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Approval voting without ballot restrictions

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We axiomatically study voting rules without making any assumption on the ballots that voters are allowed to cast. In this setting, we characterize the family of "endorsement rules," which includes approval voting and the plurality rule, via the imposition of three normative conditions. The first condition is the well known social-theoretic principle of *consistency*; the second one, *unbiasedness*, roughly requires social outcomes not to be biased toward particular candidates or voters; the last one, dubbed *no single voter overrides*, demands that the addition of a voter to an electorate cannot radically change the social outcome. Building on this result, we provide the first axiomatic characterization of approval voting without the approval balloting assumption. The informational basis of approval voting as well as its aggregative rationale are jointly derived from a set of conditions that can be defined on most of the ballot spaces studied in the literature.

Keywords. Approval voting, balloting procedures, informational basis, endorsement.

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1. Introduction

Approval voting is a voting rule where voters submit a ballot that specifies which candidates they support for office, and the candidates that are supported by the largest number of voters win the election. Since the seminal publications by Brams and Fishburn (Brams and Fishburn (1978), Fishburn (1979)), a burgeoning literature has been devoted to the analysis of the axiomatic properties of approval voting.[1] In the tradition

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[1]A comprehensive survey of axiomatic characterizations of approval voting (up to 2010) is to be found in Xu (2010); more recent contributions include Núñez and Valletta (2015), Maniquet and Mongin (2015), Sato (2019), and Brandl and Peters (2019).

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of social choice theory, these studies share a common methodology. First, it is assumed that information concerning the voters' opinions comes in a specific dichotomous format, namely approval ballots or dichotomous preferences.[2] Then it is asked how such information should be aggregated so as to make a decision.

Once approval ballots are exogenously assumed, it is hard to formally compare approval voting with voting methods that process qualitatively different ballot information, such as the Borda rule, which aggregates profiles of strict rankings over the set of candidates. This is an undesirable feature, as comparability across voting methods is needed to assess their relative merits. Moreover, if the axiomatic analysis is only conducted on a given informational environment, questions concerning whether the choice of balloting procedures carries some normative relevance cannot be asked.

The purpose of this article is to provide an axiomatic characterization of approval voting without fixing the approval balloting procedure (or, crucially, any balloting procedure) from the start. To do so, we define voting rules as mappings that take profiles of individual "signals" as input and return a set of winning candidates. These signals, which we call ballots, are abstract objects in a given set, interpreted as the admissible opinions that voters are allowed to express in a particular voting context. Specific typologies of ballots include approval ballots, preferential ballots, evaluative ballots, or interesting combinations of the two (e.g., the preference-approval ballots introduced by Brams and Sanver (2009)). We adopt a variable-population framework that is comparable to the one proposed by Myerson (1995), though we do not impose his anonymity requirement.

Our first contribution, Theorem 1, is to characterize the family of "endorsement rules," i.e., voting rules that translate every individual signal into a hypothetical dichotomous ballot and then select the candidate who receives the largest number of hypothetical endorsements. This family includes approval voting, but also variants thereof where the voters can only express a limited number of approval opinions, like the plurality rule. Endorsement rules are shown to be characterized by three axiomatic conditions. The first is the classical axiom of "consistency," which roughly requires that when a candidate is selected by two different constituencies, she must also be selected by their union. The second condition is a notion of unbiasedness of a voting rule, which essentially demands that all candidates and voters are treated equally. The last condition, dubbed "no single voter overrides," captures the idea that a voter cannot single-handedly control the electoral outcome, in the sense that the sets of winners chosen by a voting population with or without the presence of an additional voter cannot be disjoint.

Our second contribution, Theorem 2, characterizes approval voting by enriching the axiom set of Theorem 1 with two additional requirements. The first requirement demands that every ballot cast by a voter can be countered by another voter in such a way that the joint submission of opposite opinions results in the complete indecisiveness of

[2] An approval ballot is a subset of the set of available candidates, interpreted to represent those candidates of which a voter approves. Dichotomous preferences are weak orders over the set of candidates that admit at most two indifference classes, with the top indifference class representing the approved candidates.

the voting rule. The second one requires that a voter's endorsement of multiple candidates is not due to the lack of expressiveness of the ballot space, in the sense that if a voter can endorse a set of candidates for election, she can also endorse any subset thereof. Since the axioms characterizing approval voting are not explicitly linked to the informational environment of approval ballots, Theorem 2 enhances the comparability of approval voting with other well known voting methods. For instance, the Borda rule satisfies all conditions that characterize approval voting except "no single voter overrides."

Based on the absence of agreement about what type of information is to be asked of voters or used in social choice procedures, we believe that our results are promising and could be meaningfully extended to the analysis of other voting methods. For example, our approach could be used to elucidate the consequences of replacing the dichotomous informational basis of approval voting with a trichotomous one—allowing voters to express approval, disapproval, and neutrality toward candidates—as required by an alternative to approval voting, the dis&approval voting rule, that has received some attention in recent years (see Hillinger (2005), Alcantud and Laruelle (2014) or Gonzalez et al. (2019)).

2. Framework

Let C be the set of candidates (or social alternatives) and denote by $\mathcal{P}(C)$ the set of all nonempty subsets of C. For every $A \in \mathcal{P}(C)$, let $\mathcal{P}(A)$ be defined similarly. Let $\mathbb{V}$ be an infinite set, interpreted to represent the universe of voters. An electorate is a nonempty, finite subset of $\mathbb{V}$. We denote by $\mathcal{E}$ the set of all electorates, with typical element V. Let X be the (possibly infinite) set of ballots that a voter can submit to the election chair in a given voting context. For every electorate $V \in \mathcal{E}$, a ballot profile

$$B^V = (B^i)_{i\in V} \in X^V$$

is a collection of individual ballots. By this very definition, there is a formal difference between an individual ballot B^i and the ballot profile $B^{\{i\}}$. Nevertheless, when there is no risk of confusion, we slightly abuse notation and write B^i for $B^{\{i\}}$. Let $\mathcal{B}$ be the set of all ballot profiles that can be constructed from $\mathcal{E}$ and X. Given two disjoint electorates $V, W \in \mathcal{E}$ and two ballot profiles $B^V, B^W \in \mathcal{B}$, we denote by $B^{V\cup W} = (B^i)_{i\in V\cup W}$ the ballot profile obtained by merging B^V and B^W. Similarly, given two electorates $V, W \in \mathcal{E}$ such that $W \subsetneq V$ and a ballot profile $B^V \in \mathcal{B}$, we denote by $B^{V\setminus W} = (B^i)_{i\in V\setminus W}$ the ballot profile obtained by removing the ballots cast by voters in W from B^V. A voting rule f on $\mathcal{B}$ associates to any ballot profile $B^V \in \mathcal{B}$ a (possibly empty) subset of the available candidates $f(B^V) \subseteq C$ (the "winners" of the election). Denote by $\mathcal{R}$ the set of voting rules on $\mathcal{B}$. A voting rule f is *non-coarse* if there exists $V \in \mathcal{E}$ and B^V such that $f(B^V) \neq C$.

