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# The implementation of stabilization policy

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In locally linearized dynamic stochastic rational-expectations models, I introduce the concepts of *feasible* paths (paths on which the policy instrument can be expressed as a function of the policymaker's observation set) and *implementable* paths (paths that can be obtained, in a minimally robust way, as the unique local equilibrium under a policy-instrument rule consistent with the policymaker's observation set). I show that, for relevant observation sets, the optimal feasible path under monetary policy can be non-implementable in the new Keynesian model, while constant-debt feasible paths under tax policy are always implementable in the real business cycle model. The first result sounds a note of caution about one of the main lessons of the new Keynesian literature, namely the importance for central banks to track some key unobserved exogenous rates of interest, while the second result restores to some extent the role of income or labor-income taxes in safely stabilizing public debt. For any given implementable path, I show how to design arithmetically a policy-instrument rule consistent with the policymaker's observation set and implementing this path as the robustly unique local equilibrium.

Keywords. Stabilization policy, local-equilibrium determinacy, observation set, feasible path, implementable path, optimal monetary policy, debt-stabilizing tax policy.

JEL classification. E32, E52.

## 1. Introduction

Two traditions stand alongside each other in the literature on macroeconomic stabilization policy in locally linearized dynamic stochastic rational-expectations models. One tradition studies some specific exogenous-shock-contingent paths of interest for the endogenous variables (e.g., the path that they follow under Ramsey-optimal policy) without asking whether and how these paths could be implemented as the unique local equilibrium given the policymaker's observation set. The other tradition considers some specific policy-instrument rules ensuring local-equilibrium determinacy and involving only observed variables (e.g., the interest-rate rule proposed by Taylor 1993),

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without requiring these rules to implement a given exogenous-shock-contingent path of interest.

My aim, in this paper, is to build a bridge between these two separate traditions. The starting point of my analysis consists of three inputs: (i) a given model of the private sector's behavior, i.e., a given system of equilibrium conditions excluding the policy-instrument rule, (ii) a given targeted path for all the endogenous variables as functions of current and past exogenous shocks, which the policymaker would like to implement, and (iii) a given observation set for the policymaker, made of the history of some endogenous variables and/or exogenous shocks until some current or past date. Given these inputs, I ask two questions: does there exist a policy-instrument rule consistent with the policymaker's observation set and implementing the targeted path as the unique local equilibrium in the model? And, if there exists such a rule, how to design it? The first question is the question of the *implementability* of the targeted path; the second is the question of its *implementation*.

There are, of course, two trivial ways in which a given path may not be implementable. First, the path may be inconsistent with at least one structural equation, i.e., one equilibrium condition describing the private sector's behavior. Second, it may be inconsistent with the policymaker's observation set, in the sense that the policy instrument cannot be expressed, on this path, as a function of only elements of this observation set. (Such is the case, for instance, when the path makes the policy instrument depend on current exogenous shocks, while the observation set includes only past exogenous shocks or past endogenous variables.) To rule out these two uninteresting cases, I focus on the paths that are consistent both with the structural equations and with the observation set. I call them *feasible paths*. So the questions I ask in this paper are, more specifically, those of the implementability and implementation of *feasible* paths.

I make two main contributions. First, I show, through two case studies, that the (non-)implementability of feasible paths can be an issue in textbook models for standard policy instruments, relevant observation sets, and interesting feasible paths, with important policy implications. Second, I develop an arithmetic method of designing a policy-instrument rule consistent with a given observation set and implementing a given implementable path as the unique local equilibrium. I fully characterize this method in a class of univariate models (i.e., models with only one endogenous variable outside the policy instrument), and I illustrate how it can be applied to larger models.

For simplicity, I start the analysis with the class of univariate models. In this class, when there are at least as many unobserved exogenous shocks as observed endogenous variables, there is a limited set of policy-instrument rules consistent with the observation set and the feasible path considered. I show that this set is spanned by a single rule. If the system made of the structural equation and this rule satisfies Blanchard and Kahn's (1980) conditions, then the feasible path considered is implementable, and it is implemented by this rule. Alternatively, if this system does not satisfy Blanchard and Kahn's (1980) conditions, then there does not exist any rule in that set such that the system made of the structural equation and this rule satisfies these conditions, and the path is not implementable.

The path may be non-implementable because the system has *fewer* "unstable eigenvalues" (i.e., eigenvalues outside the unit circle of the complex plane) than non-predetermined variables, and because it has, therefore, an infinity of local equilibria, one of which coincides with the path. But it may also be non-implementable because the system has *more* unstable eigenvalues than non-predetermined variables. In the latter case, because of a stochastic singularity, the system has, nevertheless, a unique local equilibrium, which coincides with the path. However, if an exogenous policy shock of arbitrarily small variance (capturing, e.g., the policymaker's "trembling hand" or round-off errors) were added to the rule, then the system would no longer have a local equilibrium. In this sense, the rule does not *robustly* ensure local-equilibrium determinacy. I say that the path is not implementable in this case so as to rule out knife-edged and practically useless implementability results.

Alternatively, when there are fewer unobserved exogenous shocks than observed endogenous variables, there are additional degrees of freedom in the choice of a policy-instrument rule consistent with the observation set and the feasible path considered. I show that it is always possible to find one such rule that robustly ensures local-equilibrium determinacy. Thus, all feasible paths are implementable. Moreover, for any feasible path, I show how to design *arithmetically*, i.e., with a finite number of arithmetic operations (addition, subtraction, multiplication, and division), a policy-instrument rule consistent with the observation set and implementing this path as the robustly unique local equilibrium. The coefficients of the rule are thus explicitly expressed as *rational functions* of the structural and feasible-path parameters, i.e., as fractions of polynomial functions of these parameters. These functions are particularly easy to manipulate analytically. For instance, their derivatives can be easily computed to determine how the coefficients of the rule respond to an arbitrarily small change in the value of the structural or feasible-path parameters.

These (non-)implementability and implementation results, obtained in a class of univariate models, usefully prepare the ground for the two case studies, which involve multivariate models.

In the first case study, I consider (i) the new Keynesian (NK) model, with the central bank as the policymaker and the interest rate as the policy instrument, (ii) the central bank's observation set made of all past endogenous variables, and (iii) the optimal feasible path, i.e., the path that maximizes welfare subject to the structural equations and the central bank's observation-set constraint. In this bivariate framework with as many unobserved shocks as observed variables, some feasible paths may be non-implementable essentially in the same ways and for the same reasons as in the univariate analysis. I find that, for some values of the structural parameters, the optimal feasible path is not implementable because all the interest-rate rules consistent with the observation set and this path lead to local-equilibrium multiplicity. For other values, it is not implementable because adding an exogenous monetary-policy shock (even of arbitrarily small variance) to any of these rules leads to nonexistence of a local equilibrium.

These non-implementability results sound a note of caution about one of the main lessons of the NK literature, namely the importance for central banks to track some key unobserved exogenous rates of interest, such as, for instance, the counterfactual "natural rate of interest" (as emphasized by, e.g., Galí 2015, Chapter 9, and Woodford 2003, Chapter 4). From a normative perspective, the most important of these rates of interest is, ultimately, the exogenous-shock-contingent value taken by the interest rate on the optimal feasible path. As my results show, however, even when this value can be inferred in different ways, on the optimal feasible path, from the variables observed by the central bank, there may be no way of setting the interest rate as a function of these variables that implements this path as the robustly unique local equilibrium. In this case, any attempt to track this rate of interest and implement the optimal feasible path inevitably results, in the presence of exogenous policy shocks of arbitrarily small variance, in either local-equilibrium multiplicity or nonexistence of a local equilibrium.

The second case study is about debt-stabilizing tax policy in the real business cycle (RBC) model. Schmitt-Grohé and Uribe (1997) consider, in this model, a labor-income tax-rate or income tax-rate rule that stabilizes the stock of public debt both in and out of equilibrium. They find that this rule leads to local-equilibrium multiplicity for many empirically relevant values of the structural parameters. This finding has largely been interpreted as an argument against the use of labor-income or income taxes to stabilize debt.

Building on my univariate analysis, however, I show that in the same RBC model, for the same alternative tax instruments, and for a reasonable observation set of the tax authority (with more observed variables than unobserved shocks), all constant-debt feasible paths are implementable for all structural-parameter values. I also show how to design arithmetically, for any given constant-debt feasible path, an income tax-rate or labor-income tax-rate rule that is consistent with the tax authority's observation set and implements this path as the robustly unique local equilibrium.

These implementability and implementation results show that labor-income or income taxes can always be used to stabilize debt *in equilibrium* without generating local-equilibrium multiplicity. Pursuing the debt-stabilization objective also *out of equilibrium*, as in Schmitt-Grohé and Uribe (1997), may seem natural, but it may prevent the tax authority from putting the economy on an explosive path following a given deviation from the targeted feasible path, simply because all explosive paths may involve an explosive debt. In this case, it is only by threatening to put the economy on an explosive-debt path that the tax authority can implement the targeted constant-debt feasible path as the robustly unique local equilibrium. Despite making debt explode out of equilibrium, this tax policy is "locally Ricardian" in the sense of Woodford (2003, Chapter 4), because it makes debt explode only if the other endogenous variables explode as well. But it requires that the debt-stabilization objective be conditional on other variables being on target, so that this objective can be abandoned out of equilibrium.

This paper is the first one to raise and study the issue of feasible-path (non-)implementability. In the literature, the only result that can be interpreted as a feasible-path non-implementability result is that no feasible path may be implementable when the policymaker observes only exogenous shocks (as first shown by Sargent and Wallace 1975). However, the case in which the policymaker observes only exogenous shocks seems practically irrelevant. Bassetto (2002, 2004, 2005) is a precursor in the study of implementability problems in a broader sense. But the constraints faced out of equilibrium by the policymaker are of a different nature in his papers: e.g., physical (impossibility of spending resources that do not exist), not informational as in my paper (impossibility of setting the policy instrument as a function of unobserved variables).

This paper is also the first one to propose a method to design analytically a policy-instrument rule consistent with the policymaker's observation set and implementing a given implementable path as the (robustly) unique local equilibrium. Evans and Honkapohja (2003), Svensson and Woodford (2005), and Woodford (2003, Chapter 7) design analytically a policy-instrument rule implementing a specific path as the unique local equilibrium in a specific model. But these rules are not required to be consistent with a given observation set for the policymaker, and the method used to design them can be applied only to simple paths in simple models, as it requires checking whether the system made of the structural equations and a candidate rule satisfies Blanchard and Kahn's (1980) conditions for all structural-parameter values. Giannoni and Woodford (2017), building on their earlier work reported in Woodford (2003, Chapter 8), design analytically, in a general framework, "target criteria" that are consistent with a given path and ensure local-equilibrium determinacy. However, these target criteria do not address the issue of (operational) implementation, as they are typically not formulated as policy-instrument rules, let alone as policy-instrument rules consistent with a given observation set for the policymaker.

The rest of the paper is organized as follows. Section 2 introduces the concepts of feasible path and implementable path in a simple framework. Section 3 studies the implementability and implementation of feasible paths in a class of univariate models. Sections 4 and 5 are devoted to the two case studies: one about optimal monetary policy in the NK model; the other about debt-stabilizing tax policy in the RBC model. I then conclude and provide a technical appendix.

## 2. Feasibility and implementability in a simple model

In this section, I use a simple monetary-policy model to introduce the concepts of feasible path and implementable path, and to illustrate the two ways in which a feasible path may not be implementable. For now, I set aside the issue of implementation; I explain why at the end of the section and I address this issue in the next section.

### 2.1 *Structural equation and observation set*

I consider an endowment economy whose agents are a private sector ($\mathcal{PS}$) and a central bank ($\mathcal{CB}$). At each date $t \in \mathbb{Z}$, the $\mathcal{PS}$ sets the inflation rate $\pi_t$ and the $\mathcal{CB}$ sets the nominal interest rate $i_t$. The behavior of the $\mathcal{PS}$ obeys the (locally log-linearized) structural equation

$$i_t = \mathbb{E}_t\{\pi_{t+1}\} + \xi_t, \tag{1}$$

which is a Fisher equation derived from the consumption Euler equation, the goods-market-clearing condition, and the endowment assumption. The exogenous term $\xi_t$

can be interpreted as a preference disturbance. It is assumed to follow a first-order moving-average process,

$$\xi_t = \varepsilon_t + \theta \varepsilon_{t-1}, \tag{2}$$

where $\theta \in \mathbb{R}$ and $\varepsilon_t$ is an independent and identically distributed (i.i.d.) exogenous shock of mean zero realized at date $t$.[1] The operator $\mathbb{E}_t\{\cdot\}$ denotes the rational-expectations operator conditionally on the observation set of the $\mathcal{PS}$ when it sets $\pi_t$. For simplicity, this observation set is assumed to be made of all current and past endogenous variables and exogenous shocks (including $\pi_t$ itself, following the standard convention). I thus abstract from any observation constraint for the $\mathcal{PS}$, so as to focus on the implications of the $\mathcal{CB}$'s observation constraints.

The observation set of the $\mathcal{CB}$ when she sets $i_t$ is assumed to be

$$O_t \equiv \{\pi^t, i^{t-1}\},$$

where, for any variable $z$ and any date $t$, $z^t \equiv \{z_{t-k} | k \in \mathbb{N}\}$ denotes the history of variable $z$ until date $t$ included.[2] Thus, the $\mathcal{CB}$ observes current and past inflation rates and past interest rates, but no current or past shocks (arguably a reasonable assumption for preference shocks). The behavior of the $\mathcal{CB}$ is described by a rule that expresses $i_t$ as a (locally log-linearized) function of elements of $O_t$. For the sake of practical relevance and convenience, I impose the constraint that this function should have a finite (but unbounded) number of arguments.[3]

## 2.2 *Implementable paths*

At this stage of the exposition, a well established tradition in the literature would specify a given interest-rate rule for the $\mathcal{CB}$—a rule that is consistent with the observation set $O_t$ and ensures local-equilibrium determinacy (i.e., existence and uniqueness of a local equilibrium). Consider, for instance, the Taylor rule

$$i_t = \phi \pi_t, \tag{3}$$

with $\phi > 1$. This rule is consistent with $O_t$ (since it expresses $i_t$ as a function of $\pi_t$, which belongs to $O_t$) and delivers a unique local equilibrium (as is well known and shown in, e.g., Woodford 2003, Chapter 2). It is straightforward to check that, in this unique local equilibrium, the endogenous variables follow the exogenous-shock-contingent path

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \dfrac{\theta + \phi}{\phi^2} \varepsilon_t + \dfrac{\theta}{\phi} \varepsilon_{t-1} \\ \dfrac{\theta + \phi}{\phi} \varepsilon_t + \theta \varepsilon_{t-1} \end{bmatrix}. \tag{4}$$

[1] I discuss the moving-average assumption at the end of Section 2.4.

[2] For convenience, throughout the paper, I refer to the policymaker with the female pronoun "she."

[3] This constraint, which I impose more generally throughout the paper, enables me to work with polynomials (rather than power series).

In this paper, I do not follow this tradition. I do not start from a given rule (as the input) to get a local path (as the output). Instead, I start from a given local path (as the input), which I assume the policymaker would like the economy to follow, and I ask the question of whether a policy-instrument rule exists that is consistent with the policymaker's observation set and implements, in a minimally robust way, this given local path as the unique local equilibrium. The "minimal-robustness" requirement that I impose is that the addition of an exogenous policy shock of arbitrarily small variance to the policy-instrument rule in question (capturing, e.g., the policymaker's trembling hand or round-off errors) should still result in a unique local equilibrium, arbitrarily close to the path considered, rather than no local equilibrium at all. In Section 2.5, I explain why this minimal-robustness requirement matters and why I impose it.

If such a rule exists, I say that the path is *implementable*; otherwise, I say it is *not implementable*. In my simple setup, for instance, the path (4) is implementable because an interest-rate rule exists, namely (3), that has the following three properties: (i) it is consistent with $O_t$; (ii) it implements this path as the unique local equilibrium; (iii) when added to an exogenous policy shock, it still delivers a unique local equilibrium (which converges to the path considered as the variance of the policy shock goes to zero).

## 2.3 *Feasible paths*

There are two trivial ways in which a given local path may not be implementable: it may be inconsistent with at least one structural equation or with the policymaker's observation set. To illustrate these two cases, consider the two paths

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \varepsilon_{t-1} \\ \theta\varepsilon_{t-1} \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_t + \theta\varepsilon_{t-1} \end{bmatrix}. \tag{6}$$

The path (5) is consistent with the $\mathcal{CB}$'s observation set $O_t$, since (5) implies $i_t = \theta\pi_t$ and $\pi_t \in O_t$. However, it is not consistent with the structural equation (1), since (5) implies $i_t - \mathbb{E}_t\{\pi_{t+1}\} = -\varepsilon_t + \theta\varepsilon_{t-1} \neq \xi_t$. Alternatively, the path (6) is consistent with the structural equation (1), since (6) implies (1). However, it is not consistent with the $\mathcal{CB}$'s observation set $O_t$, since, on the path (6), $i_t$ depends on $\varepsilon_t$, but no element of $O_t$ does, so that $i_t$ cannot be expressed as a function of only elements of $O_t$.

