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# A model of weighted network formation

Leonie Baumann
Department of Economics, McGill University

This paper proposes a game of weighted network formation in which each agent has a limited resource to form links of possibly different intensities with other agents and to use for private purposes. We show that every equilibrium is either "reciprocal" or "nonreciprocal." In a reciprocal equilibrium, any two agents invest equally in the link between them. In a nonreciprocal equilibrium, agents are partitioned into "concentrated" and "diversified" agents, and a concentrated agent is only linked to diversified agents and vice versa. For every link, the concentrated agent invests more in the link than the diversified agent. The unweighted relationship graph of an equilibrium, in which two agents are linked if they both invest positively in each other, uniquely predicts the equilibrium values of each agent's network investment and utility level, as well as the ratio of any two agents' investments in each other. We show that equilibria are not pairwise stable and are not efficient due to the positive externalities of investing in a link.

Keywords. Weighted networks, network formation, continuous link strength.

JEL classification. C70, D85, L14.

## 1. Introduction

A network is a graph that describes the relationships between the network's members. A link between two members of a network can represent, for example, friendship, coauthorship, trade, or communication between them. Most of the literature on network formation, following the seminal papers by Jackson and Wolinsky (1996) and Bala and Goyal (2000), assumes that an agent decides whether to form a link, but does not determine its intensity. However, in many situations agents must choose not only with whom to interact, but also the intensity of that interaction.

We analyze a symmetric game in which each agent has a limited resource that she can keep for herself (self-investment) and invest in forming links with other agents. A strategy of an agent specifies an allocation of her resource across all agents (including herself). We say that *two agents are linked* if they both invest positively in each other. An agent's utility is the sum of her benefits from self-investment and from each of her relationships. The benefit from self-investment is represented by an increasing and strictly concave function. The benefit from her relationship with another agent is increasing

Leonie Baumann: leonie.baumann@mcgill.ca

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and strictly concave in the two agents' investments in each other and is represented by a function that exhibits strategic complementarity and is homogenous of degree 1.

In the main analysis, we investigate the game's Nash equilibria. Special attention is devoted to the (unweighted and undirected) relationship graphs that are induced by equilibria and that include a link between two agents if they both invest positively in each other.

We show that every equilibrium is one of two types: reciprocal or nonreciprocal. In a *reciprocal equilibrium*, any two linked agents invest the same amount in the link between them, and all agents choose the same self-investment and derive the same utility. Using a result from graph theory, we characterize the full set of relationship graphs associated with reciprocal equilibria. This set includes, for example, graphs in which every agent is linked to more than half of the other agents or in which every agent has the same number of links. The set excludes, for example, graphs in which there is an agent with only one link. It is possible that two agents have a different number of links and that an agent has links of varying intensities.

In a *nonreciprocal equilibrium*, agents are partitioned into two sets: the set of concentrated agents and the set of diversified agents. We will see that a *diversified agent* maintains more relationships on average than a concentrated agent and can be interpreted as more actively networking, outgoing, and free-riding on others' relationship efforts. A *concentrated agent*, to the contrary, can be said to be more introverted and dedicated to fewer relationships in which she provides the bulk of the relationship effort. Every link that is formed is between a concentrated and a diversified agent, and never between two agents of the same set. For all links, the concentrated agent invests more in the link than the diversified agent. The ratio between the investment of a concentrated agent and that of a diversified agent in their link is the same across all links (and denoted by $q^t$). All concentrated agents choose the same level of self-investment, which is higher than the level of self-investment chosen by all diversified agents. Diversified agents derive greater utility than concentrated agents. The ratio of the number of concentrated agents to the number of diversified agents is positively correlated with $q^t$.

We show that the relationship graphs of reciprocal and nonreciprocal equilibria are entirely distinct. Thus, knowing only the equilibrium relationship graph is sufficient to determine whether the equilibrium is reciprocal or nonreciprocal. Furthermore, the relationship graph of a nonreciprocal equilibrium uniquely determines the partition into concentrated and diversified agents, the value of $q^t$, and each agent's level of self-investment and utility. However, the relationship graph does not always pin down the equilibrium investments in a link. We demonstrate that many equilibria with different levels of investment in links can induce the same relationship graph.

We examine the comparative statics of equilibria when relationships become more valuable relative to self-investment and when each agent's resource endowment increases. In view of the multiplicity of equilibria, we restrict ourselves to investigate how the equilibrium values that are uniquely determined by the relationship graph and the model parameters change when the corresponding model parameter is varied and the relationship graph is held fixed.

Additionally, we show that equilibria are not stable against pairwise deviations and are not efficient, in the sense that they do not maximize the sum of agents' utilities. This is due to the positive externality of an agent's investment in a link that benefits the other agent in the link. We characterize efficient networks and find that in an efficient network, any two agents invest the same amount in each other, but choose a lower self-investment than in a reciprocal equilibrium. We show that, nevertheless, the set of relationship graphs of efficient networks coincides with the set of relationship graphs of reciprocal equilibria.

*Related literature* This paper adds to the literature on network formation with weighted links.

The most closely related articles are Salonen (2015), Griffith (2017), and Brueckner (2006), which analyze the formation of weighted social networks, and Goyal et al. (2008), which analyzes a two-stage game in which firms first form weighted links in research and development (R&D) networks and then compete in a market. These authors focus on symmetric equilibria. Restricting the analysis in this way limits the possibility of differences in link intensities in equilibrium. We extend beyond symmetric equilibria and identify asymmetric equilibrium structures.

Bloch and Dutta (2009) and Deroïan (2009) analyze the formation of communication networks, in which agents also derive utility from indirect links, with budget constraints and without self-investment. Thus, the amount invested in the network is determined exogenously and is the same for all agents. The possibility of self-investment in our model gives rise to equilibria in which agents choose different levels of network investment. Another difference is our assumption that two agents' investments in their link are strategic (imperfect) complements. For the main part of their analysis, Bloch and Dutta (2009) assume that link quality is an additively separable function of two agents' investments in their link. Deroïan (2009) assumes that an agent's link investment benefits her, but not her link partner.

Rogers (2006) suggests a different type of network formation game in which agents invest in links so as to pursue a higher status. An agent's status is increasing in the status of agents she is linked to and in the intensity of those links.

Finally, Golub and Livne (2010), Cabrales et al. (2011), Durieu et al. (2011), and Galeotti and Merlino (2014) assume that agents can choose one parameter (quality, effort, or investment level), which then affects the intensities of all their links equally. Such a constraint limits the set of weighted networks that can form in equilibrium.

*Road map* Section 2 introduces the model. Section 3 presents the equilibrium analysis and is divided into the following subsections: Section 3.1 characterizes the equilibrium investment strategy profiles and utility levels; Section 3.2 analyzes the relationship graphs of reciprocal and nonreciprocal equilibria; Section 3.3 discusses the multiplicity of equilibria; Section 3.4 presents comparative statics results. Section 4 discusses the pairwise stability of equilibria and characterizes the efficient networks.

## 2. The model

There is a set of agents $N = \{1, \dots, n\}$. Each agent $i$ possesses resource $T > 0$ that she can invest in relations with other agents and in private activity. Her investment in a relation with agent $j \neq i$ is denoted by $t_{ij}$ and her investment in private activity (*self-investment*) is denoted by $t_{ii}$. An investment strategy of agent $i$ is $t_i = (t_{i1}, \dots, t_{in})$ such that $t_{ij} \geq 0$ for all $j$ and $\sum_j t_{ij} \leq T$. The analysis is restricted to pure strategies. A strategy profile is represented by a matrix $t = [t_{ij}]_{i,j}$ and can be interpreted as a weighted directed graph, with $t_{ij}$ being the weight on the link from $i$ to $j$. We also refer to strategy profile $t$ as *network* $t$.

Agent $i$'s utility given network $t$ is the sum of her utilities from relations with others and from self-investment,

$$u_i(t) = \sum_{j \neq i} av(t_{ij}, t_{ji}) + f(t_{ii}),$$

where $av(t_{ij}, t_{ji})$, $a > 0$, is $i$'s utility from her relation with $j$ and $f(t_{ii})$ is her utility from self-investment. The parameter $a$ determines the value of relationships relative to the value of self-investment.

The relationship utility $v$ is continuously differentiable. The partial derivative of $v$ with respect to argument $k = 1, 2$ is denoted by $v_k$, and the second-order partial derivative of $v$ with respect to arguments $k = 1, 2$ and $l = 1, 2$ is denoted by $v_{kl}$. Apart from continuous differentiability, $v$ satisfies the following properties.

Property 1. $v(x, 0) = v(0, y) = 0$ for all $x, y \geq 0$. A relationship yields zero benefit if one agent does not invest in the relationship.

Property 2. For all $x, y > 0$, $v(x, y)$ is increasing and strictly concave, and $\lim_{x \to 0} v_1(x, y) = \infty$ for all $y > 0$. Agent $i$'s utility from her relationship with $j$ is increasing and strictly concave in $i$'s and $j$'s investments. Marginal utility is infinite if $i$'s investment goes to zero and $j$ invests positively.

Property 3. $v_{12}(x, y) > 0$, $v_{21}(x, y) > 0$ for all $x, y > 0$. Two agents' investments in their relationship are strategic complements.

Property 4. $v(\gamma x, \gamma y) = \gamma v(x, y)$ for all $\gamma > 0$. The relationship utility $v$ is homogenous of degree 1 and exhibits constant returns to scale.

