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# Mechanism design without quasilinearity

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This paper studies a model of mechanism design with transfers where agents' preferences need not be quasilinear. In such a model, (i) we characterize dominant strategy incentive compatible mechanisms using a monotonicity property, (ii) we establish a revenue *uniqueness* result (for every dominant strategy implementable allocation rule, there is a unique payment rule that can implement it), and (iii) we show that every dominant strategy incentive compatible, individually rational, and revenue-maximizing mechanism must charge zero payment for the *worst* alternative (outside option). These results are applicable in a wide variety of problems (single object auction, multiple object auction, public good provision, etc.) under suitable richness of type space. In particular, our results are applicable to two important type spaces: (a) type space containing an arbitrarily small perturbation of quasilinear type space and (b) type space containing all positive income effect preferences.

Keywords. Incentive compatibility, individual rationality, monotonicity, non-quasilinear preferences, revenue equivalence.

JEL classification. D82, D44, D40.

## 1. Introduction

We analyze a model of mechanism design with transfers where preferences of agents over transfers are not necessarily quasilinear. We assume agents have *classical* preferences over the set of consumption bundles, where a consumption bundle consists of an alternative and a transfer amount. A preference (type) is classical if it is monotone and continuous in transfers.

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We provide results that cover a variety of important problems: the single object auction problem, the multi-object auction problem, the public good provision problem, and so on. If the type space satisfies a richness property, we provide a simple monotonicity condition that along with a *posted-price property*, is necessary and sufficient for a mechanism to be (dominant strategy) incentive compatible. The posted-price property requires the payment decisions at various types to agree whenever the allocation decisions at these types agree. Further, we establish a *revenue uniqueness* result: for every implementable allocation rule, there is a unique payment rule such that the corresponding mechanism is incentive compatible. Though we do not have revenue equivalence in our model, our results can be interpreted as a counterpart of the monotonicity and revenue equivalence results with quasilinearity.

There are at least two important type spaces where our results apply. Let $\mathcal{Q}$ be the set of all quasilinear preferences in some standard problem. The first example of a type space where our results apply is any superset of $\mathcal{Q}$ containing "small perturbations" of types in $\mathcal{Q}$. Our second example is a type space containing all positive income effect preferences.

We study the implications of individual rationality without quasilinearity. Recall that a straightforward consequence of revenue equivalence in a quasilinear environment is that a *revenue-maximizing*, incentive compatible, and individually rational mechanism charges zero payment for allocating the *worst* alternative. In other words, individual rationality "binds" for the worst alternative. We show that this result continues to remain valid in arbitrary non-quasilinear type spaces.

We apply our results to provide a template for identifying an optimal contract in a principal–agent model. An important insight from the mechanism design literature with quasilinearity is that the optimization problem for finding an optimal contract can be reduced to an optimization problem over the set of monotone allocation rules. The objective function can be rewritten in terms of the allocation rule using the revenue equivalence formula. Furthermore, the incentive and individual rationality constraints can be replaced by a monotonicity constraint on the allocation rule. This approach is key to solving tractable optimal contract problems (Mussa and Rosen 1978, Myerson 1981). We show that this approach can be extended to our model with non-quasilinear preferences. We now proceed to the details.

## 2. The model

Let $A$ be a finite set of alternatives with $|A| \geq 2$. We endow $A$ with a strict partial order $\succ$. This partial order reflects a possible ex ante ordering of alternatives if there is no monetary compensation. The partial order $\succ$ may be empty; for instance, if $A$ is the set of public goods and there is no ex ante distinction among the public goods. In the problem of selling multiple units of a homogeneous good, a standard assumption is that more units are preferred to less. If there are two alternatives $a$ and $b$ with $a$ being $k$ units of the good and $b$ being $\ell$ units of the good, we may impose $a \succ b$ if and only if $k > \ell$. In combinatorial auctions for selling multiple objects, $\succ$ may be the partial order induced by the set inclusion relation over subsets of objects.

There is a single agent in our model. Our main results extend to a model with multiple agents in a straightforward manner as our solution concept is dominant strategy equilibrium. We allow for monetary transfers or payments. Hence, a consumption bundle for the agent is a pair $(a, p)$, where $a \in A$ is an alternative and $p \in \mathbb{R}$ is the payment of the agent. The set of all such consumption bundles is denoted by $\mathcal{Z} := \{(a, p) : a \in A, p \in \mathbb{R}\}$.

The agent has a complete and transitive preference over $\mathcal{Z}$. A typical preference is denoted by $R$ with its strict and indifferent components denoted by $P$ and $I$, respectively. We call a preference $R$ over $\mathcal{Z}$ *classical* if it satisfies the following conditions.

- *Money monotonicity.* For all $p, p' \in \mathbb{R}$ with $p > p'$ and for all $a \in A$, $(a, p')P(a, p)$.
- *Respect for* $\succ$. For all $p \in \mathbb{R}$ and for all $a, b \in A$ with $a \succ b$, $(a, p)P(b, p)$.
- *Continuity.* For all $z \in \mathcal{Z}$, the sets $\{z' \in \mathcal{Z} : zRz'\}$ and $\{z' \in \mathcal{Z} : z'Rz\}$ are closed.
- *Possibility of compensation.* For every $z \in \mathcal{Z}$ and for every $a \in A$, there exists $p$ and $p'$ such that $(a, p)Rz$ and $zR(a, p')$.

While monotonicity and continuity are natural restrictions, we impose the possibility of a compensation condition for technical reasons. We are only concerned with classical preferences in this paper, so whenever we write “preference,” we mean a classical preference.

### 2.1 *Indifference vectors*

It is convenient to think of a preference in terms of its indifference vectors. A vector $v \in \mathbb{R}^{|A|}$ is an *indifference vector* of preference $R$ if for all $a, b \in A$, we have $(a, v_a)I(b, v_b)$. Denote the set of all indifference vectors of $R$ by $\mathcal{I}(R)$. For every preference, its indifference vectors satisfy many natural properties. So as to discuss them, we introduce the notion of a valuation.

The *valuation* of the agent with preference $R$ for alternative $a$ at consumption bundle $z$ is denoted by $V^R(a, z)$, where $(a, V^R(a, z))Iz$. In other words, $V^R(a, z)$ is the unique payment that makes the agent indifferent between consumption bundle $z$ and $(a, V^R(a, z))$. The existence of $V^R(a, z)$ and its uniqueness is guaranteed by the assumptions on preferences. As a result, for every preference $R$ and for every pair of distinct indifference vectors $v, v' \in \mathcal{I}(R)$, we have either $v > v'$ or $v' > v$, i.e., $v$ and $v'$ do not intersect.[1] Further, every indifference vector of $R$ respects $\succ$: for every $v \in \mathcal{I}(R)$, we have $v_a > v_b$ if $a \succ b$.

A typical indifference vector $v$ of a preference $R$ can be represented by the diagram shown in Figure 1. We use such figures throughout our paper and we refer to them as *indifference diagrams*.[2] Figure 1 shows how a indifference diagram can be constructed for four alternatives. Each horizontal line in the indifference diagram corresponds to a

[1] Whenever, we write $v > v'$, we mean $v_x > v'_x$ for all $x \in A$.

[2] William Thomson popularized such diagrams to represent preferences in private goods models with indivisible goods and money.

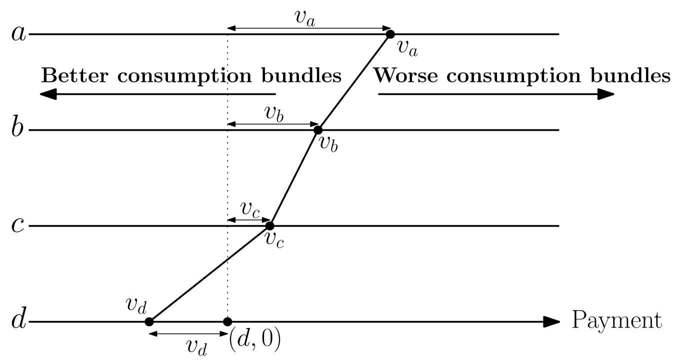


FIGURE 1. An indifference vector for the case $a \succ b \succ c \succ d$.

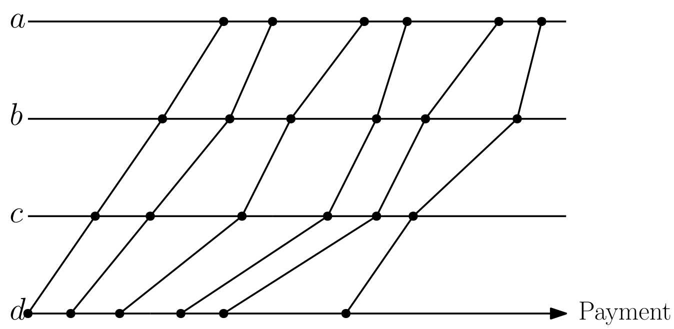


FIGURE 2. A preference and its indifference diagram for the case $a \succ b \succ c \succ d$.

unique alternative. Every point on a horizontal line corresponds to a payment level. So the four horizontal lines in this indifference diagram comprise the set of all consumption bundles $\mathcal{Z}$. As we move rightward along a horizontal line, the payment of the agent increases. Hence, consumption bundles to the right (left) of the indifference vector $v$ in Figure 1 are worse (respectively, better) than the four consumption bundles corresponding to $v$.

An equivalent way to think of a preference $R$ is through its infinite set of indifference vectors $\mathcal{I}(R)$. Hence, the indifference diagram of a preference consists of an infinite collection of such vectors; a representative indifference diagram is shown in Figure 2.

## 2.2 *Positive income effect*

Two special kinds of preferences are worth highlighting. Vectors $v, v' \in \mathbb{R}^{|A|}$ are *parallel* if for all $a, b \in A$, we have $v_a - v_b = v'_a - v'_b$. Vectors $v, v' \in \mathbb{R}^{|A|}$ with $v > v'$ satisfy *decreasing differences* (DD) if for all $a, b \in A$, we have

$$[v_a > v_b] \quad \Rightarrow \quad \left[v'_a - v'_b > v_a - v_b\right].$$

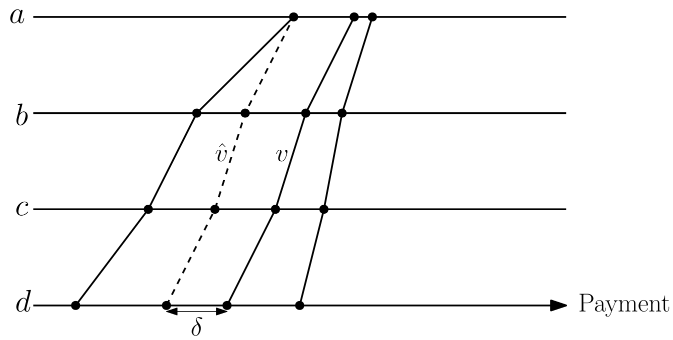


FIGURE 3. Indifference diagram of a preference satisfying PIE.

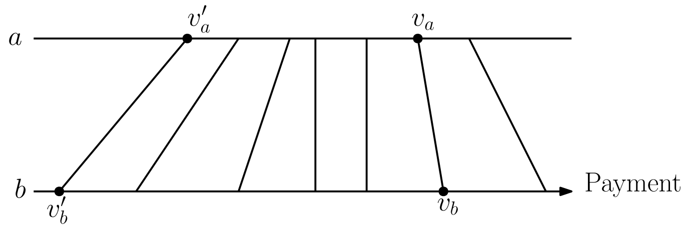


FIGURE 4. Indifference diagram of a preference that does not satisfy PIE.

DEFINITION 1. A preference $R$ is *quasilinear* if every pair of indifference vectors $v, v' \in \mathcal{I}(R)$ is parallel. A preference $R$ satisfies *positive income effect* (PIE) if every pair of indifference vectors $v, v' \in \mathcal{I}(R)$ satisfies DD.

We denote the set of all quasilinear preferences and the set of all preferences satisfying PIE by $\mathcal{Q}$ and $\mathcal{R}^+$, respectively.[3] Note that $\mathcal{Q} \cap \mathcal{R}^+ = \varnothing$. For any $R \in \mathcal{R}^+$, the indifference diagram corresponding to $R$ is shown in Figure 3.

To understand PIE, take any $R \in \mathcal{R}^+$ and $v \in \mathcal{I}(R)$. Consider another vector $\hat{v}$ defined by $\hat{v}_x = v_x - \delta \; \forall x \in A$, where $\delta > 0$. Suppose that for some $a \in A$, we have $v_a > v_x$ for all $x \neq a$ (as in Figure 3). Then we see that for any $x \neq a$, the consumption bundle $(a, \hat{v}_a)$ is strictly preferred to $(x, \hat{v}_x)$ according to $R$. As we decrease the payment by a constant amount $\delta$ from $v$ to $\hat{v}$ (i.e., increase the wealth level by $\delta$), the highest payment consumption bundle at $v$ is now the most preferred. This is the idea of a positive income effect, which is usually observed in normal goods. Contrast this with a quasilinear preference, where the agent will be indifferent between all the bundles on $\hat{v}$.

The decreasing differences property imposes restrictions on admissible indifference vectors. To further highlight these restrictions, consider $A = \{a, b\}$ and that $\succ$ is empty. The indifference diagram for a preference $R$ for this example is shown in Figure 4. We

[3]To be precise, $\mathcal{Q}$ is the set of all preferences that respect $\succ$ and are quasilinear. For notational simplicity, we suppress the dependence of $\mathcal{Q}$ on $\succ$. Further, we refer to $\mathcal{Q}$ as *the* quasilinear domain, meaning the quasilinear domain corresponding to a given $\succ$. A similar comment applies to $\mathcal{R}^+$ and other domains discussed henceforth.

argue that $R$ *does not* satisfy PIE. This is because the indifference vectors $v, v' \in \mathcal{I}(R)$ satisfy $v \succ v'$ but fail DD: $v_b > v_a$ but $v'_b - v'_a < 0 < v_b - v_a$. This is the intuition for Fact 1 below.[4]

Fact 1 (Remark 2 in Zhou and Serizawa (2018)). *Suppose $R$ is a preference that satisfies PIE. Then there exists a weak ordering $\trianglerighteq$ on the set of alternatives such that for every $a, b \in A$ and every $v \in \mathcal{I}(R)$,*

$$[a \trianglerighteq b] \Rightarrow [v_a \geq v_b].$$

Fact 1 implies that even if $\succ$ is empty, there is an underlying *weak* order for every positive income effect preference. Notice that, potentially, such a weak order may be different for different positive income effect preferences. To understand the interaction between $\trianglerighteq$ and $\succ$, fix an $R$ that satisfies PIE. Then (a) $\trianglerighteq$ is weak but complete (but $\succ$ may be empty); (b) $\trianglerighteq$ must match $\succ$ whenever the latter is nonempty, i.e., for any $a, b \in A$, $a \succ b$ implies $a \trianglerighteq b$ holds but $b \trianglerighteq a$ does not hold. Hence, if $\succ$ is a complete order, $\trianglerighteq = \succ$.