To define specific voting rules in this abstract setting, it is important to specify both their informational basis and aggregative rationale at once, as we now illustrate by introducing a number of voting methods that are used in the sequel. Given a set C, we say that the family $\mathcal{F} \subseteq \mathcal{P}(C)$ is *permutable on* C if, for all $k \in \mathbb{N}$ and $S \in \mathcal{P}(C)$ such that

$|S| = k$, whenever $S \in \mathcal{F}$, then for all $S \in \mathcal{P}(C)$ such that $|S| = k$, $S \in \mathcal{F}$. We say that a voting rule f is an *endorsement rule* if, for every $i \in \mathbb{V}$, there exists a surjection $\varphi_i : X \to \mathcal{F}$, where $\mathcal{F}$ is a family that is permutable on C, such that for all $V \in \mathcal{E}$ and for all $B^V \in \mathcal{B}$,

$$f(B^V) = \operatorname*{argmax}_{a \in C} \left|\left\{i \in V : a \in \varphi_i(B^i)\right\}\right|. \tag{1}$$

We say that an endorsement rule f is *approval voting* if f satisfies $\mathcal{F} \setminus \{C\} = \mathcal{P}(C) \setminus \{C\}$. Similarly, we say that an endorsement rule f is the *plurality rule* if f satisfies $\mathcal{F} \setminus \{C\} = \{\{a\} : a \in C\}$. Observe that when defining approval voting and the plurality rule, we allow the inclusion or exclusion of a (hypothetical) trivial ballot in which all candidates are endorsed, this being irrelevant to the determination of the winner.

Let $\mathcal{L}(C)$ denote the set of linear orders over C. We say that f is the *Borda rule* if, for every voter $i \in \mathbb{V}$, there exists a surjection $\varphi_i : X \to \mathcal{L}(C)$ such that, for every finite set of voters $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$,

$$f(B^V) = \operatorname*{argmax}_{a \in C} \sum_{i \in V} s_a(B^i), \tag{2}$$

where, for every $a \in C$, $i \in \mathbb{V}$, and $B^i \in \mathcal{B}$, $s_a(B^i) = |\{b \in C : a\varphi_i(B^i)b\}|$.

We say that f is an *evaluative voting rule* if, for every voter $i \in \mathbb{V}$, there exist a nonempty $Y \subseteq \mathbb{R}$ and a surjection $\varphi_i : X \to Y^C$ such that, for every finite set of voters $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$,

$$f(B^V) = \operatorname*{argmax}_{a \in C} \sum_{i \in V} \varphi_i(B^i)[a]. \tag{3}$$

Let $\mathcal{W}(C)$ denote the set of weak orders over C. We say that f is the *Condorcet rule* if, for every voter $i \in \mathbb{V}$, there exists a surjection $\varphi_i : X \to \mathcal{W}(C)$ such that, for every finite set of voters $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$,

$$f(B^V) = \left\{a \in C : K_{ab}(B^V) \geq K_{ba}(B^V), \forall b \in C\right\},$$

where, for every a and b in C, $K_{ab}(B^V) = |\{i \in V : a\varphi_i(B^i)b\}|$.

Observe that, mathematically, each of the above definitions is an extension of its canonical counterpart, which is obtained by identifying X with $\varphi_i(X)$ and letting φ_i be the identity mapping, for every i. Such extension includes, more generally, all voting rules that map the information contained in the signals cast by voters into the relevant typology of ballots, and then aggregate such profiles of ballots as their canonical counterpart would. The informational basis of a voting procedure, captured by the signal-interpretation mappings $\{\varphi_i\}_{i \in \mathbb{V}}$, is what matters for determining the winner of an election, meaning that, for every $V \in \mathcal{E}$ and $B^V, \tilde{B}^V \in \mathcal{B}$,

$$\left\{\varphi_i(B^i)\right\}_{i \in V} = \left\{\varphi_i(\tilde{B}^i)\right\}_{i \in V} \Longrightarrow f(B^V) = f(\tilde{B}^V).$$

Finally, voting rules as defined here are unique up to relabeling of the mappings $\{\varphi_i\}_{i \in \mathbb{V}}$ through a collection of bijections $\{\mu_i : \varphi_i(X) \to \varphi_i(X)\}_{i \in \mathbb{V}}$. Hence, while several voting rules can satisfy the same definition, each of them is "expressively" equivalent: from the voters' point of view, what ultimately matters is the procedural interpretation of a signal rather than the signal itself.

3. Endorsement rules

In this section, we characterize the family of "endorsement rules" (see (1)), i.e., voting rules that translate ballot information into a hypothetical dichotomous ballot specifying whether a voter endorses each candidate, and then select the candidate who receives the largest number of endorsements. Endorsement rules include approval voting but also variants thereof where the set of admissible endorsement opinions that voters can express is restricted, e.g., when voters are allowed to endorse at most k candidates.

The first axiom is the well known requirement of "consistency,"[3] which demands that when merging two constituencies, if some candidate is selected by both constituencies, she shall remain selected.

Consistency For every $V, V' \in \mathcal{E}$ and $B^V, B^{V'} \in \mathcal{B}$, if $V \cap V' = \varnothing$ and $f(B^V) \cap f(B^{V'}) \neq \varnothing$, then $f(B^{V \cup V'}) = f(B^V) \cap f(B^{V'})$.

The next axiom demands that for any social outcome determined by an electorate, the outcome can be permuted by a possibly distinct replica of the electorate. This intuitively means that social outcomes are not biased in favor of some candidates or voters.

Unbiasedness For every $V \in \mathcal{E}$, $B^V \in \mathcal{B}$, permutation $\pi : C \to C$ and injective mapping $\mu : V \to \mathbb{V}$, there exists $B_\pi^{\mu(V)} \in \mathcal{B}$ such that, for every $V' \subseteq V$, $f(B_\pi^{\mu(V')}) = \pi(f(B^{V'}))$.

