Theoretical Economics 16 (2021), 101–128 1555-7561/20210101

Voting in corporations

Alan D. Miller
Faculty of Law, Western University

I introduce a model of shareholder voting. I describe and provide characterizations of three families of shareholder voting rules: ratio rules, difference rules, and share majority rules. The characterizations rely on two key axioms: merger consistency, which requires consistency in voting outcomes following stock-for-stock mergers, and reallocation invariance, which requires the shareholder voting rule to be immune to certain manipulative techniques used by shareholders to hide their ownership. The paper also extends May's theorem.

Keywords. Shareholder voting, axioms, share majority rule, merger consistency, reallocation invariance, one share–one vote, difference rules, ratio rules.

JEL classification. D71, G34, K22.

1. Introduction

I introduce a model of shareholder voting; that is, voting by individuals with ownership stakes in a corporation. I then use the model to define and characterize three important families of shareholder voting rules: ratio rules, difference rules, and share majority rules. In doing so, I provide normative justifications for the "one share–one vote" principle, according to which each shareholder receives a number of votes proportional to the size of her holding.

The shareholder franchise is understood to be an essential element of corporate governance. Corporations are owned by shareholders, and shareholders exercise their power through voting. They vote to elect the board of directors, which runs the corporation directly. They vote to approve major corporate decisions such as mergers and

Alan D. Miller: alan.miller@uwo.ca
For their comments, I would like to thank two anonymous referees, as well as Marco Becht, Bernie Black, Chris Chambers, Itzhak Gilboa, Lewis Kornhauser, Kate Litvak, Hervé Moulin, Murat Mungan, Ilan Nehama, Christopher Nicholls, Alessio Pacces, and participants at seminars at the Technion, the Hebrew University, George Mason University, Virginia Tech, the University of Haifa, Western University, the University of Cyprus, Bar Ilan University, the University of Glasgow, the Université Libre du Bruxelles, Tel Aviv University, the Bucerius Law School, the University of Guelph, and at the 34th Annual Conference of the European Association of Law and Economics, the 2018 Meeting of the American Law and Economics Association, the Eighth Annual Law and Economic Theory Conference, the Haifa Social Choice Workshop, the 19th SAET Conference on Current Trends in Economics, and the CRM Workshop on Voting Systems. This research was undertaken, in part, thanks to funding from the Canada Research Chairs program.

© 2021 The Author. Licensed under the Creative Commons Attribution-NonCommercial License 4.0.
Available at https://econtheory.org. https://doi.org/10.3982/TE3668

acquisitions, equity issuances, and executive compensation plans.[1] They vote on shareholder resolutions.

The most common rule is that shareholders receive one vote per share owned[2] and that shareholder votes are decided according to the majority (or supermajority) of votes cast.[3] However, this is not the only possibility. In the early nineteenth century, shareholders often received one vote regardless of the number of shares owned (Ratner 1970).[4] Today, many corporations issue multiple classes of voting stock to allow the founders to sell their shares without losing control of the corporation or to provide extra voting power to long-term investors. Others use "voting rights ceilings" to limit the voting power of larger shareholders. Posner and Weyl (2014) propose "square-root voting," under which a shareholder receives a number of votes proportional to the square root of her holdings.

The positive effects of the one share–one vote rule have been studied extensively, including by Grossman and Hart (1988) and Harris and Raviv (1988), who analyze conditions under which a single class of equity stock and majority voting are optimal, and Ritzberger (2005), who provides conditions under which pure-strategy equilibria exist.[5] However, despite the large literature in social choice theory devoted to voting and despite the economic importance of the rules of corporate governance, to my knowledge there has never been a model to evaluate the normative properties of shareholder voting rules.[6]

This paper contributes to the literature on shareholder voting in several ways. First, it introduces a formal model of shareholder voting, through which different voting rules can be compared and evaluated in terms of their normative characteristics. Second,

---

[1]The specific decisions on which shareholders vote vary widely across jurisdictions. For more on acquisitions, see Becht et al. (2016). For more on equity issuances, see Holderness (2018). There is a growing movement to require shareholder approval of executive compensation plans known as "say on pay"; for more, see Thomas and der Elst (2015).

[2]The Delaware General Corporation Law provides, as a default, that "[u]nless otherwise provided in the certificate of incorporation and subject to § 213 of this title, each stockholder shall be entitled to 1 vote for each share of capital stock held by such stockholder." 8 Del. C. 1953, § 212(a).

[3]The default is generally majority rule. For example, when a firm does not specify otherwise in its certificate of incorporation or bylaws, the Delaware code provides that "[i]n all matters other than the election of directors, the affirmative vote of the majority of shares present in person or represented by proxy at the meeting and entitled to vote on the subject matter shall be the act of the stockholders." 8 Del. C. 1953, § 216(2).

[4]Senator Elizabeth Warren's proposed Accountable Capitalism Act would require corporations to obtain shareholder consent, as determined by the use of a one person–one vote shareholder voting rule, before making certain corporate expenditures. Accountable Capitalism Act, S. 3348, 115th Cong. § 8(b) (2018).

[5]A justification for two-class voting is provided by Maug and Yilmaz (2002). For surveys of the theoretical and empirical literature, see Burkart and Lee (2008) and Adams and Ferreira (2008), respectively. For more on shareholder voting generally, see Easterbrook and Fischel (1983) and Thompson and Edelman (2009). The problem is also considered in Barzel and Sass (1990).

[6]Of course, Grossman and Hart (1988), Harris and Raviv (1988), Maug and Yilmaz (2002), and related papers may be understood as making normative claims in terms of consequentialist welfare. The relevance of social choice to shareholder voting has been recognized; see, for example, Easterbrook and Fischel (1983). Nitzan and Procaccia (1986) applies results in aggregation theory (Nitzan and Paroush 1984) to corporate voting, but does not explicitly model shareholdings.

the paper identifies several axioms that represent normatively desirable properties of shareholder voting rules. Third, it defines three important families of shareholder voting rules that implement the one share–one vote principle. Fourth, the paper characterizes these families of rules in terms of axioms, and extends the classic result of May (1952) on majority rule.

In the model, there is a set of shareholders, each of whom has a preference on a shareholder resolution and each of whom owns a portion of the firms' common stock.[7] One may think of this as a model of weighted voting among two alternatives. Preferences are defined with respect to a binary decision: individuals may favor or oppose the resolution, or they may be indifferent between these two alternatives. A shareholder voting rule takes into account the preferences of the individuals and their shareholdings, and then uses this information to determine whether the resolution passes or fails, or whether the vote results in a tie.

This model of a binary choice can be justified by the view that shareholders have a common interest in maximizing profits and that voting serves the purpose of error correction, as suggested by Thompson and Edelman (2009), or by the fact that, in practice, most shareholder votes concern only binary decisions. The possibility of a tied outcome represents the idea that a voting rule need not always provide an answer to every question. This follows Arrow (1963) and May (1952), each of whom worked in a setting of weak preference. One might question the relevance of ties in the corporate setting; arguably, shareholder resolutions must either pass or fail. However, a tie may be interpreted as a a third outcome, different from either the passing or complete failure of a resolution. For example, under rules promulgated by the Securities and Exchange Commission (SEC), it is easier to reintroduce a failed shareholder resolution if it receives a certain percentage of the vote.[8] Similarly, the U.K. Corporate Governance Code places disclosure obligations on firms when more than 20% of votes are cast against a resolution.[9] The possibility of ties can easily be eliminated by imposing a "no-tie" axiom; the implications of such an axiom on the results are straightforward.

The results in this paper revolve around two key axioms: *merger consistency* and *reallocation invariance*. The merger consistency axiom requires a certain type of consistency in connection with votes related to stock-for-stock mergers. Consider a thought experiment in which there are two firms (firm A and firm B) that plan to merge in a stock-for-stock transaction (forming firm A+B). The shareholders must approve a resolution;

---

[7] That is, this paper considers a firm with a single class of stocks and studies why it would be desirable for that firm to give each shareholder one vote per share or, equivalently, why control rights in a corporation should be allocated proportionately to cash flow rights. The model assumes, implicitly, that the outcome of the vote depends only the shareholders' names, preferences, and proportions of the cash flow rights. This assumption is limiting in that it does not allow us to address the desirability of dual-class stock, of other voting rules that depend on more than the distribution of cash-flow rights among shareholders, or of giving control rights to holders of other securities, such as bonds or preferred stock. Methods by which the model can be generalized to allow multiple classes of stock are discussed in the conclusion.

[8] See SEC Rule 14a-8(i)(12).

[9] The U.K. Corporate Governance Code (2018), Provision 4. Under this particular interpretation, the strategyproofness axiom would not be appropriate.

for example, the resolution might be to approve the post-merger executive compensation plan.[10] One may assume that preferences are formed in anticipation of the merger and, thus, do not change, and that no shares are sold in the interim.[11] The managements of firms A and firm B could ask their shareholders to approve the compensation plan before the merger or they could agree to wait and have the shareholders of firm A+B approve the compensation plan afterward. If ex ante voting leads to a predictably different result than ex post voting, we can imagine that the management might strategically choose the alternative that is best for their interests. To eliminate this possibility, the merger consistency axiom requires that, if the outcome of the shareholder vote held in firm A is the same as the outcome of the shareholder vote held in firm B, then this must also be the outcome of a (hypothetical) shareholder vote in the combined firm A+B. The merger consistency axiom is formally related to the consistency axiom of Young (1974, 1975).

The reallocation invariance axiom is motivated by the idea that individuals may be able to manipulate the identity of their shares' owners to the extent that ownership is relevant as far as voting rights are concerned.[12] For example, if large blocks of shares were to receive disproportionately strong voting rights, like-minded shareholders may be able to combine their shares into a holding company, which becomes the sole owner of the shares.[13] The shareholders would then receive stock in the holding company. The transaction could be structured so that these shareholders could leave the holding company and take their stock with them in case that they wish to sell it or wish to vote differently from their fellow holding company participants. Alternatively, if small blocks of shares were to receive disproportionately strong voting rights, then a larger shareholder could partition her shares into several holding corporations. These are but a few of a wide variety of techniques that can be used to disguise the true ownership of the shares; for more, see Hu and Black (2005, 2006, 2007, 2008). For more on the effect of this vote trading on information aggregation, see Christoffersen et al. (2007) and Esö et al. (2015). Reallocation invariance is formally related to the no advantageous reallocation axiom introduced by Moulin (1985, 1987) in the context of bargaining and cost-sharing problems.

These axioms are then combined with several others to establish several results. The *repurchase invariance* axiom requires that the outcome of the vote not be affected by the

[10]Other examples include the possibilities that the resolution is to approve the merger or to issue stock in the combined firm.

[11]That is, the axiom applies only in the (possibly rare) case where the agents' preferences are not affected by the merger and shares are not traded. If either of these assumptions is violated, the axiom is silent.

[12]This was perceived as a problem as far back as the eighteenth century, when the Parliament of the United Kingdom attempted to stop it by forbidding shareholders from voting unless they possessed stock for a period of six months prior to the vote. *Public Companies Act 1767*, 7 Geo III, c 48. This form of manipulation may have led to the adoption of the one share–one vote standard in the nineteenth century (see Hilt 2013).

[13]Depending on its size, such a transaction may trigger SEC reporting requirements.

corporate repurchases of stock held by an indifferent shareholder.[14] The *cancellation* axiom (see Young 1974, 1975) requires that two shareholders with equal shareholdings and opposed preferences will cancel out each other's vote. The *anonymity* axiom (May 1952) requires the shareholder voting rule to treat each voter equally; it accomplishes this by requiring the result to be invariant to changes in the names of the individuals. The *unanimity* axiom (called *weak Pareto* in Arrow 1963) requires the resolution to pass when all shareholders are in favor and to fail when all are opposed. The *strategyproofness* axiom (see Dummett and Farquharson 1961, Gibbard 1973, Satterthwaite 1975) requires that shareholders not be able to benefit by misrepresenting their preferences. It ensures that interested shareholders will not make the strategic choice to pretend to be indifferent or have the opposite interest.

There are five pairs of dual results; each pair contains one result with merger consistency and one result with reallocation invariance. First, a shareholder voting rule is defined as *minimally one share–one vote* if it is a function of two numbers: the proportion of shares owned by supporters of the resolution and the proportion of shares owned by opponents. This definition reflects the idea that one share–one vote rules involve the counting of votes, and it rules out voting methods such as one person–one vote, square-root voting, and voting rights ceilings. The minimally one share– vote property is equivalent to reallocation invariance and is implied by the combination of merger consistency and anonymity.

Second, the paper defines and characterizes three families of shareholder voting rules: ratio rules, difference rules, and share majority rules. *Ratio rules* determine the outcome of the vote on the basis of the ratio of (i) the number of shares owned by supporters to (ii) the number of shares owned by opponents. *Difference rules* determine the outcome of the vote on the basis of the difference between these two quantities. *Share majority rules* determine the outcome based on the sign of the difference, i.e., whether supporters own more shares than the opponents or vice versa. These are the families of rules most closely connected with the idea of "one share–one vote," and in this sense the axiomatic characterizations provide normative justifications for this principle.

Third, the paper revisits May (1952), which characterized a form of one person–one vote using three axioms: anonymity, neutrality, and positive responsiveness. These axioms are used to characterize the share majority rule with ties, and in this sense the paper extends this classic result to the context of shareholder voting.

