \circ \gamma_1 \circ \mu)$*
- *$\gamma_n \circ \cdots \circ \gamma_1 \circ \mu = \eta$.*

[32] Clearly, there are multiple graphs that one can construct following this method, but any of these graphs is sufficient to demonstrate the existence of a set of improvement cycles.

Proof. By Lemma 6, one can construct a set of improvement cycles $\Phi = \{\phi_1, \dots, \phi_m\}$. The result is trivial for the case where all the cycles in Φ appear in $G(\mu)$: it follows that there are disjoint cycles in $G(\mu)$, and solving them in any order leads to η. To prove the alternative case, we assume that none of the cycles in Φ appears in $G(\mu)$. This assumption is without loss of generality because of the following observation: if a cycle $\phi \in \Phi$ appears in $G(\mu)$, then this cycle is solved first and $\mu' = \phi \circ \mu$ is obtained; if another cycle $\phi' \in \Phi$ also appears in $G(\mu)$, by the fact that all cycles in Φ are disjoint and by Remark 4, it also appears in $G(\mu')$. Thus, we can solve ϕ' and obtain $\mu'' = \phi' \circ \mu'$. Following this logic, whenever a subset of cycles in Φ appears in $G(\mu)$, these cycles are solved first. Consequently, we focus on the case where none of the improvement cycles appears in $G(\mu)$.

To show the existence of a cycle in $G(\mu)$, first we prove that for any $\phi \in \Phi$ and any $ij \in \phi$, there exists some $k \in I$ such that $kj \in G(\mu)$ and $\ell k \in \phi'$ for some $\ell \in I$, $\phi' \in \Phi$. Take any $\phi \in \Phi$ and $ij \in \phi$.

- If $i \in X_\mu(\mu(j))$, then this edge also appears on $G(\mu)$ by construction. Moreover, i is a part of ϕ, which implies that there exists some $\ell \in I$ with $\ell i \in \phi$.

- If $i \notin X_\mu(\mu(j))$, there exists a student i' such that $i' \in D_\mu(\mu(j))$, $(i', i) \notin C(\mu(j))$, and $i' \succ_{\mu(j)} i$. Let k be the student with the highest priority for school $\mu(j)$ among such students. Note that[33] $k \in X_\mu(\eta(i))$ and, thus, $kj \in G(\mu)$. Moreover, we claim that k is in an improvement cycle in Φ; that is, there exists $\phi' \in \Phi$ such that $\ell k \in \phi'$ for some $\ell \in I$. If $\eta(k) = \mu(k)$, then since $\mu(j)\ P_k\ \eta(k)$, $(k, i) \notin C(\mu(j))$, and $k \succ_{\mu(j)} i$, there is a priority violation at $\eta(i)$, which is not allowed, contradicting partial stability of η. Thus, $\eta(k)\ P_k\ \mu(k)$, which implies that k is in an improvement cycle in Φ. Consequently, $\ell k \in \phi'$ for some $\ell \in I$, $\phi' \in \Phi$.

Thus, for any student j who is in an improvement cycle $\phi \in \Phi$, there exists another student k such that $kj \in G(\mu)$ and k is in an improvement cycle $\phi' \in \Phi$. But since the set of students in improvement cycles is finite and each student is pointed to by another student in this set, there exists a cycle denoted by γ_1 in $G(\mu)$.

We next show that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and is (weakly) Pareto dominated by η. For any $kj \in \gamma_1$, note that $(\gamma_1 \circ \mu)(k) = \mu(j)$.

- If $kj \in \phi$ for some $\phi \in \Phi$, then $\mu(j) = \eta(k)$.

- If $kj \notin \phi$ for any $\phi \in \Phi$, we claim that $\eta(k)\ R_k\ \mu(j)$. Suppose $\mu(j)\ P_k\ \eta(k)$, that is, $k \in D_\eta(\mu(j))$. Consider the student $i \in I$ such that $ij \in \phi$ for some $\phi \in \Phi$. Since $ij \notin G(\mu)$, we must have $(k, i) \notin C(\eta(i))$ and $k \succ_{\eta(i)} i$. Thus, matching η *violates the priority* of student k at school $\eta(i)$, a contradiction.

Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than under the matching μ and weakly worse off than under the matching η. Each remaining student is assigned

[33] Note the subtle role played by Assumption 1 here. If $k \notin X_\mu(\eta(i))$, then there must be another student $\ell \in I$ such that $\ell \in D_\mu(\mu(j))$, $(\ell, k) \notin C(\mu(j))$, and $\ell \succ_{\mu(j)} k$. But by Assumption 1, $(\ell, k) \notin C(\mu(j))$ and $k \succ_{\mu(j)} i$ imply that $(\ell, i) \notin C(\mu(j))$. This is a contradiction to i being the student with the highest priority for $\mu(j)$ among $i' \in D_\mu(\mu(j))$ with $(i', i) \notin C(\mu(j))$ and $i' \succ_{\mu(j)} i$.

to the same school to which she is assigned under μ, which implies that the matching $\gamma_1 \circ \mu$ Pareto dominates μ and is weakly Pareto dominated by η. Moreover, by the same argument in Lemma 2, $\gamma_1 \circ \mu$ is partially stable. If the matching $\gamma_1 \circ \mu$ is equivalent to η, the proof is complete. If not, we use the same argument inductively: by Lemma 6, there is a set of distinct improvement cycles, such that the matching η is obtained by solving these cycles over $\gamma_1 \circ \mu$ and one can construct a cycle that appears in the graph used by SEPF. □

Appendix D: Proof of Lemma 1

Let I^t_μ denote the set of temporarily matched students at μ. *Only if.* Since there is a cycle in $G(\mu)$, the set of temporarily matched students, I^t_μ, is nonempty. By definition of the graph $G^T(\mu)$, each student in I^t_μ is pointed to by a unique student in I^t_μ. Thus, by finiteness of I^t_μ, there exists a cycle in $G^T(\mu)$. In particular, each cycle in $G^T(\mu)$ is formed by the students in I^t_μ. *If.* The "if" part follows from Remark 1.

Appendix E: Proof of Proposition 2

As argued in Section 4.4, Lemma 1 and Remark 1 imply that the TP algorithm is a member of the SEPF class.