Observe that *Unbiasedness* is equivalent to the joint imposition of a notion of neutrality and a notion of anonymity of voting rules, obtained by letting μ and π be the identity mappings in the statement of the axiom, respectively. Finally, the last condition demands that adding a voter to the electorate can, in no circumstance, exclude all previously winning candidates. As such, the condition can be understood as a democratic principle requiring the addition of a single voter not to overrule the decision of the majority.

No single voter overrides For every $V \in \mathcal{E}$, $i \in \mathbb{V} \setminus V$, and $B^V, B^i \in \mathcal{B}$, $f(B^V) \cap f(B^{V \cup \{i\}}) \neq \varnothing$.

The above condition is comparable to the (logically independent, but intuitively much weaker) property of "no minority overrides" introduced by Pivato (2014), which essentially requires the existence of at least one ballot profile in which an additional voter cannot single-handedly change the electoral outcome. *No single voter overrides* implies that the voting rule always returns a nonempty set of candidates (see the first paragraph of the proof of Theorem 1 in Section A below). This is not an assumption of substance: our results can alternatively be derived by explicitly assuming the nonemptiness of f and modifying *No single voter overrides* to apply only to profiles $B^V, B^i \in \mathcal{B}$ such that both $f(B^V)$ and $f(B^{V \cup \{i\}})$ are nonempty.

Theorem 1. *A voting rule f satisfies Consistency, Unbiasedness, and No single voter overrides if and only if it is an endorsement rule.*

[3] This terminology is due to Young (1974), Young (1975), and Saari (1990). Smith (1973) calls this condition "separability," while Myerson (1995), Pivato (2013), and Pivato (2014) call it "reinforcement." Consistency has been used in most of the existing axiomatic characterizations of scoring rules with variable electorate and so, in particular, of approval voting.

Theorem 1 shows that any voting rule satisfying the stated axioms treats the voters' ballots "as if" they were dichotomous ballots and discards any non-dichotomous information that may be available when computing the winner of the election. Several voting rules satisfy *Consistency* and *Unbiasedness*. Notable examples, which process qualitatively different ballot information than endorsement rules, include the Borda rule and evaluative voting (as defined in (2) and (3), respectively). Therefore, the result indicates that the *No single voter overrides* condition is instrumental for the binarization of ballot information. In some sense, it can be thought of as being what separates rules that process dichotomous ballot information from those that do not. As further evidence of this fact, notice that endorsement rules necessarily satisfy the following condition.

Abstract disjoint equality For every $i \in \mathbb{V}$, $j \in \mathbb{V} \setminus \{i\}$, and $B^i, B^j \in \mathcal{B}$, if $f(B^i) \cap f(B^j) = \varnothing$, then $f(B^{\{i,j\}}) = f(B^i) \cup f(B^j)$.

This property is an abstract rendering of the condition of "disjoint equality" that has often been used to characterize the approval voting rule in the domain of approval ballots (see Fishburn (1978) or Brandl and Peters (2019)). It is obtained from its conventional counterpart by replacing approval ballots with social outcomes for single voter profiles in its statement. The same procedure could be used to derive an "abstract" counterpart of Fishburn's axiom of "cancellation" (that is, the requirement that whenever every candidate receives exactly the same number of approvals, all candidates are elected; see Fishburn (1979) and Alós-Ferrer (2006)) and of the various conditions studied in Brandl and Peters (2019). These properties, satisfied by the family of endorsement rules, typically fail for procedures that take into account richer ballot information. While there is no straightforward way to adapt the proof of Theorem 1 by replacing *No single voter overrides* with *Abstract disjoint equality* (or the abstract rendering of "cancellation"), we conjecture that it would be possible to obtain a characterization of the family of endorsement rules by following this path. We nevertheless prefer to rely on the *No single voter overrides* condition because it says something non-obvious and, in our view, compelling about voting rules that treat dichotomous ballot information. Importantly, it does so without artificially building upon such dichotomous information, as it does not feature single voter profiles in its statement.

An immediate consequence of Theorem 1 is that it is possible to derive axiomatic characterizations of the plurality rule and approval voting by imposing richness restrictions on the admissible outcomes of the voting rule for single voter electorates. This is because every endorsement rule can essentially be identified by the permutable family $\mathcal{F} \subseteq \mathcal{P}(C)$ that coincides both with the range of each individual signal interpretation mapping φ_i and with the set of social outcomes that the single voter electorate $\{i\}$ can generate, that is, $f(X^{\{i\}}) := \{f(B^i) : B^i \in X^{\{i\}}\}$. We now show that to characterize the plurality rule, it is sufficient to impose a condition that demands that a voter can at most endorse one candidate (or abstain), while approval voting can be characterized by demanding that voters can endorse any number of candidates. This is formally stated as follows.

One voter–one vote For every $i \in \mathbb{V}$, for every $B^i \in \mathcal{B}$, $|f(B^i)| = 1$ or $|f(B^i)| = |C|$.

One voter–any vote For every $i \in \mathbb{V}$, for every $A \in \mathcal{P}(C) \setminus \{C\}$, there exists $B^i \in \mathcal{B}$ such that $f(B^i) = A$.

Corollary 1. *The plurality rule is the only non-coarse voting procedure that satisfies Consistency, Unbiasedness, No single voter overrides, and One voter–one vote.*

In fact, to characterize approval voting under *One voter–any vote*, it is possible to weaken *Unbiasedness* to the following condition.[4]

Procedural anonymity For every $i, j \in \mathbb{V}$, for every $B^i \in \mathcal{B}$, there exists $B^j \in \mathcal{B}$ such that $f(B^i) = f(B^j)$.

Corollary 2. *Approval voting is the only voting procedure that satisfies Consistency, Procedural anonymity, No single voter overrides, and One voter–any vote.*

Corollary 1 implies that *if* one assumes that X is the set of plurality ballots $\{\{a\} : a \in C\}$ or the domain $\{\{a\} : a \in C\} \cup \{C\}$, *then* the plurality rule is characterized by the axioms of *Consistency, Unbiasedness,* and *No single voter overrides,* plus the following condition.

Faithfulness For every $i \in \mathbb{V}$ and $B^i \in \mathcal{P}(C) \setminus \{C\}$, $f(B^i) = B^i$.