We are also concerned with the following weaker notion of decreasing differences. A pair of vectors $v, v' \in \mathbb{R}^{|A|}$ with $v > v'$ satisfies $\succ$ *decreasing differences* ($\succ$DD) if for all $a, b \in A$, we have

$$[a \succ b] \quad \Rightarrow \quad \left[v'_a - v'_b > v_a - v_b\right].$$

If $v$, $v'$ respect $\succ$ and satisfy DD, they satisfy $\succ$DD. Based on this, we can define a weaker notion of PIE. A preference $R$ satisfies $\succ$ *positive income effect* ($\succ$PIE) if every pair of indifference vectors $v, v' \in \mathcal{I}(R)$ satisfies $\succ$DD. Let $\mathcal{R}^+_{\succ}$ denote the set of all preferences satisfying $\succ$PIE. Obviously, $\mathcal{R}^+ \subseteq \mathcal{R}^+_{\succ}$ for each $\succ$. If $\succ$ is complete and $v$ respects $\succ$, then $a \succ b$ if and only if $v_a > v_b$. Hence, $\mathcal{R}^+_{\succ} = \mathcal{R}^+$ if $\succ$ is complete. This equivalence is not necessarily true if $\succ$ is not complete. For instance, the preference shown in Figure 4 satisfies $\succ$PIE vacuously since $\succ$ is empty (though it does not satisfy PIE).

### 2.3 *Domains, mechanisms and questions*

Let $\mathcal{R}$ be the set of all (classical) preferences. A *type space* or *domain* $\mathcal{D}$ is any subset of $\mathcal{R}$. A *mechanism* on a domain $\mathcal{D}$ is a pair of maps $(f, p)$, where $f : \mathcal{D} \to A$ is an *allocation rule* and $p : \mathcal{D} \to \mathbb{R}$ is a *payment rule*. Throughout the paper, we consider only allocation rules that are onto: for every $a \in A$, there exists some preference $R \in \mathcal{D}$ such that $f(R) = a$.[5] We investigate the implications of two desiderata for our mechanisms.

Definition 2. A mechanism $(f, p)$ on $\mathcal{D}$ is *dominant strategy incentive compatible* (DSIC) if for every $R, R' \in \mathcal{D}$, we have $(f(R), p(R)) R (f(R'), p(R'))$.

In addition to incentive compatibility, we consider individual rationality. Whenever we talk about individual rationality, we assume the existence of an alternative $a_0 \in A$

[4] Its proof is available in Appendix A of Zhou and Serizawa (2018).

[5] In some sense, "ontoness" is without loss of generality. If we have an allocation rule $f$ whose range is $A' \subsetneq A$, then we can consider the entire model where $A$ is replaced by $A'$.

such that $a \succ a_0$ for all $a \neq a_0$. Alternative $a_0$ can be interpreted as the outside option, for instance, not getting any object in a multiple object auction model or not providing any public good in a public good model. We do not require the existence of such an alternative for our results involving only DSIC.

Definition 3. A mechanism $(f, p)$ on $\mathcal{D}$ satisfies *individual rationality* (IR) if for every $R \in \mathcal{D}$, we have $(f(R), p(R))R(a_0, 0)$.

We answer three questions in our model of preferences.

- In the quasilinear domain, a monotonicity condition of allocation rule and a revenue equivalence formula characterize a DSIC mechanism. What is the analogue of such a characterization without quasilinearity?
- What happens to revenue equivalence without quasilinearity?
- Does the IR constraint bind for the worst alternative without quasilinearity?

## 3. Richness

Our results require our domains to be suitably rich. The weakest richness condition that we require is the following.

Definition 4. A domain of preferences $\mathcal{D}$ satisfies *one-point (OP) richness* if for every $v \in \mathbb{R}^{|A|}$ that respects $\succ$, there exists $R \in \mathcal{D}$ such that $v \in \mathcal{I}(R)$.

OP richness is satisfied by $\mathcal{Q}$, the set of all quasilinear preferences. It is therefore satisfied by any domain $\mathcal{D} \supseteq \mathcal{Q}$. It is also satisfied by $\mathcal{R}^+$, the set of all positive income effect preferences (and supersets of that). OP richness allows us to construct a preference that has *one* specific indifference vector. It is silent on the other indifference vectors in the preference.

The quasilinear domain $\mathcal{Q}$ satisfies OP richness. By definition, $\mathcal{Q}$ contains *all* quasilinear preferences respecting $\succ$. OP richness is violated if we take a strict subset of $\mathcal{Q}$. For example, suppose $A = \{a_0, a_1, a_2\}$ with $a_2 \succ a_1 \succ a_0$. A quasilinear preference is represented by a *valuation* vector $v \in \mathbb{R}^3_+$ with $v_{a_2} > v_{a_1} > v_{a_0} = 0$. The quasilinear domain $\mathcal{Q}$ contains preferences corresponding to *all* such vectors in $\mathbb{R}^3_+$. Now consider a specific kind of quasilinear preference whose valuation vector is $v_{a_2} = 2\theta$, $v_{a_1} = \theta$, $v_{a_0} = 0$ for some $\theta \in \mathbb{R}_{++}$. The set of all such preferences is

$$\mathcal{Q}^1 := \big\{R \in \mathcal{Q} : \exists\, \theta \in \mathbb{R}_{++} \text{ such that } V^R\big(a_1, (a_0, 0)\big) = \theta, V^R\big(a_2, (a_0, 0)\big) = 2\theta\big\}.$$

It is clear that $\mathcal{Q}^1$ is a strict subset of $\mathcal{Q}$ and it contains very specific "one-dimensional" quasilinear preferences. The domain $\mathcal{Q}^1$ fails OP richness. Though our richness conditions exclude domains like $\mathcal{Q}^1$, they still admit interesting domains where a large class of DSIC mechanisms exist. For instance, the family of Groves mechanisms are DSIC in $\mathcal{Q}$.

Our next notion of richness requires the existence of a preference for some specific *pair* of indifference vectors.

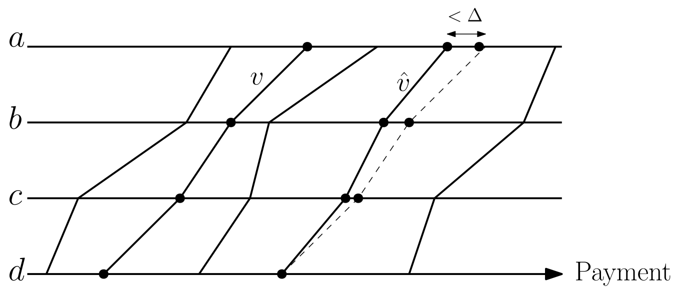


FIGURE 5. Indifference diagram of a candidate preference for satisfying $TP_+^\Delta$ richness.

DEFINITION 5. Let $\Delta > 0$. Vectors $v$, $\hat{v}$ with $\hat{v} > v$ are $\Delta_+$ *parallel* if (i) they satisfy DD and (ii) $|(\hat{v}_a - \hat{v}_b) - (v_a - v_b)| < \Delta$ for all $a, b \in A$.

Our next richness condition gives us the flexibility to construct some preference using every pair of $\Delta_+$ parallel vectors.

DEFINITION 6. Let $\Delta > 0$. A domain $\mathcal{D}$ satisfies $\Delta_+$ *two-point richness (*$TP_+^\Delta$ *richness)* if for every $v$, $\hat{v}$ with $\hat{v} > v$ such that $v$, $\hat{v}$ are $\Delta_+$ parallel and $v$, $\hat{v}$ respect $\succ$, there exists $R \in \mathcal{D}$ such that $v, \hat{v} \in \mathcal{I}(R)$.

Notice that $TP_+^\Delta$ richness is for a fixed $\Delta$. Obviously, if we choose $\Delta' > \Delta$ and $\mathcal{D}$ satisfies $TP_+^{\Delta'}$ richness, it also satisfies $TP_+^\Delta$ richness. Next, $TP_+^\Delta$ richness is about existence of some $R \in \mathcal{D}$ for every pair $v$, $\hat{v}$ satisfying the two conditions in the definition above. This preference may be different for a different pair of vectors. Further, the richness condition is silent about the properties of other indifference vectors in this preference. For illustration, consider the indifference diagram in Figure 5. The indifference diagram shows two indifference vectors $v$ and $\hat{v}$ that are $\Delta_+$ parallel and respect $\succ$. The vectors $v$ and the one represented by dashed lines are parallel. It also shows a preference $R$ (showing a subset of indifference vectors of $R$) such that $v, \hat{v} \in \mathcal{I}(R)$. Notice that there are other indifference vectors of $R$ that are *not* necessarily $\Delta_+$ parallel. For this reason, a domain satisfying $TP_+^\Delta$ richness need not contain the quasilinear domain. Example 1, which is given later in this section, highlights this point.

We now introduce our final notion of richness.

DEFINITION 7. Let $\Delta > 0$. Vectors $v$, $\hat{v}$ with $\hat{v} > v$ are $\Delta_\succ$ *parallel* if (i) they satisfy $\succ$-DD and (ii) $|(\hat{v}_a - \hat{v}_b) - (v_a - v_b)| < \Delta$ for all $a, b \in A$.

Notice that $\Delta_\succ$ parallel requires a pair of vectors to satisfy $\succ$-DD but $\Delta_+$ parallel requires them to satisfy DD. If $v$, $\hat{v}$ satisfy decreasing differences, the first condition in Definition 7 is automatically satisfied. Hence, any pair of vectors $v$, $\hat{v}$ that are $\Delta_+$ parallel are also $\Delta_\succ$ parallel. Further, if $\succ$ is a *complete* order, for every vector $v$ that respects $\succ$, we have $a \succ b$ if and only if $v_a > v_b$. Hence, if $\succ$ is complete, a pair of vectors $v$, $\hat{v}$ that respect $\succ$ are $\Delta_+$ parallel if and only if they are $\Delta_\succ$ parallel. Our final richness requires that for every pair of vectors that are $\Delta_\succ$ parallel, there is a preference that contains them as indifference vectors.

DEFINITION 8. Let $\Delta > 0$. A domain $\mathcal{D}$ satisfies $\Delta_{\succ}$ *two-point richness (*$\text{TP}^{\Delta}_{\succ}$ *richness)* if for every $v, \hat{v}$ with $\hat{v} > v$ such that $v, \hat{v}$ are $\Delta_{\succ}$ parallel and $v, \hat{v}$ respect $\succ$, there exists $R \in \mathcal{D}$ such that $v, \hat{v} \in \mathcal{I}(R)$.

We have argued earlier that if $v, \hat{v}$ are $\Delta_+$ parallel, they are also $\Delta_{\succ}$ parallel. Hence, $\text{TP}^{\Delta}_{\succ}$ richness implies $\text{TP}^{\Delta}_{+}$ richness. Further, these two notions of richness coincide if $\succ$ is complete.

*Examples of domains*

Though $\mathcal{Q}$ satisfies OP richness, it fails both the notions of TP richness. Obviously, the set of *all* (classical) preferences satisfy all the notions of richness that we have introduced. The set of all positive income effect preferences $\mathcal{R}^+$ satisfies $\text{TP}^{\Delta}_{+}$ richness for every $\Delta > 0$. However, $\mathcal{R}^+$ may fail $\text{TP}^{\Delta}_{\succ}$ richness for all $\Delta > 0$ and for some $\succ$. When $\succ$ is complete, $\text{TP}^{\Delta}_{+}$ richness and $\text{TP}^{\Delta}_{\succ}$ richness coincide for all $\Delta > 0$. Hence, $\mathcal{R}^+$ satisfies $\text{TP}^{\Delta}_{\succ}$.

We now give more examples of domains that satisfy our richness. Pick any $a^* \in A$. For every pair of preferences $R, R'$, define the *distance* between $R$ and $R'$ as

$$d(R, R') := \sup_{t} \max_{a \neq a^*} \left| V^R\big(a, (a^*, t)\big) - V^{R'}\big(a, (a^*, t)\big)\right|.$$

Note that $d(R, R') \in \mathbb{R}_+ \cup \{+\infty\}$. Further, (i) $d(R, R') = d(R', R)$, (ii) $d(R, R') = 0$ if and only if $R = R'$, and (iii) $d(R, R') + d(R', R'') \geq d(R, R'')$ for all $R, R', R''$.

Pick any $\Delta > 0$. Define $\mathcal{Q}^{\Delta}_{\succ}$ and $\mathcal{Q}^{\Delta}_{+}$ as

$$\mathcal{Q}^{\Delta}_{\succ} := \{R \in \mathcal{R}^+_{\succ} : \exists R' \in \mathcal{Q} \text{ such that } d(R, R') < \Delta\}$$
$$\mathcal{Q}^{\Delta}_{+} := \{R \in \mathcal{R}^+ : \exists R' \in \mathcal{Q} \text{ such that } d(R, R') < \Delta\}.$$

Since $\mathcal{R}^+ \subseteq \mathcal{R}^+_{\succ}$, we have $\mathcal{Q}^{\Delta}_{+} \subseteq \mathcal{Q}^{\Delta}_{\succ}$. Intuitively, $\mathcal{Q}^{\Delta}_{\succ}$ and $\mathcal{Q}^{\Delta}_{+}$ are domains of preferences that are "$\Delta$-close" to the domain of quasilinear preferences (and satisfy $\succ$PIE and PIE, respectively). Since $\mathcal{Q}^{\Delta}_{\succ} \subsetneq \mathcal{R}^+_{\succ}$ and $\mathcal{Q} \cap \mathcal{R}^+_{\succ} = \varnothing$, we see that $\mathcal{Q}^{\Delta}_{\succ} \cap \mathcal{Q} = \varnothing$. Also, $\mathcal{Q}^{\Delta}_{+} \subsetneq \mathcal{Q}^{\Delta}_{\succ}$ and, hence, $\mathcal{Q}^{\Delta}_{+} \cap \mathcal{Q} = \varnothing$. Further, $\mathcal{Q}^{\Delta}_{+} \subsetneq \mathcal{R}^+$. We show below that any domain that contains $\mathcal{Q}^{\Delta}_{\succ}$ satisfies $\text{TP}^{\Delta}_{\succ}$ richness.

LEMMA 1. *Let $\Delta > 0$. Suppose domain $\mathcal{D} \supseteq \mathcal{Q}^{\Delta}_{\succ}$. Then $\mathcal{D}$ satisfies* $\text{TP}^{\Delta}_{\succ}$ *richness. Similarly, suppose domain $\mathcal{D} \supseteq \mathcal{Q}^{\Delta}_{+}$. Then $\mathcal{D}$ satisfies* $\text{TP}^{\Delta}_{+}$ *richness.*

The proof of Lemma 1 is given in Appendix B. An important consequence of this lemma is that even though $\mathcal{Q}$ fails our richness requirements, there are domains arbitrarily close to $\mathcal{Q}$ that satisfy our richness. Further, such domains *need not* contain $\mathcal{Q}$. The following corollary provides a statement of domains that satisfy various notions of richness.