To rule out these two uninteresting cases, I focus on the local paths that are consistent both with the structural equations and with the policymaker's observation set. I say that these paths are *feasible*, and that the other paths are *not feasible*. So the implementability question that I ask is, more specifically, the question of the implementability of *feasible* paths.

### 2.4 *Feasible-path (non-)implementability*

To illustrate the two different ways in which a feasible path may not be implementable, consider the path

$$\begin{bmatrix} \pi_t \\ i_t \end{bmatrix} = \begin{bmatrix} \psi \varepsilon_t \\ \varepsilon_t + \theta \varepsilon_{t-1} \end{bmatrix}, \tag{7}$$

where $\psi \in \mathbb{R} \setminus \{0\}$. This path is consistent with the structural equation (1), since (7) implies (1). Moreover, it is also consistent with the observation set $O_t$, since (7) implies $i_t = \psi^{-1}\pi_t + \theta\psi^{-1}\pi_{t-1}$ and $\{\pi_t, \pi_{t-1}\} \subset O_t$. Therefore, the path (7) is feasible. The question I now ask is whether it is implementable.

To answer this question, I start by noting that the (locally log-linearized) interest-rate rules consistent with the observation set $O_t$ are rules of type

$$\mathcal{P}(L) i_t + \mathcal{Q}(L)\pi_t = 0, \tag{8}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, and $\mathcal{Q}(X) \in \mathbb{R}[X]$, where $L$ denotes the lag operator and $\mathbb{R}[X]$ denotes the set of polynomials in $X$ with real-number coefficients. The restriction $\mathcal{P}(0) \neq 0$ ensures that the rule involves $i_t$. If $\mathcal{P}(X)$ has no roots inside the unit circle of the complex plane, then the operator $\mathcal{P}(L)$ is invertible, and the rule (8) can be equivalently rewritten as $i_t = -\mathcal{P}(L)^{-1}\mathcal{Q}(L)\pi_t$ (where $\mathcal{P}(X)^{-1}\mathcal{Q}(X)$ is typically a power series, not a polynomial). But I also allow $\mathcal{P}(X)$ to have some roots inside the unit circle, implying that the operator $\mathcal{P}(L)$ is not invertible.[4]

Among the rules consistent with $O_t$, i.e., among the rules of type (8), the rules that are also consistent with the path (7) are those such that $\mathcal{P}(X)(1 + \theta X) + \mathcal{Q}(X)\psi = 0$. Replacing $\mathcal{Q}(L)$ by $-\psi^{-1}\mathcal{P}(L)(1 + \theta L)$ in (8), I can rewrite these rules as those of type

$$\mathcal{P}(L)\left(i_t - \frac{1}{\psi}\pi_t - \frac{\theta}{\psi}\pi_{t-1}\right) = 0. \tag{9}$$

So the question of whether the path (7) is implementable can be equivalently restated as the question of whether a $\mathcal{P}(X) \in \mathbb{R}[X]$ exists with $\mathcal{P}(0) \neq 0$ such that the rule (9) robustly ensures local-equilibrium determinacy. By "robustly," I mean again that the addition of an exogenous policy shock to the rule in question should still result in a unique local equilibrium, rather than no local equilibrium at all.

To answer the restated question, I add an exogenous policy shock $e_t$ to the rule (9) and ask whether a $\mathcal{P}(X) \in \mathbb{R}[X]$ exists with $\mathcal{P}(0) \neq 0$ such that the resulting rule,

$$\mathcal{P}(L)\left(i_t - \frac{1}{\psi}\pi_t - \frac{\theta}{\psi}\pi_{t-1}\right) = e_t, \tag{10}$$

ensures local-equilibrium determinacy. Clearly, if $\mathcal{P}(X)$ has (at least) one root inside the unit circle, then (10) has no stationary solution in the variable $i_t - (1/\psi)\pi_t - (\theta/\psi)\pi_{t-1}$, and, hence, no stationary solution in the variables $\pi_t$ and $i_t$. I can, therefore, restrict my

[4]In this case, the rule (8) is said to be *superinertial* in the terminology used in the literature (e.g., Woodford 2003, Chapter 8). I discuss superinertial rules in Sections 3 and 4.

search to the polynomials $\mathcal{P}(X)$ that have no roots inside the unit circle. In this case, $\mathcal{P}(L)^{-1}$ exists and (10) can be equivalently rewritten as

$$i_t = \frac{1}{\psi}\pi_t + \frac{\theta}{\psi}\pi_{t-1} + \mathcal{P}(L)^{-1}e_t. \tag{11}$$

Replacing $i_t$ in (1) by the right-hand side of (11), I then get the dynamic equation in $\pi_t$:

$$\mathbb{E}_t\{(\psi - L - \theta L^2)(\pi_{t+1} - \psi\varepsilon_{t+1})\} = \psi\mathcal{P}(L)^{-1}e_t. \tag{12}$$

It is straightforward to solve (12) for $\pi_t$ by applying Blanchard and Kahn's (1980) analysis. Whether (12) has a unique stationary solution in $\pi_t$ does not depend on the polynomial $\mathcal{P}(X)$. It depends, however, on the structural parameter $\theta$ and the feasible-path parameter $\psi$. The following three alternative cases are possible.

First, $\theta$ and $\psi$ may be such that one root of the characteristic polynomial $\mathcal{C}(X) \equiv \psi X^2 - X - \theta$ lies *inside* the unit circle and the other lies *outside*. In this case, with one outside root for one non-predetermined variable, (12) has a unique stationary solution in $\pi_t$ for any $\mathcal{P}(X)$. So there exists an infinity of rules consistent with the observation set $O_t$ and the path (7), and robustly ensuring local-equilibrium determinacy. Therefore, the path (7) is implementable.

Second, $\theta$ and $\psi$ may be such that both roots of $\mathcal{C}(X)$ lie *inside* the unit circle. In that case, with zero outside root for one non-predetermined variable, (12) has an infinity of stationary solutions in $\pi_t$ for any $\mathcal{P}(X)$. So there does not exist any rule consistent with the observation set $O_t$ and the path (7), and robustly ensuring local-equilibrium determinacy. Therefore, the path (7) is not implementable.

Finally, $\theta$ and $\psi$ may be such that both roots of $\mathcal{C}(X)$ lie *outside* the unit circle. In this last case, with two outside roots for only one non-predetermined variable, (12) has no stationary solution in $\pi_t$ for any $\mathcal{P}(X)$. So no rule consistent with the observation set $O_t$ and the path (7), and robustly ensuring local-equilibrium determinacy exists. Therefore, the path (7) is not implementable.

The last two cases illustrate two different ways in which a feasible path may be non-implementable. One is that all the rules consistent with the observation set and this path may lead to local-equilibrium multiplicity. The other is that all these rules, when added to an exogenous policy shock $e_t$, may lead to nonexistence of a local equilibrium.

When $\theta = 0$, we have $\mathcal{C}(X) = (\psi X - 1)X$, so that the first two cases may arise, but not the third case. Thus, in this simple framework, a nondegenerate moving-average disturbance (i.e., $\theta \neq 0$) is not needed to illustrate implementable and non-implementable feasible paths, but it is needed to illustrate the two different ways in which a feasible path may be non-implementable. In general, however, moving-average disturbances are not necessary for feasible-path non-implementability of either kind, as we will see in Section 4.5.

### 2.5 *Minimal-robustness requirement*

The minimal-robustness requirement that I impose for implementability does matter. If I did not impose it, the set of rules consistent with the observation set $O_t$ and the path

(7) would still be the set of rules of type (9) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$. But the path (7) would then be implementable if and only if there exists one rule in this set that ensures local-equilibrium determinacy (instead of "*robustly* ensures local-equilibrium determinacy," i.e., instead of "ensures local-equilibrium determinacy even when added to an exogenous policy shock"). As a consequence, the path (7) would be implementable in the third case, i.e., when both roots of $\mathcal{C}(X)$ lie outside the unit circle. Indeed, by construction, all rules of type (9) are consistent with the path (7), so that the system made of the structural equation (1) and any one of these rules has at least one stationary solution, namely, the path (7). In the third case, with more unstable eigenvalues than non-predetermined variables, this system has no other stationary solution. Therefore, the path (7) would be implementable in the third case if I did not impose the minimal-robustness requirement for implementability.

Such an implementability result would, however, be knife-edged and practically useless. In general, a system with more unstable eigenvalues than non-predetermined variables "is overdetermined and thus has *almost always*" no stationary solution (Blanchard and Kahn 1980, p. 1310, with my emphasis in Italics). The reason why the specific system made of the structural equation (1) and the rule (9) has, nevertheless, one stationary solution in the third case is that one equation of this system—namely, the rule—is precisely designed to make the system have one stationary solution—namely, the targeted feasible path (7). To achieve this goal, the rule must involve no exogenous policy shock whatsoever. If a policy shock were added to the rule (9), then this rule would turn into (10), and the system would no longer have a stationary solution in the third case. In particular, adding a policy shock of arbitrarily small variance (capturing, e.g., the policymaker's trembling hand or round-off errors) would not result in a unique local equilibrium arbitrarily close to the targeted feasible path, but would, instead, result in no local equilibrium at all. Because arbitrarily small policy shocks are unavoidable in practice, declaring the targeted feasible path "implementable" in this case would not be palatable. It is for this reason that I impose the minimal-robustness requirement for implementability.[5]

The third case, in which non-implementability is due to non-robustness, might be viewed at this stage as a mathematical curiosity with little economic significance, on the grounds that the path (7) is arbitrary from an economic point of view. In this flexible-price endowment economy, indeed, there is no particular reason why the central bank would want to implement this specific path rather than another one, since inflation and the interest rate do not matter for welfare. In Section 4, however, I show that non-implementability due to non-robustness can also arise for interesting paths (the optimal feasible path) in textbook models (the NK model), and that it can, therefore, also be economically relevant.

[5]Alternatively and equivalently, I could consider a finite starting date and some initial conditions prior to that date, and require that the rule (without policy shock) ensure local-equilibrium determinacy for any initial conditions. However, paths and rules would then have to take transitional dynamics into account, which would substantially and fruitlessly burden the exposition.

### 2.6 *Discussion*

In this illustrative section, I have focused on a specific model and a specific observation set, and I have studied the implementability of a specific feasible path. I have not explicitly addressed the question of how to implement this path when it is implementable, simply because the answer is trivial: if the path (7) is implementable, then, for instance, the rule $i_t = \psi^{-1}\pi_t + \theta\psi^{-1}\pi_{t-1}$, corresponding to (9) with $\mathcal{P}(X) = 1$, is consistent with the observation set and implements this path as the robustly unique local equilibrium.

The implementability results (and the trivial implementation result) that I have obtained may well, however, be driven by the specificities of my illustrative framework. Beyond the simplicity of the model (1)–(2) and the poor dynamics of the feasible path (7), two specificities stand out in particular: there are as many unobserved exogenous shocks (one: $\varepsilon_t$) as observed endogenous variables outside the policy instrument (one: $\pi_t$), and the unobserved shock can be inferred from the observed variable on the feasible path considered ($\varepsilon_t = \psi^{-1}\pi_t$). These two specificities imply that one rule consistent with the observation set and the path (7) can be obtained simply by replacing $(\varepsilon_t, \varepsilon_{t-1})$ with $(\psi^{-1}\pi_t, \psi^{-1}\pi_{t-1})$ in the second line of (7), and that this rule spans the set of rules consistent with the observation set and the path (7), i.e., the set of rules of type (9).

The analysis conducted so far, thus, leaves many implementability and implementation questions unanswered. What if there are still as many unobserved shocks as observed variables, but some unobserved shocks cannot be inferred from the observed variables on the feasible path because this path involves non-invertible autoregressive moving-average (ARMA) processes? What if some unobserved shocks cannot be inferred from the observed variables because there are more unobserved shocks than observed variables? What if there are fewer unobserved shocks than observed variables, and, therefore, a larger set of rules consistent with the observation set and the feasible path? What if the model is more complex and the feasible path has richer dynamics? In the next section, I answer these questions by generalizing the analysis in terms of model, observation set, and feasible path, and by studying not only the implementability of feasible paths, but also their implementation.

## 3. Implementability and implementation in univariate models

I now turn to a more general (but abstract) class of models in which I study the implementability and implementation of all feasible paths, depending on the policymaker's observation set. Based on my previous discussion, I consider three cases in turn, depending on whether the number of unobserved exogenous shocks is equal to, higher than, or lower than the number of observed endogenous variables. In the first two cases, I derive a simple necessary and sufficient condition for a feasible path to be implementable; when the feasible path is implementable, its implementation is straightforward. In the third case, I show that all feasible paths are implementable, and I develop an arithmetic method to design a policy-instrument rule consistent with the observation set and implementing a given feasible path as the robustly unique local equilibrium.

Although more general than the simple model of the previous section, the class of models that I consider in this section is, for simplicity, restricted to univariate models, i.e., models with only one variable set by the private sector and, therefore, only one structural equation. As I explain at the end of the section, nonetheless, this univariate analysis usefully lays the groundwork for the multivariate analyses of the next two sections.

### 3.1 *Structural equation*

The agents are a private sector ($\mathcal{PS}$) and a policymaker ($\mathcal{PM}$). The behavior of the $\mathcal{PS}$ consists in setting, at each date $t \in \mathbb{Z}$, an endogenous variable $z_t$ according to the (locally log-linearized) structural equation

$$\mathbb{E}_t\{L^{-\delta}A(L)z_t\} + B(L)i_t + \xi_t + c\xi'_t = 0, \tag{13}$$

where $i_t$ denotes the policy instrument set by the $\mathcal{PM}$ at date $t$, $\xi_t$ and $\xi'_t$ are two exogenous disturbances, and $\mathbb{E}_t\{\cdot\}$ is the rational-expectations operator conditional on the observation set of the $\mathcal{PS}$ when it sets $z_t$. As previously, I assume for simplicity that the $\mathcal{PS}$'s observation set is made of all current and past endogenous variables and exogenous shocks.

The structural equation (13) is parametrized by $\delta \in \mathbb{N} \setminus \{0\}$, $(A(X), B(X)) \in \mathbb{R}[X]^2$, and $c \in \{0, 1\}$. I rule out the uninteresting case $\delta = 0$, in which the structural equation involves no expectation terms. Without any loss of generality, I assume that $A(0) \neq 0$ (so that $\delta$ is the highest horizon of the expectation terms in the structural equation) and that $B(X) \neq 0$ (so that the policy instrument affects the endogenous variable set by the $\mathcal{PS}$). I also assume that $A(X)$ and $B(X)$ have no common roots, so as to rule out a tedious zero-measure case. The parameter $c$ enables me to consider either one exogenous disturbance only (when $c = 0$) or two (when $c = 1$). Finally, the disturbances $\xi_t$ and $\xi'_t$ follow some stationary ARMA processes driven, respectively, by the i.i.d. shocks $\varepsilon_t$ and $\varepsilon'_t$, which are of mean zero and orthogonal to each other. The simple model of the previous section corresponds to the particular case in which $A(X) = 1$, $B(X) = -1$, $c = 0$, $\delta = 1$, and $\xi_t$ follows a (moving-average) MA(1) process.

Let $O_t$ denote the observation set of the $\mathcal{PM}$ when she sets $i_t$. I assume that the $\mathcal{PM}$ observes the current and past values of the endogenous variable set by the $\mathcal{PS}$ ($z^t \subset O_t$), as well as the past values of her policy instrument ($i^{t-1} \subset O_t$), but may or may not observe current and past exogenous shocks. In the rest of this section, I consider three alternative cases in turn, depending on whether the number of unobserved shocks (which can be zero, one, or two) is equal to, higher than, or lower than the number of observed variables outside the policy instrument (which is one).

### 3.2 *As many unobserved shocks as observed variables*

I start with the case in which there is only one shock, and this shock is not observed by the $\mathcal{PM}$: $c = 0$ and $O_t = \{z^t, i^{t-1}\}$. In this case, there are as many unobserved shocks (one: $\varepsilon_t$) as observed variables outside the policy instrument (one: $z_t$). This is the case on which I focused in the previous section.

