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Enforcing social norms: Trust-building and community enforcement

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We study impersonal exchange and ask how agents can behave honestly in anonymous transactions without contracts. We analyze repeated anonymous random matching games, where agents observe only their own transactions. Little is known about cooperation in this setting beyond the prisoner's dilemma. We show that cooperation can be sustained quite generally, using community enforcement and "trust-building." The latter refers to an initial phase in which one community builds trust by not deviating despite a short-run incentive to cheat; the other community reciprocates trust by not punishing deviations during this phase. Trust-building is followed by cooperative play, sustained through community enforcement.

Keywords. Community enforcement, contagion, anonymous random matching, repeated games.

JEL classification. C72, C73, D82, D83.

1. Introduction

In many economic settings, impersonal exchange occurs in the absence of contractual enforcement. Buyers and sellers trade essentially anonymously. These settings motivate the central question of this paper: How do agents achieve cooperative outcomes and act in good faith in transactions with strangers without formal contracts? We model impersonal exchange as an infinitely repeated random matching game, in which players from two different communities are randomly and anonymously matched to each

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other to play a two-player game. Each player observes only his own transactions: He does not receive any information about the identity of his opponent or about how play proceeds in other transactions. With such "minimal information transmission," we ask what payoffs can be achieved in equilibrium. Can agents be prevented from behaving opportunistically?

Two early papers by Kandori (1992) and Ellison (1994) showed that in this setting cooperation can be sustained for the prisoner's dilemma (PD) by grim trigger strategies, also known as community enforcement or contagion. If a player faces a defection, he punishes all future rivals by switching to defection forever (Nash reversion). This defection spreads the information that someone has defected, more people get *infected* and start defecting, and cooperation breaks down completely. The credible threat of such a breakdown deters players from defecting in the first place. These arguments rely critically on properties of the PD: Since its Nash equilibrium is in strictly dominant actions, punishing gives a current gain even if it lowers continuation payoffs. In general games, punishing can lower both present and future payoffs, and so it is harder to provide incentives to punish. We establish that it is still possible to sustain a wide range of payoffs in equilibrium in a large class of games if players are sufficiently patient and the population is not too small.

We show that, for stage games with a strict Nash equilibrium, the ideas of community enforcement coupled with "trust-building" can be used to sustain cooperation. In equilibrium, play proceeds in two blocks: an initial phase that we call *trust-building*, followed by a cooperative phase that lasts forever, as long as nobody deviates. In the initial phase, players of one community build trust by not deviating even though they have a short-run incentive to do so, and players in the other community reciprocate the trust by not starting punishments during this phase even if they observe a deviation. This initial phase is crucial to sustaining cooperation in the long run. Deviations in the cooperative phase are punished by Nash reversion (or community enforcement).

To our knowledge, this is the first paper to sustain cooperation in a random matching game beyond the PD without extra informational assumptions. Some papers introduce verifiable information about past play to sustain cooperation. Kandori (1992) considers a mechanism that assigns labels to players based on past play, so players who have deviated or have seen a deviation can be recognized. This enables transmission of information, and cooperation can be sustained in a specific class of games.[1] More recently, Deb (forthcoming) proves a general folk theorem in the anonymous random matching setting, but allows players to send unverifiable messages to their partners just before playing the stage game.[2]

[1]For related approaches, see Dal Bó (2007), Hasker (2007), Okuno-Fujiwara and Postlewaite (1995), and Takahashi (2010).

[2]Specifically, Deb (forthcoming) uses the cheap talk messages to partially authenticate player identities, and then applies a block belief-free approach to achieve the target equilibrium payoff. In contrast, this paper examines the possibility of cooperation in the absence of any kind of communication. Recently, Sugaya (n.d., 2019) established general folk theorems under imperfect private monitoring. These results do not apply here, since our setting violates full-support monitoring and other identifiability assumptions of Sugaya's work.

An important feature of our equilibrium is that the strategies are plausible, and players have strict incentives on and off the equilibrium path. Unlike recent work on games with imperfect private monitoring (Ely and Välimäki 2002, Piccione 2002, Ely et al. 2005, Hörner and Olszewski 2006) and repeated random matching games (Takahashi 2010, Deb forthcoming), we do not rely on belief-free ideas or block strategies. Also, unlike existing literature, our strategies are robust to changes in the discount factor.

This paper relates to the literature on building trust in repeated interactions (e.g., Ghosh and Ray 1996 and Watson 2002), which focuses on "gradual" building of trust, where the stakes in a relationship grow over time. Our equilibrium does not feature gradualism. Rather, we have an initial phase in which players cooperate despite having an incentive to deviate, and this phase is exactly what helps sustain cooperation. Our model can be seen as capturing the intuitive idea that long-term relationships start out by building trust.[3]

The main challenge to sustaining cooperation through Nash reversion is that punishing may be costly for both current and future payoffs. Our construction ensures that, when a player is required to punish by Nash reversion, he believes that most players are already playing Nash (which gives him a short-run incentive to play Nash). To see the idea, suppose that players may entertain the possibility of correlated deviations. Then, upon observing a deviation, a player may think that all players in the rival community have simultaneously deviated and that everybody will start punishing, making Nash reversion optimal. Yet, this way to get the desired beliefs is not consistent with sequential equilibrium.[4] Without coordinated deviations, a player who faces a deviation early in the game will know that there are few affected players and Nash reversion may not be optimal. This suggests that, to induce appropriate beliefs, Nash reversion cannot be prescribed in the initial periods.[5]

Working with beliefs is fundamental to our approach. We develop new methodological tools, using Markov chains, to analyze incentives in belief-based equilibria in repeated games with private monitoring.

The rest of the paper is organized as follows. In Section 2, we illustrate the strategies and the intuition behind our main result using the product-choice game. Section 3 contains the model and the main result. In Section 4, we define off-path beliefs and present our methodology for computing beliefs. In Sections 5 and 6, we establish optimality of the equilibrium strategies. Section 7 discusses robustness of our results.

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[3]There is also recent literature on repeated games and community enforcement on networks (see, for instance, Ali and Miller 2013, Lippert and Spagnolo 2011, and Nava and Piccione 2014). However, this literature is substantively different because players are not anonymous on a network.

[4]Our solution concept is a natural generalization of sequential equilibrium (Kreps and Wilson 1982), which requires that off-path beliefs are the limit of the conditional beliefs obtained from a sequence of completely mixed strategy profiles converging to the strategy profile under consideration. In particular, this implies that player's deviations are independent. Therefore, in our setting, simultaneous deviations by multiple players cannot be inferred from the observation of a single deviation.

[5]We have not been able to construct strategies such that every player, at each information set, has a best reply that is independent of his beliefs (as in Kandori (1992) and Ellison (1994)).

		Buyer	
		B_H	B_L
Seller	Q_H	$2, 2$	$-1, 1$
	Q_L	$3, -1$	$0, 0$

FIGURE 1. The product-choice game.

2. Cooperation beyond the PD

2.1 *A negative result*

We present a simple example to show that a straightforward adaptation of grim trigger strategies (or contagion strategies as in Kandori (1992) or Ellison (1994)) cannot be used to support cooperation in general. The main difficulty is that players may not have the incentive to punish deviations, since punishing may be costly in both the short run and the long run.

Suppose that the product-choice game in Figure 1 is played by a community of M buyers and a community of M sellers in the repeated anonymous random matching setting.[6] In each period, every seller is randomly matched with a buyer and they play the product-choice game. The seller can exert either high effort (Q_H) or low effort (Q_L) in the production of his output. The buyer, without observing the seller's choice, can buy either a high-priced product (B_H) or a low-priced one (B_L). The buyer prefers the high-priced product if the seller has exerted high effort. For the seller, exerting low effort is a dominant action. The efficient outcome is (Q_H, B_H), while the unique Nash equilibrium is (Q_L, B_L). Hereafter, we refer to (Q_L, B_L) as the Nash action.

PROPOSITION 1. *Consider the product-choice game in the repeated random matching setting. If $M > 2$, then, regardless of the discount factor δ, there is no sequential equilibrium in which, in every period, (Q_H, B_H) is played on the equilibrium path and the Nash action is played off the equilibrium path.*

PROOF. Suppose that there is an equilibrium in which, in every period, (Q_H, B_H) is played on the equilibrium path and the Nash action is played off the equilibrium path. Suppose that a seller deviates in period 1. We argue that for the buyer who faces this deviation, it is not optimal to switch to B_L from period 2 onward. In particular, we show that playing B_H in period 2 and switching to B_L from period 3 onward gives her a higher payoff if $M > 2$. By the strategic independence implied by sequential equilibrium (see Definition 1 in Section 3.1), the buyer who faced the deviation believes that, with probability 1, there was no other deviation in period 1. Hence, she believes that, in period 2, with probability $\frac{M-1}{M}$ she will face a different seller who will play Q_H. Consider this buyer's incentives.

Short run. The buyer's payoff in period 2 from playing B_H is $\frac{-1}{M} + \frac{2(M-1)}{M} = \frac{2M-3}{M}$. Her payoff if she switches to B_L is $\frac{M-1}{M}$. Hence, if $M > 2$, she has no short-run incentive to switch to the Nash action.

[6]See Section 3.1 for a formal presentation of the random matching setting and the corresponding definition of sequential equilibrium.

Long run. With probability $\frac{1}{M}$, the buyer meets the deviant seller (who is already playing Q_L) in period 2. In this case, her action does not affect this seller's future behavior, and so her continuation payoff is the same regardless of her action.

With probability $\frac{M-1}{M}$, the buyer meets a different seller. Note that a buyer always prefers to face a seller playing Q_H. So, regardless of the buyer's strategy, the larger is the number of sellers who have already switched to Q_L, the lower is her continuation payoff. Hence, playing B_L in period 2 gives her a lower continuation payoff than playing B_H, because action B_L makes a new seller switch permanently to Q_L.

Since there is no short-run or long-run incentive to switch to the Nash action in period 2, the buyer will not punish. Therefore, playing (Q_H, B_H) in every period on path and (Q_L, B_L) off path does not constitute a sequential equilibrium, regardless of the discount factor. □

Proposition 1 states that play of the cooperative action in *every* period cannot be sustained with grim trigger. It does not rule out the possibility of doing so with other strategies.

2.2 *How to achieve cooperation: An illustration*

Next we show informally how to approximate the efficient payoff in equilibrium in the product-choice game. Section 3 formalizes this construction for general games.

2.2.1 *Equilibrium strategies*

Equilibrium play. Phase I: Action (Q_H, B_H) is played for the first T^{I} periods. Phase II: For the next T^{II} periods, (Q_L, B_H) is played. Phase III: Action (Q_H, B_H) is played thereafter.

Off-equilibrium play. If a player faces a deviation in either Phase II or Phase III, he switches to playing the Nash action (Q_L or B_L) forever. If a buyer faces a deviation in Phase I, she continues to play as if on path for the rest of Phase I and then switches to playing B_L from the start of Phase II. If a seller faces a deviation in Phase I, he continues to play as if on path.

The proof of Proposition 1 shows that grim trigger cannot sustain cooperation because a buyer who faces a deviation at the start of the game is not willing to punish. The main insight of this paper is that "delayed grim trigger strategies" can work: A buyer who observes a deviation at the start of the game delays playing the Nash action until the start of Phase II.

2.2.2 *On-path incentives* For patient players, the payoff from the strategy profile is close to $(2, 2)$. Since any short-run profitable deviation eventually triggers Nash reversion and brings continuation payoffs down to zero, sufficiently patient players do not deviate from the equilibrium path.

2.2.3 *Incentive to punish deviations faced early in the game* To prevent deviations by sellers in Phase I, a buyer who faces Q_L in Phase I must be willing to switch to the Nash action at the start of Phase II. This will trigger Nash reversion by other players and lower continuation payoffs. We start with two observations:

(i) The optimality of Nash reversion for a buyer who faces a deviation depends on her beliefs about how many sellers are playing the Nash action. If she believes that most sellers are playing Nash, then doing so herself is optimal: the Nash action would be the stage-game best reply and the effect on her continuation payoff would be insignificant. In particular, the earlier she thinks the contagion started, the more widespread she will think it is. This observation drives how we specify off-path beliefs: On facing a deviation, players believe that the first deviation was by a seller in period 1.

(ii) If a seller deviates in period 1, he will find it optimal to play Nash reversion immediately. Given the strategies, this seller knows that his opponent will start spreading the contagion by playing Nash from period $T^{\mathrm{I}} + 1$ on. Further, from period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ on, both buyers and sellers will be spreading the contagion and so it will spread exponentially fast. Thus, if he deviates in period 1, his continuation payoff after $T^{\mathrm{I}} + T^{\mathrm{II}}$ will be low, regardless of what he does in the remainder of Phase I. Therefore, if Phase I is long enough, no matter how patient this seller is, he will want to make as much profit as possible for the rest of Phase I, i.e., play Q_L.[7]

Consider now a buyer who faces a deviation in Phase I. She will believe that a seller deviated in period 1 and that he will play Q_L throughout Phase I. If Phase I is long enough she will think that, with very high probability, every buyer will face the deviating seller during Phase I. Thus, since all these buyers will revert to Nash at the start of Phase II, Nash reversion will also be optimal for her. Finally, since only one seller is playing Q_L during Phase I, such a buyer would not have an incentive to start punishing before Phase II.

2.2.4 *Role of Phase II* Phase I ensures that a buyer who faces a deviation early in the game is willing to start Nash punishments in Phase II. Phase II matters only for incentives after some histories that arise with low probability. Consider a buyer who faces Q_L in period 1 and also in all other periods of Phase I. In this case, the buyer realizes that she has met the same deviating seller throughout Phase I and that no other buyer has faced a deviation. Will it be optimal for her to revert to Nash in Phase II? The key now is that the deviating seller does not know that he has met the same buyer in every period, and so he will keep playing the Nash action, even when Phase III starts. Thus, regardless of what the buyer does, she expects her continuation payoff to drop at the start of Phase

[7]For this deviant seller's incentives, not only must T^{I} be large, but also $\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}$ must be large enough. This is important for two reasons. First, a seller who deviates in period 1 will find it optimal to keep deviating and making short-run profits in Phase I, without caring about potential losses in Phase II. Second, this seller will believe that he has infected all buyers by playing Q_L throughout Phase I and will be willing to play Nash throughout Phase II, regardless of the history he observes.

III, since contagion will spread exponentially fast from then on. Now, if Phase II is long enough, this buyer would try to make some short-term gains during Phase II, i.e., she would play the Nash action.

2.2.5 *Nash reversion after getting infected in Phase III* Finally, suppose that a player faces a deviation for the first time in Phase III. He believes that a seller deviated in period 1 and contagion has been spreading since then. However, the fact that he has not faced any deviation so far may indicate that, possibly, not so many people are infected. A crucial element of our construction is that if both T^{I} and $\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}$ are large enough, this player believes that, with high probability, contagion is widely spread and most players are playing the Nash action, making Nash reversion optimal for him.

3. Model, definitions, and main result

3.1 *The repeated anonymous random matching setting*

There are $2M$ players, with $M > 1$, divided into two *communities*, $C_1 = C_2 = \{1, 2, \dots, M\}$. In each period $t \in \mathbb{N}$, players are randomly matched into pairs, with each player $i \in C_1$ facing a player $j \in C_2$. The matching is independent over time, following a uniform distribution. After being matched, each pair plays a finite two-player game G. Players observe only the transactions they are personally engaged in, i.e., each player knows only the history of action profiles played in each of his stage games in the past. Matching is anonymous, i.e., a player never observes his opponent's identity and gets no information about how other players have been matched or about the actions chosen by any other pair. We refer to arbitrary players and players in C_1 as male and to those in C_2 as female.

The stage game. The action sets of G are denoted by A_1 and A_2, and $A := A_1 \times A_2$ denotes the set of action profiles. Generic elements are given by a_1, a_2, and a, respectively. The stage-game payoffs are given by $u : A \to \mathbb{R}^2$.

The repeated game. Given a two-player game G, a community size $M > 1$, and a discount factor $\delta \in (0, 1)$, the corresponding *repeated anonymous random matching game* is denoted by G_δ^M.

Histories. The set of t-period *personal histories* is given by $\mathcal{H}^t := A^t$. Given a player i, a personal history $h^t := \{a^1, a^2, \dots, a^t\}$ contains, for each period $\tau \le t$, the action profile observed by player i in period τ. The set of all personal histories is $\mathcal{H} := \bigcup_{t=0}^{\infty} \mathcal{H}^t$, where $\mathcal{H}^0 := \{\varnothing\}$. Given histories $h^t \in \mathcal{H} \backslash \mathcal{H}^0$ and $h^\tau \in \mathcal{H} \backslash \mathcal{H}^0$, $h^t h^\tau \in \mathcal{H}$ is the concatenation of histories h^t and h^τ. In particular, given an action profile $a \in A$, $h^t a$ is the history obtained as the concatenation of h^t and a. Throughout this paper we use the word "observed" to refer to actions that a player may have *played* or *faced* in his past matches.

Strategies. Given a player $i \in C_k$, with $k \in \{1, 2\}$, a (pure) strategy for i is a mapping $\sigma_i : \mathcal{H} \to A_k$. Let Σ_1 and Σ_2 denote the sets of strategies of players in C_1 and C_2, respectively. The set of strategy profiles is given by $\Sigma_1^M \times \Sigma_2^M$.

Continuation strategies. Given a player i, for each history $h^t \in \mathcal{H} \backslash \mathcal{H}^0$ and each strategy σ_i, player i's *continuation strategy* given history h^t, $\sigma_i|_{h^t}$, is defined, for each $h^\tau \in \mathcal{H}$, by $\sigma_i|_{h^t}(h^\tau) = \sigma_i(h^t h^\tau)$.

Outcomes and payoffs. A *personal outcome* or a *personal path of play* for player i is an element of A^∞, denoting the actions played in the matches in which he was involved. Given an outcome $(a^1, a^2, \dots) \in A^\infty$ and a player $i \in C_k$, i's *discounted payoff* in G_δ^M is given by $U_i(a^1, a^2, \dots) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_k(a^t)$.

Equilibrium. We consider a straightforward extension of sequential equilibrium (Kreps and Wilson 1982) to games of infinite length. A *system of beliefs* is a function μ that assigns, to each information set w of the game tree, a distribution of probability over its nodes or, equivalently, over the histories that may have led to w being reached. Given a strategy profile σ, a system of beliefs μ is *consistent* if there is a sequence of completely mixed strategy profiles $\{\sigma_n\}_{n\in\mathbb{N}}$ converging pointwise to σ and such that the associated conditional beliefs $\{\mu_n\}_{n\in\mathbb{N}}$ converge pointwise to the system of beliefs μ. A strategy profile is a sequential equilibrium if, after every personal history, player i is playing a best response given beliefs that are consistent with player i's personal history.

Definition 1. A strategy profile σ is a *sequential equilibrium* if there is a system of beliefs μ such that the following hold:

(i) The strategy profile σ is sequentially rational given μ, i.e., for each player i and each personal history h, player i is best replying at h given σ and μ.

(ii) The system of beliefs μ is consistent with σ.

3.2 *The main result*

Let $\mathcal{G}$ be the class of finite two-player games with two properties:

P1. There exists a *strict Nash equilibrium*, denoted by $a^* = (a_1^*, a_2^*)$.

