Theoretical Economics 16 (2021), 1095–1137 1555-7561/20211095

Market power and welfare in asymmetric divisible good auctions

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We analyze a divisible good uniform-price auction that features two groups, each with a finite number of identical bidders, who compete in demand schedules. In the linear-quadratic-normal framework, this paper presents conditions under which the unique equilibrium in linear demands exists and derives novel comparative statics results that highlight the interaction between payoff and information parameters with asymmetric groups. We find that the strategic complementarity in the slopes of traders' demands is reinforced by inference effects from prices, and we display the role of payoff and information asymmetries in explaining deadweight losses. Furthermore, price impact and the deadweight loss need not move together, and market integration may reduce welfare. The results are consistent with the available empirical evidence.

Keywords. Demand/supply schedule competition, private information, liquidity auctions, treasury auctions, electricity auctions, market integration.

JEL classification. D44, D82, G14, E58.

1. Introduction

Divisible good auctions are common in many markets, including government bonds, liquidity (refinancing operations), electricity, and emission markets. In those auctions, both market power (price impact) and asymmetries among the participants are impor-

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For helpful comments, we are grateful to Roberto Burguet, Maryam Farboodi, Vitali Gretschko, Jakub Kastl, Leslie Marx, Meg Meyer, Antonio Miralles, Stephen Morris, Andrea Prat, and Tomasz Sadzik as well as seminar participants at the BGSE Summer Forum, Chicago, Columbia, Como Information Economics Workshop, EARIE, ESSET, Federal Reserve Board, Jornadas de Economía Industrial, Northwestern, Princeton, UPF, Queen Mary Theory Workshop, Stanford, UC Berkeley, UC San Diego, and Wharton. Special thanks to three anonymous referees for detailed comments that helped to improve the presentation of our results. We are also indebted to Jorge Paz for excellent research assistance. Manzano acknowledges financial support from the Research Centre on Economics and Sustainability (ECO-SOS), the Ministry of Economy and Competitiveness (ECO 2016-75410-P AEI/FEDER, UE), the Spanish Ministry of Science and Innovation (PID2019-105982GB-I00 AEI/FEDER, UE), the Universitat Rovira i Virgili, and the Generalitat de Catalunya (2019PFR-URV-B2-53 and 2018PFR-URV-B2-53). Vives acknowledges financial support from the Ministry of Science, Innovation and Universities PGC2018-096325-B-I00 (MCIU/AEI/FEDER, UE) and the Generalitat de Catalunya, AGAUR Grants 2014SGR1496 and 2017SGR1244.

 https://doi.org/10.3982/TE3675

tant; asymmetries can make price impact relevant even in large markets. However, theoretical work in this area has been hampered by the difficulties of dealing with bidders who are asymmetric, have market power, and are competing in terms of demand or supply schedules in the presence of private information. This paper helps to fill that research gap by analyzing uniform-price auctions in which there are two asymmetric groups of bidders with interdependent values. Our aims are to characterize the equilibrium, derive novel comparative statics results that highlight the interaction between payoff and information parameters with asymmetric agents, perform a welfare analysis (from the standpoint of revenue and deadweight loss), and, finally, draw implications for policy.

Divisible good auctions are typically populated by heterogeneous participants in a concentrated market, and often we can distinguish a core group of bidders together with a fringe. Bidders from the former have better information, endure lower transaction costs,[1] and are more oligopsonistic (or oligopolistic) than members of the fringe. Treasury auctions are a leading example of the application of our model. Uniform-price auctions are often used in such auctions. Liquidity auctions and wholesale electricity markets provide other applications of our modelling. Wholesale electricity markets tend to use uniform-price auctions, are concentrated with asymmetric sellers exercising significant market power, and incomplete information on costs (e.g., on plant outages) is relevant (Cramton and Stoft 2007, Holmberg and Wolak 2018).

Treasury auctions have bidders with significant market shares who exercise market power; typically participants in these auctions can be divided into two distinct groups, which differ in terms of transaction costs and quality of information. These features are present in systems with a primary dealership, where participation is limited to a fixed number of bidders (this occurs, for example, in 29 out of 39 countries surveyed by Arnone and Iden 2003).[2] In particular, primary dealers enjoy an information advantage because they aggregate the information of indirect bidders and face lower transaction costs (see Hortaçsu and Kastl 2012 for evidence from Canadian treasury auctions). There is also evidence that there is bid synchronization among bidders of a certain group (see Armantier and Sbaï 2006 for French treasury auctions, Hamao and Jegadeesh 1998 for Japanese treasury auctions, and Kastl 2011 for Czech treasury auctions).[3] Furthermore, according to Hortaçsu et al. (2018), primary dealers systematically bid lower prices than

[1]We use the term "transaction costs" to refer to costs of changing the asset position of a trader, which encompass inventory, adjustment, opportunity, or limit to arbitrage costs. See, e.g., Du and Zhu (2017a), Rostek and Weretka (2015), and Vives (2011).

[2]In U.S. Treasury auctions, which are uniform-price auctions since 1998, the top five bidders typically purchase close to half of U.S. Treasury issues (see Malvey and Archibald 1998). Primary dealers underwent a substantial reduction going from 46 in 1998 to 23 presently. Those account for a very substantial portion of volume (from 69 to 88% of tendered quantities in the sample of Hortaçsu et al. 2018 for the years 2009–2013). Indirect bidders place their bids through the primary dealers and other direct bidders tender from 6 to 13%.

[3]Hamao and Jegadeesh (1998) show bid synchronization among Japanese banks in the Japanese government bond primary market. They argue that a plausible explanation for this bidding behavior is the fact that the Japanese investment banks have similar information or apply similar models to analyze information. Cao and Lu (2004) also find bid synchronization among large bidders in Canadian treasury auctions.

the other participants in the auction, not because they have a lower valuation of the securities, but because they exercise market power.

Our paper makes progress within the linear-Gaussian family of models by incorporating bidders' asymmetries with regard to payoffs and information. We model a uniform-price auction where asymmetric strategic bidders compete in terms of demand schedules for an inelastic supply (we can easily accommodate supply schedule competition for an inelastic demand as well as a double auction). We consider a model in which the equilibrium is privately revealing, that is, where the signal received by a trader and the price are a sufficient statistic for the trader. This allows us to focus the analysis on the inefficiencies derived from private information and market power, with no information externality present. Our modelling allows us to disentangle the price impact from the inference effects of traders, who have market power and private information, and who use price-contingent strategies.

Bidders may differ in their valuations, transaction costs, and/or the precision of their private information. With an empirical basis, we reduce heterogeneity to two groups; within each group, agents are identical and receive the same signal. This information structure is consistent with the above-mentioned empirical evidence in Hamao and Jegadeesh (1998) and Cao and Lu (2004), which tends to suggest the presence of a group with very correlated signals and high precision, and another group with low correlation and poor or uninformative signals. We seek to identify the conditions under which there exists a linear equilibrium with symmetric treatment of agents in the same group (i.e., we are looking for equilibria such that demand functions are both linear and identical among individuals of the same type). After showing that any such equilibrium must be unique, we derive comparative statics results.

We identify two basic forces that drive the comparative statics of a parameter change: a basic strategic effect of strategic complementarity in the slopes of demands submitted by traders, which is present with complete information (e.g., Back and Zender 1993), and a price inference effect, when there is incomplete information and learning from the price, which tends to reinforce the first effect. Our contribution is to characterize novel comparative statics across groups, and to identify the co-movements of payoff and information parameters (e.g., in a crisis situation) that magnify the impact of parameter changes.

More specifically, our analysis establishes that the number of group members, the transaction costs, the extent to which bidders' valuations are correlated, and the precision of private information affect the sensitivity of traders' demands to private information and prices. For example, we find that when valuations are more correlated, all groups react less to the private signal and to the price. Furthermore, if the transaction costs or the noise in the signal of a group increase, then the traders of the other group respond by diminishing their reaction to private information and submitting steeper demand schedules.[4] As discussed later in the paper, this result is consistent with the empirical findings of Cassola et al. (2013) in European post subprime crisis liquidity auctions. Increases in transaction costs, correlation of values, and noise in the signals, all

[4]A "steeper demand schedule" should be interpreted, as usual in economics, as a higher slope of *inverse* demand.

descriptive of a crisis situation, result in steepening demand schedules and illiquidity. We also find that if there is a core group of bidders with more precise private information, lower transaction costs, and more oligopsonistic, then the members of that group react more (than the bidders of the other group) to the private signal and to the price, and have more price impact. This result is consistent with the evidence in Hortaçsu et al. (2018) that primary dealers exercise market power.

When there is both a small and a large group of bidders, then the former (oligopsonistic) group has more price impact and, yet, even the latter (the large group) does not behave competitively, since it retains some price impact due to incomplete information, whenever there is learning from prices. However, the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders of both groups becomes large.

Finally, we provide a welfare analysis. First, we characterize the deadweight loss at the equilibrium and show how a subsidy scheme may induce an efficient allocation. We find that if there is a core group of bidders (as previously defined), then it should garner a higher per capita subsidy rate; the reason is that traders in the core group behave more strategically and so must be compensated more to become competitive. The paper also underscores how bidder heterogeneity (in terms of information, preferences, or group size documented in previous work) may increase deadweight losses. In particular, when the core group values the asset at least as much as the fringe, the deadweight loss increases with the quantity auctioned and also with the extent of expected valuation asymmetries. We also find that price impact need not move together with deadweight losses under asymmetry as is usually implicitly assumed in applied work. Furthermore, we provide conditions under which market integration increases or decreases welfare. Market integration is always welfare improving if bidders behave competitively or if the bidder groups are symmetric. However, the result may not hold if bidders have market power, the amount auctioned is large, and the groups are asymmetric. In such a case, gains from trade of integration may be overwhelmed by the inefficiency generated by group asymmetries and price impact.

Our work is related to the literature on divisible good auctions. Results in symmetric pure common value models are obtained by Wilson (1979), Back and Zender (1993), and Wang and Zender (2002), among others. Kastl (2011) extends the Wilson model to consider discrete bids in an independent values context. This model is extended in Hortaçsu and Kastl (2012) and Hortaçsu et al. (2018).

Results in interdependent values models with symmetric bidders are obtained by Vives (2011, 2014) and Ausubel et al. (2014), for example. Vives (2011), while focusing on the tractable family of linear-Gaussian models, shows how increased correlation in traders' valuations increases the price impact of those traders. Bergemann et al. (2021) generalize the information structure in Vives (2011), while retaining the assumption of symmetry. Rostek and Weretka (2012, 2015) partially relax that assumption and replace it with a weaker "equicommonality" assumption on the matrix correlation among the agents' values.[5] Du and Zhu (2017a) consider a dynamic auction model with expost

[5]This assumption states that the sum of correlations in each column of this matrix (or, equivalently, in each row) is the same and that the variances of all traders' values are also the same. Unlike our model,

equilibria. For the case of complete information, progress has been made in divisible good auction models by characterizing linear supply function equilibria (e.g., Klemperer and Meyer 1989, Akgün 2004, Anderson and Hu 2008). Kyle (1989) incorporates incomplete information by considering a Gaussian model of a divisible good *double* auction in which some bidders are privately informed and others are uninformed. Andreyanov and Sadzik (2021) study the design of robust exchange mechanisms in a two-type model similar to the one we present here.

To sum up, the two papers closest to ours are Vives (2011) and Kyle (1989). The novelty of our paper with respect to Vives (2011) is that in our model we allow asymmetries among bidders; with respect to Kyle (1989), that we consider interdependent values instead of a common value setup with non-optimizing liquidity traders in a double auction.

The rest of our paper is organized as follows. Section 2 outlines the model. Section 3 characterizes the equilibrium, analyzes its existence and uniqueness, and derives comparative statics results. We develop the welfare analysis in Section 4 and address the case of an oligopsony with a large fringe in Section 5. Section 6 concludes. Proofs are gathered in the Appendix, and the reader can find more results and details of the analysis in our working paper (Manzano and Vives 2019).

2. The model

Traders, of whom there are a finite number, face an inelastic supply for a risky asset. Let Q denote the aggregate quantity supplied in the market. In this market, there are buyers of two types: type 1 and type 2. We use i to refer to a generic type of bidders and use j for the other type. Thus, in what follows, $i, j = 1, 2$ and $j \neq i$. Suppose that there are n_i traders of type i. In that case, if the asset's price is p, then the profits of a representative type-i trader who buys x_i units of the asset are given by

$$\pi_i = (\theta_i - p)x_i - \lambda_i x_i^2/2, \quad x_i \in \mathbb{R}.$$

So, for any trader of type i, the marginal benefit of buying x_i units of the asset is $\theta_i - \lambda_i x_i$, where θ_i denotes the valuation of the asset and $\lambda_i > 0$ reflects an adjustment for transaction costs or opportunity costs (or a proxy for risk aversion). Traders maximize expected profits and submit demand schedules, after which an auctioneer selects a price that clears the market.[6]

We assume that θ_i is normally distributed with mean $\overline{\theta}_i$ and variance σ_θ^2. The random variables θ_1 and θ_2 may be correlated, with correlation coefficient $\rho \in [0, 1]$. Therefore, $\text{Cov}(\theta_1, \theta_2) = \rho\sigma_\theta^2$.[7] All type-$i$ traders receive the same noisy signal $s_i = \theta_i + \varepsilon_i$,

Rostek and Weretka's (2012) model maintains the symmetry assumption with regard to transaction costs and the precision of private signals. The equilibrium they derive is, therefore, still symmetric because all traders use identical strategies.

[6]The case of supply schedule competition for an inelastic demand is easily accommodated by considering negative demands ($x_i < 0$) and a negative inelastic supply ($Q < 0$). In this case, a producer of type i has a quadratic production cost $-\theta_i x_i + \lambda_i x_i^2/2$.

[7]The value of ρ depends on the type of security. In this sense, Bindseil et al. (2009) argue that the common value component is less important in a central bank repossession (repo) auction than in a treasury bill (T-bill) auction.

where ε_i is normally distributed with null mean and variance $\sigma^2_{\varepsilon_i}$. Error terms in the signals are uncorrelated across groups ($\mathrm{Cov}(\varepsilon_1, \varepsilon_2) = 0$) and are also uncorrelated with valuations of the asset ($\mathrm{Cov}(\varepsilon_i, \theta_j) = 0$ and $\mathrm{Cov}(\varepsilon_i, \theta_j) = 0$). In what follows, let $\widehat{\sigma}^2_{\varepsilon_i} \equiv \sigma^2_{\varepsilon_i}/\sigma^2_{\theta}$.