I consider the set of paths on which each endogenous variable follows a stationary (but possibly non-invertible) ARMA process driven by the exogenous shock $\varepsilon_t$, i.e., the set of paths of type

$$S_z(L)z_t = T_z(L)\varepsilon_t, \tag{14}$$

$$S_i(L)i_t = T_i(L)\varepsilon_t, \tag{15}$$

where $(S_z(X), S_i(X), T_z(X), T_i(X)) \in (\mathbb{R}[X] \setminus \{0\})^4$, $S_z(0) \neq 0$, $S_i(0) \neq 0$, and $S_z(X)$ and $S_i(X)$ have no roots inside the unit circle. Without any loss of generality, I impose that the ARMA representations (14) and (15) be minimal, i.e., that $S_z(X)$ and $T_z(X)$ have no common roots and that $S_i(X)$ and $T_i(X)$ have none either. For simplicity, I also impose that $T_z(X)$ and $T_i(X)$ have no common roots; relaxing this restriction would not affect the results substantially, but would make the analysis more tedious.

A path of type (14)–(15) is *feasible* if and only if two conditions are met. First, the structural equation (13) must be satisfied on this path. Second, $i_t$ must be expressible as a function of $O_t$ on this path. This second condition is equivalent to

$$T_z(0) \neq 0.$$

To show this equivalence, suppose first that $T_z(0) \neq 0$. Then, multiplying (14) by $T_i(L)$ and using (15) leads to $S_i(L)T_z(L)i_t = S_z(L)T_i(L)z_t$. The coefficient of $i_t$ in this equation is $S_i(0)T_z(0) \neq 0$. Therefore, this equation expresses $i_t$ only as a function of elements of $O_t$. Now suppose, alternatively, that $T_z(0) = 0$. Then $z_t$ is independent of $\varepsilon_t$ in (14). Because $T_z(X)$ and $T_i(X)$ are coprime, however, $T_z(0) = 0$ implies $T_i(0) \neq 0$, so that $i_t$ depends on $\varepsilon_t$ in (15). Since $i_t$ depends on $\varepsilon_t$ and since $z_t$ is independent of $\varepsilon_t$, there is no way to express $i_t$ only as a function of elements of $O_t$ on the path (14)–(15).

Now consider an arbitrary feasible path of type (14)–(15), denoted by $P$. To see whether $P$ is *implementable*, I proceed along the same lines as in Section 2.4. I start by noting that the (locally log-linearized) policy-instrument rules consistent with the observation set $O_t$ are the rules of type

$$\mathcal{P}(L)i_t + \mathcal{Q}(L)z_t = 0, \tag{16}$$

with $(\mathcal{P}(X), \mathcal{Q}(X)) \in \mathbb{R}[X]^2$ and $\mathcal{P}(0) \neq 0$. Among these rules, those that are also consistent with the path $P$ are those such that

$$\mathcal{P}(X)S_i(X)^{-1}T_i(X) + \mathcal{Q}(X)S_z(X)^{-1}T_z(X) = 0. \tag{17}$$

Let $S(X)$ denote the greatest common divisor of $S_z(X)$ and $S_i(X)$, defined up to a nonzero real-number multiplicative factor. Multiplying (17) by $S_z(X)S_i(X)/S(X)$ leads to

$$\mathcal{P}(X)\widetilde{S}_z(X)T_i(X) + \mathcal{Q}(X)\widetilde{S}_i(X)T_z(X) = 0, \tag{18}$$

where $\widetilde{S}_z(X) \equiv S_z(X)/S(X)$ and $\widetilde{S}_i(X) \equiv S_i(X)/S(X)$ are coprime polynomials. Multiplying (16) by $\widetilde{S}_i(L)T_z(L)$ and using (18) then leads to

$$\mathcal{P}(L)\big[\widetilde{S}_i(L)T_z(L)i_t - \widetilde{S}_z(L)T_i(L)z_t\big] = 0. \tag{19}$$

The coefficient $\mathcal{P}(0)\widetilde{S}_i(0)T_z(0)$ of $i_t$ in (19) is nonzero, notably because of the feasibility condition $T_z(0) \neq 0$. Therefore, all the equations of type (19) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$ are policy-instrument rules consistent with the observation set $O_t$ and the path $P$. Conversely, all the policy-instrument rules consistent with $O_t$ and $P$ are of type (19) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$. The reason is that $\widetilde{S}_i(X)T_z(X)$ and $\widetilde{S}_z(X)T_i(X)$ have no common roots.[6] They have no common roots because each of the pairs $\widetilde{S}_i(X)$ and $\widetilde{S}_z(X)$, $T_i(X)$ and $T_z(X)$, $\widetilde{S}_i(X)$ and $T_i(X)$, and $\widetilde{S}_z(X)$ and $T_z(X)$ is, by assumption or construction, a pair of coprime polynomials. So the question of whether the path $P$ is implementable can be equivalently restated as the question of whether $\mathcal{P}(X) \in \mathbb{R}[X]$ with $\mathcal{P}(0) \neq 0$ exists such that the rule (19) robustly ensures local-equilibrium determinacy. I can then conduct exactly the same reasoning as in Section 2.4 and conclude that a rule of type (19) exists that robustly ensures local-equilibrium determinacy if and only if the specific rule

$$\widetilde{S}_i(L)T_z(L)i_t = \widetilde{S}_z(L)T_i(L)z_t, \tag{20}$$

corresponding to $\mathcal{P}(X) = 1$, robustly ensures local-equilibrium determinacy in the first place. So the question of whether the path $P$ is implementable eventually boils down to the question of whether the system made of the structural equation (13) and the specific rule (20) satisfies Blanchard and Kahn's (1980) conditions. Moreover, when the path $P$ is implementable, the question of its implementation is trivially answered: the rule (20), whose coefficients are simple arithmetic functions of the feasible-path parameters, is consistent with $O_t$ and implements $P$ as the robustly unique local equilibrium.

These implementability and (trivial) implementation results are more general, but very similar to those of the previous section. In particular, allowing the ARMA process (14) to be non-invertible, i.e., allowing $T_z(X)$ to have roots inside the unit circle (so that the unobserved shock $\varepsilon_t$ cannot be inferred from the history of the observed variable $z_t$ on the feasible path), does not substantially change the analysis. The only consequence of this non-invertibility is that all the rules consistent with the observation set and the feasible path, i.e., all the rules of type (19), are then "superinertial" in the terminology of Woodford (2003, Chapter 8), in the sense that the polynomial $\mathcal{P}(L)\widetilde{S}_i(L)T_z(L)$ in factor $i_t$ in these rules has some roots inside the unit circle (since this polynomial is a multiple of $T_z(L)$). Because the superinertial nature of a rule is neither necessary nor sufficient for indeterminacy, however, the issues of non-invertibility and non-implementability of feasible paths seem largely unrelated to each other. I will illustrate this point in Section 4.

### 3.3 *More unobserved shocks than observed variables*

I now turn to the case in which there are two shocks, and these shocks are not observed by the $\mathcal{PM}$: $c = 1$ and $O_t = \{z^t, i^{t-1}\}$. In this case, there are more unobserved shocks (two: $\varepsilon_t$ and $\varepsilon'_t$) than observed variables outside the policy instrument (one: $z_t$).

[6]If they had a common root, say a real number $r$, then the equation $\widetilde{S}_i(L)T_z(L)(r - L)^{-1}i_t = \widetilde{S}_z(L)T_i(L)(r-L)^{-1}z_t$ would also express $i_t$ as a function of a *finite* number of elements of $O_t$ and, therefore, would also be a policy-instrument rule consistent with $O_t$ and $P$.

I consider the set of paths on which each endogenous variable follows a stationary ARMA process driven by the exogenous shocks $\varepsilon_t$ and $\varepsilon'_t$, i.e., the set of paths of type

$$S_z(L)z_t = \mathbf{T}_z(L)\boldsymbol{\varepsilon}_t \quad \text{and} \quad S_i(L)i_t = \mathbf{T}_i(L)\boldsymbol{\varepsilon}_t, \tag{21}$$

where $(S_z(X), S_i(X)) \in \mathbb{R}[X]^2$, $S_z(0) \neq 0$, $S_i(0) \neq 0$, $S_z(X)$ and $S_i(X)$ have no roots inside the unit circle, $\boldsymbol{\varepsilon}_t \equiv [\varepsilon_t \varepsilon'_t]^T$, and $(\mathbf{T}_z(X), \mathbf{T}_i(X)) \in (\mathbb{R}^{1\times 2}[X] \setminus \{\mathbf{0}\})^2$.[7] Without any loss of generality, I impose that the ARMA representations in (21) be minimal, i.e., that $S_z(X)$ and $\mathbf{T}_z(X)$ have no common roots and that $S_i(X)$ and $\mathbf{T}_i(X)$ have none either. For simplicity, I also impose that $\mathbf{T}_z(X)$ and $\mathbf{T}_i(X)$ have no common roots.

A path of type (21) is *feasible* if and only if two conditions are met. First, the structural equation (13) must be satisfied on this path. Second, $i_t$ must be expressible as a function of $O_t$ on this path. This second condition requires in particular that

$$d \equiv \det \begin{bmatrix} \mathbf{T}_z(X) \\ \mathbf{T}_i(X) \end{bmatrix} = 0,$$

where $\det[\cdot]$ denotes the determinant operator. Indeed, for $i_t$ to be expressible as a function of $O_t$ on the path (21), there must exist $(P(X), Q(X)) \in \mathbb{R}[X]^2$ with $P(0) \neq 0$ such that $P(L)i_t = Q(L)z_t$ on this path. Multiplying this equation by $S_z(L)S_i(L)$ and using (21) leads to $P(L)S_z(L)\mathbf{T}_i(L)\boldsymbol{\varepsilon}_t = Q(L)S_i(L)\mathbf{T}_z(L)\boldsymbol{\varepsilon}_t$ and, hence, to $[P(X)S_z(X)]\mathbf{T}_i(X) = [Q(X)S_i(X)]\mathbf{T}_z(X)$. So feasibility requires that the vectors $\mathbf{T}_i(X)$ and $\mathbf{T}_z(X)$ be collinear, i.e., that $d = 0$.

In turn, $d = 0$ implies that $(T_z(X), T_i(X)) \in (\mathbb{R}[X] \setminus \{0\})^2$ and $\mathbf{T}(X) \in \mathbb{R}^{1\times 2}[X] \setminus \{\mathbf{0}\}$ exist such that $\mathbf{T}_z(X) = T_z(X)\mathbf{T}(X)$ and $\mathbf{T}_i(X) = T_i(X)\mathbf{T}(X)$. Therefore, if the path (21) is feasible, then it can be rewritten as

$$S_z(L)z_t = T_z(L)\nu_t \quad \text{and} \quad S_i(L)i_t = T_i(L)\nu_t, \tag{22}$$

where $\nu_t \equiv \mathbf{T}(L)\boldsymbol{\varepsilon}_t$. Equation (22) is the same as (14)–(15), except for the replacement of $\varepsilon_t$ by $\nu_t$. So I can analyze the feasibility and implementability of the path (22) in the same way as I analyzed the feasibility and implementability of the path (14)–(15) in the previous subsection. Whether the shock is $\varepsilon_t$ or $\nu_t$ plays no role in this analysis. Therefore, defining $(\widetilde{S}_i(X), \widetilde{S}_z(X))$ from $(S_i(X), S_z(X))$ in the same way as in the previous subsection, I straightforwardly get the following three results. First, the path (21) is feasible if and only if $d = 0$ and $T_z(0) \neq 0$. Second, when it is feasible, the path (21) is implementable if and only if the specific rule (20) robustly ensures local-equilibrium determinacy. Third, when it is implementable, the path (21) can be implemented by the rule (20), in the sense that (20) is consistent with the observation set and implements this path as the robustly unique local equilibrium.

[7]Throughout the paper, boldface letters denote vectors and matrices that have (at least potentially) more than one element; in particular, $\mathbf{0}$ denotes a vector or a matrix whose elements are all equal to 0 and whose dimension depends on the specific context in which it is used. The superscript $T$ denotes the transpose operator. For any $(m, n) \in (\mathbb{N} \setminus \{0\})^2$, $\mathbb{R}^{m\times n}[X]$ denotes the set of polynomials in $X$ that have coefficients that are $m \times n$ matrices with real-number elements.

These implementability and (trivial) implementation results are essentially the same as those of the previous subsection. When there are two unobserved shocks for only one observed variable (outside the policy instrument), the policymaker cannot infer the two shocks ($\varepsilon_t$ and $\varepsilon'_t$) separately, but on any feasible path, only a combination of these shocks ($\nu_t$) matters. So the questions of whether and how feasible paths can be implemented receive essentially the same answers as when there is only one unobserved shock.

### 3.4 *Fewer unobserved shocks than observed variables*

The last case I consider is the case in which there is only one shock, and this shock is observed by the $\mathcal{PM}$: $c = 0$ and $O_t = \{z^t, i^{t-1}, \varepsilon^t\}$. In this case, there are fewer unobserved shocks (zero) than observed variables outside the policy instrument (one: $z_t$).

I consider again the set of paths on which each endogenous variable follows a stationary (but possibly non-invertible) ARMA process driven by the exogenous shock $\varepsilon_t$, i.e., the set of paths of type (14)–(15). Since (15) already expresses $i_t$ only as a function of elements of $O_t$, a path of type (14)–(15) is *feasible* simply if and only if the structural equation (13) is satisfied on this path.

Now consider an arbitrary feasible path, characterized by (14)–(15) for some $S_z(X)$, $S_i(X)$, $T_z(X)$, and $T_i(X)$. This path is *implementable* if and only a policy-instrument rule exists that is consistent with $O_t$ and such that the path is the robustly unique stationary solution of the system made of the structural equation (13) and that rule. In this subsection, I show that such a rule exists and I design it *arithmetically,* i.e., with a finite number of arithmetic operations (addition, subtraction, multiplication, and division).

To that aim, consider the class of rules of type

$$S_z(L)S_i(L)\big[\mathcal{R}_i(L)i_t - \mathcal{R}_z(L)z_t\big] - \mathcal{R}_\varepsilon(L)\varepsilon_t = 0, \tag{23}$$

with $(\mathcal{R}_i(X), \mathcal{R}_z(X), \mathcal{R}_\varepsilon(X)) \in \mathbb{R}[X]^3$, $\mathcal{R}_i(0) \neq 0$, and $\mathcal{R}_z(X) \neq 0$. Since the coefficient $S_z(0)S_i(0)\mathcal{R}_i(0)$ of $i_t$ in (23) is nonzero, (23) expresses $i_t$ only as a function of elements of $O_t$, i.e., it is a policy-instrument rule consistent with $O_t$. Moreover, since all the roots of $S_z(X)$ and $S_i(X)$ lie outside the unit circle, (23) is equivalent to

$$\mathcal{R}_i(L)i_t = \mathcal{R}_z(L)z_t + S_z(L)^{-1}S_i(L)^{-1}\mathcal{R}_\varepsilon(L)\varepsilon_t. \tag{24}$$

The system made of the structural equation (13) and the rule (24) can easily be written in Blanchard and Kahn's (1980) form with exactly $\delta$ non-predetermined variables (corresponding to the expectations $\mathbb{E}_t\{z_{t+\delta}\}, \ldots, \mathbb{E}_t\{z_{t+1}\}$, since $A(0) \neq 0$). For this system to satisfy Blanchard and Kahn's (1980) root-counting condition, we need its characteristic polynomial to have exactly $\delta$ roots outside the unit circle (as many as there are non-predetermined variables in the system). The characteristic polynomial of this system is the same, up to a multiplicative factor of type $X^p$ with $p \in \mathbb{N}$, as the characteristic polynomial $\mathcal{C}(X)$ of the corresponding perfect-foresight deterministic system, which is made of the equations $A(L)z_t + L^\delta B(L)i_t = 0$ and $\mathcal{R}_i(L)i_t = \mathcal{R}_z(L)z_t$. The *reciprocal* polynomial of $\mathcal{C}(X)$ is, straightforwardly,

$$\mathcal{R}(X) \equiv A(X)\mathcal{R}_i(X) + X^\delta B(X)\mathcal{R}_z(X).$$

For Blanchard and Kahn's (1980) root-counting condition to be satisfied, therefore, we need $\mathcal{R}(X)$ to have exactly $\delta$ roots *inside* the unit circle, all of them nonzero.