Lemma 8. *Let φ be a rule within the SEPF class. For a given problem (R, C) and partially stable initial matching μ_0, suppose student i is assigned to an underdemanded school s at Step k of the underlying algorithm of φ. Then, at each further step, she is not part of any cycle; thus, she is assigned to s at any further step.*

Proof. Let K be the last step of the algorithm underlying the rule φ and let μ_k be the matching selected at step $k \in \{1, \dots, K-1\}$. Remember that if a school s is underdemanded at μ_k, either the school has no demand at μ_k or no student on a path to a student in $\mu_k^{-1}(s)$ is part of a cycle in $G(\mu_k)$. Let $S^u_k \subseteq S$ denote the set of underdemanded schools at μ_k. We first argue that S^u_k has a specific structure. Let S^1_k be the set of schools that has no demand at μ_k. By definition, $S^1_k \subseteq S^u_k$. Let S^2_k be the set of schools such that, for a school $s_2 \in S^2_k$, the students in $\mu_k^{-1}(s_2)$ are pointed to only by students in $\bigcup_{s_1 \in S^1_k} \mu_k^{-1}(s_1)$. Let S^3_k be the set of schools such that, for $s_3 \in S^3_k$, the students in $\mu_k^{-1}(s_3)$ are pointed to only by students in $\bigcup_{s \in S^1_k \cup S^2_k} \mu_k^{-1}(s)$. Continuing in this manner, we can decompose S^u_k into groups such that $S^u_k = S^1_k \cup S^2_k \cup \cdots \cup S^m_k$ for some $m \geq 1$. In this decomposition, given a $\widetilde{m} \in \{2, \dots, m\}$, for any $s_{\widetilde{m}} \in S^{\widetilde{m}}_k$, the students in $\mu_k^{-1}(s^{\widetilde{m}}_k)$ are pointed to only by students in $\bigcup_{s \in S^1_k \cup \cdots \cup S^{\widetilde{m}-1}_k} \mu_k^{-1}(s)$.

Take any $s_1 \in S^1_k$. Since s_1 has *no demand* at μ_k, $X_{\mu_k}(s_1) = \varnothing$. By Lemma 3, $D_{\mu_k}(s_1) = \varnothing$. Moreover, by Remark 3, $D_{\mu_k}(s_1) = \varnothing$ implies that, for each $k' \geq k$, $D_{\mu_{k'}}(s_1) = \varnothing$. Since by definition, $X_\mu(s_1) \subseteq D_\mu(s_1)$, for each $k' \geq k$, $X_{\mu_{k'}}(s_1) = \varnothing$. Consequently, any $i \in \mu_k^{-1}(s_1)$ is not part of any cycle in steps $k' \geq k$.

If $S^u_k = S^1_k$, the argument is complete. If not, there exists at least one school in $s_2 \in S^2_k$. Otherwise, some students in $(S^u_k \setminus S^1_k)$ must be part of a cycle, a contradiction. Let $s_2 \in S^2_k$.

Suppose, for some student $i \in I$, $i \in (D_{\mu_k}(s_2) \setminus X_{\mu_k}(s_2))$. Then there exists $j \in (D_{\mu_k}(s_2) \setminus \{j' \in I : (j', i) \in C(s_2)\})$ such that $j \succ_{s_2} i$ and j points to the students in $\mu_k^{-1}(s_2)$. But we know that $j \in \mu_k^{-1}(s_1)$ for some $s_1 \in S_k^1$ (recall that students in $\mu_k^{-1}(s_2)$ are pointed to only by students in $\bigcup_{s_1 \in S_k^1} \mu_k^{-1}(s_1)$) and, by the argument above, j is not a part of a cycle in steps $k' \geq k$. Therefore, at each step k', $i \notin X_{\mu_k}(s_2)$. Thus, at each step $k' \geq k$, the students assigned to s_2 are pointed to only by the students who are assigned to schools with no demand and, thus, they are not part of any cycle.

Now we can continue in the same manner. For any $\widetilde{m} \in \{2, \dots, m\}$, if $S_k^u \setminus (S_k^1 \cup \dots \cup S_k^{\widetilde{m}-1})$ is nonempty, then there is at least one school in $S_k^{\widetilde{m}}$. Otherwise, some students in $S_k^u \setminus (S_k^1 \cup \dots \cup S_k^{\widetilde{m}-1})$ must be part of a cycle, a contradiction. Let $s_{\widetilde{m}} \in S_k^{\widetilde{m}}$. By the same argument above, at each step $k' \geq k$, the students assigned to $s_{\widetilde{m}}$ are pointed to only by the students who are assigned to underdemanded schools and, thus, they are not part of any cycle. Repeating the same argument once more, we conclude that all students who are assigned to a school in S_k^u are not part of any cycle at any step $k' \geq k$; thus, they are not part of any cycle. The result follows. □

Lemma 8 justifies the language we use for the students assigned to underdemanded schools at some matching μ: by definition, a permanently matched student i at μ is assigned to an underdemanded school at μ and she is not part of any cycle through an algorithm within the SEPF class that selects μ at some step. Thus, at each constrained efficient matching that weakly Pareto dominates μ, i is assigned to $\mu(i)$.

Lemma 9. *For a given problem (R, C) and partially stable initial matching μ_0, let μ_{k-1} be the matching selected at Step $k-1$ of the TP algorithm. In the graph $G^T(\mu_{k-1})$, let cycle $\phi_k = \{i_1i_2, i_2i_3, \dots, i_ni_1\}$ be solved by the TP algorithm such that $\mu_k = \phi_k \circ \mu_{k-1}$. Then, for each $i \notin \{i_1, i_2, \dots, i_n\}$, if i points to j in $G^T(\mu_{k-1})$, then i points to j in $G^T(\mu_k)$.*

Proof. A student can be pointed to by at most one student in $G^T(\mu_{k-1})$. Since i points to j in $G^T(\mu_{k-1})$, $i \notin \{i_1, i_2, \dots, i_n\}$ implies $j \notin \{i_1, i_2, \dots, i_n\}$. Since $\mu_{k-1}(i) = \mu_k(i)$ and $\mu_{k-1}(j) = \mu_k(j)$, by Remark 4 and Lemma 1, i points to j in $G(\mu_k)$. Suppose student $i' \neq i$ points to j in $G^T(\mu_k)$. Thus, $i' \in D_{\mu_k}(\mu_k(j))$ and i' is temporarily matched at μ_k. Since $\mu_k(j) = \mu_{k-1}(j)$, by Remark 3, $i' \in D_{\mu_{k-1}}(\mu_{k-1}(j))$. Moreover, since i is temporarily matched at μ_k, i' has a higher priority at the school $\mu_{k-1}(j)$ than i. Thus, that i (but not i') points to j in the graph $G^T(\mu_{k-1})$ implies that i' is permanently matched at μ_{k-1}. This contradicts Lemma 8, which implies that for each $k' \geq k-1$, i' must be permanently matched at $\mu_{k'}$. □