In our model, two traders of distinct types may differ in several respects:

- different willingness to possess the asset ($\theta_1 \neq \theta_2$)
- different transaction costs ($\lambda_1 \neq \lambda_2$)
- different levels of precision of private information ($\sigma^2_{\varepsilon_1} \neq \sigma^2_{\varepsilon_2}$).

Applications of this model are treasury auctions and liquidity auctions. For treasury auctions, θ_i is the private value of the securities to a bidder of type i; that value incorporates not only the resale value, but also idiosyncratic elements, as different liquidity or portfolio immunization needs of bidders in the two groups. Financial intermediaries may assign different values to the treasury instruments according to their use as collateral. In particular, primary dealers may attach a value to a bond beyond resale value to be used as collateral in operations with the Federal Reserve (Fed). For liquidity auctions, θ_i is the price (or interest rate) that group i commands in the secondary interbank market (which is over-the-counter). Here λ_i reflects the structure of a counterparty's pool of collateral in a repo auction. A bidder bank prefers to offer illiquid collateral to the central bank in exchange for funds; as allotments increase, however, the bidder must offer more liquid types of collateral, which have a higher opportunity cost. This yields a declining marginal utility (see Ewerhart et al. 2010).

3. Equilibrium

Denote by X_i the strategy of a type-i bidder, which is a mapping from the signal space to the space of demand functions. Thus, $X_i(s_i, \cdot)$ is the demand function of a type-i bidder that corresponds to a given signal s_i. Given her signal s_i, each bidder in a Bayesian equilibrium chooses a demand function that maximizes her conditional expected profit (while taking as given the other traders' strategies).[8] Our attention is restricted to anonymous linear Bayesian equilibria in which strategies are linear and identical among traders of the same type (for short, equilibria).

Definition. An *equilibrium* is a linear Bayesian equilibrium such that the demand functions for traders of type i are identical and equal to

$$X_i(s_i, p) = b_i + a_i s_i - c_i p,$$

where b_i, a_i, and c_i are constants.

The equilibrium is characterized in Section 3.1, together with some particular cases, and the equilibrium comparative statics properties are examined in Section 3.2.

[8] As in Kyle (1989), demands may be considered in the class of upper-hemicontinuous, convex-valued correspondences mapping prices p into nonempty subsets of the interval $[-\infty, \infty]$. If there is no market clearing price, the market shuts down, and if there are multiple clearing prices, the auctioneer chooses the one that maximizes volume traded.

3.1 *Equilibrium characterization*

Consider a trader of type i. If rivals' strategies are linear and identical among traders of the same type and if the market clears, that is, if $(n_i-1)X_i(s_i,p)+x_i+n_jX_j(s_j,p)=Q$, then this trader faces the inverse residual supply $p=I_i(s_i,s_j)+d_ix_i$, where

$$I_i(s_i,s_j)=\big((n_i-1)(b_i+a_is_i)+n_j(b_j+a_js_j)-Q\big)/\big((n_i-1)c_i+n_jc_j\big) \tag{1}$$

$$d_i=1/\big((n_i-1)c_i+n_jc_j\big). \tag{2}$$

The expression for the inverse residual supply disentangles the capacity of a bidder to influence the market price (d_i) from learning from the price ($I_i(s_i,s_j)$). Thus, the slope of the inverse residual supply (d_i) is an index of the trader's market power or price impact.[9] Indeed, by putting one more unit in the market, a trader of type i moves the price by d_i. A competitive trader would face a flat inverse residual supply ($d_i=0$). The slope d_i increases and the inverse residual supply becomes less elastic, the steeper are the demand functions submitted by the other traders (i.e., the lower c_i and c_j are).

From the expression of the inverse residual supply, we see that the intercept is random and the slope is deterministic. As a consequence, this trader's information set (s_i,p) is informationally equivalent to $(s_i,I_i(s_i,s_j))$. In addition, using (1) and assuming that $a_j\neq 0$, it is immediate that $(s_i,I_i(s_i,s_j))$ is informationally equivalent to (s_i,s_j).[10]

The bidder of type i, therefore, chooses x_i to maximize

$$\mathbb{E}[\pi_i|s_i,p]=\big(\mathbb{E}[\theta_i|s_i,s_j]-I_i(s_i,s_j)-d_ix_i\big)x_i-\lambda_ix_i^2/2,$$

since $\mathbb{E}[(\theta_i-p)x_i|s_i,p]=(\mathbb{E}[\theta_i|s_i,p]-p)x_i$. The first-order condition (FOC) is given by $\mathbb{E}[\theta_i|s_i,s_j]-I_i(s_i,s_j)-2d_ix_i-\lambda_ix_i=0$, which implies that

$$X_i(s_i,p)=\big(\mathbb{E}[\theta_i|s_i,p]-p\big)/(d_i+\lambda_i). \tag{3}$$

The second-order condition (SOC) that guarantees a maximum is $2d_i+\lambda_i>0$, which implies that $d_i+\lambda_i>0$. Therefore, a trader of type i has a speculative motive to trade, which is reflected in the numerator of (3), according to which he buys (sells) the asset when its price is lower (higher) than his conditional expected valuation. Furthermore, the bidder trades less aggressively when he has higher transaction costs (λ_i) or higher price impact (d_i).

In our framework

$$\mathbb{E}[\theta_i|s_i,p]=\mathbb{E}[\theta_i|s_i,s_j]. \tag{4}$$

According to Gaussian distribution theory,

$$\mathbb{E}[\theta_i|s_i,s_j]=\overline{\theta}_i+\Xi_i(s_i-\overline{\theta}_i)+\Psi_i(s_j-\overline{\theta}_j), \tag{5}$$

[9] The inverse residual supply for a trader of type i is well defined provided that $(n_i-1)c_i+n_jc_j\neq 0$. This inequality is satisfied in equilibrium.

[10] This would not be the case if there were more than two groups or if the traders in each group were to receive idiosyncratic signals. In this case, an information externality would appear, inducing additional inefficiencies in the market. The situation would be similar to the case of a noisy equilibrium (e.g., Vives 2017).

where

$$\Xi_i = \frac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_j}}{(1+\widehat{\sigma}^2_{\varepsilon_i})(1+\widehat{\sigma}^2_{\varepsilon_j})-\rho^2} \quad \text{and} \quad \Psi_i = \frac{\rho\widehat{\sigma}^2_{\varepsilon_i}}{(1+\widehat{\sigma}^2_{\varepsilon_i})(1+\widehat{\sigma}^2_{\varepsilon_j})-\rho^2}.$$

We remark that (5) has the following implications.

(a) The private signal s_i is useful for predicting θ_i ($\Xi_i \neq 0$) whenever $1-\rho^2+\widehat{\sigma}^2_{\varepsilon_j} \neq 0$, that is, when either the liquidation values are not perfectly correlated ($\rho \neq 1$) *or* type-j traders are imperfectly informed about θ_j ($\sigma^2_{\varepsilon_j} \neq 0$).

(b) The private signal s_j is useful for predicting θ_i ($\Psi_i \neq 0$) whenever $\rho\widehat{\sigma}^2_{\varepsilon_i} \neq 0$, that is, when the private liquidation values are correlated ($\rho \neq 0$) and type-i traders are imperfectly informed about θ_i ($\sigma^2_{\varepsilon_i} > 0$). Note that the weight given to s_j in estimating θ_i, Ψ_i, increases with the correlation coefficient of valuations (ρ).

From (3), the coefficients in the demand function (i.e., b_i, a_i, and c_i) are identified. For example, let Λ_i be the coefficient of the price in $\mathbb{E}[\theta_i|s_i, p]$; then $c_i = (1-\Lambda_i)/(d_i+\lambda_i)$. For a given Λ_i, higher transaction costs or price impact attenuates the response of a trader of type i to the price (c_i).

Our first proposition summarizes the characterization of an anonymous linear equilibrium. It shows the relationship between a_i and c_i in equilibrium and also indicates that these coefficients are positive (see Lemmas A1 and A2 in Appendix A for more details).

Proposition 1. *Let $\rho < 1$. If equilibrium exists, then it is unique and the demand function of a type-i trader is given by $X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i, p] - p)/(d_i + \lambda_i)$. In addition, we have that signal and price responsiveness (a_i and c_i) move together, $a_i = \Delta_i c_i > 0$, where $\Delta_i = 1/(1+(1+\rho)^{-1}\widehat{\sigma}^2_{\varepsilon_i})$, with*

$$c_i = (1-\Lambda_i)/(d_i+\lambda_i), \tag{6}$$

where $\Lambda_i = \Psi_i(\frac{n_i c_i}{n_j c_j}+1)/\Delta_j$, $d_i = 1/((n_i-1)c_i + n_j c_j)$, and the ratio c_1/c_2 is the unique positive solution of a cubic polynomial.

Remark 1. Since $a_i > 0$ and $c_i > 0$, it follows that in equilibrium, the higher is the value of the trader's observed private signal (or the lower the price), the higher is the quantity she demands. When a type-i bidder is imperfectly informed, $\sigma^2_{\varepsilon_i} > 0$, we have that its signal responsiveness is less than its price responsiveness, $a_i < c_i$, since $\Delta_i < 1$ in this case; when she is perfectly informed, $\sigma^2_{\varepsilon_i} = 0$, we have $\Delta_i = 1$ and $a_i = c_i$. In the latter case the speculative buying pointer happens when $s_i - p > 0$. Observe that we can write the demand as $X_i(s_i, p) = b_i + c_i(\Delta_i s_i - p)$, so that a trader responds to $\Delta_i s_i - p$, that is, to the difference between a recalibrated signal according to Δ_i and the price. The signal s_i is corrected by the term Δ_i, which increases with the precision in the signal $\widehat{\sigma}^{-2}_{\varepsilon_i}$ and with the correlation of the signals ρ. A higher Δ_i implies that the speculative trading pointer $\Delta_i s_i - p$ may be positive with a lower realization of the signal s_i.

Given that (4) holds in equilibrium, it follows that the equilibrium price is privately revealing. In other words, the private signal and the price enable a type-i trader to learn about θ_i as much as if she had access to all the information available in the market, (s_i, s_j).

How informative is the price for a bidder of type i? This depends on how much s_j adds to that bidder in the estimation of θ_i. A measure of price informativeness for bidder i is, therefore, $\frac{\text{var}[\theta_i|s_i]-\text{var}[\theta_i|s_i,s_j]}{\text{var}[\theta_i|s_i]}$. It is easily seen that this measure equals $\rho\Psi_i$. The more informative is the price for bidder i, the higher will be the weight of the price Λ_i in $\mathbb{E}[\theta_i|s_i, p]$ given (c_i, c_j). It can also be shown that, provided that $\rho\widehat{\sigma}^2_{\varepsilon_i} > 0$, $\frac{\Psi_i}{\Delta_j}$ increases with ρ and with $\widehat{\sigma}^2_{\varepsilon_i}$, and decreases with $\widehat{\sigma}^2_{\varepsilon_j}$. For given (c_i, c_j), we have that, as expected, in this case Λ_i increases with ρ and with $\widehat{\sigma}^2_{\varepsilon_i}$, and decreases with $\widehat{\sigma}^2_{\varepsilon_j}$.

Let us see how the slope of the demand for a trader of group i (c_i) varies (i) with the weight of the price in $\mathbb{E}[\theta_i|s_i, p]$, Λ_i, and (ii) with the slope of the demand for bidders of type j, c_j.

(i) From (6), we have that the larger is Λ_i, the lower is the responsiveness of the demand to the price (c_i). To understand this result, note that, from the perspective of a bidder in group i, a high price conveys the news that the realization of s_j is high and, therefore, that the value θ_i will tend to be high because of the positive correlation between θ_i and s_j. Consequently, if the price is more informative about θ_i, then the reduction in the quantity demanded by a bidder in group i due to an increase in p is smaller.

(ii) Next, we study how the slope of the demand for a trader of group i (c_i) varies due to a change in the slope in the demand for bidders of type j (c_j), with the slope of demands for other bidders of group i remaining fixed at $\bar{c}_i$. Note that price impact for this trader is $d_i = 1/((n_i-1)\bar{c}_i + n_j c_j)$. Combining (6) and the expression for Λ_i given in Proposition 1, it follows that

$$c_i = \left(1 - \frac{\Psi_i}{\Delta_j}\left(\frac{(n_i-1)\bar{c}_i}{n_j c_j} + 1\right)\right) \Big/ \left(d_i + \lambda_i + \frac{\Psi_i}{\Delta_j}\frac{1}{n_j c_j}\right). \tag{7}$$

This expression shows how the slope of the demand of a type-i trader (c_i) depends on its price impact (d_i) and the slope of the demand functions of bidders of the rival group (c_j) as well as information parameters Ψ_i and Δ_j.

When $\Psi_i = 0$ (that is, when either the valuations are uncorrelated $(\rho = 0)$ or the private signal s_i is perfectly informative $(\sigma^2_{\varepsilon_i} = 0)$), prices are uninformative for this bidder, $\Lambda_i = 0$, and we have that $c_i = 1/(d_i + \lambda_i)$. In this case, the equilibrium coincides with the full-information equilibrium (denoted by superscript f). In the full (shared) information setup, traders can access (s_1, s_2) and, consequently, the price does not provide any additional information. Given that d_i is decreasing in c_j, as shown in (2), we observe a basic *strategic complementarity* in the slopes of the demands submitted by the traders. According to this strategic effect, if the type-j rivals of a type-i trader bid a demand function with a lower c_j, then the slope of the inverse residual supply d_i for this trader increases (that is, its price impact increases) and this trader also has an incentive to bid a demand function with a lower c_i.