Similarly, Corollary 2 implies that *if* one assumes that X is the set of approval ballots $\mathcal{P}(C)$ (or $\mathcal{P}(C) \setminus \{C\}$), *then* approval voting is characterized by the axioms of *Consistency*, *No single voter overrides*, and *Faithfulness* (observe that *Procedural anonymity* becomes redundant in this case). The axiom of *Faithfulness* has been used in several existing axiomatizations of approval voting and, in our setting, makes it possible to pin down the "canonical" voting rules among the collection of their abstract counterparts (obtained via relabeling of the signal interpretation mappings).[5] In fact, by appropriately defining the notion of faithfulness so as to require that each voter's favorite candidate be elected if society is reduced to a single voter, one can extend the results just stated to most of the ballot domains used in the literature. For instance, if X is the set of linear orders and one requires only the top-ranked alternative to be selected in single voter electorates, the voting rule satisfies *One voter–one vote* and so Corollary 1 can be used to characterize the plurality rule on this domain. Alternatively, when X is the set of weak orders or the set of evaluative ballots, one can require only the candidates with highest ranking or score to be selected in single voter electorates, and obtain a voting rule that satisfies *One voter–any vote.* Hence, Corollary 2 can be used to characterize approval voting on these domains. The upshot of this discussion is that, while *One voter–one vote* and *One voter–any vote* do restrict the ballot spaces that one can study, they do not pin down a unique domain, and so the characterizations obtained in the above corollaries are fairly robust to changes in the informational environment. Nevertheless, for the purpose of

[4]Roughly, this is because an important step in the proof of Theorem 1 consists in showing that there exist voting profiles where exactly two candidates are elected. This property, that relies on *Unbiasedness,* is implied by *One voter–any vote.* However, it is not implied by *One voter–one vote*; hence, we require *Unbiasedness* in Corollary 1.

[5]See the discussion at the end of Section 2.

enhancing comparability of voting rules defined over different informational environments, such conditions are not fully satisfactory. This is why in the next section we characterize approval voting through axioms that can equally well apply to most of the ballot spaces studied in the literature.

4. Approval voting

We now enrich the axiom set of Theorem 1 by adding two conditions. The first one demands that every opinion B^i that can be expressed by a voter i can be "neutralized" by another voter j, meaning that there exist B^j such that $f(B^{\{i,j\}}) = C$.[6] The second one requires that the potential indecisiveness of individual opinions is not due to some lack of expressiveness of the balloting procedure. More precisely, if a voter i can elect a set of candidates $f(B^i)$ by casting some ballot B^i, she can also elect, by changing her ballot, any subset A of $f(B^i)$.

Opposite opinions For every $i \in \mathbb{V}$, $j \in \mathbb{V} \setminus \{i\}$, and $B^i \in \mathcal{B}$, there exists $B^j \in \mathcal{B}$ such that $f(B^{\{i,j\}}) = C$.

Refinement For every $i \in \mathbb{V}$, for every $B^i \in \mathcal{B}$, and for every nonempty $A \subseteq f(B^i)$, there exists $\tilde{B}^i \in \mathcal{B}$ such that $A = f(\tilde{B}^i)$.

The following theorem is the final contribution of this paper.

Theorem 2. *Approval voting is the only voting rule that satisfies Consistency, Unbiasedness, No single voter overrides, Opposite opinions, and Refinement.*

Theorem 2 provides an axiomatic characterization of approval voting without the approval balloting assumption.[7] It jointly derives the informational basis of approval voting and its aggregative rationale. Since the proposed axioms do not explicitly rely on the structure of approval ballots, this result makes it possible to directly compare approval voting with other well known voting rules, irrespective of differences in the ballot information that they process. For instance, the Borda rule or the family of evaluative voting rules satisfy all axioms of Theorem 2 except *No single voter overrides*; the plurality rule (when abstention ballots are not allowed) satisfies all axioms except *Opposite opinions*; the constant rule that always returns the set C satisfies all axioms except *Refinement*; the Condorcet rule satisfies all axioms except *Consistency* and *No single voter overrides*; and so on.[8]

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[6]This condition is reminiscent of "disjoint equality" and "cancellation" (see Section 3). Contrary to the latter, it is satisfied by scoring rules and evaluative voting methods.

[7]On a parenthetical note, the axioms of Theorem 2 are independent. See Section B.

[8]Of course these statements hold whenever $|C| > 2$. If $|C| = 2$, all mentioned rules coincide with approval voting and, hence, satisfy all axioms of Theorem 2. Similarly, by Definition (3), the family of evaluative voting rules contains the constant rule that always selects C (obtained by letting $|Y| = 1$) and approval voting (obtained by letting $|Y| = 2$). Hence, a more precise statement is, "Every evaluative voting rule with evaluation scale Y that contains strictly more than two points satisfies all axioms of Theorem 2 except *No single voter overrides.*"

While the proposed axiomatic conditions effectively impose the reduction of whatever information is available in the ballots to approval ballots, this does not rule out the possibility of collecting richer information from the voters or of effectively using such richer information when available. For example, approval ballots could emerge from evaluative ballots in which voters are asked to assign a numerical grade between 0 and 10 to candidates by considering the candidate(s) whose grade is above 6 to be approved. This highlights the consequentialist perspective of our analysis: Once one accepts the axioms of Theorem 2, any two ballots that induce the same set of approved candidates will have equivalent instrumental value, yet they may well be substantially different with regard to other types of considerations that can be invoked for the practical purpose of designing a voting rule. For instance, experimental evidence suggests that the labeling of ballots has a psychological meaning to voters that can affect the output of a voting rule, and that voters value the possibility of expressing non-dichotomous opinions even though the additional information is not exploited when determining the winner of the election (see Baujard et al. (2018)). Similarly, legal or political considerations may also point in the direction of enhancing the expressive freedom of voters, e.g., when explicitly introducing an “abstention” ballot that allows voters to exercise their right not to vote, while maintaining their right to secrecy.[9] Another related feature of our characterization is that the translation mappings may depend on the identity of the voter, so that the same ballot could, in principle, receive different interpretations depending, for example, on the age of the voter who casts it. Therefore, the approval voting rule (as we define it) does not necessarily satisfy the usual property of anonymity, i.e., it is possible that for two electorates $V, W \in \mathcal{E}$ and ballot profiles $B^V, B^W \in \mathcal{B}$, $B^V = B^W$ and yet $f(B^V) \neq f(B^W)$. Nevertheless, it satisfies anonymity with respect to ballot interpretations rather than ballots, in line with our consequentialist approach to ballot information.