Lemma 10. *For a given problem (R, C) and partially stable initial matching μ_0, let μ_{k-1} be the matching selected at Step $k-1$ of the TP algorithm. If a cycle ϕ in the graph $G^T(\mu_{k-1})$ is not solved at Step k, then ϕ appears in the graph $G^T(\mu_k)$.*

Proof. Let $ij \in \phi$. By Remark 4, i points to j in $G(\mu_k)$. Since this holds for each edge in ϕ, ϕ is a cycle in $G(\mu_k)$. Thus, i is temporarily matched at μ_k. By Lemma 9, this implies that i points to j in $G^T(\mu_k)$. Since this holds for each $ij \in \phi$, the graph $G^T(\mu_k)$ contains cycle ϕ. □

Lemma 11. *Let μ_0 be a partially stable matching for problem (R, C). Consider a cycle selection order denoted by $\Phi = (\phi_1, \phi_2, \dots, \phi_K)$ such that ϕ_k appears in $G^T(\mu_{k-1})$ for all $k \in \{1, 2, \dots, K\}$, where $\mu_k = \phi_k \circ \mu_{k-1}$. Let μ be the outcome of the TP algorithm under Φ. If there exists $\tilde{k} \in \{1, \dots, K-1\}$ such that $\phi_{\tilde{k}+1}$ appears in $G^T(\mu_{\tilde{k}-1})$, then the TP algorithm selects μ for the cycle selection order $\hat{\Phi} = (\phi_1, \dots, \phi_{\tilde{k}-1}, \phi_{\tilde{k}+1}, \phi_{\tilde{k}}, \phi_{\tilde{k}+2}, \dots, \phi_K)$.*

Proof. Let ν_k be the matching selected at Step k of the TP algorithm under $\hat{\Phi}$. Since in the first $\tilde{k}-1$ steps, the same cycles are solved under both $\hat{\Phi}$ and Φ, we have $\mu_k = \nu_k$ for all $k \le \tilde{k}-1$. Thus, $G^T(\mu_{\tilde{k}-1}) = G^T(\nu_{\tilde{k}-1})$. That is, $\phi_{\tilde{k}+1}$ and $\phi_{\tilde{k}}$ exist in $G^T(\nu_{k-1})$. Moreover, $\phi_{\tilde{k}+1}$ and $\phi_{\tilde{k}}$ are disjoint. When $\phi_{\tilde{k}+1}$ is solved at Step k, by Lemma 10, $\phi_{\tilde{k}}$ exists in $G^T(\nu_k)$. We have $\mu_{\tilde{k}+1} = \nu_{\tilde{k}+1}$ and $G^T(\mu_{\tilde{k}+1}) = G^T(\nu_{\tilde{k}+1})$ since the cycles are disjoint and only the students in $\phi_{\tilde{k}+1}$ and $\phi_{\tilde{k}}$ improved to the same schools. Then ϕ_k appears in $G^T(\nu_k)$ and $\mu_k = \nu_k$ for all $k \ge \tilde{k}+1$. □

Lemma 12. *For a given problem (R, C) and partially stable initial matching μ_0, the outcome of the TP algorithm is independent of the order of cycles solved.*

Proof. We prove the result by constructing a cycle selection order, Φ, which generates the same outcome as any other cycle selection order, $\hat{\Phi}$, under the TP algorithm. The construction of Φ first requires a tie-breaker vector. Let $\pi = (\pi_i)_{i \in I}$ be such a vector, where π_i is the number assigned to student $i \in I$ and $\pi_j \neq \pi_k$ for all $j \neq k$. Given the π, the order Φ is determined as follows.

At Step $k \ge 0$, given matching μ_k:

(i) Let A_k be the set of cycles in $G^T(\mu_k)$.

(ii) Consider the cycles in $\bigcup_{\tilde{k} \le k} A_{\tilde{k}}$ that are not solved yet. (By Lemma 10, this equals A_k.) Among those, select the cycle to solve according to the following (lexicographic) cycle selection order:

  (a) For all m and m' such that $m < m' \le k$, all cycles in A_m are solved before the cycles in $A_{m'} \setminus A_m$.

  (b) For all $m \le k$, the cycles in A_m are solved according to the highest tie-breaker number of the student in the cycle.

(iii) Solve the cycle according to this order and obtain the new matching μ_{k+1}.

Suppose that under Φ, the TP algorithm terminates at step K and yields μ_{K-1}. Take any other cycle selection order $\hat{\Phi}$ one can have when the TP algorithm is applied to the problem (R, C). First note that all cycles in A_0 necessarily appear under any cycle selection order. By Lemma 10, they are solved under $\hat{\Phi}$. By Lemma 11, we can rearrange the order of cycles such that the first $|A_0|$ steps are the same as those of Φ, and the final outcome of $\hat{\Phi}$ is unchanged. This produces, say, $\tilde{\Phi}$, whose final outcome is the same as $\hat{\Phi}$ and whose first $|A_0|$ steps are the same as Φ. Since the first $|A_0|$ steps are the same, the cycles in $\bigcup_{\tilde{k} \le |A_0|} A_{\tilde{k}}$ all appear under $\tilde{\Phi}$. By Lemma 10, these cycles are solved when the

TP algorithm is run under $\tilde{\Phi}$. One can then reapply Lemma 11 and obtain another cycle selection order that yields the same outcome as when the TP algorithm is run under $\hat{\Phi}$, and yields the same matchings in the first $|A_0| + |A_{|A_0|}|$ steps. One can then continue until the cycle selection order whose final outcome is the same as $\hat{\Phi}$ and whose first K steps are the same as Φ. Thus, the TP algorithm produces the same outcome under $\hat{\Phi}$ and Φ. □

By Lemma 1 and Remark 1, the TP algorithm is in the SEPF class. By Lemma 12, any cycle selection order gives the same matching under the TP algorithm. Thus, the TP algorithm produces a unique matching and it defines a rule in the SEPF class.

Appendix F: Proof of Theorem 2

We provide the proof for the TP rule starting with an arbitrary stable matching μ_0.

F.1 *Proof of the "if" part*

The TP rule yields a constrained efficient matching that improves over the initial stable matching μ_0.[34] Thus, we need to show only that the TP rule provides incentives to consent. We use the following lemma in the proof.