Property 4 implies that $v_k$ is homogenous of degree 0.

For example, a Cobb–Douglas function $v(x, y) = x^{\beta} y^{1-\beta}$ with $\beta \in (0, 1)$ satisfies Properties 1–4.

The utility function from self-investment, $f$, is increasing, strictly concave, and continuously differentiable, with $\lim_{x \to 0} f'(x) = \infty$ and $\lim_{x \to T} f'(x) = 0$.

A network $t$ induces an unweighted and undirected (relationship) graph $g(t)$ on $N$ that describes the relationships with mutual positive investments in $t$. That is, agents $i$ and $j$ are linked in $g(t)$ (link $ij \in g(t)$) if $t_{ij} > 0$ and $t_{ji} > 0$.

We introduce some graph-related definitions that are necessary for the analysis of the game. In what follows, *graph* always means an unweighted and undirected graph. Consider a graph $g$ on $N$. Agent $i$'s set of *neighbors* is $N_i := \{j|ij \in g\}$. A *walk* between agents $i$ and $j$ is a sequence of links $i_1i_2, i_2i_3, \dots, i_{K-1}i_K$ such that $i_{k-1}i_k \in g$ for all $k = 2, \dots, K$, and $i_1 = i$ and $i_K = j$. Two *agents are connected* if there exists a walk between them, and $g$ *is connected* if all agents in $N$ are connected. A *component of* $g$ is a maximal connected subgraph of $g$. This means that all agents in one component are connected to each other and not linked to any agent outside the component. An agent without any links (component of size 1) is called an *isolated* agent. To avoid unnecessary complications, we often refer to the links, components, etc. of a network $t$, when we mean the links, components, etc. of its graph $g(t)$.

## 3. Equilibrium networks

The analysis focusses on the Nash equilibria of the network formation game in which all agents simultaneously choose their investment strategies. A strategy profile $t$ is a Nash equilibrium if no agent $i$ can strictly increase her utility by deviating to another strategy, given all other agents' strategies.

In Section 3.1, we show that every equilibrium is either reciprocal or nonreciprocal. In a reciprocal equilibrium, any two agents invest the same amount in each other, and all agents have the same self-investment and utility level. In a nonreciprocal equilibrium, agents can be partitioned into two sets $C$ (concentrated agents) and $D$ (diversified agents). Links exist only between the sets, and never within them. For every link, the concentrated agent invests more in the link than the diversified agent. The ratio between the concentrated agent's investment in the link and the diversified agent's is the same across all links. All agents within same set have the same self-investment and utility level.

In Section 3.2, we characterize the relationship graphs of equilibria. We show that simply by observing an equilibrium relationship graph we can uniquely determine each agent's equilibrium self-investment and utility level as well as the ratio of any two agents' equilibrium investments in each other. In particular, the graph can be used to determine whether the equilibrium that induced it is reciprocal or nonreciprocal.

In Section 3.3, we discuss the multiplicity of the equilibria. A given relationship graph can be induced by many equilibria, which feature different link investments. We propose a simple mechanism by which we can construct multiple equilibria from a given equilibrium.

In Section 3.4, we investigate the comparative statics of equilibria for the case that relationships become relatively more valuable (i.e., increase in $a$) and for the case that the total resource endowment increases (i.e., increase in $T$). Given the multiplicity of equilibria, we restrict ourselves to analyze the change in the equilibrium values that are uniquely determined by the relationship graph and the model parameters when the corresponding model parameter is varied and the relationship graph remains the same.

### 3.1 *Investment strategy profiles and utility levels*

Note that $t_{ij} = 0$ is the unique optimal choice of agent $i$ if agent $j$ chooses $t_{ji} = 0$ because self-investment is always utility-enhancing and $v(t_{ij}, 0) = 0$ for all $t_{ij}$. Thus, a trivial equilibrium is the *empty* network where all agents invest only in themselves.[1] More generally, a network is an equilibrium if and only if the investment choices of the agents in each component of the network are an equilibrium of the network formation game reduced to the agents in that component. Therefore, so as to characterize the full set of equilibrium networks, we restrict the analysis from now on to connected equilibrium networks with $n > 1$.[2]

The next proposition requires the following definitions. Let $\sigma : \mathbb{R}^{>0} \to (0, T)$ be the function defined by the equation $f'(\sigma(x)) = av_1(x, 1)$. Function $\sigma$ is related to an agent's equilibrium self-investment. Note that the properties of $f$ guarantee that the function $\sigma$ is well defined. Let $\mu : \mathbb{R}^{>0} \to \mathbb{R}$ be the function defined by $\mu(x) = (T - \sigma(x))av(1, 1/x) + f(\sigma(x))$. Function $\mu$ is related to an agent's equilibrium utility level.

Lemma 1. *The function $\sigma$ is strictly increasing and the function $\mu$ is strictly decreasing.*

The proof of Lemma 1 is relegated to the Appendix.

Proposition 1. *For every equilibrium $t$, there exists $q^t \geq 1$ such that for every $i \in N$, there exists $q_i$, where $q_i \in \{q^t, 1/q^t\}$ and $t_{ij}/t_{ji} = q_i$ for all $j \in N_i$, $t_{ii} = \sigma(q_i)$, and $u_i(t) = \mu(q_i)$. Thus, every equilibrium $t$ fulfills one of the following alternatives:*

*(i) It is reciprocal ($q^t = 1$), where $q_i = 1$ for all $i \in N$.*

*(ii) It is nonreciprocal ($q^t > 1$), where there is a bipartition $(C, D)$ of $N$ such that if $i$ is linked to $j$, then $i$ and $j$ are in different sets. For all $i \in C$ and $j \in D$, $q_i = q^t$ and $q_j = 1/q^t$.*

Proof. We start with a lemma that establishes necessary and sufficient conditions on $t$ for it to be a Nash equilibrium.

Lemma 2. *A network $t$ is a Nash equilibrium if and only if, for all $i \in N$ and all $j \neq i$,*

*(a)* $\sum_k t_{ik} = T$

[1] This equilibrium always exists in our model.

[2] This is without loss of generality: (i) Any equilibrium network with $n > 1$ is the union of its isolated agents and its components of size larger than 1. Zooming in on a component, the self-investments of agents in the component and their investment choices toward others in the component constitute an equilibrium of the game reduced to the agents in the component. Thus, the agents in the component form a connected equilibrium network among themselves, when viewed in isolation from the rest of the network. (ii) Since zero investments across components are optimal choices, the union of $k$ connected equilibrium networks constitutes an equilibrium network with $k$ components.

From (i) and (ii) it follows that a network is an equilibrium if and only if the agents in each of its components of size larger than 1 form a connected equilibrium network among themselves, when viewed in isolation from the rest of the network.

*(b) if* $t_{ji} = 0$*, then* $t_{ij} = 0$

*(c) if* $t_{ji} > 0$*, then* $t_{ij} > 0$ *and* $av_1(t_{ij}, t_{ji}) = f'(t_{ii})$.

The proof of Lemma 2 is immediate from the standard conditions on each agent's utility maximization problem given all other agents' strategies and, hence, it is omitted. In any equilibrium, each agent $i$ invests her entire resource and invests positively in $j$ if and only if $j$ invests positively in $i$. An agent's positive investment levels are such that her marginal utility from investing in any of her links is equal to her marginal utility from self-investment.

Now consider an equilibrium $t$ and $i \in N$. Note first that $v_1(t_{ij}, t_{ji}) = v_1(t_{ij}/t_{ji}, 1)$ for all $j \in N_i$ by Property 4. By Lemma 2(c), $v_1(t_{ij}/t_{ji}, 1) = v_1(t_{ik}/t_{ki}, 1)$ for all $j, k \in N_i$. Thus, $t_{ij}/t_{ji} = t_{ik}/t_{ki}$ for all $j, k \in N_i$ because $v$ is strictly concave. Hence, there is $q_i > 0$ such that $t_{ij}/t_{ji} = q_i$ for all $j \in N_i$. Then $q_j = 1/q_i$ for all $j \in N_i$. Let $q^t = \max\{q_i, 1/q_i\}$. Since all agents are connected, $q_k \in \{q^t, 1/q^t\}$ for all $k \in N$.

By Lemma 2(c), it then follows with regard to agent $i$'s self-investment that $f'(t_{ii}) = av_1(q_i, 1)$ and, hence, $t_{ii} = \sigma(q_i)$. Regarding agent $i$'s utility, observe that $u_i(t) = \sum_{j \neq i} av(t_{ij}, t_{ji}) + f(t_{ii}) = \sum_{j \neq i} t_{ij} av(1, t_{ji}/t_{ij}) + f(t_{ii}) = (T - t_{ii}) av(1, 1/q_i) + f(t_{ii}) = \mu(q_i)$ because $v$ is homogenous of degree 1 and $t_{ii} = \sigma(q_i)$.