Appendix A: Proofs

Proof of Theorem 1. We prove only the sufficiency of the axioms.[10] Let f be a voting rule that satisfies *Consistency*, *Unbiasedness*, and *No single voter overrides*. By *No single voter overrides*, for every $V \in \mathcal{E}$, $i \in \mathbb{V} \setminus V$, and $B^V, B^i \in \mathcal{B}$, $f(B^V) \cap f(B^{V\cup\{i\}}) \neq \varnothing$. It follows that $f(B^V) \neq \varnothing$, for every $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$. Next, for every $a \in C$, $V \in \mathcal{E}$, and $B^V \in \mathcal{B}$, let

$$I_a\left(B^V\right) := \left\{i \in V \,:\, a \in f\left(B^i\right)\right\}$$

and

$$A\left(B^V\right) = \operatorname*{argmax}_{a\in C}\left|I_a\left(B^V\right)\right|.$$

[9]This argument (among others) was used in 2014 by the Indian constitutional court to introduce “none of the above” ballots, which, although counted, do not impact the result of the election process.

[10]We are indebted to an anonymous referee whose ingenious suggestions were crucial in developing the argument presented here, which is more direct than our original argument.

Moreover, for every $i \in \mathbb{V}$, let the mapping $\varphi_i : X \to \mathcal{P}(C)$ be defined as follows: for every $x \in X$, $\varphi_i(x) = f(B^i)$, where $B^i = x$. By *Unbiasedness*, $f(X^{\{i\}})$ is independent of i. Moreover, it is a permutable family on C. Hence, to establish the theorem, it suffices to show that, for every $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$,

$$f(B^V) = A(B^V). \tag{P}$$

Our proof is by induction. Let $P(n)$ be the statement, "property (P) holds for every electorate of size n."

First notice that whenever $V \in \mathcal{E}$ is such that $|V| = 1$, nonemptiness of f implies that property (P) holds, i.e., $P(1)$ holds. Next, fix a natural number $n \geq 1$ and suppose that $P(n)$ holds. We show that this implies that $P(n+1)$ also holds. The proof is divided into several steps.

STEP 1. *If there exist $V' \in \mathcal{E}$ and $B^{V'} \in \mathcal{B}$ such that $f(B^{V'}) \neq C$ and $|f(B^{V'})| \geq 2$, then, for every $a \in C$ and $b \in C \setminus \{a\}$, there exist $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$ such that $f(B^V) = \{a, b\}$.*

Suppose that for some $V' \in \mathcal{E}$ and $B^{V'} \in \mathcal{B}$, $f(B^{V'}) \neq C$ and $|f(B^{V'})| \geq 2$. Then there are candidates $a \in C$ and $b \in C \setminus \{a\}$ such that $\{a, b\} \subseteq f(B^{V'})$. By *Unbiasedness*, to establish our claim, it suffices to identify some electorate V and ballot profile $B^V \in \mathcal{B}$ such that $f(B^V) = \{a, b\}$. If $f(B^{V'}) = \{a, b\}$, the result follows. Hence, suppose that $\{a, b\} \subsetneq f(B^{V'})$. By assumption, $f(B^{V'}) \neq C$, so there exists some $d \in C \setminus f(B^{V'})$. Next, for every $c \in f(B^{V'}) \setminus \{a, b\}$, let π_c be the permutation on C defined by $\pi_c(c) = d$, $\pi_c(d) = c$, and $\pi_c(a) = a$ for every $a \in C \setminus \{c, d\}$. Moreover, let $(W_c)_{c \in f(B^{V'}) \setminus \{a,b\}}$ be a family of electorates such that, for every $c, c' \in f(B^{V'}) \setminus \{a, b\}$, $|W_c| = |V'|$, $W_c \cap V' = \varnothing$, and $W_c \cap W_{c'} = \varnothing$. By *Unbiasedness* (applied to some bijection $\mu_c : V' \to W_c$ and π_c), for every $c \in f(B^{V'}) \setminus \{a, b\}$, there exists B^{W_c} such that

$$f(B^{W_c}) = \big(f(B^{V'}) \setminus \{c\}\big) \cup \{d\}.$$

Let $W = \bigcup_{c \in f(B^v) \setminus \{a,b\}} W_c$. By *Consistency*,

$$f(B^W) = \{a, b, d\}. \tag{4}$$

Since $d \notin f(B^{V'})$ and $\{a, b\} \subseteq f(B^{V'})$, (4) implies that $\{a, b\} = f(B^W) \cap f(B^{V'})$. Therefore, by *Consistency* again,

$$\{a, b\} = f\big(B^{W \cup \{V'\}}\big),$$

as desired.

STEP 2. *For every $V \in \mathcal{E}$ of size $n+1$ and $B^V \in \mathcal{B}$, if there exists some $i \in V$ such that $f(B^{V \setminus \{i\}}) \cap f(B^i) \neq \varnothing$, then $f(B^V) = A(B^V)$.*

Let V be an electorate of size $n+1$ and fix a ballot profile B^V such that for some $i \in V$, $f(B^{V \setminus \{i\}}) \cap f(B^i) \neq \varnothing$. By *Consistency*, $f(B^{V \setminus \{i\}}) \cap f(B^i) = f(B^V)$. Meanwhile, by the induction hypothesis, $f(B^{V \setminus \{i\}}) = A(B^{V \setminus \{i\}})$. Finally, using the definition of A, $A(B^V) = A(B^{V \setminus \{i\}}) \cap f(B^i)$. Hence, $f(B^V) = A(B^V)$.

Step 3. *For every* $V \in \mathcal{E}$ *of size* $n+1$ *and* $B^V \in \mathcal{B}$, *if* $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ *for every* $i \in V$, *then* $\bigcup_{i\in V} f(B^{V\setminus\{i\}}) = A(B^V)$.