COROLLARY 1. *Suppose $\Delta > 0$. Then the following statements are true:*

*(i) $\mathcal{Q}$, $\mathcal{Q}^{\Delta}_{+}$, $\mathcal{Q}^{\Delta}_{\succ}$, $\mathcal{R}^+$, $\mathcal{R}^+_{\succ}$, $\mathcal{R}$ satisfy OP richness*

*(ii)* $\mathcal{Q}^{\Delta}_{+}$, $\mathcal{R}^{+}$, $\mathcal{R}^{+}_{\succ}$, $\mathcal{R}$ *satisfy* $\text{TP}^{\Delta}_{+}$ *richness*

*(iii)* $\mathcal{Q}^{\Delta}_{\succ}$, $\mathcal{R}^{+}_{\succ}$, $\mathcal{R}$ *satisfy* $\text{TP}^{\Delta}_{\succ}$ *richness;* $\mathcal{R}^{+}$ *and* $\mathcal{Q}^{\Delta}_{+}$ *satisfy* $\text{TP}^{\Delta}_{\succ}$ *richness if* $\succ$ *is complete.*

Our results require one of the three notions of richness. Corollary 1 shows the applicability of our results to specific economic domains of interest.

We emphasize here that $\text{TP}^{\Delta}_{+}$ richness (or $\text{TP}^{\Delta}_{\succ}$ richness) is weaker than the requirement that a domain $\mathcal{D}$ includes $\mathcal{Q}^{\Delta}_{+}$ (or $\mathcal{Q}^{\Delta}_{\succ}$). Indeed, a domain satisfying $\text{TP}^{\Delta}_{+}$ richness (or $\text{TP}^{\Delta}_{\succ}$ richness) can be very different from a small perturbation of the quasilinear domain. This is illustrated by the following example.

Example 1. Suppose $A = \{0, 1\}$ and $1 \succ 0$. Consider the preference represented by the utility function

$$u(a, p; w, \theta) = w \cdot a \cdot e^{-\theta p} - p \quad \forall a \in A,$$

where $w \in \mathbb{R}_{++}$ and $\theta \in \mathbb{R}_{++}$. The domain $\mathcal{D}$ contains all such preferences. It is easy to see that $\mathcal{D}$ satisfies $\text{TP}^{\Delta}_{+}$ richness and $\text{TP}^{\Delta}_{\succ}$ richness, but it does not include $\mathcal{Q}^{\Delta}_{+}$ or $\mathcal{Q}^{\Delta}_{\succ}$ or $\mathcal{Q}$.[6] Note that preferences in $\mathcal{D}$ are parameterized by only two parameters: $w, \theta \in \mathbb{R}_{++}$. On the other hand, $\mathcal{Q}^{\Delta}_{+}$ (or $\mathcal{Q}^{\Delta}_{\succ}$) contains a greater variety of preferences. To highlight this point further, we provide another example, which shows that $\mathcal{Q}^{\Delta}_{+}$ contains far greater variety of preferences than the domain in Example 1. ◊

Example 2. We fix some $\Delta > 0$ and explicitly construct a subset of preferences in $\mathcal{Q}^{\Delta}_{+}$ for $A = \{0, 1\}$ with $1 \succ 0$. Let $H$ denote the set of all decreasing and continuous functions from $\mathbb{R}$ to $(-1, 1)$. For every $\theta \in \mathbb{R}_{++}$ and for every $h \in H$, define a preference $R(\theta, h)$ by defining $V^{R(\theta,h)}(1, (0, p))$ as

$$V^{R(\theta,h)}\big(1, (0, p)\big) := \theta + p + \min(\theta, \Delta) \cdot h(p) \quad \forall p \in \mathbb{R}.$$

Readers can verify that for every $\theta \in \mathbb{R}_{++}$ and every $h \in H$, we have $R(\theta, h) \in \mathcal{Q}^{\Delta}_{+}$.[7] So $\mathcal{Q}^{\Delta}_{+}$ contains preferences that cannot be parameterized by a finite number of parameters. Thus, it contains a greater variety of preferences than the domain in Example 1. ◊

## 4. Monotonicity and incentive compatibility

Incentive compatibility is usually equivalent to some form of monotonicity in various models of mechanism design. Though the precise nature of monotonicity may differ

[6] In fact, this domain satisfies a stronger notion of two-point richness. Take *any* two vectors $v$ and $v'$ that respect $\succ$ and satisfy DD. In particular, let $v = (v_0, v_1 = v_0 + \delta)$, where $\delta > 0$ and $v' = (v'_0, v'_1 = v'_0 + \delta')$, where $\delta' > 0$. Let $v'_0 > v_0$ and $\delta' < \delta$ to ensure that $v' > v$ and $(v, v')$ satisfy DD. Then we can choose $\theta := \frac{\log \frac{\delta}{\delta'}}{v'_0 + \delta' - v_0 - \delta}$ and $w := \frac{\delta}{e^{-(v_0+\delta)\theta}}$ to have a utility function in this domain that has indifference vectors $v$ and $v'$. Hence, for *any* $\Delta > 0$, this domain satisfies $\text{TP}^{\Delta}_{+}$ richness.

[7] Fix $\theta \in \mathbb{R}_{++}$ and $h \in H$. For every $p$, since $h(p) \in (-1, 1)$, we have $V^{R(\theta,h)}(1, (0, p)) > p$. Since $h$ is decreasing, the indifference vectors in $R(\theta, h)$ satisfy DD. So $R(\theta, h)$ respects $\succ$ and satisfies PIE. Finally, for every $R(\theta, h)$, if we consider the quasilinear preference $R^0$ defined by $V^{R^0}(1, (0, p)) = \theta + p$, where $\theta$ is interpreted as the valuation of the agent, we note that $d(R(\theta, h), R^0) < \Delta$.

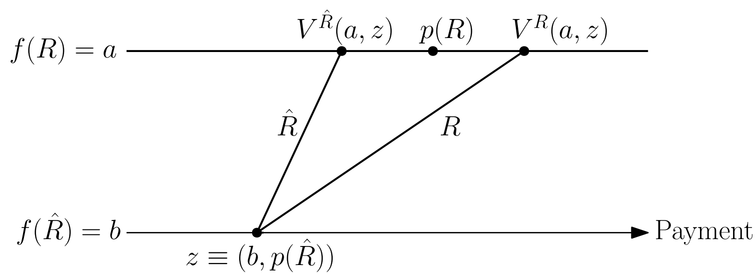


Figure 6. Illustration of necessity of monotonicity for DSIC.

from model to model, its usefulness is beyond doubt. For instance, in optimal single object auction design, Myerson (1981) uses it to simplify the optimization problem of revenue maximization; we elaborate further on such an application in Section 7.

We begin with an informal description of the notion of monotonicity relevant for our model. Let $(f, p)$ be a DSIC mechanism on a domain $\mathcal{D}$. Consider two preferences $R, \hat{R} \in \mathcal{D}$. Let $f(R) = a$ and $f(\hat{R}) = b$. Consider the two indifference vectors corresponding to $R$ and $\hat{R}$, and passing through $z \equiv (b, p(\hat{R}))$, i.e., the consumption bundle assigned to the agent with preference $\hat{R}$. Figure 6 shows the indifference diagram for this situation. DSIC implies that $z\hat{R}(a, p(R))$. Hence, $p(R) \geq V^{\hat{R}}(a, z)$ as shown in Figure 6. Similarly, DSIC implies that $(a, p(R))Rz$. Hence, $p(R) \leq V^{R}(a, z)$ as shown in Figure 6. This implies that $V^{R}(a, z) \geq V^{\hat{R}}(a, z)$. We chose the consumption bundle $z$ at preference $\hat{R}$ and alternative $a$ at $R$. Monotonicity requires that the valuation for $a$ at $z$ must weakly increase from $\hat{R}$ to $R$. This is a necessary condition for DSIC.

Definition 9. A mechanism $(f, p)$ on $\mathcal{D}$ is *monotone* if for every $R$, $\hat{R}$ with $f(R) = a$ and $z \equiv (f(\hat{R}), p(\hat{R}))$, we have $V^{R}(a, z) \geq V^{\hat{R}}(a, z)$.

We discuss the precise connections between our notion of monotonicity and the monotonicity condition used in the quasilinear domain in the literature in Section 8. Besides monotonicity, we use the following condition on payments.

Definition 10. A mechanism $(f, p)$ on $\mathcal{D}$ satisfies the *posted-price property* if there exists a map $\kappa : A \to \mathbb{R}$ such that $p(R) = \kappa(f(R))$ for all $R \in \mathcal{D}$.

The posted-price property is a trivial consequence of DSIC. It says that if two preferences are assigned the same alternative, they cannot be assigned different payments. It is obvious that DSIC implies the posted-price property. We are now ready to state our first main result.

Theorem 1. *Let $\Delta > 0$. Suppose $\mathcal{D}$ satisfies $\mathrm{TP}^{\Delta}_{\succ}$ richness and $(f, p)$ is a mechanism on $\mathcal{D}$. Then the following statements are equivalent.*

*(i) Mechanism $(f, p)$ is DSIC.*

*(ii) Mechanism $(f, p)$ is monotone and satisfies the posted-price property.*

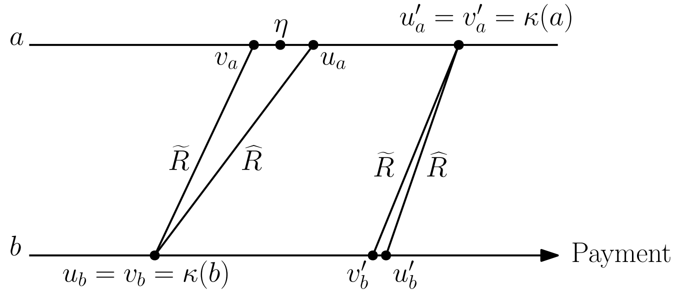


FIGURE 7. Illustration of proof of Theorem 1 for two alternatives with $a \succ b$.

The necessity of the posted-price property and monotonicity has been already argued. The proof of the other direction (ii) $\Rightarrow$ (i) is rather involved and is provided in Appendix A. We present a sketch of the proof for the case of two alternatives.

Consider the two alternatives case: $A = \{a, b\}$. Suppose $a \succ b$ (if $\succ$ is empty, the proof is simpler). Fix some $\Delta > 0$ and a domain $\mathcal{D}$ that satisfies $\text{TP}^{\Delta}_{\succ}$ richness. Let $(f, p)$ be a mechanism on this domain that satisfies monotonicity and the posted-price property. Hence, there are two numbers $\kappa(a)$ and $\kappa(b)$ such that for every $R \in \mathcal{D}$, we have $p(R) = \kappa(f(R))$. Assume, to the contrary, that there is a preference $R \in \mathcal{D}$ such that $f(R) = a$ and $(b, \kappa(b))P(a, \kappa(a))$. Then it must be that $V^R(a, (b, \kappa(b))) < \kappa(a)$. Define

$$\eta := \inf_{R' : f(R') = a} V^{R'}\big(a, \big(b, \kappa(b)\big)\big).$$

By ontoness, there is some $R''$ with $f(R'') = b$. By monotonicity, we have $V^{R'}(a, (b, \kappa(b))) \geq V^{R''}(a, (b, \kappa(b)))$ for every $R'$ with $f(R') = a$. Hence, $\eta \geq V^{R''}(a, (b, \kappa(b))) > \kappa(b)$, where the strict inequality follows from the fact that $a \succ b$ and $R''$ respects $\succ$. Since $V^R(a, (b, \kappa(b))) < \kappa(a)$, we have $\eta < \kappa(a)$. We conclude that $\eta$ is a real number that satisfies

$$\kappa(b) < \eta < \kappa(a). \tag{1}$$

We construct two pairs of vectors $(u, u')$ and $(v, v')$ as

$$u_b = v_b = \kappa(b), \qquad u_a = \eta + \delta, \qquad v_a = \eta - \delta$$

$$u'_a = v'_a = \kappa(a), \qquad u'_b = u_b + \kappa(a) - u_a + 3\delta, \qquad v'_b = v_b + \kappa(a) - v_a + \frac{1}{2}\delta,$$

where $\delta > 0$ but sufficiently small. The vectors $u$, $u'$, $v$, and $v'$ are shown in Figure 7 using an indifference diagram. Using the inequality (1), it is routine to verify that $u$, $u'$, $v$, and $v'$ respect $\succ$, $(u, u')$ are $\Delta_{\succ}$ parallel, and $(v, v')$ are $\Delta_{\succ}$ parallel. This uses the fact that for all values of $\Delta > 0$, we can always find $\delta > 0$ but sufficiently small such that these pairs of vectors are $\Delta_{\succ}$ parallel. Using $\text{TP}^{\Delta}_{\succ}$ richness, we construct two preferences $\widehat{R}$ and $\widetilde{R}$ such that $u, u' \in \mathcal{I}(\widehat{R})$ and $v, v' \in \mathcal{I}(\widetilde{R})$. By definition of $\eta$, there exists a preference $R^*$ such that $f(R^*) = a$ and $V^{R^*}(a, (b, \kappa(b))) < \eta + \delta$. Since $V^{\widehat{R}}(a, (b, \kappa(b))) = \eta + \delta$, monotonicity implies that $f(\widehat{R}) = a$. Similarly, by the definition of $\eta$, we see that $f(\widetilde{R}) = b$. But by the definition of $u'$ and $v'$, we have $V^{\widehat{R}}(b,(a,\kappa(a))) = u'_b > v'_b = V^{\widetilde{R}}(b,(a,\kappa(a)))$, which contradicts monotonicity. A similar argument shows that no manipulation is possible from a preference where $b$ is chosen. The general proof uses similar ideas but it is more involved because $A$ can have more than two alternatives and $\succ$ need not be a complete order.

We make two remarks about Theorem 1 before stating our next result.

Remark 1. Theorem 1 is the analogue of monotonicity-based characterization found in the literature on quasilinear domain. However, there are significant differences as we explain in Section 8. Informally, such a characterization for the quasilinear domain requires monotonicity of the *allocation rule* and a revenue equivalence formula (which is significantly stronger than the posted-price property). Theorem 1 shows that under $\text{TP}^{\Delta}_{\succ}$ richness, monotonicity of the *mechanism* and the posted-price property are equivalent to DSIC. Of course, monotonicity of the mechanism is different from the monotonicity of the allocation rule alone (though they coincide in the quasilinear domain; see Section 8 for details).

Remark 2. As discussed earlier, $\text{TP}^{\Delta}_{\succ}$ richness is a stronger richness requirement than $\text{TP}^{\Delta}_{+}$ richness. We give a simple example to show that Theorem 1 may not hold with $\text{TP}^{\Delta}_{+}$ richness; of course, if $\succ$ is a complete order, then the two richness notions are equivalent.