1.1 *Related literature*

Young (1974, 1975) characterizes Borda rules using several axioms including consistency and cancellation. The consistency axiom is studied extensively in the literature (see Smith 1973, Fine and Fine 1974a, 1974b), and is sometimes referred to as "reinforcement" (see Moulin 1988, Duddy et al. 2016, Brandt et al. 2016). It differs from merger consistency in two key respects. First, it applies only to combinations of disjoint sets of

[14]The axiom does not suggest that firms take the preferences of a shareholder into account when repurchasing shares, but only that if the firm was to repurchase shares and if the repurchase is done from a shareholder who is indifferent, then the repurchase should not affect the outcome of the vote.

voters; the merger consistency axiom, by contrast, applies even if the sets of shareholders overlap. Second, the consistency axiom requires that when one group is indifferent and the other group has a strict preference, the combined group must follow the strict preference; merger consistency requires nothing in this specific case. Young's cancellation axiom also differs in that it applies only to profiles that are completely balanced, so that there are an equal number of supporters and opponents. (Neither indifference nor weights play a role in his model.) In the two person case, the Borda rule is equivalent to the form of majority rule in which each voter gets one vote. The merger consistency and cancellation axioms are used in this paper to characterize difference rules and share majority rules, in which each voter gets a number of votes equal to the size of her shareholdings.

Shareholder voting is one of several settings in which voters with externally determined weights need to decide on a binary issue. For example, member states of the European Union differ in the size of their state populations; it has long been recognized that these differences in populations are relevant to the design of a voting rule. A naive approach would be to use a form of weighted majority rule where weights are equal to each state's share of the population. However, as recognized by Banzhaf (1964), such a rule would give a disproportionate amount of power to large states. In practice, the Council of the European Union uses a form of weighted majority rule where the weights were chosen according to a political compromise. There is a significant body of research on weighted voting and voting power (for more, see Taylor and Zwicker 1992, Felsenthal and Machover 1998, Chalkiadakis and Wooldridge 2016), but it is possible to go further and ask whether other voting rules could be used.

Another example involves the study of collective welfare with interpersonal comparisons of utility (see Roberts 1980, Eguia and Xefteris 2019). In this context, the external weight is the strength of preference intensity, rather than the size of a shareholding. Utilitarianism (see Harsanyi 1955) may be thought of as the equivalent of the share majority rule with ties. Roberts (1980) and Eguia and Xefteris (2019) offer characterizations of "exponent rules" in this setting (see Section 2.5.1 below).

2. The model

Let $\mathbb{N}$ be the set of all possible shareholders and let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. Let $\mathcal{R} \equiv \{-1, 0, 1\}$ be a set of preferences with preferences R_i. For a set $N \in \mathcal{N}$, let $\mathbf{x} \in \Delta(N)$ be a distribution of shares.[15] For $N \in \mathcal{N}$, let $\mathcal{Q}_N \equiv \mathcal{R}^N \times \Delta(N)$. The class of problems is the set $\mathcal{Q} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{Q}_N$.

For $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $N' \subseteq N$ for which $\sum_{i \in N'} \mathbf{x}_i = 1$, let $(R, \mathbf{x})|_{N'}$ denote the restriction of $(R, \mathbf{x})$ to $\mathcal{Q}_{N'}$. A function $f : \mathcal{Q} \to \mathcal{R}$ is *invariant to nonshareholders* if for $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $\sum_{i \in N'} \mathbf{x}_i = 1$ for $N' \subseteq N$ implies that $f(R, \mathbf{x}) = f((R, \mathbf{x})|_{N'})$. A *shareholder voting rule* is a function $f : \mathcal{Q} \to \mathcal{R}$ that is invariant to nonshareholders.

[15]The results would hold if all $\mathbf{x}_i$ were required to be rational.

The main results rely on seven axioms. The first axiom, merger consistency, requires a certain type of consistency in merged firms, as described in the Introduction. The parameter λ represents the portion of the new firm that will be owned by the shareholders of first firm, while $1-\lambda$ represents the portion that will be owned by the shareholders of the second firm. Because the model allows for non-shareholders, the axiom can be limited to the case where the sets of shareholders are the same.

Merger consistency. For $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $\lambda \in (0,1)$, if $f(R, \mathbf{x}) = f(R, \mathbf{x}')$, then $f(R, \mathbf{x}) = f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}')$.

The second axiom, reallocation invariance, is motivated by the idea that individuals may be able to manipulate the identity of their shares' owners to the extent that ownership affects voting rights. Several techniques that can be used to accomplish this result are described in the Introduction. Formally, if there is a group $S \subseteq N$ of like-minded individuals (so that $R_j = R_k$ for all $j, k \in S$) and there are no changes in the ownership of stock among individuals outside of this group (so that $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \notin S$), then the outcome of the vote must not change. (In other words, the outcome must not depend on how shares are allocated among the members of S.)

Reallocation invariance. For $N \in \mathcal{N}$, $(R, \mathbf{x})$, $(R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$, if $R_j = R_k$ for all $j, k \in S$ and $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \notin S$, then $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

The third axiom, repurchase invariance, requires that the outcome of the vote be invariant to the corporate repurchase of stock held by an indifferent shareholder. When a firm repurchases stock and places it in the corporate treasury, the number of outstanding shares decreases. As a consequence, each remaining share has a greater claim on the assets of the firm. In the formal definition of the axiom, $\mathbf{x}$ refers to the allocation of shares before the repurchase and $\mathbf{x}'$ refers to the allocation of shares after all indifferent shares have been purchased.[16] Note that the axiom only applies in the case where $\sum_{i:R_i \neq 0} \mathbf{x}_i > 0$; that is, some shares are held by non-indifferent shareholders.

Repurchase invariance. For every $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_{i:R_i \neq 0} \mathbf{x}_i > 0$, if for all $j \in N$ such that $R_j \neq 0$, $\mathbf{x}_j = \mathbf{x}'_j(\sum_{i:R_i \neq 0} \mathbf{x}_i)$, then $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

The fourth axiom, cancellation, requires that supporters and opponents with equal shareholdings cancel each other out. I implement the concept of "cancel each other out" by requiring the result of the vote to be invariant to whether the supporter and opponent were to both become indifferent. Formally, let R be the profile in which shareholders j and k have opposite views, and let R' be the profile where both are indifferent.

Cancellation. For $N \in \mathcal{N}$, $j, k \in N$, and $(R, \mathbf{x})$, $(R', \mathbf{x}) \in \mathcal{Q}_N$ if $R_j = -R_k$, $R'_j = R'_k = 0$, $\mathbf{x}_j = \mathbf{x}_k$, and $R_\ell = R'_\ell$ for $\ell \neq j, k$, then $f(R, \mathbf{x}) = f(R', \mathbf{x})$.

[16] For simplicity, I assume that the corporation has repurchased the shares of all indifferent shareholders. This simplification is without loss of generality as the set of shareholders is finite.

The fifth axiom, anonymity, requires that the result of the vote be independent of the names of the shareholders. For $N \in \mathcal{N}$, let Π_N refer to the set of permutations of N. For $\pi \in \Pi_N$, define $\pi R \equiv (R_{\pi(1)}, \ldots, R_{\pi(n)})$ and $\pi \mathbf{x} \equiv (\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(n)})$.

Anonymity. For every $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$, $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

Reallocation invariance implies anonymity. The proof of the following lemma is given in the Appendix.

Lemma 1. *A shareholder voting rule satisfies reallocation invariance only if it satisfies anonymity.*

The sixth axiom, unanimity, requires that a resolution must pass when all shareholders support it and must fail when it is opposed by all.

Unanimity. For every $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, if there exists $\kappa \in \{1, -1\}$ such that $R_i = \kappa$ for all $i \in N$, then $f(R, \mathbf{x}) = \kappa$.

The seventh axiom, strategyproofness, requires that an individual shareholder not be able to benefit from misrepresenting her preferences. For $i \in N$, $R \in \mathcal{R}^N$, and $\kappa \in \mathcal{R}$, let $[R_{-i}, \kappa] \equiv (R_1, \ldots, R_{i-1}, \kappa, R_{i+1}, \ldots, R_n)$.[17]

Strategyproofness. For $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $\kappa \in \mathcal{R}$, neither $R_i \leq f([R_{-i}, \kappa], \mathbf{x}) < f(R, \mathbf{x})$ nor $R_i \geq f([R_{-i}, \kappa], \mathbf{x}) > f(R, \mathbf{x})$.

2.1 *One share–one vote*

As explained in the Introduction, a necessary (but not sufficient) condition for a shareholder voting rule to qualify as one share–one vote is that it be a function of two numbers: the proportion of votes owned by supporters and the proportion of votes owned by opponents.

Let $\sigma : \mathcal{Q} \to \Delta(\mathcal{R})$ be such that, for all $\kappa \in \mathcal{R}$, $\sigma_\kappa(R, \mathbf{x}) \equiv \sum_{i: R_i = \kappa} \mathbf{x}_i$. Let $\mathcal{G}$ be the set of all functions $\mathbf{g} : \Delta(\mathcal{R}) \to \mathcal{R}$.

Minimally One Share–One Vote. There exists a function $\mathbf{g} \in \mathcal{G}$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$.

I prove two results. First, a necessary and sufficient condition for a shareholder voting rule to be minimally one share–one vote is that it satisfies reallocation invariance.

Proposition 1. *A shareholder voting rule f is minimally one share–one vote if and only if it satisfies reallocation invariance.*

Second, a sufficient condition for a shareholder voting rule to be minimally one share–one vote is that it satisfies both merger consistency and anonymity. Neither condition is sufficient on its own.

[17] All results in this paper would hold if this axiom was weakened to apply only in the case where a shareholder pretends to be indifferent; that is, where $\kappa = 0$.

Proposition 2. *A shareholder voting rule f is minimally one share–one vote if it satisfies merger consistency and anonymity.*

The proofs of these two results are provided in the Appendix.

2.2 *Ratio rules*

The first family of shareholder voting rules that I introduce is the ratio rules, so called because they determine the outcome of the vote on the basis of the ratio of the number of shares held by supporters of the resolution, $\sigma_1(R, \mathbf{x})$, to the number of shares held by opponents, $\sigma_{-1}(R, \mathbf{x})$.

Ratio rules are defined by a function and a parameter. The function ϕ maps from this ratio (more precisely, from $\sigma_1(R, \mathbf{x})[\sigma_1(R, \mathbf{x}) + \sigma_{-1}(R, \mathbf{x})]^{-1}$, the proportion of shares held by non-indifferent shareholders that are voted in favor of the resolution) to outcomes (elements of $\mathcal{R}$: pass, fail, or tie). The function ϕ must be monotonic and respect a basic unanimity condition. Formally, let Φ be the set of monotonically increasing functions $\phi : [0, 1] \to \mathcal{R}$ such that $\phi(0) = -1$ and $\phi(1) = 1$.

The ratio (and the proportion) is not well defined in the special case where all shareholders are indifferent, that is, when $\sigma_1(R, \mathbf{x}) = \sigma_{-1}(R, \mathbf{x}) = 0$. The parameter, $\kappa \in \mathcal{R}$, describes the default outcome in this special case. All outcomes are possible—that is, the resolution may pass, fail, or tie—but a ratio rule must treat all cases of complete indifference the same way, regardless of the identities of the shareholders or the distribution of the shares between them.

A ratio rule is pictured in Figure 1. The top of the figure explains the result in the case when all shares are held by indifferent shareholders. In this example, $\kappa = -1$; consequently, complete indifference leads to the failure of the resolution. Below is a representation of a function ϕ; one can see that the function has three levels (passes, ties, and fails), and that, as we move from left to right, it starts as failing (at 0%), ends as passing (at 100%), and is weakly increasing. As a consequence, there must be one or

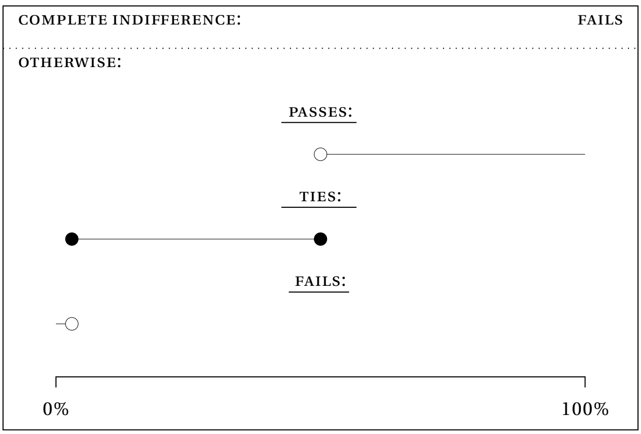

Figure 1. Ratio rules.

two points of discontinuity where the function "jumps" to a higher state. The depicted function jumps twice, once at 3% and once again at 50%. I have marked the points of discontinuity with closed and open circles to indicate what happens when the proportion of non-indifferent shares voted in favor of the resolution falls exactly on one of these points. Thus, according to the depicted rule, the resolution fails if the proportion of non-indifferent shares voted in favor of the resolution is less than 3%, passes if the proportion of non-indifferent shares voted in favor of the resolution is greater than 50%, and otherwise results in a tie. This example describes majority rule subject to SEC Rule 14a-8(i)(12), where a tie represents the outcome under which a resolution fails but is easier to reintroduce.

To simplify the definition of ratio rules, I make use of the fact that $\sigma_1(R,\mathbf{x}) + \sigma_{-1}(R,\mathbf{x}) = 1 - \sigma_0(R,\mathbf{x})$.

Ratio rules. A shareholder voting rule f is a *ratio rule* if there exist $\phi \in \Phi$ and $\kappa \in \mathcal{R}$ such that, for all $(R,\mathbf{x}) \in \mathcal{Q}$, $f(R,\mathbf{x}) = \phi(\frac{\sigma_1(R,\mathbf{x})}{1-\sigma_0(R,\mathbf{x})})$ if $\sigma_0(R,\mathbf{x}) < 1$ and $f(R,\mathbf{x}) = \kappa$ otherwise.