However, if $\Psi_i > 0$ (that is, when the valuations are correlated ($\rho > 0$) and type-i traders are imperfectly informed about θ_i ($\sigma^2_{\varepsilon_i} > 0$)), then there is also an *inference* effect from the information conveyed by the price. Now, a lower c_j increases the terms $\frac{\Psi_i}{\Delta_j}(\frac{(n_i-1)\bar{c}_i}{n_j c_j}+1)$ and $\frac{\Psi_i}{\Delta_j}\frac{1}{n_j c_j}$ in (7), which also tend to depress c_i, reinforcing the basic strategic complementarity in the slopes. This is so since a lower c_j induces a bidder of type i to take the price more into consideration when predicting θ_i. The market clearing condition indicates that the lower is the reaction to the price by group j (the lower c_j), the higher s_j should be to cause a certain increase in the price of the asset. This inference of the change in s_j is more relevant for a type-i trader when this private signal is more useful when predicting θ_i (higher Ψ_i).

Next we analyze when an equilibrium exists. If an equilibrium does exist, then Proposition 1 implies that it is unique.

Proposition 2. *Equilibrium exists if and only if* $c_i, c_j > 0$.

In Appendix A (see Proposition A1) we state a necessary and sufficient condition on parameters for $c_i, c_j > 0$. As indicated below in Corollary 1, such a condition greatly simplifies when prices are uninformative. In addition, Corollary 2 specifies instances where the existence of equilibrium is guaranteed when prices are informative for at least one type of bidders. Basically, it shows that the number of bidders and the correlation of the valuations are key parameters for the existence of equilibrium.

Corollary 1 (Uninformative prices). *When valuations are uncorrelated* ($\rho = 0$)*, or when private signals are perfectly informative* ($\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = 0$) *or uninformative* ($\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = \infty$)*, equilibrium exists if and only if* $n_i + n_j \geq 3$.

Corollary 2 (Informative prices). *When the private signal* s_j *is useful for predicting* θ_i ($\rho\sigma^2_{\varepsilon_i} > 0$ *and* $\sigma^2_{\varepsilon_j} \geq 0$) *and valuations are not perfectly correlated* ($\rho < 1$)*, equilibrium exists if any of the following conditions holds:*

(i) Both groups of bidders are large enough (n_i *and* n_j *are large enough).*

(ii) Given the number of bidders in group i (n_i)*, the number of bidders in the other group* (n_j) *is large enough and the correlation coefficient between valuations* (ρ) *is low enough.*

(iii) When $\sigma^2_{\varepsilon_j} = 0$ *and* $n_j \geq 2$*, or* $n_j = 1$*,* n_i *large enough, and* ρ *low enough.*

Remark 2. Equilibrium does not exist for ρ close to 1 and low n_i. This is so because in such cases the market power of traders explodes and the demand schedules would become vertical (with $c_i \to 0$, $i = 1, 2$). As ρ increases, the informational role of the price is more important and traders submit steeper demand schedules (see Proposition 3 below). An equilibrium also does not exist when $\rho = 1$. If the price reveals a sufficient statistic for the common valuation, then no trader has an incentive to place any weight on her signal. But if traders put no weight on signals, then the price contains no information about the common valuation. This conundrum is related to the Grossman–Stiglitz

(1980) paradox. In fact, $\rho < 1$ and $n_1 + n_2 \geq 3$ are necessary conditions for the existence of equilibrium with incomplete information (in line with Kyle 1989 and Vives 2011).[11]

Let us illustrate the existence of equilibrium result in the particular case of symmetric groups, i.e., $n_i = n$, $\lambda_i = \lambda$, and $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon}$, $i = 1, 2$.[12] We find that equilibrium exists if and only if $n > 1 + \rho\widehat{\sigma}^2_{\varepsilon}/((1-\rho)(1+\rho+\widehat{\sigma}^2_{\varepsilon}))$, where we recall that $\widehat{\sigma}^2_{\varepsilon} = \sigma^2_{\varepsilon}/\sigma^2_{\theta}$. Therefore, the equilibrium's existence is guaranteed provided either that n is high enough or that ρ or $\widehat{\sigma}^2_{\varepsilon}$ is low enough. In the model of Vives (2011), bidders receive different private signals and the condition that guarantees existence of an equilibrium is $n > 1 + n\rho\widehat{\sigma}^2_{\varepsilon}/((1-\rho)(1+(2n-1)\rho+\widehat{\sigma}^2_{\varepsilon}))$. Direct computation yields that the condition derived in the model of Vives is more stringent than the condition derived in our setup. The reason is that, in Vives (2011), the degree of asymmetry in information (and induced market power) is greater because each of the $2n$ traders receives a private signal.

3.2 *Comparative statics*

We start by considering how the model's underlying parameters affect the equilibrium and, in particular, price impact (Proposition 3). We then explore how the equilibrium is affected when there are two distinct groups of traders, that is, a core and a fringe (Corollary 3). Our theme is to explore the interaction between strategic and inference effects when a payoff or an information parameter changes. (See our 2019 working paper for additional comparative statics results.)

PROPOSITION 3. *Let $\rho\sigma^2_{\varepsilon_1}\sigma^2_{\varepsilon_2} > 0$. Then the following statements hold.*

(i) An increase in transaction costs (λ_i or λ_j), a decrease in the precision of private signals (i.e., an increase in $\sigma^2_{\varepsilon_i}$ or $\sigma^2_{\varepsilon_j}$), or an increase in the correlation coefficient between valuations (ρ) makes demand less responsive to private signals and prices (lower a_i and c_i) and increases price impact (d_i).

(ii) If the number of bidders (n_i or n_j) increases, then d_i decreases. Furthermore, d_i is not affected by the quantity offered in the auction (Q) or the prior mean of the valuations ($\overline{\theta}_i$ and $\overline{\theta}_j$).

REMARK 3 (Uninformative prices and price impact). Prices do not convey information when $\rho = 0$, with d_i independent of $\sigma^2_{\varepsilon_i}$ and $\sigma^2_{\varepsilon_j}$, and when $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = 0$ or $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = \infty$, with d_i and d_j independent of ρ. Those cases correspond to a full-information equilibrium, and comparative statics of d^f_i and d^f_j on λ_i and n_i hold as in the previous proposition.[13] That is, for the information parameters to matter for price impact, it is necessary

[11]Du and Zhu (2017b) consider ex post nonlinear equilibria in a bilateral divisible double auction and show that with more than three symmetric traders, there are no nonlinear equilibria in the class of smooth demands sloping downward in price and sloping upward in signals.

[12]See our working paper (Manzano and Vives 2019) for the cases of a monopsony competing with a fringe and of an informed group facing an uninformed group as in Grossman and Stiglitz (1980).

[13]Indeed, if $\rho = 0$, then (a) both c_i and d_i (as well as c_j and a_j) are independent of $\sigma^2_{\varepsilon_i}$, and (b) a_i decreases with $\sigma^2_{\varepsilon_i}$. If $\sigma^2_{\varepsilon_i} = 0$ for $i = 1, 2$, then c_i, c_j, a_i, a_j, d_i, and d_j are independent of ρ. Akgün (2004) considers a

that prices convey information. Proposition 3(i) implies that if $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$, then $d_i^f < d_i$. Thus, asymmetric information increases the price impact of traders in both groups beyond the full-information level.

Remark 4 (Symmetric groups). When groups are symmetric, the results hold when $\lambda_i = \lambda_j$, $\sigma_{\varepsilon_i}^2 = \sigma_{\varepsilon_j}^2$, and $n_i = n_j$ move together (Vives 2011). Proposition 3 disentangles the impact of, say, $\sigma_{\varepsilon_i}^2$ on equilibrium coefficients, keeping $\sigma_{\varepsilon_j}^2$ constant.

We discuss next the comparative statics results derived in Proposition 3. We also provide instances where those predictions are consistent with the empirical literature.

Transaction costs If the transaction costs for a bidder of type j (λ_j) increase, then that bidder sets lower a_j and c_j. This is so since a higher transaction cost makes bidders of type j less responsive to the price in their bidding, as pointed out in Section 3.1 and, from Proposition 1, we know that a_j and c_j move together. Moreover, any increase in a group's transaction costs also affects the behavior of traders in the other group. If λ_j increases, then the decrease in c_j results in an increase of the slope of the inverse residual supply for group i (higher d_i) as well as the terms related to the inference of θ_i from the price ($\frac{\Psi_i}{\Delta_j}(\frac{(n_i-1)\bar{c}_i}{n_j c_j}+1)$ and $\frac{\Psi_i}{\Delta_j}\frac{1}{n_j c_j}$) in (7) whenever $\rho\widehat{\sigma}_{\varepsilon_i}^2 \neq 0$. As both the strategic and the inference effects work in the same direction, an increase in λ_j leads group-i traders to reduce their demand sensitivity to the price (lower c_i). We can, therefore, see how an increase in the transaction costs for group-j traders (say, a deterioration of their collateral in liquidity auctions that raises λ_j) leads not only to steeper demand schedules for bidders in group j, but also, as a reaction, to steeper demands for group-i traders.

Precision of private signals If the private signal of type-j bidders is less precise (higher $\sigma_{\varepsilon_j}^2$), then their demand is less sensitive to private information (lower a_j). A private signal of reduced precision also gives the type-j bidder more incentive to consider prices when predicting θ_j (higher Λ_j). This leads, in turn, to this bidder having a demand function less responsive to the price, i.e., with lower c_j. This is so since a high price conveys the good news that the private signal received by other group's traders is high. When valuations are positively correlated, a bidder infers from the high private signal of the other group that her own valuation is high. The same can be said for a bidder of type i because of strategic complementarity in the slopes of demand functions (the decrease in c_j due to a rise in $\sigma_{\varepsilon_j}^2$ leads to lower c_i in turn). This result (in the supply competition model interpretation) may help explain why, in the Texas balancing market, small firms use steeper supply functions than predicted by theory (Hortaçsu and Puller 2008). Indeed, smaller firms may receive lower-quality signals owing to economies of scale in information gathering.

Correlation coefficient between valuations The more highly are the valuations correlated (higher ρ), the less is traders' responsiveness to private signals (lower a_i) and the steeper are demand schedules (lower c_i). As ρ increases, the private signal becomes less

linear equilibrium in a certainty common value model and shows (in our notation) that an increase in λ_i reduces c_i and c_j.

relevant for a type-i bidder to estimate θ_i, in which case demand is less sensitive to private information, while the price is more relevant. In fact, given (c_i, c_j), the information-sensitivity weight on the price (Λ_i) is higher when ρ is larger, which implies a lower responsiveness of the demand to the price because of (6).

Quantity offered in the auction and the prior mean of valuations Lemma A2 in Appendix A shows that the only equilibrium coefficient affected by the quantity offered in the auction (Q) and by the prior mean of the valuations $(\overline{\theta}_i$ and $\overline{\theta}_j)$ is b_i. In particular, price impact (d_i) is independent of these parameters.

Number of bidders Proposition 3(ii) formalizes the anticipated result that an increase in the number of auction participants (higher n_i or n_j) reduces the price impact of traders in both groups.[14]

Our comparative statics results highlight the interaction between the strategic and inference effects resulting from a parameter change. We have seen how a steepening of demand schedule by one group leads to the steepening of demand schedule by another group because of a strategic effect, which is reinforced by an inference effect. The result is strategic complementarity in the slopes of demands. The presence of private information and learning from prices compounds the strategic effect that would be present with full information and makes the impact of the change of a parameter larger.

Our results can shed light on the impact of a crisis in central bank liquidity auctions. Cassola et al. (2013) analyze the evolution of bidding data from the European Central Bank's weekly refinancing operations before and during the early part of the financial crisis in 2007. The authors find that one-third of bidders experienced no change in their costs of short-term funds from alternative sources. This means that their altered bidding behavior was mainly strategic: bids were increased as a response to the higher bids of rivals. Distressed bidders after the August 2007 shock suffered a large decline in the valuation of their collateral in the interbank market (which, in terms of our model, shows up in an increased λ_i). Those banks also had an increase in the valuation for liquidity (which, in our model, shows up as an increased $\overline{\theta}_i$).[15] In Cassola et al. (2013), it is assumed that the private valuations of the traders are independent, since the common component is known. This means that there are no information effects. However, if the common value component is not known (as is plausible), if the signals of the groups become noisier (in particular for those of the group hit by the shock), and the correlation of valuations increases (as happens in a crisis), then all these effects reinforce the steepening of the demand schedules (as found in Cassola et al. 2013).

[14] Rostek and Weretka (2015) address the question of whether encouraging trader participation enhances market competitiveness and liquidity also in a linear Gaussian, uniform-price double auction with a finite number of traders whose valuations are potentially asymmetrically correlated. They assume that each trader's value is, on average, correlated with other traders' values in the same way and find that, in general, the price impact is not monotone in market size. This is so because the arrival of an additional trader may change the informativeness of the market price so that the market power of all traders increases and the gains to trade are lower. In our model, since the equilibrium price is privately revealing, the informativeness of the market price does not change as the number of bidders increases.

[15] The marginal valuation of a bidder of type i is $\theta_i - \lambda_i x_i$. This is akin to the marginal valuation in Figure 4 in Cassola et al. (2013), where a decreased collateralized borrowing capacity of a bidder (K) makes the slope of the marginal valuation steeper.

Corollary 3 (Core and fringe). *Suppose that group* 1 *is less informed, has higher transaction costs, and is more numerous than group* 2 *(i.e.,* $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$, $\lambda_1 \geq \lambda_2$, *and* $n_1 \geq n_2$*), and suppose that at least one of these inequalities is strict. Then, in equilibrium, bidders from the core (here, group* 2*) react more both to private information and to prices (*$a_1 < a_2$, $c_1 < c_2$*) and have more price impact (*$d_1 < d_2$*) than do bidders from the fringe.*

Corollary 3 shows that if a group of traders is less informed, has higher transaction costs, and is more numerous, then it reacts less both to private signals and to prices. Observe in particular that group-1 traders, having less precise private information, rely more on the price for information (higher Λ_1); as a result, their overall price response ($c_1 = (1 - \Lambda_1)/(d_1 + \lambda_1)$) is smaller. Similarly, group-1 traders, for whom n_1 is larger, put a higher information-sensitivity weight on the price (Λ_1).[16]

As we see below, some results depend on the comparison between the "total transaction costs" $d_i + \lambda_i$ of the two groups. While with full information we have that $d_i + \lambda_i > d_j + \lambda_j$ whenever $\lambda_i > \lambda_j$, in our model we may have $d_i + \lambda_i < d_j + \lambda_j$ with $\lambda_i > \lambda_j$. This, in fact, happens whenever ρ is large, since then price impact induced by private information is large (see our 2019 working paper for the details and proof).