Example 3. Suppose $A = \{a, b\}$ and $\succ$ is empty. Suppose $\mathcal{D} = \mathcal{R}^{+}$, i.e., the domain is the set of all positive income effect preferences. Fix two numbers $\kappa(a)$ and $\kappa(b)$ with $\kappa(a) > \kappa(b)$. Define a subset of PIE preferences as $\mathcal{D}^* := \{R \in \mathcal{D} : V^R(a,(b,\kappa(b))) < \kappa(b)\}$. Now consider the following mechanism $(f, p)$ on $\mathcal{D}$: for every $R \in \mathcal{D}$:

$$\big(f(R), p(R)\big) = \begin{cases} \big(b, \kappa(b)\big) & \text{if } R \in \mathcal{D}^* \\ \big(a, \kappa(a)\big) & \text{otherwise.} \end{cases}$$

By definition, $(f, p)$ satisfies the posted-price property. To verify monotonicity, pick $R$ such that $f(R) = a$ and $R'$ such that $f(R') = b$. The indifference vectors of preference $R$ are shown in the indifference diagram in Figure 8. Since $R' \in \mathcal{D}^*$ but $R \notin \mathcal{D}^*$, we get $V^{R'}(a,(b,\kappa(b))) < \kappa(b) \le V^R(a,(b,\kappa(b)))$. This shows that one of the monotonicity conditions holds: $V^{R'}(a,(b,\kappa(b))) < V^R(a,(b,\kappa(b)))$. For the other monotonicity condition, we use the fact that $R, R'$ satisfy PIE. Using Fact 1, (a) $V^{R'}(a,(b,\kappa(b))) < \kappa(b)$ implies $V^{R'}(b,(a,\kappa(a))) \ge \kappa(a)$ and (b) $V^R(a,(b,\kappa(b))) \ge \kappa(b)$ implies $V^R(b,(a,\kappa(a))) \le \kappa(a)$ (since $R$ satisfies PIE). Combining these two, we get the other monotonicity condition: $V^R(b,(a,\kappa(b))) \le V^{R'}(b,(a,\kappa(a)))$.

However, this mechanism is not DSIC. Since $\kappa(a) > \kappa(b)$, there is $R \notin \mathcal{D}^*$ such that $V^R(b,(a,\kappa(a))) = \kappa(a) > \kappa(b)$. As shown in Figure 8, since $(f(R), p(R)) = (a, \kappa(a))$, we see that $R$ is better off manipulating to a preference in $\mathcal{D}^*$ to get the consumption bundle $(b, \kappa(b))$.

Note that if $\succ$ is empty, $\mathcal{R}^{+}_{\succ} = \mathcal{R}$ (the entire set of preferences). Consider the earlier mechanism $(f, p)$ in Example 3 on $\mathcal{D} = \mathcal{R}$. The earlier argument again shows that $f$ is

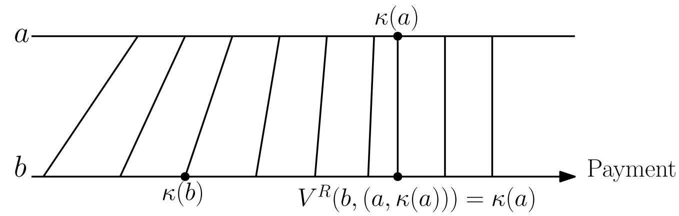


FIGURE 8. Indifference diagram of a preference $R$ satisfying PIE with $V^R(b,(a,\kappa(a))) = \kappa(a)$.

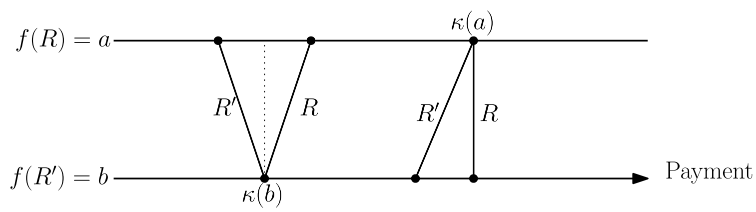


FIGURE 9. Illustration of violation of monotonicity of $f$ when $\succ$ is empty.

not DSIC. We know that Theorem 1 holds in $\mathcal{R}$. Hence, $f$ must violate monotonicity. To see this, consider $R$ and $R'$ as shown in Figure 9. Note that $R'$ violates PIE and $f(R) = a$, $f(R') = b$ by definition of $f$. However, $V^{R'}(b,(a,\kappa(a))) < V^R(b,(a,\kappa(a)))$. Hence, monotonicity is violated. $\Diamond$

## 5. A revenue uniqueness theorem

One of the fundamental results in mechanism design with quasilinearity is the revenue equivalence. It helps us to understand why seemingly different auction formats generate the same expected revenue. From a methodological point of view, revenue equivalence allows significant simplification of the optimization problem for maximizing revenue.

DEFINITION 11. A domain $\mathcal{D}$ satisfies *revenue equivalence* if for every pair of DSIC mechanisms $(f, p)$ and $(f, \hat{p})$ on $\mathcal{D}$, we have

$$p(R) - \hat{p}(R) = p(\hat{R}) - \hat{p}(\hat{R}) \quad \forall R, \hat{R} \in \mathcal{D}.$$

In other words, if two mechanisms differ from each other *only* in their payment rules, then the payment rules must be translations of each other. It is well known that $\mathcal{Q}$ is a revenue equivalence domain.

What happens to revenue equivalence without quasilinearity? Consider an example with four alternatives. Let $(f, p)$ and $(f, \hat{p})$ be two DSIC mechanisms such that $\hat{p}$ is a translation of $p$. By the posted-price property, we have two maps $\kappa : A \to \mathbb{R}$ and $\hat{\kappa} : A \to \mathbb{R}$ corresponding to these two mechanisms. Assume that $\kappa$ and $\hat{\kappa}$ are translations of each other. The indifference diagram is shown in Figure 10. As shown in Figure 10, we can construct two vectors $v$ and $\hat{v}$ such that $v$ is to the left of $\kappa$ and $\hat{v}$ is to

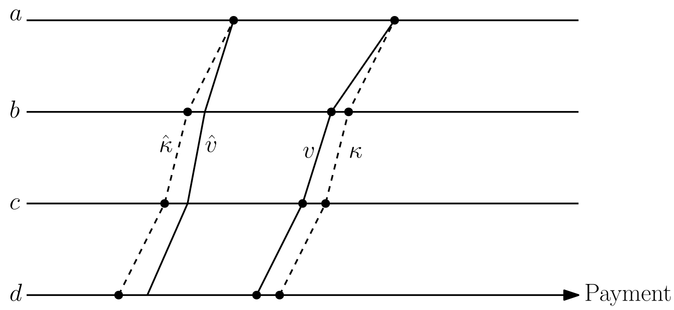


FIGURE 10. Failure of revenue equivalence.

the right of $\hat{\kappa}$. If we can ensure that there exists a preference $R$ such that $v, \hat{v} \in \mathcal{I}(R)$, then we get a contradiction from DSIC as follows: DSIC of $(f, p)$ implies that $f(R) = a$ (since $(a, \kappa(a))P(x, \kappa(x))$ for all $x \neq a$) but DSIC of $(f, \hat{p})$ implies that $f(R) \neq a$ (since $(x, \hat{\kappa}(x))P(a, \hat{\kappa}(a))$ for all $x \neq a$). We show that the existence of such a preference $R$ is guaranteed by $\mathrm{TP}_+^\Delta$ richness even if $\kappa$ and $\hat{\kappa}$ are not parallel. Hence, we get the following result.

Definition 12. A domain $\mathcal{D}$ satisfies *revenue uniqueness* if for every pair of DSIC mechanism $(f, p)$ and $(f, \hat{p})$ on $\mathcal{D}$, we have $p = \hat{p}$.

Theorem 2. *Let* $\Delta > 0$. *Every domain that satisfies* $\mathrm{TP}_+^\Delta$ *richness satisfies revenue uniqueness.*

Proof. Let $\mathcal{D}$ be a domain that satisfies $\mathrm{TP}_+^\Delta$ richness for some $\Delta > 0$. Assume, to the contrary, that there exist two DSIC mechanisms $(f, p)$ and $(f, p')$ such that $p \neq p'$. DSIC implies that the posted-price property holds; see the discussion immediately after Definition 10. Hence, there exist maps $\kappa$ and $\kappa'$ such that $\kappa(f(R)) = p(R)$ and $\kappa'(f(R)) = p'(R)$ for all $R \in \mathcal{D}$. Since $p \neq p'$, we get $\kappa \neq \kappa'$. We now complete the proof in three steps.

Step 1. In this step, we show that $\kappa$ and $\kappa'$ respect $\succ$.[8] Pick any $a, b \in A$ and suppose $b \succ a$. Pick $R$ such that $f(R) = a$ (by ontoness, this is possible). By incentive compatibility, $V^R(b, (a, \kappa(a))) \leq \kappa(b)$. Since $R$ respects $\succ$, we have $\kappa(a) < V^R(b, (a, \kappa(a)))$. Combining the two inequalities, we get $\kappa(b) > \kappa(a)$ as desired. A similar proof works for $\kappa'$.

Step 2. In this step, we show that either $\kappa > \kappa'$ or $\kappa' > \kappa$. Without loss of generality, assume that $\kappa(a) > \kappa'(a)$ for some $a \in A$. We show that $\kappa(b) > \kappa'(b)$ for all $b \in A$. Assume, to the contrary, that $\kappa(b) \leq \kappa'(b)$ for some $b \in A$. Let $\delta > 0$ but sufficiently close to zero. Define $v$ as $v_x := \kappa(x) - \delta$ for all $x \neq b$ and $v_b := \kappa(b)$. Since $\delta$ is chosen sufficiently small and $\kappa$ respects $\succ$ (by Step 1), $v$ respects $\succ$. Further, $v_a > \kappa'(a)$. By OP richness (which is implied by $\mathrm{TP}_+^\Delta$ richness), there is $R \in \mathcal{D}$ such that $v \in \mathcal{I}(R)$.

[8]Of course, if $\succ$ is empty, then there is nothing to prove.

For each $x \neq b$, we have $(b, \kappa(b))P(x, \kappa(x))$ by construction of $v$. Thus, DSIC of $(f, p)$ implies $f(R) = b$. However, since $v_a > \kappa'(a)$ and $\kappa(b) \leq \kappa'(b)$, we have

$$\big(a, \kappa'(a)\big)P\big(b, \kappa(b)\big)R\big(b, \kappa'(b)\big).$$

Then DSIC of $(f, p')$ implies $f(R) \neq b$. This is a contradiction. Hence, $\kappa(b) > \kappa'(b)$ for all $b \in A$.

Step 3. We now complete the proof. Since $\kappa$ and $\kappa'$ respect $\succ$ and $\kappa > \kappa'$, Lemma 4 (whose statement and proof are given in Appendix B) implies the existence of vectors $v, v' \in \mathbb{R}^{|A|}$ that respect $\succ$ and are $\Delta_+$ parallel. Further, by Lemma 4, there exists $a^* \in A$ such that

$$v_{a^*} = \kappa\big(a^*\big), \qquad v_x < \kappa(x) \quad \forall x \neq a^* \quad \text{and} \quad v'_{a^*} = \kappa'\big(a^*\big), \qquad v'_x > \kappa'(x) \quad \forall x \neq a^*.$$

By $\mathrm{TP}_+^\Delta$ richness, there exists a preference $R$ such that $v, v' \in \mathcal{I}(R)$. By definition of $v'$, $(x, \kappa'(x))P(a^*, \kappa'(a^*))$ for all $x \neq a^*$. Since $(f, p')$ is DSIC, we must have $f(R) \neq a^*$. Alternatively, $v_x < \kappa(x)$ for all $x \neq a^*$. But $v_{a^*} = \kappa(a^*)$ implies that $(a^*, \kappa(a^*))P(x, \kappa(x))$ for all $x \neq a^*$. Since $(f, p)$ is DSIC, we must have $f(R) = a^*$. This is a contradiction. □

Theorem 2 provides an analogue to a fundamental result in the quasilinear domain. Once an allocation rule is fixed, in a domain satisfying $\mathrm{TP}_+^\Delta$ richness, DSIC leaves little flexibility in the construction of a payment rule.

## 6. Does individual rationality bind?

A well known result in the screening problem with quasilinearity is that the individual rationality constraint binds for "low types" in the expected revenue-maximizing contract for the principal (Mussa and Rosen 1978). This result leads to a major simplification on obtaining a solution to the optimal contract. In this section, we explore the validity of this result on non-quasilinear domains.

We assume that $a_0 \in A$ is the worst alternative according to $\succ$ and that the IR constraint is defined with $(a_0, 0)$ being the outside consumption bundle (Definition 3). We state a straightforward lemma on IR below.

Lemma 2. *Suppose $(f, p)$ is a DSIC mechanism. Then it is IR if and only if $p(R) \leq 0$ for all $R$ with $f(R) = a_0$.*

Proof. If $(f, p)$ is IR, then $(a_0, p(R))R(a_0, 0)$ for all $R$ with $f(R) = a_0$, and this implies that $p(R) \leq 0$. For the converse, pick any $R$ with $f(R) = a$. There exists a $R'$ with $f(R') = a_0$. By DSIC, $(a, p(R))R(a_0, p(R'))R(a_0, 0)$, where the last relation comes from the fact that $p(R') \leq 0$. Hence, $(f, p)$ is IR. □

Lemma 2 assumes that the range of $f$ includes $a_0$. If the range of $f$ does not include $a_0$, our assumption implies the agent pays zero whenever she exercises her outside option $(a_0, 0)$. The point of this section is to show that if $a_0$ is in the range of a DSIC and IR mechanism, we can construct another DSIC and IR mechanism such that *IR binds* in the following sense.

Definition 13. *IR binds* in a domain $\mathcal{D}$ if for every DSIC and IR mechanism $(f, p)$ on $\mathcal{D}$, there exists a DSIC and IR mechanism $(\hat{f}, \hat{p})$ on $\mathcal{D}$ such that

$$\hat{p}(R) = 0 \quad \forall R \in \mathcal{D} \quad \text{such that } \hat{f}(R) = a_0$$
$$\hat{p}(R) \geq p(R) \quad \forall R \in \mathcal{D}.$$

If IR binds, we can convert any DSIC and IR mechanism to another DSIC and IR mechanism such that payment for $a_0$ is zero and revenue does not fall.

The IR constraint binds in $\mathcal{Q}$. This can be seen as follows. Pick $(f, p)$ that is DSIC and IR. By Lemma 2, we see that $p(R) \leq 0$ for all $R$ with $f(R) = a_0$. By the posted-price property, $\kappa(a_0) = p(R) \leq 0$ for all $R$ with $f(R) = a_0$. Define $\hat{p}(R) = p(R) - \kappa(a_0)$ for all $R$. Since $\hat{p}$ is a translation of $p$ and the domain is the quasilinear domain, we see that $(f, \hat{p})$ is DSIC and IR. Since $\kappa(a_0) \leq 0$, we have $\hat{p}(R) \geq p(R)$ for all $R$. Hence, IR binds in $\mathcal{Q}$. A crucial aspect of this simple argument is that we only change $p$ to $\hat{p}$ while constructing the new mechanism. The allocation rule $f$ was unchanged. Such an argument does not work if the domain is not $\mathcal{Q}$. Even then, we can establish that IR binds in any domain of preferences.

