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Chain stability in trading networks

John William Hatfield
McCombs School of Business, University of Texas at Austin

Scott Duke Kominers
Entrepreneurial Management Unit, Harvard Business School, Department of Economics and Center of Mathematical Sciences and Applications, Harvard University, and National Bureau of Economic Research

Alexandru Nichifor
Department of Economics, University of Melbourne

Michael Ostrovsky
Graduate School of Business, Stanford University and National Bureau of Economic Research

Alexander Westkamp
Department of Management, Economics, and Social Sciences, University of Cologne

In a general model of trading networks with bilateral contracts, we propose a suitably adapted chain stability concept that plays the same role as pairwise stability in two-sided settings. We show that chain stability is equivalent to stability if all agents' preferences are jointly fully substitutable and satisfy the Laws of Aggregate Supply and Demand. In the special case of trading networks with transferable utility, an outcome is consistent with competitive equilibrium if and only if it is chain stable.

John William Hatfield: john.hatfield@utexas.edu
Scott Duke Kominers: kominers@fas.harvard.edu
Alexandru Nichifor: nichifor@unimelb.edu.au
Michael Ostrovsky: ostrovsky@stanford.edu
Alexander Westkamp: westkamp@wiso.uni-koeln.de
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1. Introduction

Cooperative solution concepts in game theory often rely on coordinated deviations by large groups of agents, including, in some cases, all the agents in the economy. A natural question when considering coordinated deviations is how (and whether) such coalitions can, in fact, form. Do all the agents in the economy need to consider all the possible deviations by all the possible coalitions? Alternatively, is it perhaps sufficient for agents to consider only smaller or more structured types of deviations? Do the agents need to reason about the structure of the entire economy to discover a profitable deviation, or is it sufficient for each of them to consider only his or her "local" environment?

Shapley and Shubik (1971), Crawford and Knoer (1981), Kelso and Crawford (1982), and Roth (1984) have shown that in two-sided matching environments with substitutable preferences, one does not need to consider coordinated deviations by large groups of agents to determine the overall stability of a matching: In two-sided one-to-one and many-to-one matching markets, overall stability, along with competitive equilibrium, are both essentially equivalent to *pairwise stability*. Pairwise stability does not require considering coordinated deviations by complex coalitions; neither does it require specifying prices for trades that are not carried out, in contrast to competitive equilibrium for two-sided matching markets, which formally require that all trades—even those not carried out—be priced. Given that pairwise deviations are much easier for agents to discover, the equivalence results for pairwise stability mitigate potential concerns about solution concepts that are either based on discovering large-group deviations or that require that all trades, including those that are not carried out, be priced.

In this paper, we establish analogous results for a very rich setting—trading networks with bilateral contracts. We allow agents to be buyers in some contracts and sellers in others, and do not impose *any* restrictions on the network of possible trades. In particular, the market is neither required to have a two-sided structure nor is the network of possible trades required to have a vertical structure. The model we present here is strictly more general than any of the earlier models in the literature on matching with bilateral contracts, subsuming settings with discrete and continuous prices, with quasilinear and non-quasilinear utility functions, and with and without indifferences in agents' preferences. We prove two equivalence results. Our main result shows that if all agents' preferences jointly satisfy the *full substitutability* condition and the *Laws of Aggregate Supply and Demand* (which we make precise in Section 2.1 by way of a condition we call *monotone–substitutability*), then the concept of *stability* (under which all possible deviations by groups of agents need to be considered) is equivalent to *chain stability*, under which only deviations by *chains* of agents need to be considered.[1] We also show a corollary of the main result of the present paper and the results of Hatfield et al. (2013):

[1] Chain stability was originally introduced by Ostrovsky (2008) for a more restrictive, vertical environment in which all trade flows in one direction, from the suppliers of basic inputs to the consumers of final outputs.

In trading networks with continuously transferable utility, if all agents' preferences are fully substitutable, then an outcome is consistent with competitive equilibrium[2] if and only if it is not blocked by any chain.

After presenting our equivalence results, we quantify the simplicity of chain deviations relative to more general "blocking set" deviations. Formally, we show that as the size of the economy grows, the number of chains of trades (corresponding to possible blocking chains) is a vanishingly small fraction of the number of general sets of trades (corresponding to possible blocking sets).[3] Intuitively, just as in two-sided settings—in which it is much easier to find a pairwise block than a general blocking set—in our setting, it is much easier to find a blocking chain than a general blocking set. If the network has additional structure, the simplicity gain can be much higher than suggested by our formal counting result. For example, in the supply chain setting of Ostrovsky (2008), the number of chains grows only polynomially as a function of the number of agents, while the number of sets of contracts grows exponentially.

We also present three examples demonstrating the roles that our assumptions play in the main equivalence result. The first example shows that if the preferences of some agent do not satisfy the Laws of Aggregate Supply and Demand, then chain stable outcomes may not be stable. The second example shows that if the preferences of some agent are not fully substitutable, then chain stable outcomes may likewise not be stable. The third example illustrates that ensuring robustness to blocking chains that do not "cross" themselves (i.e., chains that involve each agent in at most two contracts) is not sufficient to ensure robustness to general blocking sets. This last example, combined with our equivalence results, illustrates that chain stability plays the same role in the trading network setting as pairwise stability does in two-sided settings: chains are the "essential" blocking sets that one needs to consider to evaluate an outcome's stability or its consistency with competitive equilibrium.

The model of trading networks that we consider is deliberately very general, encompassing many existing matching models and going beyond them. As a result, the existence of stable outcomes in our full model is not guaranteed (although it is, of course, guaranteed in many important special cases, such as the quasilinear case with transferable utility considered by Hatfield et al. 2013 and the vertical supply chain setting of Ostrovsky 2008). The motivation for considering such a general model is twofold: First, our model allows us to uncover the unifying structure underlying the equivalence between stability and chain stability. Second, and relatedly, we establish that checking whether an outcome is chain stable is sufficient to ensure that it is stable under larger and more general deviations.

In the Ostrovsky (2008) setting, any chain of contracts has a beginning and an end, and passes "through" each agent at most once. In the current, richer environment, we adapt the definition of a chain to allow a chain to end at the same node at which it began (thus becoming a "loop"), and to cross itself (potentially several times). However, as before, the essential feature of a chain is that it is a "linked" sequence of trades, such that the agent who is the buyer in a particular trade is the seller in the next trade in the sequence. We discuss our concept of chain stability in more detail in Section 2.2 after introducing it formally.

[2]That is, one can generate prices for trades that are not carried out to obtain a competitive equilibrium.

[3]That said, Fleiner et al. (2020) showed that testing stability is NP-hard in a fully general setting; combining this with our results yields that testing chain stability is NP-hard as well.

The generality of our model allows for a wide variety of special cases. For example, following the circulation of our original draft, Fleiner et al. (2019) showed that stable outcomes are guaranteed to exist in trading network settings with income effects under full substitutability, so long as there are no frictions.[4] More recently, Andersson et al. (forthcoming) developed a model of time banks in which agents exchange discrete units of time performing a particular task, and Manjunath and Westkamp (2019) considered the exchange of indivisible shifts among a group of workers. All of these applications can be embedded into our model, and our results transfer over.[5]

The remainder of the paper is organized as follows. Section 1.1 provides an overview of the related literature. Section 2 introduces our general model. Section 3 states and proves the main result on the equivalence of stability and chain stability. Section 4 discusses the correspondence between chain stable outcomes and competitive equilibria for the special case of quasilinear preferences and fully transferable utility. Section 5 assesses the simplicity of checking chain stability relative to checking stability directly. Section 6 presents the examples that show the roles of our assumptions. Section 7 concludes.

1.1 *Related literature*

The concept of blocking is fundamental in the analysis of matching markets. In the original papers of Gale and Shapley (1962) and Shapley and Shubik (1971) on stability in two-sided markets, attention is restricted to *pairwise blocks*, i.e., pairs of agents who mutually prefer each other to their assigned partners. The requirement that a two-sided matching be pairwise stable—i.e., be robust to pairwise blocks—seems much weaker than the requirement that a matching be robust to deviations by arbitrary sets of agents. Indeed, in general, in markets in which some agents are allowed to match with multiple partners, a matching that is robust to deviations by pairs may not be robust to richer deviations.[6] However, as we discussed in the Introduction, key results in the theory of two-sided, many-to-one matching show that when agents' preferences are *substitutable* (Kelso and Crawford 1982, Roth 1984), pairwise relationships are, in fact, the essential blocking sets: any pairwise stable matching is also robust to larger deviations.[7]

Ostrovsky (2008) introduced a generalization of two-sided matching to "supply chain" environments. In supply chain matching, goods flow downstream from initial producers to end consumers, potentially with numerous intermediaries in between. In the Ostrovsky (2008) framework, attention is restricted to blocking *chains*—sequences

[4]Their work is a strict generalization of the model of Hatfield et al. (2013) in that it goes beyond quasilinearity and allows for certain income effects in agents' utility functions.

[5]Our work immediately implies that chain stability is equivalent to stability in the settings of Andersson et al. (forthcoming) and Manjunath and Westkamp (2019); meanwhile, the equivalence applies in the setting of Fleiner et al. (2019) whenever agents' preferences are monotone–substitutable.

[6]For example, if every firm in an economy is only interested in hiring an even number of workers, then an empty matching will always be pairwise stable, even in the cases in which another, nonempty matching makes all agents in the economy strictly better off.

[7]Hatfield and Kominers (2017) prove this result in a general two-sided matching setting with contracts, and provide an overview of earlier literature on related results in other two-sided settings.

of agents who could benefit from recontracting with each other along a vertical chain. Outcomes robust to chain deviations are said to be *chain stable*. Chain stability is a natural extension of pairwise stability to the setting in which an agent can be both a buyer and a seller; for example, an agent may be willing to sell a unit of output only if he can buy a unit of input required to produce that output. Ostrovsky (2008) showed that when the preferences of all agents in the economy are fully substitutable (see Definition 1 in Section 2.1), chain stable outcomes are guaranteed to exist. Again, chain stability appears to be a much weaker condition than the requirement that an outcome be robust to deviations by arbitrary sets of agents. However, as in the case of pairwise stability, under the assumption that agents' preferences are fully substitutable, chains are the essential blocking sets in the supply chain setting: Hatfield and Kominers (2012) showed that in that setting, any chain stable outcome is *stable*, in the sense that it is robust to blocks by arbitrary sets of agents.[8]

Hatfield et al. (2013) dispensed with the vertical structure of the supply chain environment and instead considered arbitrary trading networks. They also assumed that prices can vary freely (instead of being restricted to a finite discrete set) and that agents' preferences are quasilinear.[9] In their analysis, Hatfield et al. (2013) considered a stability concept analogous to that of Hatfield and Kominers (2012), allowing for recontracting by arbitrary groups of agents. They showed that when agents' preferences are fully substitutable, stable outcomes exist and are essentially equivalent to competitive equilibria with personalized prices. Our model includes the setting of Hatfield et al. (2013) as a special case—and for that special case, a corollary of our main result is that an outcome is consistent with competitive equilibrium if and only if it is not blocked by any chain of contracts.

Our paper contributes to the literature on the relationships between different solution concepts in matching environments (see, e.g., Echenique and Oviedo 2006, Klaus and Walzl 2009, Westkamp 2010, and Hatfield and Kominers 2017). It also has parallels in the operations research literature on flows in networks (see, e.g., a textbook treatment by Ahuja et al. 1993); the "flow decomposition lemma" in that literature states that any

[8]The setting of Hatfield and Kominers (2012) is a special case of our framework, and for that special case, the Hatfield and Kominers (2012) definition of stability coincides with ours (Definition 4 in our Section 2.2). Note, however, that even in the case of vertical networks, our setting is substantially more general than that of Ostrovsky (2008) and Hatfield and Kominers (2012): we allow for arbitrary sets of contracts (as opposed to just finite ones) and explicitly incorporate the case in which an agent may be indifferent between two different sets of contracts (as opposed to having strict preferences); these generalizations are necessary to define the concept of competitive equilibrium and to establish the connections between chain stable outcomes and competitive equilibria.

[9]If one dispenses with supply chain structure without assuming that prices can vary freely, then stable outcomes may not exist (Hatfield and Kominers 2012). Fleiner et al. (2018) introduced a weaker concept, *trail stability*, for settings without supply chain structure. As Fleiner et al. (2018) explained (emphasis in original): "In a trail-stable outcome, no agent wants to drop his contracts and there exists no sequence of *consecutive* bilateral contracts [...] such that any intermediate agent who is offered a downstream (upstream) contract [...] wants to choose it alongside the subsequent upstream (downstream) contract [...]. Importantly, [trail stability] require[s] that the first (final) agent wants to unilaterally offer (accept) the first (final) contract [...]." Fleiner et al. (2018) showed that trail-stable outcomes are guaranteed to exist under full substitutability (in arbitrary trading networks).

"flow" in a network can be "decomposed" into a collection of simple "paths" and "cycles," resembling the decomposition of any blocking set into a collection of blocking chains in our Theorem 2. Note, however, that paths and cycles in the flow decomposition lemma cannot cross themselves, while in our environment, we need to allow for the possibility of self-crossing chains (see Example 3 in Section 6). The difficulty is due to the fact that in the "network flows" environment, there is a single type of good "flowing" through the network, and the objective function is the maximization or minimization of the aggregate flow, whereas in our setting many different types of goods may be present and the preferences of agents in the market may be more complex. For the case of quasilinear environments with transferable utility, Candogan et al. (2019) provided a detailed analysis of the connections between results on stability and competitive equilibrium in trading networks and the literature on network flows.[10]

2. Model

There is an economy with a finite set I of *agents*. Pairs of agents can participate in bilateral *trades*. Each trade ω is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. The trade ω specifies all the nonpecuniary terms and conditions associated with a relationship between $b(\omega)$ and $s(\omega)$; for instance, ω could specify the transfer of a single unit of an indivisible good or service from $s(\omega)$ to $b(\omega)$.[11] The set of possible trades, denoted Ω, is finite and exogenously given. Note that we require that the buyer and the seller associated with a trade be distinct agents, but we allow Ω to contain multiple trades associated with the same agents, and allow for the possibility of trades $\omega \in \Omega$ and $\psi \in \Omega$ such that $s(\omega) = b(\psi)$ and $s(\psi) = b(\omega)$.

To capture the purely financial aspect of a transaction associated with a trade, we augment each trade by introducing a price. Formally, a *contract* x is a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$ that specifies a trade and an associated price. For a contract $x = (\omega, p_\omega)$, we denote by $b(x) \equiv b(\omega)$ and $s(x) \equiv s(\omega)$ the buyer and the seller associated with the trade ω of x. If $b(x) = i$ for some contract x, then x is *upstream* of, or on the *buy-side* for, i; similarly, if $s(x) = i$ for some contract x, then x is *downstream* of, or on the *sell-side* for, i. We denote by $X \subseteq \Omega \times \mathbb{R}$ the set of all contracts available to the agents; this set is fixed and exogenously given. The set X can be infinite (as, e.g., in the settings of Hatfield et al. 2013 and Fleiner et al. 2019, where all prices are allowed for all trades and, thus, $X = \Omega \times \mathbb{R}$) or finite (as, e.g., in the settings of Ostrovsky 2008, Hatfield and Kominers 2012, and Fleiner et al. 2018).

For each agent $i \in I$ and set of contracts $Y \subseteq X$, we let $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$ denote the set of contracts in Y in which i is the buyer, i.e., the set of upstream contracts for i, and we let $Y_{i\to} \equiv \{y \in Y : i = s(y)\}$ denote the set of contracts in Y in which i is the

[10]Beyond their conceptual interest, our results may contribute to the emerging empirical and econometric literature on matching and trading networks (see, e.g., Fox 2017).