Let $d_A$ and $d_B$ denote the degrees of $A(X)$ and $B(X)$. Consider an arbitrary polynomial $\Phi(X) \in \mathbb{R}[X]$ with the following three properties: (i) $\Phi(X)$ has exactly $\delta$ roots inside the unit circle, none of which is zero, (ii) the degree of $\Phi(X)$ is lower than or equal to $\max(d_A - 1, 0) + d_B + \delta$, and (iii) $\Phi(X)$ is not a multiple of $A(X)$, except, of course, if $d_A = 0$. In the following, I arithmetically design $(\mathcal{R}_i^*(X), \mathcal{R}_z^*(X)) \in \mathbb{R}[X]^2$ such that

$$\mathcal{R}^*(X) \equiv A(X)\mathcal{R}_i^*(X) + X^\delta B(X)\mathcal{R}_z^*(X) = \Phi(X).$$

Start with the case in which $d_A = 0$. In this case, the arithmetic design of $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ is trivial: since $A(X) = A(0) \neq 0$, one can choose, e.g., $\mathcal{R}_z^*(X) = 1$ and $\mathcal{R}_i^*(X) = [\Phi(X) - X^\delta B(X)]/A(0)$. This choice is, straightforwardly, such that $\mathcal{R}^*(X) = \Phi(X)$.

Now turn to the alternative case in which $d_A \geq 1$. In this case, I use the Sylvester matrix of $A(X)$ and $B(X)$ to design arithmetically some $\mathcal{R}_i^*(X)$ of degree lower than or equal to $d_B + \delta - 1$ and some $\mathcal{R}_z^*(X)$ of degree lower than or equal to $d_A - 1$. More specifically, let $(a_k)_{0\leq k\leq d_A}$, $(b_k)_{0\leq k\leq d_B}$, and $(\phi_k)_{0\leq k\leq d_A+d_B+\delta-1}$ denote the coefficients of $A(X)$, $B(X)$, and $\Phi(X)$ (which are known), and let $(r_k^{z*})_{0\leq k\leq d_A-1}$ and $(r_k^{i*})_{0\leq k\leq d_B+\delta-1}$ denote those of $\mathcal{R}_z^*(X)$ and $\mathcal{R}_i^*(X)$ (which are unknown):

$$\begin{aligned}
A(X) &= a_0 + \cdots + a_{d_A} X^{d_A}, \\
B(X) &= b_0 + \cdots + b_{d_B} X^{d_B}, \\
\Phi(X) &= \phi_0 + \cdots + \phi_{d_A+d_B+\delta-1} X^{d_A+d_B+\delta-1}, \\
\mathcal{R}_z^*(X) &= r_0^{z*} + \cdots + r_{d_A-1}^{z*} X^{d_A-1}, \\
\mathcal{R}_i^*(X) &= r_0^{i*} + \cdots + r_{d_B+\delta-1}^{i*} X^{d_B+\delta-1}.
\end{aligned}$$

The equation $\mathcal{R}^*(X) = \Phi(X)$ can then be rewritten as

$$\mathbf{S}\begin{bmatrix} r_{d_A-1}^{z*} \\ \vdots \\ r_0^{z*} \\ r_{d_B+\delta-1}^{i*} \\ \vdots \\ r_0^{i*} \end{bmatrix} = \begin{bmatrix} \phi_{d_A+d_B+\delta-1} \\ \vdots \\ \phi_0 \end{bmatrix}, \quad \text{where } \mathbf{S} \equiv \Bigg[\underbrace{\begin{matrix} b_{d_B} & & \\ \vdots & \ddots & \\ b_0 & \ddots & b_{d_B} \\ & \ddots & \vdots \\ & & b_0 \\ & & \end{matrix}}_{(d_A+d_B+\delta)\times d_A} \left| \underbrace{\begin{matrix} a_{d_A} & & \\ \vdots & \ddots & \\ a_0 & \ddots & \ddots \\ & \ddots & \ddots & a_{d_A} \\ & & \ddots & \vdots \\ & & & a_0 \end{matrix}}_{(d_A+d_B+\delta)\times(d_B+\delta)} \right.\Bigg]$$

is the transpose of the Sylvester matrix of the polynomials $A(X)$ and $X^\delta B(X)$.[8] A Sylvester matrix of two polynomials (with real-number coefficients) is invertible if and only if these polynomials have no common (real or complex) roots. Now the polynomials $A(X)$ and $X^\delta B(X)$ have no common roots, since $A(X)$ and $B(X)$ are coprime

[8]To lighten the exposition, I display only the elements of $\mathbf{S}$ that may be nonzero.

and $A(0) \neq 0$. Therefore, their Sylvester matrix is invertible and so is its transpose $\mathbf{S}$, so that the coefficients of $\mathcal{R}_z^*(X)$ and $\mathcal{R}_i^*(X)$ can be arithmetically obtained as

$$\begin{bmatrix} r_{d_A-1}^{z*} & \dots & r_0^{z*} & r_{d_B+\delta-1}^{i*} & \dots & r_0^{i*} \end{bmatrix}^T = \mathbf{S}^{-1} \begin{bmatrix} \phi_{d_A+d_B+\delta-1} & \dots & \phi_0 \end{bmatrix}^T.$$

By construction, these polynomials $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ are such that $\mathcal{R}^*(X) = \Phi(X)$.

Whether $d_A = 0$ or $d_A \geq 1$, the polynomials $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ that I just designed are admissible choices for $\mathcal{R}_i(X)$ and $\mathcal{R}_z(X)$ in (23), because they are such that $\mathcal{R}_i^*(0) \neq 0$ and $\mathcal{R}_z^*(X) \neq 0$. Indeed, $\mathcal{R}^*(X) = \Phi(X)$ implies $\mathcal{R}^*(0) = A(0)\mathcal{R}_i^*(0) = \Phi(0)$ (since $\delta \geq 1$) and, hence, $\mathcal{R}_i^*(0) = \Phi(0)/A(0) \neq 0$ (since $\Phi(0) \neq 0$). If $d_A \geq 1$, then $\Phi(X)$ is not a multiple of $A(X)$, so that $\mathcal{R}^*(X) = \Phi(X)$ implies $\mathcal{R}_z^*(X) \neq 0$. If $d_A = 0$, then $\mathcal{R}_z^*(X) = 1 \neq 0$.

So consider the rule (23) with $(\mathcal{R}_i(X), \mathcal{R}_z(X)) = (\mathcal{R}_i^*(X), \mathcal{R}_z^*(X))$ for an arbitrary $\mathcal{R}_\varepsilon(X)$. As mentioned above, the property $\mathcal{R}^*(X) = \Phi(X)$, together with the fact that $\Phi(X)$ has exactly $\delta$ roots inside the unit circle (none of which is zero), implies that the system made of the structural equation (13) and this rule satisfies Blanchard and Kahn's (1980) *root-counting* condition.

In addition, this system also satisfies Blanchard and Kahn's (1980) *no-decoupling* condition, except possibly for a zero-measure set of polynomials $\Phi(X)$.[9] More specifically, the rule in this system has two properties that preclude two variants of decoupling. First, $\mathcal{R}_z^*(X) \neq 0$ ensures that the dynamics of $i_t$ are not decoupled from the dynamics of $z_t$ in the rule (which is, in this sense, a feedback rule). Second, as I show in Appendix A.1, $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ have no common roots—in particular, no common roots inside the unit circle—except possibly for a zero-measure set of polynomials $\Phi(X)$. This second property matters for the following reason. If $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ had a common root inside the unit circle, say a real number $r \in (-1, 1) \setminus \{0\}$, then the rule could be rewritten as $S_z(L)S_i(L)(1-L/r)v_t = \mathcal{R}_\varepsilon(L)\varepsilon_t$, where the variable $v_t \equiv (1-L/r)^{-1}[\mathcal{R}_i^*(L)i_t - \mathcal{R}_z^*(L)z_t]$ would be well defined as it would involve a finite number of elements of $O_t \cup \{i_t\}$. Therefore, the rule would generate explosive dynamics for the variable $v_t$, and, hence, also for $z_t$ and/or $i_t$.[10] In essence, the system would have an "unstable eigenvalue" $(1/r)$, the eigenvector of which would be a predetermined variable $(v_t)$.

Since they make the system satisfy both the root-counting and the no-decoupling conditions of Blanchard and Kahn (1980), all the rules of type (23) with $(\mathcal{R}_i(X), \mathcal{R}_z(X)) = (\mathcal{R}_i^*(X), \mathcal{R}_z^*(X))$ robustly ensure local-equilibrium determinacy. Among them, in particular, the rule

$$S_z(L)S_i(L)\big[\mathcal{R}_i^*(L)i_t - \mathcal{R}_z^*(L)z_t\big] - \mathcal{R}_\varepsilon^*(L)\varepsilon_t = 0, \tag{25}$$

[9]The no-decoupling condition requires that the system should not be "decoupled" in the sense of Sims (2007). It is formulated as a matrix-rank condition in Blanchard and Kahn (1980, p. 1308) and is often called the rank condition in the literature. Sims' (2007) bare-bones example of a system meeting the root-counting condition but not the no-decoupling condition is $x_t = 1.1x_{t-1} + \varepsilon_t$ and $\mathbb{E}_t\{y_{t+1}\} = 0.9y_t + \nu_t$.

[10]More precisely, the rule would generate explosive dynamics for $v_t$ unless $r$ is also a root of $\mathcal{R}_\varepsilon(X)$. But if $r$ is also a root of $\mathcal{R}_\varepsilon(X)$, then explosive dynamics would reemerge as soon as an exogenous policy shock, even of arbitrarily small variance, is added to the rule.

where $\mathcal{R}_\varepsilon^*(X) \equiv S_z(X)T_i(X)\mathcal{R}_i^*(X) - S_i(X)T_z(X)\mathcal{R}_z^*(X)$, is satisfied on the path (14)–(15). This rule, therefore, implements that path as the robustly unique local equilibrium. So, to sum up, (25) is a policy-instrument rule consistent with $O_t$ and implementing the path (14)–(15) as the robustly unique local equilibrium. As a consequence, this path is implementable and the rule (25) that I have arithmetically designed implements it.

These implementability and implementation results differ substantially from those of the previous subsections. When there are fewer unobserved shocks than observed variables (outside the policy instrument), there are additional degrees of freedom in the choice of a rule consistent with the observation set and the feasible path, and it is always possible to find one such rule that robustly ensures local-equilibrium determinacy. Thus, all feasible paths are implementable. Moreover, for any feasible path, I have shown how to design arithmetically a policy-instrument rule that is consistent with the policymaker's observation set and implements this path as the robustly unique local equilibrium.

This arithmetic method of designing rules does not require, in particular, determining any polynomial roots (except trivially roots of polynomials of degree 1) or any inequality condition for determinacy. Instead, it directly transforms (i) the polynomials and parameter characterizing the structural equation $(A(X), B(X), \delta)$, (ii) the polynomials characterizing the targeted feasible path $(S_z(X), T_z(X), S_i(X), T_i(X))$, and (iii) a polynomial, the roots of which include the inverses of the "unstable eigenvalues" chosen to match the non-predetermined variables $(\Phi(X))$, into the polynomials characterizing the rule ($\mathcal{R}_j^*(X)$ for $j \in \{i, z, \varepsilon\}$). Because the rule is arithmetically designed, its coefficients are explicitly expressed as *rational functions* of the structural and feasible-path parameters, i.e., as fractions of polynomial functions of these parameters. Such functions are particularly easy to manipulate analytically. For instance, their derivatives can be easily computed—with the help of a symbolic-computation software—to determine how the coefficients of the rule respond to an arbitrarily small change in the value of the structural or feasible-path parameters.

### 3.5 *Discussion*

In this section, I have focused on a class of univariate models, i.e., a class of models with only one variable set by the private sector. The implementability and implementation analysis conducted in this class of models will prove useful for the next two sections, even though the models in these sections are multivariate.

In Section 4, I will consider a framework with as many unobserved shocks as observed variables (outside the policy instrument), as in Section 3.2. One difference is that there will be two unobserved shocks and two observed variables, not one and one. This difference will not fundamentally change the analysis, as we will see. Another difference is that I will assume that the policymaker does not observe *current* variables. This difference will not play a major role either: in Section 3.2, replacing $O_t = \{z^t, i^{t-1}\}$ by $O_t = \{z^{t-1}, i^{t-1}\}$ without changing the structural equation (13) is equivalent to replacing $B(X)$ by $B(X)X$ in (13) without changing $O_t$, and does not fundamentally change the analysis.

In Section 5, I will consider a framework with fewer unobserved shocks than observed variables (outside the policy instrument), as in Section 3.4. One difference is that there will be two unobserved shocks and seven observed variables, not zero and one. This difference will not fundamentally change the analysis. In essence, I will first follow the same steps as in Section 3.4 to design a rule involving only one observed variable and the two unobserved shocks. Then I will use the structural equations to replace the unobserved shocks in this rule by some functions of observed variables, in a way that is neutral for robust local-equilibrium determinacy.[11]

## 4. Optimal monetary policy in the NK model

In this section, I study the implementability of the welfare-maximizing feasible path in the NK model, with the central bank as the policymaker and the interest rate as the policy instrument. Rotemberg and Woodford (1999, p. 103) once wrote that "the construction of a feedback rule for the funds rate that implements the optimal allocation—that is not only consistent with it but also renders it the unique stationary equilibrium consistent with the proposed policy rule—remains a nontrivial problem." In essence, this statement is still valid today when the optimal allocation and the feedback rule are required to be consistent with a given observation set of the central bank. In fact, as I show in this section, such a feedback rule may simply not exist for a reasonable observation set of the central bank.

In most of the section, I focus on the *basic* NK model, presented in detail in Woodford (2003, Chapters 2, 4, and 6) and Galí (2015, Chapter 3). At the end of the section, I extend the analysis to Svensson and Woodford's (2005) NK model with monetary-policy transmission lags.

### 4.1 *Structural equations and observation set*

I consider an economy described by the basic NK model and hit by two exogenous disturbances, one affecting the discount factor and the other the elasticity of substitution between differentiated goods. In this economy, at each date $t \in \mathbb{Z}$, the private sector ($\mathcal{PS}$) sets the inflation rate $\pi_t$ and the output level $y_t$ according to the (locally log-linearized) investment-saving (IS) equation and Phillips curve,

$$y_t = \mathbb{E}_t\{y_{t+1}\} - \frac{1}{\sigma}\big(i_t - \mathbb{E}_t\{\pi_{t+1}\}\big) + \eta_t, \tag{26}$$

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa y_t + u_t, \tag{27}$$

[11] In essence, I will illustrate how my arithmetic method of designing rules in univariate models can be extended to multivariate models with observed shocks or with shocks that are unobserved but can be recovered from observed variables using only the structural equations. This method, however, cannot be easily extended to multivariate models with both (i) fewer unobserved shocks than observed variables and (ii) some unobserved and nonrecoverable shocks. Extending my implementability and implementation analysis to such models is beyond the scope of this paper.

where $i_t$ denotes the interest rate set by the central bank ($\mathcal{CB}$) at date $t$. I assume that the exogenous disturbances $\eta_t$ and $u_t$ follow stationary ARMA(1, 1) processes

$$\eta_t = \rho_\eta \eta_{t-1} + \varepsilon_t^\eta + \theta_\eta \varepsilon_{t-1}^\eta, \tag{28}$$

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u + \theta_u \varepsilon_{t-1}^u, \tag{29}$$

where $\varepsilon_t^\eta$ and $\varepsilon_t^u$ are two orthogonal i.i.d. exogenous shocks of mean zero. The structural parameters satisfy $0 < \beta < 1$, $\sigma > 0$, $\kappa > 0$, $-1 < \rho_\eta < 1$, and $-1 < \rho_u < 1$ (while $\theta_\eta$ and $\theta_u$ may take any real-number value).

The observation set that I consider for the $\mathcal{CB}$ is

$$O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}.$$

This observation set has two notable features. First, it contains no exogenous shocks, which seems reasonable given the nature of the two shocks considered. Second, it contains no current endogenous variables. This second feature can be viewed as a consequence of the timing in which the $\mathcal{CB}$ plays before the $\mathcal{PS}$ within each period. This timing is arguably better suited than the reverse timing to capture the fact that, due to information-collecting, information-processing, and decision-making frictions, central banks take their decisions at a lower frequency than the private sector considered as a whole (though not necessarily than each individual private agent). Many papers that study the conduct of monetary policy, from Poole (1970) and Sargent and Wallace (1975) to Svensson and Woodford (2005) and Atkeson et al. (2010), explicitly assume that the central bank plays before the private sector at each date and, hence, does not observe current endogenous variables when setting the interest rate. I follow them.[12]

## 4.2 *Optimal feasible path*

The optimal feasible path is the path that maximizes welfare subject to the structural equations (26)–(27) and to the $\mathcal{CB}$'s observation-set constraint. I assume for simplicity that the steady state of the model is efficient, due to an employment or production subsidy offsetting the monopolistic-competition distortion. Therefore, the welfare-loss function, i.e., the opposite of the second-order approximation of households' intertemporal utility function in the neighborhood of the steady state, can be written as $L_t = \mathbb{E}_t\{\sum_{k=0}^{+\infty} \beta^k[(\pi_{t+k})^2 + \lambda(y_{t+k})^2]\}$, where $\lambda > 0$.