Theorem 3. *IR binds in every domain of preferences.*

Proof. Suppose $\mathcal{D}$ is a domain of preferences. Let $(f, p)$ be a DSIC and IR mechanism defined on $\mathcal{D}$. By the posted-price property, there exists $\kappa : A \to \mathbb{R}$ such that $\kappa(f(R)) = p(R)$ for all $R \in \mathcal{D}$. By Lemma 2, we know that $\kappa(a_0) \leq 0$. Let $A \setminus \{a_0\} = \{a_1, \dots, a_K\}$ and without loss of generality, assume that $\kappa(a_1) \leq \cdots \leq \kappa(a_K)$. We now define another map $\tilde{\kappa} : A \to \mathbb{R}$ as $\tilde{\kappa}(a_0) := 0$ and

$$\tilde{\kappa}(a_1) := \inf_{R: f(R) = a_1} V^R\big(a_1, \big(a_0, \tilde{\kappa}(a_0)\big)\big).$$

Having defined $\tilde{\kappa}(a_1), \dots, \tilde{\kappa}(a_k)$, we define

$$\tilde{\kappa}(a_{k+1}) := \min_{a_j \in \{a_0, \dots, a_k\}} \inf_{R: f(R) = a_{k+1}} V^R\big(a_{k+1}, \big(a_j, \tilde{\kappa}(a_j)\big)\big).$$

The mechanism $(\tilde{f}, \tilde{p})$ is defined as follows. For every $R$, the agent chooses her best consumption bundle according to $R$ from the set $\{(a_0, \tilde{\kappa}(a_0)), \dots, (a_K, \tilde{\kappa}(a_K))\}$, with ties broken in favor of the *higher indexed alternative*. More formally, for every $R$, define

$$M(R) := \big\{\big(a_j, \tilde{\kappa}(a_j)\big) : \big(a_j, \tilde{\kappa}(a_j)\big) R \big(a_k, \tilde{\kappa}(a_k)\big) \ \forall a_k \in A\big\}.$$

Then $\tilde{f}(R) := a_\ell$ if $a_\ell \in M(R)$ and $\ell > j$ for all $a_j \in M(R) \setminus \{a_\ell\}$. Further, $\tilde{p}(R) := \tilde{\kappa}(a_\ell)$.

Trivially, $(\tilde{f}, \tilde{p})$ is DSIC. Since $\tilde{\kappa}(a_0) = 0$, by Lemma 2, $(\tilde{f}, \tilde{p})$ is IR. We now complete the proof by showing that $\tilde{p}(R) \geq p(R)$ for all $R$.

Step 1. In this step, we show that $\tilde{\kappa}(a_j) \geq \kappa(a_j)$ for all $a_j \in A$. Since $(f, p)$ is IR, we know that $\kappa(a_0) \leq 0$. By definition, $0 = \tilde{\kappa}(a_0) \geq \kappa(a_0)$. Now, we prove the step using induction on the indices: suppose $\tilde{\kappa}(a_j) \geq \kappa(a_j)$ for all $a_j \in \{a_0, \dots, a_k\}$. Then we observe that

$$\begin{aligned}\tilde{\kappa}(a_{k+1}) &= \min_{a_j \in \{a_0,\dots,a_k\}} \inf_{R: f(R)=a_{k+1}} V^R\big(a_{k+1}, \big(a_j, \tilde{\kappa}(a_j)\big)\big) \\ &\geq \min_{a_j \in \{a_0,\dots,a_k\}} \inf_{R: f(R)=a_{k+1}} V^R\big(a_{k+1}, \big(a_j, \kappa(a_j)\big)\big) \quad \text{(by the induction hypothesis)} \\ &\geq \kappa(a_{k+1}),\end{aligned}$$

where the last inequality follows from the fact that for every $R$ with $f(R) = a_{k+1}$, DSIC implies that $V^R(a_{k+1}, (a_j, \kappa(a_j))) \geq \kappa(a_{k+1})$ for all $a_j \in A$.

Step 2. We now complete the proof. Pick any $R \in \mathcal{D}$. Let $f(R) = a_k$. By definition, $\tilde{\kappa}(a_k) \leq V^R(a_k, (a_j, \tilde{\kappa}(a_j)))$ for all $j < k$. Hence, $(a_k, \tilde{\kappa}(a_k))R(a_j, \tilde{\kappa}(a_j))$ for all $j < k$. But then by the tie-breaking rule, $\tilde{f}(R) \neq a_j$ for all $j < k$. So $\tilde{f}(R) = a_\ell$ for some $\ell \geq k$. Hence, we get

$$p(R) = \kappa(a_k) \leq \kappa(a_\ell) \leq \tilde{\kappa}(a_\ell) = \tilde{p}(R),$$

where the first inequality follows from the fact that $\ell \geq k$ and the second inequality follows from Step 1. □

Remark 3. The allocations rules $f$ and $\tilde{f}$ in the proof of Theorem 3 can be quite different on an arbitrary domain. They are identical if the domain is $\mathcal{Q}$. However, we can establish a relationship between $f$ and $\tilde{f}$. As in the proof of Theorem 3, let $A \setminus \{a_0\} = \{a_1, \dots, a_K\}$, and without loss of generality, assume that $\kappa(a_1) \leq \cdots \leq \kappa(a_K)$. We can show that for any $R$, $f(R) = a_j$ and $\tilde{f}(R) = a_k$ implies $k \geq j$. The claim is obvious if $j = 0$. So assume that $j > 0$. Now pick $a_\ell$, where $\ell < j$. By the definition of $\tilde{\kappa}(a_j)$, we have $\tilde{\kappa}(a_j) \leq V^R(a_j, (a_\ell, \tilde{\kappa}(a_\ell)))$. Hence, $(a_j, \tilde{\kappa}(a_j))R(a_\ell, \tilde{\kappa}(a_\ell))$. Further, $\ell < j$ and the tie-breaking rule for $\tilde{f}$ implies $\tilde{f}(R) \neq a_\ell$. In particular, when $\succ$ is complete with $a_K \succ a_{K-1} \succ \cdots \succ a_1 \succ a_0$, we have $\kappa(a_1) < \cdots < \kappa(a_K)$. This implies that $\tilde{f}(R)$ is assigned an alternative $a_k$ such that $\kappa(a_k) > \kappa(a_j)$, where $f(R) = a_j$. Thus, $\tilde{f}$ assigns "higher valued" alternatives (i.e., higher in the rank of $\succ$) at each preference to increase payments.[9]

Theorem 3 implies that IR binds in $\mathcal{Q}$ and all the other domains discussed in Corollary 1. This has implications in solving optimization problems with incentive and individual rationality constraints as we discuss in the next section.

## 7. A template for optimal contract design

In this section, we demonstrate the usefulness of the results in the previous section in finding the solution to the optimal contract problem in non-quasilinear environments. We assume the existence of $a_0$ such that $a \succ a_0$ for all $a \in A$ and the IR constraint is defined with $(a_0, 0)$ as the outside option. We fix a domain $\mathcal{D}$. Let $\mu$ be a probability

[9]We are grateful to an anonymous referee for suggesting this discussion.

measure on $\mathcal{D}$. A mechanism $(f^*, p^*)$ is *optimal* if it is a solution to the program (in all the optimization programs below, we assume that a maximum exists)

$$\max_{f:\mathcal{D}\to A,\, p:\mathcal{D}\to\mathbb{R}} \int_{\mathcal{D}} p(R)\, d\mu(R)$$

$$\text{subject to } (f, p) \text{ is DSIC and IR.}$$

Using our results, we show that the optimization program above can be simplified. To do so, we define the notion of a canonical mechanism. For every allocation rule $f : \mathcal{D} \to A$, define its *canonical payment rule* $p^f : \mathcal{D} \to \mathbb{R}$ as

$$p^f(R) = \begin{cases} 0 & \text{if } f(R) = a_0 \\ \inf\limits_{R': f(R')=a} V^{R'}\big(a, (a_0, 0)\big) & \text{if } f(R) = a \neq a_0. \end{cases}$$

A mechanism $(f, p)$ is a *canonical mechanism* if $p = p^f$. A canonical mechanism is uniquely defined by the allocation rule *only*. A canonical mechanism is not necessarily DSIC. However, if it is DSIC, then it is IR too, since the payment is zero when $a_0$ is allocated. We show below that an optimal mechanism is a canonical mechanism.

Theorem 4. *Suppose $\mathcal{D}$ satisfies OP richness. Then an optimal mechanism on $\mathcal{D}$ is a canonical mechanism.*

Proof. The proof uses the following lemma that characterizes DSIC mechanisms in domains satisfying OP richness.

Lemma 3. *Suppose $\mathcal{D}$ satisfies OP richness and $(f, p)$ is a mechanism defined on $\mathcal{D}$. Then $(f, p)$ is DSIC if and only if, for every $R, R' \in \mathcal{D}$, we have*

$$p(R) = \inf_{\hat{R}: f(R)=f(\hat{R})} V^{\hat{R}}\big(f(R), \big(f(R'), p(R')\big)\big).$$

Proof. Suppose $(f, p)$ satisfies the condition in the statement of the lemma. Then pick any $R, R'$ and we get

$$p(R) \leq V^R\big(f(R), \big(f(R'), p(R')\big)\big).$$

Hence, $(f(R), p(R))R(f(R'), p(R'))$, which implies that $(f, p)$ is DSIC.

Now suppose $(f, p)$ is DSIC. Then DSIC implies, from the necessity of the posted-price property and Step 1 in the proof of Theorem 2, that there exists a map $\kappa : A \to \mathbb{R}$ such that $\kappa$ respects $\succ$ and $\kappa(f(R)) = p(R)$ for all $R$. Then pick any $a, b \in A$. Choose $\epsilon > 0$ but arbitrarily close to zero and consider the following vector $v \in \mathbb{R}^{|A|}$:

$$v_a = \kappa(a) + \epsilon, \qquad v_x = \kappa(x) \quad \forall x \neq a.$$

Since $\kappa$ respects $\succ$ and $\epsilon$ is sufficiently close to zero, $v$ respects $\succ$. By OP richness, there is a preference $\hat{R}$ such that $v \in \mathcal{I}(\hat{R})$. Since $v_a > \kappa(a)$ and $v_x = \kappa(x)$ for all $x \neq a$, DSIC implies that $f(\hat{R}) = a$. Hence, we have $V^{\hat{R}}(a, (b, \kappa(b))) = \kappa(a) + \epsilon$. Taking $\epsilon \to 0$, we get

$$\kappa(a) = \inf_{\hat{R}: f(\hat{R})=a} V^{\hat{R}}\big(a, \big(b, \kappa(b)\big)\big). \qquad \square$$

We can now complete the proof of the theorem. By Theorem 3, IR binds. Hence, every optimal mechanism $(f, p)$ must have $p(R) = 0$ if $f(R) = a_0$. Then, by Lemma 3, OP richness guarantees that for every $R$ with $f(R) = a \neq a_0$, we have $p(R) = \inf_{R':f(R')=a} V^{R'}(a, (a_0, 0))$. Hence, $p = p^f$, and every optimal mechanism is a canonical mechanism. □

Remark 4. Theorem 4 assumes that $a_0$ lies in the range of the optimal mechanism. This is without loss of generality for any probability measure that assigns zero probability to a set containing one preference. To see this, suppose $(f, p)$ is an optimal mechanism such that $a_0$ is not in the range of $f$. Suppose the range of $f$ is $A' := A \setminus \{a_0\}$. By the posted-price property, there is a map $\kappa : A' \to \mathbb{R}$ such that for all $R$ with $f(R) = a$, we have $p(R) = \kappa(f(R))$. Then consider the vector $v$, where $v_{a_0} = 0$ and $v_a = \kappa(a)$ for all $a \in A'$. By OP richness, there is a preference $R$ such that $v \in \mathcal{I}(R)$. By definition, $(f(R), p(R)) I (a_0, 0)$. Consider a new mechanism $(f', p')$ such that $f'(R) = a_0$ and $p'(R) = 0$, and $f'(R') = f(R')$, $p'(R') = p(R')$ for all $R' \neq R$. Since $(f, p)$ is DSIC and IR, the new mechanism $(f', p')$ is also DSIC. By Lemma 2, the new mechanism $(f', p')$ satisfies IR. Further, the two mechanisms differ at only *one* preference. By our assumption on the probability measure, the expected revenue from both mechanisms is the same.[10]

Theorem 4 and Theorem 1 have an immediate corollary that allows us to reduce the search space of our optimal mechanism. First, Theorem 4 shows that we need to search only over canonical mechanisms. Second, Theorem 1 says that if the domain satisfies $\mathrm{TP}^{\Delta}_{\succ}$ richness for some $\Delta > 0$, we need to search only over canonical mechanisms that satisfy monotonicity. By construction, every DSIC canonical mechanism is IR and, hence, we can get rid of the IR constraint. Further, every canonical mechanism satisfies the posted-price property. Therefore, DSIC can just be replaced by the monotonicity condition of the *canonical* mechanism. Since a canonical mechanism is uniquely defined by the corresponding allocation rule, the search space of the optimal mechanism problem becomes the space of all allocation rules such that the canonical mechanism is monotone. This leads to the following corollary.

Corollary 2. *Let $\Delta > 0$. Suppose $\mathcal{D}$ satisfies $\mathrm{TP}^{\Delta}_{\succ}$ richness. Then every optimal mechanism is a solution to the program*

$$\max_{f:\mathcal{D}\to A} \int_{\mathcal{D}} p^f(R)\, d\mu(R)$$

$$\text{subject to } \big(f, p^f\big) \text{ is monotone.}$$

This result parallels a similar result in the quasilinear domain. As argued earlier, Theorem 4 holds in the quasilinear domain. We say an allocation rule $f$ is *implementable* if there exists a payment rule $p$ such that $(f, p)$ is DSIC. It is difficult to verify implementability of an allocation rule without quasilinearity (because of the nonseparable nature of allocation and payment in the utility function of the agent). However, if the

[10] We are grateful to an anonymous referee for motivating this discussion.

domain is $\mathcal{Q}$, an allocation rule is implementable if and only if it is *weakly monotone* (Bikhchandani et al. 2006). An allocation rule $f$ (defined on the quasilinear domain) is weakly monotone if for every quasilinear preferences $R$ and $R'$ with $f(R) = a$ and $f(R') = b$ such that valuation vectors $v$ and $v'$ represent $R$ and $R'$, respectively, we have $v_a - v_b \geq v'_a - v'_b$. As we show later, in the quasilinear domain, a mechanism $(f, p)$ is monotone if and only if $f$ is weakly monotone. We also know that if $f$ is implementable on $\mathcal{Q}$, then, by revenue equivalence, we can construct $p^f$ such that $(f, p^f)$ is DSIC and IR (Heydenreich et al. 2009). Hence, the analogue of the program in Corollary 2 is well known to be the following program on $\mathcal{Q}$:

$$\max_{f:\mathcal{Q}\to A} \int_{\mathcal{Q}} p^f(R)\, d\mu(R)$$

$$\text{subject to } f \text{ is weakly monotone.}$$

Such a simplification is useful for computing an optimal contract. For instance, in the single object allocation problem, it is standard to solve the unconstrained optimization problem and then verify that the unconstrained optimum satisfies weak monotonicity.