Corollary 3 is consistent with the results in Armantier and Sbaï (2006), who find in French treasury auctions that the group consisting mostly of smaller financial institutions, characterized by a higher level of risk aversion and receiving significantly noisier private signals, submits steeper demand functions than those submitted by the core group.

4. Welfare analysis

We identify factors that affect, in equilibrium, quantities, expected price, and revenue in the auction in Section 4.1; the equilibrium and efficient allocations in Section 4.2 to be used as a benchmark; deadweight losses in Section 4.3; and market integration in Section 4.4. In these subsections, we assume that the quantity auctioned (Q) is large enough (or, equivalently, the expected valuations of groups are not much different) so that bidders from both groups are expected to be buyers. Finally, Section 4.5 examines the polar case of a double auction ($Q = 0$).

4.1 *Quantities, prices, and revenue*

Let $t_i = \mathbb{E}[\theta_i | s_1, s_2]$ be the predicted value with full information (s_1, s_2) for group i and let $t = (t_1, t_2)$. After some algebra, it follows that equilibrium quantity for a bidder of this group as function of t is given by

$$x_i(t) = \frac{n_j(t_i - t_j)}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)} + \frac{d_j + \lambda_j}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)} Q. \tag{8}$$

[16] This follows since it can be shown that $\frac{n_1 c_1}{n_2 c_2}$ is increasing in n_1. The following scenario is an heuristic argument for the result. Consider a symmetric setting to start with and let the number of bidders of group 1 increase. Then the price will depend more (resp., less) strongly on s_1 (resp., s_2). As a result, type-1 bidders infer a higher value of the signal (s_2) of the other group, due to a given increase in the price, than type-2 bidders do about signal s_1. The positive correlation of the valuations implies then that $\Lambda_1 > \Lambda_2$.

Observe that, according to this expression, the equilibrium quantities can be decomposed into two terms: a *valuation* trading term (the first), which depends on the relative valuations of the groups, and a *clearing* trading term (the second), which is related to the absorption of Q by the traders. With regard to the valuation term, it vanishes when both groups have the same conditional expected valuation, $t_1 = t_2$, and it is positive (resp., negative) for the group with the higher (resp., lower) value of t_i. Higher total transaction costs $(d_i + \lambda_i)$ lower the response to valuation differences $t_i - t_j$. As for the clearing trading term, it is lower (resp., higher) for the group with higher (resp., lower) $d_i + \lambda_i$ and total clearing demands add up to Q.

Let $\tilde{t} = (n_1 t_1 + n_2 t_2)/(n_1 + n_2)$. Using the optimal demand of bidders, it follows that $p(t) = t_i - (d_i + \lambda_i)x_i(t)$, $i = 1, 2$. Therefore,

$$p(t) = \tilde{t} - \big((d_1 + \lambda_1)n_1 x_1(t) + (d_2 + \lambda_2)n_2 x_2(t)\big)/(n_1 + n_2).$$

From the above expressions, we can derive the following expression for expected price:

$$\mathbb{E}\big[p(t)\big] = \left(\frac{n_1}{d_1 + \lambda_1}\overline{\theta}_1 + \frac{n_2}{d_2 + \lambda_2}\overline{\theta}_2 - Q\right)\bigg/\left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}\right). \tag{9}$$

Proposition 4. *Let $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$ and suppose that bidders from both groups are expected to be buyers (i.e., Q large enough or $|\overline{\theta}_2 - \overline{\theta}_1|$ small enough). In equilibrium, the expected price is increasing in the number of bidders (n_i), but is decreasing in transaction costs (λ_i), the variances of error terms in private signals ($\sigma_{\varepsilon_i}^2$), and the correlation coefficient between valuations (ρ).*

When both groups are expected to be buyers, we confirm that the expected price increases when the number of bidders increases or when the asset becomes more attractive for the traders because of a reduction in their transaction costs or an increase in the precision of their private signals. To understand the negative relationship between the expected price and the correlation coefficient between valuations (ρ), recall that Proposition 3 indicates that an increase in ρ increases price impact (d_i), which makes buyers bid more cautiously (the inverse demand, $p(t) = t_i - (\lambda_i + d_i)x_i$, shifts inward), and this leads to a reduction in the expected price. In addition, all the results derived in Proposition 4 also apply to the expected revenue since it is equal to $Q\mathbb{E}[p(t)]$.

4.2 *Characterizing the equilibrium and efficient allocations*

Recall that $t = (t_1, t_2)$ denotes the vector of predicted values with full information (s_1, s_2). The strategies in the equilibrium induce outcomes as functions of t, given in (8). One can easily show that the equilibrium outcome solves the *distorted* benefit maximization program[17]

$$\max_{x_1, x_2} \mathbb{E}\big[n_1\big(\theta_1 x_1 - (d_1 + \lambda_1)x_1^2/2\big) + n_2\big(\theta_2 x_2 - (d_2 + \lambda_2)x_2^2/2\big)|t\big]$$
$$\text{s.t. } n_1 x_1 + n_2 x_2 = Q,$$

[17]See Lemma B1 in Appendix B.

where d_1 and d_2 are the equilibrium parameters. The efficient allocation is obtained if we set $d_1 = d_2 = 0$, which corresponds to a price-taking equilibrium (denoted by superscript o).[18] The efficient quantity for a bidder of group i as function of t is given by

$$x_i^o(t) = \frac{n_j(t_i - t_j)}{n_i\lambda_j + n_j\lambda_i} + \frac{\lambda_j}{n_i\lambda_j + n_j\lambda_i} Q. \tag{10}$$

In addition, the equilibrium strategy of a type-i bidder under perfect competition is of the form $X_i^o(s_i, p) = b_i^o + a_i^o s_i - c_i^o p$ and is derived by maximizing the program

$$\max_{x_i} \big(\mathbb{E}[\theta_i | s_i, p] - p\big) x_i - \lambda_i x_i^2/2,$$

while taking prices as given. The FOC of this optimization problem yields

$$\mathbb{E}[\theta_i | s_i, p] - p - \lambda_i x_i = 0.$$

After identifying coefficients and solving the corresponding system of equations, we find that there exists a unique equilibrium in this setup that we can characterize in closed form (see Proposition B1 in Appendix B).[19]

Remark 5 (Convergence to a price-taking equilibrium). It can be shown that, with strategic agents, if n_1 and n_2 both approach infinity and $n_i/(n_1 + n_2)$ converges to μ_i ($0 < \mu_i < 1$), then the demand equilibrium coefficients converge to the equilibrium coefficients of a continuum economy setup, which coincide with the equilibrium coefficients of the price-taking equilibrium (see our 2019 working paper). In the continuum economy, there is a mass of bidders along the interval $[0, 1]$, a fraction μ_i ($0 < \mu_i < 1$) of these bidders are traders of type i, $i = 1, 2$, and q represents the aggregate (average) quantity supplied in the market.

Provided $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$, Proposition B1 implies that c_i^o is increasing in the proportion of bidders of group j; c_i^o is decreasing in the proportion of bidders of group i, transaction costs (λ_i or λ_j), the correlation coefficient between valuations (ρ), and the variance of the error term in the private signal of group i ($\sigma_{\varepsilon_i}^2$); c_i^o is not affected by the precision of the private signal of group j ($\sigma_{\varepsilon_j}^2$).

Thus, as outlined at the beginning of this subsection, the auction outcome can be obtained as the solution to a maximization problem with a more concave objective function than the expected total surplus, which suggests that inefficiency may be eliminated by quadratic subsidies ($\kappa_i x_i^2/2$, $i = 1, 2$) that compensate for the distortions. The per capita subsidy rate (κ_i) to a trader of type i must be such that it compensates for the distortion $d_i(\kappa_i)$ while accounting for the subsidy. Since the aim is to induce competitive

[18]The efficient allocation maximizes the expected total surplus. Here, the revenue collected is just a transfer from the bidders to the auctioneer and washes out. If the social objective is just the surplus of the bidders or the revenue of the auctioneer, the objective function should be modified accordingly.

[19]When all bidders are expected to be buyers, it holds that market power lowers expected prices from the price-taking benchmark. This is easily seen since, in this case, $\mathbb{E}[p]$ is decreasing in d_i and d_j, and the price-taking benchmark is obtained when $d_1 = d_2 = 0$.

behavior, the trader should be led to respond with c_i^o to the price. This means that the exact amount of κ_i must be $d_i(c_1^o, c_2^o)$, since that would be the distortion arising when traders use the competitive linear strategies. The following proposition shows that if subsidies are selected properly, then bidders behave competitively and so the equilibrium allocation is efficient.

Proposition 5. *The efficient allocation is induced by the quadratic subsidies $\kappa_i x_i^2/2$, where $\kappa_i = d_i(c_i^o, c_j^o) = 1/((n_i - 1)c_i^o + n_j c_j^o)$. If $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$, then the per capita subsidy rate for group i (κ_i) increases with transaction costs (λ_i and λ_j), the variances of error terms in private signals ($\sigma_{\varepsilon_i}^2$ and $\sigma_{\varepsilon_j}^2$), and the correlation coefficient between valuations (ρ), but decreases with the number of bidders (n_i and n_j). Moreover, we have that $\kappa_i < \kappa_j$ if and only if $c_i^o < c_j^o$.*

Proposition 5 implies that the optimal subsidy rates with incomplete information and learning from prices are higher than with full information: $\kappa_i > \kappa_i^f$ if (a) $\rho > 0$ and (b) at least one of $\widehat{\sigma}_{\varepsilon_1}^2$ or $\widehat{\sigma}_{\varepsilon_2}^2$ is strictly positive.[20]

The optimal subsidy rates are decreasing in the number of traders, because when there are many agents, competitive behavior is already being approached in the market without subsidies. Moreover, the fact that $\kappa_i = 1/((n_i - 1)c_i^o + n_j c_j^o)$ implies that (i) the remainder of the comparative statics results stated in Proposition 5 simply follows from the comparative statics of c_i^o previously outlined, and (ii) $\mathrm{sgn}\{\kappa_1 - \kappa_2\} = \mathrm{sgn}\{c_1^o - c_2^o\}$. Hence, $\kappa_1 < \kappa_2$ if and only if $c_1^o < c_2^o$, i.e., the bidders who require a higher per capita subsidy rate are those whose demands are more sensitive to price. Moreover, one can conclude that if there is a group with more precise private information, with lower transaction costs, and that is less numerous, then it is the group meriting a higher per capita subsidy rate. The reason is that the strategic behavior of bidders from the core is more pronounced and so it must receive more compensation so as to become competitive. These conclusions would have to be revised if other considerations come into play (e.g., systemic or redistributive).[21]

Our result has policy implications. It implies, for example, that a central bank seeking an efficient distribution of liquidity among banks should relax collateral requirements (i.e., provide a larger subsidy) to the core group. This prescription sounds apparently counterintuitive because the efficiency motive may conflict with the central bank's function as lender of last resort to preserve systemic stability, which often involves shoring up weak banks (e.g., the European Central Bank relaxing the collateral requirements for Greek banks to avoid a meltdown of that country's banking system). However, the prescription is what is needed *solely* for efficiency purposes in distributing liquidity when there is no bank failure externality. Another example is that of a wholesale electricity market characterized by a small (oligopsonistic) core group and a fringe;

[20]See the proof of Proposition 5 in Appendix B where closed-form expressions for the optimal subsidy rates are displayed.

[21]Athey et al. (2013) find with regard to U.S. Forest Service timber auctions that restricting entry increases small business participation, but substantially reduces efficiency and revenue. In contrast, subsidizing small bidders directly increases revenue and the profits of small bidders without much cost in efficiency. See also Loertscher and Marx (2017) and Pai and Vohra (2012).

in this case, a regulator looking to improve productive efficiency should set a higher subsidy rate for the oligopsonistic group. This could be accomplished by offering differential subsidies to renewable energy technologies, for instance, that lower the marginal cost of production.

It is worth noting that primary dealers in the U.S. Treasury are required to bid at least the pro-rate share of those dealers present in the auction ("demonstrate substantial presence") and in exchange enjoy privileges such as exclusive intermediation of open market operations (OMO) and, in the crisis period, access to the quantitative easing (QE) auction mechanism as well as to the Primary Dealer Credit Facility. This may be interpreted as a subsidy that lowers the effective transaction cost of the dealers since they have an obligation to bid a minimum amount. In terms of our model, the marginal transaction cost (with lambda slope) would shift outward and no longer be linear, but affine.[22]

4.3 *Deadweight loss*

The expected deadweight loss, $\mathbb{E}[\mathrm{DWL}]$, at an anonymous allocation $(x_1(t), x_2(t))$ is the difference between expected total surplus at the efficient allocation, $\mathbb{E}[\mathrm{TS}^o]$, and at the baseline allocation, denoted by $\mathbb{E}[\mathrm{TS}]$. Lemma B2 in Appendix B shows that

$$\mathbb{E}[\mathrm{DWL}] = \frac{1}{2}\lambda_1 n_1 \mathbb{E}\big[\big(x_1(t) - x_1^o(t)\big)^2\big] + \frac{1}{2}\lambda_2 n_2 \mathbb{E}\big[\big(x_2(t) - x_2^o(t)\big)^2\big]. \tag{11}$$

Using (8) and (10), it follows that

$$\begin{aligned}\mathbb{E}[\mathrm{DWL}] = \phi\big(&(n_2 d_1 + n_1 d_2)^2 \mathbb{E}\big[(t_1 - t_2)^2\big] \\ &+ 2(n_2 d_1 + n_1 d_2)(\lambda_1 d_2 - \lambda_2 d_1)(\overline{\theta}_2 - \overline{\theta}_1)Q + (\lambda_1 d_2 - \lambda_2 d_1)^2 Q^2\big),\end{aligned} \tag{12}$$

where $\phi = n_1 n_2/(2(n_2\lambda_1 + n_1\lambda_2)(n_1(d_2 + \lambda_2) + n_2(d_1 + \lambda_1))^2)$, and

$$\mathbb{E}\big[(t_1 - t_2)^2\big] = (\overline{\theta}_1 - \overline{\theta}_2)^2 + (1-\rho)^2 \sigma_\theta^2 \frac{2(1+\rho) + \widehat{\sigma}_{\varepsilon_1}^2 + \widehat{\sigma}_{\varepsilon_2}^2}{\big(1 + \widehat{\sigma}_{\varepsilon_1}^2\big)\big(1 + \widehat{\sigma}_{\varepsilon_2}^2\big) - \rho^2}. \tag{13}$$

Differences between the optimal allocation and the equilibrium allocation can come from differences in the valuation trading terms or in the clearing trading terms. To highlight the role of asymmetries, let us start analyzing $\mathbb{E}[\mathrm{DWL}]$ with symmetric groups.