Let V be an electorate of size $n+1$ and fix a ballot profile B^V such that $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ for every $i \in V$. Fix $i \in V$ and $a \in f(B^{V\setminus\{i\}})$ (a exists by nonemptiness of f). Since, by the induction hypothesis, $f(B^{V\setminus\{i\}}) = A(B^{V\setminus\{i\}})$, while $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ by assumption, on the one hand,

$$\forall c \notin f\big(B^{V\setminus\{i\}}\big), \quad \big|I_c\big(B^V\big)\big| \leq \big|I_c\big(B^{V\setminus\{i\}}\big)\big| + 1 \leq \big|I_a\big(B^{V\setminus\{i\}}\big)\big| = \big|I_a\big(B^V\big)\big|; \tag{5}$$

on the other,

$$\forall c \in f\big(B^{V\setminus\{i\}}\big), \quad \big|I_c\big(B^V\big)\big| = \big|I_a\big(B^V\big)\big|. \tag{6}$$

Combining (5) and (6), we obtain that $a \in A(B^V)$. Since i was chosen arbitrarily,

$$\bigcup_{i\in V} f\big(B^{V\setminus\{i\}}\big) \subseteq A\big(B^V\big). \tag{7}$$

To prove the reverse inclusion, consider $a \in A(B^V)$. If $a \in \bigcap_{i\in V} f(B^i)$, then by *Consistency*, $a \in f(B^i) \cap f(B^{V\setminus\{i\}})$ for every $i \in V$, in contradiction to our hypothesis. Hence, let $i \in V$ be such that $a \notin f(B^i)$. Then

$$\forall c \in C, \quad \big|I_a\big(B^{V\setminus\{i\}}\big)\big| = \big|I_a\big(B^V\big)\big| \geq \big|I_c\big(B^V\big)\big| \geq \big|I_c\big(B^{V\setminus\{i\}}\big)\big|,$$

from which we deduce that $a \in A(B^{V\setminus\{i\}})$. Using the induction hypothesis, $a \in f(B^{V\setminus\{i\}})$, so $a \in \bigcup_{i\in V} f(B^{V\setminus\{i\}})$. Since a was chosen arbitrarily,

$$A\big(B^V\big) \subseteq \bigcup_{i\in V} f\big(B^{V\setminus\{i\}}\big). \tag{8}$$

Finally, combining (7) and (8) gives the desired result.

Step 4. *For every* $V \in \mathcal{E}$ *of size* $n+1$ *and* $B^V \in \mathcal{B}$, $A(B^V) \subseteq f(B^V)$.

By Steps 2 and 3, it suffices to prove that for every $V \in \mathcal{E}$ of size $n+1$ and $B^V \in \mathcal{B}$, if $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ for every $i \in V$, then $\bigcup_{i\in V} f(B^{V\setminus\{i\}}) \subseteq f(B^V)$. To do so, let $V \in \mathcal{E}$ be an electorate of size $n+1$ and fix a ballot profile $B^V \in \mathcal{B}$ such that $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ for every $i \in V$. Suppose by contradiction that for some $i \in V$ and $a \in C$, $a \in f(B^{V\setminus\{i\}}) \setminus f(B^V)$. Since $a \notin f(B^V)$,

$$f\big(B^V\big) \neq C. \tag{9}$$

Alternatively, using the induction hypothesis, $f(B^{V\setminus\{i\}}) = A(B^{V\setminus\{i\}})$. This, together with the nonemptiness of f, implies that there exists some $i_a \in V \setminus \{i\}$ such that $a \in f(B^{i_a})$. By assumption, $a \notin f(B^{V\setminus\{i_a\}})$. By *No single voter overrides*, there exists some $b \in f(B^{V\setminus\{i_a\}}) \cap f(B^V)$. Since $b \in f(B^{V\setminus\{i_a\}})$, by assumption $b \notin f(B^{i_a})$. Meanwhile, by Step 3, $\bigcup_{i\in V} f(B^{(V\setminus\{i\})}) = A(B^V)$, so

$$\{a, b\} \subseteq A\big(B^V\big). \tag{10}$$

In particular, this implies that $|I_a(B^V)| = |I_b(B^V)|$ and so, using the fact that $i_a \in I_a(B^V) \setminus I_b(B^V)$, we obtain that there exists $i_b \in I_b(B^V) \setminus I_a(B^V)$. By the induction hypothesis, $f(B^{V\setminus\{i_b\}}) = A(B^{V\setminus\{i_b\}})$, and so, again by (10),

$$a \in f\big(B^{V\setminus\{i_b\}}\big) \text{ and } b \notin f\big(B^{V\setminus\{i_b\}}\big). \tag{11}$$

Now, since $b \in f(B^V)$, $b \notin f(B^{V\setminus\{i_b\}})$, and, by *No single voter overrides*, $f(B^V) \cap f(B^{V\setminus\{i_b\}}) \neq \varnothing$,

$$\big|f\big(B^V\big)\big| \geq 2. \tag{12}$$

By Step 1 (since (9) and (12) hold), there exists some $W \in \mathcal{E}$ and $B^W \in \mathcal{B}$ such that $f(B^W) = \{a, b\}$. Using *Unbiasedness* if necessary, we can suppose without loss of generality that $W \cap V = \varnothing$. Then, on the one hand,

$$\{b\} = f\big(B^V\big) \cap f\big(B^W\big),$$

so by *Consistency*, $\{b\} = f(B^{V\cup W})$. On the other hand, using (11),

$$\{a\} = f\big(B^{V\setminus\{i_b\}}\big) \cap f\big(B^W\big),$$

so again by *Consistency*, $\{a\} = f(B^{(V\setminus\{i_b\})\cup W})$. It follows that

$$f\big(B^{(V\setminus\{i_b\})\cup W}\big) \cap f\big(B^{V\cup W}\big) = \varnothing,$$

in contradiction to *No single voter overrides*.

Step 5. *For every $V \in \mathcal{E}$ of size $n+1$ and $B^V \in \mathcal{B}$, $f(B^V) \subseteq A(B^V)$.*

By Steps 2 and 3, it suffices to prove that for every $V \in \mathcal{E}$ of size $n+1$ and $B^V \in \mathcal{B}$, if $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ for every $i \in V$, then $f(B^V) \subseteq A(B^V)$. To do so, let $V \in \mathcal{E}$ be an electorate of size $n+1$ and fix a ballot profile $B^V \in \mathcal{B}$ such that $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ for every $i \in V$. Suppose by contradiction that there is some $a \in f(B^V) \setminus A(B^V)$. Let $b \in A(B^V)$ (b exists by nonemptiness of A). By Step 4, $b \in f(B^V)$. Moreover, $|I_b(B^V)| > |I_a(B^V)|$, and so there exists $i \in I_b(B^V) \setminus I_a(B^V)$. By hypothesis, $b \in f(B^i)$ implies that $b \notin f(B^{V\setminus\{i\}})$. Let $\pi : C \to C$ be defined by $\pi(a) = b$, $\pi(b) = a$, and $\pi(d) = d$ for every $d \in C \setminus \{a, b\}$. Let $\mu : V \to \mathbb{V} \setminus V$ be an injection such that $\mu(V) = W$ and $\mu(i) = j$. By *Unbiasedness*, there exists $B^W \in \mathcal{B}$ such that $f(B^W) = \pi(f(B^V))$, $f(B^j) = \pi(f(B^i))$, and $f(B^{W\setminus\{j\}}) = \pi(f(B^{V\setminus\{i\}}))$. By construction,

$$f\big(B^j\big) = \big(f\big(B^i\big) \setminus \{b\}\big) \cup \{a\}, \qquad f\big(B^W\big) = f\big(B^V\big), \quad \text{and} \quad f\big(B^{W\setminus\{j\}}\big) = f\big(B^{V\setminus\{i\}}\big).$$