Our program in Corollary 2 has similar features: the search space of the optimal mechanism is only over allocation rules such that the corresponding canonical mechanism is monotone. Thus, we provide a template for designing optimal mechanisms without quasilinearity. Such a template is useful if the underlying optimization problem is tractable. This depends on the specific domain considered. We now briefly outline how it can be applied to a model with two alternatives.

Suppose $A = \{a_0, a_1\}$ with $a_1 \succ a_0$. In a model, where a single object needs to be allocated, $a_0$ can be thought of as the alternative where the object is not allocated and $a_1$ is the alternative where the object is allocated. To find a revenue-maximizing mechanism in such a model, Theorem 4 says that we need to focus on canonical mechanisms that correspond to monotone allocation rules. Given an allocation rule $f$, the canonical mechanism in this model has a payment rule characterized by

$$\kappa^f(a_0) = 0$$

$$\kappa^f(a_1) = \inf_{R: f(R)=a_1} V^R\big(a_1, (a_0, 0)\big).$$

Since $a_1 \succ a_0$, we have $V^R(a_1, (a_0, 0)) > 0$ for all $R$ and, hence, $\kappa^f(a_1) \geq 0$. A canonical mechanism $(f, \kappa^f)$ requires only that an agent with preference $R$ submits her valuation $V^R(a_1, (a_0, 0))$.

Monotonicity of $(f, \kappa^f)$ requires that for any $R$ with $f(R) = a_0$ and $R'$ with $f(R') = a_1$, we must have

$$V^R\big(a_0, \big(a_1, \kappa^f(a_1)\big)\big) \geq V^{R'}\big(a_0, \big(a_1, \kappa^f(a_1)\big)\big)$$

$$V^{R'}\big(a_1, (a_0, 0)\big) \geq V^R\big(a_1, (a_0, 0)\big).$$

These conditions are equivalent to requiring that $f(R) = a_1$ if $V^R(a_1, (a_0, 0)) > \kappa^f(a_1)$ and $f(R) = a_0$ if $V^R(a_1, (a_0, 0)) < \kappa^f(a_1)$. So every canonical mechanism can be constructed by a *cutoff* $\kappa^*$ such that for all $R$, if $V^R(a_1, (a_0, 0)) > \kappa^*$, we have $f(R) = a_1$, and if $V^R(a_1, (a_0, 0)) < \kappa^*$, we have $f(R) = a_0$ (if $V^R(a_1, (a_0, 0)) = \kappa^*$, we can assign either $a_1$ or $a_0$ in the allocation rule). Further, if $f(R) = a_1$ for any $R$, then $p(R) = \kappa^*$. Otherwise, $p(R) = 0$ if $f(R) = a_0$.

The revenue from such a canonical mechanism is $\kappa^*(1 - G(\kappa^*))$, where $G$ is the cumulative probability distribution of valuation for $a_1$ at $(a_0, 0)$ (i.e., distribution of $V^R(a_1, (a_0, 0))$). Hence, finding an optimal mechanism is choosing an optimal value of $\kappa^*$ to maximize $\kappa^*(1 - G(\kappa^*))$. It is well known that this expression is maximized under the *monotone hazard rate* condition at a value of $\kappa^*$ that satisfies $\kappa^* = \frac{1-G(\kappa^*)}{g(\kappa^*)}$, where $g$ is the probability density function associated with $G$.

## 8. Relationship with the quasilinear domain results

In this section, we formally compare our results to analogous results in the quasilinear domain. The comparison below highlights how our results extend and generalize existing results with quasilinearity. As defined earlier, indifference vectors in every pair in a quasilinear preference are parallel. Hence, any one indifference vector represents the entire quasilinear preference. We call it the *valuation* vector. Denote the valuation (vector) attached to quasilinear preferences $R, R', R'', \ldots$ by $v, v', v'', \ldots$, respectively. Let $\mathcal{V} \subseteq \mathbb{R}^{|A|}$ be the domain of valuations in the quasilinear preference domain. Notice that two valuation vectors that differ by the same constant in each component represent the same preference over consumption bundles.

### *Monotonicity in the quasilinear domain*

Our monotonicity condition reduces to the following condition in quasilinear domains. Take two valuation vectors $v, v' \in \mathbb{R}^{|A|}$ and denote the underlying quasilinear preferences corresponding to them by $R$ and $R'$, respectively. Consider a mechanism $(f, p)$ and let $f(R) = a$ and $f(R') = b$. Note that

$$V^R\big(a, \big(b, p\big(R'\big)\big)\big) = v_a - v_b + p\big(R'\big) \quad \text{and} \quad V^{R'}\big(a, \big(b, p\big(R'\big)\big)\big) = v'_a - v'_b + p\big(R'\big).$$

Hence, our monotonicity condition $V^R(a, (b, p(R'))) \geq V^{R'}(a, (b, p(R')))$ is equivalent to requiring $v_a - v_b \geq v'_a - v'_b$. This is the familiar *weak monotonicity* or *two-cycle monotonicity* condition from the quasilinear domain literature (Bikhchandani et al. 2006, Saks and Yu 2005, Ashlagi et al. 2010). Hence, in the quasilinear domain, monotonicity of the mechanism reduces to weak monotonicity of the *allocation rule* $f$.

### *Mechanism characterization*

The monotonicity condition is too weak in the quasilinear domain and, hence, the posted-price property needs to be significantly strengthened to a *revenue equivalence formula* to imply incentive compatibility of a mechanism. We consider a result for the quasilinear domain to compare it with our Theorem 1. The result is owing to Jehiel et al. (1999), but we report a version of this result given in Ashlagi et al. (2010). To state this result conveniently, we slightly abuse notation to write a mechanism $(f, p)$ on a quasilinear type space with domain of valuations $\mathcal{V} \subseteq \mathbb{R}^{|A|}$ as maps $f : \mathcal{V} \to A$ and $p : \mathcal{V} \to \mathbb{R}$.

FACT 2 (Proposition 1 in Jehiel et al. 1999). *Suppose $(f, p)$ is a mechanism defined on a convex domain of valuations $\mathcal{V} \subseteq \mathbb{R}^{|A|}$. Then the following statements are equivalent.*

*(i) The mechanism $(f, p)$ is DSIC.*

*(ii) The rule $f$ is weakly monotone and for every $v, v' \in \mathcal{V}$, we have*

$$p(v) = p(v') + [v_{f(v)} - v'_{f(v')}] - \int_0^1 \psi^{v',v}(x)\,dx,$$

*where $\psi^{v',v}(x) = (v - v') \cdot \mathbb{I}_{f(v'+x(v-v'))}$ for all $x \in [0, 1]$. Here, $\mathbb{I}_a$ is an indicator vector in $\mathbb{R}^{|A|}$, where the component that corresponds to $a$ is set to 1 and all other components are zero.*

Fact 2 generalizes analogous results for two alternatives case in Myerson (1981), where he develops this result in the single object auction framework. Statement (ii) in Fact 2 implies that if we have two incentive compatible mechanisms with the same allocation rule, $(f, p)$ and $(f, p')$, then $p(v) = p'(v)$ for *some* $v$ implies that $p = p'$. This is usually referred to as the *revenue equivalence* formula or the envelope formula (Milgrom and Segal 2002, Krishna and Maenner 2001). Contrast this to our Theorem 2, which states that in domains that satisfy $\mathrm{TP}_{+}^{\Delta}$ richness (for any $\Delta > 0$), we have revenue uniqueness.

We can contrast our Theorem 1 with Fact 2. Theorem 1 characterizes incentive compatibility by monotonicity of the *mechanism* and the posted-price property. On the other hand, Fact 2 characterizes incentive compatibility using weak monotonicity of the *allocation rule* and the revenue equivalence formula. The richness of our non-quasilinear type space allows us to use the monotonicity of the mechanism along with a simple condition on payments in Theorem 1 to characterize incentive compatibility.

### *Allocation rule characterization*

In the quasilinear domain, due to the separability (and linearity) of the payment from the allocation rule, one can focus attention on the *implementability* question; if we know that $f$ is implementable, then due to Fact 2, there is an explicit way to compute payments using $f$ (up to an additive constant). The following fact answers the question, "When is an allocation rule $f$ implementable?"; i.e., there exists a $p$ such that $(f, p)$ is DSIC.

FACT 3 (Saks and Yu 2005, Ashlagi et al. 2010). *Suppose closure of $\mathcal{V} \subseteq \mathbb{R}^{|A|}$ is convex and $f$ is an allocation rule defined on $\mathcal{V}$. Then $f$ is implementable if and only if it is weakly monotone.*

Fact 3 was first shown for the *quasilinear* domain (i.e., $\mathcal{V}$ consists of all valuation vectors in $\mathbb{R}^{|A|}$ respecting $\succ$) in Bikhchandani et al. (2006). Though Theorem 1 seems to be an extension of their result to non-quasilinear domains, it should not be confused as an analogue of Fact 3. There are two important differences: (i) Theorem 1 is a characterization of incentive compatible *mechanisms* using monotonicity of the *mechanism* and Fact 3 is a characterization of implementable *allocation rules* in the quasilinear domain using monotonicity of the *allocation rule*; (ii) Theorem 1 uses richness of the non-quasilinear domain and it does not apply to the quasilinear domain.

## 9. Related literature

The literature on mechanism design with quasilinearity is extensive, and almost impossible to describe exhaustively. We discussed some relevant papers in detail in Section 8, but refer the reader to two excellent books on this topic (Vohra 2011, Börgers 2015). Following Myerson (1981), several papers have extended his monotonicity and revenue equivalence characterizations to various models with multidimensional types, where more than two alternatives are allocated and agents have values for each of those alternatives. Because of quasilinearity, the allocation rule and the payment rule of a mechanism appear in separable form in the utility function of the agent. This allows one to provide separate characterizations of implementable allocation rules and the class of payment rules that implement an implementable allocation rule. The former is characterized by (weak) monotonicity (Bikhchandani et al. 2006, Saks and Yu 2005, Ashlagi et al. 2010, Cuff et al. 2012, Mishra and Roy 2013, Mishra et al. 2014, Carbajal and Müller 2015).[11] The latter is characterized by a revenue equivalence formula ((Krishna and Maenner, 2001), Milgrom and Segal 2002, Chung and Olszewski 2007, Heydenreich et al. 2009). These results exploit the geometric structure induced by quasilinearity, and make use of convex analysis machinery (Rockafellar 1970). A paper by Carbajal and Ely (2013) establishes that revenue equivalence may not hold in some problems even with quasilinearity; this is due to the nonconvex and nondifferentiable nature of valuation function. They further show that in specific problems where revenue equivalence fails, one can still do revenue maximization using revenue *inequalities*.

In the absence of quasilinearity, it is not possible to provide separate characterizations of monotonicity and revenue equivalence since the allocation and payment are no longer separable. Our monotonicity condition is indeed a condition on the mechanism, but reduces to the weak monotonicity condition on the allocation rule in the quasilinear domain. Unlike the quasilinear domain, we find revenue uniqueness of incentive compatible mechanisms in our benchmark rich type space.

There is a small but relevant literature on mechanism design with non-quasilinear preferences. The closest paper to ours is Kos and Messner (2013). They derive a necessary condition for incentive compatibility in a model with non-quasilinearity. Their condition is a generalization of the cycle monotonicity condition in Rochet (1987) for quasilinear preferences. They show that their condition is not sufficient for incentive compatibility. In contrast, our monotonicity condition is significantly weaker than their condition and is a necessary and sufficient condition for incentive compatibility along with the posted-price property. However, we focus on deterministic mechanisms that

[11] These papers (a) extend Myerson's monotonicity condition to a multidimensional environment and (b) simplify a complicated monotonicity characterization of Rochet (1987).

assume some form of richness in type space, while Kos and Messner (2013) do not make such assumptions.

Baisa (2017) considers the single object auction model with non-quasilinear preferences but allows for randomization. He introduces a novel mechanism in his setting and studies its optimality properties (in terms of revenue maximization). Garratt and Pycia (2016) study the bilateral trading model with non-quasilinear preferences, where they allow for randomization. Further, their solution concept is Bayesian incentive compatibility. Their main finding is that for a generic set of non-quasilinear preferences, the Myerson–Satterthwaite impossibility result (Myerson and Satterthwaite 1983) on bilateral trading disappears. Unlike both of these papers, we do not consider randomization and our solution concept is different from theirs. More importantly, we have a more general model of non-quasilinear preferences, albeit with deterministic mechanisms, that covers many problems of interest.

In a recent paper, Nöldeke and Samuelson (2018) analyze the principal–agent problem and the matching problem with non-quasilinear preferences. They establish an implementation duality to characterize the *implementable utility profiles*, which allows them to extend results from the quasilinear domain. Our results are specific to deterministic mechanisms, which is not assumed in Nöldeke and Samuelson (2018). Alternatively, Nöldeke and Samuelson (2018) impose topological assumptions on the set of alternatives to derive their results. The objectives and the results in both the papers are quite different.

There is a literature on axiomatic treatment of mechanisms in models without quasilinearity. The broad conclusion of this literature is that in a variety of problems, dominant strategy incentive compatibility, individual rationality, and efficiency (along with some other mild axioms) are incompatible without quasilinearity. The literature also identifies problems where these properties are compatible: see, for instance, Saitoh and Serizawa (2008), Hashimoto and Saitoh (2010), Morimoto and Serizawa (2015), Kazumura and Serizawa (2016), Zhou and Serizawa (2018), Baisa (2019), Ma et al. (2016) and references therein.

There is a literature in auction theory and algorithmic game theory on single object auctions with budget-constrained bidders; see Che and Gale (2000), Pai and Vohra (2014), Ashlagi et al. (2010), Lavi and May (2012). The budget constraint in these papers introduces a particular form of non-quasilinearity in the preferences of agents. Further, the budget constraint in these models is *hard*, i.e., the utility from any payment above the budget is minus infinity. This assumption is not satisfied by the preferences considered in our model. Hence, our results do not apply directly to these models.

In a companion paper (Kazumura et al. 2020), we investigate the revenue-maximizing auction by a seller who is selling multiple objects to agents who can buy at most one object. The *minimum Walrasian equilibrium price* mechanism is shown to be revenue-optimal in a class of mechanisms that satisfy dominant strategy incentive compatibility, individual rationality, and some other reasonable axioms.