In the case of a reciprocal equilibrium in which $q^t = 1$, obviously $q_k = 1$ for all $k \in N$. In the case of a nonreciprocal equilibrium with $q^t > 1$, there exists an agent $i \in N$ for whom $q_i = q^t$. For all $j \in N_i$, $q_j = 1/q_i = 1/q^t$ and so on. Thus, because $q_i = 1/q_j$ for all $i$ and all $j \in N_i$, there exists a partition $(C, D)$ of $N$ in which all $i$ with $q_i = q^t$ are in $C$ and all $j$ with $q_j = 1/q^t$ are in $D$, and there are only links across the sets. $\square$

It is worthwhile to summarize the observations about equilibria that follow from Proposition 1. Every equilibrium $t$ is associated with a number $q^t$, which we call the *investment ratio* of $t$. In equilibrium $t$, for any link that agent $i$ has, the ratio of $i$'s investment to her neighbor's investment in the link is equal to $q_i$. This ratio $q_i$ is either $q^t$ or $1/q^t$. Agent $i$'s equilibrium self-investment level is a strictly increasing function of $q_i$, while her equilibrium utility level is a strictly decreasing function of $q_i$.

In any reciprocal equilibrium, every agent's $q_i$ is equal to 1, and every agent chooses the same level of self-investment and derives the same level of utility. We call the agents in a reciprocal equilibrium *balanced*, and denote their self-investment and utility by $t_{bb} := \sigma(1)$ and $u_b := \mu(1)$, respectively.

In any nonreciprocal equilibrium $t$, there exists a partition of $N$ into two sets $C$ and $D$ such that links exist only between agents in different sets. We call the agents in $C$ *concentrated* and the agents in $D$ *diversified*. For every concentrated agent $i$, $q_i = q^t$, and for every diversified agent $i$, $q_i = 1/q^t$. This means that for any link, the concentrated agent invests more in the link than the diversified agent. Moreover, all agents in the same set choose the same level of self-investment and derive the same level of utility. We denote the self-investment of concentrated agents and diversified agents by $t_{cc} := \sigma(q^t)$ and $t_{dd} := \sigma(1/q^t)$, and denote their utility by $u_c := \mu(q^t)$ and $u_d := \mu(1/q^t)$, respectively.

Since $\sigma$ is strictly increasing and $\mu$ is strictly decreasing, the equilibrium levels of self-investment and utility are unambiguously ordered for different values of $q_i$. For any reciprocal equilibrium $t$ and any nonreciprocal equilibrium $t'$, $t'_{dd} < t_{bb} < t'_{cc}$ and $u'_d > u_b > u'_c$. In other words, diversified agents have the lowest self-investment and highest utility, concentrated agents have the highest self-investment and lowest utility, and balanced agents have both a self-investment and utility somewhere in between. Note that the ordering of self-investment levels trivially imposes an ordering on agents' total equilibrium network investment. A diversified agent chooses the highest total network investment, a concentrated agent chooses the lowest investment, and a balanced agent chooses somewhere in between.

The divergence of $t'_{dd}$ and $t'_{cc}$ from $t_{bb}$ is strictly increasing in $q^{t'}$, as is the divergence of $u'_d$ and $u'_c$ from $u_b$. Thus, for any equilibrium $t$, the investment ratio $q^t$ is an indication of the degree of inequality between neighbors in $t$. Define the degree of inequality between neighbors $i$ and $j$ in equilibrium $t$ as a weighted sum of the absolute differences between their self-investments, $|t_{ii} - t_{jj}|$, their utility levels, $|u_i(t) - u_j(t)|$, and between the investment ratio and 1: $q^t - 1$. Note that the degree of inequality is the same across all pairs of neighbors in a given equilibrium $t$. If $q^t = 1$, then the degree of inequality between neighbors is 0, and if $q^t > 1$, then it is strictly positive and strictly increasing in $q^t$.

Example 1 illustrates a reciprocal equilibrium and two nonreciprocal equilibria with different investment ratios for a specific configuration of the model.

Example 1. Let $n = 5$, $T = 2$, and $u_i(t) = \sum_{j \neq i} t_{ij}^{\beta} t_{ji}^{1-\beta} + t_{ii}^{\beta}$ with $\beta \in (0, 1)$. By Lemma 2, in equilibrium, agent $i$'s marginal utilities from investing in link $ij$ and from self-investment are equal:

$$av_1(t_{ij}, t_{ji}) = f'(t_{ii}) \quad \Leftrightarrow \quad \beta t_{ij}^{\beta-1} t_{ji}^{1-\beta} = \beta t_{ii}^{\beta-1} \quad \Leftrightarrow \quad t_{ii} = t_{ij}/t_{ji}.$$

Thus, in every reciprocal equilibrium $t$, $t_{bb} = 1$ and $u_b = 2$. Figure 1 shows an example of a reciprocal equilibrium $t$.

In every nonreciprocal equilibrium $t$, $t_{cc} = q^t$, $t_{dd} = 1/q^t$, $u_c = (T - q^t)(1/q^t)^{1-\beta} + {q^t}^{\beta}$, and $u_d = (T - 1/q^t){q^t}^{1-\beta} + (1/q^t)^{\beta}$. An example of a nonreciprocal equilibrium $t$, where $q^t = 3/2$, $C = \{1, 2, 3, 4\}$, and $D = \{5\}$, is shown in Figure 2(a). An example of a nonreciprocal equilibrium $t'$ with a lower inequality between neighbors, where $q^{t'} = 8/7$, $C' = \{1, 3, 5\}$, and $D' = \{2, 4\}$, is shown in Figure 2(b). ◊

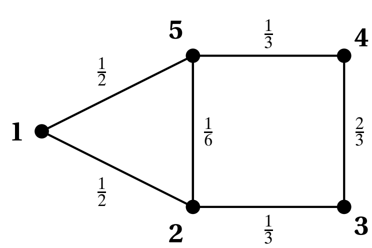
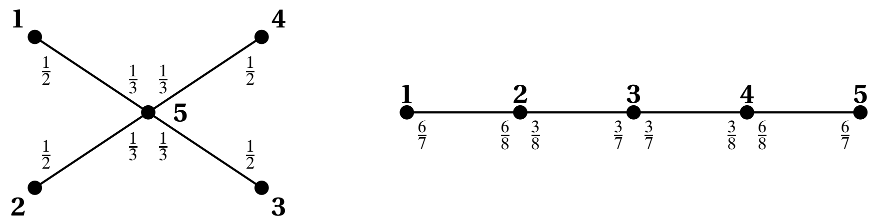



Figure 1. A reciprocal equilibrium $t$. Bold numbers indicate the identity of each agent (node). The number at link $ij$ is $t_{ij} = t_{ji}$.




(a) $q^t = 3/2$ (b) $q^{t'} = 8/7$

Figure 2. Two nonreciprocal equilibria $t$ and $t'$ with different investment ratios. Bold numbers indicate the identity of each agent (node). The number next to agent $i$ at link $ij$ is $t_{ij}$.

We conclude this subsection by showing that both a reciprocal and a nonreciprocal equilibrium always exist in our model.

First, we show that the complete network with equal investments across all links is a reciprocal equilibrium that always exists. Consider $t$ such that $t_{ii} \in (0, T)$, $f'(t_{ii}) = av_1(1, 1)$, and $t_{ij} = (T - t_{ii})/(n-1)$ for all $j \neq i$ and all $i$. The assumptions on $f$ and $v$ guarantee that $t$ exists, and $t$ is a reciprocal equilibrium.

Second, we show that if $n \geq 3$, then the star network where the center node invests equally in all her links is a nonreciprocal equilibrium that always exists. Without loss of generality (w.l.o.g.) let $D = \{1\}$ and $C = N \setminus \{1\}$. Construct a network $t$ as follows. Take $t_{11}$ and $t_{jj}$ for all $j \in C$ such that $f'(t_{11}) = av_1((T - t_{11})/(n-1), T - t_{jj})$ and $f'(t_{jj}) = av_1(T - t_{jj}, (T - t_{11})/(n-1))$. By Brouwer's fixed point theorem, $t_{11}$ and $t_{jj}$ exist. Our assumptions on $f$ and $v$ guarantee that $0 < t_{11} < t_{jj} < T$ and $(T - t_{jj})(n-1)/(T - t_{11}) > 1$. Finally, let $t_{1j} = (T - t_{11})/(n-1)$, $t_{j1} = T - t_{jj}$, and $t_{jk} = 0$ for all $k \neq 1, j$ and all $j \in C$. Then $t$ is a nonreciprocal equilibrium.

### 3.2 *Relationship graphs*

In this section, we investigate the graphs of equilibrium networks where a link between two agents means that both invest positively in each other. We show that simply by observing the graph of an equilibrium we can uniquely determine $q_i$, $t_{ii}$, and $u_i(t)$ for each agent $i$, without any other information about the investment profile.

Let $G^R = \{g \mid g = g(t)$ for some reciprocal equilibrium $t\}$, that is, $G^R$ is the set of all graphs that are induced by some reciprocal equilibrium, and let $G^{\mathrm{NR}} = \{g \mid g = g(t)$ for some nonreciprocal equilibrium $t\}$, that is, $G^{\mathrm{NR}}$ is the set of all graphs that are induced by some nonreciprocal equilibrium.

We first provide a full characterization of $G^R$.[3] Let $g[N \setminus U]$ with $U \subseteq N$ be the subgraph induced in $g$ by $N \setminus U$. Denote by $W(U)$ the set of isolated agents in $g[N \setminus U]$ and denote by $|X|$ the cardinality of a set $X$.

Proposition 2. *A connected graph $g$ on $N$ is in $G^R$ if and only if for every $U \subseteq N$,*

[3] Proposition 2 was established in personal discussions with Henning Bruhn-Fujimoto.