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[22] A simplified representation of transaction costs is

$$\begin{cases} \lambda_i (x_i - \underline{x})^2/2 & \text{if } x_i > \underline{x} \\ k + \lambda_i (x_i - \underline{x})^2/2 & \text{if } x_i \le \underline{x}, \end{cases}$$

where k is a constant sufficiently high such that bidders decide to bid at least the minimum amount, denoted by $\underline{x}$. One can show that the unique equilibrium coefficient of the demand function for a bidder of group i affected by this modification of the model is b_i.

Symmetric groups When groups are symmetric ($n_2 = n_1 = n$, $\lambda_1 = \lambda_2 = \lambda$, and $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \sigma^2_\varepsilon$), $d_1 = d_2 = d$ and $\lambda_2 d_1 - \lambda_1 d_2 = 0$ or, equivalently, $d_1/d_2 = \lambda_1/\lambda_2$. This condition means that the ratio of the price impacts of the two types of bidders is aligned with the ratio of the slopes of their respective marginal transaction costs. In this case, differences between the equilibrium allocation and the optimal allocation only come from the valuation trading terms, which are independent of Q. Indeed, (8) and (10) indicate that the clearing trading terms corresponding to the equilibrium with imperfect competition and those corresponding to the competitive equilibrium are equal. Moreover, if group i values more (interim) the asset, i.e., $t_i > t_j$, then the valuation trading term for group i is positive and is lower under imperfect competition. Hence, the group with the higher value of the asset obtains a lower quantity with imperfect competition in relation to the optimal allocation. Therefore, strategic behavior generates distributive inefficiency. Note that distributive efficiency would be ensured provided that $\overline{\theta}_1 = \overline{\theta}_2$ and $\rho \to 1$, and, given that supply is fixed, this would coincide with overall efficiency.

While price impact (d_i) and the conditional expected deadweight loss ($\mathbb{E}[\mathrm{DWL}|t]$) move together for changes in information parameters, this need not be the case with d and the (ex ante) expected deadweight loss ($\mathbb{E}[\mathrm{DWL}]$). This point is relevant since in the empirical literature, price impact is typically a measure of deadweight loss, because there is an implicit assumption that price impact and $\mathbb{E}[\mathrm{DWL}]$ move together. But this need not hold. When groups are symmetric, (12) becomes

$$\mathbb{E}[\mathrm{DWL}] = \frac{nd^2}{4(d+\lambda)^2\lambda}\mathbb{E}\big[(t_1 - t_2)^2\big],$$

where $\frac{nd^2}{4(d+\lambda)^2\lambda}$ increases in d, which in turn increases in ρ and in σ^2_ε, and the difference in predicted values, $\mathbb{E}[(t_1 - t_2)^2]$, decreases when values are more correlated (higher ρ) or signals are noisier (higher σ^2_ε).[23] Hence, it follows that $\mathbb{E}[\mathrm{DWL}]$ may increase or decrease in ρ and σ^2_ε. In particular, under full information (i.e., $\sigma^2_\varepsilon = 0$), d_i is independent of ρ and then $\mathbb{E}[\mathrm{DWL}]$ decreases with ρ. By continuity, when σ^2_ε is small enough, price impact slightly increases when ρ increases, while $\mathbb{E}[\mathrm{DWL}]$ decreases. Consequently, in this case, price impacts and $\mathbb{E}[\mathrm{DWL}]$ move in opposite directions when ρ changes.

Asymmetric groups Suppose now that Q is large enough and $\lambda_2 d_1 \neq \lambda_1 d_2$. Then the differences between the equilibrium and efficient quantities mainly arise in the clearing trading terms. For example, suppose group 2 is the core and group 1 is the fringe, with $\lambda_1 > \lambda_2$, $n_1 > n_2$, and $\sigma^2_{\varepsilon_1} > \sigma^2_{\varepsilon_2}$; then $d_1/d_2 < \lambda_1/\lambda_2$ (and, therefore, $\lambda_1 d_2 - \lambda_2 d_1 > 0$). In this case, (8) and (10) imply that group 1 gets more in the equilibrium than in the efficient allocation, $x_1(t) > x^o_1(t)$ (and group 2 gets less, $x_2(t) < x^o_2(t)$). Suppose, furthermore, that group strength and preference strength are aligned (i.e., $\overline{\theta}_1 \leq \overline{\theta}_2$). This is so with primary dealers in a treasury auction who may value the bonds more than other direct bidders because they have more clout in reselling them. In Hortaçsu et al. (2018), it is found that the willingness of primary dealers to pay is no lower than that of other

[23] The term $\mathbb{E}[(t_1 - t_2)^2]$ vanishes when ρ approaches 1 or when there is no uncertainty ($\sigma^2_\theta = 0$), provided $\overline{\theta}_1 = \overline{\theta}_2$.

direct bidders (as well as of indirect bidders). In this case $(\lambda_1 d_2 - \lambda_2 d_1), (\overline{\theta}_2 - \overline{\theta}_1) > 0$. Thus, the differences between the valuation trading terms (in expected terms) and the clearing trading terms go in the same direction and inefficiency tends to increase (the second term in (12) is positive). Moreover, (12) and (13) imply that in this case, $\mathbb{E}[\mathrm{DWL}]$ increases with expected valuation asymmetry (i.e., $|\overline{\theta}_2 - \overline{\theta}_1|$) and with the quantity offered in the auction (Q).

Finally, the impact of a small amount of asymmetry may be large. Suppose, for example, that the initial situation is symmetric for the groups and that the variance of valuations (σ_θ^2) is low. Then $\mathbb{E}[\mathrm{DWL}]$ is close to zero, since we have that $d_1/d_2 = \lambda_1/\lambda_2$. However, if λ_2 is lowered, then we can check that d_1/d_2 decreases and, therefore, $d_1/d_2 < \lambda_1/\lambda_2$, in which case $\mathbb{E}[\mathrm{DWL}]$ may be substantial if Q is large enough, since $\mathbb{E}[\mathrm{DWL}]$ is increasing quadratically in Q. This is consistent with the results in Hortaçsu et al. (2018), who document a significant amount of efficiency losses due to heterogeneity at long maturities in U.S. Treasury auctions.

4.4 *Market integration*

Our analysis can also shed light on the effects of integrating separated markets. Suppose that groups 1 and 2 operate in separate markets (auctions), that is, in market i all the buyers (n_i) are of type i and supply is $n_i Q/(n_1 + n_2)$. In this framework, given that all the individuals are identical in market i, the market clearing condition implies that the equilibrium quantities are given by $Q/(n_1 + n_2)$. Hence, the expected total surplus in market i, denoted by $\mathbb{E}[\mathrm{TS}]_{\text{Market } i}$, satisfies

$$\mathbb{E}[\mathrm{TS}]_{\text{Market } i} = \frac{n_i \overline{\theta}_i}{n_1 + n_2} Q - \frac{\lambda_i n_i}{(n_1 + n_2)^2} \frac{Q^2}{2}$$

and, consequently, the global expected total surplus is given by

$$\mathbb{E}[\mathrm{TS}]_{\text{Market } 1} + \mathbb{E}[\mathrm{TS}]_{\text{Market } 2} = \frac{n_1 \overline{\theta}_1 + n_2 \overline{\theta}_2}{n_1 + n_2} Q - \frac{n_1 \lambda_1 + n_2 \lambda_2}{(n_1 + n_2)^2} \frac{Q^2}{2}.$$

Note that the previous expression is equal to the expected total surplus at the equally distributed allocation in the integrated market.[24] As the allocation of the perfect competitive equilibrium maximizes $\mathbb{E}[\mathrm{TS}]$ in this setup, then market integration increases expected total surplus, $\mathbb{E}[\mathrm{TS}]$, if bidders behave strictly as price-takers, except if $\overline{\theta}_i = \overline{\theta}_j$, $\sigma_{\varepsilon_i}^2 = \sigma_{\varepsilon_j}^2 = \infty$, and $\lambda_i = \lambda_j$. In the latter case, payoffs are symmetric among bidders of the two groups and there is no information on values. Therefore, there are no gains from trade among the groups.

[24] Wittwer (2021) compares "connected" with "disconnected" financial markets in which agents trade two perfectly divisible assets. In a connected market, traders can make their demand for one security contingent on the price of the other. By contrast, interlinking demands across assets is not possible when each asset is traded in a separate disconnected market. This paper shows the conditions under which both market structures generate the same allocation.

The expression of expected deadweight loss given in (11) allows us to analyze the effect of integrating separated markets under imperfect competition. This expression implies that if the optimal allocation is expected to be closer to the equilibrium allocation than the equally distributed allocation, then the expected deadweight loss at the equilibrium allocation will be lower than at the equally distributed allocation. This leads us to conclude that $\mathbb{E}[\mathrm{TS}]$ is higher when the market is integrated. This is the case with symmetric bidders ($\overline{\theta}_1 = \overline{\theta}_2$, $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2}$, $\lambda_1 = \lambda_2 = \lambda$, and $n_1 = n_2 = n$). The optimal allocation, the equilibrium allocation, and the equally distributed (EQ) allocation are, respectively, given by

$$x_i^o(t) = \frac{t_i - t_j}{2\lambda} + \frac{Q}{2n}, \qquad x_i(t) = \frac{t_i - t_j}{2(d+\lambda)} + \frac{Q}{2n}, \quad \text{and} \quad x_i^{\mathrm{EQ}} = \frac{Q}{2n}.$$

Notice that $x_i^o(t) > x_i(t) > x_i^{\mathrm{EQ}}$ and $x_j^o(t) < x_j(t) < x_j^{\mathrm{EQ}}$ whenever $t_i > t_j$, resulting in a positive effect of market integration on the expected total surplus.

On the basis of the above statement, market integration may only decrease $\mathbb{E}[\mathrm{TS}]$ if bidders behave strategically and are asymmetric (apart from the potential asymmetry in expected valuations). An illustrative example is the following. Suppose that $\overline{\theta}_1 = \overline{\theta}_2$, $\sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \infty$, $\lambda_1 = \lambda_2 = \lambda$, and $n_1 > n_2$. In this case,

$$x_i^o(t) = x_i^{\mathrm{EQ}} = \frac{Q}{n_1 + n_2} \quad \text{and} \quad x_i(t) = \frac{(d_j - d_i)Qn_j}{\big(n_i(d_j + \lambda) + n_j(d_i + \lambda)\big)(n_1 + n_2)} + \frac{Q}{n_1 + n_2}.$$

Hence, $x_1(t) > x_1^{\mathrm{EQ}} = x_1^o(t)$ and $x_2(t) < x_2^{\mathrm{EQ}} = x_2^o(t)$, i.e., the optimal allocation coincides with the equally distributed allocation and differs from the equilibrium allocation. In this case, we conclude that integrating separated markets reduces the expected total surplus. With asymmetric precision of private signals ($\sigma^2_{\varepsilon_1} \neq \sigma^2_{\varepsilon_2}$) and informative prices ($\rho > 0$) but otherwise symmetric groups, integration may be also welfare decreasing for large Q. Note that this would not happen with uninformative prices, $\rho = 0$.

In summary, under symmetry or under perfect competition, market integration increases the expected total surplus. To find that market integration decreases the expected total surplus, we have to restrict our attention to a setup with strategic behavior and asymmetric groups with bidders of both groups expected to be buyers when markets are unified. In such a case, gains from trade of integration may be overwhelmed by the inefficiency generated by group asymmetries and price impact.[25]

[25]The results derived in this section are in line with Malamud and Rostek (2017). In a model with independent private information, these authors show that if traders are symmetric, then an integrated market maximizes welfare. By contrast, if traders have different risk preferences, then fragmented markets can allocate risk more efficiently, thus realizing gains from trade that cannot be reproduced in an integrated market. Babus and Kondor (2018) examine the effect of trade decentralization, comparing a centralized market as described in Vives (2011) and a decentralized market in which dealers can engage in bilateral transactions with other dealers. The paper shows that the effect of trade decentralization on welfare and liquidity is, in general, ambiguous.

4.5 *The double auction case*

Until now, welfare analysis has been performed assuming that bidders from both groups are expected to be buyers. Some of our results do not hold if bidders of one group are expected to be sellers. To illustrate this point, let us consider a double auction ($Q=0$). As shown in (8), in this case, the clearing trading terms of the equilibrium quantities vanish and bidders from the group that values the asset less are expected to be sellers, while bidders of the other group are expected to be buyers. Concerning the expected price, note that when $Q=0$, (9) implies that $\mathbb{E}[p]$ is a convex combination of $\overline{\theta}_1$ and $\overline{\theta}_2$, and comparative statics results given in Proposition 4 might not hold. Concretely, when bidders of one group turn into sellers instead of buyers of the asset, then decreasing their transaction costs, increasing the precision of their private signal, or increasing their number can lead to an increase in supply and, hence, a lower price is expected.

Another result that should be nuanced is related to the relationship between market integration and welfare. In this case, market integration always increases the expected total surplus, although bidders behave strategically and are asymmetric. We have that in the integrated market, the bidders of the group that values the asset less become sellers, while in separated markets there is no trade. Thus, in the integrated market, the group that values the asset more keeps a higher quantity of the asset than in separated markets. Consequently, in this case, market integration increases welfare.

5. Oligopsony with a large fringe

We have claimed in Remark 5 that the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders of both groups becomes large. We examine here what happens when only one group (group 1) is large. Let q denote the fixed per capita supply, that is, $Q=(n_1+n_2)q$.