The optimal feasible path that I consider is, more specifically, the optimal feasible path under Woodford's (1999) *timeless perspective*. This path can be defined as the limit of the date-$t_0$ Ramsey-optimal feasible path as $t_0 \to -\infty$. I consider the timeless-perspective optimal feasible path, rather than the date-$t_0$ Ramsey-optimal feasible path, to avoid having to deal with initial conditions (as explained in footnote 5).

To determine this path, denoted by $P$, I proceed in three steps. First, I show that all the non-sunspot-driven paths that are feasible under the observation set $O_t \equiv$

[12]In addition, this second feature of the observation set is necessary for the existence of a well defined optimal feasible path, as I explain in Appendix A.2.

$\{\pi^{t-1}, y^{t-1}, i^{t-1}\}$, including the path $P$, are also feasible under the observation set $\widetilde{O}_t \equiv \{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$. Second, I determine the timeless-perspective optimal feasible path under $\widetilde{O}_t$. Third, I show that the latter path, denoted by $\widetilde{P}$, is feasible under $O_t$. I conclude that the two paths $P$ and $\widetilde{P}$ coincide with each other. The advantage of this three-step procedure is that $\widetilde{P}$, contrary to $P$, can be easily obtained with the undetermined-coefficients method.

In the first step, I consider an arbitrary path that is feasible under $O_t$ and does not involve sunspot shocks. On this path, $i_t$ can be expressed only as a function of elements of $O_t$; I denote by $(E)$ the corresponding equation. Moreover, on this path, the forecast errors $\pi_{t+1} - \mathbb{E}_t\{\pi_{t+1}\}$ and $y_{t+1} - \mathbb{E}_t\{y_{t+1}\}$ depend only on the fundamental innovation $\boldsymbol{\varepsilon}_{t+1} \equiv [\varepsilon^{\eta}_{t+1} \quad \varepsilon^{u}_{t+1}]^T$. Therefore, the structural equations (26) and (27) can be rewritten as $\pi_t + \sigma(1-L)y_t = i_{t-1} - \sigma\eta_{t-1} + \mathbf{v}_1\boldsymbol{\varepsilon}_t$ and $(\beta - L)\pi_t + \kappa y_{t-1} = -u_{t-1} + \mathbf{v}_2\boldsymbol{\varepsilon}_t$, where $(\mathbf{v}_1, \mathbf{v}_2) \in (\mathbb{R}^{1\times 2})^2$. These two equations, together with (28)–(29) and $(E)$, form a backward-looking system that can be solved recursively to get $\pi_t$ and $y_t$ as functions of $\{\varepsilon^{\eta,t}, \varepsilon^{u,t}\}$, and get $i_t$ as a function of $\widetilde{O}_t$. Thus, all non-sunspot-driven feasible paths under $O_t$ are also feasible under $\widetilde{O}_t$.

In the second step, to determine $\widetilde{P}$, I specify the interest rate $i_t$ as a linear function of the elements of $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, and specify the inflation rate $\pi_t$ and output $y_t$ as linear functions of the elements of $\{\varepsilon^{\eta,t}, \varepsilon^{u,t}\}$. I look for the values of the coefficients of these linear functions that minimize $L_t$ subject to the structural equations (26) and (27). The computations, which are standard and of no particular interest, are available upon request. Defining $\mu \equiv (2\beta\lambda)^{-1}[\lambda + \beta\lambda + \kappa^2 - \sqrt{(\lambda + \beta\lambda + \kappa^2)^2 - 4\beta\lambda^2}] \in (0,1)$ and focusing on the generic case $\rho_u \neq \mu$, I get the minimal ARMA representation for $\widetilde{P}$,

$$(1-\rho_u L)(1-\mu L)\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & (1-\rho_\eta L) \end{bmatrix}\begin{bmatrix} \mathbf{Z}_t \\ i_t \end{bmatrix} = \begin{bmatrix} \mathbf{T}_Z(L) \\ \mathbf{T}_i(L)L \end{bmatrix}\boldsymbol{\varepsilon}_t, \tag{30}$$

where $\mathbf{Z}_t \equiv [\pi_t \quad y_t]^T$, $\mathbf{I}_2$ denotes the $2\times 2$ identity matrix, $\mathbf{T}_Z(X) \in \mathbb{R}^{2\times 2}[X]$, with $\det[\mathbf{T}_Z(0)] \neq 0$, and $\mathbf{T}_i(X) \in \mathbb{R}^{1\times 2}[X]$.

In the third step, to show that $\widetilde{P}$ is feasible under $O_t$, I rewrite the first two lines of (30) as

$$\det\big[\mathbf{T}_Z(L)\big]\boldsymbol{\varepsilon}_t = (1-\rho_u L)(1-\mu L)\,\mathrm{adj}\big[\mathbf{T}_Z(L)\big]\mathbf{Z}_t \tag{31}$$

by using Laplace's expansion $\mathrm{adj}[\mathbf{T}_Z(X)]\mathbf{T}_Z(X) = \det[\mathbf{T}_Z(X)]\mathbf{I}_2$, where $\mathrm{adj}[\mathbf{T}_Z(X)] \in \mathbb{R}^{2\times 2}[X]$ denotes the adjugate of $\mathbf{T}_Z(X)$ (i.e., the transpose of its cofactor matrix). I then multiply the left- and right-hand sides of the last line of (30) by $(1-\rho_u L)^{-1}(1-\mu L)^{-1}\det[\mathbf{T}_Z(L)]$ and use (31) to get

$$(1-\rho_\eta L)\det\big[\mathbf{T}_Z(L)\big]i_t = \mathbf{T}_i(L)\,\mathrm{adj}\big[\mathbf{T}_Z(L)\big]\mathbf{Z}_{t-1}. \tag{32}$$

Since $\det[\mathbf{T}_Z(0)] \neq 0$, (32) expresses $i_t$ only as a function of elements of $O_t$. Therefore, the path $\widetilde{P}$ is feasible under $O_t$.

So, to sum up, $P$ is optimal over the set $S$ of non-sunspot-driven feasible paths under $O_t$, this set $S$ is included in the set $\widetilde{S}$ of feasible paths under $\widetilde{O}_t$, $\widetilde{P}$ is optimal over $\widetilde{S}$, and $\widetilde{P}$ belongs to $S$. I conclude that $P = \widetilde{P}$ and, hence, that $P$ is characterized by (30).[13]

### 4.3 *(Non-)implementability of the optimal feasible path*

I now turn to the question of whether the optimal feasible path $P$ is *implementable* under $O_t$. To answer this question, I proceed in the same way as in Sections 2.4 and 3.2. I start by noting that the (locally log-linearized) interest-rate rules consistent with the observation set $O_t$ are the rules of type

$$\mathcal{P}(L)i_t + \mathcal{Q}(L)\mathbf{Z}_{t-1} = 0, \tag{33}$$

with $\mathcal{P}(X) \in \mathbb{R}[X]$, $\mathcal{P}(0) \neq 0$, and $\mathcal{Q}(X) \in \mathbb{R}^{1\times 2}[X]$. Among these rules, the rules that are also consistent with the path $P$—characterized by (30)—are those such that

$$\mathcal{P}(X)\mathbf{T}_i(X) + (1-\rho_\eta X)\mathcal{Q}(X)\mathbf{T}_Z(X) = 0. \tag{34}$$

Multiplying (34) by $\operatorname{adj}[\mathbf{T}_Z(X)]$ and using Laplace's expansion $\mathbf{T}_Z(X)\operatorname{adj}[\mathbf{T}_Z(X)] = \det[\mathbf{T}_Z(X)]\mathbf{I}_2$ leads to

$$\mathcal{P}(X)\mathbf{T}_i(X)\operatorname{adj}\big[\mathbf{T}_Z(X)\big] + (1-\rho_\eta X)\det\big[\mathbf{T}_Z(X)\big]\mathcal{Q}(X) = 0. \tag{35}$$

Multiplying (33) by $(1-\rho_\eta L)\det[\mathbf{T}_Z(L)]$ and using (35) then leads to

$$\mathcal{P}(L)\big\{(1-\rho_\eta L)\det\big[\mathbf{T}_Z(L)\big]i_t - \mathbf{T}_i(L)\operatorname{adj}\big[\mathbf{T}_Z(L)\big]\mathbf{Z}_{t-1}\big\} = 0. \tag{36}$$

Since $\det[\mathbf{T}_Z(0)] \neq 0$, all the equations of type (36) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$ are interest-rate rules consistent with $O_t$ and $P$. Conversely, all the interest-rate rules consistent with $O_t$ and $P$ are (generically) of type (36) with $\mathcal{P}(X) \in \mathbb{R}[X]$ and $\mathcal{P}(0) \neq 0$, given that $(1-\rho_\eta X)\det[\mathbf{T}_Z(X)]$ and $\mathbf{T}_i(X)\operatorname{adj}[\mathbf{T}_Z(X)]$ have no common roots (except in zero-measure cases).

So the question of whether the path $P$ is implementable can be equivalently restated as the question of whether $\mathcal{P}(X) \in \mathbb{R}[X]$ with $\mathcal{P}(0) \neq 0$ exists such that the rule (36) robustly ensures local-equilibrium determinacy. I can then conduct exactly the same reasoning as in Section 2.4 and conclude that a rule of type (36) exists that robustly ensures local-equilibrium determinacy if and only if the specific rule (32), corresponding to $\mathcal{P}(X) = 1$, robustly ensures local-equilibrium determinacy in the first place. So, the question of whether the path $P$ is implementable eventually boils down to the question of whether the system made of the structural equations (26)–(27) and the specific rule (32) satisfies Blanchard and Kahn's (1980) conditions.

[13]On this path, interestingly, the endogenous variables respond persistently to both $\eta_t$ and $u_t$, even when these disturbances are i.i.d. As is well known since Clarida et al. (1999) and Woodford (1999), a persistent response to $u_t$ is optimal because, by making $\mathbb{E}_t\{\pi_{t+1}\}$ depend negatively on $u_t$, it relaxes the constraint imposed on $(\pi_t, y_t)$ by the Phillips curve (27). In my setup, similarly, a persistent response to $\eta_t$ is optimal because, by making $\mathbb{E}_t\{y_{t+1}\} + (1/\sigma)\mathbb{E}_t\{\pi_{t+1}\}$ depend negatively on $\eta_t$, it relaxes the constraint imposed on $y_t$ by the IS equation (26) and by $\eta_t \notin O_t$.

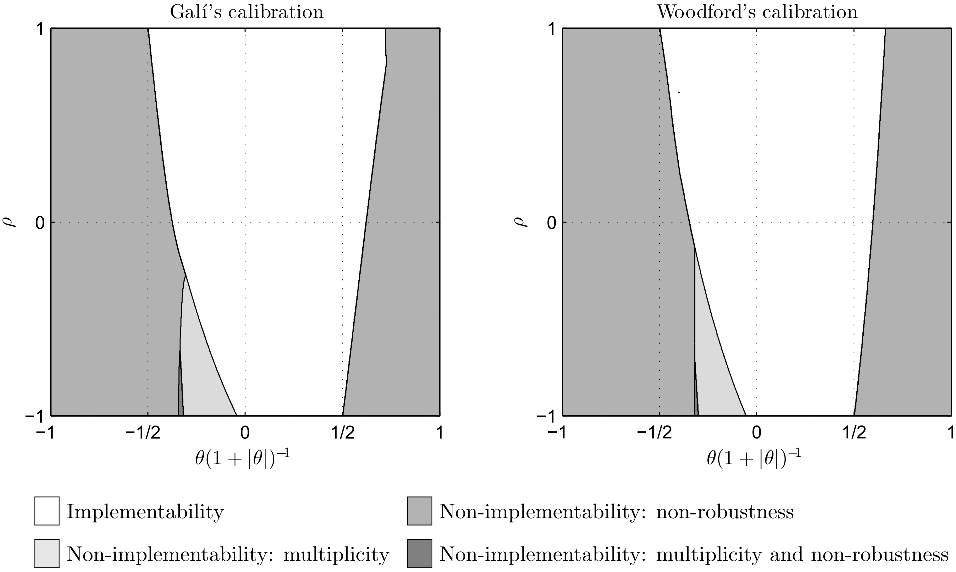


Figure 1. Implementability of the optimal feasible path in the basic NK model when the $\mathcal{CB}$ observes only past endogenous variables.

This question can be straightforwardly answered numerically. For instance, let me consider Galí's (2015, Chapter 3) and Woodford's (2003, Chapter 4) calibrations of the basic NK model, respectively characterized by $(\beta, \sigma, \kappa, \lambda) = (0.99, 1.00, 0.125, 0.021)$ and $(\beta, \sigma, \kappa, \lambda) = (0.99, 0.16, 0.022, 0.003)$, and let me focus on the values of $\rho_\eta$, $\rho_u$, $\theta_\eta$, and $\theta_u$ such that $\rho_\eta = \rho_u \equiv \rho$ and $\theta_\eta = \theta_u \equiv \theta$ (so that the two disturbances follow identical stochastic processes). As shown in Figure 1, I then obtain that $P$ is not implementable for many values of $\rho$ and $\theta$, broadly the same values under both calibrations.

For some values of $\rho$ and $\theta$ (light-gray areas in Figure 1), $P$ is not implementable because all the interest-rate rules consistent with $O_t$ and $P$ lead to local-equilibrium multiplicity. For other values of $\rho$ and $\theta$ (dark-gray areas in Figure 1), it is not implementable because adding an exogenous monetary-policy shock (even of arbitrarily small variance) to any interest-rate rule consistent with $O_t$ and $P$ leads to nonexistence of a local equilibrium. For still other values of $\rho$ and $\theta$ (very-dark-gray areas in Figure 1), $P$ is not implementable because all the interest-rate rules consistent with $O_t$ and $P$ lead to local-equilibrium multiplicity in the absence of exogenous monetary-policy shocks and to nonexistence of a local equilibrium in the presence of such shocks.

In the first two cases, the system made of the structural equations (26)–(27) and the specific rule (32) does not meet Blanchard and Kahn's (1980) root-counting condition because it has strictly fewer (in the first case) or strictly more (in the second case) eigenvalues outside the unit circle than non-predetermined variables. In the third case, this system meets Blanchard and Kahn's (1980) root-counting condition but not their no-decoupling condition (which I discuss in Section 3.4). Sims (2007) claims that systems meeting the root-counting condition but not the no-decoupling condition "can easily arise in economic research." This third case provides a concrete economic example in support of that claim.[14]

As is apparent from Figure 1, $P$ is not implementable if and only if the moving-average parameter $\theta$ is sufficiently large in absolute value. More specifically, for any of the two calibrations and any value of the autoregressive parameter $\rho$, two threshold values $\underline{\theta} < 0$ and $\overline{\theta} > 0$ exist such that $P$ is not implementable if and only if $\theta < \underline{\theta}$ or $\theta > \overline{\theta}$. In particular, $P$ is not implementable at the limit when $\theta \to -\infty$ or $\theta \to +\infty$. This limit case can be interpreted as a situation in which *news shocks* perfectly inform the $\mathcal{PS}$ about one-period-ahead disturbances. Indeed, as $\theta$ goes to minus or plus infinity, and the variance of $\varepsilon_t^\eta$ and $\varepsilon_t^u$ go to zero at speed $\theta^2$ (so that the variances of $\widetilde{\varepsilon}_t^\eta \equiv \theta\varepsilon_t^\eta$ and $\widetilde{\varepsilon}_t^u \equiv \theta\varepsilon_t^u$ are constant), the stochastic processes of $\eta_t$ and $u_t$ converge to $\eta_t = \rho\eta_{t-1} + \widetilde{\varepsilon}_{t-1}^\eta$ and $u_t = \rho u_{t-1} + \widetilde{\varepsilon}_{t-1}^u$, so that $\widetilde{\varepsilon}_t^\eta$ and $\widetilde{\varepsilon}_t^u$ can be interpreted as news shocks that perfectly inform the $\mathcal{PS}$ about $\eta_{t+1}$ and $u_{t+1}$.