*(i)* $|U| > |W(U)|$ *or*

*(ii)* $|U| = |W(U)|$ *and for every link* $ij \in g$*, if* $i \in U$*, then* $j \in W(U)$*.*

Proposition 2 states that a connected graph $g$ is induced by some reciprocal equilibrium if and only if for every $U \subseteq N$ either (i) the number of agents in $U$ is strictly larger than the number of isolated agents in $g[N \setminus U]$, or (ii) the number of agents in $U$ and the number of isolated agents in $g[N \setminus U]$ are the same, and in $g$, agents in $U$ are only linked to agents in $W(U)$. We refer to conditions (i) and (ii) of Proposition 2 as conditions 2(i) and 2(ii).

Proof of Proposition 2: Necessity. Let a connected graph $g$ be in $G^R$ and let $t$ be a reciprocal equilibria $t$ such that $g(t) = g$.

The total network investment by agents in $U$ is $|U|(T - t_{bb})$ and by agents in $W(U)$ is $|W(U)|(T - t_{bb})$. In $g$, every $i \in W(U)$ is only linked to agents in $U$; otherwise, $i \in W(U)$ would not be isolated in $g[N \setminus U]$. Thus, the total network investment by agents in $W(U)$ must be fully reciprocated by agents in $U$. Then either $|U|(T - t_{bb}) > |W(U)|(T - t_{bb})$, which means $|U| > |W(U)|$, or $|U|(T - t_{bb}) = |W(U)|(T - t_{bb})$, which means $|U| = |W(U)|$ and agents in $U$ must be linked only to agents in $W(U)$ in $g$. Otherwise, the network investment by agents in $U$ would not be sufficient to fully reciprocate that by agents in $W(U)$. $\square$

The sufficiency proof of Proposition 2 relies on an existence result for a particular type of matching in a graph found in Schrijver (2003, p. 584). Because it is largely technical, the proof is relegated to the Appendix, and only a short outline and intuition are provided here. In the proof, we first show that for a given graph, there exists a reciprocal equilibrium that induces the graph if there exists a certain *perfect b-matching with lower bounds* in that graph. The *perfect b-matching with lower bounds* is an assignment of numbers to the links of the graph such that (i) each number is above a certain lower bound and (ii) the sum of numbers assigned to the links incident with a node is equal to the value $b$ for each node. By appropriately scaling this matching, we obtain a reciprocal equilibrium. Schrijver (2003, p. 584) provides necessary and sufficient conditions on a graph for this matching to exist. The second step in the proof shows that if a connected graph $g$ satisfies the conditions of Proposition 2, then $g$ satisfies the existence conditions in Schrijver (2003, p. 584). Thus, the matching in $g$ exists, as does a reciprocal equilibrium that induces $g$.

Proposition 2 provides a tool to determine whether a connected graph $g$ is induced by some reciprocal equilibrium. For this, it is sufficient to determine whether condition 2(i) or 2(ii) is satisfied when $|U| < n/2+1$, since, for $|U| \geq n/2+1$, condition 2(i) is trivially satisfied. Some straightforward graph properties simplify this task, as shown below in Corollary 1.

An agent is *a leaf* in graph $g$ if she has only one link. A graph $g$ *is bipartite* or *has a bipartition* if there exists a bipartition $(A, B)$ of $N$ such that if $ij \in g$, then $i$ and $j$ are in different sets of the bipartition.

Corollary 1. *Let $g$ be a connected graph on $N$.*

*(a) If $|N_i| > n/2$ for all $i \in N$, then $g \in G^R$.*

*(b) If $|N_i| = d > 0$ for all $i \in N$, then $g \in G^R$.*

*(c) If $n > 2$ and $g$ contains a leaf, then $g \notin G^R$.*

*(d) If $g$ is bipartite with $|A| \neq |B|$, then $g \notin G^R$.*

Proof. (a) Given $g$, $|U| > n/2$ is necessary to have at least one isolated agent in $g[N \setminus U]$. Hence, $|U| > |W(U)|$ for all $U$ and $g \in G^R$ by Proposition 2.

(b) Given $g$, every agent in $W(U)$ is linked to $d$ agents in $U$ and, thus, there exist $d|W(U)|$ links between $U$ and $W(U)$. If every $i \in U$ is only linked to agents in $W(U)$, then $d|U| = d|W(U)|$ and condition 2(ii) is satisfied. If not every $i \in U$ is only linked to agents in $W(U)$, then $d|U| > d|W(U)|$ and condition 2(i) is satisfied. Thus, $g \in G^R$ by Proposition 2.

(c) Given $g$, let $i$ be a leaf. Take $U = N_i$. Then $|W(U)| \geq |U|$ and the only neighbor of agent $i$ is not only linked to $i$, but also to other agents because $n > 2$. Thus, $g \notin G^R$ by Proposition 2.

(d) Given $g$, where w.l.o.g. $|A| > |B|$, take $U = B$. Then $W(U) = A$ and $|W(U)| > |U|$. Thus, $g \notin G^R$ by Proposition 2. □

Hence, by Corollary 1(a) and (b), any connected graph that is "dense" or "regular" is induced by some reciprocal equilibrium. By Corollary 1(c) and (d), graphs that contain leaves (for example, trees) or graphs that are bipartite with two unequally sized sets are never induced by a reciprocal equilibrium.

We next turn to analyze $G^{\mathrm{NR}}$. In Proposition 3, we present necessary conditions for a graph to be in $G^{\mathrm{NR}}$. Let $\rho : \mathbb{R}^{>1} \to \mathbb{R}^{>1}$ be the function defined by $\rho(x) = x(T - \sigma(1/x))/(T - \sigma(x))$. Given that $\sigma$ is strictly increasing, it is straightforward to show that $\rho(x) > 1$ for all $x$ and that $\rho$ is strictly increasing.

Proposition 3. *If $g \in G^{\mathrm{NR}}$, then $g$ has a unique bipartition $(A, B)$. W.l.o.g. let $|A| \geq |B|$. For any nonreciprocal equilibrium $t$ with $g(t) = g$, $|A|/|B| = \rho(q^t) > 1$, for every $i \in A$, $q_i = q^t$, and for every $j \in B$, $q_j = 1/q^t$ and $|N_j| > 1$.*

Proposition 3 makes several statements. Consider any $g \in G^{\mathrm{NR}}$. The graph $g$ has exactly one bipartition and the two sets in that bipartition are of unequal size. For any nonreciprocal equilibrium that induces $g$, all concentrated agents are in the larger set of the bipartition and all diversified agents are in the smaller one. Any leaf is in the larger set and, thus, is a concentrated agent. There is a strictly increasing correspondence between the investment ratio $q^t$ and the ratio of concentrated to diversified agents.

Proof of Proposition 3. Let $g$ be in $G^{\mathrm{NR}}$ and let $t$ be an equilibrium with $g(t) = g$. By Proposition 1, each link is between a member of $C$ (concentrated agents) and a member of $D$ (diversified agents). Thus, $(C, D)$ is a bipartition of $g$. Since $g$ is connected, a standard result from graph theory implies that the bipartition of $g$ is unique.

For all $i \in C$ and $j \in D$, $t_{ij}/t_{ji} = q^t$ if $ij \in g$ by Proposition 1. Thus, $\sum_{i\in C, j\in D} t_{ij} = q^t \sum_{i\in C, j\in D} t_{ji}$, which is equivalent to $\sum_{i\in C}(T - t_{ii}) = q^t \sum_{j\in D}(T - t_{jj})$. Hence, $|C|(T - t_{cc}) = q^t|D|(T - t_{dd})$ and $|C|/|D| = q^t(T - t_{dd})/(T - t_{cc}) = q^t(T - \sigma(1/q^t))/(T - \sigma(q^t)) = \rho(q^t)$.

Suppose, to the contrary, that agent $i$ is a leaf and is in $D$. Then $i$'s only link is with some $j \in C$. This implies, using Lemma 2 and Proposition 1, that $t_{ii} + t_{ij} = T < t_{jj} + t_{ji}$ and $j$'s resource constraint is violated. □

The fact that each $g \in G^{\mathrm{NR}}$ is bipartite implies that there exists no graph in $G^{\mathrm{NR}}$ that includes an odd cycle.[4] Proposition 3 also provides further insight regarding the investment strategies of concentrated and diversified agents in a nonreciprocal equilibrium: A diversified agent has, on average, more links than a concentrated agent, since the network is connected and $|C| > |D|$.

The next result shows that a reciprocal equilibrium and a nonreciprocal equilibrium never induce the same graph. Moreover, some graphs cannot be induced by any equilibrium.

Proposition 4. *For every $n \geq 2$, $G^R \cap G^{\mathrm{NR}} = \varnothing$, and for every $n \geq 4$, there exists a connected graph $g$ on $N$ such that $g \notin G^R \cup G^{\mathrm{NR}}$.*

Proof. By Corollary 1(d), there exists no $g \in G^R$ with a bipartition where the two sets of the bipartition are of unequal size. By Proposition 3, every $g \in G^{\mathrm{NR}}$ has a bipartition with the two sets of unequal size. Thus, $G^R \cap G^{\mathrm{NR}} = \varnothing$.

Let $n \geq 4$ and consider the following graph. Agents 1, 2, and 3 form a triangle. Every other agent is only linked to agent 1. Thus, $g$ includes an odd cycle and, hence, $g \notin G^{\mathrm{NR}}$. Moreover, $g$ contains a leaf and, hence, $g \notin G^R$. □

Another family of graphs (in addition to the one described in the proof above) that cannot be induced by any equilibrium is one in which two leaves are connected via an odd number of links: If such a graph were in $G^R$, it would not include a leaf, and if it were in $G^{\mathrm{NR}}$, both leaves would be in $C$ and, thus, would have to be connected via an even number of links.

