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Bounds on price-setting

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I study a class of macroeconomic models in which all firms can costlessly choose any price at each date from an interval (indexed to last period's price level) that includes a positive lower bound. I prove three results that are valid for any such half-closed interval (regardless of how near zero the left endpoint is). First, given any output sequence that is uniformly bounded from above by the moneyless equilibrium output level, that bounded output sequence is an equilibrium outcome for a (possibly time-dependent) specification of monetary and fiscal policy. Second, given any specification of monetary and fiscal policy in which the former is time-invariant and the latter is Ricardian (in the sense of Woodford 1995), there is a sequence of equilibria in which consumption converges to zero on a date-by-date basis. These first two results suggest that standard macroeconomic models without pricing bounds may provide a false degree of confidence in macroeconomic stability and undue faith in the long-run irrelevance of monetary policy. This paper's final result constructs a non-Ricardian nominal framework (in which the long-run growth rate of nominal government liabilities is sufficiently high) that pins down a unique stable real outcome as an equilibrium.

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1. Introduction

Most macroeconomic models assume that price-setting firms are able to choose any element of the positive reals. This paper instead considers a class of otherwise standard macroeconomic models in which all firms can choose their prices from a common positive interval that includes its lower bound. I show that regardless of how near to zero that lower bound is, these models with bounds on price-setting give rise to dramatically different answers to key macroeconomic questions about the long-run relevance of monetary policy and long-run macroeconomic stability.

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The specifics are as follows. I study the implications of a class of simple infinite-horizon macroeconomic models in which monopolistically competitive firms can all costlessly choose prices at each date from a positive interval (indexed to the prior period's price level) that includes its lowest value. The numeraire is money, which is an asset issued (and withdrawn via lump-sum taxation) by the government. The models feature time- and state-invariant specifications of technology and preferences, so that equilibrium quantities would be time-invariant in a moneyless economy (with labor, for example, being the numeraire).

I obtain three main results. They are all valid regardless of how close the lower bound is to zero. The first result is that, given *any* output sequence that is uniformly bounded from above by the time-invariant moneyless equilibrium output level, that bounded sequence is an equilibrium outcome for some (possibly time-dependent) nominal framework (time paths of interest rate rules and nominal liabilities). This result is straightforward to prove: given a consumption sequence, we need only pick a path of nominal interest rates that is consistent with that sequence and its associated inflation rate path. But it has important implications. Scientifically, the result implies that it is impossible to model the real economy accurately without having some minimal information about monetary policy. From a policy point of view, the result implies that even if prices are highly flexible, economies cannot obtain desirable real outcomes if monetary policy interventions are poorly designed.

One response to this "extreme relevance of monetary policy" result is that its proof relies on the possibility that the central bank could follow "crazy" policies (specifically, time-dependent interest rate pegs). The second main result of the paper restricts attention to "sensible" nominal frameworks in which the central bank follows a time-invariant monetary policy rule and in which fiscal policy is Ricardian[1] in the sense of Woodford (1995) (so that the intertemporal government budget constraint is satisfied for all time paths of inflation rates). I show that for any such nominal framework, there is a sequence of equilibria in which consumption (which equals output and labor) converges datewise to zero.[2] The corresponding sequence of household utilities, as evaluated at the initial date, converges to zero (which is the households' utility level from a time path that delivers zero consumption and zero labor at all dates). This result implies that even if governments use what might appear to be sensible nominal frameworks, the dynamic complementarities in monetary economies can give rise to deviations from macroeconomic stability that are large in terms of both quantities and welfare.[3]

The third and final result describes how to design a nominal framework that uniquely implements the constant moneyless equilibrium real outcome. Again, the unique implementation is valid for any positive interval of firm pricing choices that

[1] See also Leeper's (1991) highly related discussion of active fiscal policy.

[2] A sequence of consumption paths $\{(C_t^k)_{t=1}^{\infty}\}_{k=1}^{\infty}$ converges datewise to zero if $\lim_{k\to\infty} C_t^k = 0$ for all t.

[3] This result (about the arbitrarily bad equilibria that are possible when fiscal policy is Ricardian) applies for *any* interest rate rule. Consequently, it has nothing to do with the existence of a lower bound—zero or otherwise—on the nominal interest rate, and so is not related to the results of Benhabib et al. (2001). Indeed, the range of indeterminacy exhibited in this paper is much broader than is established in those authors' work.

includes its lower bound. We know from the second result (described in the prior paragraph) that the framework must be non-Ricardian. As in prior work on non-Ricardian fiscal policies by Benhabib et al. (2002), I use a fiscal policy that targets the (long-run) growth rate of nominal liabilities.[4] Specifically, I consider any nominal framework in which the following conditions are fulfilled:

(i) The monetary policy rule is active (the implied real interest rate is a strictly increasing function of the inflation rate) when the inflation rate is above, at, or *slightly* below target.

(ii) The growth rate of the government's nominal liabilities converges over time from below to the target nominal interest rate (expected inflation is at target and the real interest rate equals the rate of time preference).

(I do not impose any restrictions on monetary policy when the inflation rate is more than slightly below target to allow for the possibility of a lower bound on the nominal interest rate.) Given such a nominal framework, I establish that, in any equilibrium, consumption is constant at the benchmark moneyless equilibrium level. Note that the last enumerated requirement means that fiscal policy is non-Ricardian, because the government's intertemporal budget constraint is not satisfied when the nominal interest rate is lower than the target nominal interest rate. This policy is being used to eliminate equilibria in which consumption is ever below its moneyless level.

Within the class of models studied in the paper, money has no transaction role.[5] Accordingly, the Friedman rule is always satisfied: the risk-adjusted real rate of return on money is the same as that on any other asset. The point of this paper is that even though the Friedman rule is always satisfied and prices are (arbitrarily) close to fully flexible, money can still be highly distortionary in this economy. The distortion arises because money's endogenous desirability as an asset is tied to household expectations about future inflation and output.

The policy conclusions in this paper are quite different from those reached in the New Keynesian literature (as elucidated in Gali 2015, for example). That literature takes as given that under any specification of monetary policy and Ricardian fiscal policy, aggregate outcomes remain close to a long-run zero inflation steady state. Its main conclusion is that, given this presumption, there are no equilibrium deviations from steady state under active monetary policy rules. In this paper, it is shown that *all* monetary policy rules, when combined with Ricardian fiscal policies, admit arbitrarily poor equilibrium outcomes. This finding about real outcomes echoes Cochrane's (2011) argument

[4]I restrict fiscal policies to date-contingent lump-sum nominal transfers/taxes and assume that the only government liability is (interest-bearing) money. Given this restriction, whether a fiscal policy is Ricardian has to do with the present value of the long-run nominal liabilities of the government, when calculated using the *lowest* possible time path of nominal interest rates. If that present value is zero, then the government's infinite-horizon intertemporal budget constraint is satisfied for any sequence of inflation rates, and becomes irrelevant for price level determination. Note that if the lowest time path of nominal interest rates is highly negative, then the long-run growth rate of nominal liabilities must also be highly negative (so that nominal liabilities converge to zero extremely rapidly) in a Ricardian fiscal policy.

[5]I relax this assumption in Appendix C without affecting the validity of the main results.

that if fiscal policy is Ricardian, then inflation is indeterminate in equilibrium for all monetary policy rules (active or not). So as to ensure macroeconomic stability, governments must follow non-Ricardian fiscal policy regimes (assuming that such regimes are possible).

In the next two sections, I set up the baseline model and describe the key results. In Section 4, I discuss implications of the results (with supporting technical details provided in Appendix B), connections to related literature, and sketch how to include non-interest-bearing currency in the model (with supporting technical details provided in Appendix C).

2. Models with a price-setting lower bound

In this section, I describe an infinite horizon monetary model in which all firms can choose prices from a common positive interval that includes its lower bound. The lower bound is defined relative to the prior period's price level. Hence, it ends up serving as a constraint on the time path of inflation rates. I define and characterize equilibria in this economy.

2.1 *Setup*

Consider an economy with a unit measure of households who live forever. Time is discrete and the households maximize the expected value of

$$\sum_{t=1}^{\infty} \beta^{t-1}\big(u(C_t) - v(N_t)\big), \quad 0 < \beta < 1,$$

where C_t is the consumption of a composite good in period t and N_t is labor in period t. Here, I assume that $u(0) = v(0) = 0$ and that

$$u', -u'', v', v'' > 0$$
$$\lim_{c\to 0} u'(c) = \infty$$
$$\lim_{c\to\infty} u'(c) = 0.$$

The composite good consists of a unit measure of consumption goods, indexed by j, and is defined as

$$C_t = \left(\int_0^1 c(j)^{1-1/\eta}\, dj\right)^{\frac{\eta}{\eta-1}}, \quad \eta > 1.$$

Each household's consumption of each good j is bounded from below by zero.

Each consumption good j is produced by a monopolistically competitive firm. A typical firm j has a technology at each date that converts x units of labor into x units of consumption good j for any $x \geq 0$. The households own equal shares of all firms. There is no entry or exit.

Labor markets are competitive and so, at each date, firms all hire workers at the same wage W_t (denominated in terms of dollars). Given that wage, firms simultaneously set prices for their consumption goods in terms of dollars.[6] The firms' problems are identical, and so they each choose the same price P_t in equilibrium; that price is also the aggregate price level. At date t, each firm j is constrained to choose its price subject to a lower bound:

$$p_t(j) \geq \pi_t^{\mathrm{LB}} P_{t-1}^*.$$

Here the bounds $\pi^{\mathrm{LB}} = (\pi_t^{\mathrm{LB}})_{t=1}^{\infty}$ form an exogenously specified sequence. The firm treats them and last period's (endogenously determined) price level P_{t-1}^* parametrically. I define the gross inflation rate π_t as P_t/P_{t-1} and (without loss of generality) set $P_0^* = 1$.

Monetary policy works as follows. Each household is initially endowed with $\bar{M}_0$ dollars. Like reserves at many central banks, money is interest-bearing. Specifically, at the beginning of period $(t+1)$, a household that has M_t dollars is paid $(R_t(\pi_t) - 1)M_t$ dollars. Here the interest rate rule $R = (R_t)_{t=1}^{\infty}$ is a sequence of exogenous (possibly time-dependent) weakly increasing continuous functions that map period t inflation into a period t gross nominal interest rate.[7] The range of R_t for any date t is restricted to be nonnegative (which translates into a zero lower bound on the *gross* nominal interest rate).

Finally, fiscal policy works as follows. The government's only liability is interest-bearing money. Let $\{\bar{M}_t\}_{t=1}^{\infty}$ be an arbitrary sequence of positive real numbers. At each date $(t+1)$, the government levies a lump-sum tax, in dollars, equal to

$$\tau_t(\pi_t) = \big(R_t(\pi_t) - 1\big)\bar{M}_t + (\bar{M}_t - \bar{M}_{t+1}).$$

This tax ensures that the per-household level of nominal government liabilities at the end of period $(t+1)$ is equal to $\bar{M}_{t+1}$.

2.2 *Equilibrium*

In this subsection, I define an equilibrium in this economy. To simplify the analysis, I restrict attention to nonstochastic but possibly time-dependent equilibria.

I refer to an interest rate rule and fiscal policy $(R, \bar{M})$ collectively as a *nominal framework*. Given its specification, an equilibrium in this economy is a vector sequence $(C^*, N^*, M^*, P^*, W^*)$, where (C^*, N^*, M^*) represent per-household consumption, labor, and money-holdings, and (P^*, W^*) represent price levels and wages. Given this vector sequence, it is useful to define the implied inflation, taxes and profits as

$$\pi_t^* = P_t^*/P_{t-1}^*$$

[6]Throughout, I treat money as the numeraire. If consumption or labor is the numeraire, then there is always an equilibrium in which the price of money in terms of that real numeraire is zero in all periods.