I provide two characterizations of the ratio rules. The first theorem states that the ratio rules comprise the family of rules satisfying merger consistency, anonymity, unanimity, and repurchase invariance.

Theorem 1. *A shareholder voting rule satisfies merger consistency, anonymity, unanimity, and repurchase invariance if and only if it is a ratio rule.*

The second theorem states that the ratio rules comprise the family of rules satisfying reallocation invariance, unanimity, repurchase invariance, and strategyproofness.

Theorem 2. *A shareholder voting rule satisfies reallocation invariance, unanimity, repurchase invariance, and strategyproofness if and only if it is a ratio rule.*

The proofs of Theorems 1 and 2 are provided in the Appendix. The collections of axioms in all theorems formally stated in this paper are independent. The proof of this fact is left as an exercise for the reader.

2.3 *Difference rules*

The second family of shareholder voting rules that I introduce is the difference rules, so called because they determine the outcome of the vote on the basis of the difference between the number of shares held by supporters and the number of shares held by opponents, i.e., $\sigma_1(R,\mathbf{x}) - \sigma_{-1}(R,\mathbf{x})$.

Difference rules are defined by a function ψ that maps from this difference to outcomes. As before, the function ψ must be monotonic and respect a basic unanimity condition. Formally, let Ψ be the set of monotonically increasing functions $\psi : [-1,1] \to \mathcal{R}$ such that $\psi(-1) = -1$ and $\psi(1) = 1$.

A difference rule is depicted in Figure 2. Note that this figure looks very similar to the depiction of a ratio rule in Figure 1, but with two exceptions. First, the rule does not

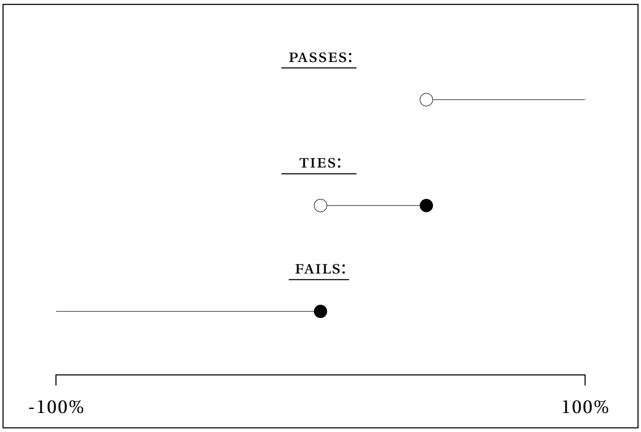

FIGURE 2. Difference rules.

specify what happens in the case of complete indifference. This is not necessary as the difference $\sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x})$ is well defined in this case (and is equal to zero). Second, the domain of the function ranges from -100% (the case where all shares are held by opponents) to 100% (the case where all shares are held by supporters).

Difference rules. A shareholder voting rule f is a *difference rule* if there exists $\psi \in \Psi$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \psi(\sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x}))$.

I provide two characterizations of the difference rules. The first theorem states that the difference rules are the family of rules satisfying merger consistency, cancellation, anonymity, and unanimity.

THEOREM 3. *A shareholder voting rule satisfies merger consistency, cancellation, anonymity, and unanimity if and only if it is a difference rule.*

The second theorem states that the difference rules are the family of rules satisfying reallocation invariance, cancellation, unanimity, and strategyproofness.

THEOREM 4. *A shareholder voting rule satisfies reallocation invariance, cancellation, unanimity, and strategyproofness if and only if it is a difference rule.*

The proofs of Theorems 3 and 4 are provided in the Appendix.

2.4 *Share majority rules*

The most common decision rule used in shareholder voting is the share majority rule, where the winning side is the one with the larger number of shares. There are in fact three such share majority rules; these differ according to the result that they prescribe in the case where the supporters and opponents have (collectively) equal numbers of shares.

Share majority rules. For $\kappa \in \mathcal{R}$ and $(R, \mathbf{x}) \in \mathcal{Q}$,

$$\mathbf{m}^{\kappa}(R, \mathbf{x}) \equiv \begin{cases} 1, & \text{if } R \cdot \mathbf{x} > 0, \\ \kappa, & \text{if } R \cdot \mathbf{x} = 0, \\ -1, & \text{if } R \cdot \mathbf{x} < 0. \end{cases}$$

Share majority rules are both ratio rules and difference rules.[18] In fact, any shareholder voting rule that is both a ratio rule and a difference rule must necessarily be a share majority rule. Theorems 1, 2, 3, and 4 thus imply that every share majority rule must satisfy merger consistency, reallocation invariance, repurchase invariance, cancellation, unanimity, and strategyproofness. The next two theorems demonstrate that a subset of these axioms (repurchase invariance, cancellation, unanimity, and either of merger consistency or reallocation invariance) is sufficient to characterize the share majority rules.

Theorem 5. *A shareholder voting rule satisfies reallocation invariance, repurchase invariance, cancellation, and unanimity if and only if it is a share majority rule.*

Theorem 6. *A shareholder voting rule satisfies merger consistency, repurchase invariance, cancellation, and unanimity if and only if it is a share majority rule.*

The proofs of Theorems 5 and 6 are provided in the Appendix.

2.4.1 *May's theorem* May (1952) characterized majority rule using three axioms: anonymity, neutrality, and positive responsiveness. Anonymity is introduced above. Neutrality requires the shareholder voting rule not to favor the passing of the resolution over its failure. For $R \in \mathcal{R}^N$, define $-R = (-R_1, \ldots, -R_n)$.

Neutrality. For every $(R, \mathbf{x}) \in \mathcal{Q}$, $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

A ratio rule satisfies neutrality if and only if the function ϕ is symmetric around 0.5 and the constant κ is equal to 0. A difference rule satisfies neutrality if and only if the function ψ is symmetric around 0.

The positive responsiveness axiom requires the rule to respond (in a specific way) to changes in the voters' preferences. If a particular resolution does not fail (that is, either it passes or there is a tie) and an individual changes her preference positively (from 0 to 1 or from -1 to 0 or 1), the result is that the resolution now passes.

Positive Responsiveness. For every $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R', \mathbf{x}) \in \mathcal{Q}_N$, if (a) there exists $i \in N$ such that $\mathbf{x}_i > 0$, $R'_i > R_i$, and $R'_j = R_j$ for all $j \neq i$, and (b) $f(R, \mathbf{x}) \neq -1$, then (c) $f(R', \mathbf{x}) = 1$.

[18] That share majority rules are difference rules follows from the fact that $R \cdot \mathbf{x} = \sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x})$. That share majority rules are ratio rules follows from the fact that $\sigma_1(R, \mathbf{x}) > \sigma_{-1}(R, \mathbf{x})$ if and only if $\frac{\sigma_1(R, \mathbf{x})}{1 - \sigma_0(R, \mathbf{x})} > 0.5$.

Positive responsiveness implies strategyproofness. Ratio rules and difference rules satisfy positive responsiveness if and only if the associated functions ϕ and ψ have a single point of discontinuity.

The model of May (1952) did not include shareholdings. To make the models formally equivalent, I introduce a share independence axiom, which requires that among shareholders with positive shareholdings, the distribution of shares is irrelevant.

Share independence. For every $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\mathbf{x}_i > 0$ if and only if $\mathbf{x}'_i > 0$ for all $i \in N$, $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

Ratio rules and difference rules satisfy share independence if and only if the associated functions ϕ and ψ are such that $\phi(z) = 0$ when $z \in (0, 1)$ and $\psi(w) = 0$ when $w \in (-1, 1)$.

May (1952) used these axioms to characterize the voter majority rule, which gives one vote to each shareholder with a positive shareholding, and then applies majority rule to decide the outcome of the vote.

Voter majority rule. For every $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $f(R, \mathbf{x}) = \textsc{sign}(\sum_{i:\mathbf{x}_i>0} R_i)$.

Theorem 7 (May, 1952). *A shareholder voting rule satisfies anonymity, neutrality, positive responsiveness, and share independence if and only if it is the voter majority rule.*

I show that the three axioms—anonymity, neutrality, and positive responsiveness—when combined with either of merger consistency or reallocation invariance, is sufficient to characterize $\mathbf{m}^0$, the share majority rule with ties.

Theorem 8. *A shareholder voting rule satisfies merger consistency, anonymity, neutrality, and positive responsiveness if and only if it is the share majority rule with ties.*

Theorem 9. *A shareholder voting rule satisfies reallocation invariance, neutrality, and positive responsiveness if and only if it is the share majority rule with ties.*

The proofs of Theorems 8 and 9 are provided in the Appendix.

2.5 *Discussion*

If shareholders are never indifferent, ratio rules and difference rules are identical; that is, for every ratio rule we can find an equivalent difference rule and vice versa. These rules differ in how they treat indifference. For example, consider a firm with 10 outstanding shares and a voting rule that states that the resolution passes if supporters own four more shares than opponents, and which otherwise fails. This is a difference rule where $\psi(w) = 1$ if $w \geq 0.4$ and where $\psi(w) = -1$ if $w < 0.4$. When no shareholders are indifferent, this is equivalent to the ratio rule where a resolution passes if supporters own 70% or more of the non-indifferent shares (i.e., $\kappa = -1$, $\phi(z) = 1$ if $z \geq 0.7$, and $\phi(z) = -1$ if $z < 0.7$). Either way, the resolution needs seven shares to pass.

Now consider a case where individuals can be indifferent, for example, let the supporters own five shares, the opponents own two, and the indifferent shareholders own three. In this case, the supporters only own three more shares than the opponents, so according to the difference rule, the resolution fails. However, the supporters own $\frac{5}{7} > 0.7$ of the non-indifferent shares, so according to the ratio rule, it passes.

This is because, under the difference rule, the relative difference becomes more "important" as more individuals become indifferent. In the example, when more than six shares are held by indifferent shareholders, the resolution always fails. One can think of ratio rules as effectively being difference rules where the difference is multiplied by $1 - \sigma_0(R, \mathbf{x})$. In the case of the share majority rules, that difference is zero, and for this reason, it is here that the ratio rules and the difference rules converge.

Whether a ratio rule or a difference rule is more appropriate for a particular firm depends on its specific needs. A ratio rule may be appropriate if the firm's founders believe that indifferent shareholders can be safely ignored, while a difference rule may be more appropriate if the founders are afraid that a small group of shareholders would otherwise determine the result (on matters where everyone else is indifferent). It is possible that a firm may wish to use an intermediate rule, where the required difference is decreasing in the number of indifferent shareholders, but not at the rate of $1 - \sigma_0(R, \mathbf{x})$. There are other ways to treat indifference; for example, it is possible to simply treat indifferent shareholders as if they were opposed. An *absolute rule* is a rule that depends entirely on $\sigma_1(R, \mathbf{x})$, and that is monotonic and satisfies a basic unanimity condition.

Absolute rules. There exists $\phi \in \Phi$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \phi(\sigma_1(R, \mathbf{x}))$.

It is not clear what is the normative justification of these rules, although the family could be characterized by replacing repurchase invariance or cancellation with an axiom that requires indifferent shareholders to be treated as if they were opponents of the resolution. These rules do seem to be used in practice, especially the majoritarian variant of this rule where the resolution passes if $\sigma_1(R, \mathbf{x}) > 0.5$ and otherwise fails.

Another possibility is to require that the vote does not take place unless enough shares are held by non-indifferent shareholders. Firms commonly require the existence of a quorum before a vote can be conducted. In this model, an indifferent shareholder is understood as one who arrives to the vote, but does not have a preference for or against the resolution to be decided on. Normally such shareholders would be counted within the quorum, but a shareholder voting rule could be defined so that a vote cannot take place if more than a certain percentage of shares are held by indifferent shareholders. As defined below, a quorum rule is one in which the outcome is determined by the share majority rule with ties, under the condition that not too many shares are held by indifferent shareholders. Otherwise, the result is a tie.

Quorum rules. There exists an $r \in (0, 1)$ such that $f(R, \mathbf{x}) = \mathbf{m}^0(R, \mathbf{x})$ if $\sigma_0(R, \mathbf{x}) < r$; otherwise $f(R, \mathbf{x}) = 0$.

The quorum rules are minimally one share–one vote rules and thus satisfy reallocation invariance. They do not, however, satisfy merger consistency or repurchase invariance. The latter axiom would eliminate rules that condition the outcome on a minimal proportion of non-indifferent votes.

2.5.1 *Not one share–one vote* Many important voting rules are not minimally one share–one vote. For example, the voter majority rule is *one person–one vote*; this rule, or something like it, was the common law rule used in the nineteenth century until it was replaced by statute (see Dunlavy 2006).[19] Posner and Weyl (2014) propose a new method they call *square-root voting*, where each shareholder receives a number of votes equal to the square root of her holdings. More generally, we may think of rules where shareholdings are transformed by an exponent α, where $\alpha = 0$ is one person–one vote, $\alpha = \frac{1}{2}$ is square-root voting, and $\alpha = 1$ is one share–one vote. Higher values of α give more voting power to larger shareholders. For $\alpha \geq 0$, let $\rho^\alpha(R, \mathbf{x}) \in \Delta(\mathcal{R})$ such that $\rho^\alpha_\kappa(R, \mathbf{x}) = \frac{\sum_{i:R_i=\kappa}(\mathbf{x}_i)^\alpha}{\sum_i(\mathbf{x}_i)^\alpha}$. Note that $\sigma(R, \mathbf{x}) = \rho^1(R, \mathbf{x})$.