Propositions 3 and 4 imply that the information about the graph $g$ induced by an equilibrium $t$ is sufficient to determine $q_i$, $t_{ii}$, and $u_i(t)$ for all $i$ in equilibrium $t$. If $g$ has a bipartition $(A, B)$, where $A$ and $B$ are of unequal size and w.l.o.g. $|A| > |B|$, then any equilibrium $t$ that induces $g$ is nonreciprocal, where $q^t = \rho^{-1}(|A|/|B|)$ and for all $i \in A$, $q_i = q^t$, and for all $i \in B$, $q_i = 1/q^t$. Otherwise, any equilibrium that induces $g$ is reciprocal, and $q_i = 1$ for all $i$. Self-investment and utility levels follow from Proposition 1.

[4]An *odd cycle* is a sequence of links $i_1i_2, \ldots, i_{K-1}i_K$, where $i_j \neq i_k$ for $k \notin \{1, K\}$, $i_1 = i_K$, and $K > 2$ is even.

### 3.3 *Equilibrium multiplicity*

We now turn to equilibrium multiplicity. Based on the previous section, equilibria that induce the same graph must feature the same values of $q_i$, $t_{ii}$, and $u_i(t)$ for all $i$ because the equilibrium graph uniquely determines those values. However, equilibria that induce the same graph may feature different levels of investments in links and those are, therefore, not always uniquely determined by the graph. In the next proposition, we capture the multiplicity with a simple mechanism that derives an equilibrium $t'$ from an equilibrium $t$. The mechanism relies on appropriately shifting link investment levels in $t$ on an even-length cycle.

The intuition behind the mechanism is the following. Take as given the self-investments and individual investment ratios $q_i$ of some equilibrium. Any allocation of link investments such that each $i$ faces the given investment ratio $q_i$ across all her links together with given self-investments constitutes an equilibrium. Our mechanism derives from one such equilibrium allocation of link investments: an alternative one. Given an equilibrium, increase two agents' investments in their link while keeping the ratio of these link investments and self-investments unchanged. Then each agent's marginal utility from investing in this link is still equal to that from self-investment since the link utility function $v$ is homogenous of degree 1. Increasing an agent's investment in one link while keeping her total investment in links unchanged requires decreasing investment in another link. Thus, by appropriately increasing and decreasing link investments alternately on an even-length cycle, we reach another equilibrium allocation of link investments.

Proposition 5. *Let $n \geq 4$ and let $t$ be an equilibrium with $g(t) = g$. Then the following strategy profile $t'$ is also an equilibrium.*

*First, let $S$ be a sequence of distinct agents $i_1, i_2, \dots, i_{K-1}$ and $i_K = i_1$ such that $K > 4$ is an odd integer, $i_k i_{k+1} \in g$ for odd $k$, and if $q^t > 1$, then $i_k \in C$ and $i_{k+1} \in D$ for odd $k$. Second, let $t'$ be equal to $t$, except for $t'_{i_k i_{k+1}} = t_{i_k i_{k+1}} + x > 0$ and $t'_{i_k i_{k-1}} = t_{i_k i_{k-1}} - x > 0$ for all $k \in \{1, \dots, K-1\}$, where $x = -h$ if $k$ is odd, $x = l$ if $k$ is even, and $h/l = q^t$.*

Proof. Consider agent $i_k$. The only change in agent $i_k$'s strategy from $t$ to $t'$ is a shift of her investment by an amount $x$ between agents $i_{k-1}$ and $i_{k+1}$. Thus, agent $i_k$'s budget constraint remains binding in $t'$. The only investments by other agents in agent $i_k$ that have changed from $t$ to $t'$ are those of agents $i_{k-1}$ and $i_{k+1}$. Thus, $t'_{i_k j} = t'_{j i_k} = 0$ if $i_k j \notin g(t)$ and $t'_{i_k j}/t'_{j i_k} = q_{i_k}$ if $i_k j \in g(t)$ for all $j \neq i_{k-1}, i_{k+1}$. We next show that $t'_{i_k j}/t'_{j i_k} = q_{i_k}$ also for $j = i_{k-1}, i_{k+1}$. If $k$ is odd and $j = i_{k+1}$, then $q_{i_k} = q^t$ and

$$
\begin{aligned}
t'_{i_k, i_{k+1}}/t'_{i_{k+1}, i_k} &= (t_{i_k, i_{k+1}} - h)/(t_{i_{k+1}, i_k} - l) \\
&= \big(q^t (t_{i_k, i_{k+1}} - h)\big)/\big(q^t (t_{i_{k+1}, i_k} - l)\big) \\
&= q^t (t_{i_k, i_{k+1}} - h)/\big((t_{i_k, i_{k+1}}/t_{i_{k+1}, i_k}) t_{i_{k+1}, i_k} - (h/l) l\big) \\
&= q^t (t_{i_k, i_{k+1}} - h)/(t_{i_k, i_{k+1}} - h) \\
&= q^t.
\end{aligned}
$$

Similarly, for $j = i_{k-1}$ and then also for even $k$.

For any agent $i$ not in the sequence $S$, $t'_i = t_i$ and the investments in $i$ are the same in both $t'$ and $t$. Thus, $t'$ is an equilibrium by Lemma 2. □

In Example 2, we apply the mechanism provided in Proposition 5.

Example 2. Consider the environment of Example 1. In Figure 3, $t$ is a reciprocal equilibrium, and in Figure 4, $t$ is a nonreciprocal equilibrium with $q^t = 8/7$. In both, $t'$ is an equilibrium obtained from $t$ by applying the mechanism described in Proposition 5. ◊

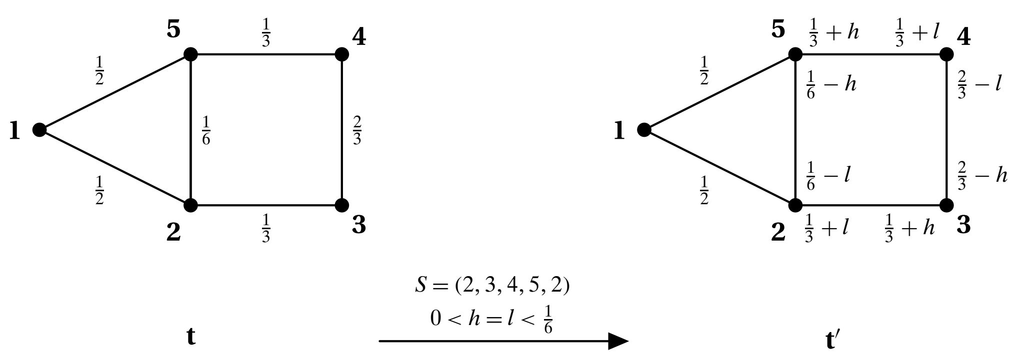



Figure 3. Deriving a reciprocal equilibrium $t'$ from the reciprocal equilibrium $t$.

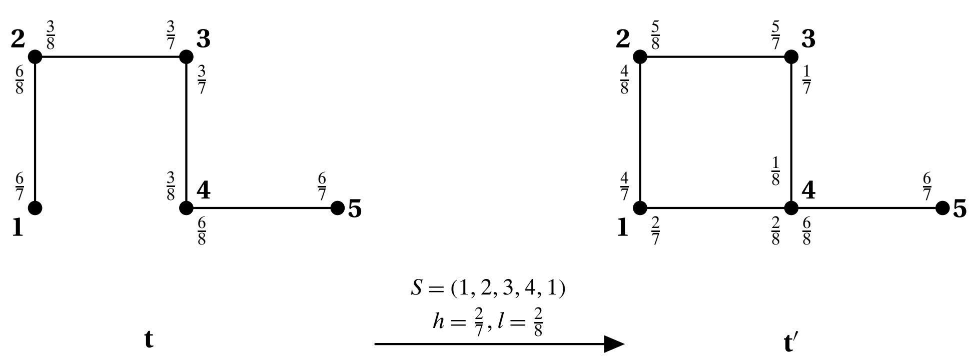



Figure 4. Deriving a nonreciprocal equilibrium $t'$ from the nonreciprocal equilibrium $t$.

### 3.4 *Comparative statics*

In this section, we present the comparative statics of the equilibria for the changes (i) when the investment in a relationship becomes more valuable relative to self-investment (i.e., an increase in $a$) and (ii) the total resources available for investment increase (i.e., an increase in $T$). Given the multiplicity of equilibria, we limit ourselves to the following comparative statics exercise: Consider a particular graph induced by an equilibrium. Assuming that the graph remains unchanged, what is the effect of a change in the model parameters on those equilibrium values of the strategy profile that are uniquely determined by the graph and the parameters?

Note that previously we assumed for simplicity that $\lim_{x\to T} f'(x) = 0$ so as to guarantee that every agent chooses a self-investment less than $T$ in any equilibrium. Keeping this assumption and altering $T$ would change the function $f$ and render the comparative statics for $T$ impossible. In this section, we assume that $f$ is fixed and that all values of $T$ satisfy $f'(T) < v_1(n-1, 1)$. That is, $f'(T)$ is low enough to guarantee equilibrium self-investments less than $T$ for all $T$.