Since $a \notin A(B^V)$, by Step 3, $a \notin f(B^{V\setminus\{i\}})$. Moreover, $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$ and $f(B^j) = (f(B^i) \setminus \{b\}) \cup \{a\}$, so

$$f\big(B^{V\setminus\{i\}}\big) \cap f\big(B^j\big) = \varnothing. \tag{13}$$

By the induction hypothesis,

$$f\big(B^{V\setminus\{i\}}\big) = A\big(B^{V\setminus\{i\}}\big). \tag{14}$$

Formulae (13) and (14) imply that

$$f\big(B^{V\setminus\{i\}}\big) \subseteq A\big(B^{(V\setminus\{i\})\cup\{j\}}\big).$$

But since $(V \setminus \{i\}) \cup \{j\}$ is an electorate of size $n+1$, by Step 4, $A(B^{(V\setminus\{i\})\cup\{j\}}) \subseteq f(B^{(V\setminus\{i\})\cup\{j\}})$, so

$$f\big(B^{V\setminus\{i\}}\big) \subseteq f\big(B^{(V\setminus\{i\})\cup\{j\}}\big). \tag{15}$$

Next, since $f(B^{W\setminus\{j\}}) = f(B^{V\setminus\{i\}})$ and $f(B^{V\setminus\{i\}}) \cap f(B^i) = \varnothing$, it follows that

$$f\big(B^{W\setminus\{j\}}\big) \cap f\big(B^i\big) = \varnothing. \tag{16}$$

By the induction hypothesis,

$$f\big(B^{W\setminus\{j\}}\big) = A\big(B^{W\setminus\{j\}}\big). \tag{17}$$

Formulae (16) and (17) imply that

$$f\big(B^{W\setminus\{j\}}\big) \subseteq A\big(B^{(W\setminus\{j\})\cup\{i\}}\big).$$

Since $(W\setminus\{j\})\cup\{i\}$ is an electorate of size $n+1$, by Step 4, $A(B^{(W\setminus\{j\})\cup\{i\}}) \subseteq f(B^{(W\setminus\{j\})\cup\{i\}})$, so

$$f\big(B^{W\setminus\{j\}}\big) \subseteq f\big(B^{(W\setminus\{j\})\cup\{i\}}\big). \tag{18}$$

By *No single voter overrides*, there exists some $c \in f(B^{V\setminus\{i\}}) \cap f(B^V)$. By (15), $c \in f(B^{(V\setminus\{i\})\cup\{j\}})$. But since $f(B^{V\setminus\{i\}}) = f(B^{W\setminus\{j\}})$, (18) implies $c \in f(B^{(W\setminus\{j\})\cup\{i\}})$. By *Consistency*,

$$f\big(B^{(V\setminus\{i\})\cup\{j\}}\big) \cap f\big(B^{(W\setminus\{j\})\cup\{i\}}\big) = f\big(B^{V\cup W}\big).$$

Since $f(B^V) = f(B^W)$, *Consistency* again implies $f(B^{V\cup W}) = f(B^V)$ and so $\{a,b\} \subseteq f(B^{V\cup W})$. Then $b \in f(B^{(V\setminus\{i\})\cup\{j\}})$. Since $b \in f(B^i)$, by *Consistency*,

$$b \in f\big(B^{V\cup\{j\}}\big). \tag{19}$$

However, $a \in f(B^V) \cap f(B^j)$, so again by *Consistency*,

$$f\big(B^{V\cup\{j\}}\big) = f\big(B^V\big) \cap f\big(B^j\big).$$

Since $b \notin f(B^j)$, we obtain $b \notin f(B^{V\cup\{j\}})$, in contradiction to (19).

Step 6. *Conclusions.*

Together, Steps 4 and 5 show that $P(n+1)$ holds, thereby completing the induction argument. We conclude that $P(n)$ holds for all $n \geq 1$, which establishes the theorem. □

Proof of Corollary 1. Let f be a non-coarse voting rule satisfying *Consistency*, *Unbiasedness*, *No single voter overrides,* and *One voter–one vote*. By Theorem 1, f is an endorsement rule. Hence, it is associated with a permutable family $\mathcal{F}$ such that $\mathcal{F} = f(X^{\{i\}})$ for every i. By *One voter–one vote*, $\mathcal{F}$ is either $\{\{a\} : a \in C\}$, or $\{\{a\} : a \in C\} \cup \{C\}$, or $\{C\}$. The latter possibility is ruled out by the assumption that f is non-coarse. □

Proof of Corollary 2. Let f be a voting rule that satisfies *Consistency, Procedural anonymity, No single voter overrides,* and *One voter–any vote.* It is immediate to verify that f satisfies the following properties.

C.1. For every $i \in \mathbb{V}$, $\mathcal{P}(C) \setminus \{C\} = f(X^{\{i\}}) \setminus \{C\}$ (hence, $f(X^{\{i\}})$ is a permutable family on C).

C.2. For every $V \in \mathcal{E}$, for every $a, b \in C$, there exists $B^V \in \mathcal{B}$ such that $f(B^V) = \{a, b\}$.

Next, we show that f also satisfies the following property.

C.3. Let $m \in \mathbb{N} \setminus \{0\}$. If for every $W \in \mathcal{E}$ such that $|W| = m$ and for every $B^W \in \mathcal{B}$, $A(B^W) \subseteq f(B^W)$, then for every $W \in \mathcal{E}$ such that $|W| = m$ and for every $B^W \in \mathcal{B}$, $f(B^W) \subseteq A(B^W)$.