P2. There exists a pure action profile $\hat{a} = (\hat{a}_1, \hat{a}_2)$ with *one-sided incentives*, in which one player has a strict incentive to deviate while the other has a strict incentive to stick to the current action. Without loss of generality, we assume that player 1 has an incentive to deviate while player 2 does not.

Let G be a game and let $\underline{a} \in A$. Let $A_{\underline{a}} := \{a \in A : a_1 = \underline{a}_1 \iff a_2 = \underline{a}_2\}$. Define $F_{\underline{a}} := \operatorname{conv}\{u(a) : a \in A_{\underline{a}}\} \cap \{v \in \mathbb{R}^2 : v > u(\underline{a})\}$.

Our main result, Proposition 2 below, says that given a game G in $\mathcal{G}$ with a strict Nash equilibrium a^*, it is possible to approximate any payoff in F_{a^*} in equilibrium in the corresponding infinitely repeated random matching game G_δ^M, if players are sufficiently patient and the communities are not too small. This result covers a large class of games that includes the PD and the product-choice game, and in both of them, F_{a^*} includes payoffs arbitrarily close to efficiency. Note that that the set of achievable payoffs F_{a^*} may not be full dimensional: e.g., for the product game, F_{a^*} is a one-dimensional subset of $\mathbb{R}^2$.

In general, we do not get a folk theorem. We conjecture that by modifying our strategies, it may be possible to support payoffs outside F_{a^*} and obtain a Nash threats folk theorem for games in $\mathcal{G}$ (see Appendix B.3 in the Supplemental Material for a discussion).

We now discuss assumptions P1 and P2. Since we consider Nash reversion, the existence of a pure Nash equilibrium is needed. We need strictness because when a player is asked to start Nash punishments, he may think that, with some probability, he will face an opponent who is not punishing, and if the short-term incentive to punish were not strict, his myopic best reply could be outside the support of the Nash action.[8]

Property P2 is a mild condition. Class $\mathcal{G}$ excludes what we call *games with strictly aligned interests*. For two-player games this means that, at each action profile, a player has a strict incentive to deviate if and only if his opponent also does. Games in $\mathcal{G}$ are generic in the class of games without strictly aligned interests with a pure Nash equilibrium.[9]

Proposition 2. *Let G be a game in $\mathcal{G}$ with a strict Nash equilibrium a^*. There exists $\underline{M} \in \mathbb{N}$ such that, for each payoff profile $v \in F_{a^*}$, each $\varepsilon > 0$, and each $M \geq \underline{M}$, there exists $\underline{\delta} \in (0, 1)$ such that there is a strategy profile in the repeated random matching game G^M_δ that constitutes a sequential equilibrium for each $\delta \in [\underline{\delta}, 1)$ and achieves a payoff within ε of v.*

Our equilibrium strategies constitute a *uniform equilibrium* (Sorin 1990): If a strategy profile constitutes an equilibrium for a given discount factor, it does so for any higher discount factor.[10] This is in contrast to existing literature, where strategies have to be fine-tuned based on the discount factor (e.g., Takahashi (2010) and Deb (forthcoming)).[11]

While cooperation with a larger population needs a higher δ, we do require a minimum community size M for our construction. A relatively large M guarantees that the off-path beliefs induce the correct incentives to punish. Yet, the lower bound $\underline{M}$ depends only on the game G and is independent of ε. Thus, Proposition 2 is not a limiting result in M.

Unlike work on games with imperfect private monitoring (Ely and Välimäki 2002, Piccione 2002, Ely et al. 2005, Hörner and Olszewski 2006) and also in repeated random matching games (Takahashi 2010, Deb forthcoming), we do not rely on complex block strategies or belief-free strategies. Our strategies give the players strict incentives on and off the equilibrium path.

[8]Unlike under perfect or imperfect public monitoring, it is not straightforward to coordinate punishments using public information in our setting.

[9]We have not been able to apply our approach to games of strictly aligned interests. We refer the reader to Appendix B.4 for an example that illustrates the difficulty with achieving cooperation in certain games in this class. However, cooperation is not an issue in commonly studied games in this class, such as battle of the sexes and chicken, since in these games, the set of Pareto efficient payoffs is spanned by the set of pure Nash payoffs (so we can alternate the pure Nash action profiles with the desired frequencies).

[10]This also implies that there exists a threshold discount factor above which our strategies are "discount robust" in the sense of Kalai and Stanford (1988). Mailath and Morris (2002) also define the related notion of "patiently strict public equilibria."

[11]Further, in Ellison (1994), the severity of punishments depends on the discount factor, which has to be common for all players. We just need all players to be sufficiently patient.

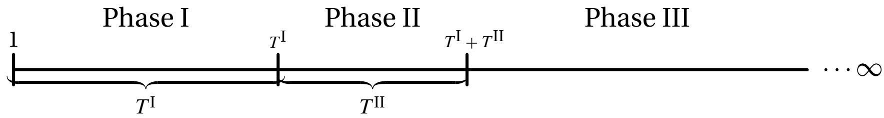

Figure 2. Different phases of the strategy profiles.

3.3 *Equilibrium strategies*

Let G be a game in $\mathcal{G}$. Recall that a^* denotes a strict Nash equilibrium of G, and that $(\hat{a}_1, \hat{a}_2)$ denotes a pure action profile in which only one player has an incentive to deviate. When we say that a player plays or faces the *Nash action*, we mean the corresponding component of a^*. Without loss of generality, we assume that, at action profile $(\hat{a}_1, \hat{a}_2)$, player 1 has an incentive to deviate while player 2 does not, and we let a_1' denote player 1's most profitable deviation. Let the target equilibrium payoff be $v \in F_{a^*}$. We maintain the convention that players 1 and 2 of the stage game belong to communities 1 and 2, respectively. Below, we present the equilibrium strategy profile that sustains v, denoted by $\bar{\sigma}$.

As we show in Figure 2, we divide the game into three phases. Phase I spans over the first T^{I} periods, Phase II spans over the next T^{II} periods, and Phase III covers the rest of the game. Phases I and II are *trust-building* phases and Phase III is the *target payoff* phase.

Equilibrium play. *Phase I.* During the first T^{I} periods, action profile $(\hat{a}_1, \hat{a}_2)$ is played. In every period in this phase, players from community 1 have a short-run incentive to deviate, but those from community 2 do not.

Phase II. During the next T^{II} periods, players play (a_1^*, a_2), an action profile where players from community 1 play their Nash action and players from community 2 do not. Player 2's action a_2 can be any action other than a_2^* in the stage game. In every period in this phase, players from community 2 have a short-run incentive to deviate.

Phase III. For the rest of the game, the players play a sequence of pure action profiles in A_{a^*} that approximates the target payoff v and such that a^* is not played in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$.

Since $\bar{\sigma}$ is pure and symmetric, on path all players observe the same personal history, denoted by $(\bar{a}^1, \bar{a}^2, \dots) \in A^\infty$.

Off-equilibrium play. Suppose that action $a_i \in A^i$ is played in period t and that $a_i \neq \bar{a}_i^t$. If $t \leq T^{\mathrm{I}}$ and $i = 2$, then a_i is *non-triggering*; otherwise, a_i is *triggering*.

Any player i, conditional on having observed a history h^t, can be in one of four moods. We define below the moods and behavior in each mood.

- *Healthy.* A player is healthy at h^t if no triggering action has been played in h^t. A healthy player continues to play as if on path. In particular, a player from Community 1 who observes a deviation in Phase I is healthy.

- *Rogue.* A player is rogue at h^t if he has played a triggering action without having faced one before. A player from community 1 who turns rogue by deviating in the first period of the game plays a'_1 until the end of Phase I. Then he switches to the Nash action and continues to play it as long as he does not observe any deviation after that. We do not describe the best response of rogue players at other histories here. We will be more specific in the proof.

- *Infected.* A player is infected at h^t if he is not rogue, he has faced a triggering action, and $t \geq T^{\mathrm{I}}$. An infected player always plays the Nash action.
- *Exposed.* A player is exposed at h^t if she is a buyer who has faced a triggering action and $t < T^{\mathrm{I}}$. An exposed player continues to play as if on path and transitions to the infected mood at the end of Phase I.

We use the term "unhealthy" to describe a player who is not in the healthy mood. Figure 3 provides a schematic for the mood transitions and behavior. These definitions imply that no player is in the infected mood in Phase I. Also, a buyer cannot turn rogue in Phase I, since her actions are not triggering in the first T^{I} periods.

Note that a profitable deviation by a player is punished (ultimately) by the whole community, with the punishment action spreading like an epidemic. This is referred to as *contagion* in the existing literature. The difference between our strategies and contagion (Kandori 1992, Ellison 1994) is that here the game starts with two initial phases in which deviations are not punished immediately. In other words, unlike the results for the PD, where the equilibria are based on trigger strategies, we have "delayed" trigger strategies.

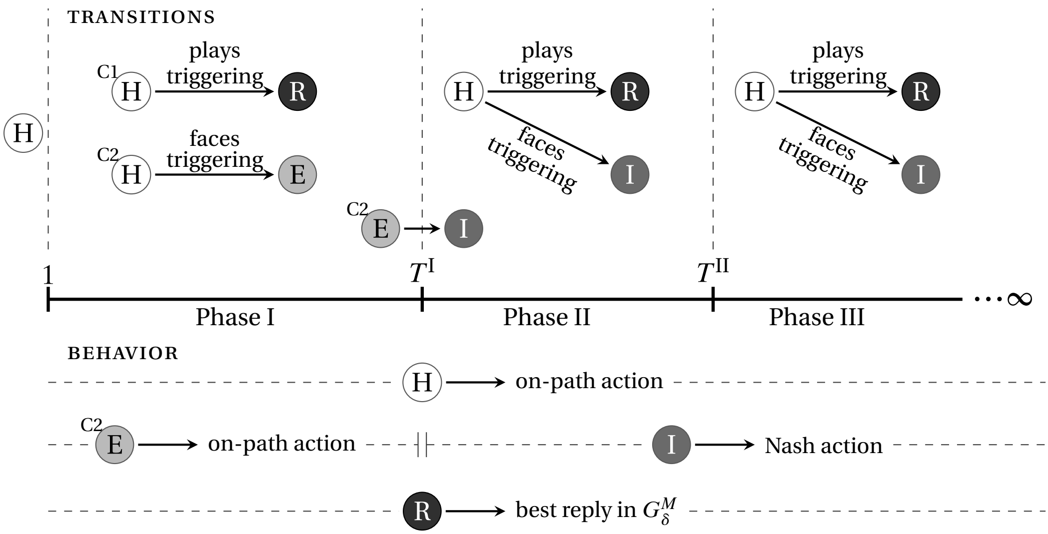

FIGURE 3. The top half describes the events that induce transitions between the four moods. Moods labeled H, E, I, and R denote healthy, exposed, infected, and rogue, respectively. A healthy player who simultaneously plays and faces a triggering action transitions to the infected mood. The bottom half describes behavior in each mood. Where needed, C1 and C2 specify the player's community.

3.4 *On-path incentives*

On-path incentives are straightforward, so we omit the formal proof. First, non-triggering deviations are never profitable, since they entail a loss in the present period and have no impact on future payoffs. Second, triggering actions start a contagion that eventually has all players playing the Nash action from some period onward. Therefore, given M, T^{I}, and T^{II}, there is $\delta_1 \in (0,1)$ such that, for each $\delta \in [\delta_1, 1)$, on-path deviations are not profitable. Moreover, since Phase III has infinite length, given T^{I}, T^{II}, and ε, there is $\delta_2 \in (0,1)$ such that, for each $\delta \in [\delta_2, 1)$, the payoff associated with $\bar{\sigma}$ is within ε of v.

3.5 *Off-path incentives: Outline of argument*

Since the proof of optimality off path is long, we first present an outline of the approach. The off-path incentives of a player depend only on his beliefs about how widespread the contagion is. Thus, establishing sequential rationality requires an analysis of off-path beliefs.

The first step is to define off-path beliefs and understand belief updating, which we do in Section 4. We specify the trembles on completely mixed strategies and present two results, Lemma 1 and Lemma 2, which characterize the ensuing beliefs. In particular, Lemma 1 is the basis for showing that beliefs evolve as simple Markov processes that can be studied using the appropriate transition matrices.

We then analyze off-path incentives in Sections 5 and 6. We classify off-path histories for a player i as follows.

H1. **Histories that can be explained by a single deviation by a seller in period** 1

  a. Histories in which player i got infected in Phase III (discussed in Section 5)

  b. Histories in which player i becomes rogue in period 1 (Section 6.1.1)

  c. Other histories, including those in which i gets exposed in Phase I or infected in Phase II (Section 6.2)

H2. **Histories that cannot be explained by a single deviation by a seller in period** 1 (Section 6.1.2 and Appendix B.2)

Crucially, the off-path beliefs defined in Section 4.1 ensure that whenever a player observes a deviation from the equilibrium path, he attaches probability 1 to the set of histories H1. In particular, Lemma 1 states that exposed and infected players always assign probability 1 to a seller having deviated in period $t = 1$. Recall that behavior has not been specified for players who became rogue at $t > 1$, but since no other player ever assigns positive probability to this event, this underspecification does not pose problems (the behavior of such a rogue player does not affect the incentives of others).[12] Only a

[12]There are some "pathological" histories that can arise and where special care is needed because of underspecification. These histories involve multiple nested off-path deviations combined with a sequence of very low probability match realizations. They are discussed in Appendix B.2.1.

rogue player can know that a history in H2 has been realized and he will play his best response there. Thus, it suffices to show that players have incentives to punish after histories in H1.

Lemma 11 in Section 5 is an important result, which presents a sufficient condition on beliefs to have incentives to punish. Informally, it states that if an infected player believes that contagion is *widely spread* (Definition 2), then he is willing to play the Nash action because he knows that his action cannot affect his continuation payoff significantly. This result reduces our problem to showing that a player, after observing a history in H1, believes that contagion is widely spread. We do this using the tools developed in Section 4.3. Beliefs in H1.a are the most complex, and are divided into three cases.

Infection early in Phase III (Section 5.1). Suppose that player i gets infected in period $t = T^{\mathrm{I}} + T^{\mathrm{II}} + 1$. Using properties of the appropriate Markov process (Lemma 8), we show that player i believes that, with high probability, everybody is unhealthy. Then, by Lemma 11, we have optimality of Nash reversion. Suppose now that, after getting infected in period t and switching to the Nash action, player i starts observing actions different from the Nash action, meaning that he is not facing infected players. In such a case, i has to revise his beliefs and two effects come into play: facing a healthy player implies that contagion was not as spread after period t, but, at the same time, player i has further spread the contagion by infecting his current opponent. For the resulting Markov process, we can still show that player i believes that, with high probability, everybody is unhealthy.

Infection late in Phase III (Section 5.2). If player i gets infected in period t, late in Phase III, then the properties of the relevant Markov process (Lemma 9) imply that player i believes that, with high probability, everybody was unhealthy at the start of Phase III. However, if after that player i starts observing actions different from the Nash action, he may no longer believe that, with high probability, everybody was infected at the start of Phase III. Section 5.2 shows that even then he believes that contagion is widely spread.

Infection in other parts of Phase III (Section 5.3). For other periods of Phase III, we use a monotonicity argument to establish that if player i observes a deviation, he still believes that contagion is widely spread.

The arguments for histories in H1.b and H1.c are more straightforward and are presented in Sections 6.1.1 and 6.2, respectively. A complete proof requires showing sequential rationality also at off-path histories in H2, which is done mostly in Appendix B.2.

4. Off-path beliefs

4.1 *Trembles and ensuing beliefs*

First, we define trembles associated with $\bar{\sigma}$ that define a sequence of completely mixed strategy profiles $\{\sigma_n\}_{n\in\mathbb{N}}$ converging (pointwise) to $\bar{\sigma}$ and such that the associated beliefs $\{\mu_n\}_{n\in\mathbb{N}}$ converge (pointwise) to a system of beliefs $\bar{\mu}$.

Fix a player i and let $D+1$ be the number of actions available to player i in the stage game $G \in \mathcal{G}$. For each $n \in \mathbb{N}$, let $\varepsilon_n := (\frac{1}{2^n})^n$. The strategy of player i in profile σ_n is denoted by $\sigma_{n,i}$. Let h^t be a personal history. Now we distinguish several cases.[13]

Player i is healthy or exposed at h^t. Player $\sigma_{n,i}(h^t)$ selects $\bar{\sigma}_i(h^t)$ with probability $(1-\varepsilon_n^{nt})$ and every other action with probability $\frac{\varepsilon_n^{nt}}{D}$.

Player i is rogue at h^t. Player $\sigma_{n,i}(h^t)$ selects $\bar{\sigma}_i(h^t)$ with probability $(1-\varepsilon_n^{1/t})$ and every other action with probability $\frac{\varepsilon_n^{1/t}}{D}$.

Player i is infected at h^t. Player $\sigma_{n,i}(h^t)$ selects $\bar{\sigma}_i(h^t)$ with probability $(1-\varepsilon_n^{1/(nt)})$ and every other action with probability $\frac{\varepsilon_n^{1/(nt)}}{D}$.

Clearly, $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to $\bar{\sigma}$. Moreover, $\{\mu_n\}_{n\in\mathbb{N}}$ converges pointwise to a system of beliefs $\bar{\mu}$. By definition, $\bar{\mu}$ is consistent with $\bar{\sigma}$ as required by sequential equilibrium.

The above sequence is chosen to ensure certain properties of the limiting beliefs. For instance, t in $(1-\varepsilon_n^{nt})$ ensures that early deviations by healthy players are regarded as infinitely more likely than late deviations. On the contrary, t in $(1-\varepsilon_n^{1/t})$ and $(1-\varepsilon_n^{1/(nt)})$ ensures that late deviations by rogue and infected players are regarded as infinitely more likely than early ones. By comparing $(1-\varepsilon_n^{nt})$ with $(1-\varepsilon_n^{1/t})$ and $(1-\varepsilon_n^{1/(nt)})$ we have that deviations by healthy players are infinitely less likely than deviations by rogue players, which are themselves infinitely less likely than deviations by infected players. Below we establish the properties of $\bar{\mu}$ needed to show that $\bar{\sigma}$ is sequentially rational given $\bar{\mu}$.

Lemma 1. *Let i be a player who is in the exposed or infected mood at some t-period history h^t. Then, according to $\bar{\mu}$, player i puts probability* 1 *on a seller having played a triggering action in period* 1.

The proof is provided in Appendix A.1. The essence of Lemma 1 is that triggering actions after period 1 are so unlikely compared to a triggering action in period 1, that regardless of the likelihood of the subsequent observations, an exposed or infected player i will always be convinced that the first triggering action occurred in period 1. For the next result, we define an *error* as an action $a_i \in A_i$ such that (i) a_i is a non-triggering action or (ii) player i is infected and does not play the Nash action. In particular, the actions of rogue players are never classified as errors.

Lemma 2. *Let i be a player who is in the infected mood at some t-period history h^t and who did not get exposed in period* 1. *Suppose, further, that h^t has probability 0 conditional on a seller playing a triggering action in period* 1 *and play proceeding according to $\bar{\sigma}$ thereafter. Then the following statements hold:*

(i) If player i faced triggering actions by sellers before period $T^{\mathrm{I}}+2$, then he assigns probability 1 *to these actions having been played by a rogue seller who also played a triggering action in period* 1.