[7]As noted in the Introduction, there is no special transaction role for money. Hence, what I term "money" could also be seen as a perpetual bond, with a coupon payment in each period that is determined with reference to the inflation rate.

$$\tau_t^* = \bar{M}_t\big(R_t(\pi_t^*) - 1\big) + \bar{M}_t - \bar{M}_{t+1}$$
$$\Phi_t^* = \big(P_t^* - W_t^*\big)N_t^*.$$

The vector sequence satisfies the usual equilibrium conditions. First, (C^*, N^*, M^*) solve the household's optimization problem, given prices, wages, taxes, and profits that it treats as exogenous:

$$\begin{aligned}
\big(C^*, N^*, M^*\big) = \underset{(C,N,M)}{\operatorname{argmax}} \sum_{t=1}^{\infty} \beta^{t-1}\big(u(C_t) - v(N_t)\big) \\
\text{s.t. } P_t^* C_t + M_t = M_{t-1}R_{t-1}\big(\pi_{t-1}^*\big) + W_t^* N_t - \tau_{t-1}^* + \Phi_t^* \quad \forall t \geq 1, \text{w.p. } 1 \\
C_t, M_t, N_t \geq 0.
\end{aligned}$$

Second, in any date, P_t^* solves firm j's pricing period t problem, given W_t^* and last period's price index (which shapes the lower bound):

$$\begin{aligned}
P_t^* = \underset{P_t}{\operatorname{argmax}}\big(P_t^{1-\eta} - W_t^* P_t^{-\eta}\big) \\
\text{s.t. } P_t \geq \pi_t^{\mathrm{LB}} P_{t-1}^*.
\end{aligned}$$

Finally, markets must clear in all dates:

$$C_t^* = N_t^*$$
$$M_t^* = \bar{M}_t.$$

2.3 *A simple characterization of equilibrium*

In this economy, there are three decisions that are made each period: consumption–savings, consumption–labor, and price-setting. The first decision gives rise to the familiar Euler equation that leaves households marginally indifferent between consumption and money:

$$u'\big(C_t^*\big) = \beta R_t\big(\pi_t^*\big)\frac{u'\big(C_{t+1}^*\big)}{\pi_{t+1}^*}.$$

If we exploit goods–market clearing, the consumption–labor decision gives rise to a standard intratemporal first order condition

$$u'\big(C_t^*\big)w_t^* = v'\big(C_t^*\big).$$

Here w_t^* represents the period t real wage

$$w_t^* \equiv W_t^* / P_t^*.$$

The household saving decision also gives rise to a transversality condition[8] that leaves households marginally indifferent to permanent increases/reductions in their money-holdings:

$$\lim_{t\to\infty} \beta^t u'(C_t^*)\bar{M}_t/P_t^* = 0.$$

Finally, the price-setting decision on the part of the firm gives rise to the condition

$$P_t^* = \max\bigl(\pi_t^{\mathrm{LB}} P_{t-1}^*, (1-1/\eta)^{-1} W_t^*\bigr).$$

In words, the firm follows the usual markup formula unless doing so violates the lower bound on prices. If we divide through by P_{t-1}^*, we can rewrite this price-setting condition as

$$\pi_t^* = \max\bigl(\pi_t^{\mathrm{LB}}, (1-1/\eta)^{-1} w_t^* \pi_t^*\bigr).$$

By combining these conditions, we can present the following conclusion.

Proposition 1. *Given a nominal framework $(R, \bar{M})$, a consumption–inflation–real-wage sequence (C^*, π^*, w^*) is part of an equilibrium if and only if it satisfies the restrictions*

$$u'(C_t^*) = \beta R_t(\pi_t^*) u'(C_{t+1}^*)/\pi_{t+1}^*$$
$$w_t^* = \frac{v'(C_t^*)}{u'(C_t^*)}$$
$$\pi_t^* = \max\bigl(\pi_t^{\mathrm{LB}}, (1-1/\eta)^{-1} w_t^* \pi_t^*\bigr)$$

in all dates and the households' transversality condition is satisfied:

$$\lim_{t\to\infty} \frac{\beta^t u'(C_t^*)\bar{M}_t}{\prod_{s=1}^{t} \pi_s^*} = 0.$$

All proofs not provided in the text are given in Appendix A.

Proposition 1 shows that the model does not imply a tight short-run or medium-run connection between money growth and inflation. (The household's transversality condition does imply that, asymptotically, money growth cannot exceed the nominal interest rate.) This disconnect has nothing to do with the lower bound on inflation; rather, it is a consequence of assuming that money has no transaction role.[9] Empirically, it is consistent with recent data from many advanced economies, as inflation has remained low even though the monetary base has grown rapidly over the past 10 or more years.

[8]In writing the transversality condition in this way, I am implicitly restricting attention to equilibria in which the limit exists. See Kocherlakota (1992) for the relevant generalization.

[9]In Appendix C, I augment the model by adding a transaction role for currency (non-interest-bearing money). In that model, there is an intratemporal equilibrium restriction between the quantity of currency and the price level.

2.4 *Ricardian versus non-Ricardian nominal frameworks*

In what follows, it will be important to distinguish between nominal frameworks $(R, \bar{M})$ that are *Ricardian* and those that are *non-Ricardian.*

2.4.1 *Definitions* As in Woodford (1995, p. 26), a nominal framework is said to be Ricardian if the limiting present value of the government's nominal liabilities is guaranteed to be zero for any possible sequence of inflation rates. Intuitively, this restriction means that, like a household in the standard definition of competitive equilibrium, the government's (intertemporal) budget constraint is satisfied for all possible price level sequences. Since the nominal interest rule consists of a sequence of weakly increasing functions, the condition

$$\lim_{t\to\infty} \frac{\bar{M}_t}{\prod_{s=1}^{t} R_s(\pi_s^{\mathrm{LB}})} = 0 \tag{1}$$

is both necessary and sufficient to ensure that the nominal framework is Ricardian. Note that, for Ricardian nominal frameworks, the household's transversality condition is implied by the other equilibrium conditions in Proposition 1, because

$$\frac{1}{R_t(\pi_t^{\mathrm{LB}})} \geq \frac{1}{R_t(\pi_t^*)} = \frac{\beta u'(C_{t+1}^*)}{u'(C_t^*)\pi_{t+1}^*}.$$

This ensures that fiscal policy (that is, the specification of the path $\bar{M}$ of nominal liabilities) plays no role in the determination of equilibrium.

A non-Ricardian nominal framework is one in which the asymptotic growth rate of nominal liabilities is sufficiently high that the limit in (1) is positive. Under a non-Ricardian fiscal policy, it is impossible for the inflation rate to equal its minimal value for all dates in an equilibrium because such a sequence fails to satisfy the household's transversality condition. Intuitively, if the nominal liabilities are growing so rapidly while paying such a low nominal return, households would find it optimal to lower their money-holdings permanently.

2.4.2 *Examples* Here are three examples that are intended to illustrate what "Ricardian" means.[10] The first example demonstrates that apparently passive fiscal policies may not be Ricardian.

Example 1. Suppose $R_t(\pi) = 1$ for all $t \geq 1$ and all π, so that money pays no interest. Suppose too that $\bar{M}_t = \bar{M}_1$ for all $t \geq 1$. Under this fiscal policy, taxes are equal to zero forever. Nonetheless, this (apparently passive) fiscal policy is not Ricardian. Indeed, the households' transversality condition is not satisfied by any price level sequence, and so there is no equilibrium.[11] Intuitively, money is an intrinsically useless object in this

[10] I thank an anonymous referee for suggesting these examples.

[11] Note that this same nonexistence result applies for any sequence $\{\bar{M}_t\}_{t=1}^{\infty}$ such that $\lim_{t\to\infty} \bar{M}_t > 0$. If consumption were the numeraire, then there would always be an equilibrium in which the price of money is constant at zero.

example. It cannot have positive value in terms of labor or consumption, and so there cannot be an equilibrium in which money is the numeraire. ◊

The next example shows that an intrinsically useless durable object can have value if the government requires that it be used to pay taxes.

Example 2. Now consider the same monetary policy as in the prior example (so that money pays no interest). The government commits to any fiscal policy in which

$$\lim_{T\to\infty} \bar{M}_T = 0. \tag{2}$$

This fiscal policy implies that the government is removing the initial supply of nominal liabilities via taxation:

$$\bar{M}_1 = \sum_{t=1}^{\infty} \tau_t.$$

No matter how quickly or slowly the path of nominal liabilities converges to zero, this fiscal policy is Ricardian because (2) implies that the limiting present value of the household's nominal liabilities is zero for any price level sequence. (Note that the stock of nominal liabilities could fall to zero in finite time, in which case the numeraire would be a pure unit of account in the following periods.) This means that the household's transversality condition is satisfied even if the price level P_t is falling much faster than $\bar{M}_t$, so that the household's real wealth is converging to infinity. In these circumstances, the household is holding a large amount of real wealth because its tax burden is equally large in real terms. ◊

The following example shows that, unlike in Example 1, fiscal policy can be Ricardian when the stock of nominal liabilities is constant over time as long as those liabilities are interest-bearing.

Example 3. Suppose that money pays positive interest in every period, so that for any $t \geq 1$, $R_t(\pi) \geq (1+\epsilon)$ for all π. Then a fiscal policy such that $\bar{M}_t = \bar{M}_1$ for all $t > 1$ is Ricardian. The constant path of nominal liabilities implies that the government simultaneously levies a lump-sum tax equal to the interest being paid on the outstanding money. The household's transversality condition is satisfied for any price level sequence because if a household deviates by spending some extra money, it will not be able to afford to pay its lump-sum taxes in some future period. ◊

2.4.3 *Are non-Ricardian fiscal policies possible?* It remains a matter of some controversy among macroeconomists whether governments can, in fact, follow non-Ricardian policies (see Buiter and Sibert 2018) to select desired equilibria (in which money is the numeraire). As Kocherlakota and Phelan (1999) discuss, it is impossible using equilibrium observations to test whether a nominal framework is Ricardian. The household's transversality condition and the government intertemporal budget constraint have to be satisfied within an equilibrium. The question is about the limiting behavior of government nominal liabilities at *unobserved inflation sequences*.

3. Results

In this section, I describe the three main results of the paper. All three are valid regardless of how small the pricing lower bound is.

By way of context for these results, it is helpful to establish a familiar benchmark for the case in which there is no lower bound on prices.