Exponent rules. A shareholder voting rule f is an *exponent rule* if there is an $\alpha \in \mathbb{R}_+$ and a $\mathbf{g} \in \mathcal{G}$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}(\rho^\alpha(R, \mathbf{x}))$.

Roberts (1980) and Eguia and Xefteris (2019) offer characterizations of exponent rules in the context of studying interpersonal comparisons of utility. Exponent rules may[20] fail to satisfy merger consistency and reallocation invariance unless $\alpha = 1$.

Another possibility is to cap the number of votes that a shareholder may receive through the use of ceiling rules, sometimes referred to as "voting rights ceilings." A study found that 15% of the firms in the FTSEurofirst 300 Index use ceiling rules (see Dunlavy 2006). For $c \in (0, 1]$, let $\tau^c_\kappa(R, \mathbf{x}) \in \Delta(\mathcal{R})$ such that $\tau^c_\kappa(R, \mathbf{x}) = \frac{\sum_{i:R_i=\kappa}\min\{\mathbf{x}_i, c\}}{\sum_i \min\{\mathbf{x}_i, c\}}$. Note that $\sigma(R, \mathbf{x}) = \tau^1(R, \mathbf{x})$.

Ceiling rules. A shareholder voting rule f is a *ceiling rule* if there is an $c \in (0, 1]$ and a $\mathbf{g} \in \mathcal{G}$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}(\tau^c(R, \mathbf{x}))$.

Ceiling rules may fail to satisfy merger consistency and reallocation invariance.[21]

A third possibility is to give extra voting power to certain shareholders. For example, startups often give founders extra voting power relative to their shareholdings to make it easier for them to retain control. One way to do this is to use a multiplier, giving extra voting power to shares in their possession. For a strictly positive set of weights $\delta \in \mathbb{R}^{\mathbb{N}}_{++}$, let $\upsilon^\delta(R, \mathbf{x}) \in \Delta(\mathcal{R})$ such that $\upsilon^\delta_\kappa(R, \mathbf{x}) = \frac{\sum_{i:R_i=\kappa}\delta_i\mathbf{x}_i}{\sum_i \delta_i\mathbf{x}_i}$. Note that $\sigma(R, \mathbf{x}) = \upsilon^\delta(R, \mathbf{x})$ when $\delta_i = \delta_j$ for all $i, j \in N$.

Weighted rules. A shareholder voting rule f is a *weighted rule* if there is a strictly positive set of weights $\delta \in \mathbb{R}^{\mathbb{N}}_{++}$ for which $f(R, \mathbf{x}) = \mathbf{g}(\upsilon^\delta(R, \mathbf{x}))$.

[19] Ratner (1970) disputes the existence of a common law rule.

[20] I write "may" because some (but not all) rules in this class fail to satisfy these axioms.

[21] Ceiling rules satisfy the anonymity axiom. However, a rule with a ceiling of 1000 votes was invalidated in Canada on the grounds of shareholder equality. Jacobsen v. United Canso Oil & Gas Ltd. (1980), 23 A.R. 512 (Can. Alta. Q.B.).

Weighted rules satisfy merger consistency but may fail reallocation invariance. Together with that of the quorum rules, this example shows that reallocation invariance and merger consistency are logically independent.

Weighted rules are not the only non-anonymous voting rule; one extreme type of non-anonymous rule is the lexicographic dictator rule. According to this rule, there is a list of individuals, and the rule proceeds by choosing the opinion of the first shareholder on the list with a strict preference and a positive holding. If no such individual exists, then the rule leads to a tie. For $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, let

$$d(R,\mathbf{x}) = \begin{cases} \min\{i : |R_i|\mathbf{x}_i > 0\}, & \text{if } \{i : |R_i|\mathbf{x}_i > 0\} \neq \varnothing, \\ \min\{i : \mathbf{x}_i > 0\}, & \text{otherwise.} \end{cases}$$

Lexicographic dictator rule. We have $f(R,\mathbf{x}) = R_{d(R,\mathbf{x})}$.

The lexicographic dictator rule satisfies merger consistency but fails reallocation invariance.

3. Conclusion

I have introduced a model of shareholder voting in which the preferences of shareholders are aggregated to decide on a shareholder resolution. I describe three important families of shareholder voting rules—the ratio rules, the difference rules, and the share majority rules—and provide two characterizations of each of these three families. Three characterizations rely on a merger consistency axiom that requires consistency in corporate decisions following mergers; the other three characterizations rely on a reallocation invariance axiom that requires the decision to be invariant to certain manipulative techniques used by shareholders to hide their ownership. I have also provided two characterizations of the share majority rule with ties, which extends the result of May (1952).

The ratio rules, difference rules, and share majority rules are closely connected with the one share–one vote principle, in that each shareholder gets a number of votes that is linear in her shareholdings. In this sense, the merger consistency and the reallocation invariance axioms may be used as independent justifications of this principle. These axioms are easiest to defend in a setting in which mergers are a realistic possibility or in which incorporation can be done at low cost. While this clearly describes the present day, this assumption would have been more questionable in the early days of corporate law. The characterization theorems suggest a normative justification for the parallel growth of the one share–one vote principle and of the corporation as a vehicle for organizing economic activity in the latter part of the nineteenth century. However, there are other potential explanations, and establishing the existence of a causal link is a matter for economic historians, outside the scope of this paper.

The results using the merger consistency axiom may serve as a partial answer to arguments made against the one share–one vote principle on the grounds of "corporate democracy" (see Ratner 1970, Dunlavy 2006). The consistency axiom is often understood as a requirement of democratic representation when two electorates may be combined. In its traditional (non-shareholder) formulation, it provides a justification of the one person–one vote rule. However, when appropriately modified for the case of shareholder voting, when two firms (with potentially overlapping sets of shareholders) may be combined, it provides a defense of the one share–one vote principle instead.

As mentioned in the Introduction, the possibility of ties can be easily eliminated by imposing a "no-tie" axiom that requires the outcome to not be a tie. As a general matter the no-tie axiom neither implies nor is implied by the axioms described above. It is, however, inconsistent with the neutrality axiom.[22] Rather than completely eliminate the possibility of ties, it is also possible to settle for a rule in which ties are "unlikely," such as with ratio rules and difference rules with a single point of discontinuity. One way to characterize these rules would be to make the merger consistency axiom more similar to the original consistency axiom of Young (1974, 1975), which requires the combined group to follow a strict preference when one of the original groups has that preference and the other group is indifferent.

In this model, shareholders are assumed to show up at the annual meeting. Indifference represents the preference of the shareholder who is not interested in the outcome. The strategyproofness axiom ensures that interested shareholders do not make the strategic choice to pretend to be indifferent. However, in practice, shareholder voting can be more complicated; shareholders may have a choice to avoid being counted in a quorum by avoiding the annual meeting. It is possible to model this choice explicitly to gain a better understanding of quorum requirements in shareholder voting.

In the model, preferences are defined on a single pair of alternatives. In practice, virtually all shareholder votes involve only a single pair of alternatives, where one of the alternatives is the status quo. However, it is theoretically possible that some corporate decisions, such as elections for the board of directors, could involve multiple alternatives. The model can be generalized to allow more complicated preferences simply by redefining $\mathcal{R}$ as appropriate.

Similarly, the model assumes a single class of stock and that shares are infinitely divisible. While this is useful as a simplifying assumption, one may wish to allow the possibility for multiple classes of stock. One simple way to do this would be to define a finite set $\mathcal{S}$ of shares, where $\mathbf{x}$ is a partition of $\mathcal{S}$, and where $\mathbf{x}_i$ then represents the shares assigned to agent i. If the model was extended further to incorporate the date the specific shares were purchased, then it might be possible to study time-phased voting rules, where the voting power of the share is increasing in the amount of time that it has been held by the shareholder.

Appendix

I begin with four lemmas. Lemma 1 (which can be found in the body of the paper) states that reallocation invariance implies anonymity. Lemma 2 states that if a shareholder voting rule f satisfies anonymity, then there is a function $\mathbf{g}$ for which $f = \mathbf{g} \circ \sigma$ in

[22] To see this, note that for $N = \{1\}$ and $(R, \mathbf{x}) = (0, 1)$, $f(R, \mathbf{x}) = f(-R, \mathbf{x})$. Neutrality implies that $f(R, \mathbf{x}) = -f(-R, \mathbf{x})$ and, therefore, $f(R, \mathbf{x}) = -f(R, \mathbf{x}) = 0$.

a special case. Lemmas 3 and 4 state that if f satisfies merger consistency or reallocation invariance, and there is a function $\mathbf{g}$ for which $f = \mathbf{g} \circ \sigma$ in that special case, then $f = \mathbf{g} \circ \sigma$ in general. Lemmas 1, 2, and 4 are used to prove Proposition 1; Lemmas 2 and 3 are used to prove Proposition 2.

Proof of Lemma 1. Let f be a shareholder voting rule that satisfies reallocation invariance. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. Without loss of generality, let $N = \{1, \dots, n\}$. Suppose, to the contrary, that $f(R, \mathbf{x}) \neq f(\pi R, \pi \mathbf{x})$.

Step 1. I show that if there is a set $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$, then for $(R', \mathbf{x}') \in \mathcal{Q}_{N'}$, if there is a one-to-one mapping $\omega : N' \to N$ such that $(R, \mathbf{x}) = (\omega R', \omega \mathbf{x}')$, then $f(R, \mathbf{x}) = f(R', \mathbf{x}')$.

Let $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$. Without loss of generality, let $N' = \{n+1, \dots, 2n\}$. Let $R' \in \mathcal{R}^{N'}$ and $\mathbf{x}' \in \Delta(N')$ such that $R'_i = R_{i-n}$ and $\mathbf{x}'_i = \mathbf{x}_{i-n}$ for $i \in N'$. For $\omega(i) = n - i$, $(R, \mathbf{x}) = (\omega R', \omega \mathbf{x}')$.

Let $R^* \in \mathcal{R}^{N \cup N'}$ such that $R^*_i = R_i$ for $i \in N$ and $R^*_i = R'_i$ for $i \in N'$. Let $\mathbf{x}^{\circ\circ} \in \Delta(N \cup N')$ such that (a) $\mathbf{x}^{\circ\circ}_i = \mathbf{x}_i$ for $i \in N$ and (b) $\mathbf{x}^{\circ\circ}_i = 0$ for $i \in N'$. Note that $(R^*, \mathbf{x}^{\circ\circ})|_N = (R, \mathbf{x})$. For $\kappa \in \mathcal{R}$, let $S^\kappa \equiv \{i \in N \cup N' : R^*_i = \kappa\}$.

Let $\mathbf{x}^{\circ\bullet} \in \Delta(N \cup N')$ such that (a) for $i \in N$, $\mathbf{x}^{\circ\bullet}_i = 0$ if $R^*_i = 1$ and $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}_i$ if $R^*_i \neq 1$, and (b) for $i \in N'$, $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}'_i$ if $R^*_i = 1$ and $\mathbf{x}^{\circ\bullet}_i = 0$ if $R^*_i \neq 1$. For all $i \notin S^1$, $\mathbf{x}^{\circ\circ}_i = \mathbf{x}^{\circ\bullet}_i$. By reallocation invariance, $f(R^*, \mathbf{x}^{\circ\circ}) = f(R^*, \mathbf{x}^{\circ\bullet})$.

Let $\mathbf{x}^{\bullet\circ} \in \Delta(N \cup N')$ such that (a) for $i \in N$, $\mathbf{x}^{\bullet\circ}_i = 0$ if $R^*_i \neq 0$ and $\mathbf{x}^{\bullet\circ}_i = \mathbf{x}_i$ if $R^*_i = 0$, and (b) for $i \in N'$, $\mathbf{x}^{\bullet\circ}_i = \mathbf{x}'_i$ if $R^*_i \neq 0$ and $\mathbf{x}^{\bullet\circ}_i = 0$ if $R^*_i = 0$. For all $i \notin S^{-1}$, $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}^{\bullet\circ}_i$. By reallocation invariance, $f(R^*, \mathbf{x}^{\circ\bullet}) = f(R^*, \mathbf{x}^{\bullet\circ})$.

Let $\mathbf{x}^{\bullet\bullet} \in \Delta(N \cup N')$ such that $\mathbf{x}^{\bullet\bullet}_i = 0$ for $i \in N$ and $\mathbf{x}^{\bullet\bullet}_i = \mathbf{x}'_i$ for $i \in N'$. For all $i \notin S^0$, $\mathbf{x}^{\bullet\circ}_i = \mathbf{x}^{\bullet\bullet}_i$. By reallocation invariance, $f(R^*, \mathbf{x}^{\bullet\circ}) = f(R^*, \mathbf{x}^{\bullet\bullet})$. It follows that $f(R, \mathbf{x}) = f((R^*, \mathbf{x}^{\bullet\bullet})|_{N'}) = f(R', \mathbf{x}')$.

Step 2. Let $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$, and let ω be a one-to-one mapping from N' to N. Let $(R', \mathbf{x}') \in \mathcal{Q}_{N'}$ such that $(R, \mathbf{x}) = (\omega R', \omega \mathbf{x}')$. It follows from Step 1 that $f(R, \mathbf{x}) = f(R', \mathbf{x}')$.

Let ω' be a one-to-one mapping from N' to N such that for all $i \in N'$, $\omega'(i) = \pi(\omega(i))$. Then $(\pi R, \pi \mathbf{x}) = (\omega' R', \omega' \mathbf{x}')$. It follows from Step 1 that $f(\pi R, \pi \mathbf{x}) = f(R', \mathbf{x}')$. Therefore, $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$. □

For a function $\mathbf{g} \in \mathcal{G}$ and a domain $\mathcal{Q}^* \subseteq \mathcal{Q}$, I define the following property.

g-Reducible on $\mathcal{Q}^*$. For all $(R, \mathbf{x}) \in \mathcal{Q}^*$, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$.