[13]See Section 7.3 for a discussion on alternative belief constructions.

(ii) If player i faced non-triggering actions in Phase I, then he assigns probability 1 *to these actions being errors made by buyers (by definition).*

(iii) If player i faced any other action that implies additional deviations from $\bar{\sigma}$*, then he assigns probability* 1 *to these deviations being errors by infected players.*

Lemma 2 implies that when an infected player i is at a history that cannot be explained just by a deviation of a seller in period 1, he will believe, if possible, that there have been as many errors by infected players as needed to explain the current history. Those deviations directly faced by player i and that cannot be attributed to infected players are covered in statements (i) and (ii), and will be attributed to the rogue seller and to buyers, respectively.

It is worth discussing why Lemma 2 is not true for a player i who gets exposed in period 1. Suppose that player i is a buyer who gets exposed in period 1 and faces off-path actions throughout Phase I. The definition of trembles ensures that deviations by rogue players are (infinitely) more likely than deviations by healthy players. Then player i will start Phase II believing that there is a rogue seller whom she has met in all periods of Phase I and that she is the only infected player. Suppose further that in period $T^{\mathrm{I}}+1$ she faces an action different from the Nash action. Then she will believe that she has met the rogue seller again and so there is no infected seller yet. If in period $T^{\mathrm{I}}+2$ she again faces an action different from the Nash action, contrary to statement (iii) in Lemma 2, she cannot attribute this deviation to an infected player since she believes there is no such player. Then she will believe that she has met the rogue seller once again. Histories like this one are what we call *pathological* histories, and the associated incentives are discussed in Appendix B.2.1. One implication of Lemma 2 is that no infected player other than i will ever assign positive probability to these pathological histories.

Importantly, Lemma 1 and Lemma 2 are crucial for the computation of off-path beliefs since they allow us to model the beliefs as Markov processes.

4.2 *Computation of off-path beliefs*

Recall that, given $\bar{\sigma}$, a player's action depends only on his mood. Therefore, all that matters for incentives are the moods of the players in each community, and so the incentives of an infected player depend only on his belief about how widespread the contagion is.[14]

When analyzing the beliefs of an infected player j, we use the term *good behavior* for actions that point toward fewer people being unhealthy. Any other action is *bad behavior*.

- **Bad behavior (*b*).** A action $a_i \in A_i$ is considered bad behavior for player j in period t if one of the following statements holds: (i) a_i is a triggering action; (ii) a_i is a non-triggering action; (iii) player j is unhealthy and $a_i = \bar{a}_i^t = a_i^*$.[15]

[14]Note that the beliefs $\bar{\mu}$ contain additional information such as whether the contagion started slow and then sped up or started fast, but this information is irrelevant for the incentives.

[15]Actions in points (ii) and (iii) are neutral: they do not point in the direction of more or less people being unhealthy. These actions are equally likely to come from healthy and unhealthy players.

- **Good behavior (g).** A action $a_i \in A_i$ is considered good behavior for a player j in period t if it is not considered bad behavior.

We slightly abuse notation and write, for instance, $h^t = g \dots gb$ to denote a history in which player i has faced good behavior during the first $t-1$ periods and bad behavior in period t.

4.2.1 *Approach to computing off-path beliefs* Suppose that I am a player who gets infected at some period $\bar{t}$ in Phase III and that I face a healthy player in period $\bar{t}+1$, i.e., $h^{\bar{t}+1} = g \dots gbg$. I will think that a seller deviated in period 1 (Lemma 1) and that in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$, all unhealthy buyers and sellers played the Nash action (which is triggering in this period). Therefore, period $T^{\mathrm{I}} + T^{\mathrm{II}} + 2$ starts with the same number of unhealthy players in both communities. Hence, it suffices to compute my beliefs about the number of unhealthy sellers. These beliefs are represented by $x^{\bar{t}+1} \in \mathbb{R}^M$, where $x_k^{\bar{t}+1}$ is the probability of exactly k sellers being unhealthy after period $\bar{t}+1$, and must be computed using Bayes rule and conditioning on my personal history. Let G^t be the event "I was healthy after period t" and let $\mathcal{U}^t$ be the random variable corresponding to the number of unhealthy sellers after period t. Then I have the following information after history $h^{\bar{t}+1}$: (i) A seller deviated at period 1, so $x^1 = (1, 0, \dots, 0)$, (ii) for each $t < \bar{t}$, event G^t holds, (iii) since I got infected at period $\bar{t}$, at least one player in the rival community got infected in the same period, and (iv) since I faced a healthy player at $\bar{t}+1$, then, for each $t < \bar{t}, \mathcal{U}^t \leq M-2$.

To compute $x^{\bar{t}+1}$, we compute a series of intermediate beliefs x^t for $t < \bar{t}+1$. We compute x^2 from x^1 by conditioning on G^2 and $\mathcal{U}^2 \leq M-2$; then we compute x^3 from x^2 and so on. Note that to compute x^2, we do not use the information that "I was healthy at the end of each period $2 < t < \bar{t}$." So, at each $t < \bar{t}$, x^t represents my beliefs when I condition on the fact that the contagion started at period 1 and that no matching that leads to more than $M-2$ people being unhealthy could have been realized.[16] Put differently, at each period, I compute my beliefs by eliminating (assigning zero probability to) the matchings I know could not have taken place. At a given period $\tau < \bar{t}$, the information that "I was healthy at the end of period t, with $\tau < t < \bar{t}$" is not used. This information is added period by period, i.e., only at period t do we add the information coming from the fact that "I was healthy at the end of period t." In Appendix B.1, we show that this method yields the correct belief $x^{\bar{t}+1}$ at period $\bar{t}+1$ conditional on the entire personal history $h^{\bar{t}+1}$.

Although from period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ onward the number of unhealthy sellers and the number of unhealthy buyers coincide, this is not the case in Phases I and II. In particular, it will be important to compute the evolution of the number of exposed buyers in Phase I.

In some abuse of notation, when it is known that a player assigns 0 probability to more than k opponents being unhealthy, we work with $x^t \in \mathbb{R}^k$. Given beliefs $x^t, \hat{x}^t \in \mathbb{R}^k$, we say that x^t *first-order stochastically dominates* $\hat{x}^t$ if x^t assigns higher probability to more people being unhealthy; i.e., for each $l \in \{1, \dots, k\}$, $\sum_{i=l}^k x_i^t \geq \sum_{i=l}^k \hat{x}_i^t$.

[16]The updating after period $\bar{t}$ is different, since I know that I was infected at $\bar{t}$ and that no more than $M-1$ people could possibly be unhealthy in the other community at the end of period $\bar{t}$.

4.3 *Modeling beliefs with contagion matrices*

4.3.1 *Contagion matrices and their properties* Beliefs evolve according to Markov processes and can be studied using appropriate transition matrices, which we call *contagion matrices*. A *contagion matrix* Q describes how contagion *spreads* in a community in a given period, with Q_{ij} denoting the probability that state "i unhealthy players" transitions to state "j unhealthy players." If we let $\mathcal{M}_k$ denote the set of $k \times k$ matrices with real entries, we say that $Q \in \mathcal{M}_k$ is a contagion matrix if it has the following properties:

(i) All the entries of Q belong to $[0, 1]$ (they represent probabilities).

(ii) Matrix Q is upper triangular (being unhealthy is irreversible).

(iii) All diagonal entries are strictly positive (with some probability, no healthy player observes a triggering action and contagion does not spread in the current period).

(iv) For each $i > 1$, $Q_{i-1,i} > 0$ (with some probability, exactly one healthy player gets exposed or infected in the current period, unless everybody is already unhealthy).

Since contagion matrices are upper triangular, their eigenvalues correspond to the diagonal entries. Given a matrix Q, let $Q_{l\rfloor}$ denote the matrix obtained by removing the last l rows and columns from Q. Similarly, $Q_{\lceil k}$ is the matrix obtained by removing the first k rows and columns and $Q_{\lceil k,l\rfloor}$ is obtained by doing both operations simultaneously. Clearly, if we perform any of these operations on a contagion matrix, we get a new contagion matrix.

Given $y \in \mathbb{R}^k$, let $\|y\| := \sum_{i\in\{1,\dots,k\}} y_i$. We are interested in the limit behavior of $y^t := \frac{yQ^t}{\|yQ^t\|}$, where Q is a contagion matrix and y is a probability vector. We present below a few results about this limit behavior for contagion matrices. The proofs are provided in Appendix A.2. Given a contagion matrix $Q \in \mathcal{M}_k$, we define the following properties.

Property Q1. We have $\{Q_{11}\} = \operatorname{argmax}_{i\in\{1,\dots,k\}} Q_{ii}$.

Property Q2. We have $Q_{kk} \in \operatorname{argmax}_{i\in\{1,\dots,k\}} Q_{ii}$.

Property Q3. For each $l < k$, $Q_{\lceil l} \in \mathcal{M}_l$ satisfies Q1 or Q2.

Lemma 3. *Let Q be a contagion matrix and let x be a left eigenvector associated with the largest eigenvalue of Q. Then x is either nonnegative or nonpositive.*

Lemma 4. *Let Q be a contagion matrix and let λ be its largest eigenvalue. Then the left eigenspace associated with λ has dimension 1; that is, the geometric multiplicity of λ is 1, irrespective of its algebraic multiplicity.*

Given a contagion matrix Q with largest eigenvalue λ, we denote by y^Q the unique nonnegative left eigenvector associated with λ such that $\|y^Q\| = 1$.

Lemma 5. *Let $Q \in \mathcal{M}_k$ be a contagion matrix. Let $l < k$ and consider vector $y^{Q_{l\rfloor}} \in \mathbb{R}^{k-l}$. If $\sum_{i=1}^{k-l} y_i^Q \neq 0$, then, for each $j \in \{1, \dots, k-l\}$, $y_j^{Q_{l\rfloor}} = \frac{y_j^Q}{\sum_{i=1}^{k-l} y_i^Q}$.*

Lemma 6. *Let $Q \in \mathcal{M}_k$ be a contagion matrix satisfying Property Q1 or Property Q2. Then, for each nonnegative vector $y \in \mathbb{R}^k$ with $y_1 > 0$, we have* $\lim_{t\to\infty} \frac{yQ^t}{\|yQ^t\|} = y^Q$. *In particular, under* Q2, $y^Q = (0, \dots, 0, 1)$.

Lemma 7. *Let $Q \in \mathcal{M}_k$ be a contagion matrix satisfying Properties Q1 and Q3. Let $y \in \mathbb{R}^k$ be a nonnegative vector. If y is close enough to* $(0, \dots, 0, 1)$, *then, for each $t \in \mathbb{N}$, y^t first-order stochastically dominates y^Q, i.e., for each $l \in \{1, \dots, k\}$, $\sum_{i=l}^k y_i^t \geq \sum_{i=l}^k y_i^Q$.*

4.4 *Relevant contagion matrices*

In this section we present the main contagion matrices that are relevant for our construction.

4.4.1 *Contagion matrix in Phase I* Let $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$ denote a history in which I am a player who gets infected in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$. Since the number of unhealthy players is the same in both communities, it suffices to compute my beliefs about the number of unhealthy buyers, $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1}$, which depends on how contagion spreads after a seller turns rogue in period 1. In Phase I, this seller continues deviating, causing buyers to get exposed. The contagion is a Markov process with state space $\{1, \dots, M\}$, representing the number of exposed buyers. This corresponds with contagion matrix $\hat{S}_M \in \mathcal{M}_M$, where a state k transitions to $k + 1$ if the rogue seller meets a healthy buyer, which has probability $\frac{M-k}{M}$. With the remaining probability, i.e., $\frac{k}{M}$, state k remains at state k. When no confusion arises, we omit subscript M in $\hat{S}_M$. Let $\hat{S}_{kl}$ be the probability that state k transitions to state l. Then

$$\hat{S}_M = \begin{pmatrix} \frac{1}{M} & \frac{M-1}{M} & 0 & 0 & \dots & 0 \\ 0 & \frac{2}{M} & \frac{M-2}{M} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \frac{M-2}{M} & \frac{2}{M} & 0 \\ 0 & 0 & 0 & 0 & \frac{M-1}{M} & \frac{1}{M} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To compute my beliefs after being infected, I must also condition on the information from my own history. Let $t < T^{\mathrm{I}}$. After observing history $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$, I know that, at the end of period $t + 1$, at most $M - 1$ buyers were exposed and I was healthy. Therefore, to compute x^{t+1}, my intermediate beliefs about the number of buyers who were exposed at the end of period $t + 1$, i.e., about $\mathcal{U}^{t+1}$, I need to condition on the following information:

(i) My beliefs about $\mathcal{U}^t$: x^t.

(ii) I was healthy at the end of $t + 1$: the event G^{t+1}.

If I am a buyer and condition on G^{t+1}, then I know I did not meet the rogue seller. The transition from state $l - 1$ to state l then requires that the rogue seller meets a healthy buyer, which has probability $\frac{M-l+1}{M}$, and that this healthy buyer is different from me, which has probability $\frac{M-l}{M-l+1}$. Alternatively, if I am a seller, conditioning on G^{t+1} is irrelevant, since sellers always observe good behavior during Phase I.

(iii) At most $M-1$ buyers were exposed by the end of period $t+1$: $\mathcal{U}^{t+1} \leq M-1$ (otherwise I would not have observed g throughout Phase II).

Therefore, given $l < M$, if I am a buyer, the probability that exactly l buyers are exposed after period $t+1$, conditional on the above information, is given by

$$\mathrm{P}\big(l^{t+1} | x^t \cap G^{t+1} \cap \mathcal{U}^{t+1} \leq M-1\big) = \frac{\mathrm{P}\big(l^{t+1} \cap G^{t+1} \cap \mathcal{U}^{t+1} \leq M-1 | x^t\big)}{\mathrm{P}\big(G^{t+1} \cap \mathcal{U}^{t+1} \leq M-1 | x^t\big)}$$

$$= \frac{x^t_{l-1} S_{l-1,l} \dfrac{M-l}{M-l+1} + x^t_l S_{l,l}}{\displaystyle\sum_{k=1}^{M-1} \left(x^t_{k-1} S_{k-1,k} \frac{M-k}{M-k+1} + x^t_k S_{k,k} \right)}.$$

The expression for a seller would be analogous, but without the $\frac{M-l}{M-l+1}$ factors. Note that we can express the transition from x^t to x^{t+1} using a *conditional transition matrix*, $\hat{Q}$. Let $\hat{Q} \in \mathcal{M}_M$ be defined, for each pair $k, l \in \{1, \ldots, M-1\}$, by $\hat{Q}_{kl} := S_{kl} \frac{M-l}{M-k}$, by $\hat{Q}_{MM} := 1$, and with all remaining entries being 0.

Since we know that $x^t_M = x^{t+1}_M = 0$, we can work in $\mathbb{R}^{M-1}$. Recall that $\hat{Q}_{1\rfloor}$ and $\hat{S}_{1\rfloor}$ denote the matrices obtained from $\hat{Q}$ and $\hat{S}$ by removing the last row and the last column of each. The truncated matrix of conditional transition probabilities $\hat{Q}_{1\rfloor}$ is

$$\hat{Q}_{1\rfloor} = \begin{pmatrix} \frac{1}{M} & \frac{M-1}{M}\frac{M-2}{M-1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{M} & \frac{M-2}{M}\frac{M-3}{M-2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \frac{M-2}{M} & \frac{2}{M}\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{M-1}{M} \end{pmatrix}.$$

We need to understand the evolution of the Markov processes associated with matrices $\hat{Q}_{1\rfloor}$ and $\hat{S}_{1\rfloor}$, starting with only one player being unhealthy. Then, for the buyer case, let $y^1_{B^0} = (1, 0, \ldots, 0) \in \mathbb{R}^{M-1}$ and define $y^{t+1}_{B^0}$ as

$$y^{t+1}_{B^0} = \frac{y^t_{B^0} \hat{Q}_{1\rfloor}}{\| y^t_{B^0} \hat{Q}_{1\rfloor} \|} = \frac{y^1_{B^0} \hat{Q}^t_{1\rfloor}}{\| y^1_{B^0} \hat{Q}^t_{1\rfloor} \|}.$$

Analogously, we define the Markov process for the seller, $y^t_{S^0}$, by using $\hat{S}_{1\rfloor}$ instead of $\hat{Q}_{1\rfloor}$. Therefore, my intermediate beliefs at the end of period T^{I}, $x^{T^{\mathrm{I}}}$, would be given by $y^{T^{\mathrm{I}}}_{B^0}$ if

I am a buyer and $y_{S^0}^{T^{\mathrm{I}}}$ if I am a seller. To compute the beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1}$, I would have to update using the contagion matrix in Phase II, but, as will be discussed in Section 5, our proof does not need to deal with it explicitly.

Suppose now that after getting infected after history $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$, during the next α periods, with $1 \leq \alpha \leq M-2$, I face good behavior while I play the Nash action, leading to a history of the form $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1+\alpha} = g \dots gbg \overset{\alpha}{\dots} g$. Suppose that I am a buyer (the arguments for a seller are analogous). After getting infected in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$, I can believe that all players in my community are unhealthy at the end of period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$. However, this is no longer possible, because I have observed the on-path action that is played only by healthy players and, moreover, I have been infecting by playing the Nash action. Thus, after $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1+\alpha} = g \dots gbg \overset{\alpha}{\dots} g$, I know that at most $M-1-\alpha$ buyers were exposed by the end of Phase I. So, for each $t \leq T^{\mathrm{I}}$ and each $k \geq M-\alpha$, $x_k^t = 0$. My beliefs are no longer computed using $\hat{Q}_{1\rfloor}$, but rather with $\hat{Q}_{\alpha+1\rfloor}$. Accordingly, denote my intermediate beliefs at the end of period T^{I} by $y_{B^\alpha}^{T^{\mathrm{I}}} \in \mathbb{R}^{M-1-\alpha}$ if I am a buyer and $y_{S^\alpha}^{T^{\mathrm{I}}} \in \mathbb{R}^{M-1-\alpha}$ if I am a seller. Below we characterize the limit behavior of $y_{B^\alpha}^t$ and $y_{S^\alpha}^t$.

Lemma 8. *For each $M > 2$ and each $\alpha \in \{0, 1, \dots, M-2\}$, we have* $\lim_{t\to\infty} y_{B^\alpha}^t = \lim_{t\to\infty} y_{S^\alpha}^t = (0, \dots, 0, 1) \in \mathbb{R}^{M-1-\alpha}$.

Proof. Since, for each $\alpha \in \{0, 1, \dots, M-2\}$, the matrix $\hat{Q}_{\alpha+1\rfloor}$ satisfies Property Q2, the result follows from Lemma 6. □

This result is intuitive. Since the largest diagonal entry in matrices $\hat{Q}_{\alpha+1\rfloor}$ and $\hat{S}_{\alpha+1\rfloor}$ is the last one, state $M-1-\alpha$ is more stable than any other state. Consequently, as more periods of contagion elapse in Phase I, state $M-1-\alpha$ becomes more and more likely.