[11]For some applications, the assignment of buyer and seller roles in a trading relationship follows immediately from the context. In other applications, one needs a convention. For instance, in a two-sided matching market without transfers, we think of agents on one side as sellers (in all possible outcomes) and agents on the other side as buyers (in all possible outcomes).

seller, i.e., the set of downstream contracts for i. We let $Y_i \equiv Y_{i\to} \cup Y_{\to i}$. We let $a(Y) \equiv \bigcup_{y\in Y}\{b(y), s(y)\}$ denote the set of agents involved in contracts in Y as either buyers or sellers. Slightly abusing notation, for a contract $x \in X$, we write $a(x) \equiv a(\{x\})$. We use analogous notation for various properties of trades $\omega \in \Omega$ and sets of trades $\Psi \subseteq \Omega$: e.g., $a(\omega) \equiv \{b(\omega), s(\omega)\}$ and $\Psi_i \equiv \{\omega \in \Psi : i \in a(\omega)\}$. Finally, we denote by $\tau(Y)$ the set of trades involved in contracts in Y: $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}$.

A set of contracts $Y \subseteq X$ is *feasible* if it does not contain two or more contracts associated with the same trade: formally, $Y \subseteq X$ is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Y$ implies that $p_\omega = \hat{p}_\omega$; equivalently, $Y \subseteq X$ is feasible if $|Y| = |\tau(Y)|$. An *outcome* is a feasible set of contracts.

2.1 *Preferences*

Each agent i has a utility function U_i over feasible sets $Y \subseteq X_i$ of contracts that involve i as the buyer or the seller. For a feasible set $Y \subseteq X_i$, we have that $U_i(Y) \in \mathbb{R} \cup \{-\infty\}$, with the value of $-\infty$ used to denote sets of contracts that are technologically impossible for the agent to undertake (e.g., selling the same object to two different buyers). We assume that $U_i(\varnothing) \in \mathbb{R}$, i.e., any agent's utility from the "outside option" of not participating in any contracts is finite.

The *choice correspondence* of agent i from a set of contracts $Y \subseteq X_i$ is defined as the collection of sets of contracts maximizing the utility of agent i:

$$C_i(Y) \equiv \{Z \subseteq Y : Z \text{ is feasible}; \forall \text{ feasible } Z' \subseteq Y,\ U_i(Z) \geq U_i(Z')\}.$$ [12]

For notational convenience, we also extend the choice correspondence to sets of contracts that do not necessarily involve agent i: for a set of contracts $Y \subseteq X$, we write $C_i(Y) \equiv C_i(Y_i)$.

We now introduce our first key condition on preferences: full substitutability.[13]

Definition 1. The preferences of agent i are *fully substitutable* if both:

(i) for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have

$$(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i}) \quad \text{and} \quad Y^*_{i\to} \subseteq Z^*_{i\to};$$

and

(ii) for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have

$$(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to}) \quad \text{and} \quad Y^*_{\to i} \subseteq Z^*_{\to i}.$$

[12] Note that $C_i(Y)$ may be empty if Y is infinite.

[13] For the case of quasilinear utility functions, the full substitutability definition we use here corresponds to the CFS condition of Hatfield et al. (2019). Thus, the results of Hatfield et al. (2019) imply that (again, for the case of quasilinear utility functions) our definition is equivalent to a number of other substitutability concepts that have originated in several distinct literatures. Ostrovsky (2008) and Hatfield et al. (2013) provide detailed discussions of the implications of full substitutability in various environments.

Informally, the choice correspondence C_i is fully substitutable if, when the set of options available to i on one side expands, i both rejects a (weakly) larger set of contracts on that side and selects a (weakly) larger set of contracts on the other side (where "larger" is understood in a set-inclusion sense). Hatfield et al. (2013, 2019) have identified several economically important examples of fully substitutable preferences.

The second property important for our results is that the preferences of all agents satisfy the Laws of Aggregate Supply and Demand.

Definition 2. The preferences of agent i satisfy the *Law of Aggregate Demand* if for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have

$$\left|Z^*_{\to i}\right| - \left|Z^*_{i\to}\right| \geq \left|Y^*_{\to i}\right| - \left|Y^*_{i\to}\right|.$$

The preferences of agent i satisfy the *Law of Aggregate Supply* if for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\to} \subseteq Z_{i\to}$, and $Y_{\to i} = Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have

$$\left|Z^*_{i\to}\right| - \left|Z^*_{\to i}\right| \geq \left|Y^*_{i\to}\right| - \left|Y^*_{\to i}\right|.$$

Informally, the choice correspondence C_i satisfies the Law of Aggregate Demand if, when the set of options available to i as a buyer expands, the *net demand of i*—i.e., the difference between the number of buy-side and sell-side contracts that i chooses—(weakly) increases.[14] Similarly, the choice correspondence C_i satisfies the Law of Aggregate Supply if, when the set of options available to i as a seller expands, the *net supply of i*—i.e., the difference between the number of sell-side and buy-side contracts that i chooses—(weakly) increases. These conditions extend the canonical Law of Aggregate Demand (Hatfield and Milgrom 2005; see also Alkan and Gale 2003) to the current setting, in which each agent can be both a buyer in some trades and a seller in others.

Intuitively, if we think of each contract as specifying the transfer of an object, the Laws of Aggregate Supply and Demand require that no object can substitute for *multiple* other objects. Thus, when i obtains access to a new buy-side contract, the total number of buy-side contracts he chooses weakly increases (for a fixed number of sell-side contracts), and, similarly, when i obtains access to a new sell-side contract, the total number of sell-side contracts he chooses weakly increases (for a fixed number of buy-side contracts).[15] For instance, in the setting of the used car market discussed by Hatfield et al. (2013), a trade represents the transfer of an automobile, and so the Laws of Aggregate Supply and Demand hold naturally: purchasing an additional car enables the dealer to sell at most one more car.

[14]That is, when an agent gains access to more buy-side contracts while holding his set of available sell-side contracts fixed, the increase in the number of buy-side contracts chosen has to be weakly larger than the increase in the number of sell-side contracts chosen.

[15]Of course, these monotonicity conditions only make sense if trades represent corresponding units of goods; see Hatfield and Kominers (2017) for a discussion of this and other issues related to contract design.

When the choice correspondence is multivalued (as can naturally arise when prices can vary continuously), the definitions of full substitutability and the Laws of Aggregate Supply and Demand must be adapted in order to account for indifferences. In particular, we need to reformulate the first part of the definition of full substitutability (Definition 1) to consider instances in which the choice correspondence is multivalued by requiring, for any set Y, for each optimal choice from Y, for a set Z that expands i's opportunities on the buy-side, i.e., a set Z such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, there exists an optimal choice from Z that satisfies the same conditions as those in the first part of Definition 1.[16] Similarly, we have to extend the Laws of Aggregate Supply and Demand to consider multivalued choice correspondences as the set of available buy-side or sell-side contracts expands. Additionally, when the choice correspondence may be multivalued, requiring both conditions to hold simultaneously becomes subtle, as we need to ensure that they apply to the *same* element of the multivalued choice correspondence. That is, for example, we need that for any set Y, for each optimal choice Y^* from Y, for a set Z that expands i's opportunities on the buy-side, there exists an optimal Z^* that simultaneously fulfills the requirements of full substitutability and the Laws of Aggregate Supply and Demand. We formalize the preceding requirements in the following definition.

Definition 3. The preferences of agent i are *monotone–substitutable* if both:[17]

(i) for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that both Y^* and Z^* are consistent with the full substitutability condition when the set of buy-side opportunities expands, i.e.,

$$(Y_{\to i} \smallsetminus Y^*_{\to i}) \subseteq (Z_{\to i} \smallsetminus Z^*_{\to i}) \quad \text{and} \quad Y^*_{i\to} \subseteq Z^*_{i\to},$$

and Y^* and Z^* are consistent with the Law of Aggregate Demand, i.e.,

$$|Z^*_{\to i}| - |Z^*_{i\to}| \geq |Y^*_{\to i}| - |Y^*_{i\to}|;$$

and

(ii) for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{\to i} = Z_{\to i}$ and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that both Y^* and Z^* are consistent with the full substitutability condition when the set of sell-side opportunities expands, i.e.,

$$(Y_{i\to} \smallsetminus Y^*_{i\to}) \subseteq (Z_{i\to} \smallsetminus Z^*_{i\to}) \quad \text{and} \quad Y^*_{\to i} \subseteq Z^*_{\to i},$$

and Y^* and Z^* are consistent with the Law of Aggregate Supply, i.e.,

$$|Z^*_{i\to}| - |Z^*_{\to i}| \geq |Y^*_{i\to}| - |Y^*_{\to i}|.$$

[16] These formulations of full substitutability are equivalent when continuous transfers are available and agents' utility functions are quasilinear (Hatfield et al. 2019).

[17] Note that when choice correspondences are single-valued, monotone–substitutability reduces to imposing full substitutability and the Laws of Aggregate Supply and Demand separately.

The following lemma illustrates one of the implications of monotone–substitutability, which plays a key role in understanding our main result.

LEMMA 1. *Suppose that the preferences of i are monotone–substitutable and $Y^* \in C_i(Y)$ for some $Y \subseteq X$.*

(i) For any contract $\bar{y} \in [X \smallsetminus Y]_{\to i}$, there exists a $\bar{Y}^* \in C_i(Y \cup \{\bar{y}\})$ such that either:

(a) we have $\bar{Y}^* = Y^*$;

(b) we have $\bar{Y}^* = Y^* \cup \{\bar{y}\}$;

(c) there exists a contract $y \in Y^*_{\to i}$ such that $\bar{Y}^*_{\to i} = (Y^*_{\to i} \cup \{\bar{y}\}) \smallsetminus \{y\}$ and $\bar{Y}^*_{i\to} = Y^*_{i\to}$; or

(d) we have $\bar{Y}^*_{\to i} = Y^*_{\to i} \cup \{\bar{y}\}$ and there exists some contract $z \in [Y \smallsetminus Y^*]_{i\to}$ such that $\bar{Y}^*_{i\to} = Y^*_{i\to} \cup \{z\}$.

(ii) For any contract $\bar{y} \in [X \smallsetminus Y]_{i\to}$, there exists a $\bar{Y}^* \in C_i(Y \cup \{\bar{y}\})$ such that either:

(a) we have $\bar{Y}^* = Y^*$;

(b) we have $\bar{Y}^* = Y^* \cup \{\bar{y}\}$;

(c) there exists a contract $y \in Y^*_{i\to}$ such that $\bar{Y}^*_{i\to} = (Y^*_{i\to} \cup \{\bar{y}\}) \smallsetminus \{y\}$ and $\bar{Y}^*_{\to i} = Y^*_{\to i}$; or

(d) we have $\bar{Y}^*_{i\to} = Y^*_{i\to} \cup \{\bar{y}\}$ and there exists some contract $z \in [Y \smallsetminus Y^*]_{\to i}$ such that $\bar{Y}^*_{\to i} = Y^*_{\to i} \cup \{z\}$.

Part (i) of Lemma 1 describes how an optimal choice by i changes when i gains access to the contract $\bar{y}$ as a buyer. There are four possibilities: In the first possibility, $\bar{y}$ is undesirable and so i's optimal choice does not change. In the second possibility, i chooses the same set of contracts along with the newly available contract $\bar{y}$. In the third possibility, the contract $\bar{y}$ "substitutes" for y, and i chooses the same set of contracts as a seller. In the fourth possibility, i chooses the same set of contracts along with the newly available contract $\bar{y}$ as a buyer, and also chooses a sell-side contract not previously chosen. Part (ii) of Lemma 1 describes the analogous behavior associated with gaining access to a new contract as a seller.

To understand the proof of Lemma 1, consider an agent with fully substitutable preferences who converts inputs into outputs and suppose that one new input $\bar{y}$ becomes available. Full substitutability ensures that the agent still rejects all of the inputs that he rejected before and does not reject any additional outputs. The Laws of Aggregate Supply and Demand ensure that when $\bar{y}$ becomes available, the intermediary's optimal choice must have a weakly higher net demand. Given the constraints imposed by full substitutability, this can only be achieved by either choosing the same set of contracts that he chose before (leaving net demand unchanged), choosing the same set of contracts along with $\bar{y}$ (which increases net demand by 1), replacing some contract y with $\bar{y}$

as a buyer (leaving net demand unchanged), or taking $\bar{y}$ and adding some contract z as a seller (also leaving net demand unchanged).

In a quasilinear setting, Hatfield et al. (2019) showed that full substitutability implies monotone–substitutability. In general, however, full substitutability does not imply the Laws of Aggregate Supply and Demand (see, e.g., Example 1 of Section 6) and, thus, does not imply monotone–substitutability.[18] Hatfield et al. (2019) provided an extended discussion of the restrictions imposed by monotone–substitutability and economically interesting classes of preferences that are monotone–substitutable in quasilinear settings.

2.2 *Stability and chain stability*

Our main result connects two solution concepts for trading network settings: *stability*, based on the concepts introduced by Hatfield and Kominers (2012) and Hatfield et al. (2013), and *chain stability*, based on the concept introduced by Ostrovsky (2008).

We begin with the definition of stability.

Definition 4. An outcome A is *stable* if it is both:

(i) *individually rational*, i.e., $A_i \in C_i(A)$ for all i; and

(ii) *unblocked*, i.e., there is no nonempty *blocking* set $Z \subseteq X$ such that

   (a) Z is feasible,

   (b) $Z \cap A = \varnothing$, and

   (c) for all $i \in a(Z)$, for all $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.

Individual rationality is a voluntary participation condition based on the idea that an agent can always unilaterally drop contracts if doing so increases his welfare. The unblockedness condition states that when presented with a stable outcome A, one cannot propose a new set of contracts such that all the agents involved in those new contracts would strictly prefer to execute all of them (and possibly drop some of their existing contracts in A) instead of executing only some of them (or none).

To introduce our second solution concept, chain stability, we first need to formalize the notion of a chain.

Definition 5. A nonempty set of trades Ψ is a *chain* if its elements can be arranged in some order $\psi^1, \ldots, \psi^{|\Psi|}$ such that $s(\psi^{\ell+1}) = b(\psi^{\ell})$ for all $\ell \in \{1, 2, \ldots, |\Psi| - 1\}$. A nonempty set of contracts Z is a *chain* if $\tau(Z)$ is a chain.

Note that because there is no vertical ordering of agents in our framework, Definition 5 adapts the "chain" concept of Ostrovsky (2008) by allowing chains to cross themselves: the buyer in contract $y^{|Z|}$ is allowed to be the seller in contract y^1 (in which case

[18] Even in the special case of a two-sided market, full substitutability does not imply the Laws of Aggregate Supply and Demand (see, e.g., Hatfield and Milgrom 2005).

the chain becomes a cycle) and a given agent can be involved in the chain multiple times. Example 3 of Section 6 illustrates the role of "self-crossing" chains in our results.

We now define chain stability.

DEFINITION 6. An outcome A is *chain stable* if it is both:

(i) *individually rational*, i.e., $A_i \in C_i(A)$ for all i; and

(ii) *not blocked by a chain*, i.e., there is no nonempty *blocking* chain $Z \subseteq X$ such that

(a) Z is feasible,

(b) $Z \cap A = \varnothing$, and

(c) for all $i \in a(Z)$, for all $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.