A function is minimally one share-one vote if and only if it is $\mathbf{g}$-reducible on $\mathcal{Q}$.

For $k \in \mathbb{N}$, define $\mathcal{N}^k \equiv \{N \in \mathcal{N} : |N| = k\}$. Let $\mathcal{Q}^3 \subseteq \bigcup_{N \in \mathcal{N}^3} \mathcal{Q}_N$ be the set of problems for which, for all $N \in \mathcal{N}^3$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $R_i \neq R_j$ for all $\{i, j\} \subseteq N$.

Lemma 2. *If f is anonymous, then f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$ for some $\mathbf{g} \in \mathcal{G}$.*

Proof. Let f satisfy anonymity. Note that for $N \in \mathcal{N}^3$, there exists $\mathbf{g} \in \mathcal{G}$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3 \cap \mathcal{Q}_N$. Let $N, N' \in \mathcal{N}^3$. Then there exists $\mathbf{g}, \mathbf{g}' \in \mathcal{G}$ such that $f(R, \mathbf{x}) =$

$\mathbf{g}(\sigma(R,\mathbf{x}))$ for all $(R,\mathbf{x}) \in \mathcal{Q}^3 \cap \mathcal{Q}_N$ and $f(R',\mathbf{x}') = \mathbf{g}'(\sigma(R',\mathbf{x}'))$ for all $(R',\mathbf{x}') \in \mathcal{Q}^3 \cap \mathcal{Q}_{N'}$. I show that $\mathbf{g} = \mathbf{g}'$.

Let $\{j,k,\ell\} \subseteq N$ and $\{j',k',\ell'\} \subseteq N'$. Let $(R,\mathbf{x}) \in \mathcal{Q}_N$ and $(R',\mathbf{x}') \in \mathcal{Q}_{N'}$ such that $R_j = R'_{j'} = 1$, $R_k = R'_{k'} = -1$, $R_\ell = R'_{\ell'} = 0$, $\mathbf{x}_j = \mathbf{x}'_{j'}$, $\mathbf{x}_k = \mathbf{x}'_{k'}$, and $\mathbf{x}_\ell = \mathbf{x}'_{\ell'}$. Note that $\sigma(R,\mathbf{x}) = \sigma(R',\mathbf{x}')$. Thus to prove that $\mathbf{g} = \mathbf{g}'$, it is sufficient to show that $f(R,\mathbf{x}) = f(R',\mathbf{x}')$.

Let $N^* = \{j^*,k^*,\ell^*\} \in \mathcal{N}$ such that $N \cap N^* = N' \cap N^* = \varnothing$. Let $\pi \in \Pi_{N\cup N^*}$ such that $\pi(j) = j^*$, $\pi(k) = k^*$, $\pi(\ell) = \ell^*$, $\pi(j^*) = j$, $\pi(k^*) = k$, and $\pi(\ell^*) = \ell$. Let $\pi' \in \Pi_{N'\cup N^*}$ such that $\pi'(j') = j^*$, $\pi'(k') = k^*$, $\pi'(\ell') = \ell^*$, $\pi'(j^*) = j'$, $\pi'(k^*) = k'$, and $\pi'(\ell^*) = \ell'$.

Let $(R^\circ,\mathbf{x}^\circ), (R^\circ,\mathbf{x}^{\circ\circ}) \in \mathcal{Q}_{N\cup N^*}$ such that (a) $(R^\circ,\mathbf{x}^\circ)|_N = (R,\mathbf{x})$, (b) $R^\circ = \pi R^\circ$, and (c) $\mathbf{x}^{\circ\circ} = \pi \mathbf{x}^\circ$. From invariance to non-shareholders, it follows that $f(R,\mathbf{x}) = f(R^\circ,\mathbf{x}^\circ)$. It follows from anonymity that $f(R^\circ,\mathbf{x}^\circ) = f(\pi R^\circ, \pi\mathbf{x}^\circ) = f(R^\circ,\mathbf{x}^{\circ\circ})$. Therefore, $f(R,\mathbf{x}) = f(R^\circ,\mathbf{x}^{\circ\circ})$.

Let $(R^\bullet,\mathbf{x}^\bullet), (R^\bullet,\mathbf{x}^{\bullet\bullet}) \in \mathcal{Q}_{N'\cup N^*}$ such that (a) $(R^\bullet,\mathbf{x}^\bullet)|_{N'} = (R',\mathbf{x}')$, (b) $R^\bullet = \pi' R^\bullet$, and (c) $\mathbf{x}^{\bullet\bullet} = \pi'\mathbf{x}^\bullet$. By the same argument, $f(R',\mathbf{x}') = f(R^\bullet,\mathbf{x}^{\bullet\bullet})$.

Because $R^\circ = \pi R^\circ$ and $R^\bullet = \pi' R^\bullet$, it follows that $R^\circ_{j^*} = R^\bullet_{j^*}$, $R^\circ_{k^*} = R^\bullet_{k^*}$, and $R^\circ_{\ell^*} = R^\bullet_{\ell^*}$. Because $\mathbf{x}^{\circ\circ} = \pi\mathbf{x}^\circ$ and $\mathbf{x}^{\bullet\bullet} = \pi'\mathbf{x}^\bullet$, it follows that $\mathbf{x}^{\circ\circ}_{j^*} = \mathbf{x}^{\bullet\bullet}_{j^*} = \mathbf{x}_j$, $\mathbf{x}^{\circ\circ}_{k^*} = \mathbf{x}^{\bullet\bullet}_{k^*} = \mathbf{x}_k$, and $\mathbf{x}^{\circ\circ}_{\ell^*} = \mathbf{x}^{\bullet\bullet}_{\ell^*} = \mathbf{x}_\ell$. Consequently, $(R^\circ,\mathbf{x}^{\circ\circ})|_{N^*} = (R^\bullet,\mathbf{x}^{\bullet\bullet})|_{N^*}$. It follows from invariance to non-shareholders that $f(R,\mathbf{x}) = f(R',\mathbf{x}')$. □

Lemma 3. *If f satisfies merger consistency and there is $\mathbf{g} \in \mathcal{G}$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$, then f is $\mathbf{g}$-reducible on $\mathcal{Q}$.*

Proof. Let f satisfy merger consistency and let $\mathbf{g} \in \mathcal{G}$ such that $f(R,\mathbf{x}) = \mathbf{g}(\sigma(R,\mathbf{x}))$ for all $(R,\mathbf{x}) \in \mathcal{Q}^3$. Let $N \in \mathcal{N}$ and let $(R,\mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\mathbf{x}_i > 0$ for all $i \in N$. I show that $f(R,\mathbf{x}) = \mathbf{g}(\sigma(R,\mathbf{x}))$.

For $\kappa \in \mathcal{R}$, define $S^\kappa \equiv \{i \in N : R_i = \kappa\}$. For $\kappa \in \mathcal{R}$ and $i \in S^\kappa$, let $\omega_i = \mathbf{x}_i[\sigma_\kappa(R,\mathbf{x})]^{-1}$ and let $\mathbf{z}^i \in [0,1]^N$ such that $\mathbf{z}^i_i = \sigma_\kappa(R,\mathbf{x})$ and $\mathbf{z}^i_j = 0$ for $j \neq i$. For $\kappa \in \mathcal{R}$, if $S^\kappa \neq \varnothing$, then let $\mathcal{S}^\kappa = S^\kappa$. If $S^\kappa = \varnothing$, let $\mathcal{S}^\kappa = \{i^\kappa\}$, where $\mathcal{S}^\kappa \cap N = \varnothing$, where $\omega_{i^\kappa} = 1$, and where $\mathbf{z}^{i^\kappa}_j = 0$ for all $j \in N$. For $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, let $\mathbf{x}^{jk\ell} \in \Delta(N)$ such that $\mathbf{x}^{jk\ell} = \mathbf{z}^j + \mathbf{z}^k + \mathbf{z}^\ell$. Note that

$$\mathbf{x} = \sum_{j\in\mathcal{S}^1} \sum_{k\in\mathcal{S}^{-1}} \sum_{\ell\in\mathcal{S}^0} \omega_j \omega_k \omega_\ell \mathbf{x}^{jk\ell}.$$

By construction, for all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $\sigma(R,\mathbf{x}) = \sigma(R,\mathbf{x}^{jk\ell})$ and, therefore, $f((R,\mathbf{x}^{jk\ell})|_{\{j,k,\ell\}\cap N}) = \mathbf{g}(\sigma(R,\mathbf{x}))$. Also by construction, the sets $\mathcal{S}^\kappa$ are finite; thus, it follows from merger consistency that for all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $f(R,\mathbf{x}^{jk\ell}) = f(R,\mathbf{x}) = \mathbf{g}(\sigma(R,\mathbf{x}))$. □

Lemma 4. *If f satisfies reallocation invariance and there is $\mathbf{g} \in \mathcal{G}$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$, then f is $\mathbf{g}$-reducible on $\mathcal{Q}$.*

Proof. Let f satisfy reallocation invariance and let $\mathbf{g} \in \mathcal{G}$ such that $f(R,\mathbf{x}) = \mathbf{g}(\sigma(R,\mathbf{x}))$ for all $(R,\mathbf{x}) \in \mathcal{Q}^3$. Let $N \in \mathcal{N}$ and let $(R,\mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\{1,2,3\} \cap N = \varnothing$. I show that $f(R,\mathbf{x}) = \mathbf{g}(\sigma(R,\mathbf{x}))$.

Let $(R^*, \mathbf{x}^*) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R^* = (1, -1, 0)$ and $\mathbf{x}^* = \sigma(R, \mathbf{x})$. Define $N^+ \equiv \{1, 2, 3\} \cup N$. Let $(R', \mathbf{x}'), (R', \mathbf{x}'') \in \mathcal{Q}_{N^+}$ such that (a) $(R', \mathbf{x}')|_{\{1,2,3\}} = (R^*, \mathbf{x}^*)$ and (b) $(R', \mathbf{x}'')|_N = (R, \mathbf{x})$.

For $\kappa \in \mathcal{R}$, define $S^\kappa \equiv \{i \in N^+ : R'_i = \kappa\}$. Let $\mathbf{x}^\circ \in \Delta(N^+)$ such that $\mathbf{x}^\circ_1 = \mathbf{x}'_1$, $\mathbf{x}^\circ_i = 0$ for $i \in S^1 \setminus \{1\}$, and $\mathbf{x}^\circ_j = \mathbf{x}''_j$ for $j \notin S^1$. Let $\mathbf{x}^\bullet \in \Delta(N^+)$ such that $\mathbf{x}^\bullet_2 = \mathbf{x}'_2$, $\mathbf{x}^\bullet_i = 0$ for $i \in S^{-1} \setminus \{2\}$, and $\mathbf{x}^\bullet_j = \mathbf{x}^\circ_j$ for $j \notin S^{-1}$.

Because $R'_i = R'_j$ for all $i, j \in S^1$ and because $\mathbf{x}^\circ_k = \mathbf{x}''_k$ for $k \notin S^1$, reallocation invariance implies that $f(R', \mathbf{x}^\circ) = f(R', \mathbf{x}'')$. Because $R'_i = R'_j$ for all $i, j \in S^{-1}$ and because $\mathbf{x}^\bullet_k = \mathbf{x}^\circ_k$ for $k \notin S^{-1}$, reallocation invariance implies that $f(R', \mathbf{x}^\bullet) = f(R', \mathbf{x}^\circ)$. Because $R'_i = R'_j$ for all $i, j \in S^0$ and because $\mathbf{x}'_k = \mathbf{x}^\bullet_k$ for $k \notin S^0$, reallocation invariance implies that $f(R', \mathbf{x}') = f(R', \mathbf{x}^\bullet)$. Hence, $f(R', \mathbf{x}') = f(R', \mathbf{x}'')$. By invariance to non-shareholders, it follows that $f(R, \mathbf{x}) = f((R', \mathbf{x}'')|_N) = f((R', \mathbf{x}')|_{\{1,2,3\}}) = f(R^*, \mathbf{x}^*)$. Because $(R^*, \mathbf{x}^*) \in \mathcal{Q}^3$, it follows that $f(R^*, \mathbf{x}^*) = \mathbf{g}(\sigma(R, \mathbf{x}))$. Therefore, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$. □

Proof of Proposition 1. If: Let f satisfy reallocation invariance. By Lemma 1, f satisfies anonymity. By Lemma 2, there is a $\mathbf{g} \in \mathcal{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}^3$ and, therefore, by Lemma 4, f is minimally one share–one vote.

Only if: Let f be a shareholder voting rule and let $\mathbf{g} \in \mathcal{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}$. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$ such that, for all $i, j \in S$, $R_i = R_j$, and for all $k \notin S$, $\mathbf{x}_k = \mathbf{x}'_k$. For all $\kappa \in \mathcal{R}$, $\sum_{i:R_i=\kappa} \mathbf{x}_i = \sum_{i:R_i=\kappa} \mathbf{x}'_i$; therefore, $\sigma_\kappa(R, \mathbf{x}) = \sigma_\kappa(R, \mathbf{x}')$. Consequently, $\mathbf{g}(\sigma(R, \mathbf{x})) = \mathbf{g}(\sigma(R, \mathbf{x}'))$. It follows that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. □

Proof of Proposition 2. Let f satisfy merger consistency and anonymity. By Lemma 2, there is a $\mathbf{g} \in \mathcal{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}^3$. By Lemma 3, there is a $\mathbf{g} \in \mathcal{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}$; therefore, f is minimally one share–one vote. □

Lemma 5 shows that under a weakened form of merger consistency, the other axioms of Theorem 1 imply that a shareholder voting rule must coincide with a ratio rule in the special case. Lemma 6 shows that reallocation invariance and strategyproofness imply the weakened form of merger consistency. (The two axioms together do not imply the full merger consistency axiom.) As a consequence, the axioms of Theorem 2 also imply that a shareholder voting rule must coincide with a ratio rule in that special case. By adding Lemmas 3 and 4, this is used to prove Theorems 1 and 2.