4.4.2 *Contagion matrix in Phase III* Suppose that I get infected after observing $h^{\bar{t}+1} = g \dots gb$, with $\bar{t} > T^{\mathrm{I}} + T^{\mathrm{II}} + 1$. My beliefs $x^{\bar{t}+1}$ also depend on how contagion spreads in Phase III. The new contagion matrix is $\bar{S} \in \mathcal{M}_M$, where, for each pair $k, l \in \{1, \dots, M\}$, if $k > l$ or $l > 2k$, $\bar{S}_{kl} = 0$; otherwise, i.e., if $k \leq l \leq 2k$, the probability of transition to state k to state l is (see Figure 4)

$$\bar{S}_{kl} = \frac{\left(\binom{k}{l-k}\binom{M-k}{l-k}(l-k)!\right)^2 (2k-l)!(M-l)!}{M!}$$
$$= \frac{(k!)^2\big((M-k)!\big)^2}{\big((l-k)!\big)^2(2k-l)!(M-l)!M!}.$$

Since I have observed history $h^{\bar{t}+1} = g \dots gb$, given t such that $T^{\mathrm{I}} + T^{\mathrm{II}} < t < \bar{t}$, I know that "at most $M-1$ people could have been unhealthy in the rival community at the end of period $t+1$," i.e., $\mathcal{U}^{t+1} \leq M-1$, and "I was healthy at the end of period $t+1$" (event G^{t+1}). As before, let x^t be my intermediate beliefs after period t. Since, for each

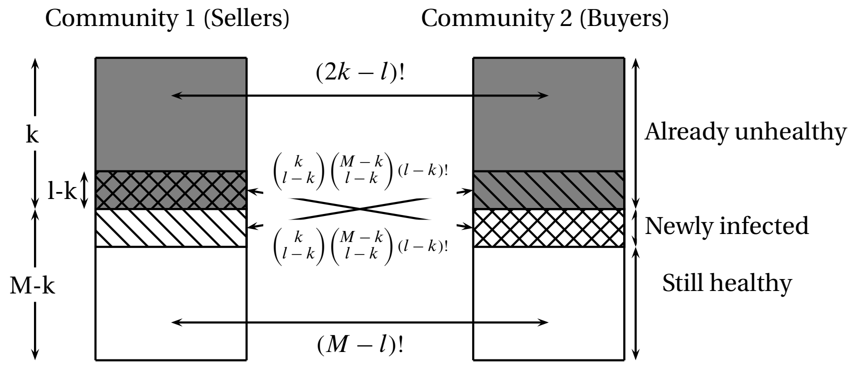

FIGURE 4. Spread of contagion in Phase III. There are $M!$ possible matchings. For state k to transition to state l, exactly $(l-k)$ unhealthy people from each community must meet $(l-k)$ healthy people from the other one. The number of ways to choose exactly $(l-k)$ buyers from k unhealthy ones is $\binom{k}{l-k}$. The number of ways to choose the corresponding $(l-k)$ healthy sellers that will get infected is $\binom{M-k}{l-k}$. Finally, the number of ways in which these sets of $(l-k)$ people can be matched is the number of permutations of $l-k$ people, i.e., $(l-k)!$. Analogously, we choose the $(l-k)$ unhealthy sellers who will be matched to $(l-k)$ healthy buyers. The number of ways in which the remaining unhealthy buyers and sellers get matched to each other is $(2k-l)!$ and, for the healthy ones, we have $(M-l)!$.

$t \leq \bar{t}$, $x^t_M = 0$, we can work with $x^t \in \mathbb{R}^{M-1}$. Thus, for each $l \in \{1, \dots, M-1\}$, we want to compute x^{t+1}_l, which is given by

$$\mathrm{P}\big(l^{t+1} | x^t \cap G^{t+1} \cap \mathcal{U}^{t+1} \leq M-1\big) = \frac{\mathrm{P}\big(l^{t+1} \cap G^{t+1} \cap \mathcal{U}^{t+1} \leq M-1\big) | x^t)}{\mathrm{P}\big(G^{t+1} \cap \mathcal{U}^{t+1} \leq M-1\big) | x^t)}$$

$$= \frac{\displaystyle\sum_{k\in\{1,\dots,M\}} x^t_k \bar{S}_{kl} \frac{M-l}{M-k}}{\displaystyle\sum_{l\in\{1,\dots,M-2\}} \Bigg(\sum_{k\in\{1,\dots,M\}} x^t_k \bar{S}_{kl} \frac{M-l}{M-k}\Bigg)}.$$

Again, we can express these probabilities using the corresponding conditional transition matrix. Let $\bar{Q} \in \mathcal{M}_M$ be defined, for each pair $k, l \in \{1, \dots, M-1\}$, by $\bar{Q}_{kl} := \bar{S}_{kl} \frac{M-l}{M-k}$, by $\bar{Q}_{MM} := 1$, and with all remaining entries being 0. Then, given a vector of beliefs at the beginning of Phase III represented by a probability vector $\bar{y}^0_{B^0}$, we are interested in the evolution of the Markov process where $\bar{y}^{t+1}_{B^0}$ is defined as

$$\bar{y}^{t+1}_{B^0} = \frac{\bar{y}^t_{B^0} \bar{Q}_{1\rfloor}}{\| \bar{y}^t_{B^0} \bar{Q}_{1\rfloor} \|}.$$

There is no need to distinguish between $\bar{y}^t_{B^0}$ and $\bar{y}^t_{S^0}$, since in Phase III the contagion spreads identically in both communities. For each $t \leq \bar{t} - T^{\mathrm{II}} - T^{\mathrm{I}}$, $\bar{y}^t_{B^0}$ coincides with the intermediate beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+t}$. Below, we characterize the limit behavior of $\bar{y}^t_{B^0}$. Importantly, provided that $(\bar{y}^0_{B^0})_1 > 0$, the limit does not depend on $\bar{y}^0_{B^0}$.

Lemma 9. *Suppose that* $(\bar{y}^0_{B^0})_1 > 0$. *Then* $\lim_{t\to\infty} \bar{y}^t_{B^0} = (0, 0, \dots, 0, 1) \in \mathbb{R}^{M-1}$.

Proof. Since $\bar{Q}_{1\rfloor}$ satisfies Property Q2, the result follows from Lemma 6. □

The logic behind the result is less straightforward than that for Lemma 8. The largest diagonal entries of $\bar{Q}_{1\rfloor}$ are the first and last ones: $\bar{Q}_{11} = \bar{Q}_{M-1,M-1} = \frac{1}{M}$. Unlike in the contagion matrix of Phase I, state $M-1$ is not the unique most stable state. Here, states 1 and $M-1$ are equally stable, and are more stable than any other state. Yet, in each period many states transition to $M-1$ with positive probability, while no state transitions to state 1 and so the ratio $\frac{(\bar{y}^t_{B^0})_{M-1}}{(\bar{y}^t_{B^0})_1}$ goes to infinity as t increases.

Suppose that I get infected after $h^{\bar{t}+1} = g \dots gb$ and in the next α periods, with $1 \le \alpha \le M-2$, I face good behavior while I play the Nash action, leading to a history $h^{\bar{t}+1+\alpha} = g \dots gbg \overset{\alpha}{\dots} g$. Then I know that fewer than $(M-1-\alpha)$ people in each community were unhealthy at the end of period $\bar{t}$ since, otherwise, I could not have faced g in α periods after getting infected. I have to recompute my beliefs using the information that, for each $t \le \bar{t}$, $\mathcal{U}^t \le M-1-\alpha$. In particular, for each $t \le \bar{t}$ and each $k \ge M-\alpha$, $x^t_k = 0$. My beliefs are computed using $\bar{Q}_{\alpha+1\rfloor}$ and we denote my intermediate beliefs at the end of period T^{I} by $\bar{y}^{T^{\mathrm{I}}}_{B^\alpha} \in \mathbb{R}^{M-1-\alpha}$ (the process for sellers, $\bar{y}^{T^{\mathrm{I}}}_{S^\alpha} \in \mathbb{R}^{M-1-\alpha}$, is the same and can be omitted).

We have the Markov process that starts with a vector of beliefs at the beginning of Phase III, represented by a probability vector $\bar{y}^0_{B^\alpha}$ and such that $\bar{y}^{t+1}_{B^\alpha}$ is computed as

$$\bar{y}^{t+1}_{B^\alpha} = \frac{\bar{y}^t_{B^\alpha}\bar{Q}_{\alpha+1\rfloor}}{\|\bar{y}^t_{B^\alpha}\bar{Q}_{\alpha+1\rfloor}\|}.$$

As before, for each $t \le \bar{t} - T^{\mathrm{I}} - T^{\mathrm{II}}$, $\bar{y}^t_{B^\alpha}$ coincides with the intermediate beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+t}$. We want to study the limit behavior of $\bar{y}^t_{B^\alpha}$ as t goes to ∞.

The extra difficulty comes from the fact that, for each α with $1 \le \alpha \le M-2$, $\bar{Q}_{M-1-\alpha,M-1-\alpha} < \bar{Q}_{11} = \frac{1}{M}$, and so matrix $\bar{Q}_{\alpha+1\rfloor}$ does not satisfy Property Q2. Therefore, the intuition behind Lemma 9 do not apply and, indeed, the limit beliefs do not converge to $(0, \dots, 0, 1)$. Yet, Property Q1 holds and we can rely on Lemma 6 to ensure convergence.

Lemma 10. *Let* $M > 2$ *and* $\alpha \in \{1, \dots, M-2\}$. *Suppose that* $(\bar{y}^0_{B^\alpha})_1 > 0$. *Then* $\lim_{t\to\infty} \bar{y}^t_{B^\alpha} = \bar{y}^M_{B^\alpha}$, *where* $\bar{y}^M_{B^\alpha}$ *is the unique nonnegative left eigenvector associated with the largest eigenvalue of* $\bar{Q}_{\alpha+1\rfloor}$ *such that* $\|\bar{y}^M_{B^\alpha}\| = 1$. *In particular,* $\bar{y}^M_{B^\alpha}\bar{Q}_{\alpha+1\rfloor} = \frac{\bar{y}^M_{B^\alpha}}{M}$.

Proof. Since, for each $\alpha \in \{1, \dots, M-2\}$, the matrix $Q_{\alpha+1\rfloor}$ satisfies Property Q1, with $(\bar{Q}_{\alpha+1\rfloor})_{11} = \frac{1}{M}$, the result follows from Lemma 6. □

The result above implies that the limit as $\bar{t}$ goes to infinity of the beliefs $x^{\bar{t}}$ is independent of T^{I} and T^{II}. Given these results on off-path beliefs, we are now equipped to study the off-path incentives of players.

5. Incentives after getting infected in Phase III

Checking incentives of players infected in Phase III is the heart of the proof. We consider three cases: First, players who gets infected at the start of Phase III. Next, players who get infected late in Phase III. Finally, we use a monotonicity argument on the beliefs to check the incentives after infection in intermediate periods in Phase III.

The main idea is that an infected player will always believe that contagion is widely spread and, therefore, will find it optimal to play the Nash action. Accordingly, we define a notion of "contagion being widely spread," and establish two preliminary results.

Definition 2. Let $x \in \mathbb{R}^M$ represent a probability distribution over the number of unhealthy people in a community, so that x_k is the probability that there are k unhealthy people. Let $p \in [0, 1]$ and $r \in [0, 1]$.

- We say that contagion is *totally* p*-spread* given x if $x_M \geq p$.
- We say that contagion is (r, p)*-spread* given x if $\sum_{j=\lceil rM \rceil}^{M} x_j \geq p$.[17]

Note that totally p-spread is equivalent to $(1, p)$-spread. Lemma 11 below relates Definition 2 with the incentives of an unhealthy player, regardless of how patient he is. This independence with respect to δ is very important because, given our equilibrium strategies, a high δ is needed for on-path incentives, but may make off-path incentives harder to satisfy. Since a seller can profitably deviate throughout Phase I, if T^{I} is large, sellers must be patient so that the potential losses in Phase III outweigh any possible gains in Phase I. Alternatively, in Phase III, a very patient infected player may not want to punish, since that would spread contagion and reduce his continuation payoff. Lemma 11 shows that if an infected player believes that contagion is widely spread, then he is willing to play the Nash action because his action cannot affect his continuation payoff significantly.

Lemma 11. *Let $G \in \mathcal{G}$. Then there are $p^G \in (0, 1)$ and $r^G \in (0, 1)$ such that, for each $p \geq p^G$ and $r \geq r^G$, the following statement holds for every game G_δ^M with $M > 2$ and $\delta \in (0, 1)$:*

An unhealthy player who, at some period $\bar{t} > T^{\mathrm{I}} + T^{\mathrm{II}}$, believes that the contagion is (r, p)-spread, finds it sequentially rational to play the Nash action at the given period.

Now, suppose that, at some point in Phase III, I am an unhealthy player who believes that at least one player is infected in each community. Suppose further that I then play the Nash action for t periods while observing only g. Thus, in each period I infect a new player and contagion keeps spreading. As the game proceeds, I will eventually believe that contagion is (r^G, p^G)-spread. The lemma below shows that the number of periods necessary for this to happen depends only on the game G and on the population size M and, we denote it by $\phi^G(M)$. Since contagion spreads exponentially fast in Phase III, for fixed G, $\phi^G(M)$ is some logarithmic function of M and the following result is straightforward.

[17] The term $\lceil z \rceil$ denotes the smallest integer not smaller than z and the term $\lfloor z \rfloor$ denotes the largest integer not larger than z.

Lemma 12. *Let $G \in \mathcal{G}$ and $\bar{r} \in (0,1)$. Then there is $\underline{M} \in \mathbb{N}$ such that, for each $M \geq \underline{M}$, we have $\phi^G(M) < (1-\bar{r})M$.*

The above result is important to study incentives in Phase III, but we also need to understand how beliefs evolve before the Nash action has been played for $\phi^G(M)$ periods. Suppose that I am an infected player who is computing his beliefs after history $h^t = g \dots gb$, with $t > T^{\mathrm{I}} + T^{\mathrm{II}}$. Contagion matrices are used to prove that after h^t, I essentially believe that $N-1$ people were infected at the end of period $t-1$ and, therefore, everybody was infected after period t. Histories of the form $h^{t+\alpha} = g \dots gbg \overset{\alpha}{\dots} g$ are more involved and can only be explained by having at most $N-\alpha$ people infected at the end of period t. Thus, the larger α is, the more people I believe were healthy at the end of period t. Alternatively, we have an effect that goes in the opposite direction: from period t to period $t+\alpha$, I am infecting healthy players (I am observing g) and contagion keeps spreading. A fundamental part of the results below consists of showing that this second effect ensures that, regardless of the value of α, I will believe that contagion is (r^G, p^G)-spread.

5.1 *Infection at the start of Phase III*

Let $h^{\bar{t}}$ be a history in which I got infected in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$, i.e., $h^{\bar{t}}$ starts with $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$. The equilibrium strategies prescribe that I play the Nash action at $\bar{t}+1$. The optimality of this action depends on my beliefs $x^{\bar{t}}$ about the number of unhealthy players in the other community. I must believe that contagion is (r,p)-spread with $p \geq p^G$ and $r \geq p^G$. Establishing this is the core of the proof of Proposition 3 below.

Proposition 3. *Let $G \in \mathcal{G}$. Fix $T^{\mathrm{II}} \in \mathbb{N}$ and $M > 2$. Let $\bar{t} \geq T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and let $h^{\bar{t}}$ be a history that starts with $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$. There is $T_1^{\mathrm{I}} \in \mathbb{N}$ such that, for each $T^{\mathrm{I}} \geq T_1^{\mathrm{I}}$, if I observe $h^{\bar{t}}$, then it is sequentially rational for me to play the Nash action at period $\bar{t}+1$.*

Proof. We show that after $h^{\bar{t}}$, I believe that contagion is totally p-spread with $p \geq p^G$. Then the result follows from Lemma 11. We analyze three cases.

Case 1. Suppose that $h^{\bar{t}}$ is a history of the form $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$. By Lemma 8, taking T^{I} large enough, the intermediate beliefs $x^{T^{\mathrm{I}}} \in \mathbb{R}^{M-1}$, which coincide with $y_{B^0}^{T^{\mathrm{I}}}$ if I am a buyer and with $y_{S^0}^{T^{\mathrm{I}}}$ if I am an seller, can be made arbitrarily close to $(0,\dots,0,1)$.

Suppose that I am a buyer. I will assign probability $p \geq p^G$ to $M-1$ players in my community being exposed at the end of Phase I. Since both healthy and unhealthy sellers play the Nash action in Phase II, I cannot learn anything from play in Phase II. I also know that there were at least as many unhealthy sellers as unhealthy buyers by the end of Phase II. Hence, if T^{I} is large enough, the intermediate beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}} \in \mathbb{R}^{M-1}$ are such that I assign probability $p \geq p^G$ to $M-1$ players being unhealthy in each community. Then, in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$, with probability at least p, I got infected by an unhealthy seller and also the last healthy seller got infected (I was the last healthy buyer). Thus, my beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} \in \mathbb{R}^M$ are such that after $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1}$, I believe that contagion is totally p-spread with $p \geq p^G$.

Next suppose that I am a seller. Since no buyer infected me in Phase II, the intermediate beliefs x^t with $t > T^{\mathrm{I}}$ are computed from $x^{T^{\mathrm{I}}}$ by factoring in this information, which will shift them toward less people being unhealthy. Yet, if $\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}$ is large enough, Lemma 8 implies that beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1}$ are such that I believe that contagion is totally p-spread with $p \geq p^G$.

Case 2. Suppose that $h^{\bar{t}}$ is a history of the form $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1+\alpha} = g \dots gbg \overset{\alpha}{\dots} g$. First, suppose that $1 \leq \alpha \leq M-2$. As we argued in the discussion preceding Lemma 8, I know that at most $M-1-\alpha$ buyers were exposed at the end of Phase I. So for each $t \leq T^{\mathrm{I}}$ and each $k \geq M-\alpha$, $x_k^t = 0$. Then we can represent the beliefs at the end of period T^{I} by $y_{B\alpha}^{T^{\mathrm{I}}} \in \mathbb{R}^{M-1-\alpha}$ if I am a buyer and $y_{S\alpha}^{T^{\mathrm{I}}} \in \mathbb{R}^{M-1-\alpha}$ if I am a seller. By Lemma 8, for T^{I} large enough, these beliefs can be made arbitrarily close to $(0, \dots, 0, 1) \in \mathbb{R}^{M-1-\alpha}$. In particular, I will assign probability $p \geq P^G$ to $M-1-\alpha$ players in my community being exposed at the end of Phase I. Suppose that I am a buyer. By the same arguments as Case 1, the intermediate beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}} \in \mathbb{R}^{M-1-\alpha}$ are such that I assign probability $p \geq P^G$ to $M-1-\alpha$ players being infected in each community. Thus, I got infected in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and at most $M-\alpha$ buyers (and sellers) remained healthy. Then, with probability at least p, in each one of the following α periods, I faced one of the remaining healthy sellers and infected him, infecting the last one in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1 + \alpha$. Therefore, my beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1+\alpha}$ are such that after $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1+\alpha}$, I believe that contagion is totally p-spread with $p \geq p^G$. If I am a seller, similar considerations to those in Case 1 are needed, where $\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}$ has to be large enough.