Proposition 2. *Suppose that all firms were able to choose their prices from the set of positive reals (so that they do not face a lower bound). Then, given any nominal framework and in any equilibrium, $C_t^* = Y^{\text{real}}$, where Y^{real} satisfies the first order condition*

$$u'(Y^{\text{real}})(1-1/\eta) = v'(Y^{\text{real}}).$$

Proof. In the absence of a lower bound on prices, the price-setting condition in Proposition 1 becomes

$$\pi_t^* = w_t^*(1-1/\eta)^{-1}\pi_t^*.$$

Since π_t^* is finite and positive, we can conclude that

$$v'(C_t^*)/u'(C_t^*) = w_t^* = (1-1/\eta)$$

and that in any equilibrium, output is always equal to Y^{real}, where

$$u'(Y^{\text{real}})(1-1/\eta) = v'(Y^{\text{real}}). \qquad \square$$

The notation Y^{real} is meant to suggest that this level of output would be the equilibrium in a moneyless static economy in which, for example, labor was the numeraire. This result shows that the equilibrium allocation is independent of monetary and fiscal policy when there are no bounds on price-setting.[12]

Once we add the lower bound on prices, equilibrium output may be lower than Y^{real}. Proposition 1 implies that if $C_t^* < Y^{\text{real}}$, then $\pi_t^* = \pi_t^{\text{LB}}$. However, equilibrium output can never be higher than Y^{real}. Optimal firm pricing (as in Proposition 1) implies that

$$1 \geq w_t^*(1-1/\eta)^{-1}.$$

Since

$$w_t^* = v'(C_t^*)/u'(C_t^*),$$

we can conclude that

$$C_t^* \leq Y^{\text{real}}.$$

[12]It can be shown via contradiction that the uniqueness of equilibrium allocations established in Proposition 2 generalizes to the situation in which firms face a lower bound on *production* (as opposed to a lower bound on prices). Suppose that instead of producing Y^{real}, all firms are at a production lower bound Y_{LB} in equilibrium. Then their pricing first order condition is $\pi_t^* \leq (1-1/\eta)^{-1}w_t^*\pi_t^*$ (because firms are unable to lower production further by *raising* prices). It follows that $v'(Y_{\text{LB}})/u'(Y_{\text{LB}}) = w_t^* \geq (1-1/\eta) = v'(Y^{\text{real}})/u'(Y^{\text{real}})$. But this contradicts the standard assumptions that $v'', -u'' > 0$. (This same logic can be used to establish uniqueness of equilibrium allocations if firms face upper bounds on production.)

3.1 *The extreme relevance of the nominal framework*

In this subsection, I prove that if an output sequence is uniformly bounded from above by the moneyless equilibrium output level Y^{real}, then that sequence is an equilibrium for some Ricardian nominal framework $(R, \bar{M})$.

Proposition 3. *Let C^* be any positive consumption sequence such that $C_t^* \leq Y^{\text{real}}$ for all $t \geq 1$. Then there exists a Ricardian nominal framework $(R, \bar{M})$ such that C^* is part of an equilibrium given that framework.*

Proof. Define the equilibrium inflation rate π_t^* to be equal to π_t^{LB} for all dates. Define R to be any interest rate rule (really, sequence of interest rate pegs) so that for all $t \geq 1$ and all π,

$$R_t(\pi) = R_t^* \equiv \left(\beta^{-1} u'\left(C_t^*\right)/u'\left(C_{t+1}^*\right)\right)/\pi_{t+1}^{\text{LB}},$$

and define $\bar{M}$ to be

$$\bar{M}_t = \frac{\prod_{s=1}^{t} R_s^*}{t}.$$

Then define

$$w_t^* = v'\left(C_t^*\right)/u'\left(C_t^*\right).$$

It is readily verified, using Proposition 1, that (C^*, π^*, w^*) is part of an equilibrium given the nominal framework $(R, \bar{M})$. □

What happens in this proposition? Consider, by way of example, any period t in which

$$\beta R_t u'\left(C_{t+1}^*\right)/\pi_{t+1}^* > u'\left(Y^{\text{real}}\right).$$

This inequality says that the nominal return on money is sufficiently high, given households' low expectations for future consumption and inflation, to lead households to demand less consumption than Y^{real}. Given the low demand for consumption, firms bid down their prices as much as possible. This same force would be at work in a model without pricing bounds, but could not be reflected in an equilibrium. In a model with a pricing lower bound, it pushes π_t^* down to its lowest possible level.

Note that Proposition 3 is independent of the specification of the inflation lower bound sequence (π^{LB}). In other words, regardless of how near zero the inflation lower bound is, a macroeconomist has no information about how bad real economic outcome can be without having at least some information about the nominal framework, and, regardless of how near zero the inflation lower bound is, sufficiently poor choices of the nominal framework can lead to (arbitrarily) poor economic outcomes.

3.2 *Real indeterminacy in Ricardian nominal frameworks*

The result in the prior subsection shows that when firms face a lower bound on prices, it is only possible to ensure desirable real outcomes in equilibrium if the government has an appropriate nominal framework. This subsection asks a converse question: How bad can equilibrium outcomes be if the government is restricted to use "sensible" nominal frameworks? We shall see that the answer is very bad indeed.

In the remainder of the paper (not just this subsection), I restrict attention to environments and nominal frameworks that are time-invariant. Specifically, I assume that the price-setting lower bound is independent of time:

$$\pi_t^{\mathrm{LB}} = \pi_{\min}, \quad t \geq 1.$$

I require that the interest rate rule is time-invariant, so that there is a weakly increasing continuous function $\hat{R}$,

$$R_t(\pi) = \hat{R}(\pi)$$

for all t, where the range of $\hat{R}$ is the nonnegative reals.

A nominal framework with a time-invariant interest rate rule is said to target an inflation rate π^{TAR} if

$$\pi^{\mathrm{TAR}} > \pi_{\min}$$
$$\hat{R}(\pi^{\mathrm{TAR}}) = \beta^{-1}\pi^{\mathrm{TAR}}.$$

The following proposition shows that inflation-targeting regimes implement the "natural" outcomes in which the real outcome is constant at Y^{real} and inflation is constant at π^{TAR}.

Proposition 4. *Suppose $(R, \bar{M})$ is a Ricardian nominal framework with a time-invariant interest rate rule that targets π^{TAR}. Then there is an equilibrium consumption–real-wage–inflation sequence (C^*, w^*, π^*) such that for all t,*

$$C_t^* = Y^{\mathrm{real}}$$
$$w_t^* = (1 - 1/\eta)$$
$$\pi_t^* = \pi^{\mathrm{TAR}}.$$

It is readily verified that (C^*, w^*, π^*) satisfy the conditions in Proposition 1.

However, the next proposition demonstrates there can be many other equilibria associated with a Ricardian nominal framework with a time-invariant interest rate rule that targets π^{TAR}. As in the literature on the so-called neo-Fisherian determination of inflation rates,[13] it considers a time-invariant interest rate peg.

[13]See, among others, Garcia-Schmidt and Woodford (2019), Cochrane (2016), and Schmitt-Grohé and Uribe (2017).

Proposition 5. *Consider a Ricardian nominal framework with a time-invariant interest rate peg $\hat{R}$ such that*

$$\hat{R}(\pi) = \bar{R} \equiv \beta^{-1}\pi^{\text{TAR}} \quad \textit{for all } \pi$$

$$\bar{M}_t = \frac{\bar{R}^t}{t} \quad \textit{for all } t.$$

Then, for any horizon $K > 0$, there is a consumption–inflation sequence (C^{K*}, π^{K*}),

$$u'\left(C_t^{K*}\right) = \frac{\left(\pi^{\text{TAR}}\right)^{K-t} u'\left(Y^{\text{real}}\right)}{(\pi_{\min})^{K-t-1}\pi_K^{K*}}, \quad t < K$$

$$C_t^{K*} = Y^{\text{real}}, \quad t \geq K$$

$$\pi_t^{K*} = \pi_{\min}, \quad t \leq (K-1)$$

$$\pi_K^{K*} \in \left[\pi_{\min}, \pi^{\text{TAR}}\right)$$

$$\pi_t^{K*} = \pi^{\text{TAR}}, \quad t \geq K+1,$$

that is part of an equilibrium. The sequence of utilities $U^{K*} \equiv \sum_{t=1}^{\infty}\beta^{t-1}[u(C_t^{K*}) - v(C_t^{K*})]$ converges to zero.

Proof. If we set the real wage $w^{K*} = v'(C^{K*})/u'(C^{K*})$, then it is easy to check that $(C^{K*}, w^{K*}, \pi^{K*})$ satisfy the conditions of Proposition 1. □

The proposition shows that a constant interest rate peg admits a class of equilibria in which the initial level of economic activity is below Y^{real}. *Within* any of these equilibria, the economy converges in *finite* time to the targeted real outcome Y^{real} and the targeted inflation rate $\pi^{\text{TAR}} = \beta\bar{R}$. This convergence is consistent with the neo-Fisherian logic that under a nominal interest rate peg, the long-run inflation rate has to increase one-for-one with the level of the peg.

However, once we look *across* equilibria, we see that this result is highly misleading because convergence can take an arbitrarily long period of time. Indeed, as we drive K large, the limiting equilibria become extremely undesirable as the long run becomes irrelevant:

$$\lim_{K\to\infty} C_t^{K*} = 0$$

$$\lim_{K\to\infty} \pi_t^{K*} = \pi_{\min}.$$

Technically, given any date and any ϵ, we can find an equilibrium in which, at that date (and all earlier ones), consumption/output is less than ϵ and inflation is equal to its lowest possible level $\pi_{\min}$.

Intuitively, these near-zero equilibria are generated by the dynamic strategic complementarities within the model. For example, in the equilibrium indexed by K, firms set their prices in period K so that inflation is less than π^{TAR}. As a result, money has a high gross real rate of return (relative to $1/\beta$) from period $(K-1)$ to period K. Given that high real return for money, agents' demand for consumption in period $(K-1)$ is lower than Y^{real}. Firms respond to that low demand by bidding inflation down to its lowest possible level ($\pi_{\min}$) in period $(K-1)$. We can recurse backward using the same logic to generate the low-output–inflation outcomes in periods prior to K.

Of course, interest rate pegs are well known to have undesirable properties relative to interest rate rules that respond aggressively to the inflation rate (Sargent and Wallace 1975). But the following proposition shows that we can generalize Proposition 5 to *any* Ricardian nominal framework with a time-invariant interest rate rule

Proposition 6. *Consider any Ricardian nominal framework* $(R, \bar{M})$ *that has a time-invariant interest rate rule* $\hat{R}$ *that targets* π^{TAR}. *Then there exists a sequence (indexed by k) of consumption–inflation sequences* $(C^{k*}, \pi^{k*})_{k=1}^{\infty}$ *that are parts of equilibria and such that for all* $t \geq 1$,

$$C_t^{k*} \leq Y^{\text{real}}, \quad k \geq 1$$

$$\lim_{k\to\infty} C_t^{k*} = 0$$

$$\lim_{k\to\infty} U^{k*} = 0$$

$$\lim_{k\to\infty} \pi_t^{k*} = \pi_{\min},$$

where $U^{k*} \equiv \sum_{t=1}^{\infty} \beta^{t-1}[u(C_t^{k*}) - v(C_t^{k*})]$.

Proposition 6 shows that for a given Ricardian nominal framework, equilibrium outcomes can be arbitrarily bad in a welfare sense. Note that, like Proposition 3, it is valid for *any* specification of the lower bound $\pi_{\min}$ on inflation.

The proposition covers two possible scenarios. In the first, the real rate of return on money when inflation is constant at $\pi_{\min}$ is higher than the rate of time preference. This kind of monetary policy rule induces equilibria that resemble those described in Proposition 5. In the second scenario, the real rate of return on money when inflation is always at its lowest level $\pi_{\min}$ is less than or equal to the rate of time preference. In this situation, there is a class of equilibria in which consumption starts at some initial level C_1^* below Y^{real} and then stays at or below C_1^*. This class of equilibria converges to zero datewise if we take C_1^* to zero.

3.3 *Unique implementation*

The prior section emphasized that under Ricardian nominal frameworks, the model economy described in Section 2 gives rise to low-output–low-inflation equilibria that can be arbitrarily close to zero. In this subsection, I describe a class of nominal frameworks that, despite the dynamic complementarities, serve to implement a unique real outcome. It is independent of the lower bound on inflation (except for the knowledge that target inflation is above that lower bound).