The essential difference between the definitions of stability (Definition 4) and chain stability (Definition 6) consists of just one word: "set" in requirement (ii) in the definition of stability versus "chain" in requirement (ii) in the definition of chain stability. Substantively, however, the two definitions are very different. Blocking sets considered in Definition 4 can be arbitrarily complex, involving any sets of contracts and agents. By contrast, blocking chains considered in Definition 6 have a well-defined linear structure. As Ostrovsky (2008) argued, blocking chains are much easier to identify and organize than arbitrary blocking sets. An agent can contact a potential supplier and propose a possible contract. That supplier would then contact one of his suppliers, and so on, and the process would proceed in a linear fashion until a blocking chain is identified. An important difference in our setting is that in the case of loops, the agent initiating the communication may need to make his initial offer tentative: instead of proposing a contract outright, he would have to say something along the lines of, "I may be interested in signing the contract x with you, if I can subsequently sign a contract y with a customer for one of the outputs I am offering." An initial agent may also try initiating the deviation in both directions at the same time, making tentative offers to a supplier and a customer. While identifying chains in our trading network setting is more complicated than identifying pairwise blocks in two-sided settings or blocking chains in the setting of Ostrovsky (2008), it is still relatively simple and natural compared to trying to identify grand coalitions, which may require considering large, complex sets of blocking contracts (see Section 5).

3. Main result: Equivalence of stability concepts

Stability appears substantively different and noticeably stronger than chain stability: the former requires robustness to all blocking sets, while the latter requires robustness only to specific blocking sets—chains of contracts. It is immediate that any stable outcome is chain stable, regardless of whether agents' preferences are fully substitutable or satisfy the Laws of Aggregate Supply and Demand. Our main result shows that when agents' preferences are monotone–substitutable, the two solution concepts are, in fact, equivalent.

Theorem 1. *If all agents' preferences are monotone–substitutable, then any chain stable outcome is stable.*

Theorem 1 is an immediate corollary of a stronger result: when agents' preferences are monotone–substitutable, any set blocking an outcome A can be "decomposed" into blocking chains.

Theorem 2. *Suppose that all agents' preferences are monotone–substitutable. If a feasible outcome A is blocked by a set Z, then for some $K \geq 1$ we can partition the set Z into a collection of K chains $\{W^k\}_{k=1}^K$ such that A is blocked by W^1 and for any $k \leq K-1$ the set of contracts $A \cup W^1 \cup \cdots \cup W^k$ is blocked by W^{k+1}.*

In particular, Theorem 1 follows from Theorem 2 by noting that if an outcome A is not stable, then either A is not individually rational (and so A cannot be chain stable by definition) or there exists a blocking set Z; Theorem 2 then implies that we can construct a chain W^1 that blocks A.

We prove Theorem 2 by way of the following lemma, which shows that, for any set of contracts A blocked by some set Z, either Z is a chain or we can remove a chain W from Z and still have that $Z \setminus W$ blocks A.

Lemma 2. *Suppose that all agents' preferences are monotone–substitutable. For any feasible outcome A blocked by a set Z, if Z is not itself a chain, then there exists a chain $W \subsetneq Z$ such that A is blocked by $Z \setminus W$ and $A \cup (Z \setminus W)$ is blocked by W.*

Lemma 2 implies that for any set Z blocking A, if Z is not a chain, then there exists a chain $\tilde{W}^1 \subsetneq Z$ such that A is blocked by $Z \setminus \tilde{W}^1$ and $A \cup (Z \setminus \tilde{W}^1)$ is blocked by $\tilde{W}^1$. But then, applying Lemma 2 again, if $Z \setminus \tilde{W}^1$ is not a chain, there exists a chain $\tilde{W}^2 \subsetneq Z \setminus \tilde{W}^1$ such that A is blocked by $(Z \setminus \tilde{W}^1) \setminus \tilde{W}^2 = Z \setminus (\tilde{W}^1 \cup \tilde{W}^2)$ and $A \cup (Z \setminus (\tilde{W}^1 \cup \tilde{W}^2))$ is blocked by $\tilde{W}^2$. Iterating the preceding logic, we obtain a sequence of chains $\tilde{W}^1, \tilde{W}^2, \ldots$ such that, for each ℓ, $Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^\ell)$ blocks A, $\tilde{W}^\ell$ blocks $A \cup (Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^\ell))$, and, if $Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^\ell)$ is not a chain, we can extend the sequence by another chain $\tilde{W}^{\ell+1}$. As Z is finite (since it is feasible), the sequence of chains $\tilde{W}^1, \tilde{W}^2, \ldots$ must be finite; consequently, there must be some L such that $Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^L)$ is a chain that—by construction—blocks A. Setting $W^1 \equiv Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^L)$ and $W^\ell \equiv \tilde{W}^{L+2-\ell}$ for all $\ell \in \{2, \ldots, L+1\}$, we see that Lemma 2 implies Theorem 2: By construction of $\tilde{W}^1, \ldots, \tilde{W}^L$, we have that $\tilde{W}^1, \ldots, \tilde{W}^L$ is a partition of Z, A is blocked by the set $W^1 = Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^L)$, and, for all $\ell \leq L$,

$$A \cup W^1 \cup \cdots \cup W^\ell = A \cup (Z \setminus (\tilde{W}^1 \cup \cdots \cup \tilde{W}^{L+1-\ell}))$$

is blocked by the set $W^{\ell+1} = \tilde{W}^{L+1-\ell}$.

Before proceeding to the formal proof of Lemma 2, we explain the intuition behind this result and highlight where our assumptions on preferences are used. For simplicity, assume that there are no indifferences in agents' preferences over any relevant sets of contracts, i.e., that all agents' choice correspondences over subsets of $Z \cup A$ are single-valued. Our goal is to "peel off" a chain W from the set Z in such a way that the remaining set $Z \smallsetminus W$ still blocks A.

We start the chain "in the middle," by choosing an arbitrary contract $z^0 \in Z$. Since Z is a blocking set, we have that $Z_{b(z^0)} \subseteq Z^*_{b(z^0)}$, where $\{Z^*_{b(z^0)}\} = C_{b(z^0)}(Z \cup A)$. Thus, as the preferences of $b(z^0)$ are monotone–substitutable, case (i) of Lemma 1 implies that, for the unique $Y^*_{b(z^0)} \in C_{b(z^0)}((Z \smallsetminus \{z^0\}) \cup A)$, either:[19]

(i) $Z^*_{b(z^0)} = Y^*_{b(z^0)} \cup \{z^0\}$;

(ii) $Z^*_{\to b(z^0)} = (Y^*_{\to b(z^0)} \cup \{z^0\}) \smallsetminus \{y\}$ for some $y \in [(A \cup Z) \smallsetminus Z^*]_{\to b(z^0)}$ and $Z^*_{b(z^0)\to} = Y^*_{b(z^0)\to}$; or

(iii) $Z^*_{\to b(z^0)} = Y^*_{\to b(z^0)} \cup \{z^0\}$ and $Z^*_{b(z^0)\to} = Y^*_{b(z^0)\to} \cup \{\bar{y}\}$ for some $\bar{y} \in Z^*_{b(z^0)\to}$.

Rewriting the preceding equalities to describe Y^*, we find that either:

(i′) $Y^*_{b(z^0)} = Z^*_{b(z^0)} \smallsetminus \{z^0\}$;

(ii′) $Y^*_{\to b(z^0)} = (Z^*_{\to b(z^0)} \smallsetminus \{z^0\}) \cup \{y\}$ for some $y \in (A \cup Z) \smallsetminus Z^*$ and $Y^*_{b(z^0)\to} = Z^*_{b(z^0)\to}$; or

(iii′) $Y^*_{\to b(z^0)} = (Z^*_{\to b(z^0)} \smallsetminus \{z^0\})$ and $Y^*_{b(z^0)\to} = Z^*_{b(z^0)\to} \smallsetminus \{\bar{y}\}$ for some $\bar{y} \in Z^*_{b(z^0)\to}$.

Thus, with respect to the contracts in Z, there are two possibilities:

(a) The agent $b(z^0)$ chooses all of his remaining contracts in $[Z \smallsetminus \{z^0\}]_{b(z^0)}$, i.e., we have $[Z \smallsetminus \{z^0\}]_{b(z^0)} \subseteq Y^*$. This possibility corresponds to the above cases (i′) and (ii′) (since $(A \cup Z) \smallsetminus Z^* \subseteq A$ as $b(z^0)$ chooses all of the contracts in Z from $A \cup Z$), and (iii′) when $\bar{y} \in A$.

(b) The agent $b(z^0)$ chooses all of his remaining contracts in $[Z \smallsetminus \{z^0\}]_{b(z^0)}$ except for one contract $z^1 = \bar{y}$ for which he is the seller, i.e., we have $[Z \smallsetminus \{z^0, z^1\}]_{b(z^0)} \subseteq Y^*$ for some $z^1 \in Z_{b(z^0)\to}$. This possibility corresponds to case (iii′) when $\bar{y} \in Z$.

In case (a), we have found the "downstream end" of the chain; in case (b), we extend the chain by adding z^1 and now consider $b(z^1)$. By assumption, the preferences of $b(z^1)$ are also monotone–substitutable, and so the same analysis applies: either z^1 is the downstream end of the chain or we can extend the chain by adding a contract $z^2 \in Z_{b(z^1)\to}$ and considering $b(z^2)$. Since the number of contracts in Z is finite, by iterating this process, we must eventually reach a contract z^N such that the agent $b(z^N)$ chooses all of his remaining contracts in $[Z \smallsetminus \{z^0, \dots, z^N\}]_{b(z^N)}$; thus, z^N is the downstream end of the chain. An analogous process allows us to grow the chain "upstream," adding contracts $z^{-1}, z^{-2}, \dots$ until we reach the "upstream end," z^{-M}. The chain $W = \{z^{-M}, \dots, z^N\}$ satisfies the requirements of Lemma 2: First, W blocks $(Z \cup A) \smallsetminus W$

[19]Note that the first possibility described in the first case of Lemma 1 cannot happen, as we know $z^0 \in Z^*_{b(z^0)}$ since Z is a blocking set.

because Z blocks A, i.e., every contract in Z is chosen from $Z \cup A$, and so every contract in $W \subseteq Z$ is chosen from $((Z \cup A) \smallsetminus W) \cup W = Z \cup A$. Second, by construction, every agent chooses all of their contracts in $Z \smallsetminus W$ from $(Z \cup A) \smallsetminus W$; thus, $Z \smallsetminus W$ blocks A.

The full proof of Lemma 2 follows the sketch just described, but the execution is much more challenging due to the need to account for multivalued choice correspondences.[20]

3.1 *Proof of* Lemma 2

We first define the *A-endowed utility function* $\hat{U}_i(\cdot; A)$ for each $i \in I$ as

$$\hat{U}_i(Y; A) \equiv \max_{\bar{A} \subseteq A} \{U_i(Y \cup \bar{A})\};$$

that is, $\hat{U}_i(Y; A)$ is the maximum utility that agent i can obtain by combining Y with elements of A. This gives rise to an *A-endowed choice correspondence* $\hat{C}_i(\cdot; A)$ for each $i \in I$, given by

$$\hat{C}_i(Y; A) \equiv \underset{\bar{Y} \subseteq Y}{\operatorname{argmax}} \{\hat{U}_i(\bar{Y}; A)\} = \{\tilde{Y} \smallsetminus A : \tilde{Y} \in C_i(Y \cup A)\};$$

that is, an element of $\hat{C}_i(Y; A)$ is a set of contracts that i "chooses" from Y when he has access to all the contracts in A. Note that since Z is a blocking set, $Z_i \subseteq Y$ for all $Y \in C_i(Z \cup A)$ for all $i \in I$ and, thus, $\hat{C}_i(Z; A) = \{Z_i\}$ for all $i \in I$.

Take any contract $z^0 \in Z$. We algorithmically "grow" a chain W containing z^0 by proceeding upstream and downstream from z^0. Specifically, in a sequence of steps from z^0, we grow a *quasi-removable chain*, i.e., a chain $W = \{z^{-m}, \dots, z^0, \dots, z^n\}$ such that $Z \smallsetminus W$ is a blocking set except (possibly) for the buyer of z^n and the seller of z^{-m}. We first proceed downstream, showing that after each step, either $Z \smallsetminus W$ behaves like a blocking set for the buyer of z^n, in which case z^n is a *terminal contract*, or we can extend the quasi-removable chain W at least one step further. We then proceed upstream analogously. Once we have found the downstream and upstream terminal contracts, our quasi-removable chain W is in fact "removable" from the blocking set Z, in the sense that $Z \smallsetminus W$ blocks A, as desired. We now formally define what it means for a chain to be quasi-removable.

Definition 7. A chain $W^{-m,n} = \{z^{-m}, \dots, z^n\}$ is *quasi-removable* under the following conditions:

(i) For all $i \in I \smallsetminus \{s(z^{-m}), b(z^n)\}$, we have that $\{[Z \smallsetminus W^{-m,n}]_i\} = \hat{C}_i(Z \smallsetminus W^{-m,n}; A)$.

[20]Our formal proof follows the sketch just presented, but allows for cases in which the choice correspondence is not single-valued. In particular, we cannot use Lemma 1, as it does not allow us to characterize $C_{b(z^0)}((Z \smallsetminus \{z^0\}) \cup A)$ if the choice correspondence $C_{b(z^0)}(Z \cup A)$ is not single-valued; rather, we need to prove an analogue to the conclusion of Lemma 1 that accounts for the fact that $C_{b(z^0)}(Z \cup A)$ may be multivalued. Similarly, we need to prove an analogue to the conclusion of Lemma 1 for the case in which a chain "self-crosses."

(ii) If $b(z^n) \neq s(z^{-m})$, then, when choosing from $Z \setminus W^{-m,n}$, we have that both:

(a) the buyer of z^n never drops a contract for which he is the buyer and drops at most one contract for which he is the seller, i.e., we have for all

$$\hat{Z}^* \in \hat{C}_{b(z^n)}(Z \setminus W^{-m,n}; A)$$

that

$$\hat{Z}^*_{\to b(z^n)} = [Z \setminus W^{-m,n}]_{\to b(z^n)}$$

and either

$$\hat{Z}^*_{b(z^n)\to} = [Z \setminus W^{-m,n}]_{b(z^n)\to}$$

or there exists a $z^{n+1} \in Z_{b(z^n)\to}$ such that

$$\hat{Z}^*_{b(z^n)\to} = [Z \setminus (W^{-m,n} \cup \{z^{n+1}\})]_{b(z^n)\to};$$

and

(b) the seller of z^{-m} never drops a contract for which he is the seller and drops at most one contract for which he is the buyer, i.e., we have for all

$$\hat{Z}^* \in \hat{C}_{s(z^{-m})}(Z \setminus W^{-m,n}; A)$$

that

$$\hat{Z}^*_{s(z^{-m})\to} = [Z \setminus W^{-m,n}]_{s(z^{-m})\to}$$

and either

$$\hat{Z}^*_{\to s(z^{-m})} = [Z \setminus W^{-m,n}]_{\to s(z^{-m})}$$

or there exists a $z^{-m-1} \in Z_{\to s(z^{-m})}$ such that

$$\hat{Z}^*_{\to s(z^{-m})} = [Z \setminus (W^{-m,n} \cup \{z^{-m-1}\})]_{\to s(z^{-m})}.$$

(iii) If $b(z^n) = s(z^{-m}) = k$, then when choosing from $Z \setminus W^{-m,n}$, agent k drops at most one contract for which he is the buyer and at most one contract for which he is the seller, i.e., we have for all $\hat{Z}^* \in \hat{C}_k(Z \setminus W^{-m,n}; A)$ that both:

(a) either

$$\hat{Z}^*_{\to k} = [Z \setminus W^{-m,n}]_{\to k}$$

or there exists a $z^{-m-1} \in Z_{\to k}$ such that

$$\hat{Z}^*_{\to k} = [Z \setminus (W^{-m,n} \cup \{z^{-m-1}\})]_{\to k};$$

and

(b) either

$$\hat{Z}^*_{k\to} = [Z \smallsetminus W^{-m,n}]_{k\to}$$

or there exists a $z^{n+1} \in Z_{k\to}$ such that

$$\hat{Z}^*_{k\to} = [Z \smallsetminus (W^{-m,n} \cup \{z^{n+1}\})]_{k\to}.$$

The first condition of Definition 7 ensures that each agent not associated with either end of the chain chooses all of the contracts in $Z \smallsetminus W^{-m,n}$ that he is associated with. The second condition of Definition 7 ensures that when each end of the chain is associated with a different agent, the agent at each end chooses all but one contract in $Z \smallsetminus W^{-m,n}$ that he is associated with. The third condition of Definition 7 ensures that when each end of the chain is associated with the same agent, that agent chooses all of the contracts in $Z \smallsetminus W^{-m,n}$ that he is associated with except for possibly one contract as a buyer and possibly one contract as a seller.