Finally, suppose that $\alpha > M-2$. In this case, by statement (iii) in Lemma 2, I must assign probability 1 to the following history: The seller who deviated in period 1 met the same buyer throughout Phases I and II, so that Phase III started with only one infected player in each community; then, I got infected in period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and I infected healthy players in the next $M-2$ periods; from period $T^{\mathrm{I}} + T^{\mathrm{II}} + M - 1$ onward, I met infected players who were making errors. In particular, I believe that contagion is totally 1-spread.

Case 3. Now consider histories where, after getting infected, I observe a sequence of actions that may include both g and b, i.e., histories starting with $h^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} = g \dots gb$ and where I faced b in one or more periods after getting infected. By definition, every observation of b shifts my beliefs toward more people being unhealthy. Therefore, since the beliefs in the two cases above are such that after $h^{\bar{t}}$, I believe that contagion is totally p-spread with $p \geq p^G$, the same also holds in this third case. □

5.2 *Infection late in Phase III*

We now analyze histories in which I get infected in period $\bar{t} > T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and study beliefs and incentives as $\bar{t}$ goes to infinity. We start with the result for histories of the form $g \dots gb$ and then move to the most challenging case, $g \dots gbg \overset{\alpha}{\dots} g$ with $1 \leq \alpha \leq M-2$.

Proposition 4. *Let $G \in \mathcal{G}$. Fix $T^{\mathrm{I}} \in \mathbb{N}$, $T^{\mathrm{II}} \in \mathbb{N}$, and $M > 2$. Let $\bar{t} > T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and let $h^{\bar{t}} = g \dots gb$. There is $\hat{t} \in \mathbb{N}$ such that, if $\bar{t} > \hat{t}$ and I observe $h^{\bar{t}}$, then it is sequentially rational for me to play the Nash action at period $\bar{t}+1$.*

Proof. First consider $x^{\bar{t}-1} \in \mathbb{R}^{M-1}$, my intermediate beliefs given history $h^{\bar{t}}$ just before getting infected. There is a positive probability that the rogue seller who deviated in period 1 has met the same buyer throughout the first two phases. Thus, $x_1^{T^{\mathrm{I}}+T^{\mathrm{II}}} > 0$. Then, when computing $x^{\bar{t}-1}$ from $x^{T^{\mathrm{I}}+T^{\mathrm{II}}}$, we can apply Lemma 9 with $\bar{y}^0_{B^0} = x^{T^{\mathrm{I}}+T^{\mathrm{II}}}$ and get that $\lim_{t\to\infty} \bar{y}^t_{B^0} = (0, 0, \dots, 0, 1) \in \mathbb{R}^{M-1}$. Therefore, if $\bar{t}$ is large enough, the intermediate beliefs $x^{\bar{t}-1}$, which coincide with $\bar{y}^{\bar{t}-1}_{B^0}$, are such that I assign probability $p \geq p^G$ to $M-1$ being infected in each community.[18] Then, with probability at least p, in period $\bar{t}$, I got infected by an unhealthy player and the last healthy player in the rival community also got infected. Therefore, my beliefs $x^{\bar{t}}$ are such that after $h^{\bar{t}}$, I believe that contagion is totally p-spread with $p \geq p^G$. The result follows from Lemma 11. □

Next suppose that I get infected in period $\bar{t} > T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and after that I face good behavior for α periods, i.e., I observe a history $h^{\bar{t}+\alpha}$ of the form $g \dots gbg \overset{\alpha}{\dots} g$ with $1 \leq \alpha \leq M-2$. After these histories, updating of beliefs builds upon the $\bar{Q}_{\alpha+1\rfloor}$. By Lemma 10, as long as the intermediate beliefs at the start of Phase III, $\bar{y}^0_{B^\alpha} \in \mathbb{R}^{M-1-\alpha}$, are such that $(\bar{y}^0_{B^\alpha})_1 > 0$, then $\lim_{t\to\infty} \bar{y}^t_{B^\alpha} = \bar{y}^M_{B^\alpha}$, where $\bar{y}^M_{B^\alpha}$ is such that $\|\bar{y}^M_{B^\alpha}\| = 1$ and $\bar{y}^M_{B^\alpha} = M\bar{y}^M_{B^\alpha}\bar{Q}_{\alpha+1\rfloor}$.[19] The difficulty comes from the fact that now $\bar{y}^M_{B^\alpha} \neq (0, \dots, 0, 1)$.

The core of the current section consists of establishing that, for each $r \in (0, 1)$ and each $p \in (0, 1)$, if M is large enough, I believe that contagion is (r, p)-spread after history $h^{\bar{t}+\alpha}$. To do so, the crucial step is to show the following result: let $r \in (0, 1)$ and $m \in \mathbb{N}$; then, if M is large enough, for each $k < \lceil rM \rceil$, there are $\bar{r} \in (r, 1)$ and $\bar{k} \in [\lceil rM \rceil, \lfloor \bar{r}M \rfloor]$ such that $(\bar{y}^M_{B^\alpha})_{\bar{k}} > M^{m+1}(\bar{y}^M_{B^\alpha})_k$.

Two opposing forces affect how my beliefs evolve after I observe $g \dots gbg \overset{\alpha}{\dots} g$. On the one hand, each observation of g suggests that not too many people are unhealthy, making me step back in my beliefs and assign higher weight to lower states (fewer unhealthy people). On the other hand, since I believe that contagion started at $t = 1$ and that it is spreading during Phase III, every elapsed period makes me assign more weight to higher states (more unhealthy people). The intuition behind the magnitudes of these two effects is as follows. First, each time I observe g, my beliefs get updated with more weight assigned to lower states and, roughly speaking, this step back in beliefs turns out to be on the order of M. Second, the state k' arising after the most likely transition from a given state k is about $\sqrt{M}$ times more likely than the state k. Then, by taking M large enough, we can find $\bar{r} \in (r, 1)$ such that, given $k < \lceil rM \rceil$, the number of "most likely transitions" needed to get from state k to a state $k' > \lfloor \bar{r}M \rfloor$ is as large as needed. In turn, there will be a state $\bar{k} \in [\lceil rM \rceil, \lfloor \bar{r}M \rfloor]$ that can be made arbitrarily more likely than k.

[18] Recall that there is no need to distinguish between $\bar{y}^t_{B^0}$ and $\bar{y}^t_{S^0}$, since in Phase III an equal number of players are infected in each community and contagion spreads identically in both communities.

[19] In our construction, the condition $\bar{y}^0_1 > 0$ follows from the fact that, with positive probability, the rogue seller may meet the same buyer in all the periods in Phases I and II.

We need some preliminaries before proving formally the above observations. Recall that

$$(\bar{Q}_{\alpha+1\rfloor})_{k,k+j} = \bar{S}_{k,k+j}\frac{M-k-j}{M-k} = \frac{(k!)^2\big((M-k)!\big)^2}{(j!)^2(k-j)!(M-k-j)!M!}\frac{M-k-j}{M-k}.$$

Given a state $k \in \{1, \dots, M-2\}$, let $\mathrm{tr}(k) := \lfloor \frac{k(M-k)}{M} \rfloor$, which, for large M, is such that $k + \mathrm{tr}(k)$ is a good approximation of the most likely transition from state k. Next, we temporarily switch to the case where there is a continuum of states, i.e., we think of the set of states as the interval $[0, M]$. In the continuous setting, a state $z \in [0, M]$ can be represented as rM, where $r = z/M$ can be interpreted as the proportion of unhealthy people at state z. Let $\gamma \in \mathbb{R}$ and let $f_\gamma : [0, 1] \to \mathbb{R}$ be defined as

$$f_\gamma(r) := \frac{rM(M-rM)}{M} + \gamma = \big(r - r^2\big)M + \gamma.$$

Note that all f_γ functions are continuous and that $\mathrm{tr}(rM) = \lfloor f_0(r) \rfloor$, so f_0 is the extension to the continuous case of function $\mathrm{tr}(\cdot)$. We want to understand what the likelihood of the transition from state rM to $rM + f_0(r)$ is. Let $g : [0, 1] \to [0, 1]$ be defined as

$$g(r) := 2r - r^2.$$

The function g is continuous and strictly increasing. Given $r \in [0, 1]$, $g(r)$ represents the proportion of unhealthy people if, at state rM, $f_0(r)$ healthy people get infected, since $rM + f_0(r) = rM + (r - r^2)M = (2r - r^2)M$. Let $g^2(r) := g(g(r))$ and define analogously any other power of g. Hence, for each $r \in [0, 1]$, $g^n(r)$ represents the fraction of unhealthy people after n steps starting at rM when transitions are made according to $f_0(\cdot)$.

Lemma 13. *Let $M \in \mathbb{N}$ and $a, b \in (0, 1)$, with $a > b$. Then $aM + f_0(a) > bM + f_0(b)$.*

Proof. Note that $aM + f_0(a) - bM - f_0(b) = (g(a) - g(b))M$, and the result follows from the fact that $g(\cdot)$ is strictly increasing on $(0, 1)$. □

Let $h^M_\gamma : (0, 1) \to (0, \infty)$ be defined as

$$h^M_\gamma(r) := \frac{(rM!)^2\big((M-rM)!\big)^2}{\big(f_\gamma(r)!\big)^2\big(rM - f_\gamma(r)\big)!\big(M - rM - f_\gamma(r)\big)!M!}\frac{M - rM - f_\gamma(r)}{M - rM}.$$

This function is the continuous version of the transitions given by the matrix $\bar{Q}_{\alpha+1\rfloor}$. In particular, given $\gamma \in \mathbb{R}$ and $r \in [0, 1]$, the function $h^M_\gamma(r)$ represents the conditional probability of transition from state rM to state $rM + f_\gamma(r)$. With some abuse of notation, we apply the factorial function to non-integer real numbers. In such cases, the factorial can be interpreted as the corresponding gamma function, i.e., $a! = \Gamma(a + 1)$.

Lemma 14. *Let $\gamma \in \mathbb{R}$ and $r \in (0, 1)$. Then $\lim_{M\to\infty} M h^M_\gamma(r) = \infty$. More precisely,*

$$\lim_{M\to\infty} \frac{M h^M_\gamma(r)}{\sqrt{M}} = \frac{1}{r\sqrt{2\pi}}.$$

Proof. We prove the result in two steps.

Step 1: $\gamma = 0$. Stirling's formula implies that $\lim_{n\to\infty}(e^{-n}n^{n+\frac{1}{2}}\sqrt{2\pi})/n! = 1$. Given $r \in (0,1)$, to study $h_\gamma^M(r)$ in the limit, we use the approximation $n! = e^{-n}n^{n+\frac{1}{2}}\sqrt{2\pi}$. Substituting and simplifying, we get

$$\begin{aligned} Mh_0^M(r) &= M\frac{\big((rM)!\big)^2\big(((1-r)M)!\big)^2}{M!\big(r^2M\big)!\big(\big((r-r^2)M\big)!\big)^2\big((1-r)^2M\big)!}(1-r) \\ &= \frac{M(rM)^{1+2rM}\big((1-r)M\big)^{1+2(1-r)M}(1-r)}{\sqrt{2\pi}M^{\frac{1}{2}+M}\big((1-r)^2M\big)^{1+2(1-r)^2M}\big((r-r^2)M\big)^{\frac{1}{2}+(r-r^2)M}\big(r^2M\big)^{\frac{1}{2}+r^2M}} \\ &= \frac{\sqrt{M}}{r\sqrt{2\pi}}. \end{aligned}$$

Step 2: Let $\gamma \in \mathbb{R}$ and $r \in (0, 1)$. Now

$$\frac{h_0^M(r)}{h_\gamma^M(r)} = \frac{\big(r^2M-\gamma\big)!\big(\big((r-r^2)M+\gamma\big)!\big)^2\big((1-r)^2M-\gamma\big)!}{\big(r^2M\big)!\big(\big((r-r^2)M\big)!\big)^2\big((1-r)^2M\big)!}\frac{(1-r)^2M}{(1-r)^2M-\gamma}.$$

Applying Stirling's formula again, the above expression becomes

$$\begin{aligned} &\frac{\big(r^2M-\gamma\big)^{\frac{1}{2}+r^2M-\gamma}}{\big(r^2M\big)^{\frac{1}{2}+r^2M}}\frac{\big((r-r^2)M+\gamma\big)^{1+2(r-r^2)M+2\gamma}}{\big((r-r^2)M\big)^{1+2(r-r^2)M}} \\ &\quad\times\frac{\big((1-r)^2M-\gamma\big)^{\frac{1}{2}+(1-r)^2M-\gamma}}{\big((1-r)^2M\big)^{\frac{1}{2}+(1-r)^2M}}\frac{(1-r)^2M}{(1-r)^2M-\gamma}. \end{aligned} \tag{1}$$

To compute the limit of the above expression as $M\to\infty$, we analyze the four fractions above separately. Clearly, $((1-r)^2M)/((1-r)^2M-\gamma)\to 1$ as $M\to\infty$. So we restrict attention to the first three fractions. Take the first one,

$$\frac{\big(r^2M-\gamma\big)^{\frac{1}{2}+r^2M-\gamma}}{\big(r^2M\big)^{\frac{1}{2}+r^2M}} = \left(1-\frac{\gamma}{r^2M}\right)^{\frac{1}{2}}\cdot\left(1-\frac{\gamma}{r^2M}\right)^{r^2M}\cdot\big(r^2M-\gamma\big)^{-\gamma} = A_1\cdot A_2\cdot A_3,$$

where $\lim_{M\to\infty}A_1 = 1$ and $\lim_{M\to\infty}A_2 = e^{-\gamma}$. Similarly, the second fraction decomposes as $B_1\cdot B_2\cdot B_3$, where $\lim_{M\to\infty}B_1 = 1$, $\lim_{M\to\infty}B_2 = e^{2\gamma}$, and $B_3 = ((r-r^2)M+\gamma)^{2\gamma}$. The third fraction can be decomposed as $C_1\cdot C_2\cdot C_3$, where $\lim_{M\to\infty}C_1 = 1$, $\lim_{M\to\infty}C_2 = e^{-\gamma}$, and $C_3 = ((1-r)^2M-\gamma)^{-\gamma}$. Thus, the limit of expression (1) as $M\to\infty$ reduces to

$$\begin{aligned} &\lim_{M\to\infty}\frac{1}{e^\gamma\big(r^2M-\gamma\big)^\gamma}\cdot e^{2\gamma}\big((r-r^2)M+\gamma\big)^{2\gamma}\cdot\frac{1}{e^\gamma\big((1-r)^2M-\gamma\big)^\gamma} \\ &\quad=\lim_{M\to\infty}\left(\frac{\big((r-r^2)M+\gamma\big)^2}{\big(r^2M-\gamma\big)\big((1-r)^2M-\gamma\big)}\right)^\gamma = 1. \end{aligned}$$

□

We are now ready to present the results regarding the properties of $\bar{y}^M_{B^1}$, which, relying on Lemma 5, can be used to get properties of the other $\bar{y}^M_{B^\alpha}$ vectors.

Lemma 15. *Let $r \in (0,1)$ and $m \in \mathbb{N}$. Then there are $\bar{r} \in (r,1)$ and $\underline{M} \in \mathbb{N}$ with the following property: for each $M \geq \underline{M}$ and each $k < \lceil rM \rceil$, there is $\bar{k} \in [\lceil rM \rceil, \lfloor \bar{r}M \rfloor]$ such that $(\bar{y}^M_{B^1})_{\bar{k}} > M^{m+1}(\bar{y}^M_{B^1})_k$.*

Proof. Fix $r \in (0,1)$ and $m \in \mathbb{N}$. We start with state $k_0 = \lceil rM \rceil - 1$. Let $\rho := 2m+3$ and $\bar{r} := g^\rho(r)$. Recall that functions f_0 and g are such that $r < \bar{r} < 1$. Let M' be such that, for each $M \geq M'$, $\bar{r}M \leq M-2$. Let $\bar{k}$ be the number of unhealthy people after ρ steps according to function $\operatorname{tr}(\cdot)$, starting from state k_0. Clearly, $\bar{k} > \lceil rM \rceil$ and, since $k_0 < rM$, Lemma 13 implies that $\bar{k} < \bar{r}M$. Thus, $\bar{k} \in [\lceil rM \rceil, \lfloor \bar{r}M \rfloor]$.

For each $j \in \{1,\ldots,\rho\}$, let $k_j := k_{j-1} + \operatorname{tr}(\cdot)$. In particular, $\bar{k} = k_\rho$. Recall that, for each $\hat{r} \in (0,1)$, $\operatorname{tr}(\hat{r}M) = \lfloor f_0(\hat{r}) \rfloor$. Then, for each $j \in \{1,\ldots,\rho\}$, there is $\gamma_j \in (-1,0]$ such that $\operatorname{tr}(k_{j-1}) = f_{\gamma_j}(\frac{k_{j-1}}{M})$. By Lemma 10, $\bar{y}^M_{B^1} = M\bar{y}^M_{B^1}\bar{Q}_{2\rfloor}$. Then

$$(\bar{y}^M_{B^1})_{k_1} = M\sum_{k=1}^{M-2}(\bar{y}^M_{B^1})_k(\bar{Q}_{2\rfloor})_{kk_1} > M(\bar{y}^M_{B^1})_{k_0}(\bar{Q}_{2\rfloor})_{k_0k_1} = (\bar{y}^M_{B^1})_{k_0}Mh^M_{\gamma_1}(r),$$

which, by Lemma 14, can be approximated by $\frac{\sqrt{M}}{r\sqrt{2\pi}}(\bar{y}^M_{B^1})_{k_0}$ if M is large enough. Repeating the same argument for the other intermediate states that are reached in each of the ρ steps, we get that there is M_{k^0} such that, for each $M \geq M_{k^0}$,

$$(\bar{y}^M_{B^1})_{\bar{k}} > \frac{M^{\frac{\rho}{2}}}{(r\sqrt{2\pi})^\rho}(\bar{y}^M_{B^1})_{k_0} = M^{m+1}\frac{M^{\frac{1}{2}}}{(r\sqrt{2\pi})^\rho}(\bar{y}^M_{B^1})_{k_0} > M^{m+1}(\bar{y}^M_{B^1})_{k_0}.$$

The proof for an arbitrary state $k < \lceil rM \rceil - 1$ is very similar; the only difference is that more than ρ steps might be needed to get to a state $\bar{k} \in [\lceil rM \rceil, \lfloor \bar{r}M \rfloor]$. Yet, the extra number of steps makes the difference between $(\bar{y}^M_{B^1})_k$ and $(\bar{y}^M_{B^1})_{\bar{k}}$ even larger. Then it suffices to define $\underline{M} := \max\{M', \max_{k \leq k_0}\{M_k\}\}$. □

The following result is an immediate consequence of Lemma 15.

Corollary 1. *Let $r \in (0,1)$ and $m \in \mathbb{N}$. Then there are $\bar{r} \in (r,1)$ and $\underline{M} \in \mathbb{N}$ such that, for each $M \geq \underline{M}$,*

(i) $\sum_{j=\lceil rM \rceil}^{M-2}(\bar{y}^M_{B^1})_j > 1 - \frac{1}{M^m}$ *and*

(ii) for each α such that $M-1-\alpha \geq \lfloor \bar{r}M \rfloor$, $\frac{\sum_{j=\lceil rM \rceil}^{\lfloor \bar{r}M \rfloor}(\bar{y}^M_{B^\alpha})_j}{\sum_{j=1}^{\lfloor \bar{r}M \rfloor}(\bar{y}^M_{B^\alpha})_j} > 1 - \frac{1}{M^m}$.