Proposition 6 tells us that any such class must be restricted to non-Ricardian nominal frameworks. As noted earlier, many economists are uncomfortable with non-Ricardian fiscal policies. For those economists, Proposition 6 is really the end of the story of what happens once firms' pricing choices are restricted by a positive lower bound. Others (such as Cochrane 2011) have argued that non-Ricardian fiscal policy is essential for price level determinacy; the following proposition is congruent with this thinking.[14]

Proposition 7. *Consider a nominal framework $(R, \bar{M})$ with a time-invariant interest rate rule $\hat{R}$ that targets π^{TAR} and such that for some $\epsilon > 0$, the gross real interest rate*

$$\hat{R}(\pi)/\pi$$

is strictly increasing for $\pi \in [\pi^{\mathrm{TAR}} - \epsilon, \pi^{\mathrm{TAR}}]$. Suppose that fiscal policy $\bar{M}$ takes the form

$$\bar{M}_t = \frac{M_0}{t} \hat{R}\big(\pi^{\mathrm{TAR}}\big)^t$$

so that the rate of growth of nominal liabilities asymptotes from below to $(\hat{R}(\pi^{\mathrm{TAR}}) - 1)$. Then, in any equilibrium, $C_t^* = Y^{\mathrm{real}}$ for all $t \geq 1$.

The proposition shows that is possible to eliminate the bad equilibria with a two-pronged approach. First, fiscal policy is non-Ricardian: nominal liabilities grow so rapidly over time that households can improve their welfare by permanently reducing their money-holdings unless the long-run nominal return on money is at least $\hat{R}(\pi^{\mathrm{TAR}}) = \beta^{-1}\pi^{\mathrm{TAR}}$. Second, monetary policy is active when inflation is slightly below target. This ensures that if $C_t^* < Y^{\mathrm{real}}$, inflation converges to some rate that is strictly below π^{TAR}, so that the long-run nominal return on money has to be less than $\beta^{-1}\pi^{\mathrm{TAR}}$.

4. Discussion

In this section, I discuss three aspects of the above analysis: the broader lessons of the results (with supporting technical details in Appendix B), the relationship of the analysis to other work on adverse equilibria in macroeconomic models, and the robustness of the results to adding currency (with supporting technical details provided in Appendix C).[15]

4.1 *Two lessons*

In this subsection, I describe two kinds of conclusions that can be drawn from the above analysis.

[14] Proposition 7 can be readily extended to eliminate real outcomes other than Y^{real} in stochastic (sunspot) equilibria.

[15] It is possible to generalize the results in Section 3 to a model with variable demand elasticity of the kind used by Arkolakis et al. (2019). That generalization is available on request.

4.1.1 *Lesson 1: Change in policy implications* The most direct conclusion from the results in Section 3 is that the normative and positive implications of a standard class of macroeconomic models are highly sensitive to the inclusion of a lower bound (regardless of how small) into the constraint set of price-setting firms. Is this kind of lower bound a feature of actual economies? There is no clear evidence with which to address this question—one way or another—about whether boundary points are included in firm action sets. But that lack of evidence surely suggests that we should be willing to put *some* weight on the possibility that firm price-setting choice sets do contain a positive lower bound. Once we do so, the models' implications for appropriate monetary/fiscal policy change radically.

More explicitly, suppose a policymaker is choosing between any Ricardian nominal framework and a specific non-Ricardian nominal framework of the kind constructed in Proposition 7. If there is no lower bound included in the price-setting firms' constraint sets, the policymaker can choose either of these nominal frameworks, because (per Proposition 2) they both lead to the same unique (stable) real outcome. But if it is even *possible* that the firms' constraint sets include a lower bound, then the former (Ricardian) nominal framework *could* give rise to highly adverse macroeconomic outcomes. The policymaker would then find the non-Ricardian framework to be strictly optimal.

4.1.2 *Lesson 2: Revelation of otherwise hidden strategic forces* Proposition 2 proves that, regardless of the choice of nominal framework, there is a unique equilibrium allocation once we drop the lower bound on prices. In this subsection, I argue that this result is a (highly) misleading description of the possible strategic interactions among the firms in the model. Even if we remove the bounds, the model retains the powerful dynamic complementarities that play such a key role in Proposition 6. Those forces continue to push firms and households to coordinate on low-output–low-inflation outcomes, even though they are not formally in the equilibrium set.

The following game is a simple illustration of this line of reasoning. Suppose two players simultaneously choose actions from the set $(-\infty, \infty)$, and player i receives a payoff $4a_i a_j - a_i^2$, where $j \neq i$. There is a unique (pure strategy Nash) equilibrium in this game, in which $a_i = a_j = 0$.

But this apparent uniqueness result is misleading. The best response function to this game is that player i chooses $2a_j$, where $j \neq i$. If we iterate this function k times, starting at an initial vector (ϵ, ϵ) of actions, we arrive at the action vector $(2^k\epsilon, 2^k\epsilon)$. In this sense, the players find it desirable to coordinate on very large outcomes in absolute value, even though those outcomes are not in the equilibrium set. It is not surprising that if we add an upper bound to the action sets, so that they become $[0, a_{\max}]$, there is another equilibrium $\{a_{\max}, a_{\max}\}$ and it is the robust one.

In Appendix B, I show how this same logic applies in (a numerical parameterization of) the model in Sections 2 and 3 when we drop the lower bound on firm pricing decisions. I focus on the case in which the interest rate rule is time-invariant and active (so that there is no lower bound on the net nominal interest rate). In this dynamic environment, an iterated best response involves simultaneous choices at all dates by all households and firms. I show that iterating on the firms' best response function results in a limit in which consumption is zero at all dates. Note that this is the same extreme limit that we found in Proposition 6.

In summary, (standard) macro models feature strong downward pricing complementarities, as households' perceptions about the real return to money depend on their expectations about inflation and their shadow real interest rates depend on their expectations about future consumption. These forces serve to create the possibility of a macroeconomic "death spiral" of sorts. Appendix B demonstrates that these forces are present even when the firms do not face pricing lower bounds. But Propositions 2 and 6 together show that the full effect of these powerful complementarities becomes manifest *in the equilibrium set* only if firms' pricing decisions are bounded from below.

This is not the first paper to note the strong power of the intertemporal complementarities in representative agent macroeconomic models. They are, for example, the source of the so-called forward guidance puzzle (Del Negro et al. 2015 and McKay et al. 2016).[16]

4.2 *Other related literature*

In this subsection, I discussed the connections between this paper and other recent work on the potential for highly adverse outcomes in macroeconomic models.

Werning (2011) studies a continuous time version of a linearized Calvo model with an initial shock to the discount factor that persists over a finite horizon. As in Eggertsson and Woodford (2003), he considers an equilibrium in which output and inflation return to target after the shock ends. He shows that, conditional on this equilibrium, the initial response of output and inflation to a given shock becomes increasingly large as the fraction of price-changers grows closer to 1. (This is a sharp analytical characterization of what Eggertsson and Krugman 2012 call the paradox of flexibility.)

Cochrane (2017) reconsiders the impact of the zero lower bound on nominal interest rates in linearized continuous-time Calvo models. Like Werning (2011), he points out that in response to a negative shock to the natural real interest rate, there are multiple Pareto-ranked equilibria for a given interest rate path. Unlike Werning, Cochrane argues in favor of a selection based on bounding the size of the endogenous response to an initial adverse shock. Applying Cochrane's desideratum in this model would imply restricting attention to equilibria of the kind described in Proposition 4 (in which output equals Y^{real} and inflation is always equal to target).

The current paper can be seen as an extension of Werning (2011) and Cochrane (2017) in the following ways:

* The main Proposition 6 applies for any interest rate rule, regardless of whether it has a zero lower bound.

[16] Farhi and Werning (2019) and Gabaix (2020) analyze bounded rationality modifications of the standard model in which these intertemporal complementarities are dampened. There is an ongoing debate about how the magnitude of these intertemporal complementarities is affected by the introduction of incomplete financial markets; see, among others, Werning (2015) and Kaplan et al. (2018).

* As discussed in Section 4.1.2, this paper shows that the potential for highly adverse low-output outcomes is not tied to Calvo pricing per se. Rather, those outcomes are a natural consequence of the basic strategic complementarities associated with firms' pricing decisions.

* Cochrane suggests that non-Ricardian fiscal policy can be used to support his "muted response" equilibrium selection criterion. In contrast, I prove in Propositions 6 and 7 that unique implementation of desirable equilibrium outcomes *requires* the use of non-Ricardian fiscal policy.

Like this paper, Bassetto and Phelan (2015) show how adding plausible constraints to a macroeconomic model can give rise to additional undesirable equilibria. However, their focus is different: they show how central bank limits on household borrowing can give rise to *hyperinflationary* outcomes. As in this paper, the additional undesirable equilibria can be eliminated through the appropriate use of a non-Ricardian fiscal policy.

4.3 *Money and currency*

In the models described in Section 2, money has no liquidity role. Money is held only to pay lump-sum taxes levied by the government and pays the same real return as all other assets in the economy. It is nonetheless potentially distorting because households can contemplate off-equilibrium trades of consumption for money.

In reality, households do hold non-interest-bearing currency, and banks can always trade their interest-bearing reserves with the government for that currency. How would adding currency to the model affect the results obtained in Section 3? Suppose, in particular, that households get momentary utility from the real value of their currency holdings X according to a function

$$u_m(X/P).$$

This function is strictly increasing for $x \leq \bar{x}$ and satisfies

$$u_m(x) = u_m(\bar{x})$$

for all $x \geq \bar{x}$ (so that $\bar{x}$ can be viewed as a satiation level of real currency-holdings).

In Appendix C, I extend the model of Sections 2 and 3 by adding non-interest-bearing currency in this fashion. I show that Propositions 3 and 6 generalize to this extended model. I also prove a version of Proposition 7, in which the targeted gross inflation rate is above, but arbitrarily close to, the discount factor β (the Friedmanian rate of deflation). This last restriction ensures that real balances are approximately optimal in equilibrium.

5. Conclusion

Most macroeconomic models treat price-setting firms as being able to make their choices from the set of positive reals. In this paper, I instead consider a class of macroeconomic models in which firms choose prices from an interval that includes its positive lower bound. I find that the normative and positive implications of the models are highly sensitive to this perturbation, regardless of how close to zero the lower bound is. If the models have no pricing lower bound, there is a strong (albeit well known) irrelevance result: the equilibrium level of output is completely independent of the specification of monetary and fiscal policy. In contrast, in models with some positive lower bound on price-setting, the nature of the nominal framework matters greatly for the set of equilibrium real allocations. In particular, governments can then ensure macroeconomic stability only if they follow (appropriate) non-Ricardian fiscal policies.

Some readers may question the empirical relevance of the assumption that firms' pricing sets include positive lower bounds. But the analysis in this paper contains a more general message: even without the bounds, there are strong strategic forces in this (standard macro) model that are inducing firms to coordinate on arbitrarily low-output outcomes (see the discussion in Section 4 and Appendix B). Basically, if we omit boundary points from firm action sets, we risk "hiding" what would otherwise be completely natural equilibria (in the sense of being generic limit points of iterations of the best response function). The model's implications for appropriate policy should take these "missing" equilibria into account.

Accordingly, the results in this paper place new emphasis on an old question: can governments follow arbitrary non-Ricardian fiscal policies? If they have this capability, do they exploit it in reality? Addressing these questions in a compelling fashion will likely require a deeper modeling and understanding of fiscal policy than is incorporated into current macroeconomic theory.[17]

Appendix A

In this appendix, I gather the proofs of Propositions 1, 6, and 7.