We say that a quasi-removable chain $W^{-m,n} = \{z^{-m}, \dots, z^n\}$ is

(i) *downstream terminal* if $b(z^n)$ strictly demands all of the contracts for which he is a seller, i.e., for all $\hat{Z}^* \in \hat{C}_{b(z^n)}(Z \smallsetminus W^{-m,n}; A)$, we have that

$$\hat{Z}^*_{b(z^n)\to} = [Z \smallsetminus W^{-m,n}]_{b(z^n)\to},$$

and

(ii) *upstream terminal* if $s(z^{-m})$ strictly demands all of the contracts for which he is a buyer, i.e., for all $\hat{Z}^* \in \hat{C}_{s(z^{-m})}(Z \smallsetminus W^{-m,n}; A)$, we have that

$$\hat{Z}^*_{\to s(z^{-m})} = [Z \smallsetminus W^{-m,n}]_{\to s(z^{-m})}.$$

We now present a series of five claims, all proven in Appendix A, that we combine to establish Lemma 2.

Claim 1. *Consider any $z^0 \in Z$. Then $W^{0,0} \equiv \{z^0\}$ is a quasi-removable chain.*

Claim 1 shows that for any arbitrary element $z^0 \in Z$, the set $W^{0,0} \equiv \{z^0\}$ is a quasi-removable chain. Our next claim shows that any blocking chain that is not downstream terminal can be extended into a longer quasi-removable chain through the addition of a downstream contract.

Claim 2. *Suppose that $W^{-m,n} = \{z^{-m}, \dots, z^n\}$ is a quasi-removable chain that is not downstream terminal. Then there exists a z^{n+1} such that $s(z^{n+1}) = b(z^n)$ and such that $W^{-m,n+1} \equiv W^{-m,n} \cup \{z^{n+1}\}$ is a quasi-removable chain. Moreover, if $W^{-m,n}$ is upstream terminal, then $W^{-m,n+1}$ is upstream terminal.*

An analogous result holds upstream: any blocking chain that is not upstream terminal can be extended into a longer quasi-removable chain through the addition of an upstream contract.

Claim 3. *Suppose that $W^{-m,n} = \{z^{-m}, \dots, z^n\}$ is a quasi-removable chain that is not upstream terminal. Then there exists a z^{-m-1} such that $b(z^{-m-1}) = s(z^{-m})$ and such that $W^{-m-1,n} \equiv W^{-m,n} \cup \{z^{-m-1}\}$ is a quasi-removable chain. Moreover, if $W^{-m,n}$ is downstream terminal, then $W^{-m-1,n}$ is downstream terminal.*

Our next claim ensures that once we have found a quasi-removable chain that is both downstream and upstream terminal, then that quasi-removable chain is, in fact, a blocking chain.

Claim 4. *If $W^{-m,n} = \{z^{-m}, \dots, z^n\}$ is a downstream and upstream terminal quasi-removable chain, then $Z \setminus W^{-m,n}$ blocks A.*

Our last claim verifies that any subset W of a blocking set Z blocks $A \cup (Z \setminus W)$.

Claim 5. *Any nonempty $W \subseteq Z$ blocks $A \cup (Z \setminus W)$.*

We now complete the proof of Lemma 2 by way of our claims. Consider any $z^0 \in Z$; by Claim 1, we have that $W^{0,0} = \{z^0\}$ is a quasi-removable chain. If $W^{0,0}$ is not downstream terminal, then by Claim 2, there exists a z^1 such that $s(z^1) = b(z^0)$ and such that $W^{0,1} = \{z^0, z^1\}$ is a quasi-removable chain. Proceeding inductively, any quasi-removable chain $W^{0,n} = \{z^0, \dots, z^n\}$ that is not downstream terminal can be extended to a quasi-removable chain $W^{0,n+1} = W^{0,n} \cup \{z^{n+1}\}$ by adding one sell-side contract z^{n+1} for the buyer of z^n. Since Z is finite and all the quasi-removable chains are contained in Z, this downstream extension process must eventually end at a quasi-removable chain $W^{0,N}$ that is downstream terminal. Similarly, if $W^{0,N}$ is a quasi-removable chain that is downstream but not upstream terminal, then by Claim 3, there exists a z^{-1} such that $W^{-1,N} = W^{0,N} \cup \{z^{-1}\}$ is a downstream terminal quasi-removable chain. Again proceeding inductively, we can extend any downstream but not upstream terminal quasi-removable chain $W^{-m,N}$ to a downstream terminal quasi-removable chain $W^{-m-1,N}$, until we reach a quasi-removable chain $W^{-M,N}$ that is downstream and upstream terminal. Finally, by Claims 4 and 5, $Z \setminus W^{-M,N}$ must block A and $W^{-M,N}$ must block $A \cup (Z \setminus W^{-M,N})$.

4. Chain stability and competitive equilibrium

The results of Section 3 hold for general sets of contracts under monotone–substitutable preferences. For an environment in which both

- prices are continuous and unrestricted, i.e., $X = \Omega \times \mathbb{R}$, and
- agents' preferences are quasilinear in prices,

Hatfield et al. (2013) showed that when agents' preferences are fully substitutable, an outcome is stable if and only if it is consistent with competitive equilibrium. Thus, a corollary of Theorem 1 is that in the trading network setting of Hatfield et al. (2013), an outcome is consistent with competitive equilibrium if and only if it is not blocked by a chain of contracts; for a formal statement of this result, see Appendix B.

5. Quantifying the simplicity gain

Theorem 1 implies that under monotone–substitutability, checking whether an outcome is stable (and in the quasilinear case, consistent with competitive equilibrium) reduces to checking whether that outcome is chain stable. In this section, we examine the extent to which Theorem 1 simplifies checking stability. We first consider asymptotics as the economy grows large; then we discuss computational complexity aspects.

5.1 *Asymptotic simplicity gains*

We show that while checking directly whether a given outcome Y is stable requires checking $2^{|X \smallsetminus Y|}$ possible blocking sets, the reduction to chain stability leads to a significant asymptotic simplicity gain, in the sense that the proportion of possible blocking sets that are chains goes to 0 as the economy grows large.[21]

Formally, we define a *sequence of economies* $(I, \Omega^m)_{m=1}^{\infty}$ as having a fixed set of agents I and a sequence of finite sets of trades $\Omega^1, \Omega^2, \ldots$ such that $|\Omega^m| = m$. For a given $\omega \in \bigcup_{m=1}^{\infty} \Omega^m$, let $\mathbb{P}(\omega) \subseteq \mathbb{R}$ be the set of possible prices for ω; that is, we assume that the set of possible prices associated with a given trade ω does not vary with m. For the economy (I, Ω^m), the set of contracts is given by $X^m \equiv \bigcup_{\omega \in \Omega^m} \bigcup_{p \in \mathbb{P}(\omega)} \{(\omega, p)\}$.[22]

Note that checking the stability of an outcome Y for the economy (I, Ω^m) may require checking blocking sets corresponding to any set of trades in

$$\mathfrak{B}^m(Y) \equiv \{\Psi \subseteq \Omega^m \smallsetminus \tau(Y)\}.$$

By contrast, checking the chain stability of an outcome Y for the economy (I, Ω^m) requires checking blocking chains corresponding to any chain of trades in

$$\mathfrak{C}^m(Y) \equiv \{\Psi \subseteq \Omega^m \smallsetminus \tau(Y) : \Psi \text{ is a chain}\}.$$

We show that, for any fixed set of contracts Y, the ratio of [the number of distinct sets of chains of trades corresponding to possible blocking chains] to [the number of distinct sets of trades corresponding to possible blocking sets], i.e., $\frac{|\mathfrak{C}^m(Y)|}{|\mathfrak{B}^m(Y)|}$, goes to 0 as m grows large.

Theorem 3. *For any sequence of economies $(I, \Omega^m)_{m=1}^{\infty}$ such that $|\Omega^m| = m$ for all m, for any Y, we have that*

$$\frac{|\mathfrak{C}^m(Y)|}{|\mathfrak{B}^m(Y)|} = O\left(\frac{\sqrt{\log_2 m}}{\sqrt{m}}\right).$$

In particular, $\frac{|\mathfrak{C}^m(Y)|}{|\mathfrak{B}^m(Y)|} \to 0$ *as* $m \to \infty$.

[21]Intuitively, we show that as we randomly add trades and the economy grows large, the probability that an arbitrary blocking set is a chain goes to 0.

[22]Our modeling in this section is deliberately parsimonious. Since the results in this section rely exclusively on combinatorial arguments regarding the number of chains and sets of trades that need to be considered, our requirements that the set of agents I is fixed and that the set of possible prices associated with a trade is invariant across economies could both be relaxed, e.g., allowing the sets of agents and prices to vary with m would not affect our results.

Theorem 3 follows from a general graph-theoretic result proven by Shayani (2018). In Appendix C, we present a formal proof of Theorem 3, adapting the argument of Shayani (2018) to our setting.

To understand the intuition behind Theorem 3, consider a directed multi-graph with the set of vertices I and the set of edges having one edge for each trade in Ω^m, directed from the seller to the buyer of that trade. To determine what proportion of the subsets of Ω^m consists of chains, we proceed via a probabilistic argument: We consider a random set of trades Ψ chosen from Ω^m by including each trade $\omega \in \Omega^m$ in Ψ independently with probability $\frac{1}{2}$. For a random set of trades Ψ to be a chain, the following two "balancedness" requirements have to be satisfied:

(i) For each agent $i \in a(\Psi)$, the number of contracts in which that agent is the buyer differs by at most 1 from the number of contracts in which he is the seller, i.e., $||\Psi_{\to i}| - |\Psi_{i\to}|| \leq 1$.

(ii) There are at most two distinct agents $j \in a(\Psi)$ who sign different numbers of contracts as a buyer and as a seller, i.e., there are at most two distinct agents $j \in a(\Psi)$ for whom $|\Psi_{\to j}| - |\Psi_{j\to}| \neq 0$.

These balancedness requirements follow directly from the definition of a chain (Definition 5), as the buyer of the first trade is the seller of the second trade, the buyer of the second trade is the seller of the third trade, ..., and the buyer of the $(|\Psi| - 1)$-st trade is the seller of the $|\Psi|$-th trade; thus, only the seller of the first trade and the buyer of the $|\Psi|$-th trade can sign different numbers of contracts as a buyer and a seller.

For the random set of trades Ψ, there are two cases to consider.

Large agent case: In this case, there is one "large" agent i, who is involved in many of the m trades in Ψ.

Many small agents case: In this case, there are many "small" distinct agents, each of whom is involved in a few trades in Ψ.

In the large agent case, we show that the probability that the first balancedness condition is satisfied for a "big" agent i is small because it is unlikely that i will have roughly equal numbers of contracts in which he is a buyer and in which he is a seller, as he is involved in many trades in the random set Ψ. In the many small agents case, we show that the probability that Ψ is such that $|\Psi_{\to j}| - |\Psi_{j\to}| = 0$ for all but two agents j is small, so it is unlikely that the second balancedness condition is satisfied. Combining the preceding two results shows that, in the limit, very few random sets of trades will be chains.

Theorem 3 implies that for general trading networks, the ratio of chains to the total number of subsets converges to 0 as the number of trades grows large. Thus, the set of chains is asymptotically a vanishingly small fraction of the set of potential blocking sets; consequently, checking stability by considering each possible blocking chain is asymptotically much simpler than checking stability by considering each possible blocking set.[23] In fact, even for settings for which the existence of stable outcomes is not guaranteed,[24] Theorem 1 implies that checking for the existence of chain stable outcomes is sufficient when agents' preferences are monotone–substitutable, and Theorem 3 implies that checking chain stability is substantially easier than checking stability.

If the trading network has additional structure, then the simplicity gain can be much higher than that implied by the bound in Theorem 3. For instance, consider the case of *multilayered supply chains*, á la Ostrovsky (2008). In a multilayered supply chain, there are $L+1$ *layers* $(I^\ell)_{\ell=1}^{L+1}$, which partition the set of agents; each trade "flows" one layer down the supply chain, i.e., for any trade $\omega \in \Omega$, if $s(\omega) \in I^\ell$, then $b(\omega) \in I^{\ell+1}$. Thus, there are L *bands* of trades, $\Omega^1, \ldots, \Omega^L$, in between the layers of agents, such that $s(\Omega^\ell) \subseteq I^\ell$ and $b(\Omega^\ell) \subseteq I^{\ell+1}$. In this case, the total number of chains of trades is bounded by $\prod_{\ell=1}^{L}(|\Omega^\ell|+1)$, while the total number of sets of trades is given by $2^{|\Omega^1|+\cdots+|\Omega^L|}$.

Our results also imply (by combining Corollary 1 and Theorem 3) that checking whether an outcome is consistent with competitive equilibrium becomes straightforward in the Sun and Yang (2006, 2009) environment with gross substitutes and complements. In such an environment, one side of the market is a set of buyers while the other side of the market consists of two distinct groups of objects. Buyers view objects in the same group as substitutes for each other, but view objects in different groups as complements; such preferences arise naturally when a firm has two types of complementary inputs. As Hatfield et al. (2013) showed, the Sun and Yang (2006, 2009) environment is a special case of the Hatfield et al. (2013) trading network framework.[25] Moreover, chains in the Sun and Yang (2006, 2009) environment are particularly simple: they consist either of one buyer and one object (or, more formally, one contract between a buyer and an object) or of one buyer and one object from each of the two groups (again, more formally, two contracts, involving the same buyer and two objects from different groups). Thus, checking for consistency with competitive equilibrium reduces to checking one- and two-contract blocking chains. In the two-sided setting of Kelso and Crawford (1982), which is itself a special case of the Sun and Yang (2006, 2009) framework, our results imply that checking for consistency with competitive equilibrium reduces to checking for single-contract blocks.

5.2 *Computational complexity*

Subsequent to the first version of this paper, a number of settings that are special cases of our model have been studied.

For the special case of trading networks called *flow networks*, in which agents' preferences are strict (and, thus, continuous transfers are not allowed) and there are exactly two so-called *terminal agents* who always choose all of the contracts that they

[23]However, for arbitrarily complex trading networks, Shayani (2018) showed that the bound in Theorem 3 is almost tight.

[24]For example, if prices are not allowed to vary freely and preferences are not quasilinear, monotone–substitutability is not, in general, sufficient to guarantee the existence of stable outcomes; see, e.g., Hatfield and Kominers (2012).