Proof. The proof of statement (i) is straightforward. Moreover, by Lemma 5, for each $\alpha \in \{2,\ldots,M-2\}$ and each $j \leq M-1-\alpha$, $(\bar{y}^M_{B^\alpha})_j = \frac{(\bar{y}^M_{B^1})_j}{\sum_{i=1}^{M-1-\alpha}(\bar{y}^M_{B^1})_i}$. Then, for each α such that $M-1-\alpha \geq \lfloor \bar{r}M \rfloor$, we have

$$\frac{\sum_{j=\lceil rM \rceil}^{\lfloor \bar{r}M \rfloor} (\bar{y}^M_{B^\alpha})_j}{\sum_{j=1}^{\lfloor \bar{r}M \rfloor} (\bar{y}^M_{B^\alpha})_j} = \frac{\sum_{j=\lceil rM \rceil}^{\lfloor \bar{r}M \rfloor} (\bar{y}^M_{B^1})_j}{\sum_{j=1}^{\lfloor \bar{r}M \rfloor} (\bar{y}^M_{B^1})_j}$$

and the proof of statement (ii) is also straightforward. The condition $M-1-\alpha \geq \lfloor \bar{r}M \rfloor$ is important, since $\bar{y}^M_{B^\alpha} \in \mathbb{R}^{M-1-\alpha}$. □

Proposition 5. *Let $G \in \mathcal{G}$. Fix $T^{\mathrm{I}} \in \mathbb{N}$ and $T^{\mathrm{II}} \in \mathbb{N}$. Let $\bar{t} > t > T^{\mathrm{I}} + T^{\mathrm{II}} + 1$ and let $h^{\bar{t}}$ be a history that starts with $h^t = g \ldots gb$. There are $\hat{t} \in \mathbb{N}$ and $M_1^G \in \mathbb{N}$ such that, for each $M \geq M_1^G$, if $t > \hat{t}$ and I observe $h^{\bar{t}}$, then it is sequentially rational for me to play the Nash action at period $\bar{t}+1$.*

Proof. The logic of the proof is similar to that of Proposition 3. We divide the proof into three cases for which we show that, after $h^{\bar{t}}$, I believe that contagion is (r^G, p^G)-spread. Then the result follows from Lemma 11.

The case $\bar{t} = t$, i.e., $h^{\bar{t}} = h^t = g \ldots gb$, follows from Proposition 4.

Case 1. Suppose that $h^{\bar{t}}$ is a history of the form $h^{t+1} = g \ldots gbg$, so $\bar{t} = t+1$. Similarly to the proof of Proposition 4, we are interested in my beliefs $x^{\bar{t}} \in \mathbb{R}^M$, but we start studying $x^{\bar{t}-2} \in \mathbb{R}^{M-2}$, my intermediate beliefs given history $h^{\bar{t}}$ right before getting infected. There is positive probability that the rogue seller who deviated in period 1 has met the same buyer throughout the first two phases. Thus, $x_1^{T^{\mathrm{I}}+T^{\mathrm{II}}} > 0$. Then, when computing my intermediate beliefs $x^{\bar{t}-2}$ from $x^{T^{\mathrm{I}}+T^{\mathrm{II}}} \in \mathbb{R}^{M-2}$, we can apply Lemma 10 with $\bar{y}^0_{B^1} = x^{T^{\mathrm{I}}+T^{\mathrm{II}}}$ and get that $\lim_{t\to\infty} \bar{y}^t_{B^1} = \bar{y}^M_{B^1}$. Thus, if $\bar{t}$ is large enough, $x^{\bar{t}-2}$, which coincides with $\bar{y}_{B^1}^{\bar{t}-2-T^{\mathrm{I}}-T^{\mathrm{II}}}$, is very close to $\bar{y}^M_{B^1}$. In particular, by taking $r \in (0,1)$, $r \geq r^G$, and $m=1$ in statement (i) of Corollary 1, we have that there are t' and M' such that, for each $\bar{t} > t'$ and each $M > M'$,

$$\sum_{j=\lceil rM \rceil}^{M-2} x_j^{\bar{t}-2} = \sum_{j=\lceil rM \rceil}^{M-2} \left(\bar{y}_{B^1}^{\bar{t}-2-T^{\mathrm{I}}-T^{\mathrm{II}}}\right)_j > 1 - \frac{1}{M} \geq p^G.$$

Now we use $x^{\bar{t}-2}$ to compute $x^{\bar{t}}$.

- After period $\bar{t}-1$. I compute $x^{\bar{t}-1}$ by updating $x^{\bar{t}-2}$, conditioning on (i) I observed b in period $\bar{t}-1$ and (ii) at most $M-1$ people were unhealthy after $\bar{t}-1$ (I observed g at $\bar{t}$). Let $\tilde{x}^{\bar{t}-1}$ be the belief computed from $x^{\bar{t}-2}$ by conditioning instead on (i) I observed g in period $\bar{t}-1$ and (ii) at most $M-2$ people are unhealthy. Clearly, $x^{\bar{t}-1}$ first-order stochastically dominates $\tilde{x}^{\bar{t}-1}$, in the sense of placing higher probability on more people being unhealthy. Moreover, $\tilde{x}^{\bar{t}-1}$ coincides with $\bar{y}_{B^1}^{\bar{t}-1-T^{\mathrm{I}}-T^{\mathrm{II}}}$, which also satisfies that $\sum_{j=\lceil rM \rceil}^{M-2} (\bar{y}_{B^1}^{\bar{t}-1-T^{\mathrm{I}}-T^{\mathrm{II}}})_j > p^G$.

- After period $\bar{t}$. I compute $x^{\bar{t}}$ based on $x^{\bar{t}-1}$ and conditioning on (i) I observed g, (ii) I infected my opponent by playing the Nash action at $\bar{t}$, and (iii) at most M people are unhealthy after $\bar{t}$. Again, this updating leads to beliefs that first-order stochastically dominate $\tilde{x}^{\bar{t}}$, the beliefs we would obtain if we instead conditioned on (i) I observed g and (ii) at most $M-2$ people are unhealthy after $\bar{t}$. Again, $\tilde{x}^{\bar{t}}$ coincides with $\bar{y}_{B^1}^{\bar{t}-T^{\mathrm{I}}-T^{\mathrm{II}}}$, which also satisfies that $\sum_{j=\lceil rM\rceil}^{M-2}(\bar{y}_{B^1}^{\bar{t}-T^{\mathrm{I}}-T^{\mathrm{II}}})_j > p^G$.

Hence, contagion is (r^G, p^G)-spread given $x^{\bar{t}}$.

Case 2. Suppose that $h^{\bar{t}}$ is a history of the form $h^{t+\alpha} = g\dots gbg \overset{\alpha}{\dots} g$, so $\bar{t} = t+\alpha$. Again, we start with $x^{\bar{t}-1-\alpha} \in \mathbb{R}^{M-1-\alpha}$, my intermediate beliefs given history $h^{\bar{t}}$ right before getting infected. Similarly to Case 1, relying on Lemma 10 with $\bar{y}_{B^\alpha}^0 = x^{T^{\mathrm{I}}+T^{\mathrm{II}}} \in \mathbb{R}^{M-1-\alpha}$, we get that $\lim_{t\to\infty} \bar{y}_{B^\alpha}^t = \bar{y}_{B^\alpha}^M$. Thus, if $\bar{t}$ is large enough, $x^{\bar{t}-1-\alpha}$, which coincides with $\bar{y}_{B^1}^{\bar{t}-1-\alpha-T^{\mathrm{I}}-T^{\mathrm{II}}}$, is very close to $\bar{y}_{B^\alpha}^M$. Now, by taking $r\in(0,1)$, $r\geq r^G$, and $m=1$ in statement (ii) of Corollary 1, we have that there are t'' and M'' such that, for each $\bar{t} > t''$ and each $M > M''$, for each α such that $M-1-\alpha \geq \lfloor \bar{r}M\rfloor$,

$$\frac{\sum_{j=\lceil rM\rceil}^{\lfloor \bar{r}M\rfloor}\left(\bar{y}_{B^\alpha}^{\bar{t}-1-\alpha-T^{\mathrm{I}}-T^{\mathrm{II}}}\right)_j}{\sum_{j=1}^{\lfloor \bar{r}M\rfloor}\left(\bar{y}_{B^\alpha}^{\bar{t}-1-\alpha-T^{\mathrm{I}}-T^{\mathrm{II}}}\right)_j} > 1-\frac{1}{M} \geq p^G.$$

Next we use $\phi^G(M)$, defined after Lemma 11. By Lemma 12, there is M'' such that, for each $M > M''$, $\phi^G(M) < (1-\bar{r})M$. Suppose that $M \geq M''$ and $\bar{t} > t''$. We distinguish two subcases, depending on the value of α.

(a) $\boldsymbol{M-1-\alpha \geq \lfloor \bar{r}M\rfloor}$. In this case, if we let $t^* := \bar{t}-1-\alpha-T^{\mathrm{I}}-T^{\mathrm{II}}$, we have

$$\begin{aligned}\sum_{j=\lceil rM\rceil}^{M-1-\alpha} x_j^{\bar{t}-1-\alpha} &= \sum_{j=\lceil rM\rceil}^{M-1-\alpha}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j = \frac{\sum_{j=\lceil rM\rceil}^{M-1-\alpha}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j}{1}\\ &= \frac{\sum_{j=\lceil rM\rceil}^{\lfloor \bar{r}M\rfloor}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j + \sum_{\lfloor \bar{r}M\rfloor+1}^{M-1-\alpha}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j}{\sum_{j=1}^{\lfloor \bar{r}M\rfloor}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j + \sum_{\lfloor \bar{r}M\rfloor+1}^{M-1-\alpha}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j} \geq \frac{\sum_{j=\lceil rM\rceil}^{\lfloor \bar{r}M\rfloor}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j}{\sum_{j=1}^{\lfloor \bar{r}M\rfloor}\left(\bar{y}_{B^\alpha}^{t^*}\right)_j} > p^G.\end{aligned}$$

Therefore, $\sum_{j=\lceil rM\rceil}^{M-1-\alpha} x_j^{\bar{t}-1-\alpha} > p^G$. We can repeat the arguments of Case 1 to show that my beliefs $x^{\bar{t}}$ first-order stochastically dominate $x^{\bar{t}-1-\alpha}$, obtaining again that contagion is (r^G, p^G)-spread given $x^{\bar{t}}$.

(b) $\boldsymbol{M-1-\alpha < \lfloor \bar{r}M\rfloor}$. Since $\phi^G(M) < (1-\bar{r})M$, we have $\alpha > M-1-\lfloor \bar{r}M\rfloor \geq (1-\bar{r})M > \phi^G(M)$ and we have, by definition of $\phi^G(M)$, that I believe that contagion is (r^G, p^G)-spread given $x^{\bar{t}}$.

Case 3. Now consider histories where, after getting infected, I observe a sequence of actions that may include both g and b, i.e., histories starting with $h^t = g\dots gb$ and where I faced b in one or more periods after getting infected. By definition, every observation of b shifts my beliefs toward more people being unhealthy. Therefore, since the beliefs in the two cases above are such that, after $h^{\bar{t}}$, I believe that contagion is (r^G, p^G)-spread given $x^{\bar{t}}$, the same also holds in this third case.

To conclude the proof, just let $M_1^G := \max\{M', M''\}$ and $\hat{t} := \max\{t', t''\}$. □

5.3 *Infection in other periods of Phase III*

In Section 5.1 we proved that if I get infected at the start of Phase III, I will believe that contagion is totally p^G-spread. In Section 5.2 we proved that if I get infected late in Phase III, I will believe that contagion is totally (r^G, p^G)-spread. Next we show that if I get infected in other periods of Phase III, my beliefs will lie in between. In some sense, as a function of the period in which I get infected, my beliefs will move "monotonically" from the kind of beliefs characterized in Section 5.1 to those characterized in Section 5.2.

Proposition 6. *Let $G \in \mathcal{G}$ and let $M \geq M_1^G$. Fix $T^{\mathrm{II}} \in \mathbb{N}$. There is $T_2^{\mathrm{I}} \in \mathbb{N}$ such that, for each $T^{\mathrm{I}} \geq T_2^{\mathrm{I}}$, it is sequentially rational for me to play the Nash action after each history in which I get infected in Phase III.*

Proof. The cases in which I get infected at the start of Phase III and late in Phase III are covered by Proposition 3, Proposition 4, and Proposition 5. What remains to be shown is that the same is true if I get infected at some intermediate period in Phase III. We prove this for histories in Phase III of the form $h^{\bar{t}} = g\dots gbg$. The proof can be extended to include other histories, just as the proofs of the above propositions. We want to compute my belief $x^{\bar{t}}$ after $h^{\bar{t}}$. We first compute the intermediate beliefs $x^{\bar{t}-2}$.

Beliefs are computed using matrix $\hat{Q}_{2\rfloor}$ in Phase I and $\bar{Q}_{2\rfloor}$ in Phase III. We know from Section 5.1 (the arguments in Proposition 3 that build upon Lemma 8) that by taking T^{I} large enough, we can make the intermediate beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1} \in \mathbb{R}^M$ arbitrarily close to $(0, \dots, 0, 1)$. Since $\bar{Q}_{2\rfloor}$ satisfies Q1 and Q3, by Lemma 7, if I start Phase III with such beliefs $x^{T^{\mathrm{I}}+T^{\mathrm{II}}+1}$, $x^{\bar{t}-2}$ first-order stochastically dominates $\bar{y}^M_{B^1}$. I still need to update my beliefs from $x^{\bar{t}-2}$ to $x^{\bar{t}-1}$ and then from $x^{\bar{t}-1}$ to $x^{\bar{t}}$. The arguments to show that the resulting beliefs are such that I believe that contagion is (r^G, p^G)-spread are analogous to those used when proving Case 1 in Proposition 4. □

6. Off-path incentives at other histories

In this section, we discuss the incentives at histories not covered in the preceding section. For the sake of brevity, the exposition here is informal. The incentives at the histories discussed here are straightforward after the foregoing analysis in Sections 4 and 5. In Section 6.4, we conclude our analysis by specifying the order in which the different parameters of the construction, M, T^{I}, T^{II}, and δ are fixed.

6.1 *Incentives after becoming rogue*

6.1.1 *A seller becomes rogue in period* 1 Exposed and infected players believe that a seller became rogue in period 1. Thus, the behavior of such a seller is important for the off-path incentives of infected players.

Recall that the equilibrium strategies prescribe that a seller who turns rogue in period 1 of the game plays a_1' until the end of Phase I and then switches to the Nash action forever. Upon deviating in period 1, the rogue seller knows that one buyer is exposed and that this buyer will start playing the Nash action from the start of Phase II. Moreover, there is a $T_1^{\mathrm{II}} \in \mathbb{N}$ such that if $T^{\mathrm{II}} \geq T_1^{\mathrm{II}}$, this buyer will almost certainly infect all sellers during Phase II. Then, from the start of Phase III, all infected sellers will be playing the Nash action and, therefore, everybody will almost certainly be infected after period $T^{\mathrm{I}} + T^{\mathrm{II}} + 1$. Now given the length T^{II} of Phase II, there is $T_3^{\mathrm{I}} \in \mathbb{N}$ such that, for each $T^{\mathrm{I}} \geq T_3^{\mathrm{I}}$, the following statements hold:

- The time $\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}$ is large enough so that the rogue seller will have an incentive to keep deviating in Phase I, since his short-run gains in Phase I will be larger than the potential losses in Phase II and Phase III. This is the case independently of the discount factor $\delta \in (0, 1)$, and the logic is analogous to that behind Lemma 11.

- The time $\frac{T^{\mathrm{I}}}{T^{\mathrm{II}}}$ is large enough so that even if the rogue seller faces the on-path action many times in Phase II, he still believes that, with high probability, $M - 1$ buyers got exposed in Phase I and he has been repeatedly meeting the only remaining healthy buyer in Phase II. Thus, regardless of what he observes after becoming rogue in period 1, if he plays as prescribed by the strategy from that period onward, he will start Phase III believing that, with very high probability, at most one buyer is healthy:
  - If he thinks that everybody is infected at the start of Phase III, then playing the Nash action at the start of Phase III is optimal. In the remainder of Phase III, no matter what actions he faces, he will always believe that, with very high probability, everybody is infected. This is so even after observing good behavior, since after any such observation, he will believe that he has just infected the last healthy opponent (this argument was discussed more formally during some parts of the analysis in Section 5).
  - Even if he thinks that there is one uninfected buyer, there is $M_2^G \in \mathbb{N}$ such that, for each $M \geq M_2^G$, the probability of meeting such a buyer in the given period is so small that the potential gain the seller might get by facing her when not playing Nash would not compensate the losses when facing any other buyer (who would be playing the Nash action).

6.1.2 *A player becomes rogue after period* 1 The behavior of these players has not been specified but, since no other player would ever assign positive probability to such a player existing, their behavior is irrelevant for the incentives of other players.

6.2 *Incentives after facing deviations in Phases I and II*

6.2.1 *A buyer gets exposed in Phase I* The strategy prescribes that, during Phase I, an exposed buyer plays the on-path action and reverts to the Nash action at the start of Phase II. Since deviations of buyers during Phase I are non-triggering, her incentives at a given period of Phase I just depend on her expected payoff in that period. Since the action profile played in Phase I has one-sided incentives, the exposed buyer could profit only by deviating from the on-path action if she happened to meet the rogue seller. Then there is $M_3^G \in \mathbb{N}$ such that, for each $M \geq M_3^G$, the probability of meeting the rogue seller in the given period is so small that the potential profit the buyer might get by facing him when deviating would not compensate the losses when facing any other seller. Therefore, playing as if on path during Phase I is optimal for her.

Once Phase II starts, two things can happen:

(i) The buyer has observed an off-path action in every period of Phase I. Then she knows that she has met the rogue seller in every period of Phase I and that no other buyer is infected. Moreover, she knows that the rogue seller believes that, almost certainly, he has infected all buyers in Phase I, and is playing Nash and will spread the contagion in Phase III. Then there is $T_2^{\mathrm{II}} \in \mathbb{N}$ such that, for each $T^{\mathrm{II}} \geq T_2^{\mathrm{II}}$, she will have an incentive to play Nash in Phase II, since her short-run gains in Phase II will be larger than the potential losses in Phase III. This is the case independently of the discount factor $\delta \in (0, 1)$ (the logic is analogous to that of Lemma 11).

(ii) The buyer has observed the on-path action at least once in Phase I. In this case, Phase II starts with at least two infected buyers and, regardless of the actions of this buyer, contagion would spread during this Phase II. Thus, the incentives to play Nash and make short-run gains during Phase II are even larger than in the case above.

Finally, once Phase III starts, the buyer will believe that everybody is infected and so she has the incentive to keep playing Nash. As before, observations of good behavior during Phase III would not change these beliefs, because after every such observation, the buyer would think that she has just infected the last healthy opponent.