Proposition 6. *Let $\rho\sigma^2_{\varepsilon_1}>0$. Suppose that $n_1\to\infty$ and $n_2<\infty$. Then the following results hold:*

(i) An equilibrium exists if and only if $n_2>\bar{n}_2(\rho,\widehat{\sigma}^2_{\varepsilon_1},\widehat{\sigma}^2_{\varepsilon_2})$, where $\bar{n}_2$ is increasing in ρ and $\widehat{\sigma}^2_{\varepsilon_1}$ and where $\bar{n}_2$ is decreasing in $\widehat{\sigma}^2_{\varepsilon_2}$ whenever $(2\rho-1)\widehat{\sigma}^2_{\varepsilon_1}<1-\rho^2$.[26]

(ii) An agent in the large group absorbs the inelastic per capita supply in the limit ($\lim_{n_1\to\infty}b_1=q$, $\lim_{n_1\to\infty}a_1=\lim_{n_1\to\infty}c_1=0$) and retains some price impact ($\lim_{n_1\to\infty}d_1>0$), while an agent in the small group commands higher impact ($\lim_{n_1\to\infty}d_2>\lim_{n_1\to\infty}d_1$).[27]

(iii) In the limit, the price depends only on the valuations and price impact of agents in the large fringe: $\lim_{n_1\to\infty}p=\mathbb{E}[\theta_1|s_1,s_2]-(\lim_{n_1\to\infty}d_1+\lambda_1)q$.

[26] In the particular case where $n_2=1$, the existence condition boils down to $(2\rho-1)\widehat{\sigma}^2_{\varepsilon_1}<1-\rho^2$.

[27] The limit expected quantity of a bidder of group 2 is given by

$$\lim_{n_1\to\infty}\mathbb{E}\big[x_2(t)\big]=\Big(\overline{\theta}_2-\overline{\theta}_1+\Big(\lim_{n_1\to\infty}d_1+\lambda_1\Big)q\Big)\Big/\Big(\lim_{n_1\to\infty}d_2+\lambda_2\Big).$$

Equation (50) in Appendix B shows that when $n_2 = \bar{n}_2(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2})$, the demand functions for bidders in group 2 are completely inelastic ($\lim_{n_1\to\infty} c_2 = 0$). This explains why the inequality $n_2 > \bar{n}_2(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2})$ is required for the existence of equilibrium, since otherwise we would have $\lim_{n_1\to\infty} c_i = 0$, $1 = 1, 2$.

Part (iii) of Proposition 6 highlights the fact that in the limit, the reduced number of type-2 bidders in relation to the large fringe makes the price independent of the predicted value of the asset for group 2 and its price impact. Thus, we have that in the limit, the equilibrium price reflects only the element of information that is common to a large number of traders. In fact, this is the ex post valuation of the fringe. Indeed, the price in the limit is a linear function of $\mathbb{E}[\theta_1|s_1, s_2]$ and, therefore, a sufficient statistic for θ_1 with the information (s_1, s_2).

We see that an agent in the large group just absorbs the inelastic per capita supply, behaving like a "Cournot quantity setter," and keeping some price impact ($\lim_{n_1\to\infty} d_1 > 0$), while bidders in the small group command relatively more market power ($\lim_{n_1\to\infty} d_2 > \lim_{n_1\to\infty} d_1$). The result for the small group is in line with Baisa and Burkett (2018). These authors obtain that in a uniform-price auction with independent private values, a single large bidder (with multi-unit demand) retains market power when he competes against many small bidders, each with single-unit demands. However, in our case, with both groups competing in demand schedules and correlated values, the fringe also retains market power when this group learns from the price (i.e., with $\rho\sigma^2_{\varepsilon_1} > 0$).

The intuition for the result is that when the fringe learns from the price, it reacts less to the price, as a high price provides good news about the valuation of the asset, and it reacts less and less to the price as $n_1 \to \infty$, not reacting at all in the limit. Indeed, recall that price impact for an agent of group 1 is given by $d_1 = ((n_1 - 1)c_1 + n_2c_2)^{-1}$, which is the slope of the residual supply that the agent faces. Thus, $\lim_{n_1\to\infty} d_1 > 0$ is due to the fact that when group 1 learns from the price, the inverse residual supply that this agent faces does not become flat as $n_1 \to \infty$. This is so since the aggregate demand of this group does not become flat, $\lim_{n_1\to\infty} n_1c_1 < \infty$. Note that as $n_1 \to \infty$, the weight of the price in $\mathbb{E}[\theta_1|s_1, p]$, Λ_1, tends to 1 (at the rate of $1/n_1$), since, in the limit, the price is a sufficient statistic for θ_1 with the information (s_1, s_2). Since $c_1 = (1 - \Lambda_1)/(d_1 + \lambda_1)$, we have that price responsiveness of group 1 (c_1) converges to zero, $c_1 \to 0$ at the rate of $1/n_1$, leading to $\lim_{n_1\to\infty} d_1 > 0$. However, if the large group does not learn from the price ($\rho\sigma^2_{\varepsilon_1} = 0$), then the weight of the price in $\mathbb{E}[\theta_1|s_1, p]$ is null ($\Lambda_1 = 0$) and, consequently, c_1 does not tend to 0 as $n_1 \to \infty$, which implies that $\lim_{n_1\to\infty} n_1c_1 = \infty$. In the limit, the aggregate demand of group 1 is flat; then taking into account that $d_2 = (n_1c_1 + (n_2 - 1)c_2)^{-1}$, it is easy to see that in this case, there is no price impact in the limit to any group: $\lim_{n_1\to\infty} d_1 = \lim_{n_1\to\infty} d_2 = 0$. It is worth noting that if the large group does not have price impact, then the small group cannot have it either, the reason being that, in this case, both face flat inverse residual supply curves.

Example. If the small group is fully informed ($\sigma^2_{\varepsilon_2} = 0$) and the large group is entirely uninformed ($\sigma^2_{\varepsilon_1} \to \infty$), then $\bar{n}_2 = 2\rho$, an equilibrium always exists for $n_2 > 2$, and the equilibrium coefficients for group 2 are $\lim_{n_1\to\infty} b_2 = 0$, and $\lim_{n_1\to\infty} a_2 = \lim_{n_1\to\infty} c_2 =$

$(n_2 - 2\rho)/((n_2 - \rho)\lambda_2)$. In this case, the groups' relative price impact is given by $\lim_{n_1\to\infty}(d_2/d_1) = 1 + \rho/(n_2 - \rho)$. As ρ increases, the relative price impact of group 2 also increases. This is so since a higher ρ makes the price more informative for group 1 and, in consequence, this group tends to react less to the price. $\Diamond$

6. Concluding remarks

The comparative statics results obtained provide testable predictions. For example, an increase in transaction costs or noise in the signals in any group, or an increase in correlation of values across groups, should increase the price impact of traders in both groups. Furthermore, co-movements in those parameters magnify the impact. The core group (because it has more precise private information, faces lower transaction costs, and is more oligopsonistic) has more price impact. This result is consistent with the evidence of U.S. Treasury auctions (Hortaçsu et al. 2018), where primary dealers exercise market power and earn significant surplus, on top of having privileges in exchange for bidding minimum amounts in the auctions. The expected deadweight loss increases with the quantity auctioned and with the degree of expected valuation asymmetry, provided the core values the asset no less than does the fringe. A small amount of asymmetry may generate large deadweight losses. The link between heterogeneity and efficiency losses is corroborated empirically for treasury auctions by Hortaçsu et al. (2018).

Our findings have policy implications. Consider a regulator who wants to reduce inefficiency in an industry with two groups of firms (e.g., a small oligopsonistic group and a large fringe). This regulator must bear in mind that any intervention directed toward one group also affects the other's behavior. In addition, for the regulator to induce competitive behavior, it should set a higher subsidy rate for the group that has better information, is more oligopsonistic, and has lower transaction costs. The framework developed here can be adapted to study competition policy, analyzing the effects of mergers and industry capacity redistribution.

Several extensions could be considered. A primary extension is to see how the results would be modified in a discriminatory auction.[28] A secondary extension is to allow for traders in each group to receive different signals. The latter is not a minor departure, since, in general, the equilibrium would be no longer privately revealing.

Appendix

Proofs of results in Sections 3, 4, and 5 are displayed, respectively, in Appendixes A, B, and C.

Appendix A: Proofs of results in Section 3

Proposition 1 follows from Lemmata A1 and A2.

[28] Ausubel et al. (2014) find that in symmetric auctions with decreasing linear marginal utility, the seller's revenue is greater in a discriminatory auction than in a uniform-price auction. Pycia and Woodward (2017) demonstrate that a discriminatory pay-as-bid auction is revenue-equivalent to the uniform-price auction provided that supply and reserve prices are set optimally.

LEMMA A1. *Let $\rho < 1$. In equilibrium, the demand function for a trader of type i is given by $X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i, p] - p)/(d_i + \lambda_i)$, with $d_i + \lambda_i > 0$. The equilibrium coefficients satisfy the system of equations*

$$b_i = \left((1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j - \frac{\Psi_i(n_ib_i + n_jb_j - Q)}{n_ja_j}\right)\Big/(d_i+\lambda_i) \tag{14}$$

$$a_i = \left(\Xi_i - \frac{n_ia_i}{n_ja_j}\Psi_i\right)\Big/(d_i+\lambda_i) \tag{15}$$

$$c_i = \left(1 - \frac{\Psi_i(n_ic_i + n_jc_j)}{n_ja_j}\right)\Big/(d_i+\lambda_i), \tag{16}$$

with $i, j = 1, 2$, $j \neq i$. Moreover, in equilibrium, $a_i > 0$.

PROOF. Consider a trader of type i. Recall that at the beginning of Section 3.1 we derive (3) and (4). Since we are looking for strategies of the form $X_i(s_i, p) = b_i + a_is_i - c_ip$, from the market clearing condition, we get

$$s_j = \frac{(n_ic_i + n_jc_j)p + Q - n_i(b_i + a_is_i) - n_jb_j}{n_ja_j}.$$

Thus, from (5), it follows that

$$\begin{aligned}\mathbb{E}[\theta_i|s_i, p] = (1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j + \Psi_i\left(\frac{Q - n_ib_i - n_jb_j}{n_ja_j}\right)\\ + \left(\Xi_i - \frac{n_ia_i}{n_ja_j}\Psi_i\right)s_i + \Psi_i\left(\frac{n_ic_i + n_jc_j}{n_ja_j}\right)p.\end{aligned}$$

Substituting the foregoing expression in (3) and then identifying coefficients, we obtain the expressions for the demand coefficients given in (14)–(16).

Finally, we show the positiveness of the coefficients a_i. From (15), we get $a_i = \Xi_i/(d_i + \lambda_i + n_i\Psi_i/(n_ja_j))$ and $a_j = \Xi_j/(d_j + \lambda_j + n_j\Psi_j/(n_ia_i))$. Combining the previous expressions, we have

$$a_i = \frac{n_j(\Xi_i\Xi_j - \Psi_i\Psi_j)}{n_i\Psi_i(d_j+\lambda_j) + \Xi_jn_j(d_i+\lambda_i)}. \tag{17}$$

Direct computation yields $\Xi_i\Xi_j - \Psi_i\Psi_j = (1-\rho^2)/((1+\widehat{\sigma}^2_{\varepsilon_1})(1+\widehat{\sigma}^2_{\varepsilon_2}) - \rho^2) > 0$ whenever $\rho < 1$. Moreover, using the positiveness of $d_i + \lambda_i$, $d_j + \lambda_j$, Ξ_j, and Ψ_i, we conclude that, in equilibrium, the coefficient a_i is strictly positive. □

LEMMA A2. *In equilibrium,*

$$b_i = \frac{\Psi_i}{n_in_j}\frac{n_i\Xi_j\dfrac{a_i}{a_j} - n_j\Psi_j}{\Xi_i\Xi_j - \Psi_i\Psi_j}Q + a_i\left(\frac{\Xi_j\overline{\theta}_i - \Psi_i\overline{\theta}_j}{\Xi_i\Xi_j - \Psi_i\Psi_j} - \overline{\theta}_i\right) \tag{18}$$

$$a_i = \Delta_ic_i \tag{19}$$

$$c_1 = \left(\Xi_1\Delta_1^{-1} - \frac{n_1}{n_2}(1 - \Xi_1\Delta_1^{-1})z - \frac{z}{(n_1-1)z + n_2}\right)\Big/\lambda_1 \tag{20}$$

$$c_2 = \left(\Xi_2\Delta_2^{-1} - \frac{n_2}{n_1}(1 - \Xi_2\Delta_2^{-1})\frac{1}{z} - \frac{1}{n_1 z + n_2 - 1}\right)\Big/\lambda_2, \tag{21}$$

where $\Delta_i = 1/(1 + (1+\rho)^{-1}\widehat{\sigma}^2_{\varepsilon_i})$*. Moreover,* $z \equiv c_1/c_2$ *is the unique positive solution to the cubic polynomial* $G(\zeta) = g_3\zeta^3 + g_2\zeta^2 + g_1\zeta + g_0$*, with*

$$\begin{aligned}
g_3 &= n_1^2(n_1-1)\big(n_2\Xi_2\Delta_2^{-1}\lambda_1 + n_1\big(1 - \Xi_1\Delta_1^{-1}\big)\lambda_2\big)\\
g_2 &= n_1\big((3n_2n_1 - n_1 - 2n_2 + 1)\big(n_2\Xi_2\Delta_2^{-1}\lambda_1 - n_1\Xi_1\Delta_1^{-1}\lambda_2\big)\\
&\qquad + \lambda_2 n_1(2n_2n_1 - n_1 + 1) - (n_1-1)(n_2+1)n_2\lambda_1\big),\\
g_1 &= n_2\big((3n_2n_1 - 2n_1 - n_2 + 1)\big(n_2\Xi_2\Delta_2^{-1}\lambda_1 - n_1\Xi_1\Delta_1^{-1}\lambda_2\big)\\
&\qquad + \lambda_2 n_1(n_2-1)(n_1+1) - (2n_2n_1 - n_2 + 1)n_2\lambda_1\big)\\
g_0 &= -n_2^2(n_2-1)\big(n_2\big(1 - \Xi_2\Delta_2^{-1}\big)\lambda_1 + n_1\Xi_1\Delta_1^{-1}\lambda_2\big).
\end{aligned}$$

Proof. In relation to the expression for b_i, notice that (15) implies

$$d_i + \lambda_i = \left(\Xi_i - \frac{n_ia_i}{n_ja_j}\Psi_i\right)\Big/a_i. \tag{22}$$

Substituting this expression in (14), it follows that

$$b_i = a_i\frac{(1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j - \dfrac{\Psi_i(n_ib_i + n_jb_j - Q)}{n_ja_j}}{\Xi_i - \dfrac{n_ia_i}{n_ja_j}\Psi_i}. \tag{23}$$

Thus,

$$\begin{aligned}
n_ib_i + n_jb_j &= n_ia_i\frac{(1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j - \dfrac{\Psi_i(n_ib_i + n_jb_j - Q)}{n_ja_j}}{\Xi_i - \dfrac{n_ia_i}{n_ja_j}\Psi_i}\\
&\quad + n_ja_j\frac{(1-\Xi_j)\overline{\theta}_j - \Psi_j\overline{\theta}_i - \dfrac{\Psi_j(n_ib_i + n_jb_j - Q)}{n_ia_i}}{\Xi_j - \dfrac{n_ja_j}{n_ia_i}\Psi_j}.
\end{aligned}$$

Isolating $n_ib_i + n_jb_j$ in the previous formula and substituting the resulting expression in (23), (18) is obtained.