Lemma 13. *Let i be a permanently matched student at μ for (R, C). Then i is permanently matched at μ for each problem (R, C'), where C and C' coincide except for i's consent.*

Proof. If i does not point to j in the graph $G(\mu)$, then i's consent for $\mu(j)$ does not affect who points to j in $G(\mu)$.[35] Since i is permanently matched at μ, by the definition of an underdemanded school, either $\mu(i)$ has no demand at μ or each path to i ends in a student assigned to a school with no demand at μ. For the former case, a school having no demand depends only on the students' preferences. Thus, a student cannot change it through her consent decision. Now assume the latter case. First, realize that the only way to change the underdemanded status of the school is through changing the directed edges in the paths leading to i. Nevertheless, if i changes her consent for a school that no student in a path leading to i is assigned to, she cannot affect the edges in the paths leading to i. This means that we can restrict attention to changes in consents to schools to which some students in the paths leading to i are assigned. Let j be a student on a path to i. Clearly, i does not point to j (otherwise, there would be a cycle involving i and i would not be permanently matched at μ). Thus, each path to i remains the same regardless of the consents of i for the schools to which the students on these

[34] This is a direct consequence of Theorem 1 and Proposition 2.

[35] The following argument clarifies this. If $\mu(i)\ R_i\ \mu(j)$, then $i \notin D_\mu(\mu(j))$. Thus, i's consent for $\mu(j)$ is never used in the construction of $G(\mu)$. Therefore, it does not determine who points to j. If $\mu(j)\ P_i\ \mu(i)$, then there is another student i' pointing to j such that i' has a higher priority than i at $\mu(j)$, and does not consent for priority violation by i and students with lower priority than i to be assigned to $\mu(j)$. But then the consent of i does not determine who points to j, because there is a higher priority and nonconsenting student.

paths are assigned. This means that i remains permanently matched at μ regardless of her consenting behavior for the schools of students in the paths leading to i. Thus, $\mu(i)$ remains underdemanded at μ for each problem (R, C'), where C and C' coincide except i's consent. We conclude that i is permanently matched at μ for such a problem. □

Next, we show that, under the TP rule, the placement of a student does not depend on her consent decision. Let (R, C) be a problem and let μ_0 be an initial partially stable matching. Let K be the last step of the TP algorithm and let μ_k be the matching selected at step $k \in \{1, \dots, K-1\}$. A temporarily matched student can potentially affect the graph $G(\mu_{k-1})$ by her consenting decision, but not $G^T(\mu_{k-1})$. This is because the only way in which a temporarily matched student i affects $G^T(\mu_{k-1})$ by not consenting for s is by being the top priority student among those who are temporarily matched and who point to $\mu_{k-1}^{-1}(s)$ in $G(\mu_{k-1})$. But in this case, i points to $\mu_{k-1}^{-1}(s)$ under $G^T(\mu_{k-1})$ anyway. Thus, her consenting decision is irrelevant. Moreover, the consent decision of a temporarily matched student does not affect the set of underdemanded schools. Thus, at each step k, the consent of only the permanently matched students at μ_{k-1} is relevant for the graph $G^T(\mu_{k-1})$. Therefore, the only students who can be better off by not consenting under the TP rule are the permanently matched students. Nevertheless, by Lemma 8, each permanently matched student at μ_{k-1} is assigned to the same school under the matching given by the TP rule. Thus, not consenting would not make her better off. Moreover, by Lemma 13, a permanently matched student at μ_{k-1} is permanently matched at μ_{k-1} regardless of her consenting decisions. Thus, she can not change her status through consenting decisions either. That is, whenever a student's consent matters at some step k of the TP rule, then that student is already assigned to her school under the matching given by the TP rule at an earlier step $k' < k$, and her consenting decision cannot affect her matching under the TP rule.

F.2 *Proof of the "only if" part*

Lemma 14. *Let (R, C) be a problem and let μ_0 be an initial stable matching. Let K be the last step of the TP algorithm and let μ_k be the matching selected at step $k \in \{1, \dots, K-1\}$. Let ψ be a rule that provides incentives to consent and gives a constrained efficient matching that weakly Pareto dominates μ_0. For each $k \in \{1, \dots, K-1\}$, $\psi_{(R,C)}$ weakly Pareto dominates μ_k.*

Proof. We prove this result by contradiction. We construct a consent profile C^* for which $\psi_{(R,C^*)}$ is not a constrained efficient matching.

Let $A_0 = \varnothing$ and let ϕ_k be the cycle solved in $G^T(\mu_{k-1})$ for each $k \in \{1, \dots, K-1\}$ under the TP rule. Suppose that there is a step $\tilde{k} \in \{1, \dots, K\}$, where $\psi_{(R,C)}$ does not weakly Pareto dominate $\mu_{\tilde{k}}$. Let k be the first such step. That is, for each $k' < k$ and $i \in I$, $\psi_{(R,C)}(i)\ R_i\ \mu_{k'}(i)$. Let $\phi_k = \{i_1 i_2, i_2 i_3, \dots, i_n i_1\}$. Since we chose k to be the *first* step at which the matching chosen is not weakly Pareto dominated, there exists a student in $\{i_1, i_2, \ldots, i_n\}$ who prefers her assignment under μ_k to $\psi_{(R,C)}$. Without loss of generality, suppose $\mu_k(i_1)\ P_{i_1}\ \psi_{(R,C)}(i_1)$. Note that $\mu_k(i_1) = \mu_{k-1}(i_2)$, and denote $\mu_{k-1}(i_2)$ with s_1.

Let $A_1 := A_0 \cup \{(i_1, s_1)\}$. Construct the following consent profile C^1: $(i_1, j) \notin C^1(s_1)$ for all $j \in I$ and the *consent* profile for the remaining schools/students is the same as C, i.e., $C^1(s) = C(s)$ for all $s \neq s_1$ and $C^1(s_1) = C(s_1) \setminus \bigcup_{j\in I}(i_1, j)$. Now, we consider two possible cases:

Case 1. Suppose $(i_1, j) \notin C(s_1)$ for all $j \in I$. In this case, $C = C^1$. Thus, we do not change the original consent profile. Since $s_1\ P_{i_1}\ \psi_{(R,C^1)}(i_1)$ and i_1 has the highest priority at s_1 among the temporarily matched students who prefer s_1 to their assignment at μ_{k-1}, assigning a temporarily matched student, who is not in $\mu^{-1}_{k-1}(s_1)$ and prefers s_1 to her matched school under μ_{k-1}, to s_1 would violate the partial fairness of $\psi_{(R,C^1)}$. Consequently, the temporarily matched students at μ_{k-1} cannot be assigned to s_1 under $\psi_{(R,C^1)}$. Moreover, the permanently matched students at μ_{k-1} must be assigned to their schools in μ_{k-1} at any constrained efficient matching that weakly Pareto dominates μ_{k-1}.[36] Thus, they cannot be assigned to s_1 under $\psi_{(R,C^1)}$ either. At each partially stable matching weakly Pareto dominating μ_{k-1}, the number of students assigned to s_1 is $|\mu_0^{-1}(s_1)| = q_{s_1}$ (Lemma 4). Thus, the students in $\mu^{-1}_{k-1}(s_1)$ cannot be assigned to a school other than s_1 under $\psi_{(R,C^1)}$. Consequently, $\mu^{-1}_{k-1}(s_1) = \psi^{-1}_{(R,C^1)}$. In particular, since $\mu_{k-1}(i_2) = s_1$, she is assigned to s_1 under $\psi_{(R,C^1)}$. That is, *if $(i_1, j) \notin C(s_1)$ for all $j \in I$, then i_2 is assigned a school worse than $\mu_k(i_2) = \mu_{k-1}(i_3)$ in $\psi_{(R,C^1)}$.*

Case 2. Suppose $(i_1, \bar{j}) \in C(s_1)$ for some $\bar{j} \in I$. Since ψ provides incentives to consent, $\psi_{(R,C)}(i_1)\ R_{i_1}\ \psi_{(R,C^1)}(i_1)$. Since $s_1\ P_{i_1}\ \psi_{(R,C)}(i_1)$, we have $s_1\ P_{i_1}\ \psi_{(R,C^1)}(i_1)$. There are two possibilities.

Case 2.1: $\psi^{-1}_{(R,C^1)}(s_1) = \mu^{-1}_{k-1}(s_1)$. Student i_2 is assigned to a school worse than $\mu_k(i_2) = \mu_{k-1}(i_3)$ in $\psi_{(R,C^1)}$ as explained in Case 1.

Case 2.2: $\psi^{-1}_{(R,C^1)}(s_1) \neq \mu^{-1}_{k-1}(s_1)$. Since both $\psi_{(R,C^1)}$ and μ_{k-1} are partially stable and Pareto dominate μ_0, by Lemma 4, $|\psi^{-1}_{(R,C^1)}(s_1)| = |\mu^{-1}_{k-1}(s_1)|$. Then there exists a $j \in (\psi^{-1}_{(R,C^1)}(s_1) \setminus \mu^{-1}_{k-1}(s_1))$. We have two subcases.

Case 2.2.a: $\mu_{k-1}(j)\ P_j\ s_1$. Then j is assigned to a school worse than $\mu_{k-1}(j)$ in $\psi_{(R,C^1)}$.

Case 2.2.b: $s_1\ P_j\ \mu_{k-1}(j)$. Since, by Lemma 4, $|\psi^{-1}_{(R,C^1)}(\mu_{k-1}(j))| = |\mu^{-1}_{k-1}(\mu_{k-1}(j))|$, there exists a $j' \in (\psi^{-1}_{(R,C^1)}(\mu_{k-1}(j)) \setminus \mu^{-1}_{k-1}(\mu_{k-1}(j)))$. If $\mu_{k-1}(j')\ P_{j'}\ \mu_{k-1}(j)$, then j' is assigned to a school worse than $\mu_{k-1}(j')$ in $\psi_{(R,C^1)}$. Alternatively, consider the case where $\mu_{k-1}(j)\ P_{j'}\ \mu_{k-1}(j')$. We claim that we can find a student $\tilde{j}$ where $\tilde{j}$ is assigned to a school worse than $\mu_{k-1}(\tilde{j})$ under $\psi_{(R,C^1)}$.

[36]Since $\psi_{(R,C^1)}$ is constrained efficient and weakly Pareto dominates μ_{k-1}, the matching $\psi_{(R,C^1)}$ is obtained by solving a sequence of cycles that all appear in the graph used by the rules in the SEPF class (Theorem 1). But, by Lemma 8, a permanently matched student at μ_{k-1} is not a part of any cycle. Thus, she must be assigned to her school under μ_{k-1}.

To see why, first note that $\mu_{k-1}(j)\ P_{j'}\ \mu_{k-1}(j')$. Thus, $j' \in D_{\mu_{k-1}}(\mu_{k-1}(j))$. But recall that $(i_1, j) \notin C^1(s_1)$ and $\psi_{(R,C^1)}(j) = s_1$. Thus, by partial fairness of ψ, $j \succ_{s_1} i_1$. Nevertheless, i_1 points to i_2 in $G^T(\mu_{k-1})$, which implies that j is permanently matched under μ_{k-1}. For this, either (i) j' is permanently matched under μ_{k-1} as well or (ii) there exists a student j'' temporarily matched under μ_{k-1}, such that $j'' \succ_{\mu_{k-1}(j)} j'$ and $(j'', j') \notin C^1(\mu_{k-1}(j))$. In case (i), take a student in $(\psi^{-1}_{(R,C^1)}(\mu_{k-1}(j')) \setminus \mu^{-1}_{k-1}(\mu_{k-1}(j')))$. In case (ii), take a student in $(\psi^{-1}_{(R,C^1)}(\mu_{k-1}(j'')) \setminus \mu^{-1}_{k-1}(\mu_{k-1}(j'')))$. Continue until finding a student $\tilde{j}$ such that $\mu_{k-1}(\tilde{j})\ P_{\tilde{j}}\ \psi_{(R,C^1)}(\tilde{j})$. Because we are always following students assigned to underdemanded schools, the process cannot cycle and eventually ends up with such a student.

Let us summarize everything we have done so far. We began with Step k, where ψ does not weakly Pareto dominate μ_k. Next, we found a student–school pair (i_1, s_1), with the property that $s_1\ P_{i_1}\ \psi_{(R,C)}(i_1)$. Then we found a consent profile C^1, where $(i_1, i') \notin C^1(s_1)$ for all $i' \in I$, and a step $k^1 \leq k$ with the following property: for some $\ell \in I$, $\mu_{k^1}(\ell)\ P_\ell\ \psi_{(R,C^1)}(\ell)$ and for all $\bar{k} < k^1$ and all $\bar{i}$, $\psi_{(R,C^1)}(\bar{i})\ P_{\bar{i}}\ \mu_{\bar{k}}(\bar{i})$.

Now, we repeat the whole argument. Let step k^1 be as defined in the previous paragraph, and let $(i_2, s_2) := (\ell, \mu_{k^1}(\ell))$ be the student–school pair. By construction, this pair satisfies the property that $s_2\ P_{i_2}\ \psi_{(R,C^1)}(i_2)$. Let $A_2 := A_1 \cup \{(i_2, s_2)\}$. Consider C^2, where $(i_2, i') \notin C^2(s_2)$ for all $i' \in I$ and the consent profile for the remaining schools/students is the same as C^1. Following the exact same argument, one can find a step $k^2 \leq k^1$ with the following property: for some $\ell \in I$, $\mu_{k^2}(\ell)\ P_\ell\ \psi_{(R,C^2)}(\ell)$.