6.2.2 *A player gets infected in Phase II* Next consider players who get infected in Phase II. The strategy prescribes that these players, buyers or sellers, should switch to play Nash forever. These players would believe that the contagion is widely spread, the logic being very similar to the case of a player getting infected at the start of Phase III, discussed in Section 5.1. In particular, a result analogous to Proposition 3 holds: Given T^{II} and $M > 2$, there is $T_4^{\mathrm{I}} \in \mathbb{N}$ such that, for each $T^{\mathrm{I}} \geq T_4^{\mathrm{I}}$, it is sequentially rational for a player to play Nash after every history in which he got infected by observing a triggering action in Phase II.

6.2.3 *A non-triggering action is played in Phase I* The equilibrium strategy prescribes that these deviations are ignored. Thus, both the seller observing this deviation and the buyer playing it believe that his opponent will continue to play as if on path. Given that the opponent will indeed ignore the deviation, the incentives for both players coincide with the on-path ones.

6.3 *Incentives after histories with multiple deviations*

A complete analysis of off-path incentives requires the study of histories that involve multiple off-path deviations. At some of these histories, behavior has not yet been specified explicitly. Since these histories are of secondary importance, we discuss them in Appendix B.2, which also contains a classification of all off-path histories that can arise and describes the relevant arguments for the incentives at each of them.

6.4 *Choice of the parameters*

To establish the intermediate results used in the proof of Proposition 2, we have used bounds on the different parameters M, T^{I}, T^{II}, and δ. Thus, it is important to specify the order in which they have to be chosen so that all the results can be applied.

(i) **Population size: $\underline{M}$.** The first parameter to be fixed is $\underline{M}$. Recollecting the different bounds obtained for M, we have M_1^G in Proposition 5, M_2^G in Section 6.1.1, and M_3^G in Section 6.2.1. Then it suffices to take $\underline{M} \geq \max\{M_1^G, M_2^G, M_3^G, 3\}$. Note that M just depends on the payoffs of G, so Proposition 2 is not a limiting result on M.

(ii) **Length of Phase II: T^{II}.** Recollecting the different bounds for T^{II}, we have T_1^{II} in Section 6.1.1 and T_2^{II} in Section 6.2.1. Then it suffices to take $T^{\mathrm{II}} \geq \max\{T_1^{\mathrm{II}}, T_2^{\mathrm{II}}\}$.

(iii) **Length of Phase I: T^{I}.** Once T^{II} has been fixed, we pick T^{I}. Regarding the bounds for T^{I}, we have T_1^{I} in Proposition 3, T_2^{I} in Proposition 6, T_3^{I} in Section 6.1.1, and T_4^{I} in Section 6.2.2. The length bT^{II} must be fixed already because some of these bounds depend on T^{II}. Then it suffices to take $T^{\mathrm{I}} \geq \max\{T_1^{\mathrm{I}}, T_2^{\mathrm{I}}, T_3^{\mathrm{I}}, T_4^{\mathrm{I}}\}$.

(iv) **Discount factor $\underline{\delta}$.** The last parameter to be chosen is the discount factor, whose role is twofold: to ensure that deviations from the equilibrium path are not profitable and to ensure that $\bar{\sigma}$ approximates the target payoff v. To this end, bounds δ_1 and δ_2 are given in Section 3.4. Thus, once $\underline{M}$, T^{I}, T^{II}, and the degree of approximation ε have been chosen, it suffices to take $\underline{\delta} \geq \max\{\delta_1, \delta_2\}$.[20]

7. Discussion

7.1 *The role of calendar time*

We implicitly assume that all players know when the game started and can perfectly coordinate using calendar time. Although this is a standard and quite innocuous assumption in game theory, it turns out to be more substantive in our setting.

[20]It is worth highlighting that Lemma 11 is crucial. It states that if an unhealthy player believes that contagion is (r, p)-spread with $r \geq r^G$ and $p \geq p^G$, then, regardless of discount factor, he will find it optimal to play the Nash action. This independence with respect to the discount factor δ is what allows us to choose δ last and ensure that this choice does not interfere with the results related to the off-path incentives.

Commonly known start of the game. The fact that all players know that the game starts at time $t = 1$ is important in our construction. When a player is required to punish a deviation by playing the Nash action, she believes that enough people are already infected, which makes the Nash action optimal. Here, we use the fact that players know how long the game has been played and can, therefore, deduce that enough people are infected. An interesting line of investigation may be to consider a model of repeated interactions in which the start date is not commonly known. For instance, one possible approach would be to consider a setting where players enter and leave the game as time unfolds and have limited information about past history. A detailed analysis of this issue is beyond the scope of this paper.

Perfectly synchronized interactions. In our setting, it is commonly known that in every period, all players participate. One could consider alternate models in which only some players are matched in every period or in which matches take place with some probabilities in a continuous time setting. An analysis of this is beyond the scope here. We think that synchronized play is not crucial, and that a result like Proposition 2 may still hold.[21]

7.2 *Introduction of noise*

Since players have strict incentives, our equilibria are robust to the introduction of some noise in the payoffs. Suppose, however, that players were constrained to make mistakes with probability at least $\varepsilon > 0$ at every history. Our equilibrium construction is not robust to this modification. Our assumption that early deviations are believed to be more likely ensures that when players are required to punish, they think that the contagion has spread enough for punishing to be optimal. If mistakes occur with positive and equal probability in all periods, this property is lost.

7.3 *Alternative systems of beliefs*

What is important for our delayed grim trigger strategies to work is that an infected player believes that almost everybody was infected after Phase I. We can guarantee this with our assumption that a player who observes a triggering action believes that some player from community 1 deviated in the first period of the game. However, our construction would work as long as the first triggering deviation is believed to have happened early enough in the game, not necessarily in the first period. We work with the extreme case for tractability.

Our extreme belief also yields the weakest bound on M. With other assumptions, for a given game $G \in \mathcal{G}$ and given T^{I} and T^{II}, the threshold population size $\underline{M}$ required to sustain cooperation would be weakly greater than the threshold we obtain. Why is this so? Formally, on getting infected at period t, let a vector $x^t \in \mathbb{R}^M$ denote my belief about the number of people who are not healthy in the other community at the end of period t,

[21] Some "problematic" histories of our setting would not arise under asynchronous matching; e.g., there could be no history in which a buyer starts Phase II knowing that she and the rogue seller have only faced each other in Phase I.

where x_k^t denotes the probability of exactly k people not being healthy. Then my belief x^t can be expressed as $x^t = \sum_{\tau=1}^{t} \mu(\tau) y^t(\tau)$, where $\mu(\tau)$ is the probability I assign to the first deviation having occurred at period τ, and $y^t(\tau)$ is my belief about the number of people who are not healthy if I know that the first deviation took place at period τ. Since contagion is not reversible, every elapsed period of contagion results in a weakly greater number of infected people. Thus, my belief if I think the first infection occurred at $t = 1$ first-order stochastically dominates my belief if I think the first infection happened later, at any $t > 1$, i.e., for each τ and each $l \in \{1, \dots, M\}$, $\sum_{i=l}^{M} y_i^t(1) \geq \sum_{i=l}^{M} y_i^t(\tau)$. Now consider any belief $\hat{x}^t$ that I might have had with differently chosen trembles. This belief will be some convex combination of the beliefs $y^t(\tau)$ for $\tau = 1, \dots, t$. Since we know that $y^t(1)$ first-order stochastically dominates $y^t(\tau)$ for all $\tau > 1$, it follows that $y^t(1)$ will also first-order stochastically dominate $\hat{x}^t$. Therefore, the belief system in this paper is the one for which players will think that the contagion is most widespread at any given time.

Appendix A: Proofs omitted in the text

A.1 *Proofs of results in Section 4.1*

Proof of Lemma 1. So as to prove a property for $\bar{\mu}$, we need to study the sequences $\{\sigma_n\}_{n\in\mathbb{N}}$ and $\{\mu_n\}_{n\in\mathbb{N}}$. Consider the following three events:

- $E^{\mathrm{Tr}} :=$ "there has been a triggering action"
- $E^1 :=$ "a seller played a triggering action in period 1"
- $E^0 :=$ "no seller played a triggering action in period 1."

For each $n \in \mathbb{N}$, we use P_n to denote probabilities of different events given σ_n. Note that E^1 and E^0 are disjoint events and that $E^{\mathrm{Tr}} = E^1 \cup E^0$. Since player i is in the exposed or infected mood at h^t, $P_n(E^{\mathrm{Tr}}|_{h^t}) = 1$ for each $n \in \mathbb{N}$. We are interested in $P_n(E^1|_{h^t})$ and $P_n(E^0|_{h^t}) = 1 - P_n(E^1|_{h^t})$. We want to prove that $\lim_{n\to\infty} P_n(E^1|_{h^t}) = 1$. Note that

$$\frac{P_n(E^0|_{h^t})}{P_n(E^1|_{h^t})} = \frac{\dfrac{P_n(E^0 \cap h^t)}{P_n(h^t)}}{\dfrac{P_n(E^1 \cap h^t)}{P_n(h^t)}} = \frac{P_n(E^0 \cap h^t)}{P_n(E^1 \cap h^t)} = \frac{1 - P_n(E^1 \cap h^t)}{P_n(E^1 \cap h^t)},$$

and, therefore, to prove that $P_n(E^1|_{h^t})$ converges to 1, we can equivalently prove that $\lim_{n\to\infty}(1 - p_n^1)/p_n^1 = 0$, where $p_n^1 = P_n(E^1 \cap h^t)$.

If $t = 0$, no player can be exposed or infected after h^t, so there is nothing to prove. If $t = 1$, no player can be infected after h^t and only a buyer can be exposed after h^t, which would happen only if she has faced a triggering action in period 1. Hence, for such a buyer, $P_n(E^1|_{h^t}) = 1$ for every $n \in \mathbb{N}$. If $t > 1$ and player i has faced a triggering action in period 1, then also $P_n(E^1|_{h^t}) = 1$ for every $n \in \mathbb{N}$.

Suppose now that $t > 1$ and that player i has neither faced a triggering action in period 1 nor a non-triggering action in h^t. The case with non-triggering actions in h^t is discussed at the end. Given M, t, and h^t, let $F^1(M,t,h^t)$ denote the number of different ways to match the $2M$ players through periods 1 to t. Next, we construct a lower bound for p_n^1 and an upper bound for $1-p_n^1$.

We start by computing a lower bound on the probability of the most unlikely complete history (not just personal history) compatible with $E^1 \cap h^t$ with the following two properties: (i) the only deviation from $\bar{\sigma}$ by a healthy or exposed player is made by a seller in period 1 and (ii) at most one player deviates from $\bar{\sigma}$ at any given period. First, since matching is uniform, $\frac{1}{F^1(M,t,h^t)}$ is the probability of the corresponding matches having been realized. Then, since a seller deviated in period 1, such a deviation had probability $\frac{\varepsilon_n^n}{D}$ (recall that $D+1$ is the number of actions available to sellers in the stage game). By (ii), no one else deviated in period 1, which has probability $(1-\varepsilon_n)^{2M-2}$. In each of the remaining $t-1$ periods, the most unlikely profile that is compatible with (i) and (ii) is that a rogue player deviated and that no one else did.[22] The probability of such a profile at a period τ is bounded below by $\frac{\varepsilon_n^{1/\tau}}{D}(1-\varepsilon_n^{1/n\tau})^{2M-2}$; the second term reflects that no other player deviated and it represents a lower bound since $1-\varepsilon_n^{1/n\tau}$ is the probability that an infected player does not deviate (and infected players are the most likely ones to do so). Thus, the probability of the complete history under discussion is bounded below by

$$\begin{aligned} &\frac{1}{F^1(M,t,h^t)}\frac{\varepsilon_n^n}{D}(1-\varepsilon_n)^{2M-2}\prod_{\tau=2}^{t}\left(\frac{\varepsilon_n^{1/\tau}}{D}(1-\varepsilon_n^{1/n\tau})^{2M-2}\right)\\ &\quad\geq \frac{1}{F^1(M,t,h^t)}G(n)\frac{\varepsilon_n^{n+1}}{D^t}, \end{aligned}$$

where $\lim_{n\to\infty} G(n)=1$. Since the above probability corresponds to just one of the possible histories compatible with $E^1\cap h^t$, we have that

$$p_n^1 \geq \frac{1}{F^1(M,t,h^t)}G(n)\frac{\varepsilon_n^{n+1}}{D^t}.$$

Now we do the opposite exercise and compute an upper bound on the probability of the most likely complete history compatible with $E^0\cap h^t$. Since such a history must contain a triggering action in a period different from period 1, the associated probability is bounded above by $\frac{\varepsilon_n^{2n}}{D}$, which is the probability of a triggering action in period 2 (and forgetting about all other terms dealt with in the case above, since all of them are bounded above by 1). Thus, we have that $1-p_n^1$ can be bounded by

$$1-p_n^1 \leq F^*(M,t,h^t,D)\frac{\varepsilon_n^{2n}}{D},$$

[22]Recall that deviations by infected players are more likely than deviations by rogue players and that (i) requires that no healthy or exposed players deviate in h^t after period 1.

where $F^*(M, t, h^t, D)$ denotes the number of complete histories compatible with $E^0 \cap h^t$. Therefore, we have

$$\frac{1-p_n^1}{p_n^1} \le \frac{F^*\big(M, t, h^t, D\big)\dfrac{\varepsilon_n^{2n}}{D}}{\dfrac{1}{F^1\big(M, t, h^t\big)}G(n)\dfrac{\varepsilon_n^{n+1}}{D^t}} = \frac{F^*\big(M, t, h^t, D\big)F^1\big(M, t, h^t\big)D^{t-1}}{G(n)} \cdot \frac{\varepsilon_n^{2n}}{\varepsilon_n^{n+1}}.$$

Since $\lim_{n\to\infty} G(n) = 1$ and all other terms not including ε_n are constant in n,

$$\lim_{n\to\infty} \frac{1-p_n^1}{p_n^1} = 0,$$

which implies that $\lim_{n\to\infty} P_n(E^1|_{h^t}) = 1$. Yet, one has to ensure that there exists a complete history compatible with $E^1 \cap h^t$ satisfying (i) and (ii), but this readily follows from the fact that the σ_n strategies are completely mixed (and all histories have positive probability of being realized).

Finally, suppose that the player i has faced some triggering action in h^t. These actions can only be made by healthy or exposed players and have no impact on the future behavior of other players. Thus, the computations of the bounds above for p_n^1 and $1-p_n^1$ can be immediately extended by requiring that the studied histories contain the observed non-triggering actions. Since this inclusion would be in the histories associated with both E^1 and E^0, with the same probabilities in both cases, the corresponding terms would cancel out when computing $\frac{1-p_n^1}{p_n^1}$. □

Proof of Lemma 2. Suppose that h^t has probability 0 conditional on a seller playing a triggering action in period 1 and play proceeding according to $\bar{\sigma}$ thereafter. Lemma 1 still guarantees that player i puts probability 1 on E^1. Yet, additional deviations from $\bar{\sigma}$ are needed to explain h^t.

We start with statements (i) and (ii). If player i is a buyer and has faced a triggering action before period $T^{\mathrm{I}}+2$, since no seller can be in the infected mood before that period, then either a healthy or a rogue seller made that deviation. Since deviations by a rogue seller become infinitely more likely as n goes to ∞, in the limit player i will put probability 1 on such a deviation coming from the rogue seller. If player i is a seller and has faced some non-triggering action in Phase I, then these deviations from $\bar{\sigma}$ are errors by definition.

Consider now statement (iii). Since player i did not get exposed in period 1, with positive probability a buyer became exposed in period 1 and then infected some seller in period $T^{\mathrm{I}}+2$. Thus, for each $n \in \mathbb{N}$, player i puts positive probability on the event "there is at least one infected player in each community after period $T^{\mathrm{I}}+1$." Suppose now that h^t has probability 0 conditional on a seller playing a triggering action in period 1 and play proceeding according to $\bar{\sigma}$ thereafter except for possibly some deviation already covered by statements (i) and (ii). Consider the following three events:

- $A^{\mathrm{Ad}} :=$ "h^t cannot be explained with a single deviation in period 1 and some deviation covered by statements (i) and (ii)"

- $A^1 :=$ "all additional deviations have been made by infected players (errors)"
- $A^0 :=$ "at least one additional deviation has been made by a healthy or rogue player."

The arguments are now similar to those in the proof of Lemma 1. We want to show that $\lim_{n\to\infty} P_n(A^1|_{h^t}) = 1$. The construction again relies on the computation of lower and upper bounds for $P_n(A^1 \cap h^t)$ and $P_n(A^0 \cap h^t)$, respectively.

Setting aside terms that are constant in n or that converge to 1 as n goes to ∞, the lower bound on $P_n(A^1 \cap h^t)$ would be on the order of ε_n^n (deviation by a seller in period 1) multiplied by $\prod_{\tau=T^{\mathrm{I}}+2}^{t} \varepsilon_n^{1/n\tau}$ (a deviation by an infected seller in each and every period from $T^{\mathrm{I}}+2$). Thus, this lower bound would be on the order of

$$\varepsilon_n^n \cdot \prod_{\tau=T^{\mathrm{I}}+2}^{t} \varepsilon_n^{1/n\tau} \geq \varepsilon_n^n \cdot \prod_{\tau=1}^{t} \varepsilon_n^{1/n} = \varepsilon_n^n \cdot \varepsilon_n^{t/n}.$$

Alternatively, the upper bound on $P_n(A^0 \cap h^t)$ would arise when considering that, apart from the deviation by a seller in period 1, there was only one additional deviation, which was made by a rogue player at period t (late deviations by rogue players are the most likely ones). Then an upper bound can be given by $\varepsilon_n^n \cdot \varepsilon_n^{1/t}$. Hence, since the terms ε_n^n in the two bounds cancel out and, as n goes to ∞, $\varepsilon_n^{1/t}$ becomes infinitely smaller than $\varepsilon_n^{t/n}$, we have

$$\lim_{n\to\infty} \frac{P_n(A^0 \cap h^t)}{P_n(A^1 \cap h^t)} = 0.$$

Therefore, $\lim_{n\to\infty} P_n(A^1|_{h^t}) = 1$. □

A.2 *Proofs of general results for contagion matrices (Section 4.3.1)*

Proof of Lemma 3. Let λ be the largest eigenvalue and let x be a left eigenvector associated with it. Suppose k is the first coordinate of x such that $x_k \neq 0$ and assume that $x_k > 0$ (the case $x_k < 0$ is analogous). We want to prove that $x_i > 0$ for all $i \geq k$. The proof is done by induction on $i - k$. The case $i - k = 0$ follows by assumption. Suppose that the result is true for $i - k = j$, i.e., $x_i = x_{k+j} > 0$. We want to show that $x_{i+1} = x_{k+j+1} > 0$.