Despite $P$ being non-implementable if and only if $|\theta|$ is sufficiently large, there is no apparent link between the (non-)implementability of $P$ and the (non-)invertibility of the ARMA processes of the exogenous disturbances (28)–(29). Indeed, the non-implementability thresholds $\underline{\theta}$ and $\overline{\theta}$ are such that $-1 < \underline{\theta} < 1 < \overline{\theta}$ (or, equivalently, $-1/2 < \underline{\theta}(1 + |\underline{\theta}|)^{-1} < 1/2 < \overline{\theta}(1 + |\overline{\theta}|)^{-1}$ in Figure 1). Thus, denoting by (N)I-(N)I the (non-)implementability of $P$ and the (non-)invertibility of (28)–(29), we have NI-NI for $\theta < -1$, NI-I for $-1 < \theta < \underline{\theta}$, I-I for $\underline{\theta} < \theta < 1$, I-NI for $1 < \theta < \overline{\theta}$, and NI-NI again for $\overline{\theta} < \theta$. Therefore, non-invertibility of the ARMA processes of the exogenous disturbances is neither necessary nor sufficient for non-implementability of the optimal feasible path.[15]

There is also no apparent link between the (non-)implementability of $P$ and the (non-)invertibility of the VARMA process (31) relating $\mathbf{Z}_t$ to $\boldsymbol{\varepsilon}_t$ on $P$. For the whole grid of values of $\rho$ and $\theta$ that I consider under both calibrations, I find that $\det[\mathbf{T}_Z(X)]$ has at least one root inside the unit circle, so that (31) is not invertible and $\boldsymbol{\varepsilon}_t$ cannot be inferred from $\mathbf{Z}_t$ on $P$. Yet, $P$ is implementable for many values of $\rho$ and $\theta$, as is clear from Figure 1. So non-invertibility of a feasible path is not a sufficient condition for non-implementability of this path.[16] As in Section 3.2, the non-invertibility of $P$ implies that all the rules consistent with $O_t$ and $P$, i.e., all the rules of type (36), are superinertial; but superinertial rules may very well ensure robust local-equilibrium determinacy.

### 4.4 *Policy implications*

This non-implementability result sounds a note of caution about one of the main lessons of the NK literature, namely the importance for central banks to track some key

[14] I obtain this numerical result using indifferently my own Matlab code, Sims' (2001) gensys.m code, or Dynare.

[15] It is true that the borderline between the implementability and non-implementability areas in Figure 1 apparently goes through the limit points $(-0.5, 1)$ and $(0.5, -1)$ in the $(\theta(1 + |\theta|)^{-1}, \rho)$ plane under both calibrations. This result, however, may reflect the peculiarity of these two limit points (at which $\theta = -\rho$, and, hence, $\eta_t = \varepsilon_t^\eta$ and $u_t = \varepsilon_t^u$), rather than a link between non-invertibility of (28)–(29) and non-implementability of $P$.

[16] It is also not a necessary condition. In Section 2.4, for instance, the first line of the path (7) is trivially invertible, yet this path may be non-implementable.

unobserved exogenous rates of interest such as, for instance, the counterfactual "natural rate of interest" (as emphasized by, e.g., Galí 2015, Chapter 9, and Woodford 2003, Chapter 4).[17] This lesson is drawn from analyses that implicitly assume that the central bank can produce an *exogenous* estimate of these exogenous rates of interest, that is to say, in effect, that it can infer them from its observation of exogenous shocks only. Under this assumption, the interest-rate rule can involve that estimate as an exogenous term, which is neutral for robust local-equilibrium determinacy.[18] Add to this term a suitably chosen out-of-equilibrium reaction, and the resulting rule implements, as the robustly unique local equilibrium, a path along which the interest rate is equal to that estimate.

In my framework, for instance, if the $\mathcal{CB}$'s observation set were $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}, \pi^{t-1}, y^{t-1}, i^{t-1}\}$ instead of $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$, then the $\mathcal{CB}$ could follow the rule

$$i_t = i_t^* + \phi\big(\pi_{t-1} - \pi_{t-1}^*\big), \tag{37}$$

with $1 < \phi < 1 + 2(1+\beta)\sigma/\kappa$, where

$$i_t^* \equiv (1-\rho_\eta L)^{-1}(1-\rho_u L)^{-1}(1-\mu L)^{-1}\mathbf{T}_i(L)\boldsymbol{\varepsilon}_{t-1},$$
$$\pi_t^* \equiv (1-\rho_u L)^{-1}(1-\mu L)^{-1}\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{T}_Z(L)\boldsymbol{\varepsilon}_t$$

denote the exogenous-shock-contingent values taken by $i_t$ and $\pi_t$ on the path $P$. This rule is consistent with the path $P$, in the sense that it is satisfied on this path. Moreover, it robustly ensures local-equilibrium determinacy, as I show in Appendix A.3. Therefore, this rule robustly implements the path $P$ as the unique local equilibrium. Thus, producing an (exact) exogenous estimate of $i_t^*$ (and of $\pi_{t-1}^*$), that is to say, inferring $i_t^*$ (and $\pi_{t-1}^*$) from the observed past exogenous shocks $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$ only, enables the $\mathcal{CB}$ to robustly implement $P$ as the unique local equilibrium.

As my analysis shows, however, things are different when the central bank can only produce *endogenous* estimates of these exogenous rates of interest, as seems more reasonable to assume, i.e., when it has to infer them from its observation of endogenous variables. The result that I obtain is that even when $i_t^*$ (and $\pi_{t-1}^*$) can be inferred in many alternative ways, on the optimal feasible path $P$, from the endogenous variables $\{\pi^{t-1}, y^{t-1}, i^{t-1}\}$ observed by the central bank, there may be no way of setting the interest rate $i_t$ as a function of these variables that implements this path as the robustly unique local equilibrium. In this case, any attempt to track the rate of interest $i_t^*$ and implement the optimal feasible path will inevitably result, in the presence of exogenous policy shocks of arbitrarily small variance, in either local-equilibrium multiplicity or nonexistence of a local equilibrium (depending on the values of the structural parameters). Central banks should, therefore, make a cautious use of the key unobserved rates of interest that the NK literature recommends that they track.

[17]In Galí's (2015, Chapter 9) words, "these new models identify tracking the natural equilibrium of the economy, which is not directly observable, as an important challenge for central banks." In Woodford's (2003, Chapter 4) words, "keeping track of its current value would be an important (and far from trivial) task of central-bank staff."

[18]Interest-rate rules involving such exogenous rates of interest can be found in, e.g., Woodford (2003, Chapters 4, 5, 7, and 8), and, more recently, Barsky et al. (2014), Cúrdia et al. (2015), and Galí (2015, Chapters 4, 5, and 8).

### 4.5 *Policy-transmission lags*

I have so far considered the *basic* NK model, and found that the optimal feasible path is non-implementable only when the stochastic process of the exogenous disturbances has a (sufficiently strong) moving-average component. In this subsection, I introduce monetary-policy-transmission lags into the basic NK model, and I show that the optimal feasible path can then be non-implementable even for a zero moving-average parameter.

More specifically, I consider Svensson and Woodford's (2005) model, which amounts to the basic NK model with one-period monetary-policy-transmission lags and two AR(1) exogenous disturbances with nonnegative autoregressive parameters. When these disturbances are interpreted as affecting the discount factor for one and the elasticity of substitution between differentiated goods for the other (as previously), the only changes to be brought to the setup described in Section 4.1 are that the IS equation (26) and the Phillips curve (27) should be replaced, respectively, by

$$y_t = \mathbb{E}_{t-1}\{y_{t+1}\} - \frac{1}{\sigma}\big(\mathbb{E}_{t-1}\{i_t\} - \mathbb{E}_{t-1}\{\pi_{t+1}\}\big) + \eta_t, \tag{38}$$

$$\pi_t = \beta \mathbb{E}_{t-1}\{\pi_{t+1}\} + \kappa \mathbb{E}_{t-1}\{y_t\} + u_t, \tag{39}$$

and that the parameters in (28) and (29) now satisfy $(\rho_\eta, \rho_u) \in [0, 1)^2$ and $\theta_\eta = \theta_u = 0$.[19] The structural equations (38) and (39) involve expectations formed at date $t-1$ because the $\mathcal{PS}$ makes its decisions one period in advance in this model. This feature generates one-period monetary-policy-transmission lags, in the sense that an unexpected announcement made by the $\mathcal{CB}$ at date $t$ can affect $(\pi_{t+1}, y_{t+1})$, but not $(\pi_t, y_t)$.

I consider the same observation set for the $\mathcal{CB}$ as previously, namely $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. Svensson and Woodford (2005) compute the timeless-perspective optimal feasible path when the $\mathcal{CB}$'s observation set is $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$ instead of $O_t$. Using their results, it is easy to get the minimal ARMA representation for this path,

$$(1-\rho_u L)(1-\mu L)\begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & (1-\rho_\eta L) \end{bmatrix}\begin{bmatrix} \mathbf{Z}_t \\ i_t \end{bmatrix} = \begin{bmatrix} \mathbf{T}_Z^{\mathrm{SW}}(L) \\ \mathbf{T}_i^{\mathrm{SW}}(L)L \end{bmatrix}\boldsymbol{\varepsilon}_t, \tag{40}$$

where $\mathbf{T}_Z^{\mathrm{SW}}(X) \in \mathbb{R}^{2\times 2}[X]$ and $\mathbf{T}_i^{\mathrm{SW}}(X) \in \mathbb{R}^{1\times 2}[X]$, with $\det[\mathbf{T}_Z^{\mathrm{SW}}(0)] \neq 0$.[20] This representation is the same as (30), except for the replacement of $\mathbf{T}_Z(X)$ and $\mathbf{T}_i(X)$ by $\mathbf{T}_Z^{\mathrm{SW}}(X)$ and $\mathbf{T}_i^{\mathrm{SW}}(X)$. So I can directly infer two results from my analysis in Sections 4.2 and 4.3. First, the path (40) is the timeless-perspective optimal feasible path not only under the observation set $\{\varepsilon^{\eta,t-1}, \varepsilon^{u,t-1}\}$, but also under the observation set $O_t$. Second, this path is implementable under $O_t$ if and only if the specific rule

$$D(L)^{-1}(1-\rho_\eta L)\det\big[\mathbf{T}_Z^{\mathrm{SW}}(L)\big]i_t = D(L)^{-1}\mathbf{T}_i^{\mathrm{SW}}(L)\,\mathrm{adj}\big[\mathbf{T}_Z^{\mathrm{SW}}(L)\big]\mathbf{Z}_{t-1} \tag{41}$$

[19] Svensson and Woodford (2005) allow the mean of $\eta_t$ to be nonzero; for simplicity and without any loss of generality, I set it to zero. They also write the IS equation and the Phillips curve in terms of the welfare-relevant output gap, rather than the output level; under my interpretation of the exogenous disturbances, these two variables coincide with each other.

[20] The superscript "SW" stands for Svensson–Woodford. The computations, which are straightforward and of no particular interest, are available upon request.

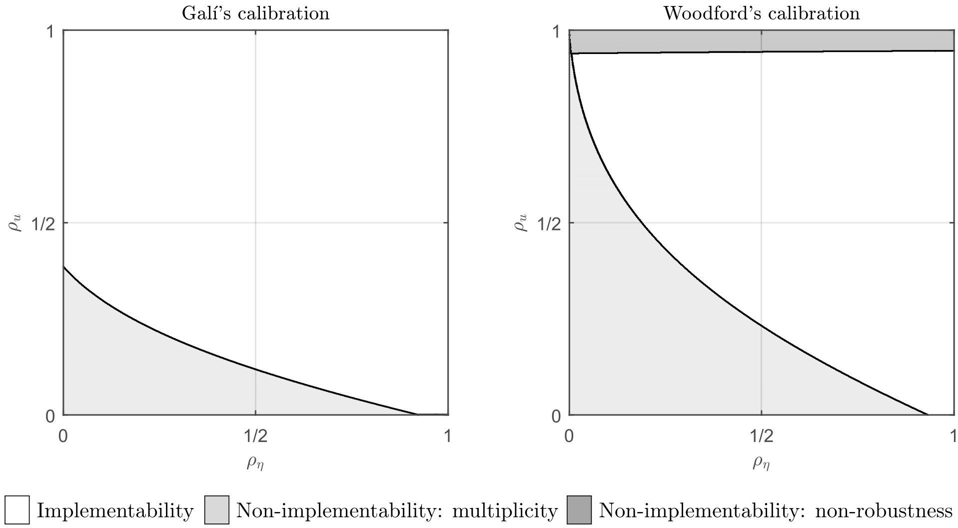


FIGURE 2. Implementability of the optimal feasible path in Svensson and Woodford's (2005) model when the $\mathcal{CB}$ observes only past endogenous variables.

robustly ensures local-equilibrium determinacy. This rule is the same as (32) except for: (i) the replacement of $\mathbf{T}_Z(X)$ and $\mathbf{T}_i(X)$ by $\mathbf{T}_Z^{\mathrm{SW}}(X)$ and $\mathbf{T}_i^{\mathrm{SW}}(X)$, and (ii) the division of the left- and right-hand sides by $D(L)$, where $D(X) \equiv (1-\rho_u X)(1-\mu X)$ is the greatest common divisor, defined up to a nonzero real-number multiplicative factor, of $(1-\rho_\eta X)\det[\mathbf{T}_Z^{\mathrm{SW}}(X)]$ and $\mathbf{T}_i^{\mathrm{SW}}(X)\,\mathrm{adj}[\mathbf{T}_Z^{\mathrm{SW}}(X)]$. So the optimal feasible path (40) is implementable if and only if the system made of the structural equations (38)–(39) and the specific rule (41) satisfies Blanchard and Kahn's (1980) conditions. Under the same two alternative calibrations as previously, I then obtain that this path is not implementable for many values of $\rho_\eta$ and $\rho_u$, as shown in Figure 2.

For some values of $\rho_\eta$ and $\rho_u$ (light-gray areas in Figure 2), and, in particular, for $\rho_\eta = \rho_u = 0$ (i.i.d. disturbances), the path (40) is not implementable because all the interest-rate rules consistent with $O_t$ and this path lead to local-equilibrium multiplicity. For some other values of $\rho_\eta$ and $\rho_u$ (dark-gray area in Figure 2), it is not implementable because adding an exogenous monetary-policy shock (even of arbitrarily small variance) to any interest-rate rule consistent with $O_t$ and this path leads to nonexistence of a local equilibrium. This result, thus, shows that moving-average disturbances are not necessary for feasible-path non-implementability of either kind.

Moreover, as in Section 4.3, I do not find any apparent link between the (non-)implementability of the optimal feasible path and the (non-)invertibility of the vector autoregressive moving-average (VARMA) process relating $\mathbf{Z}_t$ to $\boldsymbol{\varepsilon}_t$ on this path. More specifically, I find here that any of the following four possibilities can arise, depending on the calibration (Galí's or Woodford's) and the value of $(\rho_\eta, \rho_u)$: NI-NI, NI-I, I-NI, and I-I, where (N)I-(N)I denotes the (non-)implementability of the path (40) and the (non-)invertibility of the first two lines of (40). This result shows, again, that non-invertibility of a feasible path is neither necessary nor sufficient for non-implementability of this path. If the path (40) is not invertible, i.e., if $\det[\mathbf{T}_Z^{\mathrm{SW}}(X)]$ has at least one root inside the unit circle, then the rule (41) is superinertial, and so are all the other rules consistent with the observation set $O_t$ and the path (40); but again, as already discussed in Section 3.2 and illustrated in Section 4.3, superinertial rules may very well ensure robust local-equilibrium determinacy.

## 5. Debt-stabilizing tax policy in the RBC model

In the previous section, I illustrated how feasible-path *non-implementability* can be an issue in a textbook model (the NK model), for a standard policy instrument (the interest rate), a relevant observation set (made of past endogenous variables), and an interesting feasible path (the optimal feasible path). In the current section, I illustrate this time how feasible-path *implementability* can obtain, against conventional wisdom, in a textbook model (the RBC model), for a standard policy instrument (the labor-income tax or income-tax rate), a relevant observation set (made of endogenous variables), and interesting feasible paths (constant-debt feasible paths). I also address the issue of feasible-path *implementation* in this context.

Schmitt-Grohé and Uribe (1997) consider, in the RBC model, a labor-income tax-rate or income tax-rate rule that stabilizes the stock of public debt both in and out of equilibrium.[21] They find that this rule leads to local-equilibrium multiplicity for many empirically relevant values of the structural parameters. Their finding has largely been interpreted as an argument against the use of labor-income or income taxes to stabilize public debt. However, the fact that *this* (labor-)income tax-rate rule fails to ensure local-equilibrium determinacy does not imply that *all* the (labor-)income tax-rate rules that stabilize public debt in equilibrium fail to ensure local-equilibrium determinacy.

In this section, I challenge the interpretation commonly made of their finding by showing that, in the same model, for the same alternative tax instruments and for a reasonable observation set of the tax authority, all constant-debt feasible paths are implementable for all structural-parameter values.[22] I show the implementability of each of these feasible paths by designing arithmetically a (labor-)income tax-rate rule that is consistent with the tax authority's observation set and implements this path as the robustly unique local equilibrium.