Concerning the expression for a_i, substituting (22) in (16), it follows that

$$c_i = a_i\left(1 - \frac{\Psi_i(n_ic_i + n_jc_j)}{n_ja_j}\right)\Big/\left(\Xi_i - \frac{n_ia_i}{n_ja_j}\Psi_i\right). \tag{24}$$

Hence, $n_ic_i + n_jc_j = n_ia_i\frac{n_ja_j - \Psi_i(n_ic_i+n_jc_j)}{n_ja_j\Xi_i - n_ia_i\Psi_i} + n_ja_j\frac{n_ia_i - \Psi_j(n_ic_i+n_jc_j)}{n_ia_i\Xi_j - n_ja_j\Psi_j}$. Isolating $n_ic_i + n_jc_j$ in the previous formula and substituting the resulting expression in (24), we obtain a formula that is equivalent to (19). Using (19) in (16), we get the expression for c_i given in the statement of Proposition 1.

In relation to c_1 and c_2, using (2) and (19), (22) implies that

$$\lambda_i = \left(\frac{\Xi_i}{\Delta_i} - \frac{n_i\Psi_ic_i}{n_j\Delta_jc_j}\right)c_i^{-1} - \big((n_i-1)c_i + n_jc_j\big)^{-1}, \quad i, j = 1, 2, \text{ and } j \neq i,$$

or, since $\Psi_i\Delta_j^{-1} = 1 - \Xi_i\Delta_i^{-1}$,

$$\lambda_i = \left(\Xi_i\Delta_i^{-1} - \frac{n_i}{n_j}\big(1 - \Xi_i\Delta_i^{-1}\big)\frac{c_i}{c_j}\right)c_i^{-1} - \big((n_i-1)c_i + n_jc_j\big)^{-1}, \quad i, j = 1, 2, \text{ and } j \neq i,$$

which imply (20) and (21) since $z = c_1/c_2$. Moreover, dividing the previous expressions for λ_1 and λ_2, it follows that

$$\frac{\lambda_1}{\lambda_2} = \frac{\Xi_1\Delta_1^{-1} - \dfrac{n_1}{n_2}\big(1 - \Xi_1\Delta_1^{-1}\big)z - z\big((n_1-1)z + n_2\big)^{-1}}{\Xi_2\Delta_2^{-1}z - \dfrac{n_2}{n_1}\big(1 - \Xi_2\Delta_2^{-1}\big) - z(n_1z + n_2 - 1)^{-1}}. \tag{25}$$

After some algebra, (25) is equivalent to $G(z) = 0$, where $G(\zeta)$ is the polynomial given in the statement of this lemma. Notice that $G(0) < 0$ and $\lim_{\zeta\to\infty} G(\zeta) = \infty$. Consequently, there exists a positive root of $G(\zeta)$. Furthermore, we have that $g_2/n_1 > g_1/n_2$. The combination of this inequality with the fact that $g_3 > 0$ and $g_0 < 0$ allows us to conclude that there is only one sign change of the coefficients of $G(\zeta)$. To show that, we distinguish three cases:

Case 1: $0 \geq \frac{g_2}{n_1} > \frac{g_1}{n_2}$. This implies that $0 \geq g_2$ and $0 > g_1$. As $g_3 > 0$ and $g_0 < 0$, it follows that there is only one sign change of the coefficients of $G(\zeta)$.

Case 2: $\frac{g_2}{n_1} > 0 \geq \frac{g_1}{n_2}$. This implies that $g_2 > 0 \geq g_1$. As $g_3 > 0$ and $g_0 < 0$, it follows that there is only one sign change of the coefficients of $G(\zeta)$.

Case 3: $\frac{g_2}{n_1} > \frac{g_1}{n_2} > 0$. This implies that $g_2 > 0$ and $g_1 > 0$. As $g_3 > 0$ and $g_0 < 0$, it follows that there is only one sign change of the coefficients of $G(\zeta)$.

Applying Descartes' rule, we conclude that there exists a unique positive root of $G(\zeta)$. □

Proposition A1. *Let $\rho < 1$.*

(a) There exists an equilibrium if and only if $c_1, c_2 > 0$, where

$$c_1 = \frac{H_N(z)}{\big((n_1-1)z + n_2\big)n_2\lambda_1} \quad \textit{and} \quad c_2 = \frac{H_D(z)}{(n_1z + n_2 - 1)n_1z\lambda_2}, \tag{26}$$

where $z = c_1/c_2$ and the expressions of $H_N(\zeta)$ and $H_D(\zeta)$ are given by

$$H_N(\zeta) = n_2^2\Xi_1\Delta_1^{-1} + n_2\big(\Xi_1\Delta_1^{-1}(2n_1-1) - (n_1+1)\big)\zeta - (n_1-1)\big(1 - \Xi_1\Delta_1^{-1}\big)n_1\zeta^2$$

$$H_D(\zeta) = -n_2(n_2-1)\big(1 - \Xi_2\Delta_2^{-1}\big) + n_1\big(\Xi_2\Delta_2^{-1}(2n_2-1) - (n_2+1)\big)\zeta + n_1^2\Xi_2\Delta_2^{-1}\zeta^2.$$

(b) *Uninformative prices. When* $\rho = 0$, $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = 0$, *or* $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = \infty$, *equilibrium exists if and only if* $n_1 + n_2 \geq 3$.

(c) *Informative prices.*

(i) *Let* $\rho\sigma^2_{\varepsilon_1}\sigma^2_{\varepsilon_2} > 0$. *Then* $c_1, c_2 > 0$ *if and only if* $\overline{z}_N > \overline{z}_D$, *where* $\overline{z}_N$ *and* $\overline{z}_D$ *denote the highest root of* $H_N(\zeta)$ *and* $H_D(\zeta)$, *respectively.*

(ii) *Let* $\rho\sigma^2_{\varepsilon_i} > 0$ *and* $\sigma^2_{\varepsilon_j} = 0$. *Then* $c_1, c_2 > 0$ *if* $n_j \geq 2$ *or if* $n_j = 1$, n_i *large enough, and* ρ *low enough.*

Remark 6. For an equilibrium to exist, we must have $c_i, c_j > 0$, and these inequalities hold if and only if $\overline{z}_D < z < \overline{z}_N$. If $n_1 = 1$ and $n_2 = 1$, then $\overline{z}_N = 1/(2\Delta_1\Xi_1^{-1} - 1)$ and $\overline{z}_D = 2\Delta_2\Xi_2^{-1} - 1$. Since $\Delta_1\Xi_1^{-1}, \Delta_2\Xi_2^{-1} \geq 1$ and $\Delta_1\Xi_1^{-1} = \Delta_2\Xi_2^{-1} = 1$ do not hold, we can use direct computation to obtain $\overline{z}_N < \overline{z}_D$. Applying Proposition A1, we conclude that no equilibrium exists in this case. Therefore, $n_1 + n_2 \geq 3$ is a necessary condition for the existence of an equilibrium.

Remark 7. In (c)(i), we obtain that $\lim_{\lambda_1 \to 0} z = \overline{z}_N$ and $\lim_{\lambda_2 \to 0} z = \overline{z}_D$.

Remark 8. In (c)(ii), when $\sigma^2_{\varepsilon_2} = 0$, $\overline{z}_D = 1/n_1$ if $n_2 = 1$, whereas $\overline{z}_D = 0$ if $n_2 \geq 2$.

Proof. (a) Necessity. From Proposition 1, we know that $a_i, a_j > 0$ whenever $\rho < 1$. Combining this property with expressions given in (19), we have that, in equilibrium, the coefficients c_i and c_j are strictly positive. Moreover, (20) and (21) can be rewritten as the expressions given in (26).

Sufficiency. Suppose that the candidates' equilibrium coefficients c_1 and c_2 are positive and satisfy (26). Then the ratio $z = c_1/c_2 > 0$ and satisfies (25). Hence we conclude that an equilibrium exists and it is unique since we know that (25) has a unique positive solution. Finally, substituting this value of z in the expressions stated in Lemma A2, we obtain the equilibrium coefficients of the demand functions.

(b) When $\rho = 0$ or $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = 0$, the demand function for a trader of type i is given by

$$X_i(s_i, p) = \big(\mathbb{E}[\theta_i|s_i] - p\big)/(d_i + \lambda_i),$$

while when $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = \infty$, the demand function for a trader of type i holds:

$$X_i(s_i, p) = (\overline{\theta}_i - p)/(d_i + \lambda_i).$$

Moreover, recall that the SOCs imply $d_i + \lambda_i > 0$. In all these cases, we can express the coefficients of the demand functions in terms of d_i and d_j. In particular, $c_i = 1/(d_i + \lambda_i) > 0$, $i = 1, 2$. From (2), we characterize d_1 and d_2 as the positive solutions of the system of equations

$$d_i = \left(\frac{n_i - 1}{d_i + \lambda_i} + \frac{n_j}{d_j + \lambda_j}\right)^{-1}, \quad i, j = 1, 2 \text{ and } j \neq i.$$

After some algebra, we conclude that this system has positive solutions if and only if $n_1 + n_2 \geq 3$.

(c)(i) Necessity. Let $\overline{z}_N$ and $\overline{z}_D$ denote the highest root of $H_N(\zeta)$ and $H_D(\zeta)$, respectively. Notice that the positiveness of c_i and c_j is equivalent to $\overline{z}_N > z > \overline{z}_D$. Therefore, $\overline{z}_N > \overline{z}_D$.

Sufficiency. Suppose that $\overline{z}_N > \overline{z}_D$. Recall that Lemma A2 shows that there exists a unique positive value of z that solves (25), which can be rewritten as

$$\frac{\lambda_1}{\lambda_2} = \frac{n_1(n_2 - 1 + n_1 z)H_N(z)}{\big(n_2 + (n_1 - 1)z\big)n_2 H_D(z)}. \tag{27}$$

This implies that $\overline{z}_N > z > \overline{z}_D$. Notice that these inequalities guarantee the positiveness of c_i and c_j.

(c)(ii) Suppose that $\rho\sigma^2_{\varepsilon_1} > 0$ and $\sigma^2_{\varepsilon_2} = 0$. In this case, $\Xi_2\Delta_2^{-1} = 1$ and, hence, $H_D(\zeta) = \zeta n_1(n_2 + \zeta n_1 - 2)$. On the one hand, if $n_2 = 1$, then $\overline{z}_D = 1/n_1$. As in (c)(i), the condition that guarantees the existence of equilibrium is $\overline{z}_N > \overline{z}_D$, which is equivalent to $n_1(2\Xi_1\Delta_1^{-1} - 1) > \Xi_1\Delta_1^{-1}$, i.e., $\Xi_1\Delta_1^{-1} > 1/2$ and $n_1 > \Xi_1\Delta_1^{-1}/(2\Xi_1\Delta_1^{-1} - 1)$ or, using the expressions of Ξ_1 and Δ_1, $1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}^2_{\varepsilon_1} > 0$ and $n_1 > 1 + \widehat{\sigma}^2_{\varepsilon_1}\rho/(1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}^2_{\varepsilon_1})$, which applies when ρ is low enough and n_1 is large enough.

On the other hand, if $n_2 \geq 2$, $H_D(\zeta) > 0$ for all $\zeta > 0$ and, therefore, we have that $c_2 > 0$ is satisfied. The positiveness of c_1 requires that $\overline{z}_N > z$. But this inequality holds since z solves (27). To sum up, when $\sigma^2_{\varepsilon_2} = 0$, an equilibrium exists if $n_2 = 1$, n_1 large enough and ρ low enough, or if $n_2 \geq 2$.

Now suppose that $\rho\sigma^2_{\varepsilon_2} > 0$ and $\sigma^2_{\varepsilon_1} = 0$. In this case, $\Xi_1\Delta_1^{-1} = 1$ and, hence, $H_N(\zeta) = n_2^2 + n_2(n_1 - 2)\zeta$. On the one hand, if $n_1 = 1$, then $\overline{z}_N = n_2$. As in (c)(i), the condition that guarantees the existence of equilibrium is $\overline{z}_N > \overline{z}_D$, which is equivalent to $n_2(2\Xi_2\Delta_2^{-1} - 1) > \Xi_2\Delta_2^{-1}$, i.e., $\Xi_2\Delta_2^{-1} > 1/2$ and $n_2 > \Xi_2\Delta_2^{-1}/(2\Xi_2\Delta_2^{-1} - 1)$ or, using the expressions of Ξ_2 and Δ_2, $1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}^2_{\varepsilon_2} > 0$ and $n_2 > 1 + \widehat{\sigma}^2_{\varepsilon_2}\rho/(1 - \rho^2 + (1 - 2\rho)\widehat{\sigma}^2_{\varepsilon_2})$, which applies when ρ is low enough and n_2 is large enough.

On the other hand, if $n_1 \geq 2$, $H_N(\zeta) > 0$ for all $\zeta > 0$ and, therefore, we have that $c_1 > 0$ is satisfied. The positiveness of c_2 requires that $z > \overline{z}_D$. But this inequality holds since the equilibrium value, z, solves (27). To sum up, when $\sigma^2_{\varepsilon_1} = 0$, an equilibrium exists if $n_1 = 1$, n_2 large enough and ρ low enough, or if $n_1 \geq 2$. □

Lemma A3. *The condition $\overline{z}_N > \overline{z}_D$ given in the statement of Proposition A1 is satisfied in the following cases:*

(i) if $\rho < 1$ and n_1, n_2 are large enough

(ii) given n_i, n_j is large enough and ρ is low enough.