Proof of Proposition 1

The necessity of the first order conditions is straightforward. The necessity of the transversality condition follows from a standard argument. Suppose

$$\lim_{t\to\infty} \beta^t u'(C_t^*)\bar{M}_t/P_t^* = L > 0.$$

I claim that it is possible to find a budget-feasible perturbation that makes the household better off. Thus, given ε in $(0, L)$, there exists T such that

$$\beta^t u'(C_t^*)\bar{M}_t/P_t^* > \varepsilon$$

for all $t \geq T$. Consider a perturbation whereby the household increases consumption at date t by $\beta^{-t}\varepsilon/u'(C_t^*)$, lowers M_t by $\beta^{-t}\varepsilon u'(C_t^*)^{-1}/P_t^*$, and lowers M_{t+s}, $s \geq 1$, by

$$\begin{aligned}
&\left(\varepsilon\beta^{-t}u'(C_t^*)^{-1}/P_t^*\right)\prod_{\tau=1}^{s} R_{t+\tau}^* \\
&\quad = \left(\varepsilon\beta^{-t}u'(C_t^*)^{-1}/P_t^*\right)\beta^{-s}\left(u'(C_t^*)/u'(C_{t+s}^*)\right)\left(P_t^*/P_{t+s}^*\right)
\end{aligned}$$

[17] Bassetto (2002) represents an early effort along these lines.

$$
\begin{aligned}
&= \frac{\varepsilon \beta^{-t-s}}{P^*_{t+s} u'(C^*_{t+s})} \\
&< \bar{M}_{t+s}.
\end{aligned}
$$

This perturbation is budget-feasible (because the household's money-holdings remain positive in all future periods).

The sufficiency of the price-setting first order condition as a solution to the firm's problem is obvious. The sufficiency of the other conditions for household optimality is by contradiction. Suppose (C', N', M') is budget-feasible and dominates $(C^*, C^*, \bar{M})$, so that

$$
0 < \lim_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}\big\{\big(u(C'_t) - v(N'_t)\big) - \big(u(C^*_t) - v(C^*_t)\big)\big\}.
$$

We can apply the subgradient inequality for concave functions,

$$
\begin{aligned}
0 &< \liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}\big(u'(C^*_t)(C'_t - C^*_t) - v'(C^*_t)(N'_t - C^*_t)\big) \\
&= \liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} u'(C^*_t)\big((C'_t - C^*_t) - W^*_t(N'_t - N^*_t)/P^*_t\big) \\
&= \liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1} u'(C^*_t)\big((M'_{t-1} - \bar{M}_{t-1})R^*_t/P^*_t - (M'_t - \bar{M}_t)/P^*_t\big) \\
&= \liminf_{T\to\infty} \beta^{T-1}\big(u'(C^*_T)(\bar{M}_T - M'_T)/P^*_T\big) \\
&\leq \liminf_{T\to\infty} \beta^{T-1} u'(C^*_T)\bar{M}_T/P^*_T \\
&= 0,
\end{aligned}
$$

where the penultimate step comes from the nonnegativity of M'. This contradiction proves the proposition.

Proof of Proposition 6

There are two distinct cases based on the magnitude of the parameter γ, defined as

$$
\gamma = \beta^{-1}\pi_{\min}/\hat{R}(\pi_{\min}). \tag{3}
$$

Suppose first that $\gamma \geq 1$ (so that the average real return to money is no higher than $1/\beta$ when inflation is at its lowest level). Define $(\lambda_k)_{k=1}^{\infty}$ to be any strictly increasing sequence that converges to infinity with initial $\lambda_1 > 1$. Define $(C_t^{k*})_{t=1}^{\infty}$ via the Euler equation:

$$
u'(C^{k*}_{t+1}) = \lambda_k \gamma^{t-1} u'(Y^{\text{real}}), \quad t = 1, 2, \ldots.
$$

We can readily verify that for all (k, t),

$$u'\left(C_{t+1}^{k*}\right) > u'\left(Y^{\text{real}}\right).$$

Define $\pi_t^{k*} = \pi_{\min}$ and $w_t^{k*} = v'(C_t^{k*})/u'(C_t^{k*})$ for all (k, t). Then we can verify, using the conditions in Proposition 1, that $(C^{k*}, w^{k*}, \pi^{k*})$ is part of an equilibrium. Note that for any t,

$$\lim_{k\to\infty} u'\left(C_t^{k*}\right) \geq \gamma^{t-1} u'\left(Y^{\text{real}}\right) \lim_{k\to\infty} \lambda_k = \infty,$$

and so $\lim_{k\to\infty} C_t^{k*} = 0 = \lim_{k\to\infty} U^{k*}$.

The second case is that, as defined in (3), $\gamma < 1$ (intuitively, the average long-run real return to money is higher than $1/\beta$ when inflation equals $\pi_{\min}$). In that case, let $\hat{\pi}$ satisfy

$$\pi_{\min} < \hat{\pi} < \pi^{\text{TAR}}$$
$$\beta\hat{R}(\pi_{\min}) > \hat{\pi}.$$

(There is such a value for $\hat{\pi}$ because $\pi^{\text{TAR}} = \beta\hat{R}(\pi^{\text{TAR}}) \geq \beta\hat{R}(\pi_{\min}) > \pi_{\min}$.) Pick any horizon $k > 1$. Given k, define an inflation sequence π^{k*} recursively as

$$\pi_{t+1}^{k*} = \beta\hat{R}\left(\pi_t^{k*}\right), \quad t \geq k$$
$$\pi_k^{k*} = \hat{\pi}$$
$$\pi_t^{k*} = \pi_{\min}, \quad t < k$$

and define a consumption sequence C^{k*} so that

$$C_t^{k*} = Y^{\text{real}}, \quad t \geq k$$
$$u'\left(C_t^{k*}\right) = \gamma^{t-k-1} \frac{\beta\hat{R}(\pi_{\min})u'\left(Y^{\text{real}}\right)}{\pi_k^{k*}}, \quad 1 \leq t < k.$$

Since $\gamma < 1$, $u'(C_t^{k*}) > u'(Y^{\text{real}})$ for $t < k$.

We know that

$$\pi_{k+1}^{k*} = \beta\hat{R}(\hat{\pi}) \geq \beta\hat{R}(\pi_{\min}) > \hat{\pi} = \pi_k^{k*}.$$

We know too that since $\pi^{\text{TAR}} > \pi_k^{k*}$, $\pi^{\text{TAR}} \geq \pi_{k+1}^{k*}$. Since $\hat{R}$ is weakly increasing, induction implies that

$$\pi^{\text{TAR}} \geq \pi_{t+1}^{k*} \geq \pi_t^{k*} \geq \pi_{\min}$$

for all $t \geq k$ and that the sequence $(\pi_t^{k*})_{t\geq k}$ converges (as t converges to infinity) to the smallest fixed point of $\beta\hat{R}$ that is larger than $\hat{\pi}$. We can then verify using Proposition 1 that (C^{k*}, π^{k*}) is part of an equilibrium.

It is clear that $\lim_{k\to\infty} \pi_t^{k*} = \pi_{\min}$ for all t, and since $\gamma < 1$, $\lim_{k\to\infty} u'(C_t^{k*}) = \infty$ for all t. It follows too that $\lim_{k\to\infty} U^{k*} = 0$.

Proof of Proposition 7

There are two cases.

Case 1: $\beta\hat{R}(\pi_{\min}) \leq \pi_{\min}$. The proof for this case is by contradiction. Suppose C^* is part of an equilibrium and $C_t^* < Y^{\text{real}}$. Then

$$\begin{aligned} u'(C_{t+1}^*) &= \frac{\beta^{-1}u'(C_t^*)}{\hat{R}(\pi_{\min})}\pi_{t+1}^* \\ &= \frac{\beta^{-1}u'(C_t^*)}{\hat{R}(\pi_{\min})/\pi_{\min}}\frac{\pi_{t+1}^*}{\pi_{\min}} \\ &\geq u'(C_t^*). \end{aligned}$$

Hence, by induction, $\pi_{t+s}^* = \pi_{\min}$ for all $s \geq 0$. The households' transversality condition requires that

$$\begin{aligned} 0 &= u'(C_t^*) \lim_{s\to\infty} \frac{\beta^s u(C_{t+s}^*)\bar{M}_{t+s}}{u'(C_t^*)(\pi_{\min})^s P_t^*} \\ &= u'(C_t^*) \lim_{s\to\infty} \frac{M_0(t+s)^{-1}\hat{R}(\pi^{\text{TAR}})^s}{\hat{R}(\pi_{\min})^s} \\ &= \infty, \end{aligned}$$

which is a contradiction.

Case 2: $\beta\hat{R}(\pi_{\min})/\pi_{\min} > 1$. Define $\hat{\pi} \in (\pi_{\min}, \pi^{\text{TAR}} - \epsilon)$ so that it satisfies

$$\begin{aligned} \beta\hat{R}(\hat{\pi}) &= \hat{\pi} \\ \beta\hat{R}(\pi) &> \pi \end{aligned}$$

for all π in $[\pi_{\min}, \hat{\pi})$. We know such a $\hat{\pi}$ exists because $\beta\hat{R}(\pi^{\text{TAR}} - \epsilon) < (\pi^{\text{TAR}} - \epsilon)$ and $\beta\hat{R}(\pi_{\min}) > \pi_{\min}$.

Suppose $C_t^* < Y^{\text{real}}$ at some date t. I show first, by contradiction, that there is some $s \geq 0$ such that $C_{t+s+1}^* = Y^{\text{real}}$. Suppose not. Then for all $s \geq 0$,

$$u'(C_{t+s+1}^*) = [\beta^{-1}\hat{R}(\pi_{\min})^{-1}\pi_{\min}]^{s+1}u'(C_t^*).$$

But this implies that $u'(C_{t+s+1}^*)$ is lower than $u'(Y^{\text{real}})$ for s sufficiently large, which is the desired contradiction.

Hence, there is some $s \geq 0$ such that $C_{t+s+1}^* = Y^{\text{real}}$ and $C_{t+s}^* < Y^{\text{real}}$. It follows that

$$\begin{aligned} \pi_{t+s+1}^* &= \beta\hat{R}(\pi_{\min})u'(C_{t+s+1}^*)/u'(C_{t+s}^*) \\ &< \beta\hat{R}(\pi_{\min})u'(C_{t+s+1}^*)/u'(Y^{\text{real}}) \\ &\leq \beta\hat{R}(\hat{\pi}) \\ &= \hat{\pi}. \end{aligned}$$

Since $\beta\hat{R}(\pi^*_{t+s+1}) > \pi^*_{t+s+1}$, we can conclude that

$$\frac{u'(C^*_{t+s+2})}{\pi^*_{t+s+2}} = \left[\beta\frac{\hat{R}(\pi^*_{t+s+1})}{\pi^*_{t+s+1}}\right]^{-1}\frac{u'(Y^{\text{real}})}{\pi^*_{t+s+1}}$$
$$< u'(Y^{\text{real}})/\pi^*_{t+s+1}$$

and so $C^*_{t+s+2} = Y^{\text{real}}$. Hence,

$$\begin{aligned}\pi^*_{t+s+2} &= \beta\hat{R}(\pi^*_{t+s+1})\\ &< \beta\hat{R}(\hat{\pi})\\ &= \hat{\pi}.\end{aligned}$$

By induction, we can conclude that for all $r \geq 1$,

$$\begin{aligned}C^*_{t+s+r} &= Y^{\text{real}}\\ \pi^*_{t+s+r+1} &= \beta\hat{R}(\pi^*_{t+s+r}),\end{aligned}$$

where π^*_{t+s+1} is specified as above. The sequence $(\pi^*_{t+s+r})_{r=1}^{\infty}$ is strictly increasing and is bounded from above by $\hat{\pi}$. Hence, it converges to $\hat{\pi}$ (the smallest fixed point of $\beta\hat{R}$ that is greater than $\pi_{\min}$).