Proposition 6. *(a) Consider $a < \hat{a}$. Let $t$ be an equilibrium given $a$ and let $\hat{t}$ be an equilibrium given $\hat{a}$, where $t$ and $\hat{t}$ induce the same graph ($g(t) = g(\hat{t})$).*

*If $q^t = 1$, then $\hat{q}^t = 1$ and $t_{bb} > \hat{t}_{bb}$.*

*If $q^t > 1$, then $\hat{q}^t > 1$, $|C|/|D| = |\hat{C}|/|\hat{D}|$, $t_{cc} > \hat{t}_{cc}$, and $t_{dd} > \hat{t}_{dd}$.*

*(b) Consider $T < \hat{T}$ and suppose that $f = \hat{f}$ with $f'(T) < v_1(n-1, 1)$. Let $t$ be an equilibrium given $T$ and let $\hat{t}$ be an equilibrium given $\hat{T}$, where $g(t) = g(\hat{t})$.*

*If $q^t = 1$, then $\hat{q}^t = 1$ and $t_{bb} = \hat{t}_{bb}$.*

*If $q^t > 1$, then $q^t < \hat{q}^t$, $|C|/|D| = |\hat{C}|/|\hat{D}|$, $t_{cc} < \hat{t}_{cc}$, $T - t_{cc} < \hat{T} - \hat{t}_{cc}$, and $t_{dd} > \hat{t}_{dd}$.*

Before presenting the proof, it is worthwhile to restate Proposition 6. Consider part (a). Unsurprisingly, since investment in a relationship is more valuable under $\hat{a}$ than under $a$, the self-investment of all types of agents is lower under $\hat{a}$ than under $a$. If $t$ is a nonreciprocal equilibrium, then the ratio of concentrated to diversified agents remains the same, and the effect on the investment ratio is not uniquely determined.

Consider part (b). As the resource endowment increases, the self-investment level of balanced agents remains the same and, therefore, their network investment increases. If $t$ is a nonreciprocal equilibrium, then the ratio of concentrated to diversified agents remains unchanged. Both a concentrated agent's self-investment and her network investment are increasing in the resource endowment, whereas a diversified agent's self-investment decreases and, thus, her network investment increases by a larger amount than the increase in the resource endowment. An increase in the resource endowment also increases the ratio of a concentrated agent's investment to that of a diversified agent in the link between them.

Proof of Proposition 6(a). (The proof for part (b) proceeds similarly and is relegated to the Appendix.) Consider $q^t = 1$. Then $g(t) \in G^R$. The set $G^R$ is independent of $a$ by Proposition 2. Therefore, $g(\hat{t}) \in \hat{G}^R$ and $\hat{q}^t = 1$. By Proposition 1, $f'(t_{bb}) = av_1(1, 1)$.

Then, by the implicit function theorem, $\partial t_{bb}/\partial a = v_1(1,1)/f''(t_{bb})$, which is strictly negative because $f$ is strictly concave and $v$ is strictly increasing. Thus, $t_{bb} > \hat{t}_{bb}$.

Next consider $q^t > 1$. Then $g(t) = g(\hat{t}) \notin G^R = \hat{G}^R$ and, thus, $\hat{q}^t > 1$. By Proposition 3, the bipartition of $g(t)$ is unique and, thus, $|C|/|D| = |\hat{C}|/|\hat{D}|$. By Propositions 1 and 3, $\hat{t}$ is such that (i) $f'(\hat{t}_{cc}) = \hat{a}v_1(\hat{q}^t, 1)$, (ii) $f'(\hat{t}_{dd}) = \hat{a}v_1(1/\hat{q}^t, 1)$, and (iii) $|C|/|D| = \hat{q}^t(T - \hat{t}_{dd})/(T - \hat{t}_{cc})$, and the analogous conditions hold for $t$. Suppose, to the contrary, that $\hat{t}_{cc} \geq t_{cc}$. Then $\hat{q}^t > q^t$ by (i). This implies that $\hat{t}_{dd} < t_{dd}$ by (ii). Then $|C|/|D| < \hat{q}^t(T - \hat{t}_{dd})/(T - \hat{t}_{cc})$, contradicting (iii). The proof that $\hat{t}_{dd} < t_{dd}$ proceeds analogously. □

## 4. Stability and efficiency

In this section, we show that Nash equilibria are not "stable" if pairwise deviations are allowed, and are not efficient, in the sense that they do not maximize the sum of agents' utilities. This is due to the positive externalities of an agent's network investment on her neighbors, which are not incorporated into an agent's individual utility maximization.

### 4.1 *Equilibrium stability*

Following Bloch and Dutta (2009), we say that a strategy profile $t$ is strongly pairwise stable if it is a Nash equilibrium and if there are no two agents $(i, j)$ who would both be strictly better off by a joint deviation from $(t_i, t_j)$ to $(t'_i, t'_j)$, given all other agents' strategies.

Proposition 7. *No strategy profile $t$ is strongly pairwise stable.*

Proof. We show that for any equilibrium, there exist two agents who gain from reducing their self-investment and establishing or intensifying a reciprocal relationship between them. Suppose $t$ is an equilibrium. Consider any $i \in N$ and any $j \neq i$. If $i$ reduces her self-investment by $c > 0$, and $i$ and $j$ each invest $c$ so as to intensify or establish a reciprocal link between them, then $i$'s change in utility is $\Delta u_i(c) = f(t_{ii} - c) - f(t_{ii}) + cav(1,1)$. If $i$ is a balanced or concentrated agent, then there is $c > 0$ such that $\Delta u_i(c)$ is positive because $[\partial \Delta u_i/\partial c](0) = -f'(t_{ii}) + av(1,1)$ and $f'(t_{ii}) < av(1,1) = a(v_1(1,1) + v_2(1,1))$ for $t_{ii} \in \{t_{bb}, t_{cc}\}$. If $q^t = 1$, then there exist two balanced agents, and if $q^t > 1$, then there exist two concentrated agents; therefore, in each case, there is a pair with a strict incentive to jointly deviate. □

### 4.2 *Efficient networks*

Following Jackson and Wolinsky (1996), we say that a strategy profile $t$ is efficient if $t$ maximizes $\sum_{i \in N} u_i(t)$ such that $\sum_j t_{ij} = T$ for all $i$.

We see that the set of efficient networks and the set of equilibrium networks do not intersect. However, there is no distinction between the set of all graphs that are induced by some efficient network and the set of all graphs that are induced by some reciprocal equilibrium.

Proposition 8. *A network $t$ is efficient if and only if $t_{ij} = t_{ji}$, $\sum_k t_{ik} = T$, and $f'(t_{ii}) = av(1,1)$ for all $i$ and all $j \neq i$. A graph is induced by an efficient network if and only if each of its components is induced by some reciprocal equilibrium of the network formation game reduced to the agents in that component.*

Proposition 8 states that in every efficient network any two agents invest the same amount in each other. Moreover, any agent's self-investment is such that her marginal utility from self-investment equals the marginal increase in the sum of her own and her neighbor's utility from her investment in their reciprocal link. Thus, the efficient level of self-investment accounts for the positive externalities from network investment and is lower than the level of self-investment in a reciprocal equilibrium. Since every agent's self-investment is less than $T$ (as implied by the assumptions on $f$), there is no isolated agent in an efficient network. In particular, the set of all graphs of efficient networks is identical to the set of all graphs of equilibrium networks that consist only of reciprocal equilibrium components.

Proof of Proposition 8. Let $t$ be efficient. Then every agent's resource constraint is binding, since self-investment is always beneficial. Moreover, for all $i$ and $j \neq i$, $t_{ii} > 0$, and $t_{ij} = 0$ if and only if $t_{ji} = 0$. By the first-order conditions on $t$ to maximize the sum of utilities, any positive link investments $t_{ii}$, $t_{ij}$, and $t_{ji}$ must satisfy $f'(t_{ii}) = av_1(t_{ij}, t_{ji}) + av_2(t_{ji}, t_{ij}) = av_1(t_{ij}/t_{ji}, 1) + av_2(1, t_{ij}/t_{ji})$ for all $j \in N_i$ and all $i$. In other words, agent $i$'s investment in her link with agent $j$ is such that its marginal impact on the sum of utilities equals agent $i$'s marginal utility from self-investment.

We next show that any link is reciprocal, that is, $t_{ij} = t_{ji}$ for all $t_{ij}, t_{ji} > 0$. Suppose, to the contrary, that link $ij$ is nonreciprocal and w.l.o.g $t_{ij}/t_{ji} > 1$. Hence, $f'(t_{ii}) = av_1(t_{ij}/t_{ji}, 1) + av_2(1, t_{ij}/t_{ji}) < f'(t_{jj}) = av_1(t_{ji}/t_{ij}, 1) + av_2(1, t_{ji}/t_{ij})$ by the concavity of $v$ and $t_{ii} > t_{jj}$ by the concavity of $f$. Since $t_{ij} + t_{ii} > t_{ji} + t_{jj}$ and $i$'s resource constraint must bind, $j$ must have another link to some agent $k \neq i$. By efficiency and the strict concavity of $v$, $t_{ji}/t_{ij} = t_{jk}/t_{kj} < 1$, and $t_{kl}/t_{lk} = t_{im}/t_{mi} > 1$ for all $l \in N_k$ and $m \in N_i$, which implies that $i$ and $k$ are not linked and $t_{ik} = t_{ki} = 0$. Now consider strategy profile $t' \neq t$, where the self-investment of both $i$ and $k$ is reduced by $c$ and a reciprocal link between them is established with an investment of $c$ by each. As in the proof for Proposition 7, we can show that there is $c > 0$ such that $u_i(t') - u_i(t) > 0$ and $u_k(t') - u_k(t) > 0$. Moreover, $u_l(t') = u_l(t)$ for all $l \neq i, k$ and no agent's resource constraint has been affected by moving from $t$ to $t'$. Hence, a nonreciprocal link cannot exist in an efficient network.