### 5.1 *Structural equations and observation set*

In Schmitt-Grohé and Uribe's (1997) model, at each date $t \in \mathbb{Z}$, the private sector ($\mathcal{PS}$) sets output $y_t$, the capital stock $k_t$, investment $x_t$, hours worked $h_t$, consumption $c_t$, the (after-tax) rental price of capital $r_t$, the (after-tax) wage $w_t$, and the stock of public

[21] Most of Schmitt-Grohé and Uribe's (1997) analysis is conducted in continuous time. I refer here to the discrete-time analysis conducted in Section 4 and in the Appendix of their paper.

[22] Since Schmitt-Grohé and Uribe (1997), several papers have shown that policy instruments other than labor-income or income taxes (e.g., government purchases in Guo and Harrison 2004) can be used in the same model to stabilize public debt without generating local-equilibrium multiplicity. My point is that this can be done even with labor-income or income taxes.

debt $b_t$, according to the structural equations, log-linearized in the neighborhood of the zero-debt steady state,

$$y_t = a_t + (1-\alpha)k_t + \alpha h_t \tag{42}$$

$$k_t = (1-\delta)k_{t-1} + \delta x_{t-1} \tag{43}$$

$$y_t = s_g g_t + s_x x_t + (1 - s_g - s_x)c_t \tag{44}$$

$$w_t = \sigma c_t + \chi h_t \tag{45}$$

$$c_t = \mathbb{E}_t\{c_{t+1}\} - \big[1 - \beta(1-\delta)\big]\sigma^{-1}\mathbb{E}_t\{r_{t+1}\} \tag{46}$$

$$r_t = a_t - \alpha(k_t - h_t) - \omega\tau(1-\tau)^{-1}\tau_t \tag{47}$$

$$w_t = a_t + (1-\alpha)(k_t - h_t) - \tau(1-\tau)^{-1}\tau_t \tag{48}$$

$$b_t = \beta^{-1}b_{t-1} + s_g g_t - \big[\alpha + (1-\alpha)\omega\big]\tau(y_t + \tau_t), \tag{49}$$

where $\tau_t$ denotes the labor-income tax rate (when $\omega = 0$) or the income tax rate (when $\omega = 1$) set by the tax authority ($\mathcal{TA}$) at date $t$.[23] The structural parameters satisfy $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \delta < 1$, $\chi > 0$, $\sigma > 0$, $0 < s_g < 1$, $0 < s_x < 1$, $0 < s_g + s_x < 1$, $0 < \tau < 1$, and $\omega \in \{0, 1\}$. To lighten the exposition, I assume that the exogenous productivity and government-purchases disturbances $a_t$ and $g_t$ are not only orthogonal and of mean zero, but also i.i.d.[24]

The observation set that I consider for the $\mathcal{TA}$ is $O_t \equiv \{y^t, x^t, h^t, c^t, u^t, w^t, b^t, \tau^{t-1}\}$. I thus assume that the $\mathcal{TA}$ observes all endogenous variables, except the capital stock, and no exogenous disturbances.[25]

### 5.2 *Constant-debt feasible paths*

To characterize the set of constant-debt feasible paths, I replace $b_t$ and $b_{t-1}$ in (49) by 0 and get

$$\tau_t = -y_t + \frac{s_g}{\big[\alpha + (1-\alpha)\omega\big]\tau} g_t, \tag{50}$$

which corresponds to the balanced-budget tax-rate rule considered by Schmitt-Grohé and Uribe (1997). I can rewrite (42)–(45), (47)–(48), and (50) as

$$\begin{bmatrix} y_t & x_t & h_t & c_t & r_t & w_t & \tau_t \end{bmatrix}^T = \mathbf{A}\begin{bmatrix} \kappa_t & \kappa_{t-1} & a_t & g_t \end{bmatrix}^T, \tag{51}$$

where $\kappa_t \equiv k_{t+1}$ is determined at $t$ and $\mathbf{A} \in \mathbb{R}^{7\times 4}$ (the closed-form expression of $\mathbf{A}$ is available upon request). Using (51), I can in turn rewrite (46) as

$$\mathbb{E}_t\big\{S(L)\kappa_{t+1}\big\} = A_{43}a_t + A_{44}g_t, \tag{52}$$

[23] All variables are expressed in percentage deviation from their steady-state value except public debt $b_t$, which is expressed as a fraction of steady-state output (since steady-state public debt is zero).

[24] The analysis can be extended to stationary and invertible ARMA processes of arbitrary orders for these two exogenous disturbances. Schmitt-Grohé and Uribe (1997) assume that they follow AR(1) processes.

[25] The results would be identical if the $\mathcal{TA}$ were assumed to observe the capital stock and/or one or the two exogenous disturbances.

where

$$S(X) \equiv \left[A_{41} - \frac{1-\beta(1-\delta)}{\sigma} A_{51}\right] + \left[A_{42} - A_{41} - \frac{1-\beta(1-\delta)}{\sigma} A_{52}\right] X - A_{42} X^2$$

and, for any matrix $\mathbf{M}$, $M_{ij}$ denotes its row-$i$ column-$j$ element. I focus on the case in which the two roots of $S(X)$ lie outside the unit circle, implying that (52) has multiple stationary solutions for $\kappa_t$, as it is the case in which Schmitt-Grohé and Uribe's (1997) balanced-budget tax-rate rule (50) leads to indeterminacy. In this case, the set of stationary constant-debt paths that do not involve sunspot shocks is characterized block-recursively by (51) and the stationary ARMA(2, 1) process for $\kappa_t$,

$$S(L)\kappa_t = \psi_a a_t + A_{43} a_{t-1} + \psi_g g_t + A_{44} g_{t-1}, \tag{53}$$

which is parametrized by $(\psi_a, \psi_g) \in \mathbb{R}^2$.[26] On any of these paths, (44) and (50) are satisfied and together imply the relationship

$$\tau_t = -y_t + \frac{y_t - s_x x_t - (1 - s_g - s_x) c_t}{\left[\alpha + (1-\alpha)\omega\right]\tau},$$

which expresses $\tau_t$ only as a function of elements of $O_t$. As a consequence, any of these paths is feasible, and the set of constant-debt *feasible* paths is also characterized by (51) and (53). Note finally, for later use, that (51) and (53) imply the stationary ARMA(2, 2) process for $\tau_t$,

$$S(L)\tau_t = T_a(L) a_t + T_g(L) g_t, \tag{54}$$

where $T_a(X) \equiv (A_{71} + A_{72}X)(\psi_a + A_{43}X) + A_{73}S(X)$ and $T_g(X) \equiv (A_{71} + A_{72}X)(\psi_g + A_{44}X) + A_{74}S(X)$.

### 5.3 *Implementability and implementation of these paths*

Consider an arbitrary constant-debt feasible path, characterized by (51) and (53) for some $(\psi_a, \psi_g) \in \mathbb{R}^2$. This path, denoted by $P$, is implementable if and only if there exists a tax-rate rule consistent with $O_t$ and such that $P$ is the robustly unique stationary solution of the system made of the structural equations (42)–(49) and that rule. In this subsection, I show that such a rule exists, and I design it arithmetically, building notably on the analysis in Section 3.4.

I proceed in four steps. In the first step, I point out that under any tax-rate rule that does not involve the debt level, $(y_t, x_t, h_t, c_t, r_t, w_t, k_t, \tau_t)$ is (uniquely or not uniquely) determined by the system made of (42)–(48) and that rule, while $b_t$ is residually determined by (49). Now, in the presence of a tax-policy shock (even of arbitrarily small variance), (49) generates explosive dynamics for $b_t$, since $\beta^{-1} > 1$. Therefore, any tax-rate rule that does not involve the debt level fails to *robustly* ensure local-equilibrium determinacy. In essence, the system made of the structural equations and any such rule

[26] I exclude constant-debt paths involving sunspot shocks from the analysis because the tax authority is unlikely to be interested in implementing such paths.

fails to meet Blanchard and Kahn's (1980) *no-decoupling* condition: $\beta^{-1}$ is an "unstable eigenvalue" of this system, and the associated eigenvector is the predetermined variable $b_t$.[27] Thus, the rule that I am looking for necessarily involves the debt level.

In the second step, I rewrite the system of structural equations (42)–(49) in a block-recursive way. More specifically, I rewrite (42)–(45) and (47)–(48) as

$$\begin{bmatrix} y_t & x_t & h_t & c_t & r_t & w_t \end{bmatrix}^T = \mathbf{B} \begin{bmatrix} \kappa_t & \kappa_{t-1} & \tau_t & a_t & g_t \end{bmatrix}^T, \tag{55}$$

where $\mathbf{B} \in \mathbb{R}^{6\times 5}$ (the closed-form expression of $\mathbf{B}$ is available upon request). In turn, using (55), I rewrite (46) and (49) as

$$\mathbb{E}_t\{P_k(L)\kappa_{t+1} + P_\tau(L)\tau_{t+1}\} + P_a a_t + P_g g_t = 0 \tag{56}$$

$$Q_b(L)b_t + Q_k(L)\kappa_t + Q_\tau \tau_t + Q_a a_t + Q_g g_t = 0, \tag{57}$$

where $(P_k(X), P_\tau(X), Q_b(X), Q_k(X)) \in \mathbb{R}[X]^4$ and $(P_a, P_g, Q_\tau, Q_a, Q_g) \in \mathbb{R}^5$ (the closed-form expression of these polynomials and parameters is available upon request).

The third step directly draws on the analysis in Section 3.4. In this step, I design a rule that implements $P$ as the robustly unique local equilibrium, but is not consistent with $O_t$. Consider the class of rules of type

$$S(L)\big[\mathcal{R}_\tau(L)\tau_t - \mathcal{R}_b(L)b_t\big] - \mathcal{R}_a(L)a_t - \mathcal{R}_g(L)g_t = 0, \tag{58}$$

with $(\mathcal{R}_b(X), \mathcal{R}_\tau(X), \mathcal{R}_a(X), \mathcal{R}_g(X)) \in \mathbb{R}[X]^4$, $\mathcal{R}_\tau(0) \neq 0$, and $\mathcal{R}_b(X) \neq 0$. Since the coefficient $S(0)\mathcal{R}_\tau(0)$ of $\tau_t$ in (58) is nonzero, (58) expresses $\tau_t$ only as a function of elements of $O_t$, i.e., it is a tax-rate rule consistent with $O_t$. Moreover, since the two roots of $S(X)$ lie outside the unit circle, (58) is equivalent to

$$\mathcal{R}_\tau(L)\tau_t = \mathcal{R}_b(L)b_t + S(L)^{-1}\big[\mathcal{R}_a(L)a_t + \mathcal{R}_g(L)g_t\big]. \tag{59}$$

Under any rule of type (58), $(b_t, \kappa_t, \tau_t)$ is (uniquely or not uniquely) determined by the system made of (56)–(57) and (59), while $(y_t, x_t, h_t, c_t, r_t, w_t)$ is residually determined by (55).

The system made of (56)–(57) and (59) can easily be written in Blanchard and Kahn's (1980) form with exactly one non-predetermined variable, corresponding to the term $\mathbb{E}_t\{P_k(0)\kappa_{t+1} + P_\tau(0)\tau_{t+1}\}$ in (56). For this system to satisfy Blanchard and Kahn's (1980) root-counting condition, thus, we need its characteristic polynomial to have exactly one root outside the unit circle. The characteristic polynomial of this system is the same, up to a multiplicative factor of type $X^p$ with $p \in \mathbb{N}$, as the characteristic polynomial $\mathcal{C}(X)$ of the corresponding perfect-foresight deterministic system, which is made of the equations $P_k(L)\kappa_t + P_\tau(L)\tau_t = 0$, $Q_b(L)b_t + Q_k(L)\kappa_t + Q_\tau\tau_t = 0$, and $\mathcal{R}_\tau(L)\tau_t = \mathcal{R}_b(L)b_t$. The *reciprocal* polynomial of $\mathcal{C}(X)$ is, straightforwardly,

$$\mathcal{R}(X) \equiv Z_\tau(X)\mathcal{R}_\tau(X) + Z_b(X)\mathcal{R}_b(X),$$

[27] In particular, Schmitt-Grohé and Uribe's (1997) balanced-budget tax-rate rule (50) does not robustly ensure local-equilibrium determinacy, as I discuss in Section 5.4.

where $Z_\tau(X) \equiv P_k(X)Q_b(X)$ and $Z_b(X) \equiv P_k(X)Q_\tau - P_\tau(X)Q_k(X)$. Blanchard and Kahn's (1980) root-counting condition is satisfied, therefore, if $\mathcal{R}(X)$ has exactly one root *inside* the unit circle and this root is nonzero.

Except for a zero-measure set of structural-parameter values, $Z_\tau(X)$ is of degree 3, $Z_b(X)$ is of degree 2, and they have no common roots, so that their Sylvester matrix is invertible. Therefore, for an arbitrary real number $\phi \in (-1, 1) \setminus \{0\}$, I can arithmetically design $\mathcal{R}^*_\tau(X) \in \mathbb{R}[X]$ of degree at most 1 and $\mathcal{R}^*_b(X) \in \mathbb{R}[X]$ of degree at most 2 such that

$$\mathcal{R}^*(X) \equiv Z_\tau(X)\mathcal{R}^*_\tau(X) + Z_b(X)\mathcal{R}^*_b(X) = X - \phi,$$

following exactly the same procedure as in Section 3.4 with $(Z_\tau(X), Z_b(X), X - \phi, \mathcal{R}^*_\tau(X), \mathcal{R}^*_b(X))$ playing the role of $(A(X), X^\delta B(X), \Phi(X), \mathcal{R}^*_i(X), \mathcal{R}^*_z(X))$. The polynomials $\mathcal{R}^*_\tau(X)$ and $\mathcal{R}^*_b(X)$ thus designed are admissible choices for $\mathcal{R}_\tau(X)$ and $\mathcal{R}_b(X)$ in (58), because they are generically such that $\mathcal{R}^*_\tau(0) \neq 0$ and $\mathcal{R}^*_b(X) \neq 0$ (where "generically," in this subsection, means "except possibly for a zero-measure set of values for the structural parameters and the parameter $\phi$").

So consider the rule (58) with $(\mathcal{R}_\tau(X), \mathcal{R}_b(X)) = (\mathcal{R}^*_\tau(X), \mathcal{R}^*_b(X))$ for some arbitrary $\mathcal{R}_a(X)$ and $\mathcal{R}_g(X)$. As mentioned above, the property $\mathcal{R}^*(X) = X - \phi$, together with $\phi \in (-1, 1) \setminus \{0\}$, implies that the system made of the structural equations and this rule satisfies Blanchard and Kahn's (1980) *root-counting* condition. In addition, this system also generically satisfies Blanchard and Kahn's (1980) *no-decoupling* condition, as the rule generically has two properties that preclude the two variants of decoupling discussed in Section 3.4: (i) $\mathcal{R}^*_b(X) \neq 0$, i.e., the rule involves the debt level (as explained in the first step of my current analysis), and (ii) $\mathcal{R}^*_\tau(X)$ and $\mathcal{R}^*_b(X)$ have no common roots inside the unit circle.

Since they make the system satisfy both the root-counting and the no-decoupling conditions of Blanchard and Kahn (1980), all the rules of type (58) with $(\mathcal{R}_\tau(X), \mathcal{R}_b(X)) = (\mathcal{R}^*_\tau(X), \mathcal{R}^*_b(X))$ robustly ensure local-equilibrium determinacy. Among them, in particular, the rule

$$S(L)\big[\mathcal{R}^*_\tau(L)\tau_t - \mathcal{R}^*_b(L)b_t\big] - \mathcal{R}^*_a(L)a_t - \mathcal{R}^*_g(L)g_t = 0, \tag{60}$$

where $\mathcal{R}^*_a(X) \equiv \mathcal{R}^*_\tau(X)T_a(X)$ and $\mathcal{R}^*_g(X) \equiv \mathcal{R}^*_\tau(X)T_g(X)$, is satisfied on the path $P$, since it is satisfied when both $b_t = 0$ and (54) hold. This rule, therefore, implements $P$ as the robustly unique local equilibrium.