To be an equilibrium, the households' transversality condition must be satisfied:

$$\begin{aligned}0 &= \lim_{T\to\infty}\bar{M}_{t+s+T}\Big/\prod_{r=1}^{T}\hat{R}(\pi^*_{t+s+r})\\ &\geq \lim_{T\to\infty}\bar{M}_{t+s+T}/\big(\hat{R}(\hat{\pi})\big)^T\\ &= \lim_{T\to\infty}\frac{\bar{M}_0\hat{R}(\pi^{\text{TAR}})^{t+s+T}}{(t+s+T)\big(\hat{R}(\hat{\pi})\big)^T}\\ &= \infty.\end{aligned}$$

But this is a contradiction: the nominal liabilities are growing too fast to be consistent with an equilibrium in which inflation is bounded from above by $\hat{\pi}$. It follows that there cannot be any $C^*_t < Y^{\text{real}}$.

Appendix B

In this appendix, I consider the limits of iterated best responses in a numerical example of the model in Sections 2 and 3 without a lower bound on firm prices.

The utility functions (u, v) are defined as

$$\begin{aligned}u(C) &= 2(1-1/\eta)^{-1}C^{1/2}\\ v(N) &= N^2.\end{aligned}$$

Under this parameterization, $Y^{\text{real}} = 1$. There is a time-invariant interest rate rule $\hat{R}$ defined by

$$\hat{R}(\pi) = \beta^{-1}\pi^{\text{TAR}}\frac{\pi^{\alpha+1}}{(\pi^{\text{TAR}})^{\alpha+1}}, \quad \alpha > 0,$$

for some π^{TAR}. It follows that $\hat{R}$ obeys the Taylor principle and that $\hat{R}(\pi^{\text{TAR}})/\pi^{\text{TAR}} = 1/\beta$.

Within this numerical example, I explore the limits of iterated best responses (where, in this dynamic setting, each iteration specifies choices for all firms and households at all dates). In the first iteration, I suppose that all firms at date t set their prices equal to $(\bar{\pi}^1)^t$, where $\bar{\pi}^1$ is time-invariant and less than π^{TAR}. (Recall that the initial price $P_0^* = 1$.) Households then best respond to the resulting low real interest rates through falling consumption sequences, under the presumption that fiscal policy is Ricardian. Through the households' labor supply condition, those falling consumption choices give rise to implied real wages in each period, which become a sequence of nominal wages when multiplied by the price level sequence $((\bar{\pi}^1)^t)_{t=1}^{\infty}$. All firms then choose their best response $(\bar{\pi}^2)^t$ by maximizing their profits in response to this nominal wage sequence.

Suppose inductively that in a hypothetical kth iteration, all firms at date $t \geq 1$ choose their prices equal to $(\bar{\pi}^k)^t$, where $\bar{\pi}^k < \pi^{\text{TAR}}$. Suppose too that households react by demanding a falling consumption sequence that is everywhere below Y^{real}:

$$C_t^k = C_0^k\left(\frac{\bar{\pi}^k}{\pi^{\text{TAR}}}\right)^{2\alpha t}, \quad t \geq 1$$

$$C_0^k \leq Y^{\text{real}}.$$

It is readily verified that this shrinking consumption sequence satisfies the Euler equation for any specification of C_0^k. (The sequence is budget-feasible for any C_0^k because fiscal policy is Ricardian.)

The resulting implied real-wage sequence w^k is

$$w_t^k = v'(C_t^k)/u'(C_t^k) = \frac{(C_0^k)^{3/2}\left(\frac{\bar{\pi}^k}{\pi^{\text{TAR}}}\right)^{3\alpha t}}{(1-1/\eta)^{-1}}, \quad t \geq 1.$$

This real-wage sequence, combined with the firms' pricing choices in the kth iteration, implies that the nominal wage sequence in the kth iteration is

$$W_t^k = (C_0^k)^{3/2}\frac{\left(\frac{\bar{\pi}^k}{\pi^{\text{TAR}}}\right)^{3\alpha t}(\bar{\pi}^k)^t}{(1-1/\eta)^{-1}}, \quad t \geq 1.$$

The update is that a firm's pricing best response in the $(k+1)$st iteration to this nominal wage sequence is given by

$$P_t^{k+1} = (C_0^k)^{3/2}\left(\frac{\bar{\pi}^k}{\bar{\pi}^{\text{TAR}}}\right)^{3\alpha t}(\bar{\pi}^k)^t, \quad t \geq 1.$$

The implied inflation rate in the $(k+1)$st iteration is again a time-invariant constant:

$$\begin{aligned}\pi_{t+1}^{k+1} &= \bar{\pi}^{k+1} \\ &\equiv \left(\frac{\bar{\pi}^{k}}{\bar{\pi}^{\text{TAR}}}\right)^{3\alpha} \bar{\pi}^{k} \\ &= \bar{\pi}^{\text{TAR}} \left(\frac{\bar{\pi}^{k}}{\bar{\pi}^{\text{TAR}}}\right)^{3\alpha+1}.\end{aligned}$$

By recursing over k, we can conclude that

$$\frac{\bar{\pi}^{k+1}}{\pi^{\text{TAR}}} = \left(\frac{\bar{\pi}^{1}}{\pi^{\text{TAR}}}\right)^{(3\alpha+1)k},$$

where $\bar{\pi}^1$ is the initial seed (less than π^{TAR}) for the best response iteration. (Here, in the term on the right hand side, the superscript k is an exponent.) Then, as we iterate over k, the gross inflation rate converges datewise to zero, and the households' desired consumption sequence,

$$\begin{aligned}C_t^{k+1} &= C_0^k \left(\frac{\bar{\pi}^{1}}{\pi^{\text{TAR}}}\right)^{2\alpha t(3\alpha+1)k}, \quad t \geq 1 \\ &< Y^{\text{real}} \left(\frac{\bar{\pi}^{1}}{\pi^{\text{TAR}}}\right)^{2\alpha t(3\alpha+1)k},\end{aligned}$$

converges datewise to zero (not to Y^{real}) as k converges to infinity.

Appendix C

In this appendix, I describe an extension of the baseline model to include non-interest-bearing currency. I then provide generalizations of the main results (Propositions 3, 6, and 7) for this model.

C.1 *Setup*

The setup of the model is the same as in Section 2, except for the definitions of monetary and fiscal policy.

In particular, suppose a household has M_t dollars at the end of period t. At that point in time, the household can split these dollars into non-interest-bearing currency X_t and interest-bearing reserves $(M_t - X_t)$. In period t, the household derives momentary utility from its real holdings of currency at the end of the period,

$$u_m\big(X_t/P_t^*\big),$$

where u_m is bounded from below by zero and satisfies

$$u_m(x) = u_m(\bar{x}), \quad x \geq \bar{x}$$
$$u_m', -u_m'' > 0 \quad \text{if } x < \bar{x}$$
$$\lim_{x \to 0} u_m'(x) = \infty.$$

Note that the household is satiated with currency when its real currency holdings exceed the exogenous level $\bar{x}$.

At the beginning of period $(t+1)$, a household that has $(M_t - X_t)$ in reserves is paid $(R_t(\pi_t) - 1)(M_t - X_t)$ dollars in interest. As in Section 2, the interest rate rule $R = (R_t)_{t=1}^{\infty}$ is a sequence of exogenous (possibly time-dependent) weakly increasing continuous functions that map period t inflation into a period t gross nominal interest rate. I do not impose a lower bound on $(M_t - X_t)$. (Intuitively, household can borrow from the government up to a limit defined by its holdings of currency.) Hence, to ensure household optimality, the range of R_t for any date t is restricted to be $[1, \infty)$ (which translates into a zero lower bound on the *net* nominal interest rate).

Fiscal policy works as follows. Let $\{\bar{M}_t\}_{t=1}^{\infty}$ be an arbitrary sequence of positive real numbers. At each date $(t+1)$, the government levies a lump-sum tax, in dollars, equal to

$$\tau_t(\pi_t, X_t) = \big(R_t(\pi_t) - 1\big)(\bar{M}_t - X_t) + (\bar{M}_t - \bar{M}_{t+1}).$$

This tax ensures that for any (π_t, X_t), the per-household level of nominal government liabilities at the end of period $(t+1)$ is equal to $\bar{M}_{t+1}$.

C.2 *Definition of equilibrium*

In this subsection, I define an equilibrium in this economy with non-interest-bearing currency.

As in Section 2, I refer to an interest rate rule and fiscal policy $(R, \bar{M})$ collectively as a *nominal framework*. Given its specification, an equilibrium in this economy is a vector sequence $(C^*, N^*, M^*, X^*, P^*, W^*)$, where (C^*, N^*, M^*, X^*) represent per-household consumption, labor, money-holdings, and currency-holdings, and (P^*, W^*) represent price levels and wages. Given this vector sequence, it is useful to define the implied inflation, taxes, and profits as

$$\pi_t^* = P_t^* / P_{t-1}^*$$
$$\tau_t^* = \big(\bar{M}_t - X_t^*\big)\big(R_t\big(\pi_t^*\big) - 1\big) + \bar{M}_t - \bar{M}_{t+1}$$
$$\Phi_t^* = \big(P_t^* - W_t^*\big)N_t^*.$$

The vector sequence satisfies the same equilibrium conditions as in the body of the paper. First, (C^*, N^*, M^*, X^*) solve the household's optimization problem, given prices, wages, taxes, and profits that it treats as exogenous:

$$
\begin{aligned}
&\left(C^*, N^*, M^*, X^*\right) \\
&\quad = \underset{(C,N,M)}{\operatorname{argmax}} \sum_{t=1}^{\infty} \beta^{t-1}\left(u(C_t) - v(N_t) + u_m\left(X_t/P_t^*\right)\right) \\
&\qquad \text{s.t. } P_t^* C_t + M_t = (M_{t-1} - X_{t-1}) R_{t-1}\left(\pi_{t-1}^*\right) + X_{t-1} + W_t^* N_t - \tau_{t-1}^* + \Phi_t^* \quad \forall t \geq 1 \\
&\qquad\quad C_t, M_t, N_t, X_t \geq 0.
\end{aligned}
$$

Second, in any date, P_t^* solves firm j's pricing problem in period t, given W_t^* and the last period's price index (which shapes the bounds):

$$
\begin{aligned}
P_t^* &= \underset{P_t}{\operatorname{argmax}}\left(P_t^{1-\eta} - W_t^* P_t^{-\eta}\right) \\
&\quad \text{s.t. } \pi_t^{\mathrm{LB}} \leq P_t / P_{t-1}^*.
\end{aligned}
$$

Finally, markets must clear in all dates:

$$
\begin{aligned}
C_t^* &= N_t^* \\
M_t^* &= \bar{M}_t.
\end{aligned}
$$

C.3 *Characterization of equilibrium*

In this subsection, I provide an explicit simple characterization of equilibrium outcomes in this economy with currency.