Thus, $t_{ij} = t_{ji}$ for all $i$ and $j$, and $av_1(1,1) + av_2(1,1) = av(1,1) = f'(t_{ii})$ for all $i$ who have a link. It remains to show that every agent has a link.

Assume there exist at least two isolated agents $i$ and $j$. Then $t_{ii} = t_{jj} = T$ by efficiency. However, by the same argument as in the proof of Proposition 7, the sum of utilities can be increased if their self-investment is decreased and a reciprocal link between them is established.

Assume there exists only one isolated agent $i$. Let agents $j$ and $k$ be linked to each other. In this case, the sum of utilities can be increased as follows: Decrease $i$'s self-investment by $2\epsilon > 0$ and decrease the investments of $j$ and $k$ in their link $jk$ by $\epsilon$ each, and establish the reciprocal links $ij$ and $ik$, with $t'_{ij} = t'_{ji} = t'_{ik} = t'_{ki} = \epsilon$. For $\epsilon$ small enough, $i$'s utility strictly increases and the utility of no other agent changes, by $v$'s homogeneity of degree 1.

Let $t$ be such that $t_{ij} = t_{ji}$, $\sum_k t_{ik} = T$, and $f'(t_{ii}) = av(1, 1)$ for all $i$ and all $j \neq i$. Then agent $i$'s utility is $u_i(t) = \sum_{j \neq i} av(t_{ij}, t_{ji}) + f(t_{ii}) = \sum_{j \in N_i} av(t_{ij}, t_{ji}) + f(t_{ii}) = \sum_{j \in N_i} at_{ij}v(1, t_{ji}/t_{ij}) + f(t_{ii}) = \sum_{j \in N_i} a(T - t_{ii})v(1, 1) + f(t_{ii})$. Thus, the sum of utilities for any such $t$ is the same and, hence, any such $t$ is efficient. This concludes the first part of the proof of Proposition 8.

To prove the second part, first observe that we know from the first part of Proposition 8 that in an efficient network, every agent belongs to a component of at least two agents who are connected via reciprocal links.

The result then follows from the observation that a connected graph $g$ is induced by an efficient network if and only if $g$ is induced by a reciprocal equilibrium: Let $t$ be an efficient network that induces a connected graph $g$. Thus, $t_{ij} = t_{ji} > 0$ for all $ij \in g$, $t_{ij} = t_{ji} = 0$ for all $ij \notin g$, $\sum_{j \in N_i} t_{ij} = T - t_{ii}$ for all $i$, and $f'(t_{ii}) = av(1, 1)$. Then the following $t'$ is a reciprocal equilibrium that induces $g$. Let $t'$ be such that $t'_{ij} = t'_{ji} = t_{ij}(T - t'_{ii})/(T - t_{ii})$ for all $ij \in g$, $t'_{ij} = t'_{ji} = 0$ for all $ij \notin g$, and $f'(t'_{ii}) = av_1(1, 1)$.

Let $t'$ be a reciprocal equilibrium that induces a connected graph $g$. Thus, $t'_{ij} = t'_{ji} > 0$ for all $ij \in g$, $t'_{ij} = t'_{ji} = 0$ for all $ij \notin g$, $\sum_{j \in N_i} t'_{ij} = T - t'_{ii}$ for all $i$, and $f'(t'_{ii}) = av_1(1, 1)$. Then the following $t$ is a connected efficient network that induces $g$. Let $t$ be such that $t_{ij} = t_{ji} = t'_{ij}(T - t_{ii})/(T - t'_{ii})$ for all $ij \in g$, $t_{ij} = t_{ji} = 0$ for all $ij \notin g$, and $f'(t_{ii}) = av(1, 1)$. □

## 5. Concluding comments

We analyzed a game of weighted network formation in which agents simultaneously decide how to allocate a limited budget between building links of possibly different intensities with other agents and self-investment. Expanding the discussion of network formation from unweighted to weighted networks enlarges the strategy space of agents. Nevertheless, we obtained results about the structure of the game's equilibria. In particular, we showed that an equilibrium must have one of two structures, i.e., either reciprocal or nonreciprocal, and we characterized their properties.

Some of the results are consistent with empirical findings. First, note that in both reciprocal and nonreciprocal equilibria, two agents' investments in the link between them are predicted to be positively correlated. Griffith (2017) finds support for this property in his analysis of a weighted social network among school girls. He shows that the weights assigned by two girls to their relationship are positively (though not perfectly) correlated.

The presence of reciprocal and nonreciprocal relationships is investigated in Wang et al. (2013). They find that in a mobile phone communication network, 72% of all links are such that the two linked agents call each other with significantly different probabilities. They further suggest that the presence of reciprocal relations is more likely when the total network investment (number of calls made) by an agent is positively correlated across linked agents. This is in line with our theoretical findings: In a reciprocal equilibrium, each agent chooses the same total network investment that is thus predicted to be perfectly and positively correlated across agents, while in a nonreciprocal equilibrium, the level of total network investments by a concentrated agent is negatively correlated with that by a diversified agent.

We also characterized the properties of the graphs for reciprocal and nonreciprocal equilibria. We showed, for example, that any sufficiently "dense" graph, where *density* is measured by the number of links in the graph, or a "regular" graph, in which every agent has the same number of neighbors, is only induced by reciprocal equilibria (Corollary 1(a) and 1(b)). Some empirical studies provide evidence for a positive correlation between reciprocity and network density and/or regularity (for example, Kovanen et al. (2011) and Wang et al. (2013) for mobile phone communication networks). We also found that in graphs of nonreciprocal equilibria, diversified agents have, on average, more links than concentrated agents. This again resonates with Wang et al. (2013), who suggest that "networked systems that induce anti-correlation in the number of neighbors of each vertex [agent] in a dyad [link] should—all else being equal—be characterized by high levels of non-reciprocity."

On an anecdotal level, the three types of agents that arise in our model's equilibria can perhaps be observed in real life. *Diversified* agents are more popular and outgoing; they more actively network and free-ride on the efforts of other agents. *Concentrated* agents rely more on themselves, are more introverted, provide greater effort in relationships, and are exploited. *Balanced* agents are in give-and-take relationships and share responsibilities equally.

An avenue for further research would be to introduce heterogeneity between agents, and to investigate how this affects the existence and properties of reciprocal and nonreciprocal equilibria. A first step could be to differentiate between two types of agents, where linking to one of the types is more profitable than linking to the other. Griffith (2020) has started to analyze equilibrium networks in the presence of agent heterogeneity for the case that link utility is given by $v(t_{ij}, t_{ji}) = t_{ij}^{\alpha} t_{ji}^{\beta}$ with $\alpha, \beta > 0$. Heterogeneity is introduced by scaling link utilities with agent-specific weights and by allowing individuals to have different costs of self-investment. For $\alpha + \beta = 1$, a special case of our model, the author finds that in equilibrium, agent heterogeneity must be somewhat limited relative to the network graph. When heterogeneity is continuously distributed and, hence, infinite, these limits are violated with probability 1 for many graphs that occur in equilibrium in our model. Thus, the set of equilibrium networks depends on the degree of heterogeneity among agents. The effect of agent heterogeneity on equilibrium networks merits further investigation.

## Appendix

Proof of Lemma 1. Since $f$ and $v$ are both increasing and strictly concave, it immediately follows that $\sigma$ is strictly increasing.

We next show that $\partial\mu/\partial x$ is strictly negative:

$$\partial\mu/\partial x = -\sigma'(x)av(1, 1/x) - \big(T - \sigma(x)\big)av_2(1, 1/x)/x^2 + f'\big(\sigma(x)\big)\sigma'(x) \tag{1}$$

$$= -\sigma'(x)av(1, 1/x) - \big(T - \sigma(x)\big)av_2(1, 1/x)/x^2 + av_1(x, 1)\sigma'(x) \tag{2}$$

$$= -\big(T - \sigma(x)\big)av_2(1, 1/x)/x^2 + \sigma'(x)\big[av_1(x, 1) - av(1, 1/x)\big] \tag{3}$$

$$= -\big(T - \sigma(x)\big)av_2(1, 1/x)/x^2 + \sigma'(x)\big\{av_1(1, 1/x) - a\big[v_1(1, 1/x) + v_2(1, 1/x)/x\big]\big\} \tag{4}$$

$$= -\big(T - \sigma(x)\big)av_2(1, 1/x)/x^2 - \sigma'(x)av_2(1, 1/x)/x \tag{5}$$

$$< 0. \tag{6}$$

To get from (1) to (2), we use $f'(\sigma(x)) = v_1(x, 1)$; from (3) to (4), we use Euler's theorem and that $v_1$ is homogeneous of degree 0; from (5) to (6), we use $\sigma'(x) > 0$. □

Proof of Proposition 2: Sufficiency. In the following discussion we prove that there exists a reciprocal equilibrium $t$ with $g(t) = g$ and, hence, $g \in G^R$ if $g$ is connected and is such that for every $U \subseteq N$, condition 2(i) or 2(ii) is satisfied.