## Appendix A: Sufficiency part of Theorem 1

We prove the sufficiency part of Theorem 1.[12]

Proof of Theorem 1. Take any $(f, p)$ that satisfies monotonicity and the posted-price property. Then there exists a map $\kappa : A \to \mathbb{R}$ such that $p(R) = \kappa(f(R))$ for all $R$. For any $a, b \in A$, we say incentive constraint $a \to b$ holds if for every $R \in \mathcal{D}$ with $f(R) = a$, we have $(a, \kappa(a))R(b, \kappa(b))$. Assume, to the contrary, that there exists $x \in A$ and a preference $R$ such that $f(R) = x$ and $(y', \kappa(y'))P(x, \kappa(x))$ for some $y' \in A$, i.e., $x \to y'$ does not hold. Without loss of generality, assume that for all $y$ with $y \succ x$, incentive constraints $y \to b$ holds for all $b \in A$ (in other words, $x$ is a *maximal* alternative according to $\succ$ that can manipulate).

Let $Z^\kappa := \{(y, \kappa(y)) : y \in A\}$ be the set of consumption bundles in the range of the mechanism $(f, p)$. Among the consumption bundles in $Z^\kappa$, let $(y^*, \kappa(y^*))$ be a maximum according to $R$. Then we have $(y^*, \kappa(y^*))P(x, \kappa(x))$ and $(y^*, \kappa(y^*))R(y, \kappa(y))$ for all $(y, \kappa(y)) \in Z^\kappa$. The first relation implies that $V^R(x, (y^*, \kappa(y^*))) < \kappa(x)$. By ontoness, there is a preference $R'$ such that $f(R') = y^*$. Hence, monotonicity implies that $V^{R'}(x, (y^*, \kappa(y^*))) \leq V^R(x, (y^*, \kappa(y^*)))$, so we have

$$V^{R'}\big(x, \big(y^*, \kappa\big(y^*\big)\big)\big) \leq V^R\big(x, \big(y^*, \kappa\big(y^*\big)\big)\big) < \kappa(x). \tag{2}$$

Also, since $(y^*, \kappa(y^*))R(y, \kappa(y))$ for all $(y, \kappa(y)) \in Z^\kappa$, we have

$$V^R\big(y, \big(y^*, \kappa\big(y^*\big)\big)\big) \leq \kappa(y) \quad \forall y \in A. \tag{3}$$

Using this, we define a vector $v^* \in \mathbb{R}^{|A|}$ as follows: for every $y \in A$, let

$$v_y^* = \begin{cases} V^R\big(y, \big(y^*, \kappa\big(y^*\big)\big)\big) = \kappa\big(y^*\big) & \text{if } y = y^* \\ V^R\big(y, \big(y^*, \kappa\big(y^*\big)\big)\big) + \epsilon & \text{if } y = x \\ V^R\big(y, \big(y^*, \kappa\big(y^*\big)\big)\big) - \epsilon & \text{if } y \succ x \text{ and } y \neq y^* \\ \min\big\{V^R\big(y, \big(y^*, \kappa\big(y^*\big)\big)\big), V^{R'}\big(y, \big(y^*, \kappa\big(y^*\big)\big)\big)\big\} - \epsilon & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$ but is sufficiently small. An illustration of $v^*$ is provided in the indifference diagram in Figure 11 for an example with $A = \{a, b, x, y^*, c\}$, where $a \succ b \succ x \succ y^* \succ c$. Now, the proof is completed in four steps.

Step 1. We show that $v^*$ respects $\succ$, but we prove a slightly general assertion, which is useful to us in later steps. Pick any $v \in \mathbb{R}^{|A|}$ such that $v_y = v_y^*$ $\forall y \in A \setminus \{x\}$ and $v_x \in [V^{R'}(x, (y^*, \kappa(y^*))) - \delta, v_x^* + \delta]$ for sufficiently small $\delta > 0$ with $\delta < \epsilon$; note that by inequality (2), $V^{R'}(x, (y^*, \kappa(y^*))) < v_x^*$. We show that $v$ respects $\succ$. For this, pick any pair $a, b \in A$ such that $a \succ b$. Suppose $a = y^*$ or $a \succ x$. We have $v_a \geq V^R(a, (y^*, \kappa(y^*))) - \epsilon$. Alternatively, $v_b \leq V^R(b, (y^*, \kappa(y^*))) + \epsilon$. Since $R$ respects $\succ$ and $a \succ b$, then $V^R(a, (y^*, \kappa(y^*))) > V^R(b, (y^*, \kappa(y^*)))$. Since $\epsilon$ is sufficiently small, we obtain $v_a > v_b$.

[12] We are grateful to an anonymous referee whose comments led to a simpler proof.

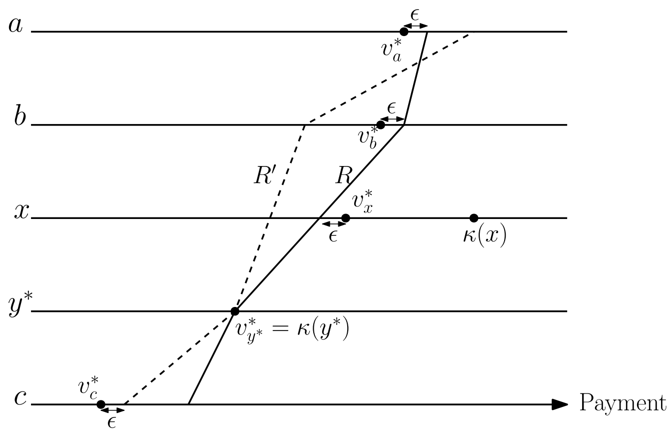


FIGURE 11. Illustration of $v^*$ for the case $a \succ b \succ x \succ y^* \succ c$.

Next suppose $a \neq y^*$ and $a \not\succ x$. We have $v_a \geq \min\{V^R(a, (y^*, \kappa(y^*))), V^{R'}(a, (y^*, \kappa(y^*)))\} - \epsilon$. If $b = y^*$, then $v_b = \kappa(y^*) = \min\{V^R(y^*, (y^*, \kappa(y^*))), V^{R'}(y^*, (y^*, \kappa(y^*)))\}$. Else, $a \not\succ x$ implies that $b \not\succ x$ and $b \neq x$; hence, $v_b = \min\{V^R(b, (y^*, \kappa(y^*))), V^{R'}(b, (y^*, \kappa(y^*)))\} - \epsilon$. Since $R$ and $R'$ respect $\succ$ and $a \succ b$,

$$\begin{aligned}&\min\{V^R(a, (y^*, \kappa(y^*))), V^{R'}(a, (y^*, \kappa(y^*)))\}\\ &\quad > \min\{V^R(b, (y^*, \kappa(y^*))), V^{R'}(b, (y^*, \kappa(y^*)))\}.\end{aligned}$$

Since $\epsilon$ is sufficiently small, $v_a > v_b$. Hence, $v$ respects $\succ$.

STEP 2. We construct the set of preferences

$$\mathcal{R}^* := \{R^* \in \mathcal{D} : V^{R^*}(y, (y^*, \kappa(y^*))) = v_y^* \ \forall y \neq x\}.$$

By Step 1, $v^*$ respects $\succ$. Thus, by OP richness (implied by $\text{TP}_{\succ}^{\Delta}$ richness), there is a preference $R^*$ such that $v^* \in \mathcal{I}(R^*)$ and, clearly, $R^* \in \mathcal{R}^*$. This implies that $\mathcal{R}^*$ is nonempty.

In this step, we show that for each $\bar{R} \in \mathcal{R}^*$, we have $f(\bar{R}) \in \{x, y^*\}$. Suppose $f(\bar{R}) = y$ and $y \notin \{x, y^*\}$. If $y \succ x$, then $V^{\bar{R}}(y, (y^*, \kappa(y^*))) = v_y^* < V^R(y, (y^*, \kappa(y^*)))$. By our assumption, $y \to y^*$ holds. Hence, we must have $(y, \kappa(y))\bar{R}(y^*, \kappa(y^*))$ or $V^{\bar{R}}(y, (y^*, \kappa(y^*))) \geq \kappa(y)$. Combining, we get $V^R(y, (y^*, \kappa(y^*))) > \kappa(y)$. This is a contradiction to inequality (3). Next, if $y \not\succ x$, then we have $V^{\bar{R}}(y, (y^*, \kappa(y^*))) < V^{R'}(y, (y^*, \kappa(y^*)))$. This is a contradiction to monotonicity since $f(\bar{R}) = y$ and $f(R') = y^*$. This completes the proof that every $\bar{R} \in \mathcal{R}^*$ satisfies the fact that $f(\bar{R}) \in \{x, y^*\}$.

STEP 3. As before, using OP richness, choose $R^* \in \mathcal{R}^*$ such that $v^* \in \mathcal{I}(R^*)$. By Step 2, $f(R^*) \in \{x, y^*\}$. Further, $V^{R^*}(x, (y^*, \kappa(y^*))) = v_x^* > V^R(x, (y^*, \kappa(y^*)))$ and monotonicity implies that $f(R^*) = x$. Hence,

$$\eta = \inf_{\hat{R} \in \mathcal{R}^* : f(\hat{R}) = x} V^{\hat{R}}(x, (y^*, \kappa(y^*)))$$

is well defined. Note that $\eta$ is a real number since monotonicity implies that $\eta \geq V^{R'}(x, (y^*, \kappa(y^*)))$. Note also that since $f(R^*) = x$, we have $\eta \leq V^{R^*}(x, (y^*, \kappa(y^*))) = v^*_x$.

Step 4. We now define two vectors $u, v \in \mathbb{R}^{|A|}$ as

$$u_x = \eta + \delta, \qquad v_x = \eta - \delta, \qquad u_y = v_y = v^*_y \quad \forall y \neq x,$$

where $\delta > 0$ but sufficiently small. Let $\hat{v} \in \mathbb{R}^{|A|}$ be such that $\hat{v}_x = \eta$ and $\hat{v}_y = v^*_y$ for every $y \neq x$. By Step 3, $V^{R'}(x, (y^*, \kappa(y^*))) \leq \eta = \hat{v}_x \leq v^*_x$. Hence, by Step 1, $\hat{v}$ respects $\succ$. Thus, since $\delta$ is sufficiently small, $u$ and $v$ also respect $\succ$. By Step 3, $\eta \leq v^*_x = V^R(x, (y^*, \kappa(y^*))) + \epsilon < \kappa(x)$, where the last inequality follows from inequality (2) and the fact that $\epsilon$ is sufficiently small. Hence, $\eta < \kappa(x)$, and using the fact that $\delta$ is sufficiently small, we get $u_x < \kappa(x)$ and $v_x < \kappa(x)$. By construction, $u_{y^*} = v_{y^*} = v^*_{y^*} = \kappa(y^*)$. Hence, by Lemma 5 (whose statement and proof are given in Appendix B), we have two vectors $u'$ and $v'$ that respect $\succ$ such that $(u, u')$ are $\Delta_\succ$ parallel, $(v, v')$ are $\Delta_\succ$ parallel with $u'_x = v'_x = \kappa(x)$, and $u'_{y*} > v'_{y^*}$.

Now, by $\text{TP}^\Delta_\succ$ richness, there is a preference $\widehat{R}$ such that $u, u' \in \mathcal{I}(\widehat{R})$ and there is a preference $\widetilde{R}$ such that $v, v' \in \mathcal{I}(\widetilde{R})$. By Step 2, $f(\widehat{R}), f(\widetilde{R}) \in \{x, y^*\}$. By definition of $\eta$ and the fact that $V^{\widetilde{R}}(x, (y^*, \kappa(y^*))) = v_x < \eta$, we get $f(\widetilde{R}) = y^*$. Also, by definition of $\eta$, there is a preference $R'' \in \mathcal{R}^*$ such that $f(R'') = x$ and $V^{R''}(x, (y^*, \kappa(y^*))) < \eta + \delta = V^{\widehat{R}}(x, (y^*, \kappa(y^*)))$. By monotonicity, $f(\widehat{R}) = x$. But then, $V^{\widetilde{R}}(y^*, (x, \kappa(x))) = v'_{y^*} < u'_{y^*} = V^{\widehat{R}}(y^*, (x, \kappa(x)))$. This is a contradiction to monotonicity. $\square$

## Appendix B: Missing lemmas and proofs

We give all the missing proofs in this section. We start by giving some elementary lemmas that we use throughout our proofs.

Proof of Lemma 1. Fix $\Delta > 0$. We show that $\mathcal{Q}^\Delta_+$ satisfies $\text{TP}^\Delta_+$ richness. This implies that every domain $\mathcal{D} \supseteq \mathcal{Q}^\Delta_+$ satisfies $\text{TP}^\Delta_+$ richness. To show that $\mathcal{Q}^\Delta_+$ satisfies $\text{TP}^\Delta_+$ richness, we pick two vector $v$, $\hat{v}$ with $\hat{v} > v$ that are $\Delta_+$ parallel and respect $\succ$. Let $R^q$ be the quasilinear preference with $v \in \mathcal{I}(R^q)$, i.e., $\mathcal{I}(R^q)$ consists of all indifference vectors parallel to $v$.

We now construct a new preference $R$ as follows. Fix $a_0 \in A$. It is enough to specify $V^R(a, (a_0, t))$ for each $t \in \mathbb{R}$ and all $a \in A \setminus \{a_0\}$. Let $v' < v$ be such that $v$ and $v'$ are $\Delta_+$ parallel. Let $\hat{v}' > \hat{v}$ be such that $\hat{v}'$ and $v$ are $\Delta_+$ parallel. For all $t \in \mathbb{R}$ and all $a \in A \setminus \{a_0\}$,

$$V^R\big(a, (a_0, t)\big) = \begin{cases} t + \alpha \cdot (v_a - v_{a_0}) + (1 - \alpha) \cdot \big(v'_a - v'_{a_0}\big), & \text{where } \alpha = \dfrac{-1}{t - v_{a_0} - 1} \quad \text{if } t < v_{a_0} \\ t + \alpha \cdot (v_a - v_{a_0}) + (1 - \alpha) \cdot (\hat{v}_a - \hat{v}_{a_0}), & \text{where } \alpha = \dfrac{\hat{v}_{a_0} - t}{\hat{v}_{a_0} - v_{a_0}} \quad \text{if } v_{a_0} \leq t \leq \hat{v}_{a_0} \\ t + \alpha \cdot (\hat{v}_a - \hat{v}_{a_0}) + (1 - \alpha) \cdot \big(\hat{v}'_a - \hat{v}'_{a_0}\big), & \text{where } \alpha = \dfrac{1}{t - \hat{v}_{a_0} + 1} \quad \text{if } t > \hat{v}_{a_0}. \end{cases}$$

Note for all $t \in \mathbb{R}$, $\alpha \in [0,1]$, that $V^R(a,(a_0,v_{a_0})) = v_a$ and $V^R(a,(a_0,\hat{v}_{a_0})) = \hat{v}_a$, and that $\alpha \to 1$ as $t \to v_{a_0}$, $\alpha \to 0$ as $t \nearrow \hat{v}_{a_0}$, and $\alpha \to 1$ as $t \searrow \hat{v}_{a_0}$. Thus, $V^R(a,(a_0,t))$ is continuous with respect to $t$. Also note $\alpha \to 0$ as $t \to -\infty$ or $t \to \infty$. Thus, since $v' < v < \hat{v} < \hat{v}'$, and $v$, $\hat{v}$, $v'$, and $\hat{v}'$ satisfy DD, every pair of vectors in $\mathcal{I}(R)$ also satisfy DD. Hence, $R$ satisfies PIE.