In general, at each step m, given A_{m-1} and k^{m-1}, take this pair and let $(i_m, s_m) := (\ell, \mu_{k^{m-1}}(\ell))$. Define $A_m = A_{m-1} \cup \{(i_m, s_m)\}$ and find a consent profile C^m, where $(i_m, i') \notin C^m(s_m)$ for all $i' \in I$, and a step $k^m \leq k^{m-1}$ with the following property: for some $\ell \in I$, $\mu_{k^m}(\ell)\ P_\ell\ \psi_{(R,C^m)}(\ell)$.

Realize that k^m is a weakly decreasing sequence and that A_m is expanding at each step. These two facts, combined with the finiteness of student and school sets, imply that eventually the next pair (i_{m+1}, s_{m+1}) will be a pair that is already in A_m. That is, the process will run into a cycle. Fix the consent profile C^m and step k^m at this moment, and denote them by C^* and k^*, respectively. Now we have a consent profile C^*, a step k^*, and a cycle of students $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_m i_1\}$ that appears in $G^T(\mu_{k^*})$, with the following property: for each $n \in \{1, \ldots, m\}$, $\mu_{k^*}(i_n)\ P_{i_n}\ \psi_{(R,C^*)}(i_n)$. Since the solution of this cycle ϕ does not violate partial fairness of $\psi_{(R,C^*)}$ and does not make any student worse off, $\psi_{(R,C^*)}$ cannot be constrained efficient. $\square$

Let (R, C) be problem and let μ_0 be an initial stable matching. Let K be the last step of the TP algorithm and let μ_k be the matching selected at step $k \in \{1, \ldots, K-1\}$. Let ψ be a rule that provides incentives to consent and gives a constrained efficient matching that weakly Pareto dominates μ_0. Note that $TP_{(R,C)} = \mu_{K-1}$. By Lemma 14, $\psi_{(R,C)}$ weakly Pareto dominates μ_k for each $k \in \{1, \ldots, K-1\}$, which implies that $\psi_{(R,C)}$ weakly Pareto dominates $TP_{(R,C)}$. Since both $\psi_{(R,C)}$ and $TP_{(R,C)}$ are constrained efficient, this implies that $\psi_{(R,C)} = TP_{(R,C)}$. This completes the proof.

Appendix G: Proof of Theorem 3

Consider the problem $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, and $q_s = 1$ for all $s \in S$. The preferences and the priorities are $s_1 \, P_{i_1} \, s_2 \, P_{i_1} \, s_3$, $s_1 \, P_{i_2} \, s_2 \, P_{i_2} \, s_3$, $s_3 \, P_{i_3} \, s_1 \, P_{i_3} \, s_2$, $i_3 \succ_{s_1} i_1 \succ_{s_1} i_2$, $i_1 \succ_{s_2} i_2 \succ_{s_2} i_3$, and $i_1 \succ_{s_3} i_2 \succ_{s_3} i_3$.

Assume that $C(s_1) = \{(i_1, i_2)\}$ and $C(s_2) = C(s_3) = \varnothing$. This problem has three partially stable matchings:[37] $\mu := \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$, $\mu' := \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$, and $\mu'' := \{(i_1, s_2), (i_2, s_3), (i_3, s_1)\}$. Note that μ'' is Pareto dominated by μ and the only constrained efficient matchings are μ and μ'.

Let ψ be a constrained efficient and strategy-proof rule. Suppose μ' is the outcome of ψ. If i_1 reveals $P'_{i_1} : s_1 \, P'_{i_1} \, s_3 \, P'_{i_1} \, s_2$, in the new problem, the only constrained efficient matching is μ.[38] Then ψ must select μ in the new problem. Thus, i_1 can gain from misreporting if ψ selects μ' in the original problem.

Alternatively, suppose ψ selects μ in the original problem. If i_2 reveals $P'_{i_2} : s_1 \, P'_{i_2} \, s_3 \, P'_{i_2} \, s_2$, in the new problem, the only constrained efficient matching is μ'.[39] Then ψ must select μ' in this problem. Thus, i_2 is better off by misreporting if ψ selects μ in the original problem.[40]

Appendix H: Strategy-proof and constrained efficient rules

This section considers Cs that, when fixed, present no tension between strategy-proofness and constrained efficiency. The two extremes mentioned at the end of Section 2.2 are two clear examples of such Cs: when all priorities are allowed to be violated, the top trading cycles (TTC) rule is strategy-proof and constrained efficient. When no priorities are allowed to be violated, the student-proposing DA rule is constrained efficient and strategy-proof. The following Proposition 5 provides other sufficient conditions that guarantee the existence of such a rule.

Given a correspondence C, we say that a student $i \in I$ is *fully consenting* if $(i, j) \in C(s)$ for each $s \in S$ and $j \in I$ such that $i \succ_s j$. A student is *nonconsenting* if $(i, j) \notin C(s)$ for each $s \in S$ and $j \in I$. A student is *partially consenting* if she is neither fully consenting nor nonconsenting. Note that, by definition, C satisfies the all-or-nothing property if and only if the number of partially consenting students is zero.

[37] Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1.

[38] Any matching where i_1 is assigned to s_2 violates the priority of i_1 for s_3, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1. The only partially stable matchings are μ and $\{(i_1, s_3), (i_2, s_2), (i_3, s_1)\}$, but the former Pareto dominates the latter.

[39] Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1. Also, under μ, i_2's priority for s_3 is violated. The only partially stable matchings are μ' and μ'', but the former Pareto dominates the latter.

[40] To prove Theorem 3, one can also modify the problem used in the proof of Proposition 1 in Ehlers and Westkamp (2011) so as to construct a counterexample. Their proof gives an example that subsumes the case where some schools are consented by all students and the remaining schools are not consented by any students.

Proposition 5. *If*

- *the correspondence C satisfies all-or-nothing property*
  or
- *the total number of partially consenting and nonconsenting students is less than or equal to 2,*

then the following rule is strategy-proof and constrained efficient: run the student-proposing DA rule by considering all the capacity and the partially consenting and non-consenting students. Then run the TTC rule for the remaining capacity and the fully consenting students.