In this economy, the household's currency-holding X_t satisfies the first order condition

$$
\begin{aligned}
\frac{u_m'\left(X_t/P_t^*\right)}{P_t^*} &= \beta \frac{u'\left(C_{t+1}^*\right)}{P_{t+1}^*}\left(R_t\left(\pi_t^*\right) - 1\right) \quad \text{if } R_t\left(\pi_t^*\right) > 1 \\
X_t/P_t^* &\geq \bar{x} \quad \text{if } R_t\left(\pi_t^*\right) = 1.
\end{aligned}
$$

Given this first order condition, we can show using the logic of Proposition 1 that given a nominal framework $(R, \bar{M})$, a consumption–inflation–real-wage sequence (C^*, π^*, w^*, X^*) is part of an equilibrium if and only if it satisfies the restrictions

$$
\begin{aligned}
u'\left(C_t^*\right) &= \beta R_t\left(\pi_t^*\right) u'\left(C_{t+1}^*\right) / \pi_{t+1}^* \\
w_t^* &= \frac{v'\left(C_t^*\right)}{u'\left(C_t^*\right)} \\
\pi_t^* &= \max\left(\pi_t^{\mathrm{LB}}, (1 - 1/\eta)^{-1} w_t^* \pi_t^*\right) \\
X_t^* &= P_t^* u_m'^{-1}\left(\frac{\beta u'\left(C_{t+1}^*\right)}{\pi_{t+1}^*}\left(R_t\left(\pi_t^*\right) - 1\right)\right) \quad \text{if } R_t\left(\pi_t^*\right) > 1 \\
&\geq P_t^* \bar{x} \quad \text{if } R_t\left(\pi_t^*\right) = 1
\end{aligned}
$$

in all dates and the households' transversality condition is satisfied:

$$\lim_{t\to\infty} \frac{\beta^t u'(C_t^*)\bar{M}_t}{\prod_{s=1}^{t} \pi_s^*} = 0.$$

As in the body of the paper, we distinguish between nominal frameworks $(R, \bar{M})$ that are *Ricardian* and those that are *non-Ricardian*. Since the nominal interest rate rules are known to be bounded below by 1, we define a framework to be Ricardian if

$$\lim_{t\to\infty} \bar{M}_t = 0.$$

C.4 *Generalization of Proposition 2*

In the following proposition, I consider what happens in the model with non-interest-bearing currency if we drop the lower bound on prices.

Proposition 8. *Suppose that all firms were able to choose their prices from the set of positive reals (so that they do not face a lower bound). Then, given any nominal framework, in any equilibrium, $C_t^* = Y^{\text{real}}$, where Y^{real} satisfies the first order condition*

$$u'(Y^{\text{real}})(1 - 1/\eta) = v'(Y^{\text{real}}).$$

Households are satiated with real balances in equilibrium if and only if $R_t(\pi_t^*) = 1$ for all dates.

Proof. In the absence of a lower bound on prices, the price-setting condition becomes

$$\pi_t^* = w_t^*(1 - 1/\eta)^{-1}\pi_t^*.$$

Since π_t^* is finite and positive, we can conclude that

$$v'(C_t^*)/u'(C_t^*) = w_t^* = (1 - 1/\eta)$$

and that, in any equilibrium, output is always equal to Y^{real}, where

$$u'(Y^{\text{real}})(1 - 1/\eta) = v'(Y^{\text{real}}).$$

It is straightforward that households are satiated with real balances at date t, so that $X_t^* = P_t^*\bar{x}$, if and only if $R_t(\pi_t^*) = 1$. □

If there are no pricing bounds, then the equilibrium allocation of consumption and labor is independent of the nominal framework. But unlike the model without currency, there is a notion of optimal monetary policy: the central bank should simply follow the Friedman rule by setting the gross nominal interest rate equal to 1. This policy ensures that households are satiated with their currency-holdings. We shall see, though, that once we add any positive lower bound for price-setting, the nominal framework will matter for the consumption–labor allocation and for currency-holdings.

C.5 *Generalization of Proposition 3*

In this subsection, I generalize Proposition 3 to the model economy in which agents can hold currency.

Proposition 9. *Let C^* be any positive consumption sequence such that $C_t^* \leq Y^{\text{real}}$ for all $t \geq 1$. Then there exists a Ricardian nominal framework $(R, \bar{M})$ such that C^* is part of an equilibrium given that framework. In any such equilibrium, real currency-holdings at date t are bounded from above by*

$$u_m'^{-1}\left(u'\left(C_t^*\right) - \frac{\beta u'\left(C_{t+1}^*\right)}{\pi_{t+1}^{\text{LB}}}\right) < \bar{x}$$

if $u'(C_t^*)\pi_{t+1}^{\text{LB}} > \beta u'(C_{t+1}^*)$.

Proof. In period 1, set $R_0(\pi) = 1$ for all π and set $\pi_1 = \pi_1^{\text{LB}}$. Consider any date $t \geq 1$. Suppose

$$u'\left(C_t^*\right) \geq \beta u'\left(C_{t+1}^*\right)/\pi_{t+1}^{\text{LB}}.$$

Then set the period t interest rate rule so that

$$R_t(\pi) = \frac{u'\left(C_t^*\right)\pi_{t+1}^{\text{LB}}}{\beta u'\left(C_{t+1}^*\right)}$$

for all π and set $\pi_{t+1}^* = \pi_{t+1}^{\text{LB}}$.

Suppose instead that

$$u'\left(C_t^*\right) < \beta u'\left(C_{t+1}^*\right)/\pi_{t+1}^{\text{LB}}.$$

Then set $R_t(\pi) = 1$ for all π and set

$$\pi_{t+1}^* = \beta u'\left(C_{t+1}^*\right)/u'\left(C_t^*\right).$$

Define the price level $P_t^* = \prod_{s=1}^t \pi_s^*$. Then we can solve for currency-holdings as

$$\begin{aligned} X_t &= \bar{x}P_t^* \quad \text{if } R_t = 1 \\ &= P_t^* u_m'^{-1}\left(\frac{\beta u'\left(C_{t+1}^*\right)}{\pi_{t+1}^*}(R_t - 1)\right) \quad \text{if } R_t > 1. \end{aligned}$$

Define the time path of nominal liabilities $\bar{M}$ to be

$$\bar{M}_t = \frac{1}{t}.$$

Then define

$$w_t^* = v'\left(C_t^*\right)/u'\left(C_t^*\right).$$

It is readily verified that (C^*, π^*, w^*, X^*) is part of an equilibrium given the nominal framework $(R, \bar{M})$.

Now consider any equilibrium in which C^* is the allocation of consumption. In that equilibrium, consider any date t such that

$$\frac{u'(C_t^*)\pi_{t+1}^{\text{LB}}}{\beta u'(C_{t+1}^*)} > 1.$$

In that date, then real currency-holdings are

$$\begin{aligned} u_m'^{-1}\left(\frac{\beta u'(C_{t+1}^*)}{\pi_{t+1}^*}\big(R_t(\pi_t^*) - 1\big)\right) &= u_m'^{-1}\left(u'(C_t^*) - \frac{\beta u'(C_{t+1}^*)}{\pi_{t+1}^*}\right) \\ &\leq u_m'^{-1}\big(u'(C_t^*) - \beta u'(C_{t+1}^*)/\pi_{t+1}^{\text{LB}}\big). \end{aligned}$$

□

C.6 *Generalization of Proposition 6*

In this subsection, I generalize Proposition 6. As I did in Section 3, I now restrict attention to environments and nominal frameworks that are time-invariant. Specifically, I assume that the price-setting lower bound is independent of time:

$$\pi_t^{\text{LB}} = \pi_{\min}, \quad t \geq 1.$$

I require that the interest rate rule is time-invariant, so that there is a weakly increasing continuous function $\hat{R}$,

$$R_t(\pi) = \hat{R}(\pi)$$

for all t. Recall that the existence of currency implies that the nominal interest rate rule is bounded from below by 1, and so I set

$$\hat{R}(\pi_{\min}) = 1$$

and assume that the time path of nominal liabilities is any sequence $\bar{M}$ that converges to zero. A nominal framework with a time-invariant interest rate rule is said to target an inflation rate π^{TAR} if

$$\pi^{\text{TAR}} > \pi_{\min}$$
$$\hat{R}(\pi^{\text{TAR}}) = \beta^{-1}\pi^{\text{TAR}} \geq 1.$$

The following proposition then generalizes Proposition 6.

Proposition 10. *Consider any Ricardian nominal framework $(R, \bar{M})$ that has a time-invariant interest rate rule $\hat{R}$ that targets π^{TAR}. Then there exists a sequence (indexed by k) of consumption–inflation sequences $(C^{k*}, \pi^{k*})_{k=1}^{\infty}$ that are parts of equilibria and such that for all $t \geq 1$,*

$$C_t^{k*} \leq Y^{\text{real}}, \quad k \geq 1$$

$$\lim_{k\to\infty} C_t^{k*} = 0$$
$$\lim_{k\to\infty} U^{k*} = 0$$
$$\lim_{k\to\infty} \pi_t^{k*} = \pi_{\min},$$

where $U^{k*} \equiv \sum_{t=1}^{\infty} \beta^{t-1}[u(C_t^{k*}) - v(C_t^{k*})]$. *In these sequences of equilibria, real currency-holdings* x^{k*} *converge datewise to satiation, so that for all* t,

$$\lim_{k\to\infty} x_t^{k*} = \lim_{k\to\infty} X_t^{k*}/P_t^{k*} = \bar{x}.$$

Proof. There are two distinct cases based on the magnitude of the parameter γ, defined as

$$\gamma = \beta^{-1}\pi_{\min}. \tag{4}$$

Suppose first that $\gamma \geq 1$ (so that the average real return to money is no higher than $1/\beta$ when inflation is at its lowest level). Define $(\lambda_k)_{k=1}^{\infty}$ to be any strictly increasing sequence that converges to infinity with initial $\lambda_1 > 1$. Define $(C_t^{k*})_{t=1}^{\infty}$ via the Euler equation

$$u'(C_{t+1}^{k*}) = \lambda_k \gamma^{t-1} u'(Y^{\text{real}}), \quad t = 1, 2, \ldots.$$

We can readily verify that for all (k, t),

$$u'(C_{t+1}^{k*}) > u'(Y^{\text{real}}).$$

Define $\pi_t^{k*} = \pi_{\min}$, $w_t^{k*} = v'(C_t^{k*})/u'(C_t^{k*})$, and

$$x_t^{k*} = \bar{x}$$

for all (k, t).

Given this definition, it is readily verified that (C^{k*}, X^{k*}, P^{k*}) is part of an equilibrium given $(\hat{R}, \bar{M})$. Note that for any t,

$$\lim_{k\to\infty} u'(C_t^{k*}) \geq \gamma^{t-1} u'(Y^{\text{real}}) \lim_{k\to\infty} \lambda_k = \infty$$

and so $\lim_{k\to\infty} C_t^{k*} = 0 = \lim_{k\to\infty} U^{k*}$.

The second case is that, as defined in (4), $\gamma < 1$ (intuitively, the average long-run real return to money is higher than $1/\beta$ when inflation equals $\pi_{\min}$). In that case, let $\hat{\pi}$ satisfy

$$\pi_{\min} < \hat{\pi} < \beta.$$

Pick any horizon $k > 1$. Given k, define an inflation sequence π^{k*} recursively as

$$\pi_{t+1}^{k*} = \beta\hat{R}(\pi_t^{k*}), \quad t \geq k$$
$$\pi_k^{k*} = \hat{\pi}$$
$$\pi_t^{k*} = \pi_{\min}, \quad t < k,$$

and define a consumption sequence C^{k*} so that

$$C_t^{k*} = Y^{\text{real}}, \quad t \geq k$$

$$u'(C_t^{k*}) = \gamma^{t-k-1} \frac{\beta u'(Y^{\text{real}})}{\pi_k^{k*}}, \quad 1 \leq t < k.$$

Since $\gamma < 1$, $u'(C_t^{k*}) > u'(Y^{\text{real}})$ for $t < k$.