[25]The embedding of Hatfield et al. (2013) allows for much more general environments than those considered by Sun and Yang (2006, 2009): e.g., "objects" may have preferences over whom they match with and may be involved in multiple contracts.

are involved in, Fleiner et al. (2020) showed that establishing whether a stable outcome exists is an NP-complete problem.[26] Note that since the result established by Fleiner et al. (2020) is for a setting that is a special case of ours, it directly implies that checking whether a stable outcome exists remains an NP-complete problem in our setting too.

For the special case of arbitrarily complex trading networks in which agents' preferences are quasilinear in the numeraire, Candogan et al. (2019) developed a polynomial-time algorithm that, for a given outcome, either constructs a blocking chain or verifies that no such chain exists; for this case, combining the result of Candogan et al. (2019) with our Theorem 1 yields a polynomial-time algorithm for checking stability.

Finally, a number of new applications can be embedded into our model for which stable outcomes can be computed efficiently. For example, Andersson et al. (forthcoming) studied the organization of time banks and Manjunath and Westkamp (2019) studied shift exchanges between workers. Both Andersson et al. (forthcoming) and Manjunath and Westkamp (2019) developed algorithms that find individually rational and Pareto-efficient outcomes in polynomial time; furthermore, these outcomes turn out to be stable as well.

6. Examples

The proof of our main equivalence result (Theorem 1) requires monotone–substitutability—the conjunction of full substitutability and the Laws of Aggregate Supply and Demand. In this section, we show that whenever some agent's preferences fail to be fully substitutable or fail to satisfy the Laws of Aggregate Supply and Demand, our equivalence result may not hold. We also show that it is essential that the definition of chain stability allow chains to cross themselves, i.e., that we allow an agent to be involved in more than two contracts in a chain.[27]

We start with an example of preferences that are fully substitutable, but for which the Laws of Aggregate Supply and Demand do not hold—and the equivalence result does not hold either.[28]

Example 1. There are two agents, i and j. There are four contracts between the two agents: x, y, z, and w. Agent i is the seller of x, y, and z, and is the buyer of w, while agent j is the buyer of x, y, and z, and the seller of w. The economy is depicted in Figure 1.

[26] Even though Fleiner et al. (2020) did not explicitly assume that agents' choice functions satisfy the Laws of Aggregate Supply and Demand, their results still apply in our setting, as the choice functions they used in their construction satisfy the Laws of Aggregate Supply and Demand.

[27] For convenience, we give our examples in terms of ordinal preference relations over sets of contracts; it is straightforward to construct corresponding cardinal utility functions over sets of contracts that give rise to these ordinal preference relations, and we omit those constructions.

[28] As shown by Hatfield and Kominers (2012), the Laws of Aggregate Supply and Demand are not necessary for the equivalence of stability and chain stability in the supply chain setting. The need for monoton–substitutability in our setting is because we need to allow for chains to be self-crossing—which cannot happen in supply chain networks.

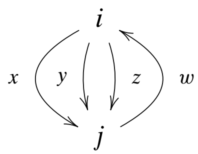

FIGURE 1. The economy of Example 1. Each arrow denotes a contract from its seller to its buyer.

The preferences of the agents are as follows. Informally, agent i is happy to sign contract w in which he is the buyer, regardless of what his options are on the other side of the market, and if (and only if) he is able to sign contract w, then he is also happy to sign any subset of the other three contracts (in which he is the seller)—the more, the better. Formally, the preferences of i over acceptable bundles of contracts are

$$\{w,x,y,z\} \succ_i \{w,x,y\} \succ_i \{w,x,z\} \succ_i \{w,y,z\} \succ_i \{w,x\} \succ_i \{w,y\} \succ_i \{w,z\} \succ_i \{w\} \succ_i \varnothing.$$

Agent j is happy to sign any subset of $\{x,y,z\}$ (in which he is the buyer)—the more, the better—no matter what his options are on the other side of the market. If (and only if) he has access to all three of x, y, and z, then he is also happy to sign contract w (in which he is the seller). Formally, the preferences of j over acceptable bundles of contracts are

$$\{w,x,y,z\} \succ_j \{x,y,z\} \succ_j \{x,y\} \succ_j \{x,z\} \succ_j \{y,z\} \succ_j \{x\} \succ_j \{y\} \succ_j \{z\} \succ_j \varnothing.$$

Note first that the preferences of agents i and j are fully substitutable but also note that the preferences of agent i do not satisfy the Law of Aggregate Demand.[29] The empty set of contracts, $\varnothing$, is not stable: it is blocked by the full set of contracts in the economy, $\{w,x,y,z\}$, which is the most preferred set of contracts for both agents. At the same time, the empty set of contracts is not blocked by any chain; hence, the empty set is chain stable. To see this, note first that any blocking set would, of course, have to involve both agents. Second, every nonempty set acceptable to agent i must include contract w, so w would have to be a part of the blocking chain. Third, the only set of contracts involving contract w that is acceptable to agent j is the full set of contracts $\{w,x,y,z\}$. Thus, $\{w,x,y,z\}$ is the only blocking set in this example—and it cannot be represented as a chain. ◊

Our second example shows that full substitutability likewise plays a critical role for the equivalence result: without it, chain stability is strictly weaker than stability, even when all agents' preferences satisfy the Laws of Aggregate Supply and Demand.

EXAMPLE 2. There are three agents: i, j, and k. There are two contracts: x and y. Agent i is the buyer of both x and y, agent j is the seller of x, and agent k is the seller of y. The economy is depicted in Figure 2.

[29] Indeed, $C_i(\{x,y,z\}) = \varnothing$ while $C_i(\{w,x,y,z\}) = \{w,x,y,z\}$, that is, the net demand of i falls (from 0 to -2) after i receives the new buy-side offer w.

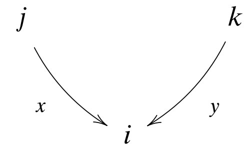

FIGURE 2. The economy of Example 2. Each arrow denotes a contract from its seller to its buyer.

The preferences of agents j and k are straightforward and are fully substitutable. Indeed, each agent desires to sign the contract in which he is the seller: $\{x\} \succ_j \varnothing$ and $\{y\} \succ_k \varnothing$. The preferences of agent i are not fully substitutable: i prefers signing both contracts to signing none, but prefers not signing any contracts to signing only one contract; that is, the preferences of agent i over acceptable bundles of contracts are given by $\{x, y\} \succ_i \varnothing$.

The empty set of contracts, $\varnothing$, is not stable: it is blocked by the full set of contracts in the economy, $\{x, y\}$. At the same time, the empty set is chain stable: Any chain involves agent i and contains exactly one contract—and agent i finds any such set of contracts unacceptable. ♢

Our third and final example shows that even when preferences are monotone–substitutable, it may not be sufficient to restrict attention to blocking chains that do not cross themselves. Specifically, if attention is restricted to chains in which each agent appears in at most two consecutive contracts, then an outcome that is robust to deviations by such chains may be blocked by richer sets of contracts.[30]

EXAMPLE 3. There are three agents: i, j, and k. There are four contracts: x^1, x^2, y^1, and y^2. Agent i is the buyer of contract x^1 and the seller of contract y^1. Agent j is the buyer of contract x^2 and the seller of contract y^2. Agent k is the seller of contracts x^1 and x^2 and the buyer of contracts y^1 and y^2. The economy is depicted in Figure 3.

The preferences of agents i and j are straightforward: Each one prefers to sign both contracts that he is associated with and is not interested in any other nonempty set of contracts; that is, $\{x^1, y^1\} \succ_i \varnothing$ and $\{x^2, y^2\} \succ_j \varnothing$. The preferences of agent k are

$$\{x^1, x^2, y^1, y^2\} \succ_k \{x^1, y^2\} \succ_k \{x^2, y^1\} \succ_k \varnothing;$$

agent k finds other nonempty sets of contracts unacceptable.

In this example, all agents' preferences are monotone–substitutable. Also, the empty set of contracts is not stable, as it is blocked by the chain $\{x^1, y^1, x^2, y^2\}$; that chain is self-crossing—it involves agent k in all four contracts. However, no chain that does not cross

[30]However, the necessity of considering self-crossing chains is only present in fully general trading networks. In particular, in the supply chain setting of Ostrovsky (2008), self-crossing chains are not even possible because each agent buys only from agents upstream and sells only to agents downstream. In the supply chain setting, stability and chain stability are also equivalent to the *tree stability* concept of Ostrovsky (2008).

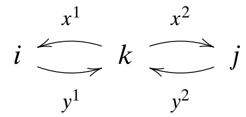

Figure 3. The economy of Example 3. Each arrow denotes a contract from its seller to its buyer.

itself blocks the empty set of contracts. For example, the chain $\{x^1, y^1\}$ does not block the empty set because $C_k(\{x^1, y^1\}) = \{\varnothing\}$, and so $\{x^1, y^1\}_k = \{x^1, y^1\} \notin C_k(\{x^1, y^1\})$. ◊

7. Conclusion

In this paper, we have shown that when all agents have monotone–substitutable preferences, every chain stable outcome is stable. As a corollary, we also have shown that in quasilinear environments with transferable utility and fully substitutable preferences, an outcome is consistent with competitive equilibrium if and only if it is chain stable.

In practice, blocking chains may be relatively easy to form: They require much less coordination than general blocking sets. Our work shows that under reasonable assumptions on preferences, ruling out these particularly natural blocks in fact guarantees that there are no possible blocks by groups of agents.

Appendix A: Proofs omitted from the main text

A.1 *Proof of Lemma 1*

To prove the first part of Lemma 1, we consider the case in which $\bar{y} \in [X \smallsetminus Y]_{\to i}$. Since the preferences of i are monotone–substitutable, we have from condition (i) of the definition of monotone–substitutability that there exists a $\bar{Y}_i^* \in C_i(Y \cup \{\bar{y}\})$ such that

$$Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq \left(Y_{\to i} \cup \{\bar{y}\}\right) \smallsetminus \bar{Y}^*_{\to i};$$

thus, we have

$$\bar{Y}^*_{\to i} \subseteq Y^*_{\to i} \cup \{\bar{y}\}. \tag{1a}$$

Moreover, we have from condition (i) of the definition of monotone–substitutability that

$$\bar{Y}^*_{i\to} \supseteq Y^*_{i\to}, \tag{1b}$$

which implies that

$$\left|\bar{Y}^*_{i\to}\right| - \left|Y^*_{i\to}\right| \geq 0.$$

Condition (i) of the definition of monotone–substitutability also then implies that

$$\left|\bar{Y}^*_{\to i}\right| - \left|Y^*_{\to i}\right| \geq \left|\bar{Y}^*_{i\to}\right| - \left|Y^*_{i\to}\right| \geq 0. \tag{2}$$

There are two cases to consider.

Case 1: $\bar{y} \notin \bar{Y}^*$. Then (1a) implies that $\bar{Y}^*_{\to i} \subseteq Y^*_{\to i}$ and so $|\bar{Y}^*_{\to i}| - |Y^*_{\to i}| \leq 0$. As (2) implies that $|\bar{Y}^*_{\to i}| - |Y^*_{\to i}| \geq 0$, we must then have that $|\bar{Y}^*_{\to i}| = |Y^*_{\to i}|$; hence, $\bar{Y}^*_{\to i} = Y^*_{\to i}$. Then (2) can only hold if $|\bar{Y}^*_{i\to}| = |Y^*_{i\to}|$, and so (1b) implies that $\bar{Y}^*_{i\to} = Y^*_{i\to}$. Thus, $\bar{Y}^* = Y^*$, which is possibility (a) in the lemma.

Case 2: $\bar{y} \in \bar{Y}^*$. Then (1a) implies that $|\bar{Y}^*_{\to i}| \leq |Y^*_{\to i}| + 1$.

If $|\bar{Y}^*_{\to i}| \leq |Y^*_{\to i}|$, then (as in the preceding case), (2) can only hold if

$$|\bar{Y}^*_{\to i}| - |Y^*_{\to i}| = |\bar{Y}^*_{i\to}| - |Y^*_{i\to}| = 0,$$

which implies that $\bar{Y}^*_{\to i} = (Y^*_{\to i} \cup \{\bar{y}\}) \setminus \{y\}$ for some $y \in Y^*_{\to i}$ (from (1a)) and $Y^*_{i\to} = \bar{Y}^*_{i\to}$ (from (1b)), which is possibility (c) in the lemma.

Otherwise, $|\bar{Y}^*_{\to i}| = |Y^*_{\to i}| + 1$ and so (1a) implies that $\bar{Y}^*_{\to i} = Y^*_{\to i} \cup \{\bar{y}\}$. Thus, $|\bar{Y}^*_{\to i}| - |Y^*_{\to i}| = 1$ and so (2) can only hold if either $|\bar{Y}^*_{i\to}| - |Y^*_{i\to}| = 0$ or $|\bar{Y}^*_{i\to}| - |Y^*_{i\to}| = 1$. In the former case, we have (from (1b)) that $\bar{Y}^*_{i\to} = Y^*_{i\to}$ and so $\bar{Y}^* = Y^* \cup \{\bar{y}\}$, which is possibility (b) in the lemma. In the latter case, we have (from (1b)) that $\bar{Y}^*_{i\to} = Y^*_{i\to} \cup \{z\}$ for some $z \in [Y \setminus Y^*]_{i\to}$, which is possibility (d) in the lemma.

The proof of the second part of Lemma 1 is completely analogous to the proof of the first part.

A.2 *Proof of Claim 1*

Condition (i) of Definition 7 holds immediately.

To prove that part (a) of condition (ii) of Definition 7 holds, we proceed as follows: Choose an arbitrary $Y^* \in C_{b(z^0)}((Z \setminus \{z^0\}) \cup A)$. Since $C_{b(z^0)}$ is monotone–substitutable, there exists a $Z^* \in C_{b(z^0)}(Z \cup A)$ such that

$$\big[\big((Z \setminus \{z^0\}) \cup A\big) \setminus Y^*\big]_{\to b(z^0)} \subseteq \big[(Z \cup A) \setminus Z^*\big]_{\to b(z^0)} \tag{3}$$

$$Y^*_{b(z^0)\to} \subseteq Z^*_{b(z^0)\to} \tag{4}$$

$$\big|Z^*_{\to b(z^0)}\big| - \big|Y^*_{\to b(z^0)}\big| \geq \big|Z^*_{b(z^0)\to}\big| - \big|Y^*_{b(z^0)\to}\big|. \tag{5}$$

Partition Y^* into $\hat{Y}^* \equiv Y^* \cap (Z \setminus \{z^0\})$ and $\check{Y}^* \equiv Y^* \cap A$, and partition Z^* into $\hat{Z}^* \equiv Z^* \cap Z$ and $\check{Z}^* \equiv Z^* \cap A$.[31] Note that since Z is a blocking set, we must have $\hat{Z}^*_{b(z^0)} = Z_{b(z^0)}$.

We argue first that when z^0 is no longer available, every optimal choice by $b(z^0)$ includes all of the remaining contracts in Z for which he is a buyer, i.e., $\hat{Y}^*_{\to b(z^0)} = [Z \setminus \{z^0\}]_{\to b(z^0)}$. Using the notation just introduced, we can rewrite (3) as

$$\big[\big((Z \setminus \{z^0\}) \cup A\big) \setminus \big(\hat{Y}^* \cup \check{Y}^*\big)\big]_{\to b(z^0)} \subseteq \big[(Z \cup A) \setminus \big(\hat{Z}^* \cup \check{Z}^*\big)\big]_{\to b(z^0)},$$

or, equivalently,

$$\big[\big((Z \setminus \{z^0\}) \setminus \hat{Y}^*\big) \cup \big(A \setminus \check{Y}^*\big)\big]_{\to b(z^0)} \subseteq \big[A \setminus \check{Z}^*\big]_{\to b(z^0)}. \tag{6}$$

[31] To see that these are partitions, recall that $A \cap Z = \varnothing$ as Z is a blocking set.