Clearly, since Q is a contagion matrix, the properties of x and λ imply that $(xQ)_{i+1} = x_iQ_{i,i+1} + x_{i+1}Q_{i+1,i+1} = \lambda x_{i+1}$. Then $x_iQ_{i,i+1} = (\lambda - Q_{i+1,i+1})x_{i+1}$. By the induction hypothesis, $x_i > 0$, and since Q is a contagion matrix, $Q_{i,i+1} > 0$. Then $x_iQ_{i,i+1} > 0$ and since $\lambda \geq Q_{i+1,i+1}$, we have $\lambda > Q_{i+1,i+1}$ and $x_{i+1} > 0$. □

Proof of Lemma 4. Let l be the largest index such that $Q_{ll} = \lambda > 0$, and let y be a nonnegative left eigenvector associated with λ. We claim that, for each $i < l$, $y_i = 0$. Suppose not and let i be the largest index smaller than l such that $y_i \neq 0$. If $i < l - 1$, we have that $y_{i+1} = 0$ and since $Q_{i,i+1} > 0$, we get $(yQ)_{i+1} > 0$, which contradicts that y is an eigenvector associated with λ. If $i = l - 1$, then $(yQ)_l \geq Q_{ll}y_l + Q_{l-1,l}y_{l-1} > Q_{ll}y_l = \lambda y_l$, which, again, contradicts that y is an eigenvector associated with λ. Then we can restrict attention to matrix $Q_{\lceil(l-1)}$. Now, λ is also the largest eigenvalue of $Q_{\lceil(l-1)}$, but, by definition of l, only one diagonal entry of $Q_{\lceil(l-1)}$ equals λ and, hence, its multiplicity is 1. Then $z \in \mathbb{R}^{k-(l-1)}$ is a left eigenvector associated with λ for matrix $Q_{\lceil(l-1)}$ if and only if $(0, \ldots, 0, z) \in \mathbb{R}^k$ is a left eigenvector associated with λ for matrix Q. □

Proof of Lemma 5. Let $l < k$ and let $z := (y_1^Q, \ldots, y_{k-l}^Q) \in \mathbb{R}^{k-l}$. Since a contagion matrix is upper triangular, we have that, for each, $j \in \{1, \ldots, k-l\}$, $(zQ_{l\rfloor})_j = (y^Q Q)_j$. Therefore, z is a left eigenvector associated with the largest eigenvalue of Q, which, therefore, is also the largest eigenvalue of $Q_{l\rfloor}$. Then, by definition, $y^{Q_{l\rfloor}} = \frac{z}{\|z\|} = \frac{z}{\sum_{i=1}^{k-l} y_i^Q}$. □

Proof of Lemma 6. Clearly, since Q is a contagion matrix, if t is large enough, all the components of y^t are positive. Then, for the sake of exposition, we assume that all the components of y are positive. We distinguish two cases.

The matrix Q satisfies Property Q1. This part of the proof is a direct application of the Perron–Frobenius theorem. First, note that $\frac{yQ^t}{\|yQ^t\|}$ can be written as $\frac{y(Q^t/\lambda^t)}{\|y(Q^t/\lambda^t)\|}$. Now using, for instance, Theorem 1.2 in Seneta (2006), we have that $\frac{Q^t}{\lambda^t}$ converges to a matrix that is obtained as the product of the right and left eigenvectors associated to λ. Since in our case the right eigenvector is $(1, 0, \ldots, 0)$, $\frac{Q^t}{\lambda^t}$ converges to a matrix that has y^Q in the first row and with all other rows being the zero vector. Therefore, the result follows from the fact that $y_1 > 0$.

The matrix Q satisfies Property Q2. We show that, for each $i < k$, $\lim_{t\to\infty} y_i^t = 0$. We prove this by induction on i. Let $i = 1$. Then, for each $t \in \mathbb{N}$,

$$\frac{y_1^{t+1}}{y_k^{t+1}} = \frac{Q_{11}y_1^t}{\sum_{l\leq k} Q_{lk}y_l^t} < \frac{Q_{11}y_1^t}{Q_{kk}y_k^t} \leq \frac{y_1^t}{y_k^t},$$

where the first inequality is strict because $y_{k-1} > 0$ and $Q_{k-1,k} > 0$ (Q is a contagion matrix); the second inequality follows from Property Q2. Hence, the ratio $\frac{y_1^t}{y_k^t}$ is strictly decreasing in t. Moreover, since all the components of y^t lie in $[0, 1]$, it is not hard to see that, as far as y_1^t is bounded away from 0, the speed at which the above ratio decreases is also bounded away from 0.[23] Therefore, $\lim_{t\to\infty} y_1^t = 0$. Suppose that the claim holds for each $i < j < k - 1$. Now

$$\begin{aligned}\frac{y_j^{t+1}}{y_k^{t+1}} &= \frac{\sum_{l\leq j} Q_{lj}y_l^t}{\sum_{l\leq k} Q_{lk}y_l^t} < \frac{\sum_{l\leq j} Q_{lj}y_l^t}{Q_{kk}y_k^t} \\ &= \sum_{l<j}\frac{Q_{lj}}{Q_{kk}}\frac{y_l^t}{y_k^t} + \frac{Q_{jj}}{Q_{kk}}\frac{y_j^t}{y_k^t} \leq \sum_{l<j}\frac{Q_{lj}}{Q_{kk}}\frac{y_l^t}{y_k^t} + \frac{y_j^t}{y_k^t}.\end{aligned}$$

[23]Roughly speaking, this is because the state k will always get some probability from state 1 via the intermediate states, and this probability will be bounded away from 0 as far as the probability of state 1 is bounded away from 0.

By the induction hypothesis, for each $l < j$, the term $\frac{y_l^t}{y_k^t}$ can be made arbitrarily small for large enough t. Then the first term in the above expression can be made arbitrarily small. Hence, it is easy to see that, for large enough t, the ratio $\frac{y_j^t}{y_k^t}$ is strictly decreasing in t. As above, this can happen only if $\lim_{t\to\infty} y_j^t = 0$. □

Proof of Lemma 7. For each $i \in \{1, \dots, k\}$, let e_i denote the ith element of the canonical basis in $\mathbb{R}^k$. By Property Q1, Q_{11} is larger than any other diagonal entry of Q. Let y^Q be the unique nonnegative left eigenvector associated with Q_{11} such that $\|y^Q\| = 1$. Clearly, $y_1^Q > 0$ and, hence, $\{y^Q, e_2, \dots, e_k\}$ is a basis in $\mathbb{R}^k$. With respect to this basis, matrix Q is of the form

$$\left(\begin{array}{c|c} Q_{11} & 0 \\ \hline 0 & Q_{\lceil 1} \end{array}\right).$$

Now, we distinguish two cases.

The matrix $Q_{\lceil 1}$ satisfies Property Q2. In this case, we can apply Lemma 6 to $Q_{\lceil 1}$ to get that, for each nonnegative vector $z \in \mathbb{R}^{k-1}$ with $z_1 > 0$, $\lim_{t\to\infty} \frac{zQ_{\lceil 1}^t}{\|zQ_{\lceil 1}^t\|} = (0, \dots, 0, 1)$. Now let $y \in \mathbb{R}^k$ be the vector in the statement of this result, since y is very close to $(0, \dots, 0, 1)$. Then using the above basis, it is clear that $y = \alpha y^Q + v$, with $\alpha \geq 0$ and $v \approx (0, \dots, 0, 1)$. Let $t \in \mathbb{N}$. Then, for each $t \in \mathbb{N}$,

$$y^t = \frac{yQ^t}{\|yQ^t\|} = \frac{\lambda^t \alpha y^Q + vQ^t}{\|yQ^t\|} = \frac{\lambda^t \alpha y^Q + \|vQ^t\| \dfrac{vQ^t}{\|vQ^t\|}}{\|yQ^t\|}.$$

Clearly, $\|yQ^t\| = \|\lambda^t \alpha y^Q + \|vQ^t\| \frac{vQ^t}{\|vQ^t\|}\|$, and since all the terms are positive,

$$\|yQ^t\| = \|\lambda^t \alpha\| \|y^Q\| + \|vQ^t\| \left\| \frac{vQ^t}{\|vQ^t\|} \right\| = \|\lambda^t \alpha\| + \|vQ^t\|$$

and, hence, we have that y^t is a convex combination of y^Q and $\frac{vQ^t}{\|vQ^t\|}$. Since $v \approx (0, \dots, 0, 1)$ and $\frac{vQ^t}{\|vQ^t\|} \to (0, \dots, 0, 1)$, it is clear that, for each $t \in \mathbb{N}$, $\frac{vQ^t}{\|vQ^t\|}$ first-order stochastically dominates y^Q in the sense of more people being unhealthy. Therefore, y^t will also first-order stochastically dominate y^Q.

The matrix $Q_{\lceil 1}$ satisfies Property Q1. By Q1, the first diagonal entry of $Q_{\lceil 1}$ is larger than any other diagonal entry. Let $y^{c_{\lceil 1}}$ be the unique associated nonnegative left eigenvector such that $\|y^{c_{\lceil 1}}\| = 1$. It is easy to see that $y^{c_{\lceil 1}}$ first-order stochastically dominates y^Q; the reason is that $y^{c_{\lceil 1}}$ and y^Q are the limit of the same contagion process, with the only difference that the state in which only one person is unhealthy is known to have probability 0 when obtaining $y^{c_{\lceil 1}}$ from $Q_{\lceil 1}$. Clearly, $y_2^{c_{\lceil 1}} > 0$ and, hence,

$\{y^Q, y^{c_{\lceil 1}}, e_3, \ldots, e_k\}$ is a basis in $\mathbb{R}^k$. With respect to this basis, the matrix Q is of the form

$$\left(\begin{array}{c|c|c} Q_{11} & 0 & 0 \\ \hline 0 & Q_{22} & 0 \\ \hline & & \\ 0 & 0 & Q_{\lceil 2} \\ & & \end{array}\right).$$

Again, we can distinguish two cases.

- **The matrix $Q_{\lceil 2}$ satisfies Property Q2.** In this case, we can repeat the arguments above to show that y^t is a convex combination of y^Q, $y^{c_{\lceil 1}}$, and $\frac{vQ^t}{\|vQ^t\|}$. Since both $y^{c_{\lceil 1}}$ and $\frac{vQ^t}{\|vQ^t\|}$ first-order stochastically dominate y^Q, y^t also does.
- **The matrix $Q_{\lceil 2}$ satisfies Property Q1.** Now we would get a vector $y^{c_{\lceil 2}}$, and the procedure would continue until a truncated matrix satisfies Property Q2 or until we get a basis of eigenvectors, one of them being y^Q and all the others first-order stochastically dominating y^Q. In both situations, the result immediately follows from the above arguments. □

A.3 *Proofs of results in Section 5*

Proof of Lemma 11. Let $G \in \mathcal{G}$, $M > 1$, $\delta \in (0, 1)$, and $r \in (0, 1)$. Consider game G^M_δ. Let $k \in \{1, 2\}$ and let $i \in C_k$ be a player who is unhealthy after some history $h^{\bar{t}}$, with $\bar{t} > T^{\mathrm{I}} + T^{\mathrm{II}}$. Suppose that exactly $\lceil rM \rceil$ people are infected in each community. Given $\sigma_i \in \Sigma_i$, the payoff associated with the continuation strategy $\sigma_i|_{h^{\bar{t}}}$ can be decomposed as $(1 - \delta)(u_{\bar{t}+1} + V(\sigma_i, r, M, \delta))$, where $u_{\bar{t}+1}$ denotes the expected payoff in period $\bar{t} + 1$ and $V(\sigma_i, r, M, \delta)$ denotes the (expected) sum of discounted continuation payoffs from period $\bar{t} + 2$ onward. Let σ^*_i be a maximizer of $V(\sigma_i, r, M, \delta)$ for given r, M, and δ. Then define

$$\Delta(r, M, \delta) := V\big(\sigma^*_i, r, M, \delta\big) - V(\bar{\sigma}_i, r, M, \delta),$$

the difference between the (expected) sums of discounted continuation payoffs associated with σ^*_i and $\bar{\sigma}_i$ (which prescribes to play the Nash action). We first establish a claim that is a consequence of the fact that contagion spreads exponentially fast during Phase III.

Claim 1. *Let $G \in \mathcal{G}$. There is $\bar{U}_G \in \mathbb{R}$ such that, for each $r > \frac{1}{2}$, each $M > 1$, and each $\delta \in (0, 1)$, if $\lceil rM \rceil > \frac{M}{2} + 1$, then $\Delta(r, M, \delta) \leq \bar{U}_G$.*

Proof. Consider a situation in which there are k unhealthy players in each community playing the Nash action in a given period t in Phase III and, hence, less than $M - k$ healthy players. Then let $P(k, M)$ be the probability that there are more than $\frac{M-k}{2}$ healthy players in each of the communities at the end of period t. We want to show that if $k > \frac{M}{2}$, then $P(k, M) < \frac{1}{2}$. Clearly, $P(k, M)$ is strictly decreasing in k, so it suffices to show that $P(\frac{M}{2}, M) \leq \frac{1}{2}$. We want to show that the probability that more than $\frac{M - \frac{M}{2}}{2} = \frac{M}{4}$ players remain healthy is not larger than $\frac{1}{2}$.

Recall that the transition matrix in Phase III, $\bar{S} \in \mathcal{M}_M$ (defined in Section 4.4), is such that for each pair $k, l \in \{1, \ldots, M\}$, $\bar{S}_{kl}$ is 0 unless $k \leq l \leq 2k$, in which case

$$\bar{S}_{kl} = \frac{(k!)^2\big((M-k)!\big)^2}{\big((l-k)!\big)^2(2k-l)!(M-l)!M!} \quad \text{and, hence,}$$

$$\bar{S}_{\frac{M}{2}l} = \frac{\left(\frac{M}{2}!\right)^2\left(\left(M-\frac{M}{2}\right)!\right)^2}{\left(\left(l-\frac{M}{2}\right)!\right)^2\big((M-l)!\big)^2 M!}.$$

The above probabilities are symmetric in the sense that transitioning from $\frac{M}{2}$ to $\frac{M}{2} + \alpha$ is as likely as transitioning from $\frac{M}{2}$ to $M - \alpha$. Thus, for each transition that results in less than $\frac{M}{4}$ new infections, there is an equally likely transition that delivers more than $\frac{M}{4}$. Thus, the probability that more than $\frac{M}{4}$ players in each community remain healthy is not larger than $\frac{1}{2}$, and so $P(k, M) < \frac{1}{2}$ whenever $k > \frac{M}{2}$.

Now recall that $\Delta(r, M, \delta) = V(\sigma_i^*, r, M, \delta) - V(\bar{\sigma}_i, r, M, \delta)$, defined in the proof of Lemma 11, is the difference between the (expected) sums of discounted continuation payoffs from period $\bar{t}+2$ onward associated with σ_i^* and $\bar{\sigma}_i$ (which prescribes to play the Nash action). Given that $\lceil rM \rceil > \frac{M}{2} + 1$, the computation behind $\Delta(r, M, \delta)$ assumes that there are at least $\lceil rM \rceil > \frac{M}{2} + 1$ unhealthy players in each community. Thus, regardless of the action of player i in period $\bar{t}+1$, more than $\frac{M}{2}$ unhealthy players will be playing the Nash action. Therefore, by the above result regarding the $P(k, M)$ probabilities, there is $\hat{p}$ such that $P(\lceil rM \rceil - 1, M) \leq \hat{p} < \frac{1}{2}$. We start by computing the probability of meeting a healthy player in future periods.

- Period $t+1$. Regardless of the action chosen by player i in period t, the probability that less than half of the healthy players got infected in period t is at most $\hat{p}$. Then the probability of meeting a healthy opponent in period $t+1$ is at most

$$\hat{p}(1-r) + (1-\hat{p})\frac{1-r}{2} < \frac{1-r}{2} + \hat{p}(1-r) = (1-r)\left(\frac{1}{2} + \hat{p}\right).$$

- Period $t+2$. Similarly, the probability of meeting a healthy opponent in period $t+2$ is at most $\hat{p}(\hat{p}(1-r) + (1-\hat{p})\frac{1-r}{2}) + (1-\hat{p})\frac{\hat{p}(1-r)+(1-\hat{p})\frac{1-r}{2}}{2}$, which reduces to

$$\hat{p}^2(1-r) + 2\hat{p}(1-\hat{p})\frac{1-r}{2} + (1-\hat{p})^2\frac{1-r}{4} = (1-r)\left(\frac{1}{2} + \hat{p}\right)^2.$$

- Period $t+\tau$. In general, regardless of the actions chosen by player i, the probability of meeting a healthy opponent in period $t+\tau$ is less than $(1-r)(\frac{1}{2} + \hat{p})^\tau$.

We turn to the computation of $\Delta(r, M, \delta) = V(\sigma_i^*, r, M, \delta) - V(\bar{\sigma}_i, r, M, \delta)$. Suppose the payoffs in G are such that (i) the payoff loss from deviating from the strict Nash a^* is at least $\underline{l} > 0$ and (ii) the maximal possible gain from not playing according to a^* against an opponent who is not playing according to a^* is at most $\bar{m}$. Then we have

$$\Delta(r, M, \delta) \leq \sum_{\tau=1}^{\infty}(1-r)\left(\frac{1}{2}+\hat{p}\right)^{\tau}\delta^{\tau}\bar{m} - \left(1-(1-r)\left(\frac{1}{2}+\hat{p}\right)^{\tau}\right)\delta^{\tau}\underline{l}$$
$$\leq \sum_{\tau=1}^{\infty}(1-r)\left(\frac{1}{2}+\hat{p}\right)^{\tau}\delta^{\tau}\bar{m} \leq \bar{m}\sum_{\tau=1}^{\infty}\left(\frac{1}{2}+\hat{p}\right)^{\tau}.$$

Since $\frac{1}{2} + \hat{p} < 1$, the above series converges. Thus, if we define $\bar{U}_G := \bar{m}\sum_{\tau=1}^{\infty}(\frac{1}{2} + \hat{p})^{\tau}$, the result follows. □

We can use Claim 1 to now prove the lemma. Claim 1 captures the fact that once the contagion has infected half of the population, no matter how patient a player is, there is not much to gain by slowing down the contagion (regardless of the value of M).

Now suppose that player i believes that contagion is (r, p)-spread and chooses a continuation strategy in which he does not play the Nash action in period $\bar{t}$. Then we have the following possibilities.

(i) Player i meets an unhealthy player. This event has probability at least rp, player i incurs some loss $\underline{l} > 0$ by not playing Nash, and does not slow down the contagion.

(ii) There are two cases in which player i can meet a healthy player:

- Case 1. At least rM people are unhealthy and player i meets a healthy player. This event has probability at most $1 - r$.
- Case 2. At most rM people are unhealthy and player i meets a healthy player. This event has probability at most $1 - p$.

In both cases above, player i makes some gain $\bar{m}$ in the current period and, provided that $\lceil rM \rceil > \frac{M}{2} + 1$, at most $\bar{U}_G$ in the future.

Hence, the gain from not playing the Nash action instead of doing so is bounded above by

$$(1-p)(\bar{m}+\bar{U}_G) + (1-r)(\bar{m}+\bar{U}_G) - rp\underline{l}.$$

Since $\bar{m}$, $\underline{l}$, and $\bar{U}_G$ just depend on the stage game G, there exist $p^G \in (0, 1)$ and $r^G \in (0, 1)$ such that, for each $p \geq p^G$ and each $r \geq r^G$, we have that the above expression is negative and, moreover, $\lceil rM \rceil > \frac{M}{2} + 1$ for all $M > 2$ (so that we can rely on bound $\bar{U}_G$). Thus, for such values, it is sequentially rational for player i to play the Nash action. □

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