We know that

$$\pi_{k+1}^{k*} = \beta \hat{R}(\hat{\pi}) \geq \beta > \hat{\pi} = \pi_k^{k*}.$$

We know too that since $\pi^{\text{TAR}} > \pi_k^{k*}$, $\pi^{\text{TAR}} \geq \pi_{k+1}^{k*}$. Since $\hat{R}$ is weakly increasing, induction implies that

$$\pi^{\text{TAR}} \geq \pi_{t+1}^{k*} \geq \pi_t^{k*} \geq \pi_{\min}$$

for all $t \geq k$ and that the sequence $(\pi_t^{k*})_{t \geq k}$ converges, with respect to t, to the smallest fixed point of $\beta\hat{R}$ that is larger than $\hat{\pi}$. We can then readily verify that (C^{k*}, π^{k*}) is part of an equilibrium. Define the equilibrium price level path to be

$$\begin{aligned} P_t^{k*} &= (\pi_{\min})^t, \quad t < k \\ &= (\pi_{\min})^{k-1}\hat{\pi}, \quad t = k \\ &= (\pi_{\min})^{k-1}\hat{\pi} \prod_{s=k+1}^{t} \pi_s^{k*}, \quad t > k. \end{aligned}$$

Then we can define currency-holdings to be

$$\begin{aligned} X_t^{k*} &= \bar{x} P_t^{k*} \quad \text{if } \hat{R}(\pi_t^{k*}) = 1 \\ &= P_t^{k*} u_m'^{-1}\left(u'(C_t^{k*}) - \frac{\beta u'(C_{t+1}^{k*})}{\pi_{t+1}^{k*}}\right) \quad \text{if } \hat{R}(\pi_t^{k*}) > 1. \end{aligned}$$

We can readily verify that (C^{k*}, X^{k*}, P^{k*}) is an equilibrium given $(\hat{R}, \bar{M})$. It is clear that $\lim_{k\to\infty} \pi_t^{k*} = \pi_{\min}$ for all t, and since $\gamma < 1$, $\lim_{k\to\infty} u'(C_t^{k*}) = \infty$ for all t. It follows too that $\lim_{k\to\infty} U^{k*} = 0$. In contrast, for all t, $\lim_{k\to\infty} X_t^{k*}/P_t^{k*} = \bar{x}$, so that real currency-holdings converge to satiation at each date. □

In this proposition, in the limit, the government is following the Friedman rule (the gross nominal interest rate equals 1) and households are satiated with real balances. But consumption and labor are zero in the limit. Thus, the economy may have extremely poor outcomes even though the central bank (approximately) follows the Friedman rule.

C.7 *Generalization of Proposition 7*

In this subsection, I provide a generalization of Proposition 7 for the model with non-interest-bearing currency. As in that proposition, unique implementation requires the interest rate rule to be active for inflation rates slightly below target. But this rules out the possibility that households are satiated with currency when inflation is at target. The following proposition deals with this issue by showing that, given any δ close to zero but positive, it is possible to find an interest rate rule that both implements Y^{real} as a unique consumption–labor equilibrium allocation and uniquely implements a constant gross nominal interest rate $(1+\delta)$ in any equilibrium in which inflation is bounded from above.

PROPOSITION 11. *Suppose $\delta > 0$ such that $(1+\delta)\beta < 1$, and suppose $\beta > \pi_{\min}$. Consider a nominal framework $(R, \bar{M})$ with a time-invariant interest rate rule $\hat{R}_\delta$ such that it targets $\pi_\delta^{\text{TAR}} = \beta(1+\delta)$:*

$$\hat{R}_\delta(\pi_\delta^{\text{TAR}}) = (1+\delta)$$
$$\pi_\delta^{\text{TAR}} = \beta(1+\delta).$$

Suppose also that interest rate rule is active for inflation rates above or slightly below target, so that the gross pseudo-real interest rate

$$\beta\hat{R}_\delta(\pi)/\pi$$

is strictly increasing for all $\pi \in [\beta(1+\delta/2), \infty)$. Finally, suppose that fiscal policy $\bar{M}$ takes the form

$$\bar{M}_t = \frac{M_0}{t}(1+\delta)^t.$$

Then, in any equilibrium, $C_t^* = Y^{\text{real}}$ for all $t \geq 1$, and in any equilibrium with bounded inflation, real currency-holdings satisfy

$$X_t^*/P_t^* = u_m'^{-1}\left(u'(Y^{\text{real}})\frac{\delta}{1+\delta}\right)$$

for all $t \geq 1$.

PROOF. We first prove that in any equilibrium, $C_t^* = Y^{\text{real}}$. Note that $\beta > \pi_{\min}$ and $\hat{R}_\delta(\pi_{\min}) = 1$. Hence,

$$\beta\hat{R}_\delta(\pi_{\min}) > \pi_{\min}$$
$$\beta\hat{R}_\delta(\beta(1+\delta/2)) < \beta(1+\delta/2).$$

This implies that we can define $\hat{\pi}_\delta \in (\pi_{\min}, \beta(1+\delta/2))$ so that it satisfies

$$\beta\hat{R}_\delta(\hat{\pi}_\delta) = \hat{\pi}_\delta$$
$$\beta\hat{R}_\delta(\pi) > \pi$$

for all π in $[\pi_{\min}, \hat{\pi}_\delta)$.

Suppose $C_t^* < Y^{\text{real}}$ at some date t. I show first, by contradiction, that there is some $s \geq 0$ such that $C_{t+s+1}^* = Y^{\text{real}}$. Suppose not. Then for all $s \geq 0$,

$$u'\left(C_{t+s+1}^*\right) = \left[\beta^{-1}\hat{R}_\delta(\pi_{\min})^{-1}\pi_{\min}\right]^{s+1} u'\left(C_t^*\right).$$

But this implies that $u'(C_{t+s+1}^*)$ is lower than $u'(Y^{\text{real}})$ for s sufficiently large, which is the desired contradiction.

Hence, there is some $s \geq 0$ such that $C_{t+s+1}^* = Y^{\text{real}}$ and $C_{t+s}^* < Y^{\text{real}}$. It follows that

$$\begin{aligned}
\pi_{t+s+1}^* &= \beta\hat{R}_\delta(\pi_{\min})u'\left(C_{t+s+1}^*\right)/u'\left(C_{t+s}^*\right) \\
&< \beta\hat{R}_\delta(\pi_{\min})u'\left(Y^{\text{real}}\right)/u'\left(Y^{\text{real}}\right) \\
&\leq \beta\hat{R}_\delta(\hat{\pi}_\delta) \\
&= \hat{\pi}_\delta.
\end{aligned}$$

Since $\beta\hat{R}_\delta(\pi_{t+s+1}^*) > \pi_{t+s+1}^*$, we can conclude that

$$\begin{aligned}
\frac{u'\left(C_{t+s+2}^*\right)}{\pi_{t+s+2}^*} &= \left[\beta\frac{\hat{R}_\delta\left(\pi_{t+s+1}^*\right)}{\pi_{t+s+1}^*}\right]^{-1}\frac{u'\left(Y^{\text{real}}\right)}{\pi_{t+s+1}^*} \\
&< u'\left(Y^{\text{real}}\right)/\pi_{t+s+1}^*,
\end{aligned}$$

and so $C_{t+s+2}^* = Y^{\text{real}}$. Hence,

$$\begin{aligned}
\pi_{t+s+2}^* &= \beta\hat{R}_\delta\left(\pi_{t+s+1}^*\right) \\
&< \beta\hat{R}_\delta(\hat{\pi}_\delta) \\
&= \hat{\pi}_\delta.
\end{aligned}$$

By induction, we can conclude that for all $r \geq 1$,

$$\begin{aligned}
C_{t+s+r}^* &= Y^{\text{real}} \\
\pi_{t+s+r+1}^* &= \beta\hat{R}_\delta\left(\pi_{t+s+r}^*\right),
\end{aligned}$$

where $\pi_{t+s+1}^* = \beta\hat{R}_\delta(\pi_{\min})$. The sequence $(\pi_{t+s+r}^*)_{r=1}^\infty$ is strictly increasing and converges to $\hat{\pi}_\delta$.

To be an equilibrium, the households' transversality condition must be satisfied:

$$\begin{aligned}
0 &= \lim_{T\to\infty}\bar{M}_{t+s+T}\Big/\prod_{r=1}^{T}\hat{R}_\delta\left(\pi_{t+s+r}^*\right) \\
&\geq \lim_{T\to\infty}\bar{M}_{t+s+T}/\left(\hat{R}_\delta(\hat{\pi}_\delta)\right)^T \\
&= \bar{M}_t\lim_{T\to\infty}\left(t/(t+T)\right)\left(\frac{(1+\delta)}{\hat{R}_\delta(\hat{\pi}_\delta)}\right)^T \\
&= \infty.
\end{aligned}$$

But this is a contradiction: the nominal liabilities are growing too fast to be consistent with an equilibrium in which inflation is bounded from above by $\hat{\pi}_\delta$. It follows that in any equilibrium, $C_t^* = Y^{\text{real}}$ for all t.

I next show that in any equilibrium in which inflation is bounded from above, $\pi_1 = \pi_\delta^{\text{TAR}} = \beta(1+\delta)$. Suppose not. Then there are three cases, depending on the initial level of inflation.

In the first case, suppose $\beta\hat{R}_\delta(\pi_1) \leq \pi_1 < \beta(1+\delta)$. Then the sequence defined by

$$\pi_{t+1} = \beta\hat{R}_\delta(\pi_t)$$

is weakly decreasing and converges to a limit that is smaller than $\beta(1+\delta)$. But this is a violation of the transversality condition.

In the second case, suppose $\pi_1 < \beta\hat{R}_\delta(\pi_1)$ and $\pi_1 < \beta(1+\delta)$. Then the sequence defined by

$$\pi_{t+1} = \beta\hat{R}_\delta(\pi_t)$$

is strictly increasing. It converges to a limit that is necessarily smaller than $\beta(1+\delta/2)$, and again this is a violation of the transversality condition.

Finally, suppose $\pi_1 > \beta(1+\delta)$. Then $\pi_1 > \beta\hat{R}_\delta(\pi_1)$ and the sequence defined by

$$\pi_{t+1} = \beta\hat{R}_\delta(\pi_t)$$

is strictly increasing. If it were bounded, then the sequence would converge to a fixed point larger than π_δ^{TAR}, but there is no such fixed point (given that $\beta R(\pi)/\pi$ is strictly increasing).

We can conclude that inflation π_t equals $\beta(1+\delta)$ for all $t \geq 1$ in any equilibrium with bounded inflation. It follows that in any equilibrium with bounded inflation, the nominal interest rate is constant at $\hat{R}(\beta(1+\delta)) = (1+\delta)$. Accordingly, in any such equilibrium, real currency-holdings are constant at

$$X_t^*/P_t^* = u_m'^{-1}\left(u'\left(Y^{\text{real}}\right)\frac{\delta}{1+\delta}\right). \qquad \square$$

The restriction to equilibria with bounded inflation in the above proposition is ad hoc. However, it is possible, as in Bassetto and Phelan (2015), to use a richer notion of non-Ricardian fiscal policy to ensure that all equilibria have bounded inflation.[18] Suppose, for example, that we specify $\pi_{\max} > \beta(1+\delta)$ and require that if $\pi_t > \pi_{\max}$, then $\bar{M}_{t+s} = \bar{M}_t \prod_{j=0}^{s-1} R_{t+j}$. Then any path for inflation that exceeds $\pi_{\max}$ cannot be an equilibrium, because it gives rise to a violation of the household's transversality condition.

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