To see why, assume that, for some given $m \in \mathbb{N}$ such that $m > 1$, $A(B^W) \subseteq f(B^W)$ for every $W \in \mathcal{E}$ such that $|W| = m$ and $B^W \in \mathcal{B}$. Next suppose by contradiction that for some $W \in \mathcal{E}$ such that $|W| = m$ and $B^W \in \mathcal{B}$, there exists some $a \in f(B^W)$ such that $a \notin A(B^W)$. Let $b \in A(B^W)$. Since $a \notin A(B^W)$ and $b \in A(B^W)$, there exists a voter $i \in W$ such that $b \in f(B^i)$ and $a \notin f(B^i)$. Let $j \in \mathbb{V} \setminus W$ be a voter and fix $B^j \in \mathcal{B}$ such that $f(B^j) = \{a, b\}$ (this is possible by *One voter–any vote*). On the one hand, it is easy to check that

$$b \in A\big(B^{(W\setminus\{i\})\cup\{j\}}\big). \tag{20}$$

Since $(W \setminus \{i\}) \cup \{j\}$ is an electorate of size m, (20) and the working hypothesis imply that

$$b \in f\big(B^{(W\setminus\{i\})\cup\{j\}}\big). \tag{21}$$

Since $b \in f(B^i)$, by (21), we obtain

$$f\big(B^{(W\setminus\{i\})\cup\{j\}}\big) \cap f\big(B^i\big) \neq \varnothing,$$

and by *Consistency,* this intersection is equal to $f(B^{W\cup\{j\}})$. It follows that

$$a \notin f\big(B^{W\cup\{j\}}\big), \tag{22}$$

because $a \notin f(B^i)$. On the other hand, since $a \in f(B^W)$ and $a \in f(B^j)$, *Consistency* implies that $a \in f(B^{W\cup\{j\}})$, contradicting (22). We conclude that for every electorate W of size m, $f(B^W) \subseteq A(B^W)$.

Properties C.1–C.3 can be used to establish the induction argument used in the proof of Theorem 1 without assuming *Unbiasedness.* To see why, let the collection of $\{\varphi_i\}_{i\in\mathbb{V}}$ be defined as follows: for every $i \in \mathbb{V}$ and every $x \in X$, $\varphi_i(x) = f(B^i)$, where $B^i = x$. By construction, for every $i \in \mathbb{V}$, $f(X^i) = \varphi_i(X)$. By *Procedural anonymity,* $\varphi_i(X)$ is independent of i; by C.1, it is permutable on C. As per the induction step of Theorem 1, observe that *Unbiasedness* is used to prove Steps 1, 4, and 5. It is easy to see that Step 1 is implied by C.2. In Step 4, *Unbiasedness* is only used (in combination with Step 1) so as to show that for every given $V \in \mathcal{E}$ and $a, b \in C$ such that $a \neq b$, one can find an electorate $W \in \mathcal{E}$ such that $V \cap W = \varnothing$ and a ballot profile $B^W \in \mathcal{B}$ such that $f(B^W) = \{a, b\}$. This again follows from C.2. Finally, to establish Step 5, observe that Step 4 implies that the antecedent of property C.3 holds for $m = n + 1$, where n is the size of the electorate used in the induction hypothesis. Then Step 5 follows from property C.3. $\square$

Proof of Theorem 2. We prove only the sufficiency of the axioms. Let f be a voting rule satisfying *Consistency, Unbiasedness, No single voter overrides, Opposite opinions,* and *Refinement*. By Theorem 1, f is an endorsement rule. Next, fix some voter $i \in \mathbb{V}$ and a candidate $a \in C$. Using the nonemptiness of f, *Refinement*, and *Unbiasedness*, there exists $B^i \in \mathcal{B}$ such that $f(B^i) = \{a\}$. Let $j \in \mathbb{V} \setminus \{i\}$. By *Opposite opinion*, there exists $B^j \in \mathcal{B}$ such that $f(B^{\{i,j\}}) = C$. Using *Consistency*, $f(B^i) \cap f(B^j) = \varnothing$. Using the definition of endorsement rule, $f(B^j) = C \setminus \{a\}$. By *Refinement*, $\mathcal{P}(C \setminus \{a\}) \subseteq f(X^{\{j\}})$. By *Unbiasedness*,

$$\bigcup_{a \in C} \mathcal{P}\big(C \setminus \{a\}\big) \subseteq f\big(X^{\{j\}}\big).$$

Since, by definition of f, $f(X^{\{j\}}) \subseteq \mathcal{P}(C)$, the previous formula shows that $\mathcal{P}(C) \setminus \{C\} = f(X^{\{j\}}) \setminus \{C\}$. Finally, again by *Unbiasedness*,

$$f\big(X^{\{j\}}\big) = f\big(X^{\{i\}}\big) \quad \text{for every } i, j \in \mathbb{V},$$

establishing that f is approval voting. □

Appendix B: Independence of the axioms of Theorem 2

Table 1 shows that the axioms of Theorem 2 are independent (the formal proof is left to the reader). To do so, we need to define some additional voting rules. We say that f is the *plurality rule without abstention* if f is the plurality rule and $\mathcal{F} = \{\{a\} : a \in C\}$. We say that f is the *unanimity rule* if, for every voter $i \in \mathbb{V}$, there exists a surjection $\varphi_i : X \to \{\{a\} : a \in C\}$ such that for every $V \in \mathcal{E}$ and $B^V \in \mathcal{B}$,

$$f\big(B^V\big) = \begin{cases} \{a\}, & \text{if } \{a\} = \varphi_i\big(B^i\big) \text{ for all } i \in \mathbb{V} \\ C, & \text{otherwise.} \end{cases}$$

We say that a voting rule is the *For/against rule* if it is an endorsement rule such that $\mathcal{F} = \{A \in \mathcal{P}(C) : A = \{a\} \text{ or } A = C \setminus \{a\}, a \in C\}$. We call *nonneutral approval voting* the voting rule f where voters can vote for every element of $\mathcal{P}(C \setminus \{a\}) \setminus \{C \setminus \{a\}\}$ and f is like approval voting, but selects the union of $\{a\}$ and the set of approval voting winners if the latter coincides with the set $C \setminus \{a\}$.

	Unbias.	Cons.	NSVO	Op. Op.	Refi.
For/against	Yes	Yes	Yes	Yes	No
Plurality w/o abstention	Yes	Yes	Yes	No	Yes
Borda rule	Yes	Yes	No	Yes	Yes
Unanimity	Yes	No	Yes	Yes	Yes
Nonneutral approval voting	No	Yes	Yes	Yes	Yes

Table 1. Independence of the axioms of Theorem 2. (NSVO stands here for “No single voter overrides.”)

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