Let $\mathcal{Q}^2 \subseteq \bigcup_{N \in \mathcal{N}^2} \mathcal{Q}_N$ be the set of problems for which, for all $N \in \mathcal{N}^2$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, there exists $j, k \in N$ such that $R_j = 1$ and $R_k = -1$. The next lemma makes use of the following property.

Merger on $\mathcal{Q}^2$. For $N \in \mathcal{N}^2$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}^2 \cap \mathcal{Q}_N$, and $\lambda \in (0, 1)$, if $f(R, \mathbf{x}) = f(R, \mathbf{x}')$, then $f(R, \mathbf{x}) = f(R, \lambda\mathbf{x} + (1 - \lambda)\mathbf{x}')$.

For $\kappa \in \mathcal{R}$ and $\phi \in \Phi$, define $g^{\kappa\phi} \in \mathcal{G}$ such that $g^{\kappa\phi}(\mathbf{x}_1, \mathbf{x}_{-1}, \mathbf{x}_0) = \kappa$ if $\mathbf{x}_0 = 1$ and $g^{\kappa\phi}(\mathbf{x}_1, \mathbf{x}_{-1}, \mathbf{x}_0) = \phi(\frac{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{x}_{-1}})$, otherwise. Let $\mathcal{G}^R \subset \mathcal{G}$ be the set of all functions $g^{\kappa\phi}$.

Lemma 5. *If f satisfies anonymity, unanimity, repurchase invariance, and merger on $\mathcal{Q}^2$, then there is $\mathbf{g} \in \mathcal{G}^R$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$.*

Proof. Step 1. Let f satisfy anonymity, unanimity, and merger on $\mathcal{Q}^2$. Let $N = \{j, k, \ell\} \in \mathcal{N}$ and let $R \in \mathcal{R}^N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$.

For $z \in [0, 1]$, let $\mathbf{x}^z \in \Delta(N)$ such that $\mathbf{x}^z_j = z$, $\mathbf{x}^z_k = 1 - z$, and $\mathbf{x}^z_\ell = 0$, and define $\phi(z) \equiv f(R, \mathbf{x}^z)$. By unanimity and invariance to non-shareholders, $\phi(0) = f(R, \mathbf{x}^0) = -1$ and $\phi(1) = f(R, \mathbf{x}^1) = 1$. Let $z, z' \in [0, 1]$ such that $z \leq z'$, and suppose, to the contrary, that $\phi(z) > \phi(z')$. Note that $\phi(z) > \phi(z')$ implies that $f(R, \mathbf{x}^z) > f(R, \mathbf{x}^{z'})$ and, therefore, that $z \neq z'$. Therefore, $z < z'$. It follows that there exists $\lambda, \lambda' \in (0, 1)$ such that $\mathbf{x}^z = \lambda \mathbf{x}^{z'} + (1 - \lambda)\mathbf{x}^0$ and $\mathbf{x}^{z'} = \lambda' \mathbf{x}^z + (1 - \lambda')\mathbf{x}^1$.

There are two cases: (a) $f(R, \mathbf{x}^z) = 1$ or (b) $f(R, \mathbf{x}^z) < 1$. If $f(R, \mathbf{x}^z) = 1$, then merger on $\mathcal{Q}^2$ and the fact that $f(R, \mathbf{x}^1) = 1$ imply that $f(R, \mathbf{x}^z) = f(R, \lambda' \mathbf{x}^z + (1 - \lambda')\mathbf{x}^1) = f(R, \mathbf{x}^{z'})$, a contradiction. If $f(R, \mathbf{x}^z) \neq 1$, then if $f(R, \mathbf{x}^{z'}) = -1$, it follows that merger on $\mathcal{Q}^2$ and the fact that $f(R, \mathbf{x}^0) = -1$ imply that $f(R, \mathbf{x}^{z'}) = f(R, \lambda \mathbf{x}^{z'} + (1 - \lambda)\mathbf{x}^0) = f(R, \mathbf{x}^z)$, a contradiction. Therefore, $\phi \in \Phi$.

Step 2. Let f additionally satisfy repurchase invariance. Let $\mathring{\mathbf{x}} \in \Delta(N)$ such that $\mathring{\mathbf{x}}_\ell = 1$. Define $\kappa \equiv f(R, \mathring{\mathbf{x}})$. It remains to be shown that for $\mathbf{x} \in \Delta(N) \setminus \{\mathring{\mathbf{x}}\}$, $f(R, \mathbf{x}) = \phi(\frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k})$. Let $\mathbf{x} \in \Delta(N) \setminus \{\mathring{\mathbf{x}}\}$ and let $z = \frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k}$. By repurchase invariance, $f(R, \mathbf{x}) = f(R, \mathbf{x}^z) = \phi(z) = \phi(\frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k})$.

This proves that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3 \cap \mathcal{Q}_N$ for some $\mathbf{g} \in \mathcal{G}^R$. Consequently, Lemma 2 implies (by anonymity) that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. $\square$

Lemma 6. *If f satisfies reallocation invariance and strategyproofness, then it satisfies merger on $\mathcal{Q}^2$.*

Proof. Let f satisfy reallocation invariance and strategyproofness. Let $N = \{j, k\} \in \mathcal{N}$ and $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Let $\mathbf{x}, \mathbf{x}', \mathbf{x}'' \in \Delta(N)$ such that $\mathbf{x}_j < \mathbf{x}'_j < \mathbf{x}''_j$ and $f(R, \mathbf{x}) = f(R, \mathbf{x}'')$. Note that $(R, \mathbf{x}), (R, \mathbf{x}'), (R, \mathbf{x}'') \in \mathcal{Q}^2 \cap \mathcal{Q}_N$.

Let $N' = N \cup \{\ell\}$. Let $R', R'' \in \mathcal{R}^{N'}$ such that $R'_j = R''_j = R''_\ell = 1$ and $R'_k = R''_k = R'_\ell = -1$. By strategyproofness, for $\dot{\mathbf{x}} \in \Delta(N')$, it cannot be that $R'_\ell \leq f([R'_{-\ell}, 0], \dot{\mathbf{x}}) < f(R', \dot{\mathbf{x}})$ or that $R''_\ell \geq f([R'_{-\ell}, 0], \dot{\mathbf{x}}) > f(R'', \dot{\mathbf{x}})$. Because $R'_\ell = -1$ and $R''_\ell = 1$, it follows that $f(R', \dot{\mathbf{x}}) \leq f([R'_{-\ell}, 0], \dot{\mathbf{x}})$ and that $f([R''_{-\ell}, 0], \dot{\mathbf{x}}) \leq f(R'', \dot{\mathbf{x}})$. Because $[R'_{-\ell}, 0] = [R''_{-\ell}, 0]$, it follows that $f(R', \dot{\mathbf{x}}) \leq f(R'', \dot{\mathbf{x}})$.[23]

Let $\mathbf{x}^*, \mathbf{x}^{**}, \mathbf{y}, \mathbf{y}', \mathbf{y}'' \in \Delta(N')$ such that $\mathbf{x}^* = (\mathbf{x}_j, \mathbf{x}'_k, \mathbf{x}'_j - \mathbf{x}_j)$, $\mathbf{x}^{**} = (\mathbf{x}'_j, \mathbf{x}''_k, \mathbf{x}''_j - \mathbf{x}'_j)$, $\mathbf{y} = (\mathbf{x}_j, \mathbf{x}_k, 0)$, $\mathbf{y}' = (\mathbf{x}'_j, \mathbf{x}'_k, 0)$, and $\mathbf{y}'' = (\mathbf{x}''_j, \mathbf{x}''_k, 0)$.

Because $R'_k = R'_\ell$ and $\mathbf{y}_j = \mathbf{x}^*_j$, reallocation invariance implies that $f(R', \mathbf{y}) = f(R', \mathbf{x}^*)$. By invariance to non-shareholders, $f(R', \mathbf{y}) = f(R, \mathbf{x})$. Because $R''_j = R''_\ell$ and $\mathbf{y}'_k = \mathbf{x}^*_k$, reallocation invariance implies that $f(R'', \mathbf{y}') = f(R'', \mathbf{x}^*)$. By invariance to non-shareholders, $f(R'', \mathbf{y}') = f(R, \mathbf{x}')$. Because $f(R', \mathbf{x}^*) \leq f(R'', \mathbf{x}^*)$, it follows that $f(R, \mathbf{x}) \leq f(R, \mathbf{x}')$.

[23]A slightly simpler proof exists; this version also shows that the lemma holds even if strategyproofness is replaced by a weaker axiom under which no agent can benefit by falsely pretending to be indifferent.

Because $R'_k = R'_\ell$ and $\mathbf{y}'_j = \mathbf{x}^{**}_j$, reallocation invariance implies that $f(R', \mathbf{y}') = f(R', \mathbf{x}^{**})$. By invariance to non-shareholders, $f(R', \mathbf{y}') = f(R, \mathbf{x}')$. Because $R''_j = R''_\ell$ and $\mathbf{y}''_k = \mathbf{x}^{**}_k$, reallocation invariance implies that $f(R'', \mathbf{y}'') = f(R'', \mathbf{x}^{**})$. By invariance to non-shareholders, $f(R'', \mathbf{y}'') = f(R, \mathbf{x}'')$. Because $f(R', \mathbf{x}^{**}) \leq f(R'', \mathbf{x}^{**})$, it follows that $f(R, \mathbf{x}') \leq f(R, \mathbf{x}'')$ and, therefore, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = f(R, \mathbf{x}'')$. Because there is $\lambda \in (0, 1)$ such that $\mathbf{x}' = \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}'$, it follows that f satisfies merger on $\mathcal{Q}^2$. □

Proof of Theorem 1. That ratio rules satisfy the four axioms is straightforward. Let f satisfy the four axioms. Because f satisfies merger consistency, it satisfies merger on $\mathcal{Q}^2$. Thus, by Lemma 5, there is $\mathbf{g} \in \mathcal{G}^R$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. By Lemma 3, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$; thus, f is a ratio rule. □

Proof of Theorem 2. That ratio rules satisfy the four axioms is straightforward. Let f satisfy the four axioms. Lemma 6, reallocation invariance and strategyproofness imply that f satisfies merger on $\mathcal{Q}^2$. By Lemma 1, reallocation invariance implies anonymity. Because f satisfies anonymity, unanimity, repurchase invariance, and merger on $\mathcal{Q}^2$, it follows from Lemma 5 that there is $\mathbf{g} \in \mathcal{G}^R$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. Consequently, by Lemma 4, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$; that is, f is a ratio rule. □

For $\psi \in \Psi$, define $g^\psi \in \mathcal{G}$ such that $g^\psi(\mathbf{x}_1, \mathbf{x}_{-1}, \mathbf{x}_0) = \psi(\mathbf{x}_1 - \mathbf{x}_{-1})$. Let $\mathcal{G}^D \subset \mathcal{G}$ be the set of all functions g^ψ.

Proof of Theorem 3. That difference rules satisfy the four axioms is straightforward. Let f satisfy the four axioms. Let $N = \{j, k, \ell\} \in \mathcal{N}$ and let $R \in \mathcal{R}^N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$. For $w \in [-1, 1]$, let $x^w \in \Delta(N)$ such that $\mathbf{x}^w_j = \frac{1+w}{2}$, $\mathbf{x}^w_k = \frac{1-w}{2}$, and $\mathbf{x}^w_\ell = 0$. Define $\psi(w) = f(R, \mathbf{x}^w)$. Because f satisfies merger consistency, it satisfies merger on $\mathcal{Q}^2$. Thus by Step 1 of the Proof of Lemma 5, using the substitution $z = \frac{1+w}{2}$, it follows that $\psi \in \Psi$.

Let $\mathbf{x} \in \Delta(N)$ and define $z = \mathbf{x}_j - \mathbf{x}_k$. I show that $f(R, \mathbf{x}) = \psi(z)$. If $\mathbf{x}_\ell = 0$, then $\mathbf{x} = \mathbf{x}^z$ and we are done. Assume that $\mathbf{x}_\ell > 0$.

Let $N' = N \cup \{m\} \in \mathcal{N}$. Let $\pi \in \Pi^N$ such that $\pi(j) = j$, $\pi(k) = k$, $\pi(\ell) = m$, and $\pi(m) = \ell$. Let $(R', \mathbf{x}'), (R'', \mathbf{x}'') \in \mathcal{Q}_{N'}$ such that (a) $(R', \mathbf{x}')|_N = (R, \mathbf{x})$, (b) $R'_m = 0$, (c) $R''_j = R''_\ell = 1$, (d) $R''_k = R''_m = -1$, and (e) $\mathbf{x}'' = \frac{1}{2}\mathbf{x}' + \frac{1}{2}\pi\mathbf{x}'$. Then by invariance to non-shareholders, $f(R, \mathbf{x}) = f(R', \mathbf{x}')$. By anonymity, $f(R', \mathbf{x}') = f(\pi R', \pi\mathbf{x}') = f(R', \pi\mathbf{x}')$. Thus, by merger consistency, $f(R', \mathbf{x}') = f(R', \mathbf{x}'')$. By cancellation, $f(R', \mathbf{x}'') = f(R'', \mathbf{x}'')$. Thus, $f(R, \mathbf{x}) = f(R'', \mathbf{x}'')$.