Next, we complete the proof by showing that $d(R,R^q) < \Delta$, thus showing that $R \in \mathcal{Q}_+^\Delta$. For every $t < v_{a_0}$,

$$\begin{aligned}
&\left|V^R\big(a,(a_0,t)\big) - V^{R^q}\big(a,(a_0,t)\big)\right| \\
&\quad = \left|\big[t + \alpha\cdot(v_a - v_{a_0}) + (1-\alpha)\cdot\big(v'_a - v'_{a_0}\big)\big] - \big[t + (v_a - v_{a_0})\big]\right| \\
&\quad = \left|(1-\alpha)\big[\big(v'_a - v'_{a_0}\big) - (v_a - v_{a_0})\big]\right| < \Delta,
\end{aligned}$$

where the inequality follows from $\alpha \in [0,1]$ and $|(v'_a - v'_{a_0}) - (v_a - v_{a_0})| < \Delta$.

For every $t \in [v_{a_0}, \hat{v}_{a_0}]$,

$$\begin{aligned}
&\left|V^R\big(a,(a_0,t)\big) - V^{R^q}\big(a,(a_0,t)\big)\right| \\
&\quad = \left|\big[t + \alpha\cdot(v_a - v_{a_0}) + (1-\alpha)\cdot(\hat{v}_a - \hat{v}_{a_0})\big] - \big[t + (v_a - v_{a_0})\big]\right| \\
&\quad = \left|(1-\alpha)\big[(\hat{v}_a - \hat{v}_{a_0}) - (v_a - v_{a_0})\big]\right| < \Delta,
\end{aligned}$$

where the inequality follows from $\alpha \in [0,1]$ and $|(\hat{v}_a - \hat{v}_{a_0}) - (v_a - v_{a_0})| < \Delta$.

For every $t > v_{a_0}$,

$$\begin{aligned}
&\left|V^R\big(a,(a_0,t)\big) - V^{R^q}\big(a,(a_0,t)\big)\right| \\
&\quad = \left|\big[t + \alpha\cdot(\hat{v}_a - \hat{v}_{a_0}) + (1-\alpha)\cdot\big(\hat{v}'_a - \hat{v}'_{a_0}\big)\big] - \big[t + (v_a - v_{a_0})\big]\right| \\
&\quad = \left|\alpha\cdot\big[(\hat{v}_a - \hat{v}_{a_0}) - (v_a - v_{a_0})\big] + (1-\alpha)\cdot\big[\big(\hat{v}'_a - \hat{v}'_{a_0}\big) - (v_a - v_{a_0})\big]\right| \\
&\quad \le \left|\alpha\cdot\big[(\hat{v}_a - \hat{v}_{a_0}) - (v_a - v_{a_0})\big]\right| + \left|(1-\alpha)\cdot\big[\big(\hat{v}'_a - \hat{v}'_{a_0}\big) - (v_a - v_{a_0})\big]\right| \\
&\quad < \alpha\cdot\Delta + (1-\alpha)\cdot\Delta = \Delta,
\end{aligned}$$

where the second inequality follows from $|(\hat{v}_a - \hat{v}_{a_0}) - (v_a - v_{a_0})| < \Delta$ and $|(\hat{v}'_a - \hat{v}'_{a_0}) - (v_a - v_{a_0})| < \Delta$. These observations imply that $d(R,R^q) < \Delta$. Hence, we have $R \in \mathcal{Q}_+^\Delta$. A similar proof can also be done for the case when $\mathcal{D} \supseteq \mathcal{Q}_\succ^\Delta$ to show that $\mathcal{D}$ satisfies $\text{TP}_\succ^\Delta$ richness. $\square$

The following technical lemma is used in the proof of Theorem 2.

Lemma 4. *Suppose $w, w' \in \mathcal{R}^{|A|}$ are two vectors that respect $\succ$ with $w > w'$. Then there exists $a^* \in A$ and two vectors $v, v' \in \mathbb{R}^{|A|}$ that respect $\succ$ and are $\Delta_+$ parallel such that the following conditions hold:*

*(i)* $v_{a^*} = w_{a^*}$, $v_x < w_x$ $\forall x \neq a^*$

*(ii)* $v'_{a^*} = w'_{a^*}$, $v'_x > w'_x$ $\forall x \neq a^*$.

Proof. Let $\eta$ be defined as

$$\eta := \min_{a \in A}(w_a - w'_a).$$

Let $A^* := \{a \in A : w_a - w'_a = \eta\}$. Let $A^1, \dots, A^K$ be the partitioning of $A$ that satisfies, for every $j \in \{1, \dots, K\}$, for every $a, b \in A^j$, $w_a = w_b$, and for every $a \in A^j$ with $j \leq K - 1$ and $b \in A^{j+1}$, $w_a > w_b$. Pick $a^* \in A^*$ such that $w_{a^*} \leq w_a$ for all $a \in A^*$. Suppose $a^* \in A^k$, hence, for all $a \in A^*$ if $a \in A^j$, then $j \leq k$.

Pick $\delta > 0$ but arbitrarily close to zero. Define two vectors $v, v' \in \mathbb{R}^{|A|}$ as

$$v_{a^*} = w_{a^*}, \qquad v_x = w_x - \frac{\delta}{4}, \quad \forall x \neq a^*$$

$$v'_{a^*} = w_{a^*} - \eta = w'_{a^*}, \qquad v'_x = w_x - \eta - \left(j - k - \frac{2}{3}\right)\delta, \quad \forall x \neq a^*, \text{ where } x \in A^j.$$

By construction, $v$ satisfies condition (i) of the claim. Further, since $\delta$ is sufficiently close to zero and $w$ respects $\succ$, $v$ respects $\succ$. Again, since $w$ respects $\succ$, and for every $y \in A$, $v'_y$ is arbitrarily close to $w_y - \eta$, $v'$ also respects $\succ$. Thus, $v$, $v'$ respect $\succ$. In the following discussion, we show that condition (ii) holds.

Condition (ii). Since $v'_{a^*} = w'_{a^*}$ by construction, we only show that $v'_x > w'_x$ for all $x \neq a^*$. Pick $x \neq a^*$ with $x \in A^j$. If $x \notin A^*$, then $\eta < w_x - w'_x$. As a result, if $\delta$ is sufficiently small, $\eta + (j - k - \frac{2}{3})\delta < w_x - w'_x$. Hence, $v'_x = w_x - \eta - (j - k - \frac{2}{3})\delta > w'_x$. If $x \in A^*$, then by construction, $k \geq j$ and $\eta = w_x - w'_x$. As a result, $v'_x = w_x - \eta - (j - k - \frac{2}{3})\delta > w'_x$.

Finally, to show that $v$, $v'$ satisfy DD, pick $x, y \in A$ and consider the following two cases. Note that $v > v'$.

Case 1: $x, y \neq a^*$. Suppose $x \in A^j$, $y \in A^h$, and $v_x > v_y$. Then $w_x > w_y$ and $j < h$. As a result, $v'_x - v'_y = w_x - w_y - (j - h)\delta > w_x - w_y = v_x - v_y$.

Case 2: $x \neq a^*$, $y = a^*$. Suppose $x \in A^j$. As a result,

$$v'_x - v'_{a^*} = w_x - w_{a^*} - \left(j - k - \frac{2}{3}\right)\delta = v_x - v_{a^*} + \frac{\delta}{4} - \left(j - k - \frac{2}{3}\right)\delta.$$

If $v_x > v_{a^*}$, then $w_x > w_{a^*}$ and $j < k$. This means that the previous expression is strictly greater than $v_x - v_{a^*}$. If $v_x < v_{a^*}$, then since $\delta$ is sufficiently small, $w_x < w_{a^*}$. Hence, $j - k \geq 1$. As a result, the previous expression is smaller than $v_x - v_{a^*}$. Hence, $v$, $v'$ satisfy DD. Further, since $v'_y$ is arbitrarily close to $w_y - \eta$ for each $y \in A$ with $\eta > 0$ and $v$ is arbitrarily close to $w$, we conclude that $v$, $v'$ are $\Delta_+$ parallel. □

The following technical lemma is used in the proof of Theorem 1.

Lemma 5. *Let $a, b \in A$. Suppose $u, v \in \mathbb{R}^{|A|}$ are two vectors that respect $\succ$ and satisfy*

$$u_a = v_a = \kappa(a), \qquad u_b = \eta + \delta, \qquad v_b = \eta - \delta,$$

*where $\eta < \kappa(b)$ and $\delta > 0$ is sufficiently small. Then there exists $u', v' \in \mathbb{R}^{|A|}$ that respect $\succ$ such that the following statements hold.*

*(i)* $u'_b = v'_b = \kappa(b)$ *and* $u'_a > v'_a$

*(ii)* $(u, u')$ *are* $\Delta_{\succ}$ *parallel and* $(v, v')$ *are* $\Delta_{\succ}$ *parallel.*

Proof. Consider the partitioning $(A_1, \dots, A_K)$ of $A$ as follows. Let $A_1 := \{x \in A : y \not\succ x \ \forall y \in A\}$. Having defined $A_1, \dots, A_{k-1}$, define $A_k := \{x \in A \setminus \bigcup_{j=1}^{k-1} A_j : y \not\succ x \ \forall y \in A \setminus \bigcup_{j=1}^{k-1} A_j\}$. Note that $x \succ y$ with $x \in A_h$ and $y \in A_\ell$ implies that $h < \ell$.

Suppose $a \in A_k$ and $b \in A_j$. We consider three cases.

Case 1: $a \succ b$. Then $k < j$. In that case, we define two vectors $u'$ and $v'$ as follows. For every $\ell \in \{1, \dots, K\}$ and for every $x \in A_\ell$, we define

$$u'_x = u_x - u_b + \kappa(b) - (j - \ell)\frac{1}{2}\delta, \qquad v'_x = v_x - v_b + \kappa(b) - (j - \ell)3\delta.$$

Since $\delta$ is sufficiently small and $\kappa(b) > u_b > v_b$, we have $u' > u$ and $v' > v$. We show that $u'$, $v'$ respect $\succ$. Pick $x \in A_h$ and $y \in A_\ell$ with $x \succ y$. Hence, we have $h < \ell$. Now $u'_x - u'_y = u_x - u_y + (h - \ell)\frac{1}{2}\delta > 0$, where the strict inequality follows since $u$ respects $\succ$ and, hence, $u_x > u_y$, and $\delta$ is sufficiently small. This shows that $u'$ respects $\succ$. An identical argument shows that $v'$ respects $\succ$. By construction, $u'_b = v'_b = \kappa(b)$ and $j > k$ implies that

$$u'_a - v'_a = (u_a - v_a) - (u_b - v_b) + (j - k)\frac{5}{2}\delta \geq \frac{1}{2}\delta > 0.$$

So property (i) of the claim holds.

Finally, we show that $(u, u')$ are $\Delta_{\succ}$ parallel and $(v, v')$ are $\Delta_{\succ}$ parallel. To do so, pick $x \in A_h$ and $y \in A_\ell$ with $x \succ y$, and, hence, $h < \ell$. Now

$$\left(u'_x - u'_y\right) - (u_x - u_y) = (h - \ell)\frac{1}{2}\delta < 0,$$

where the inequality follows since $\ell > h$ and $\delta > 0$. Hence, $(u, u')$ are $\Delta_{\succ}$ parallel. A similar argument shows that $(v, v')$ are $\Delta_{\succ}$ parallel. This completes the proof of the claim for this case.

Case 2: $b \succ a$. Then $k > j$. In this case, we define two vectors $u''$ and $v''$ as follows. For every $\ell \in \{1, \dots, K\}$ and for every $x \in A_\ell$, we define

$$u''_x = u_x - u_b + \kappa(b) - (j - \ell)3\delta, \qquad v''_x = v_x - v_b + \kappa(b) - (j - \ell)\frac{1}{2}\delta.$$

Following the same argument as in Case 1, we can show that $u''$, $v''$ respect $\succ$, $(u, u'')$ are $\Delta_{\succ}$ parallel, and $(v, v'')$ are $\Delta_{\succ}$ parallel. Further, since $u_b < \kappa(b)$ and $\delta$ is sufficiently small, we have $u'' > u$ and $v'' > v$. Finally, since $k > j$, we have $u''_a - v''_a = (v_b - u_b) + (k - j)\frac{5}{2}\delta \geq \frac{1}{2}\delta > 0$. Hence, $u''_a > v''_a$ as required by property (i). This completes the proof of the claim for this case.

Case 3: $a \not\succ b$ and $b \not\succ a$. In this case, we define two vectors $u^*$ and $v^*$ using $u''$ and $v''$ defined in Case 2 as follows. For every $\ell \in \{1, \dots, K\}$ and for every $x \in A_\ell$, we define $u^*$

as

$$u_a^* = u_a'' + \delta', \qquad u_x^* = u_x'' \quad \forall x \neq a,$$

where $\delta' > 0$ is sufficiently small; in particular, $\delta' < \delta$. Since $u''$ respects $\succ$ and $\delta'$ is sufficiently small, $u^*$ also respects $\succ$. Similarly, we define $v^*$ as

$$v_a^* = v_a'' - \delta', \qquad v_x^* = v_x'' \quad \forall x \neq a.$$

Since $v''$ respects $\succ$ and $\delta'$ is sufficiently small, $v^*$ also respects $\succ$.

By construction $u_b^* = v_b^* = \kappa(b)$. Further, $u_a^* - v_a^* = 2\delta' > 0$. Hence, property (i) holds.

We now show that $(u, u^*)$ are $\Delta_\succ$ parallel. First, since $u'' > u$, and $u''$ and $u^*$ are sufficiently close, $u^* > u$. Now pick $x \in A_h$ and $y \in A_\ell$ with $x \succ y$, and, hence, $h < \ell$. If $a \notin \{x, y\}$, then $(u_x^* - u_y^*) - (u_x - u_y) = (u_x'' - u_y'') - (u_x - u_y) < 0$, where the inequality follows from Case 2. Suppose $a = x$. Then $h = k < \ell$ and we get

$$\big(u_a^* - u_y^*\big) - (u_a - u_y) = \big(u_a'' - u_y''\big) - (u_a - u_y) + \delta' = (k - \ell)3\delta + \delta' < 0,$$

where the inequality follows since $\delta' < \delta$ and $k < \ell$. Finally, suppose $a = y$. Then $h < k = \ell$,

$$\big(u_x^* - u_a^*\big) - (u_x - u_a) = \big(u_x'' - u_a''\big) - (u_x - u_a) - \delta' = (h - k)3\delta - \delta' < 0,$$

where the inequality follows since $h < k$. □

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