Here we have used the fact that $[Z \smallsetminus \hat{Z}^*]_{\to b(z^0)} = \varnothing$ since Z is a blocking set. From (6), we can immediately infer that $[(Z \smallsetminus \{z^0\}) \smallsetminus \hat{Y}^*]_{\to b(z^0)} \subseteq A$. Given that Z is a blocking set, we must have $Z \cap A = \varnothing$ and, thus, $[(Z \smallsetminus \{z^0\}) \smallsetminus \hat{Y}^*]_{\to b(z^0)} = \varnothing$. Hence, $[Z \smallsetminus \{z^0\}]_{\to b(z^0)} \subseteq \hat{Y}^*$. Since $\hat{Y}^* = Y^* \cap (Z \smallsetminus \{z^0\})$, we obtain that $[Z \smallsetminus \{z^0\}]_{\to b(z^0)} = \hat{Y}^*_{\to b(z^0)}$.

Next we argue that when z^0 is no longer available, every optimal choice by $b(z^0)$ excludes at most one of his contracts in Z for which he is a seller, i.e., either $\hat{Y}^*_{b(z^0)\to} = Z_{b(z^0)\to}$ or there exists a $z^1 \in Z$ such that $\hat{Y}^*_{b(z^0)\to} = [Z \smallsetminus \{z^1\}]_{b(z^0)\to}$. Note first that (3) implies that $b(z^0)$ chooses at most one fewer contract as a buyer when z^0 is no longer available, i.e., $|Z^*_{\to b(z^0)}| \le |Y^*_{\to b(z^0)} \cup \{z^0\}| = |Y^*_{\to b(z^0)}| + 1$, and so $|Z^*_{\to b(z^0)}| - |Y^*_{\to b(z^0)}| \le 1$. Hence, (5) implies that $|Z^*_{b(z^0)\to}| - |Y^*_{b(z^0)\to}| \le 1$; we rewrite this last inequality as

$$\left(|\hat{Z}^*_{b(z^0)\to}| - |\hat{Y}^*_{b(z^0)\to}|\right) + \left(|\check{Z}^*_{b(z^0)\to}| - |\check{Y}^*_{b(z^0)\to}|\right) \le 1. \tag{7}$$

Now by (4), we have that $Y^*_{b(z^0)\to} \subseteq Z^*_{b(z^0)\to}$ and, thus, $\check{Y}^*_{b(z^0)\to} \subseteq \check{Z}^*_{b(z^0)\to}$; combining this with (7) implies that $|\hat{Z}^*_{b(z^0)\to}| - |\hat{Y}^*_{b(z^0)\to}| \le 1$. Moreover, by (4), we have that $Y^*_{b(z^0)\to} \subseteq Z^*_{b(z^0)\to}$ and, thus, $\hat{Y}^*_{b(z^0)\to} \subseteq \hat{Z}^*_{b(z^0)\to}$; hence, either $\hat{Y}^*_{b(z^0)\to} = \hat{Z}^*_{b(z^0)\to} = Z_{b(z^0)\to}$ or there exists a $z^1 \in Z$ such that $\hat{Y}^*_{b(z^0)\to} = [\hat{Z}^* \smallsetminus \{z^1\}]_{b(z^0)\to} = [Z \smallsetminus \{z^1\}]_{b(z^0)\to}$.

Part (b) of condition (ii) of Definition 7 follows analogously to part (a).

Condition (iii) of Definition 7 holds vacuously as $b(z^0) \neq s(z^0)$.

A.3 *Proof of Claim 2*

Since $W^{-m,n}$ is a quasi-removable chain that is not downstream terminal, there exists a set $\tilde{Z}^* \in \hat{C}_{b(z^n)}(Z \smallsetminus W^{-m,n}; A)$ and a contract $z^{n+1} \in [Z \smallsetminus W^{-m,n}]_{b(z^n)\to}$ such that $\tilde{Z}^*_{b(z^n)\to} = [(Z \smallsetminus W^{-m,n}) \smallsetminus \{z^{n+1}\}]_{b(z^n)\to}$. We argue that $W^{-m,n+1} \equiv W^{-m,n} \cup \{z^{n+1}\}$ is a quasi-removable chain.

To see that $W^{-m,n+1}$ satisfies condition (i) of Definition 7, we note that:

- For all $i \in I \smallsetminus \{s(z^{-m}), b(z^n), b(z^{n+1})\}$, we have that

$$\{[Z \smallsetminus W^{-m,n+1}]_i\} = \hat{C}_i(Z \smallsetminus W^{-m,n+1}; A),$$

  as $W^{-m,n}$ is quasi-removable and $[Z \smallsetminus W^{-m,n+1}]_i = [Z \smallsetminus W^{-m,n}]_i$ for each such i.

- We show here that if $b(z^n) \neq s(z^{-m})$, then we have that $\{[Z \smallsetminus W^{-m,n+1}]_{b(z^n)}\} = \hat{C}_{b(z^n)}(Z \smallsetminus W^{-m,n+1}; A)$.[32,33] Let $\tilde{Z} \subsetneq [Z \smallsetminus W^{-m,n+1}]_{b(z^n)}$ be arbitrary. We claim that $\tilde{Z} \notin \hat{C}_{b(z^n)}(Z \smallsetminus W^{-m,n+1}; A)$. To see this, note first that part (a) of condition (ii) of Definition 7 applied to $W^{-m,n}$ implies that $\tilde{Z} \notin \hat{C}_{b(z^n)}(Z \smallsetminus W^{-m,n}; A)$; meanwhile, $[(Z \smallsetminus W^{-m,n}) \smallsetminus \{z^{n+1}\}]_{b(z^n)} \in \hat{C}_{b(z^n)}(Z \smallsetminus W^{-m,n}; A)$ by assumption. Thus,

$$\hat{U}_{b(z^n)}\left(\left[\left(Z \smallsetminus W^{-m,n}\right) \smallsetminus \{z^{n+1}\}\right]_{b(z^n)}\right) > \hat{U}_{b(z^n)}(\tilde{Z}).$$

[32]We consider the other case where $b(z^n) = s(z^{-m})$ subsequently.

[33]If $b(z^n) \neq s(z^{-m})$, we have that $b(z^n) \in I \smallsetminus \{s(z^{-m}), b(z^{n+1})\}$ since $s(z^{n+1}) = b(z^n)$ and $s(z^{n+1}) \neq b(z^{n+1})$. Hence, we also have to establish condition (i) of Definition 7 for $b(z^n)$ if $b(z^n) \neq s(z^{-m})$.

Hence, since $[(Z \setminus W^{-m,n}) \setminus \{z^{n+1}\}]_{b(z^n)}$ is available from $[(Z \setminus W^{-m,n}) \setminus \{z^{n+1}\}]_{b(z^n)}$ and provides a higher utility than $\tilde{Z}$, we have that $\tilde{Z} \notin \hat{C}_{b(z^n)}(Z \setminus W^{-m,n+1}; A)$.

To finish the proof, we consider two cases, depending on whether both ends of the chain $W^{-m,n+1}$ are associated to the same agent. In the first case, we suppose the ends of the chain are associated with distinct agents, i.e., $b(z^{n+1}) \neq s(z^{-m})$. In the second case, we suppose the ends of the chain are associated with the same agent, i.e., $b(z^{n+1}) = s(z^{-m})$.

Case 1: $b(z^{n+1}) \neq s(z^{-m})$. If $b(z^{n+1}) \neq s(z^{-m})$, then we need to check the condition (ii) of Definition 7. We prove first that part (a) of condition (ii) is satisfied. We proceed via an argument analogous to that used to prove Claim 1: Choose an arbitrary $Y^* \in C_{b(z^{n+1})}((Z \setminus W^{-m,n+1}) \cup A)$. Note first that since $C_{b(z^{n+1})}$ is monotone–substitutable, there exists a $Z^* \in C_{b(z^{n+1})}((Z \setminus W^{-m,n}) \cup A)$ such that

$$\big[\big((Z \setminus W^{-m,n+1}) \cup A\big) \setminus Y^*\big]_{\to b(z^{n+1})} \subseteq \big[\big((Z \setminus W^{-m,n}) \cup A\big) \setminus Z^*\big]_{\to b(z^{n+1})} \tag{8}$$

$$Y^*_{b(z^{n+1})\to} \subseteq Z^*_{b(z^{n+1})\to} \tag{9}$$

$$\big|Z^*_{\to b(z^{n+1})}\big| - \big|Y^*_{\to b(z^{n+1})}\big| \geq \big|Z^*_{b(z^{n+1})\to}\big| - \big|Y^*_{b(z^{n+1})\to}\big|. \tag{10}$$

Partition Y^* into $\hat{Y}^* \equiv Y^* \cap (Z \setminus W^{-m,n+1})$ and $\check{Y}^* \equiv Y^* \cap A$, and partition Z^* into $\hat{Z}^* \equiv Z^* \cap (Z \setminus W^{-m,n})$ and $\check{Z}^* \equiv Z^* \cap A$.[34]

We argue first that when z^{n+1} is no longer available, every optimal choice by $b(z^{n+1})$ includes all of his remaining contracts in Z for which he is a buyer, i.e., $\hat{Y}^*_{\to b(z^{n+1})} = [Z \setminus W^{-m,n+1}]_{\to b(z^{n+1})}$. Using the notation just introduced, we can rewrite (8) as

$$\big[\big((Z \setminus W^{-m,n+1}) \cup A\big) \setminus \big(\hat{Y}^* \cup \check{Y}^*\big)\big]_{\to b(z^{n+1})} \subseteq \big[\big((Z \setminus W^{-m,n}) \cup A\big) \setminus \big(\hat{Z}^* \cup \check{Z}^*\big)\big]_{\to b(z^{n+1})}$$

or, equivalently,

$$\big[\big((Z \setminus W^{-m,n+1}) \setminus \hat{Y}^*\big) \cup \big(A \setminus \check{Y}^*\big)\big]_{\to b(z^{n+1})} \subseteq \big[A \setminus \check{Z}^*\big]_{\to b(z^{n+1})}. \tag{11}$$

Here, we have used the fact that $[(Z \setminus W^{-m,n}) \setminus \hat{Z}^*]_{\to b(z^{n+1})} = \varnothing$ since we have that $\hat{C}_{b(z^{n+1})}(Z \setminus W^{-m,n}; A) = \{[Z \setminus W^{-m,n}]_{b(z^{n+1})}\}$ (which follows from the fact that $W^{-m,n}$ is a quasi-removable chain and applying condition (i) of Definition 7). From (11), we can immediately infer that $[(Z \setminus W^{-m,n+1}) \setminus \hat{Y}^*]_{\to b(z^{n+1})} \subseteq A$. Given that Z is a blocking set, we have $Z \cap A = \varnothing$ and thus $[(Z \setminus W^{-m,n+1}) \setminus \hat{Y}^*]_{\to b(z^{n+1})} = \varnothing$. Hence, $[Z \setminus W^{-m,n+1}]_{\to b(z^{n+1})} \subseteq \hat{Y}^*$. Since $\hat{Y}^* = Y^* \cap (Z \setminus W^{-m,n+1})$, we obtain $[Z \setminus W^{-m,n+1}]_{\to b(z^{n+1})} = \hat{Y}^*_{\to b(z^{n+1})}$.

Next we argue that when z^{n+1} is no longer available, every optimal choice by $b(z^{n+1})$ excludes at most one of his contracts in $Z \setminus W^{-m,n+1}$ for which he is a seller, i.e., either $\hat{Y}^*_{b(z^{n+1})\to} = [Z \setminus W^{-m,n+1}]_{b(z^{n+1})\to}$ or there exists a $z^{n+2} \in Z$ such that

[34] Recall that $A \cap Z = \varnothing$ and so $A \cap (Z \setminus W^{-m,n+1}) = \varnothing$.

$\hat{Y}^*_{b(z^{n+1})\to} = [(Z \setminus W^{-m,n+1}) \setminus \{z^{n+2}\}]_{b(z^{n+1})\to}$. Note first that (8) implies that $b(z^{n+1})$ chooses at most one fewer contract as a buyer when z^{n+1} is no longer available, i.e., $|Z^*_{\to b(z^{n+1})}| - |Y^*_{\to b(z^{n+1})}| \le 1$. Hence, (10) implies that $|Z^*_{b(z^{n+1})\to}| - |Y^*_{b(z^{n+1})\to}| \le 1$; we can rewrite this last inequality as

$$\left(\left|\hat{Z}^*_{b(z^{n+1})\to}\right| - \left|\hat{Y}^*_{b(z^{n+1})\to}\right|\right) + \left(\left|\check{Z}^*_{b(z^{n+1})\to}\right| - \left|\check{Y}^*_{b(z^{n+1})\to}\right|\right) \le 1. \tag{12}$$

Now by (9), we have that $Y^*_{b(z^{n+1})\to} \subseteq Z^*_{b(z^{n+1})\to}$ and, thus, $\check{Y}^*_{b(z^{n+1})\to} \subseteq \check{Z}^*_{b(z^{n+1})\to}$; combining this with (12) implies that $|\hat{Z}^*_{b(z^{n+1})\to}| - |\hat{Y}^*_{b(z^{n+1})\to}| \le 1$. Moreover, by (9), we have that $Y^*_{b(z^{n+1})\to} \subseteq Z^*_{b(z^{n+1})\to}$ and, thus, $\hat{Y}^*_{b(z^{n+1})\to} \subseteq \hat{Z}^*_{b(z^{n+1})\to}$; hence, either we have $\hat{Y}^*_{b(z^{n+1})\to} = \hat{Z}^*_{b(z^{n+1})\to}$ or there exists a z^{n+2} such that we have $\hat{Y}^*_{b(z^{n+1})\to} = [\hat{Z}^* \setminus \{z^{n+2}\}]_{b(z^{n+1})\to}$.

To prove that part (b) of condition (ii) of Definition 7 is satisfied—as well as that extending the chain conserves upstream terminality—we distinguish two cases:

– If $s(z^{-m}) \neq b(z^n)$, we have that $W^{-m,n}_{s(z^{-m})} = W^{-m,n+1}_{s(z^{-m})}$, as we have assumed by hypothesis $b(z^{n+1}) \neq s(z^{-m})$; thus, since $W^{-m,n}$ is a quasi-removable chain by assumption, we have that part (b) of condition (ii) of Definition 7 is satisfied for $s(z^{-m})$ and $W^{-m,n+1}$.

  Also, if $W^{-m,n}$ is upstream terminal, then for all $\bar{Z}^* \in \hat{C}_{s(z^{-m})}(Z \setminus W^{-m,n}; A)$, we have that

$$\bar{Z}^*_{\to s(z^{-m})} = \left[Z \setminus W^{-m,n}\right]_{\to s(z^{-m})}. \tag{13}$$

  Combining (13) with the fact that $W^{-m,n}_{s(z^{-m})} = W^{-m,n+1}_{s(z^{-m})}$ yields that for all $\bar{Z}^* \in \hat{C}_{s(z^{-m})}(Z \setminus W^{-m,n+1}; A)$, we have that

$$\bar{Z}^*_{\to s(z^{-m})} = \left[Z \setminus W^{-m,n+1}\right]_{\to s(z^{-m})},$$

  and, thus, $W^{-m,n+1}$ is upstream terminal by definition.