Let $\pi' \in \Pi^N$ such that $\pi'(j) = \ell$, $\pi'(\ell) = j$, $\pi'(k) = k$, and $\pi'(m) = m$. Let $\mathbf{x}^*, \mathbf{x}^{**} \in \Delta(N')$ such that $f(R'', \mathbf{x}^*)|_{\{j,k\}} = f(R, \mathbf{x}^z)|_{\{j,k\}}$ and $\mathbf{x}^{**} = \frac{2\mathbf{x}_j}{2\mathbf{x}_j + \mathbf{x}_\ell}\mathbf{x}^* + \frac{\mathbf{x}_\ell}{2\mathbf{x}_j + \mathbf{x}_\ell}\pi'\mathbf{x}^*$. By invariance to non-shareholders, $\psi(z) = f(R, \mathbf{x}^z) = f(R'', \mathbf{x}^*)$. By anonymity, $f(R'', \mathbf{x}^*) = f(\pi' R'', \pi'\mathbf{x}^*) = f(R'', \pi'\mathbf{x}^*)$. It follows from merger consistency that $f(R'', \mathbf{x}^*) = f(R'', \mathbf{x}^{**})$.

Let $\pi'' \in \Pi^N$ such that $\pi''(j) = j$, $\pi''(\ell) = \ell$, $\pi''(k) = m$, and $\pi''(m) = k$. Note that $\mathbf{x}'' = \frac{2\mathbf{x}_k}{2\mathbf{x}_k + \mathbf{x}_\ell}\mathbf{x}^{**} + \frac{\mathbf{x}_\ell}{2\mathbf{x}_k + \mathbf{x}_\ell}\pi''\mathbf{x}^{**}$. By anonymity, $f(R'', \mathbf{x}^{**}) = f(\pi'' R'', \pi''\mathbf{x}^{**}) = f(R'', \pi''\mathbf{x}^{**})$. It follows from merger consistency that $f(R'', \mathbf{x}^{**}) = f(R'', \mathbf{x}'')$. Therefore, $\psi(z) = f(R, \mathbf{x})$.

Thus, there is $\mathbf{g} \in \mathcal{G}^D$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. By Lemma 3, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$; that is, f is a difference rule. $\square$

Proof of Theorem 4. That difference rules satisfy the four axioms is straightforward. Let f satisfy the four axioms. Let $N = \{j, k, \ell\} \in \mathcal{N}$ and let $R \in \mathcal{R}^N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$. For $w \in [-1, 1]$, let $x^w \in \Delta(N)$ such that $\mathbf{x}^w_j = \frac{1+w}{2}$, $\mathbf{x}^w_k = \frac{1-w}{2}$, and $\mathbf{x}^w_\ell = 0$. Define $\psi(w) = f(R, \mathbf{x}^w)$. Because f satisfies reallocation invariance and strategyproofness, Lemma 6 implies that it satisfies merger on $\mathcal{Q}^2$. Thus by Step 1 of the Proof of Lemma 5, using the substitution $z = \frac{1+w}{2}$, it follows that $\psi \in \Psi$.

Let $\mathbf{x} \in \Delta(N)$ and define $z = \mathbf{x}_j - \mathbf{x}_k$. I show that $f(R, \mathbf{x}) = \psi(z)$.

Let $N' = N \cup \{m\} \in \mathcal{N}$. Let $(R', \mathbf{x}') \in \mathcal{Q}_{N'}$ such that $(R', \mathbf{x}')|_N = (R, \mathbf{x})$ and $R'_m = 0$. Let $(R'', \mathbf{x}'') \in \mathcal{Q}_{N'}$ such that (a) $R''_j = R''_\ell = 1$, (b) $R''_k = R''_m = -1$, (c) $\mathbf{x}''_j = \mathbf{x}'_j$, (d) $\mathbf{x}''_k = \mathbf{x}'_k$, and (e) $\mathbf{x}''_\ell = \mathbf{x}''_m = \frac{\mathbf{x}'_\ell}{2}$. Let $\mathbf{x}''', \mathbf{x}'''' \in \Delta(N')$ such that $\mathbf{x}'''_j = \mathbf{x}''''_j = \mathbf{x}''_j + \mathbf{x}''_\ell$, $\mathbf{x}'''_k = \mathbf{x}''_k$, $\mathbf{x}''''_k = \mathbf{x}''_k + \mathbf{x}''_\ell$, $\mathbf{x}'''_\ell = \mathbf{x}''''_\ell = \mathbf{x}'''_m = 0$, and $\mathbf{x}'''_m = \mathbf{x}''_m$.

By invariance to non-shareholders, $f(R, \mathbf{x}) = f(R', \mathbf{x}')$. Because $R'_\ell = R'_m$, $\mathbf{x}'_j = \mathbf{x}''_j$, and $\mathbf{x}'_k = \mathbf{x}''_k$, it follows from reallocation invariance that $f(R', \mathbf{x}') = f(R', \mathbf{x}'')$. By cancellation, $f(R', \mathbf{x}'') = f(R'', \mathbf{x}'')$. Because $R'_j = R'_\ell$, $\mathbf{x}''_k = \mathbf{x}'''_k$, and $\mathbf{x}''_m = \mathbf{x}'''_m$, it follows from reallocation invariance that $f(R'', \mathbf{x}'') = f(R'', \mathbf{x}''')$. Because $R'_k = R'_m$, $\mathbf{x}'''_j = \mathbf{x}''''_j$, and $\mathbf{x}'''_\ell = \mathbf{x}''''_\ell$, it follows from reallocation invariance that $f(R'', \mathbf{x}''') = f(R'', \mathbf{x}'''')$. Note that $(R'', \mathbf{x}'''')|_N = (R, \mathbf{x}^z)$ and, therefore, by invariance to non-shareholders, $f(R'', \mathbf{x}'''') = \psi(z)$. Therefore, $f(R, \mathbf{x}) = \psi(z)$.

Thus, there is $\mathbf{g} \in \mathcal{G}^D$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. By Lemma 4, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$; that is, f is a difference rule. $\square$

Let $\mathcal{G}^M \equiv \{\mathbf{g} \in \mathcal{G} : \mathbf{g}(\sigma(R, \mathbf{x})) = \mathbf{m}^\kappa(R, \mathbf{x}) \text{ for some } \kappa \in \mathcal{R}\}$.

Proof of Theorem 5. That share majority rules satisfy the axioms is straightforward. Let f satisfy reallocation invariance, unanimity, repurchase invariance, and cancellation. I show that f is a share majority rule.

Let $N \in \mathcal{N}^3$. I show that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3 \cap \mathcal{Q}_N$ for some $\mathbf{g} \in \mathcal{G}^M$. Consequently, Lemma 2 implies (by anonymity) that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$.

Let $j, k, \ell \in N$ and let $R \in \mathcal{R}^N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$. For $i \in N$, let $\mathbf{x}^i \in \Delta(N)$ such that $\mathbf{x}^i_i = 1$. By unanimity and invariance to non-shareholders, $f(R, \mathbf{x}^j) = 1$ and $f(R, \mathbf{x}^k) = -1$. Let $\kappa = f(R, \mathbf{x}^\ell)$.

Let $\mathbf{x} \in \Delta(N)$.

Case 1: $R \cdot \mathbf{x} > 0$. Then $\mathbf{x}_j > \mathbf{x}_k$. Let $\mathbf{x}^* \in \Delta(N)$ such that $\mathbf{x}^*_j = \mathbf{x}_j - \mathbf{x}_k$ and $\mathbf{x}^*_k = 0$. Let $N', N^\circ \in \mathcal{N}$ such that $N' = N \cup \{m\}$ and $N^\circ = N \setminus \{k\}$. Let $(R', \mathbf{x}') \in \mathcal{Q}_{N'}$ such that (a) $(R', \mathbf{x}')|_N = (R, \mathbf{x})$ and (b) $R'_j = R'_m$. Let $(R'', \mathbf{x}''), (R'', \mathbf{x}''') \in \mathcal{Q}_{N'}$ such that (c) $R''_j = 1$, (d) $R''_k = R''_\ell = R''_m = 0$, (e) $\mathbf{x}''_j = \mathbf{x}'''_j = \mathbf{x}_j - \mathbf{x}_k$, (f) $\mathbf{x}''_k = \mathbf{x}''_m = \mathbf{x}'_k$, (g) $\mathbf{x}''_\ell = \mathbf{x}'_\ell$, (h) $\mathbf{x}'''_k = \mathbf{x}'''_m = 0$, and (i) $\mathbf{x}'''_\ell = \mathbf{x}_\ell + 2\mathbf{x}_k$. Let $(R^\circ, \mathbf{x}^\circ) \in \mathcal{Q}_{N^\circ}$ such that $(R^\circ, \mathbf{x}^\circ) = (R'', \mathbf{x}''')|_{N^\circ}$.

By invariance to non-shareholders, $f(R, \mathbf{x}) = f(R', \mathbf{x}')$. Because $R'_j = R'_m$, $\mathbf{x}'_k = \mathbf{x}''_k$, and $\mathbf{x}'_\ell = \mathbf{x}''_\ell$, it follows from reallocation invariance that $f(R', \mathbf{x}') = f(R', \mathbf{x}'')$. By cancellation, $f(R', \mathbf{x}'') = f(R'', \mathbf{x}'')$. Because $R''_k = R''_\ell = R''_m$ and $\mathbf{x}''_j = \mathbf{x}'''_j$, it follows from reallocation invariance that $f(R'', \mathbf{x}'') = f(R'', \mathbf{x}''')$. By invariance to non-shareholders,

$f(R'', \mathbf{x}''') = f(R^\circ, \mathbf{x}^\circ)$. Also by invariance to non-shareholders, $f(R^\circ, \mathbf{x}^\circ) = f(R, \mathbf{x}^*)$. Because $\sum_{i:R_i \neq 0} \mathbf{x}^*_i = \mathbf{x}^*_j$, it follows from repurchase invariance that $f(R, \mathbf{x}^*) = f(R, \mathbf{x}^j) = 1$.

Case 2: $R \cdot \mathbf{x} < 0$. This is dual to the first case.

Case 3: $R \cdot \mathbf{x} = 0$. Then $\mathbf{x}_j = \mathbf{x}_k$. Let $R' \in \mathcal{R}^N$ such that $R'_i = 0$ for all $i \in N$. Let $(R'', \mathbf{x}') \in \mathcal{Q}_{\{\ell\}}$ such that $R''_\ell = 0$ and $\mathbf{x}'_\ell = 1$. By cancellation, $f(R, \mathbf{x}) = f(R', \mathbf{x})$. Because $R_j = R_k = R_\ell$, it follows from reallocation invariance that $f(R', \mathbf{x}) = f(R', \mathbf{x}^\ell)$. By invariance to non-shareholders, $f(R', \mathbf{x}^\ell) = f(R'', \mathbf{x}')$. Also by invariance to non-shareholders, $f(R'', \mathbf{x}') = f(R, \mathbf{x}^\ell) = \kappa$. □

Lemma 7. *A shareholder voting rule satisfies anonymity if it satisfies repurchase invariance and cancellation.*

Proof. Let $N \in \mathcal{N}$. Without loss of generality, let $N = \{1, \ldots, n\}$. Let $(R, \mathbf{x}) \in \mathcal{Q}_N$. Let $\pi \in \Pi_N$.

Let $N', N'' \in \mathcal{N}$ such that $N' = \{n+1, \ldots, 2n\}$ and $N'' = \{2n+1, \ldots, 3n\}$. Let $\pi' \in \Pi_{N \cup N' \cup N''}$ such that for $i \in N$, $\pi'(i) = \pi(i)$, and for $i \in N' \cup N''$, $\pi'(i) = i$.

Let $R', R'', R''' \in \mathcal{R}^{N \cup N' \cup N''}$ such that for $i \in N$, $R'_i = R''_i = R_i$ and $R'''_i = 0$, such that for $i \in N'$, $R''_i = R'''_i = R_{i-n}$ and $R'_i = 0$, and such that for $i \in N''$, $R''_i = -R_{i-2n}$ and $R'_i = R'''_i = 0$. Let $\mathbf{x}', \mathbf{x}'', \mathbf{x}''' \in \Delta(N \cup N' \cup N'')$ such that for $i \in N$, $\mathbf{x}'_i = \mathbf{x}_i$, $\mathbf{x}''_i = \frac{\mathbf{x}_i}{3}$, and $\mathbf{x}'''_i = 0$, for $i \in N'$, $\mathbf{x}'_i = 0$, $\mathbf{x}''_i = \frac{\mathbf{x}_{i-n}}{3}$, and $\mathbf{x}'''_i = \mathbf{x}_{i-n}$, and for $i \in N''$, $\mathbf{x}'_i = \mathbf{x}'''_i = 0$ and $\mathbf{x}''_i = \frac{\mathbf{x}_{i-2n}}{3}$.