In the fourth and last step, I transform the rule (60) designed in the previous step into a rule consistent with $O_t$ in a way that is neutral for robust local-equilibrium determinacy. Note that (42) and (43) together imply

$$\big[1 - (1-\delta)L\big]a_t = \big[1 - (1-\delta)L\big](y_t - \alpha h_t) - (1-\alpha)\delta x_{t-1}. \tag{61}$$

Multiplying the left- and right-hand sides of (60) by $1-(1-\delta)L$, using (61) to replace $[1-(1-\delta)L]a_t$, and using (44) to replace $g_t$, I get

$$\begin{aligned}&\big[1-(1-\delta)L\big]S(L)\big[\mathcal{R}_b^*(L)b_t+\mathcal{R}_\tau^*(L)\tau_t\big]\\&\quad+\mathcal{R}_a^*(L)\big\{\big[1-(1-\delta)L\big](y_t-\alpha h_t)-(1-\alpha)\delta x_{t-1}\big\}\\&\quad+s_g^{-1}\big[1-(1-\delta)L\big]\mathcal{R}_g^*(L)\big[y_t-s_x x_t-(1-s_g-s_x)c_t\big]=0.\end{aligned}\tag{62}$$

The rule (62) expresses $\tau_t$ only as a function of elements of $O_t$. Moreover, because the only equations used to transform (60) into (62) are the structural equations (42)–(44), the system made of all structural equations and the rule (60) is equivalent to the system made of all structural equations and the rule (62). Since $P$ is the unique stationary solution of the former system, it is therefore also the unique stationary solution of the latter system. Finally, because the root of the polynomial $1-(1-\delta)X$ lies outside the unit circle, (62) still ensures local-equilibrium determinacy when one adds an exogenous tax-policy shock to it, like (60). So, to sum up, (62) is a tax-rate rule consistent with $O_t$ and implementing $P$ as the robustly unique local equilibrium. As a consequence, $P$ is implementable, and the rule (62) that I have arithmetically designed implements it.

### 5.4 *Policy implications*

I have just shown that, in Schmitt-Grohé and Uribe's (1997) model, for any of their two alternative tax instruments and under the observation set that I consider, all constant-debt feasible paths are implementable for all structural-parameter values. Thus, a tax authority *can always* stabilize public debt *in equilibrium* without generating local-equilibrium multiplicity. This result differs from Schmitt-Grohé and Uribe's (1997) result, which is that a tax authority *cannot always* stabilize public debt *both in and out of equilibrium* without generating local-equilibrium multiplicity. Let me elaborate on the difference.

If stabilizing debt is the *only* objective of the tax authority, then it seems natural to think that it will pursue this objective under whatever circumstances might arise, i.e., both in and out of equilibrium. In this case, it will follow the rule (50), as in Schmitt-Grohé and Uribe (1997). Under this rule, for some structural-parameter values, there are multiple local equilibria. These equilibria include all the constant-debt feasible paths characterized in Section 5.2, as well as sunspot-driven paths. They differ from one another in the values of the endogenous variables other than debt.

Alternatively, however, the tax authority's objective could be to stabilize debt *conditionally* on some endogenous variables (other than debt and the tax instrument) taking the same values as on a given targeted constant-debt feasible path. Were these variables to deviate from that path, the tax authority would no longer be required to stabilize debt. Under this conditional objective, the tax authority could follow the rule (62) that I have designed. Under this rule, there is a unique local equilibrium. In this equilibrium, which coincides with the targeted constant-debt feasible path, debt is stabilized and the other endogenous variables are on target.

Requiring that debt be stabilized out of equilibrium, as under the rule (50), may prevent the tax authority from putting the economy on an explosive path following a deviation from the targeted feasible path, simply because all explosive paths may involve an explosive debt. In this case, it is only by threatening to put the economy on an explosive-debt path, as under the rule (62), that the tax authority can implement the targeted constant-debt feasible path as the robustly unique local equilibrium.

Despite making debt explode out of equilibrium, the tax policy described by the rule (62) is "locally Ricardian" in the sense of Woodford (2003, Chapter 4), because it makes debt explode *only if* the other endogenous variables explode as well. Debt cannot explode alone because the rule (62) links the tax rate to the debt level, so as to make the system satisfy Blanchard and Kahn's (1980) no-decoupling condition, as discussed in Section 5.3. Adding a tax-policy shock to the rule (62) does not fundamentally change this feedback loop from the debt level to the tax rate, so this tax policy is still locally Ricardian in the presence of tax-policy shocks.

By contrast, the tax policy described by the rule (50), i.e., the balanced-budget policy studied in Schmitt-Grohé and Uribe (1997), is locally non-Ricardian in the presence of tax-policy shocks. Indeed, adding a shock $e_t$ to the rule (50) and using the resulting rule to replace $\tau_t$ in (49) leads to $b_t = \beta^{-1} b_{t-1} - [\alpha + (1-\alpha)\omega]\tau e_t$. As a result, the debt level $b_t$ explodes over time independently of the dynamics of the other endogenous variables. The basic reason is that the rule (50) does not create a feedback loop from the debt level to the tax rate, and, therefore, the system does not satisfy Blanchard and Kahn's (1980) no-decoupling condition.[28]

## 6. Concluding remarks

This paper has sought to dig deeper into the modeling of stabilization policy. The starting point for stabilization-policy modeling should be to specify the policymaker's observation set—a key feature of the environment that should be explicitly stated alongside other features such as preferences, technologies, and markets. Once this observation set is specified, two concepts naturally emerge: those of feasible paths and implementable paths. The goal of this paper has been to show, through two case studies, that feasible-path (non-)implementability can be an issue in textbook models for standard policy instruments, relevant observation sets, and interesting feasible paths (with important policy implications), and to develop and illustrate an arithmetic method of designing, for a given implementable path, a policy-instrument rule consistent with the observation set and implementing this path as the robustly unique local equilibrium.

[28]The system is made of two independent subsystems: the equation $b_t = \beta^{-1} b_{t-1} - [\alpha + (1-\alpha)\omega]\tau e_t$ determines the debt level, while the equations (42)–(48) and (50) (with the shock $e_t$ in the last equation) separately determine the other endogenous variables. The first subsystem has one excess unstable eigenvalue, while the second one may have one excess stable eigenvalue (as shown by Schmitt-Grohé and Uribe 1997). Thus, the whole system does not satisfy Blanchard and Kahn's (1980) no-decoupling condition, but it may satisfy their root-counting condition, which can be seen as another concrete economic example in support of Sims' (2007) claim discussed in Section 4.3.

As in most of the literature on stabilization policy, I have focused throughout the paper on *local*-equilibrium determinacy, that is to say that I have abstracted from the possible existence of *nonlocal* equilibria. In the context of interest-rate rules, as argued by Cochrane (2011), there is usually no solid economic reason to assume away the existence of nonlocal equilibria. The most common policy proposal to eliminate them, made initially by Christiano and Rostagno (2001) and Benhabib et al. (2002), discussed by Woodford (2003, Chapter 2), and used notably by Atkeson et al. (2010), consists in switching from an interest-rate rule ensuring local-equilibrium determinacy to a money-growth rule (possibly accompanied by a non-Ricardian fiscal policy) when the economy goes *outside* a specified neighborhood of the steady state considered. The interest-rate rules that I design (when the policy instrument is the interest rate) fit naturally into this proposal, insofar as they are followed *inside* the specified neighborhood.

# Appendix

## A.1 *Coprimeness of* $\mathcal{R}_i^*(X)$ *and* $\mathcal{R}_z^*(X)$

In this appendix, I show that the polynomials $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ designed in Section 3.4 have no common roots, except possibly for a zero-measure set of polynomials $\Phi(X)$.

If $d_A = 0$, then $\mathcal{R}_z^*(X) = 1$. Therefore, $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ have no roots in common, since $\mathcal{R}_z^*(X)$ has no roots at all.

Alternatively, if $d_A \geq 1$, then suppose that $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ have at least one common root. This root, denoted by $\gamma$, may be real or complex (nonreal). Consider first the case in which $\gamma$ is a real number. Since $A(X)\mathcal{R}_i^*(X) + X^\delta B(X)\mathcal{R}_z^*(X) = \Phi(X)$, $\gamma$ is also a root of $\Phi(X)$. Therefore, we have $A(X)\widetilde{\mathcal{R}}_i^*(X) + X^\delta B(X)\widetilde{\mathcal{R}}_z^*(X) = \widetilde{\Phi}(X)$, where $(\widetilde{\mathcal{R}}_i^*(X), \widetilde{\mathcal{R}}_z^*(X), \widetilde{\Phi}(X)) \equiv (\mathcal{R}_i^*(X)/(X-\gamma), \mathcal{R}_z^*(X)/(X-\gamma), \Phi(X)/(X-\gamma)) \in \mathbb{R}[X]^3$. The latter equation can be rewritten as

$$\mathbf{v}_r = \mathbf{S}^{-1}\mathbf{v}_\phi \quad \text{with} \quad \begin{cases} \mathbf{v}_r \equiv \begin{bmatrix} 0 & \widetilde{r}^{z*}_{d_A-2} & \cdots & \widetilde{r}^{z*}_0 & 0 & \widetilde{r}^{i*}_{d_B+\delta-2} & \cdots & \widetilde{r}^{i*}_0 \end{bmatrix}^T, \\ \mathbf{v}_\phi \equiv \begin{bmatrix} 0 & \widetilde{\phi}_{d_A+d_B+\delta-2} & \cdots & \widetilde{\phi}_0 \end{bmatrix}^T, \end{cases}$$

where $(\widetilde{r}^{z*}_k)_{0\leq k\leq d_A-2}$, $(\widetilde{r}^{i*}_k)_{0\leq k\leq d_B+\delta-2}$, and $(\widetilde{\phi}_k)_{0\leq k\leq d_A+d_B+\delta-2}$ denote the coefficients of $\widetilde{\mathcal{R}}_z^*(X)$, $\widetilde{\mathcal{R}}_i^*(X)$, and $\widetilde{\Phi}(X)$, respectively.

Let $E \equiv \mathbb{R}^{d_A+d_B+\delta}$ and, for any $k \in \{1, \ldots, d_A + d_B + \delta\}$, let $\mathbf{e}_k$ denote the vector of $E$ whose $k$th element is 1 and whose other elements are 0. The vector $\mathbf{v}_\phi$ belongs to the hyperplane $H_\phi \equiv \{\mathbf{v} \in E | \mathbf{e}_1^T\mathbf{v} = 0\}$. The image of $H_\phi$ by $\mathbf{S}^{-1}$ is the hyperplane $H_r \equiv \{\mathbf{v} \in E | b_{d_B}\mathbf{e}_1^T\mathbf{v} + a_{d_A}\mathbf{e}_{d_A+1}^T\mathbf{v} = 0\}$. Given the constraints imposed on $\Phi(X)$, the set of admissible values for $\mathbf{v}_\phi$ is an open (nonempty) subset of $H_\phi$. Therefore, the set of admissible values for $\mathbf{S}^{-1}\mathbf{v}_\phi$ is an open (nonempty) subset of $H_r$, denoted by $S_r$. However, $\mathbf{v}_r = \mathbf{S}^{-1}\mathbf{v}_\phi$ belongs to the subspace $\{\mathbf{v} \in E | \mathbf{e}_1^T\mathbf{v} = \mathbf{e}_{d_A+1}^T\mathbf{v} = 0\}$, which is one dimension smaller than $H_r$ and, hence, of measure zero relative to $H_r$, and, therefore, of measure zero relative to $S_r$. As a consequence, the set of polynomials $\Phi(X)$ such that $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$ have at least one common real root is of measure zero.

Now turn to the alternative case in which the common root $\gamma$ is a complex (nonreal) number. In this case, its conjugate $\overline{\gamma}$ is also a common root of $\mathcal{R}_i^*(X)$ and $\mathcal{R}_z^*(X)$, and both $\gamma$ and $\overline{\gamma}$ are also roots of $\Phi(X)$. Therefore, I can divide $\mathcal{R}_i^*(X)$, $\mathcal{R}_z^*(X)$, and $\Phi(X)$ by $(X-\gamma)(X-\overline{\gamma})$ and proceed in a way similar to above to show that the set of polynomials $\Phi(X)$ corresponding to this case is of measure zero.

### A.2 *Observation of current variables in the basic NK model*

To understand why no optimal feasible path would exist in Section 4 if the $\mathcal{CB}$ also observed current endogenous variables, suppose that the $\mathcal{CB}$'s observation set is $\overline{O}_t \equiv \{\pi^t, y^t, i^{t-1}\}$ instead of $O_t \equiv \{\pi^{t-1}, y^{t-1}, i^{t-1}\}$. For the sake of the argument, assume for simplicity that the exogenous disturbance $\eta_t$ is i.i.d. (i.e., $\rho_\eta = \theta_\eta = 0$ and $\eta_t = \varepsilon_t^\eta$), and that there is no disturbance $u_t$ (i.e., $u_t = 0$). Consider first the optimal path, i.e., the path that minimizes $L_t$ subject only to the structural equations (26) and (27), in the absence of any observation-set constraint. This path, denoted by $P^\star$, is trivially $[\pi_t \quad y_t \quad i_t] = [0 \quad 0 \quad \sigma\varepsilon_t^\eta]$: on this path, $i_t$ reacts to $\varepsilon_t^\eta$ so as to insulate $\pi_t$ and $y_t$ from $\varepsilon_t^\eta$ and get $L_t = 0$. Thus, on this path, $i_t$ depends on $\varepsilon_t^\eta$, but no element of $\overline{O}_t$ does, so $i_t$ cannot be expressed as a function of only elements of $\overline{O}_t$. Therefore, the path $P^\star$ is not feasible.

Now turn to the path $[\pi_t \quad y_t \quad i_t] = [\epsilon\kappa\varepsilon_t^\eta \quad \epsilon\varepsilon_t^\eta \quad (1-\epsilon)\sigma\varepsilon_t^\eta]$, where $\epsilon \in \mathbb{R} \setminus \{0\}$. This path, denoted by $P_\epsilon$, is consistent with the structural equations (26) and (27) (since it implies them), and also with the observation set $\overline{O}_t$ (since it implies $i_t = (1-\epsilon)\sigma\pi_t/(\kappa\epsilon)$, with $\pi_t \in \overline{O}_t$). Therefore, $P_\epsilon$ is feasible. As $\epsilon$ shrinks to zero, the feasible path $P_\epsilon$ converges to the non-feasible path $P^\star$ and the value taken by $L_t$ on $P_\epsilon$ goes to zero. Thus, the set of feasible paths is not closed, and the optimal path $P^\star$ lies at the boundary of this set. Therefore, there is no optimal feasible path under the observation set $\overline{O}_t$.

### A.3 *Taylor rule with lagged inflation in the basic NK model*

In this appendix, I show that the rule

$$i_t = i_t^* + \phi\left(\pi_{t-1} - \pi_{t-1}^*\right) \tag{37}$$

robustly ensures local-equilibrium determinacy in the basic NK model if and only if

$$1 < \phi < 1 + \frac{2(1+\beta)\sigma}{\kappa}. \tag{63}$$

To do so, I use the Phillips curve (27) to replace $y_t$ and $y_{t+1}$ in the IS equation (26), I use the rule (37) to replace $i_t$, and I get the dynamic equation in inflation,

$$\mathbb{E}_t\left\{\pi_{t+2} - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma}\right)\pi_{t+1} + \left(\frac{1}{\beta}\right)\pi_t + \left(\frac{\kappa\phi}{\beta\sigma}\right)\pi_{t-1}\right\} = 0,$$

where I have ignored all exogenous terms to lighten the exposition, as they do not matter for the analysis. This dynamic equation has two non-predetermined variables ($\mathbb{E}_t\{\pi_{t+2}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$), and its characteristic polynomial is

$$\mathcal{C}(X) \equiv X^3 - \left(1 + \frac{1}{\beta} + \frac{\kappa}{\beta\sigma}\right)X^2 + \left(\frac{1}{\beta}\right)X + \left(\frac{\kappa\phi}{\beta\sigma}\right).$$

Therefore, the rule (37) robustly ensures local-equilibrium determinacy if and only if $\mathcal{C}(X)$ has two roots outside the unit circle and one root inside. One *necessary* condition for that is that $\mathcal{C}(-1)$ and $\mathcal{C}(1)$ should be of opposite signs. Since

$$\mathcal{C}(-1)=\frac{\kappa\phi}{\beta\sigma}-\left(2+\frac{2}{\beta}+\frac{\kappa}{\beta\sigma}\right)<\frac{\kappa\phi}{\beta\sigma}-\frac{\kappa}{\beta\sigma}=\mathcal{C}(1),$$

$\mathcal{C}(-1)$ and $\mathcal{C}(1)$ are of opposite signs if and only if $\mathcal{C}(-1)<0<\mathcal{C}(1)$, that is to say if and only if (63) holds.

Conversely, suppose that (63) holds. Then $\mathcal{C}(-1)<0<\kappa\phi/(\beta\sigma)=\mathcal{C}(0)$. So $\mathcal{C}(X)$ has at least one real root in $(-1,0)$, which I denote by $r_1$. Since $\mathcal{C}(X)$ is of type $X^3-a_2X^2+a_1X+a_0$, we have $r_1+r_2+r_3=a_2\equiv 1+1/\beta+\kappa/(\beta\sigma)>2$, where $r_2$ and $r_3$ denote the other two roots of $\mathcal{C}(X)$. Therefore, $r_2+r_3=a_2-r_1>2$, which implies that $r_2$ and $r_3$ lie outside the unit circle. As a consequence, the rule (37) robustly ensures local-equilibrium determinacy.

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