– If $s(z^{-m}) = b(z^n) = k$, recall first that $\tilde{Z}^* \in \hat{C}_k(Z \setminus W^{-m,n}; A)$ and z^{n+1} were chosen such that $\tilde{Z}^*_{k\to} = [(Z \setminus W^{-m,n}) \setminus \{z^{n+1}\}]_{k\to}$. Since $\tilde{Z}^* \in \hat{C}_k(Z \setminus W^{-m,n}; A)$,

$$\hat{U}_i\left(\tilde{Z}^*; A\right) > \hat{U}_i(Y; A)$$

  for all $Y \subseteq Z \setminus W^{-m,n}$ such that $Y \notin \hat{C}_i(Z \setminus W^{-m,n}; A)$; that is, $\tilde{Z}^*$ provides a higher utility to i than any $Y \subseteq Z \setminus W^{-m,n}$ that was *not* chosen when $Z \setminus W^{-m,n}$ was available. Thus, because $\tilde{Z}^*_{k\to} = [(Z \setminus W^{-m,n}) \setminus \{z^{n+1}\}]_{k\to}$ (and, thus, $\tilde{Z}^* \subseteq Z \setminus W^{-m,n+1}$), we have that

$$\hat{C}_k\left(Z \setminus W^{-m,n+1}; A\right) \subseteq \hat{C}_k\left(Z \setminus W^{-m,n}; A\right). \tag{14}$$

  We now show that when z^{n+1} is no longer available, the seller of z^{-m} never drops a contract for which he is the seller, i.e., for all $\bar{Z}^* \in \hat{C}_k(Z \setminus W^{-m,n+1}; A)$, we have that

$$\bar{Z}^*_{k\to} = \left[Z \smallsetminus W^{-m,n+1}\right]_{k\to}.$$

Since $W^{-m,n}$ is a quasi-removable chain, there is no $Y \in \hat{C}_k(Z \smallsetminus W^{-m,n}; A)$ such that $|Y_{k\to}| < |[Z \smallsetminus W^{-m,n}]_{k\to}| - 1$. Thus, using (14), we have that there does not exist $Y \in \hat{C}_k(Z \smallsetminus W^{-m,n+1}; A)$ such that

$$|Y_{k\to}| < |[Z \smallsetminus W^{-m,n}]_{k\to}| - 1 = |[Z \smallsetminus W^{-m,n+1}]_{k\to}|.$$

Therefore, k must now choose, as a seller, all of the contracts in $Z \smallsetminus W^{-m,n+1}$, i.e.,

$$\bar{Z}^*_{k\to} = \left[Z \smallsetminus W^{-m,n+1}\right]_{k\to}.$$

Next we show that when z^{n+1} is no longer available, the seller of z^{-m} drops at most one contract as a buyer, i.e., for all $\bar{Z}^* \in \hat{C}_k(Z \smallsetminus W^{-m,n+1}; A)$ either

$$\bar{Z}^*_{\to k} = \left[Z \smallsetminus W^{-m,n+1}\right]_{\to k}$$

or there exists a $z^{-m-1} \in Z$ such that

$$\bar{Z}^*_{\to k} = \left[Z \smallsetminus \left(W^{-m,n+1} \cup \left\{z^{-m-1}\right\}\right)\right]_{\to k}.$$

Since $W^{-m,n}$ was a quasi-removable chain, there is no $Y \in \hat{C}_k(Z \smallsetminus W^{-m,n}; A)$ such that $|Y_{\to k}| < |[Z \smallsetminus W^{-m,n}]_{\to k}| - 1$. Thus, using (14), we have that there does not exist $Y \in \hat{C}_k(Z \smallsetminus W^{-m,n+1}; A)$ such that $|Y_{\to k}| < |[Z \smallsetminus W^{-m,n}]_{\to k}| - 1$. Therefore, $k = s(z^{-m})$ must now choose all but one of his contracts as a buyer, i.e., either

$$\bar{Z}^*_{\to s(z^{-m})} = \left[Z \smallsetminus W^{-m,n}\right]_{\to s(z^{-m})}$$

or there exists a $z^{-m-1} \in Z$ such that

$$\bar{Z}^*_{\to s(z^{-m})} = \left[Z \smallsetminus \left(W^{-m,n} \cup \left\{z^{-m-1}\right\}\right)\right]_{\to s(z^{-m})}.$$

Since $s(z^{n+1}) = b(z^n) = s(z^{-m})$, the preceding logic implies that we now have $[Z \smallsetminus W^{-m,n}]_{\to s(z^{-m})} = [Z \smallsetminus W^{-m,n+1}]_{\to s(z^{-m})}$ and, thus, we obtain the desired statement.

Finally, we show that if $W^{-m,n}$ is upstream terminal, then $W^{-m,n+1}$ is upstream terminal. If $W^{-m,n}$ is upstream terminal, then for all $\bar{Z}^* \in \hat{C}_k(Z \smallsetminus W^{-m,n}; A)$, we have that

$$\bar{Z}^*_{\to k} = \left[Z \smallsetminus W^{-m,n}\right]_{\to k}. \tag{15}$$

Combining (15) with (14) yields that for all $\bar{Z}^* \in \hat{C}_{s(z^{-m})}(Z \smallsetminus W^{-m,n+1}; A)$, we have that

$$\bar{Z}^*_{\to k} = \left[Z \smallsetminus W^{-m,n+1}\right]_{\to k}.$$

Thus, $W^{-m,n+1}$ is upstream terminal by definition.

Case 2: $b(z^{n+1}) = s(z^{-m})$. If $b(z^{n+1}) = s(z^{-m}) \equiv k$, then we need to check the condition (iii) of Definition 7. Analogously to Case 1, we choose an arbitrary set $Y^* \in C_k((Z \smallsetminus W^{-m,n+1}) \cup A)$. Note first that since $C_{b(z^{n+1})}$ is monotone–substitutable, there exists a $Z^* \in C_{b(z^{n+1})}((Z \smallsetminus W^{-m,n}) \cup A)$ such that

$$\big[\big((Z \smallsetminus W^{-m,n+1}) \cup A\big) \smallsetminus Y^*\big]_{\to k} \subseteq \big[\big((Z \smallsetminus W^{-m,n}) \cup A\big) \smallsetminus Z^*\big]_{\to k} \tag{16}$$

$$Y^*_{k\to} \subseteq Z^*_{k\to} \tag{17}$$

$$\big|Z^*_{\to k}\big| - \big|Y^*_{\to k}\big| \geq \big|Z^*_{k\to}\big| - \big|Y^*_{k\to}\big|. \tag{18}$$

Partition Y^* into $\hat{Y}^* \equiv Y^* \cap (Z \smallsetminus W^{-m,n+1})$ and $\check{Y}^* \equiv Y^* \cap A$, and partition Z^* into $\hat{Z}^* \equiv Z^* \cap (Z \smallsetminus W^{-m,n})$ and $\check{Z}^* \equiv Z^* \cap A$.[35] Note that either $\hat{Z}^*_{\to k} = [Z \smallsetminus W^{-m,n}]_{\to k}$ or $\hat{Z}^*_{\to k} = [(Z \smallsetminus W^{-m,n}) \smallsetminus \{z^{-m-1}\}]_{\to k}$ for some $z^{-m-1} \in [Z \smallsetminus W^{-m,n}]_{\to k}$, as $W^{-m,n}$ is a quasi-removable chain and $b(z^n) \neq s(z^{-m})$ in the case we consider here.[36]

We argue first that condition (iii)(a) of Definition 7 is satisfied: When z^{n+1} is no longer available, every optimal choice by k excludes at most one of his remaining contracts as a buyer, i.e., either $\hat{Y}^*_{\to k} = [Z \smallsetminus W^{-m,n+1}]_{\to k}$ or there exists a $z^{-m-1} \in Z$ such that $\hat{Y}^*_{\to k} = [(Z \smallsetminus W^{-m,n+1}) \smallsetminus \{z^{-m-1}\}]_{\to k}$. We can rewrite (16) as

$$\big[\big[(Z \smallsetminus W^{-m,n+1}) \cup A\big]_{\to k} \smallsetminus \big[\hat{Y}^* \cup \check{Y}^*\big]_{\to k}\big] \subseteq \big[\big[(Z \smallsetminus W^{-m,n}) \cup A\big]_{\to k} \smallsetminus \big[\hat{Z}^* \cup \check{Z}^*\big]_{\to k}\big]$$

or, equivalently,

$$\big[\big[(Z \smallsetminus W^{-m,n+1}) \smallsetminus \hat{Y}^*\big]_{\to k} \cup \big[A \smallsetminus \check{Y}^*\big]_{\to k}\big] \subseteq \big[\big[(Z \smallsetminus W^{-m,n}) \smallsetminus \hat{Z}^*\big]_{\to k} \cup \big[A \smallsetminus \check{Z}^*\big]_{\to k}\big];$$

given that $Z \cap A = \varnothing$, this subset relation implies that

$$\big[(Z \smallsetminus W^{-m,n+1}) \smallsetminus \hat{Y}^*\big]_{\to k} \subseteq \big[(Z \smallsetminus W^{-m,n}) \smallsetminus \hat{Z}^*\big]_{\to k}. \tag{19}$$

If $\hat{Z}^*_{\to k} = [Z \smallsetminus W^{-m,n}]_{\to k}$, then (19) implies that $\hat{Y}^*_{\to k} \supseteq [Z \smallsetminus W^{-m,n+1}]_{\to k}$; but $\hat{Y}^* \equiv Y^* \cap (Z \smallsetminus W^{-m,n+1})$ and so $\hat{Y}^*_{\to k} = [Z \smallsetminus W^{-m,n+1}]_{\to k}$. Consequently, if $W^{-m,n}$ is upstream terminal (i.e., $\hat{Z}^*_{\to k} = [Z \smallsetminus W^{-m,n}]_{\to k}$), then $W^{-m,n+1}$ is upstream terminal (i.e., $\hat{Z}^*_{\to k} = [Z \smallsetminus W^{-m,n+1}]_{\to k}$). If $\hat{Z}^*_{\to k} = [(Z \smallsetminus W^{-m,n}) \smallsetminus \{z^{-m-1}\}]_{\to k}$ for some $z^{-m-1} \in [Z \smallsetminus W^{-m,n}]_{\to k}$, then (19) implies that $\hat{Y}^*_{\to k} \supseteq [(Z \smallsetminus W^{-m,n+1}) \smallsetminus \{z^{-m-1}\}]_{\to k}$; but $\hat{Y}^* \equiv Y^* \cap (Z \smallsetminus W^{-m,n+1})$ and so either $\hat{Y}^*_{\to k} = [Z \smallsetminus W^{-m,n+1}]_{\to k}$ or $\hat{Y}^*_{\to k} = [(Z \smallsetminus W^{-m,n+1}) \smallsetminus \{z^{-m-1}\}]_{\to k}$. Hence, when z^{n+1} is no longer available, every optimal choice by k as a buyer includes all but at most one of the contracts in $[Z \smallsetminus W^{-m,n+1}]_{\to k}$.

We argue second that condition (iii)(b) of Definition 7 is satisfied: When z^{n+1} is no longer available, every optimal choice by k excludes at most one of his remaining contracts as a seller, i.e., either $\hat{Y}^*_{k\to} = [Z \smallsetminus W^{-m,n+1}]_{k\to}$ or there exists a $z^{n+2} \in Z_{k\to}$ such that $\hat{Y}^*_{k\to} = [(Z \smallsetminus W^{-m,n+1}) \smallsetminus \{z^{n+2}\}]_{k\to}$. Note first that (16) implies that k

[35] Recall that $A \cap (Z \smallsetminus W^{-m,n+1}) = \varnothing$.

[36] In this case, we have assumed that $b(z^{n+1}) = s(z^{-m})$ and so, since $b(z^n) = s(z^{n+1})$ and $s(z^{n+1}) \neq b(z^{n+1})$, we have that $b(z^n) \neq s(z^{-m})$.

chooses at most one fewer contract as a buyer when z^{n+1} is no longer available, i.e., $|Z^*_{\to k}| - |Y^*_{\to k}| \leq 1$. Hence, (18) implies that $|Z^*_{k\to}| - |Y^*_{k\to}| \leq 1$. We can rewrite this last inequality as

$$\left(\left|\hat{Z}^*_{k\to}\right| - \left|\hat{Y}^*_{k\to}\right|\right) + \left(\left|\check{Z}^*_{k\to}\right| - \left|\check{Y}^*_{k\to}\right|\right) \leq 1. \tag{20}$$

Now by (17), we have that $Y^*_{k\to} \subseteq Z^*_{k\to}$ and, thus, $\check{Y}^*_{k\to} \subseteq \check{Z}^*_{k\to}$; combining this with (20) implies that $|\hat{Z}^*_{k\to}| - |\hat{Y}^*_{k\to}| \leq 1$. Moreover, by (17), we have that $Y^*_{k\to} \subseteq Z^*_{k\to}$ and, thus, $\hat{Y}^*_{k\to} \subseteq \hat{Z}^*_{k\to}$; hence, either $\hat{Y}^*_{k\to} = \hat{Z}^*_{k\to}$ or there exists a z^{n+2} such that $\hat{Y}^*_{k\to} = [\hat{Z}^* \smallsetminus \{z^{n+2}\}]_{k\to}$.

This completes the proof of Claim 2.

A.4 *Proof of Claim 3*

The proof follows the proof of Claim 2 *mutatis mutandis*.

A.5 *Proof of Claim 4*

As $W^{-m,n}$ is quasi-removable, we know that for all $i \in I \smallsetminus \{s(z^{-m}), b(z^n)\}$, we have that $\{[Z \smallsetminus W^{-m,n}]_i\} = \hat{C}_i(Z \smallsetminus W^{-m,n})$ (from condition (i) of Definition 7). There are two cases to consider:

Case 1: $b(z^n) \neq s(z^{-m})$. Since $W^{-m,n}$ is downstream terminal, we have that

$$\left\{Z_{b(z^n)} \smallsetminus \left\{z^{-m}, \ldots, z^n\right\}\right\} = \hat{C}_{b(z^n)}\left(Z \smallsetminus \left\{z^{-m}, \ldots, z^n\right\}; A\right).$$

Furthermore, since $W^{-m,n}$ is upstream terminal, we have that

$$\left\{Z_{s(z^{-m})} \smallsetminus \left\{z^{-m}, \ldots, z^n\right\}\right\} = \hat{C}_{s(z^{-m})}\left(Z \smallsetminus \left\{z^{-m}, \ldots, z^n\right\}; A\right).$$

Thus, $Z \smallsetminus W^{-m,n}$ blocks A.

Case 2: $b(z^n) = s(z^{-m})$. In this case, since $W^{-m,n}$ is downstream and upstream terminal, we have that

$$\left\{Z_{b(z^n)} \smallsetminus \left\{z^{-m}, \ldots, z^n\right\}\right\} = \hat{C}_{b(z^n)}\left(Z \smallsetminus \left\{z^{-m}, \ldots, z^n\right\}; A\right).$$

Thus, $Z \smallsetminus W^{-m,n}$ blocks A.

This completes the proof of Claim 4.

A.6 *Proof of Claim 5*

As Z blocks A, for all $i \in a(Z)$, for each $Y \in C_i(A \cup Z)$, we have that $Z_i \subseteq Y$. Since $W \subseteq Z$, for all $i \in a(Z)$, for each $Y \in C_i(A \cup Z)$, we have that $W_i \subseteq Y$. Thus, for all $i \in a(W) \subseteq a(Z)$, for each $Y \in C_i(A \cup Z) = C_i((A \cup (Z \smallsetminus W)) \cup W)$, we have that $W_i \subseteq Y$. Thus, by definition, W blocks $A \cup (Z \smallsetminus W)$.