Proof. *Strategy-proofness.* Both the student-proposing DA rule and the TTC rule are strategy-proof. Moreover, all seats are available when the student-proposing DA rule is applied and the students considered under the TTC rule cannot affect the assignment selected by the DA rule via preference manipulation. Thus, no student can gain from misreporting her preferences.

Partial stability. Since priorities of all students considered under the student-proposing DA rule are respected and only the fully consenting students are considered under the TTC rule, the outcome is partially fair. Moreover, non-wastefulness and individual rationality of both rules imply the non-wastefulness and individual rationality of the outcome. Thus, the outcome is partially stable.

Constrained efficiency. First note that when there are at most two students, by Ergin (2002), the student-proposing DA rule selects a efficient allocation. Thus, the outcome under the latter condition is Pareto efficient. Recall that all seats are available when the student-proposing DA rule is applied. Thus, we cannot form a welfare improvement cycle including students considered under both student-proposing DA and TTC. That is, any welfare improvement under the former condition require priorities of some non-consenting students to be violated. □

Appendix I: Proof of Proposition 4

Let C be constructed according to (1). We claim that a matching μ is partially stable with respect to $\succ'$ and C if and only if it is stable with respect to $\succsim$. First, note that the definitions of individual rationality and non-wastefulness are the same across two concepts, so that the equivalence of partial fairness with respect to $\succ'$ and C and fairness with respect to $\succsim$ implies the desired result. We claim that if μ is stable with respect to $\succsim$, then it is partially stable with respect to $\succ'$ and C. Suppose it is not. If μ is not partially fair, then there exist $i, j \in I$ and $s \in S$ such that (i) $\mu(j) = s$, (ii) $s\ P_i\ \mu(i)$, (iii) $i \succ'_s j$, and (iv) $(i, j) \notin C(s)$. But by (1), $(i, j) \notin C(s)$ implies that either $i \succ_s j$ or $j \succ_s i$. Since $\succ' \in \mathcal{T}(\succsim)$, by (iii), we conclude that $i \succ_s j$. But then we have (i) $\mu(j) = s$, (ii) $s\ P_i\ \mu(i)$, and (iii) $i \succ_s j$. Thus, μ is not fair with respect to $\succsim$, a contradiction. Now, assume μ is partially stable with respect to $\succ'$ and C. If μ is not fair with respect to $\succsim$, then there exist $i, j \in I$ and $s \in S$ such that (i) $\mu(j) = s$, (ii) $s\ P_i\ \mu(i)$, and (iii) $i \succ_s j$. But (iii) implies that $i \succ'_s j$ and $(i, j) \notin C(s)$. Thus, μ cannot be partially fair, a contradiction. Due to this equivalence, we conclude that constrained efficiency with respect to $\succ'$ and C is equivalent to student-optimal stability with respect to $\succsim$ and Pareto dominating μ_0. Thus, Theorem 1 implies the desired result.

Appendix J: Examples

Example 2. (Based on Example 3 in Kesten (2010), p. 1310). Let $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $q_{s_x} = 1$ for $x = 1, \dots, 4$ and $q_{s_5} = 2$. Assume that $(i, j) \in C(s)$ for each $s \in S$ and $i, j \in I$ with $i \succ_s j$. The students' preferences and schools' priorities are

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
s_2	s_3	s_3	s_1	s_1	s_4	i_2	i_3	i_1	i_4	$\vdots$
s_1	$\boxed{s_1}$	s_4	s_2	$\boxed{s_5}$	s_1	i_1	i_6	i_6	i_3	$\vdots$
$\boxed{s_3}$	s_5	$\boxed{s_2}$	$\boxed{s_4}$	$\vdots$	s_3	i_5	i_4	i_2	i_6	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_2	i_6	i_1	i_3	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\boxed{s_5}$	i_4	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	i_3	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\diamondsuit$

The SOSM for this problem is $\mu_0 = \{(i_1, s_3), (i_2, s_1), (i_3, s_2), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$ (marked with boxes above). The sets X_{μ_0} are

$$X_{\mu_0}(s_1) = \{i_1, i_4, i_5, i_6\} \qquad X_{\mu_0}(s_2) = \{i_1, i_4, i_6\} \qquad X_{\mu_0}(s_3) = \{i_2, i_3, i_6\}$$
$$X_{\mu_0}(s_4) = \{i_3, i_6\} \qquad X_{\mu_0}(s_5) = \varnothing.$$

The graph $G(\mu_0)$ is given in Figure 1.[41]

There are four cycles in this graph: $\phi_1 = \{i_3 i_4, i_4 i_3\}$, $\phi_2 = \{i_1 i_3, i_3 i_1\}$, $\phi_3 = \{i_1 i_2, i_2 i_1\}$, and $\phi_4 = \{i_1 i_3, i_3 i_4, i_4 i_2, i_2 i_1\}$. First, as an illustration, we demonstrate how the algorithm proceeds when one follows a specific cycle selection rule. In $G(\mu_0)$, assume that the cycle selection rule requires ϕ_1 to be solved. Once ϕ_1 is solved, $\mu_1 =$

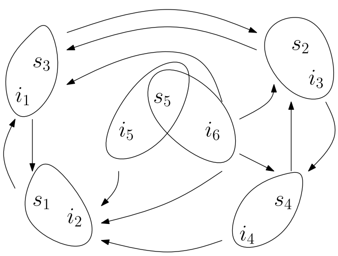

Figure 1. Graph $G(\mu_0)$.

[41] As defined in Section 3.1, each graph in the algorithm is on the set of students. For convenience and tractability, we include the school that is assigned to the student in the current matching as well.

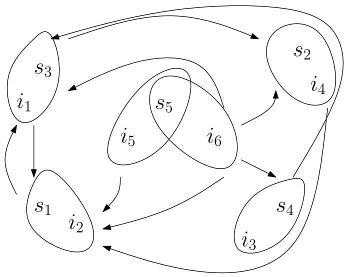

FIGURE 2. Graph $G(\mu_1)$.

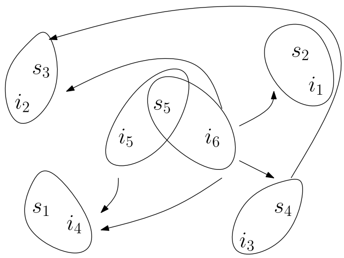

FIGURE 3. Graph $G(\mu_2)$.

$\{(i_1, s_3), (i_2, s_1), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$ is obtained. The sets X_{μ_1} are

$$X_{\mu_1}(s_1) = \{i_1, i_4, i_5, i_6\} \qquad X_{\mu_1}(s_2) = \{i_1, i_6\} \qquad X_{\mu_1}(s_3) = \{i_2, i_3, i_6\}$$
$$X_{\mu_1}(s_4) = \{i_6\} \qquad X_{\mu_1}(s_5) = \varnothing.$$

The graph $G(\mu_1)$ is given in Figure 2.