By invariance to null shareholders, $f(R', \mathbf{x}') = f(R, \mathbf{x})$. By repurchase invariance, $f(R', \mathbf{x}'') = f(R', \mathbf{x}')$. By cancellation, $f(R'', \mathbf{x}'') = f(R', \mathbf{x}'')$. Also by cancellation, $f(R''', \mathbf{x}'') = f(R'', \mathbf{x}'')$. By repurchase invariance, $f(R''', \mathbf{x}''') = f(R''', \mathbf{x}'')$, Because $R'''_i = 0$ for all $i \in N$, it follows that $\pi R''' = R'''$ and, therefore, that $f(\pi R''', \mathbf{x}''') = f(R''', \mathbf{x}''')$. Again by repurchase invariance, $f(\pi R''', \pi \mathbf{x}'') = f(\pi R''', \mathbf{x}''')$ By cancellation, $f(\pi R'', \pi \mathbf{x}'') = f(\pi R''', \pi \mathbf{x}'')$. Also by cancellation, $(\pi R', \pi \mathbf{x}'') = (\pi R'', \pi \mathbf{x}'')$. By repurchase invariance, $(\pi R', \pi \mathbf{x}') = (\pi R', \pi \mathbf{x}'')$. By invariance to null shareholders, $f(\pi R, \pi \mathbf{x}) = (\pi R', \pi \mathbf{x}')$ and, therefore, $f(\pi R, \pi \mathbf{x}) = f(R, \mathbf{x})$. □

Proof of Theorem 6. That the share majority rules satisfy the axioms is straightforward. Let f satisfy merger consistency, unanimity, repurchase invariance, and cancellation. Because f satisfies repurchase invariance and cancellation, f satisfies anonymity by Lemma 7. Because f satisfies merger consistency and anonymity, it is minimally one share–one vote, by Proposition 2. Therefore, by Proposition 1, f satisfies reallocation invariance. It follows from Theorem 5 that f is a share majority rule. □

Let $\mathbf{g}^{\mathbf{m}^0} \in \mathcal{G}$ such that $\mathbf{g}^{\mathbf{m}^0}(\sigma(R, \mathbf{x}))$ is $\mathbf{m}^0$, the share majority rule with ties.

Proof of Theorem 9. That the share majority rule with ties satisfies the axioms is straightforward. Let f satisfy reallocation invariance, neutrality, and positive responsiveness. By Lemma 1, f satisfies anonymity. Let $N \in \mathcal{N}^3$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N \cap \mathcal{Q}^3$. Let $j, k, \ell \in N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$. Let $\pi \in \Pi_N$ such that $\pi(j) = k$ and $\pi(k) = j$. Note that $\mathbf{g}^{\mathbf{m}^0}(\sigma(R, \mathbf{x})) = \textsc{sign}(\mathbf{x}_j - \mathbf{x}_k)$.

Step 1. I show that $f(R,\mathbf{x}) = -f(R,\pi\mathbf{x})$. By anonymity, $f(R,\mathbf{x}) = f(\pi R,\pi\mathbf{x})$. Because $\pi R = -R$, it follows that $f(R,\mathbf{x}) = f(-R,\pi\mathbf{x})$. By neutrality, $f(-R,\pi\mathbf{x}) = -f(R,\pi\mathbf{x})$ and, therefore, $f(R,\mathbf{x}) = -f(R,\pi\mathbf{x})$.

Step 2. I show that if $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = 0$, then $f(R,\mathbf{x}) = 0$. Let $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = 0$. Then $\mathbf{x}_j = \mathbf{x}_k$, which implies that $\mathbf{x} = \pi\mathbf{x}$. From Step 1 it follows that $f(R,\mathbf{x}) = -f(R,\mathbf{x}) = 0$.

Step 3. I show that if $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = 1$, then $f(R,\mathbf{x}) = 1$. Let $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = 1$ and assume, to the contrary, that $f(R,\mathbf{x}) \neq 1$. Then by Step 1, $f(R,\pi\mathbf{x}) \in (0,1)$. Because $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = 1$, it follows that $\mathbf{x}_j > \mathbf{x}_k$.

Let $N' = N \cup \{m\} \in \mathcal{N}$, and let $(R',\mathbf{x}'), (R'',\mathbf{x}'') \in \mathcal{Q}_N N'$ such that $(R',\mathbf{x}')|_N = (R,\pi\mathbf{x})$, $R'_m = -1$, $(R'',\mathbf{x}'')|_N = (R,\mathbf{x})$, and $R''_m = 1$. Let $\mathbf{x}^* \in \Delta(N')$ such that $\mathbf{x}^*_j = \mathbf{x}^*_k = \mathbf{x}_k$, $\mathbf{x}^*_\ell = \mathbf{x}_\ell$, and $\mathbf{x}^*_m = \mathbf{x}_j - \mathbf{x}_k$. Because $(R',\mathbf{x}')|_N = (R,\pi\mathbf{x})$, invariance to non-shareholders implies that $f(R,\pi\mathbf{x}) = f(R',\mathbf{x}')$. Because $R'_k = R'_m$, $\mathbf{x}'_j = \mathbf{x}^*_j$, and $\mathbf{x}'_\ell = \mathbf{x}^*_\ell$, it follows from reallocation invariance that $f(R',\mathbf{x}') = f(R',\mathbf{x}^*)$. Therefore, $f(R',\mathbf{x}^*) \in \{0,1\}$.

By positive responsiveness, because $\mathbf{x}^*_m > 0$, $R''_m > R'_m$, and $R''_i = R'_i$ for $i \neq m$, it follows that $f(R'',\mathbf{x}^*) = 1$. Because $R''_j = R''_m$, $\mathbf{x}''_k = \mathbf{x}^*_k$, and $\mathbf{x}''_\ell = \mathbf{x}^*_\ell$, it follows from reallocation invariance that $f(R'',\mathbf{x}'') = f(R'',\mathbf{x}^*)$. Because $(R'',\mathbf{x}'')|_N = (R,\mathbf{x})$, invariance to non-shareholders implies that $f(R,\mathbf{x}) = f(R'',\mathbf{x}'')$ and, therefore, $f(R,\mathbf{x}) = 1$, a contradiction.

Step 4. I show that if $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = -1$, then $f(R,\mathbf{x}) = -1$. Let $\mathbf{g}^{\mathbf{m}^0}(\sigma(R,\mathbf{x})) = -1$. Then $\mathbf{x}_j < \mathbf{x}_k$. By Step 3, $f(R,\pi\mathbf{x}) = 1$. By Step 1, $f(R,\mathbf{x}) = -1$.

Step 5. Thus, f is $\mathbf{g}^{\mathbf{m}^0}$-reducible on $\mathcal{Q}^3$. By Lemma 4, it follows that f is $\mathbf{g}^{\mathbf{m}^0}$-reducible on $\mathcal{Q}$; therefore, it is the share majority rule with ties. □

Proof of Theorem 8. That the share majority rule with ties satisfies the axioms is straightforward. Let f satisfy merger consistency, anonymity, neutrality, and positive responsiveness. Because f satisfies merger consistency and anonymity, it is minimally one share–one vote, by Proposition 2. Therefore, by Proposition 1, f satisfies reallocation invariance. Thus, by Theorem 9, f is $\mathbf{m}^0$, the share majority rule with ties. □

References

Adams, Renée and Daniel Ferreira (2008), "One share-one vote: The empirical evidence." *Review of Finance*, 12, 51–91. [102]

Arrow, Kenneth J. (1963), *Social Choice and Individual Values*, Second edition. Yale University Press. [103, 105]

Banzhaf, John F. III. (1964), "Weighted voting doesn't work: A mathematical analysis." *Rutgers Law Review*, 19, 317–343. [106]

Barzel, Yoram and Tim R. Sass (1990), "The allocation of resources by voting." *Quarterly Journal of Economics*, 105, 745–771. [102]

Becht, Marco, Andrea Polo, and Stefano Rossi (2016), "Does mandatory shareholder voting prevent bad acquisitions?" *Review of Financial Studies*, 29, 3035–3067. [102]

Brandt, Felix, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia (2016), *Handbook of Computational Social Choice*. Cambridge University Press. [105]

Burkart, Mike and Samuel Lee (2008), "One share-one vote: The theory." *Review of Finance*, 12, 1–49. [102]

Chalkiadakis, Georgios and Michael Wooldridge (2016), "Weighted voting games." In *Handbook of Computational Social Choice* (Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, eds.). Cambridge University Press. [106]

Christoffersen, Susan E. K., Christopher C. Geczy, David K. Musto, and Adam V. Reed (2007), "Vote trading and information aggregation." *Journal of Finance*, 62, 2897–2929. [104]

Duddy, Conal, Ashley Piggins, and William S. Zwicker (2016), "Aggregation of binary evaluations: A Borda-like approach." *Social Choice and Welfare*, 46, 301–333. [105]

Dummett, Michael and Robin Farquharson (1961), "Stability in voting." *Econometrica*, 29, 33–43. [105]

Dunlavy, Colleen A. (2006), "Social conceptions of the corporation: Insights from the history of shareholder voting rights." *Washington and Lee Law Review*, 63, 1347–1388. [115, 116]

Easterbrook, Frank H. and Daniel R. Fischel (1983), "Voting in corporate law." *Journal of Law and Economics*, 26, 395–427. [102]

Eguia, Jon X. and Dimitrios Xefteris (2019), "Social welfare with net utilities." *Public Choice*, 179, 41–49. [106, 115]

Esö, Péter, Stephen Hansen, and Lucy White (2015), "A theory of vote trading and information aggregation." Unpublished paper, Harvard Business School and CEPR. [104]

Felsenthal, Dan S. and Moshé Machover (1998), *The Measurement of Voting Power*. Edward Elgar Publishing. [106]

Fine, Benjamin and Kit Fine (1974a), "Social choice and individual rankings I." *Review of Economic Studies*, 41, 303–322. [105]

Fine, Benjamin and Kit Fine (1974b), "Social choice and individual rankings II." *Review of Economic Studies*, 41, 459–475. [105]

Gibbard, Allan (1973), "Manipulation of voting schemes: A general result." *Econometrica*, 41, 587–601. [105]

Grossman, Sanford J. and Oliver D. Hart (1988), "One share-one vote and the market for corporate control." *Journal of Financial Economics*, 20, 175–202. [102]

Harris, Milton and Artur Raviv (1988), "Corporate governance: Voting rights and majority rules." *Journal of Financial Economics*, 20, 203–235. [102]

Harsanyi, John C. (1955), "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility." *Journal of Political Economy*, 63, 309–321. [106]

Hilt, Eric (2013), "Shareholder voting rights in early American corporations." *Business History*, 55, 620–635. [104]

Holderness, Clifford G. (2018), "Equity issuances and agency costs: The telling story of shareholder approval around the world." *Journal of Financial Economics*, 129, 415–439. [102]

Hu, Henry T. C. and Bernard Black (2005), "The new vote buying: Empty voting and hidden (morphable) ownership." *Southern California Law Review*, 79, 811–908. [104]

Hu, Henry T. C. and Bernard Black (2006), "Empty voting and hidden (morphable) ownership: Taxonomy, implications, and reforms." *Business Lawyer*, 61, 1011–1070. [104]

Hu, Henry T. C. and Bernard Black (2007), "Hedge funds, insiders, and the decoupling of economic and voting ownership: Empty voting and hidden (morphable) ownership." *Journal of Corporate Finance*, 13, 343–367. [104]

Hu, Henry T. C. and Bernard Black (2008), "Equity and debt decoupling and empty voting II: Importance and extensions." *University of Pennsylvania Law Review*, 156, 625–739. [104]

Maug, Ernst and Bilge Yilmaz (2002), "Two-class voting: A mechanism for conflict resolution." *American Economic Review*, 92, 1448–1471. [102]

May, Kenneth O. (1952), "A set of independent necessary and sufficient conditions for simple majority decision." *Econometrica*, 20, 680–684. [103, 105, 112, 113, 116]

Moulin, Hervé (1985), "Egalitarianism and utilitarianism in quasi-linear bargaining." *Econometrica*, 53, 49–67. [104]

Moulin, Hervé (1987), "Equal or proportional division of a surplus, and other methods." *International Journal of Game Theory*, 16, 161–186. [104]

Moulin, Hervé (1988), *Axioms of Cooperative Decision Making*. Cambridge University Press, Cambridge. [105]

Nitzan, Shmuel and Jacob Paroush (1984), "A general theorem and eight corollaries in search of a correct decision." *Theory and Decision*, 17, 211–220. [102]

Nitzan, Shmuel and Uriel Procaccia (1986), "Optimal voting procedures for profit maximizing firms." *Public Choice*, 51, 191–208. [102]

Posner, Eric A. and E. Glen Weyl (2014), "Quadratic voting as efficient corporate governance." *University of Chicago Law Review*, 81, 251–272. [102, 115]

Ratner, David L. (1970), "Government of business corporations critical reflections on the rule of one share one vote." *Cornell Law Review*, 56, 1–56. [102, 115, 116]

Ritzberger, Klaus (2005), "Shareholder voting." *Economics Letters*, 86, 69–72. [102]

Roberts, Kevin W. S. (1980), "Interpersonal comparability and social choice theory." *Review of Economic Studies*, 47, 421–439. [106, 115]

Satterthwaite, Mark A. (1975), "Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory*, 10, 187–217. [105]

Smith, John H. (1973), "Aggregation of preferences with variable electorate." *Econometrica*, 41, 1027–1041. [105]

Taylor, Alan and William Zwicker (1992), "A characterization of weighted voting." *Proceedings of the American Mathematical Society*, 115, 1089–1094. [106]

Thomas, Randall S. and Christoph Van der Elst (2015), "Say on pay around the world." *Washington University Law Review*, 92, 653–731. [102]

Thompson, Robert B. and Paul H. Edelman (2009), "Corporate voting." *Vanderbilt Law Review*, 62, 127–174. [102, 103]

Young, H. Peyton (1974), "An axiomatization of Borda's rule." *Journal of Economic Theory*, 9, 43–52. [104, 105, 117]

Young, H. Peyton (1975), "Social choice scoring functions." *SIAM Journal on Applied Mathematics*, 28, 824–838. [104, 105, 117]

Co-editor Dilip Mookherjee handled this manuscript.

Manuscript received 2 March, 2019; final version accepted 8 May, 2020; available online 12 May, 2020.