Appendix B: Chain stability and competitive equilibrium

In this appendix, we show that in the trading network setting of Hatfield et al. (2013), an outcome is consistent with competitive equilibrium if and only if it is not blocked by a chain of contracts.

The Hatfield et al. (2013) setting is a special case of ours that requires that

- prices are continuous and unrestricted, i.e., $X = \Omega \times \mathbb{R}$, and
- agents' preferences are quasilinear in prices.

Formally, a utility function U_i is *quasilinear in prices* if there exists a valuation function u_i from the sets of trades involving agent i to $\mathbb{R} \cup \{-\infty\}$ such that for any feasible set $Y \subseteq X_i$,

$$U_i(Y) = u_i\big(\tau(Y)\big) + \sum_{(\omega, p_\omega) \in Y_{i\to}} p_\omega - \sum_{(\omega, p_\omega) \in Y_{\to i}} p_\omega .$$

Definition 8. An outcome Y is *consistent with competitive equilibrium* if there exists a vector of prices for all trades in the economy, $p \in \mathbb{R}^\Omega$, such that

- for every $\omega \in \tau(Y)$, we have $(\omega, p_\omega) \in Y$, and
- for every agent i, for every set of trades $\Phi \subseteq \Omega_i$, we have

$$U_i(Y_i) \geq u_i(\Phi) + \sum_{\omega \in \Phi_{i\to}} p_\omega - \sum_{\omega \in \Phi_{\to i}} p_\omega .$$

An outcome Y only specifies prices for the trades that are, in fact, executed under the outcome, while a competitive equilibrium specifies prices for all the trades in the economy. For an outcome to be consistent with competitive equilibrium, it must be that one can specify prices for the trades that are *not* executed so that, for each agent i, selecting the trades associated with the outcome Y is, in fact, consistent with utility maximization; Definition 8 formalizes this requirement.

We are now ready to state our competitive equilibrium equivalence result.

Corollary 1. *Suppose that the set of contracts is $X = \Omega \times \mathbb{R}$, and that all agents' preferences are fully substitutable and quasilinear in prices. Then an outcome is consistent with competitive equilibrium if and only if it is chain stable.*

Proof. Under the assumed conditions on X and agents' preferences, Theorem 10 of Hatfield et al. (2019) implies that all agents' utility functions are monotone–substitutable. Thus, by our Theorem 1, an outcome is chain stable if and only if it is stable. Moreover, by Theorems 5 and 6 of Hatfield et al. (2013), an outcome is stable if and only if it is consistent with competitive equilibrium under fully substitutable preferences. Thus, under the assumptions of the corollary, an outcome is chain stable if and only if it is consistent with competitive equilibrium. □

Appendix C: Proof of Theorem 3

Here, we implicitly use I as the set of agents who have trades in Ω^m.

For a set of agents $J \subseteq I$ and for any set of trades $\Phi \subseteq \Omega^m$, let

$$\Phi_J \equiv \bigcup_{j\in J} \Phi_j = \{\varphi \in \Phi : b(\varphi) \in J \text{ or } s(\varphi) \in J\}.$$

For a set of agents $J \subseteq I$, an agent $i \in I \setminus J$ *expands* Ω^m_J if $\Omega^m_{\{i\}\cup J} \setminus \Omega^m_J \neq \varnothing$. An *expanding* sequence of agents is a sequence of agents $(i_1, \ldots, i_R)$ such that i_r expands $\Omega^m_{\{i_1,\ldots,i_{r-1}\}}$ for all $r = 1, \ldots, R$. A *complete* expanding sequence of agents is a sequence of agents $(i_1, \ldots, i_R)$ such that $\Omega^m_{\{i_1,\ldots,i_R\}} = \Omega^m$. Note that every set of trades can be constructed by sequentially choosing sets of trades in $\Omega_{i_1,\ldots,i_r} \setminus \Omega_{i_1,\ldots,i_{r-1}}$ for $r = 1, \ldots, R$.

We complete the proof by considering two mutually exclusive cases: In the first case, we assume that there exists an agent associated with at least $\frac{m}{\log_2(m)}$ trades; in the second case, since no agent is associated with $\frac{m}{\log_2(m)}$ or more trades, any complete expanding sequence must contain at least $\log_2(m)$ agents. In each case, we construct a bound on the ratio of the number of chains to the number of potential blocking sets.

Case 1: A large agent. Here, we suppose that there exists an agent i such that $|\Omega^m_i| \geq \frac{m}{\log_2(m)}$.

For a set of trades $\Phi \subseteq \Omega^m$ to be a chain, it must be the case that either $|\Phi_{\to i}| = |\Phi_{i\to}|$, $|\Phi_{\to i}| = |\Phi_{i\to}| + 1$ (when i is at the downstream end of the chain) or $|\Phi_{\to i}| + 1 = |\Phi_{i\to}|$ (when i is at the upstream end of the chain). We compute that:

– the number of the sets of trades satisfying the first of these three criteria is

$$\sum_{n=0}^{|\Omega^m_{\to i}|} \binom{|\Omega^m_{\to i}|}{n} \binom{|\Omega^m_{i\to}|}{n} = \binom{|\Omega^m_i|}{|\Omega^m_{\to i}|} \leq \binom{|\Omega^m_i|}{\frac{|\Omega^m_i|}{2}};$$

– the number of the sets of trades satisfying the second of these three criteria is

$$\sum_{n=0}^{|\Omega^m_{\to i}|} \binom{|\Omega^m_{\to i}|}{n+1} \binom{|\Omega^m_{i\to}|}{n} = \binom{|\Omega^m_i|}{|\Omega^m_{\to i}| - 1} \leq \binom{|\Omega^m_i|}{\frac{|\Omega^m_i|}{2}};$$

– the number of the sets of trades satisfying the third of these three criteria is

$$\sum_{n=0}^{|\Omega^m_{\to i}|} \binom{|\Omega^m_{\to i}|}{n} \binom{|\Omega^m_{i\to}|}{n+1} = \binom{|\Omega^m_i|}{|\Omega^m_{\to i}| + 1} \leq \binom{|\Omega^m_i|}{\frac{|\Omega^m_i|}{2}}.$$

Summing the three previous expressions, we find that the number of sets of trades satisfying one of our three conditions is no greater than

$$3\binom{|\Omega^m_i|}{\frac{|\Omega^m_i|}{2}}.$$

Thus, using Stirling's bounds, we find that the number of chains of trades is no greater than

$$3\sqrt{\frac{2}{\pi}}\left(\frac{2^{|\Omega_i^m|}}{\sqrt{|\Omega_i^m|}}\right).$$

Thus, as the number of subsets of trades is simply $2^{|\Omega_i^m|}$, we have that $\frac{|\mathfrak{C}^m(Y)|}{|\mathfrak{B}^m(Y)|} = O(\frac{\sqrt{\log_2 m}}{\sqrt{m}})$ as $|\Omega_i^m| \geq \frac{m}{\log_2(m)}$.

Case 2: Small agents. Here we suppose that $|\Omega_i^m| < \frac{m}{\log_2(m)}$ for all $i \in I$. Thus, there must exist a complete expanding sequence of agents $(i_1, \dots, i_R)$ such that $R \geq \log_2(m)$.

It is easy to compute that, as $\Omega_{i_1}^m$ is nonempty, the following inequality holds:[37]

$$\frac{\{W^1 \subseteq \Omega_{i_1}^m : |W^1_{\to i_1}| = |W^1_{i_1\to}|\}}{\{W^1 \subseteq \Omega_{i_1}^m\}} \leq \frac{1}{2}.$$

That is, the number of subsets of $\Omega_{i_1}^m$ that are "balanced for i_1" (i.e., such that i_1 is associated with the same number of buy and sell contracts) is at most half of the number of subsets of $\Omega_{i_1}^m$. We can also compute, taking any sequence $W^1, \dots, W^{r-1}$, where $W^{r'}$ is chosen from $\Omega^m_{\{i_1,\dots,i_{r'}\}}$, that (recalling that $(i_1, \dots, i_R)$ is an expanding sequence)

$$\frac{\left\{W^r \subseteq \Omega^m_{\{i_1,\dots,i_r\}} \smallsetminus \Omega^m_{\{i_1,\dots,i_{r-1}\}} : \left|\left[\bigcup_{s=1}^r W^s\right]_{\to i_r}\right| = \left|\left[\bigcup_{s=1}^r W^s\right]_{i_r\to}\right|\right\}}{\left\{W^r \subseteq \Omega^m_{\{i_1,\dots,i_r\}} \smallsetminus \Omega^m_{\{i_1,\dots,i_{r-1}\}}\right\}} \leq \frac{1}{2}.$$

That is, taking any sequence $W^1, \dots, W^{r-1}$, where $W^{r'}$ is chosen from $\Omega^m_{\{i_1,\dots,i_{r'}\}}$, the number of subsets of $\Omega^m_{\{i_1,\dots,i_r\}} \smallsetminus \Omega^m_{\{i_1,\dots,i_{r-1}\}}$ such that $\bigcup_{s=1}^r W^s$ is "balanced for i_r" is at most half of the number of subsets of $\Omega^m_{\{i_1,\dots,i_r\}} \smallsetminus \Omega^m_{\{i_1,\dots,i_{r-1}\}}$.

Using the preceding two observations, if we construct a set by choosing trades in this way along the complete expanding sequence, the overall probability that each agent i_r is balanced is bounded by

$$\left(\frac{1}{2}\right)^R.$$

[37]This follows as we can compute $\{W^1 \subseteq \Omega_{i_1}^m : |W^1_{\to i_1}| = |W^1_{i_1\to}|\}$ as the sum over

$$\sum_{n=0}^{|\Omega^m_{\to i_1}|}\binom{|\Omega^m_{\to i_1}|}{n}\binom{|\Omega^m_{i_1\to}|}{n} = \binom{|\Omega^m_{i_1}|}{|\Omega^m_{\to i_1}|} \leq \binom{|\Omega^m_{i_1}|}{\left\lfloor\frac{|\Omega^m_{i_1}|}{2}\right\rfloor} \leq 2^{|\Omega^m_{i_1}|-1}.$$

Similarly, the overall probability that each agent i_r except one is balanced is bounded by

$$\binom{R}{1}\left(\frac{1}{2}\right)^{R-1}.$$

Finally, the overall probability that each agent i_r except for two is balanced is bounded by

$$\binom{R}{2}\left(\frac{1}{2}\right)^{R-2}.$$

Summing the three preceding expressions, we compute that the probability that a set of trades constructed by choosing each trade in Ω^m with probability $\frac{1}{2}$ is a chain is no more than

$$\left(\frac{1}{2}\right)^{R}+\binom{R}{1}\left(\frac{1}{2}\right)^{R-1}+\binom{R}{2}\left(\frac{1}{2}\right)^{R-2}\leq 4R^2\left(\frac{1}{2}\right)^{R}.$$

Recalling that $R\geq \log_2(m)$, we have that

$$\frac{|\mathfrak{C}^m(Y)|}{|\mathfrak{B}^m(Y)|}=O\left(\frac{(\log_2(m))^2}{m}\right)\ll O\left(\frac{\sqrt{\log_2 m}}{\sqrt{m}}\right).$$

References

Ahuja, Ravindra K., Thomas L. Magnanti, and James B. Orlin (1993), *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall, Englewood Cliffs, N.J. [201]

Alkan, Ahmet and David Gale (2003), "Stable schedule matching under revealed preference." *Journal of Economic Theory*, 112, 289–306. [204]

Andersson, Tommy, Ágnes Cseh, Lars Ehlers, and Albin Erlanson (forthcoming), "Organizing time exchanges: Lessons from matching markets." *American Economic Journal: Microeconomics*. [200, 218]

Candogan, Ozan, Markos Epitropou, and Rakesh V. Vohra (2019), "Competitive equilibrium and trading networks: A network flow approach." Working paper, SSRN 2738610. [202, 218]

Crawford, Vincent P. and Elsie Marie Knoer (1981), "Job matching with heterogeneous firms and workers." *Econometrica*, 49, 437–450. [198]

Echenique, Federico and Jorge Oviedo (2006), "A theory of stability in many-to-many matching markets." *Theoretical Economics*, 1, 233–273. [201]

Fleiner, Tamás, Ravi Jagadeesan, Zsuzsanna Jankó, and Alexander Teytelboym (2019), "Trading networks with frictions." *Econometrica*, 87, 1633–1661. [200, 202]

Fleiner, Tamás, Zsuzsanna Jankó, Ildikó Schlotter, and Alexander Teytelboym (2020), "Complexity of stability in trading networks." Working paper, arXiv:1805.08758. [199, 218]

Fleiner, Tamás, Zsuzsanna Jankó, Akihisa Tamura, and Alexander Teytelboym (2018), "Trading networks with bilateral contracts." Working paper, SSRN 2457092. [201, 202]

Fox, Jeremy T. (2017), "Specifying a structural matching game of trading networks with transferable utility." *American Economic Review*, 107, 256–260. [202]

Gale, David and Lloyd S. Shapley (1962), "College admissions and the stability of marriage." *American Mathematical Monthly*, 69, 9–15. [200]

Hatfield, John William and Scott Duke Kominers (2012), "Matching in networks with bilateral contracts." *American Economic Journal: Microeconomics*, 4, 176–208. [201, 202, 207, 217, 218]

Hatfield, John William and Scott Duke Kominers (2017), "Contract design and stability in many-to-many matching." *Games and Economic Behavior*, 101, 78–97. [200, 201, 204]

Hatfield, John William, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp (2013), "Stability and competitive equilibrium in trading networks." *Journal of Political Economy*, 121, 966–1005. [198, 199, 200, 201, 202, 203, 204, 207, 214, 217, 229]

Hatfield, John William, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp (2019), "Full substitutability." *Theoretical Economics*, 14, 1535–1590. [203, 204, 205, 207, 229]

Hatfield, John William and Paul R. Milgrom (2005), "Matching with contracts." *American Economic Review*, 95, 913–935. [204, 207]

Kelso, Alexander S. and Vincent P. Crawford (1982), "Job matching, coalition formation, and Gross substitutes." *Econometrica*, 50, 1483–1504. [198, 200, 217]

Klaus, Bettina and Markus Walzl (2009), "Stable many-to-many matchings with contracts." *Journal of Mathematical Economics*, 45, 422–434. [201]

Manjunath, Vikram and Alexander Westkamp (2019), "Strategy-proof exchange under trichotomous preferences." Working paper, Department of Management, Economics and Social Sciences, University of Cologne. [200, 218]

Ostrovsky, Michael (2008), "Stability in supply chain networks." *American Economic Review*, 98, 897–923. [198, 199, 200, 201, 202, 203, 207, 208, 217, 220]

Roth, Alvin E. (1984), "Stability and polarization of interests in job matching." *Econometrica*, 52, 47–58. [198, 200]

Shapley, Lloyd S. and Martin Shubik (1971), "The assignment game I: The core." *International Journal of Game Theory*, 1, 111–130. [198, 200]

Shayani, Joseph (2018), "How many subsets of edges of a directed multigraph can be represented as trails?" Working paper, arXiv:1606.09107. [216, 217]

Sun, Ning and Zaifu Yang (2006), "Equilibria and indivisibilities: Gross substitutes and complements." *Econometrica*, 74, 1385–1402. [217]

Sun, Ning and Zaifu Yang (2009), "A double-track adjustment process for discrete markets with substitutes and complements." *Econometrica*, 77, 933–952. [217]

Westkamp, Alexander (2010), "Market structure and matching with contracts." *Journal of Economic Theory*, 145, 1724–1738. [201]

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