In the graph $G(\mu_1)$, there are two cycles: $\phi'_1 = \{i_1 i_4, i_4 i_2, i_2 i_1\}$ and $\phi''_1 = \{i_1 i_2, i_2 i_1\}$. Assume that the cycle selection rule requires ϕ'_1 to be solved. Then $\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$ is obtained. The graph $G(\mu_2)$ is given in Figure 3.

Since there is no cycle in the graph $G(\mu_2)$, the algorithm stops. By considering all possible cycle selection rules, we can list all matchings obtained by the SEPF algorithm (we do not find it necessary to go through all of the cycle selection rules and we omit them here). In fact, there are two different matchings generated by the rules in the SEPF class and these matchings are depicted in the preference table (μ is marked with boxes and μ' is marked with underlines):

$$\mu = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$$
$$\mu' = \{(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$$

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
$\underline{\underline{\boxed{s_2}}}$	$\boxed{s_3}$	$\underline{s_3}$	$\boxed{s_1}$	s_1	s_4
s_1	$\underline{s_1}$	$\boxed{s_4}$	s_2	$\underline{\underline{\boxed{s_5}}}$	s_1
s_3	s_5	s_2	$\underline{s_4}$	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\underline{\underline{\boxed{s_5}}}$

Example 3. Let us consider the problem given in Example 2. The SOSM for this problem is $\mu_0 = \{(i_1, s_3), (i_2, s_1), (i_3, s_2), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$. The graph $G(\mu_0)$ is given in Figure 1. ◊

The graph $G^T(\mu_0)$ is obtained from $G(\mu_0)$ in the following way: (i) the students who are permanently matched at μ_0 are removed and (ii) if, in the remaining graph, more than one student points to j, then only the one with the highest priority for $\mu_0(j)$ points to j. The crucial point here is the order in which (i) and (ii) are conducted. Suppose that step (ii) is conducted first, such that among the students pointing to a particular student, say i, in $G(\mu_0)$, the top priority student is selected, and only this student points to i. This gives the graph in Figure 4.

This graph has no cycles, but the application graph $G(\mu_0)$ does. Thus, when step (i) is skipped, in general, we end up with a matching that is not constrained efficient. The TP algorithm ignores the permanently matched students when selecting the student with the highest priority for a given school: students i_5 and i_6 are permanently matched at μ_0 (note that $\mu_0(i_5) = \mu_0(i_6) = s_5$ has no demand at μ_0) and the edges that originate from these students are removed in step (i), resulting in the subgraph of $G(\mu_0)$ in Figure 5.

Among the students pointing to a student i in this graph, the student with the highest priority at school $\mu_0(i)$ is selected and in the graph $G^T(\mu_0)$, only that student points to i (Figure 6).

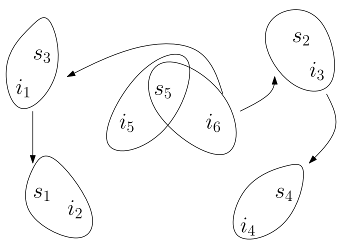

Figure 4. The highest priority students are chosen before permanently matched students are removed from the application graph.

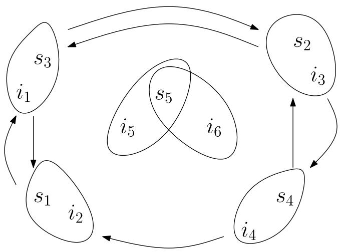

FIGURE 5. The subgraph of $G(\mu_0)$ after permanently matched students' demands are ignored.

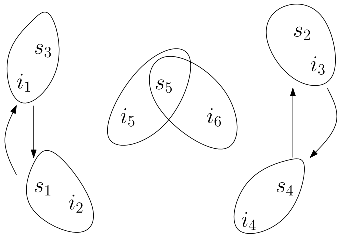

FIGURE 6. Graph $G^T(\mu_0)$.

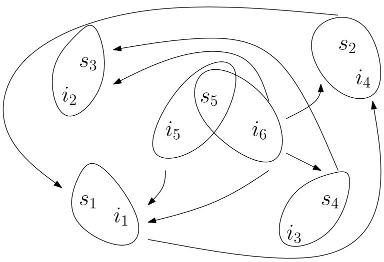

FIGURE 7. Graph $G(\mu_1)$.

There are two cycles in this graph: $\phi_1 = (i_3 i_4, i_4 i_3)$ and $\phi_3 = (i_1 i_2, i_2 i_1)$. The TP algorithm proceeds by solving both of these cycles simultaneously and the matching

$$\mu_1 = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$$

is obtained.[42] The graph $G(\mu_1)$ is given in Figure 7.

Once again, the TP algorithm proceeds by first ignoring the demands of permanently matched students, who are i_2, i_3, i_5, and i_6 (Figure 8).

[42]Actually, in the TP algorithm, only one cycle is solved at each step. But, as we argue in the proof of Proposition 2 (see Appendix E), the order of cycles solved is not consequential. Equivalently, each cycle in the graph used by the TP algorithm can be solved simultaneously.

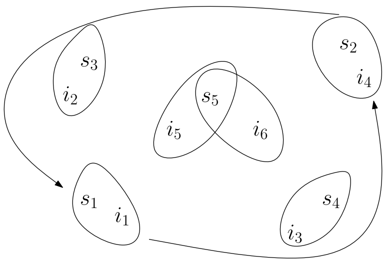

FIGURE 8. The subgraph of $G(\mu_1)$ after permanently matched students' demands are ignored.

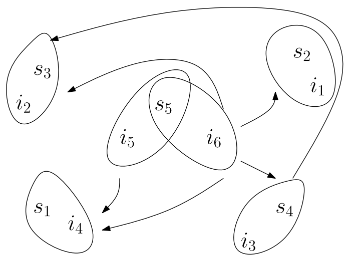

FIGURE 9. Graph $G(\mu_2)$.

Since no student is pointed to by more than one student, the graph $G^T(\mu_1)$ is the same as in Figure 8. By solving the only cycle $(i_1 i_4, i_4 i_1)$ in the graph $G^T(\mu_1)$, the matching

$$\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$$

is obtained. In the graph $G(\mu_2)$, there is no cycle (see Figure 9). Thus, the TP algorithm stops and the matching obtained by the TP algorithm is μ_2. This matching is also the one obtained by the (generalized) EADAM.

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