For the proof, we draw on Theorem 35.1 in Schrijver (2003, p. 584), which states necessary and sufficient conditions for a perfect $b$-matching to exist for a graph $g$. A *perfect $b$-matching* for $g$ is a function that assigns a value to each link such that the sum of the values of links incident at one node is equal to the *$b$-value* of that node.

We first show that if a perfect $b$-matching for a connected graph $g$ exists, then a reciprocal equilibrium $t$ with $g(t) = g$ exists, and, second, that if $g$ is connected and is such that for all $U \subseteq N$, either condition 2(i) or 2(ii) is true, then a perfect $b$-matching for $g$ exists (for the second part, we use the theorem in Schrijver (2003)). This proves sufficiency for Proposition 2.

Consider a graph $g$ on $N$. Let $E^g$ be the set of all links in $g$ and let $E^g[X, Y]$ be the set of links $xy \in g$ with $x \in X \subseteq N$, $y \in Y \subseteq N$, and $X \cap Y = \varnothing$. Let $E^g[Y]$ be the set of links $ij \in g$ with $i, j \in Y \subseteq N$. Denote by $\delta(i)$ the set of links incident at node $i \in N$. Let $g[Y]$ be the subgraph induced in $g$ by $Y \subseteq N$. For every vector $w \in \mathbb{R}^Y$ with vector components $w_y$, let $w(U) := \sum_{y \in U} w_y$ for any $U \subseteq Y$. The set of integers is denoted by $\mathbb{Z}$.

Considering just a special case, Theorem 35.1 in Schrijver (2003, p. 584) can be reduced to the following statement.

Special case of Theorem 35.1 in Schrijver (2003, p. 584). *Let $g$ be a graph on $N$, and let $b \in \mathbb{Z}^N$ and $c \in \mathbb{Z}^{E^g}$ with every $c_{ij} > 1$. Then there exists an $x \in \mathbb{Z}^{E^g}$ such that (i) $1 \leq x_{ij} \leq c_{ij}$ for all $ij \in E^g$ and (ii) $x(\delta(i)) = b_i$ for all $i \in N$ if and only if for each partition $\{T, V, Y\}$ of $N$, the number of components $K$ of $g[T]$ with*

$$b(K) + c\big(E^g[K, Y]\big) + \big|E^g[K, V]\big| \tag{35.2}$$

*odd is at most*

$$b(V) - 2\big|E^g[V]\big| - \big|E^g[T, V]\big| - b(Y) + 2c\big(E^g[Y]\big) + c\big(E^g[T, Y]\big). \tag{35.3}$$

Let every $c_{ij} = \gamma$ with $\gamma$ extremely large and every $b_i = \beta$ with $\beta$ sufficiently large. If $g$ is connected and $x$ given $g$ exists, then a reciprocal equilibrium $t$ with $g(t) = g$ is such that $t_{ij} = t_{ji} = (x_{ij}/\beta)(T - t_{bb})$ for all $ij \in E^g$ and $t_{ij} = t_{ji} = 0$ for all $ij \notin E^g$.

Let $g$ be connected and such that for all $U \subseteq N$, with $W(U)$ being the set of isolates in $g[N \backslash U]$, either

(a) $|U| > |W(U)|$ or

(b) $|U| = |W(U)|$ and for every link $ij \in E^g$, if $i \in U$, then $j \in W(U)$.

We next show by contradiction that $x$ exists given $g$.

Suppose $x$ does not exist. Then, by the theorem in Schrijver (2003, p. 584), there must be a partition $\{T, V, Y\}$ of $N$ such that the number of components $K$ of $g[T]$ with (35.2) odd is greater than (35.3); otherwise $x$ would exist.

For any partition with $E^g[Y] \neq \varnothing$ and/or $E^g[T, Y] \neq \varnothing$, the number of components $K$ with (35.2) odd is always smaller than (35.3) because $\gamma$ is extremely large and the number of components $K$ is finite. Then there must be a partition with $E^g[Y] = E^g[T, Y] = \varnothing$ with a number of components $K$ with (35.2) odd greater than (35.3).

For every partition $\{T, V, Y\}$ with $E^g[Y] = E^g[T, Y] = \varnothing$, it must be true that every $i \in Y$ has links to nodes in $V$ only and that every $i \in Y$ has at least one link to nodes in $V$ because $g$ is connected. Then $Y$ is a subset of the set of isolates in $g[N \setminus V]$. Hence, $Y \subseteq W(U)$ for $U = V$. We know that in $g$ for all $U \subseteq N$, either (a) $|U| > |W(U)|$ or (b) $|U| = |W(U)|$ and for every link $ij \in E^g$, if $i \in U$, then $j \in W(U)$. This implies that, for any $V$, either (a) $|V| > |Y|$ or (b) $|V| = |Y|$ and for every link $ij \in E^g$, if $i \in V$, then $j \in Y$. Thus, there does not exist a partition $\{T, V, Y\}$ of $N$ for which $E^g[Y] = E^g[T, Y] = \varnothing$ and $|V| < |Y|$.

Then there must be a partition $\{T, V, Y\}$ of $N$ for which $E^g[Y] = E^g[T, Y] = \varnothing$ and $|V| \geq |Y|$ such that the number of components $K$ with (35.2) odd is greater than (35.3).

For any partition with $E^g[Y] = E^g[T, Y] = \varnothing$ and $|V| > |Y|$, the number of components $K$ with (35.2) odd is always smaller than (35.3) because $\beta$ is chosen sufficiently large.

For any partition with $E^g[Y] = E^g[T, Y] = \varnothing$ and $|V| = |Y|$, we know that for every link $ij \in E^g$, if $i \in V$, then $j \in Y$. (The reason is that if $|U| = |W(U)|$, then for every link $ij \in E^g$ with $i \in U$, it is true that $j \in W(U)$, and for $U = V$ in this case $W(U) = Y$.) Then $E^g[V] = E^g[T, V] = E^g[T, Y] = \varnothing$. This implies that $T = \varnothing$. If $T$ were not empty, nodes in $T$ would not be connected to either $V$ or $Y$, and $g$ would not be connected, a contradiction. From $T = \varnothing$, it follows that the number of components $K$ is zero. Expression (35.3) is also zero. Hence, the number of components $K$ is not greater than (35.3).

Thus, there does not exist any partition $\{T, V, Y\}$ of $N$ such that the number of components $K$ with (35.2) odd is greater than (35.3). This is a contradiction and, therefore, $x$ must exist. Thus, there also exists a reciprocal equilibrium $t$ with $g(t) = g$. □

Proof of Proposition 6(b). First, suppose that $q^t = 1$. Then $g(t) \in G^R$. The set $G^R$ is independent of $T$ by Proposition 2. Thus $g(\hat{t}) \in \hat{G}^R$ and $\hat{q}^t = 1$. By Proposition 1, $f'(t_{bb}) = av_1(1, 1)$. Thus, $t_{bb}$ is independent of $T$ and, therefore, $t_{bb} = \hat{t}_{bb}$.

Second, suppose that $q^t > 1$. Then $g(t) = g(\hat{t}) \notin G^R = \hat{G}^R$ and, thus, $\hat{q}^t > 1$. By Proposition 3, the bipartition of $g(t)$ is unique and, therefore, $|C|/|D| = |\hat{C}|/|\hat{D}|$.

By Propositions 1 and 3, (i) $f'(\hat{t}_{cc}) = av_1(\hat{q}^t, 1)$, (ii) $f'(\hat{t}_{dd}) = av_1(1/\hat{q}^t, 1)$, and (iii) $|C|/|D| = \hat{q}^t(\hat{T} - \hat{t}_{dd})/(\hat{T} - \hat{t}_{cc})$. The analogous conditions hold for $t$. Applying the implicit function theorem to the equation $|C|/|D| = q^t(T - t_{dd})/(T - t_{cc})$, we get

$$\begin{aligned}\partial q^t/\partial T = -\big[\big(\big(T-\sigma\big(q^t\big)\big) - \big(T-\sigma\big(1/q^t\big)\big)\big)/\big(T-\sigma\big(q^t\big)\big)^2\big] \\ \times \big[\big(T-\sigma\big(1/q^t\big)\big)/\big(T-\sigma\big(q^t\big)\big) + q^t\big(\sigma'\big(1/q^t\big)\big(T-\sigma\big(q^t\big)\big)/{q^t}^2 \\ + \big(T-\sigma\big(1/q^t\big)\big)\sigma'\big(q^t\big)\big)/\big(T-\sigma\big(q^t\big)\big)^2\big]^{-1}.\end{aligned}$$

It is straightforward to show—keeping in mind that $\sigma$ is strictly increasing—that the numerator is negative and the denominator is positive, and, therefore, $\partial q^t/\partial T > 0$.

Thus, $\hat{q}^t > q^t$, $\hat{t}_{cc} > t_{cc}$ by (i) and $\hat{t}_{dd} < t_{dd}$ by (ii). Hence $\hat{T} - \hat{t}_{dd} > T - t_{dd}$. From $\hat{q}^t > q^t$, it follows that

$$(\hat{T} - \hat{t}_{cc})/(\hat{T} - \hat{t}_{dd}) > (T - t_{cc})/(T - t_{dd})$$

and, therefore, $\hat{T} - \hat{t}_{cc} > T - t_{cc}$. □

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