Proof. We distinguish two cases: $n_1 > 1$ and $n_1 = 1$.

Case 1: $n_1 > 1$. In this case

$$\overline{z}_N = \frac{n_2\Big((n_1 - 1)\big(2\Xi_1\Delta_1^{-1} - 1\big) - \big(2 - \Xi_1\Delta_1^{-1}\big) + \sqrt{\big(2 - \Xi_1\Delta_1^{-1}\big)^2 + (n_1 - 1)\big(n_1 + 3 - 6\Xi_1\Delta_1^{-1}\big)}\Big)}{2n_1(n_1 - 1)\big(1 - \Xi_1\Delta_1^{-1}\big)} \tag{28}$$

and

$$\overline{z}_D = \frac{n_2 + 1 - \Xi_2\Delta_2^{-1}(2n_2 - 1) + \sqrt{\left(2 - \Xi_2\Delta_2^{-1}\right)^2 + (n_2 - 1)\left(n_2 + 3 - 6\Xi_2\Delta_2^{-1}\right)}}{2\Xi_2\Delta_2^{-1}n_1}. \tag{29}$$

Proposition A1 indicates that an equilibrium exists if and only if $\overline{z}_N > \overline{z}_D$ or, equivalently, $n_1\overline{z}_N/n_2 > n_1\overline{z}_D/n_2$. Using the expressions of $\overline{z}_N$ and $\overline{z}_D$, we have that $n_1\overline{z}_N/n_2$ is increasing in n_1 and $n_1\overline{z}_D/n_2$ is decreasing in n_2. Taking limits, it follows that

$$\lim_{n_1\to\infty} n_1\overline{z}_N/n_2 = \Xi_1\Delta_1^{-1}/\left(1 - \Xi_1\Delta_1^{-1}\right) \quad \text{and} \quad \lim_{n_2\to\infty} n_1\overline{z}_D/n_2 = \left(1 - \Xi_2\Delta_2^{-1}\right)/\left(\Xi_2\Delta_2^{-1}\right).$$

Moreover, using the expressions of Ξ_i and Δ_i, we have that

$$\frac{\Xi_1\Delta_1^{-1}}{1 - \Xi_1\Delta_1^{-1}} - \frac{1 - \Xi_2\Delta_2^{-1}}{\Xi_2\Delta_2^{-1}} = \frac{\left(1 - \rho^2\right)\left(1 + \rho + \widehat{\sigma}_{\varepsilon_1}^2\right)\left(\left(1 + \widehat{\sigma}_{\varepsilon_1}^2\right)\left(1 + \widehat{\sigma}_{\varepsilon_2}^2\right) - \rho^2\right)}{\rho\widehat{\sigma}_{\varepsilon_1}^2\left(1 + \rho + \widehat{\sigma}_{\varepsilon_2}^2\right)\left(1 - \rho^2 + \widehat{\sigma}_{\varepsilon_1}^2\right)} > 0.$$

Hence, we get that, as $\rho < 1$, $\lim_{n_1\to\infty} n_1\overline{z}_N/n_2 > \lim_{n_2\to\infty} n_1\overline{z}_D/n_2$. This implies that whenever $\rho < 1$, and n_1 and n_2 are large enough, the existence of the equilibrium is guaranteed.

Consider now a fixed positive integer n_1, such that $n_1 > 1$. Using the fact that $\overline{z}_N$ is the positive root of $H_N(\zeta)$, it follows that $n_1\overline{z}_N/n_2 > \Xi_1\Delta_1^{-1}/(2 - \Xi_1\Delta_1^{-1})$. Moreover,

$$\Xi_1\Delta_1^{-1}/\left(2 - \Xi_1\Delta_1^{-1}\right) > \left(1 - \Xi_2\Delta_2^{-1}\right)/\left(\Xi_2\Delta_2^{-1}\right) \tag{30}$$

whenever ρ is low enough. Therefore,

$$n_1\overline{z}_N/n_2 > \Xi_1\Delta_1^{-1}/\left(2 - \Xi_1\Delta_1^{-1}\right) > \left(1 - \Xi_2\Delta_2^{-1}\right)/\left(\Xi_2\Delta_2^{-1}\right) = \lim_{n_2\to\infty} n_1\overline{z}_D/n_2.$$

Hence, we conclude that if n_2 is large enough, as $n_1\overline{z}_D/n_2$ is decreasing in n_2, the previous inequalities imply that $n_1\overline{z}_N/n_2 > n_1\overline{z}_D/n_2$ or, equivalently, $\overline{z}_N > \overline{z}_D$. Applying Proposition A1, it follows that in this case, there exists an equilibrium provided that n_2 is high enough and ρ is low enough.

Consider now a fixed positive integer n_2, such that $n_2 \geq 1$, and assume again that $\rho < 1$. Using the fact that $\overline{z}_D$ is the positive root of $H_D(\zeta)$, it follows that $n_1\overline{z}_D/n_2 \leq (2 - \Xi_2\Delta_2^{-1})/(\Xi_2\Delta_2^{-1})$. In addition, when ρ is low enough, then we have that

$$\left(2 - \Xi_2\Delta_2^{-1}\right)/\left(\Xi_2\Delta_2^{-1}\right) < \Xi_1\Delta_1^{-1}/\left(1 - \Xi_1\Delta_1^{-1}\right) = \lim_{n_1\to\infty} n_1\overline{z}_N/n_2.$$

Thus, we have that $n_1\overline{z}_D/n_2 < \lim_{n_1\to\infty} n_1\overline{z}_N/n_2$. Using the fact that $n_1\overline{z}_N/n_2$ increases with n_1, we have that when n_1 is high enough, $n_1\overline{z}_D/n_2 < n_1\overline{z}_N/n_2$ or, equivalently, $\overline{z}_D < \overline{z}_N$, which guarantees the existence of equilibrium. To sum up, we have that given n_2, there exists an equilibrium provided that n_1 is high enough and ρ is low enough.

Case 2: $n_1 = 1$. In this case, we have that $\overline{z}_N = n_2\Xi_1\Delta_1^{-1}/(2 - \Xi_1\Delta_1^{-1})$ and

$$\overline{z}_D = \frac{n_2 + 1 - \Xi_2\Delta_2^{-1}(2n_2 - 1) + \sqrt{\left(2 - \Xi_2\Delta_2^{-1}\right)^2 + (n_2 - 1)\left(n_2 + 3 - 6\Xi_2\Delta_2^{-1}\right)}}{2\Xi_2\Delta_2^{-1}}.$$

Furthermore, whenever ρ is low enough, (30) holds. Therefore, it follows that

$$\overline{z}_N/n_2 = \Xi_1\Delta_1^{-1}/\big(2-\Xi_1\Delta_1^{-1}\big) > \big(1-\Xi_2\Delta_2^{-1}\big)/\big(\Xi_2\Delta_2^{-1}\big) = \lim_{n_2\to\infty} \overline{z}_D/n_2.$$

Using the fact that $\overline{z}_D/n_2$ decreases with n_2, the previous inequality implies that $\overline{z}_N/n_2 > \overline{z}_D/n_2$ whenever n_2 is high enough, i.e., $\overline{z}_N > \overline{z}_D$, which guarantees the existence of equilibrium. To sum up, we have that when $n_1 = 1$, there exists an equilibrium provided that n_2 is high enough and ρ is low enough. □

The proofs of Proposition 2, Corollary 1, and Corollary 2 follow directly from Proposition A1 and Lemma A3.

Remark 9 (Symmetric groups). Let $n_i = n_j = n$, $\lambda_i = \lambda_j = \lambda$, and $\sigma^2_{\varepsilon_i} = \sigma^2_{\varepsilon_j} = \sigma^2_{\varepsilon}$. Here $z = 1$ in equilibrium. From Proposition A1, we know that if an equilibrium exists, then the value of z is in the interval $(\overline{z}_D, \overline{z}_N)$. It follows that $\overline{z}_N > 1 > \overline{z}_D$ or, equivalently, that $H_N(1) > 0$ and $H_D(1) > 0$. After performing some algebra, we find that the foregoing inequalities are satisfied if and only if $n > 1 + \rho\widehat{\sigma}^2_{\varepsilon}/((1-\rho)(1+\rho+\widehat{\sigma}^2_{\varepsilon}))$, where $\widehat{\sigma}^2_{\varepsilon} = \sigma^2_{\varepsilon}/\sigma^2_{\theta}$.

Proof of Proposition 3. Let $\rho\sigma^2_{\varepsilon_1}\sigma^2_{\varepsilon_2} > 0$. In what follows, we prove the comparative statics results

(a) $\partial a_i/\partial\lambda_i < 0$ and $\partial c_i/\partial\lambda_i < 0$

(b) $\partial a_i/\partial\lambda_j < 0$ and $\partial c_i/\partial\lambda_j < 0$

(c) $\partial a_i/\partial\rho < 0$ and $\partial c_i/\partial\rho < 0$

(d) $\partial a_i/\partial\sigma^2_{\varepsilon_i} < 0$ and $\partial c_i/\partial\sigma^2_{\varepsilon_i} < 0$

(e) $\partial a_i/\partial\sigma^2_{\varepsilon_j} < 0$ and $\partial c_i/\partial\sigma^2_{\varepsilon_j} < 0$

(f) $\partial d_i/\partial Q = 0$

(g) $\partial d_i/\partial\overline{\theta}_i = 0$ and $\partial d_i/\partial\overline{\theta}_j = 0$

(h) $\partial d_i/\partial n_i < 0$ and $\partial d_j/\partial n_i < 0$.

In what follows, without any loss of generality, let $i = 1$. First, we prove that $\partial z/\partial\lambda_1 < 0$. From Lemma A2, we know that z is the unique positive solution that satisfies

$$\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} = 0, \tag{31}$$

where

$$N(z) = \Xi_1\Delta_1^{-1} - n_1\big(1-\Xi_1\Delta_1^{-1}\big)z/n_2 - z\big((n_1-1)z+n_2\big)^{-1}$$
$$D(z) = \Xi_2\Delta_2^{-1}z - n_2\big(1-\Xi_2\Delta_2^{-1}\big)/n_1 - z(n_1z+n_2-1)^{-1},$$

with $\Xi_i\Delta_i^{-1} = (1-\rho^2+\widehat{\sigma}^2_{\varepsilon_j})(1+\rho+\widehat{\sigma}^2_{\varepsilon_i})(((1+\widehat{\sigma}^2_{\varepsilon_i})(1+\widehat{\sigma}^2_{\varepsilon_j})-\rho^2)(1+\rho))^{-1}$. Applying the implicit function theorem, we get

$$\frac{\partial z}{\partial \lambda_i} = -\frac{\partial\big(\lambda_1/\lambda_2 - N(z)/D(z)\big)/\partial\lambda_i}{\partial\big(\lambda_1/\lambda_2 - N(z)/D(z)\big)/\partial z}.$$

As $\partial(\lambda_1/\lambda_2 - N(z)/D(z))/\partial\lambda_1 > 0$, $\partial(\lambda_1/\lambda_2 - N(z)/D(z))/\partial\lambda_2 < 0$, and $\partial(\lambda_1/\lambda_2 - N(z)/D(z))/\partial z > 0$ because of $z \in (\overline{z}_D, \overline{z}_N)$, we conclude that $\partial z/\partial\lambda_1 < 0$ and $\partial z/\partial\lambda_2 > 0$.

Next, we study the relationship between cs and λ_1. Differentiating (21), we have

$$\frac{\partial c_2}{\partial\lambda_1} = \frac{\partial c_2}{\partial z}\frac{\partial z}{\partial\lambda_1} = \frac{1}{\lambda_2}\left(\frac{n_2\big(1-\Xi_2\Delta_2^{-1}\big)}{n_1 z^2} + \frac{n_1}{(n_1 z + n_2 - 1)^2}\right)\frac{\partial z}{\partial\lambda_1} < 0,$$

since $\partial z/\partial\lambda_1 < 0$. Moreover, as $c_1 = zc_2$, it follows that $\partial c_1/\partial\lambda_1 = (\partial z/\partial\lambda_1)c_2 + z(\partial c_2/\partial\lambda_1) < 0$, because of the positiveness of c_2 and z, and the negativeness of $\partial z/\partial\lambda_1$ and $\partial c_2/\partial\lambda_1$. In relation to a_1 and a_2, from (19), direct computation yields $\partial a_1/\partial\lambda_1 < 0$ and $\partial a_2/\partial\lambda_1 < 0$, since $\partial c_1/\partial\lambda_1 < 0$ and $\partial c_2/\partial\lambda_1 < 0$.

Now we study how the correlation coefficient ρ affects a_1. Let $y = a_1/a_2$. As $a_1 = \Delta_1 c_1$ and $a_2 = \Delta_2 c_2$, then $z = \Delta_2 y/\Delta_1$. Substituting this expression in (25) and after some algebra, we have that

$$\frac{\lambda_1}{\lambda_2}y = \frac{\widetilde{N}(y,\rho)}{\widetilde{D}(y,\rho)}, \tag{32}$$

where $\widetilde{N}(y,\rho) = \frac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_2} - \frac{n_1}{n_2}\widehat{\sigma}^2_{\varepsilon_1}\rho y}{(1+\widehat{\sigma}^2_{\varepsilon_1})(1+\widehat{\sigma}^2_{\varepsilon_2})-\rho^2} - \Big((n_1-1)\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_1}}{1+\rho} + n_2\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_2}}{1+\rho}\frac{1}{y}\Big)^{-1}$ and

$$\widetilde{D}(y,\rho) = \frac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_1} - \dfrac{n_2}{n_1}\widehat{\sigma}^2_{\varepsilon_2}\rho\dfrac{1}{y}}{\big(1+\widehat{\sigma}^2_{\varepsilon_1}\big)\big(1+\widehat{\sigma}^2_{\varepsilon_2}\big)-\rho^2} - \left(n_1\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_1}}{1+\rho}y + (n_2-1)\frac{1+\rho+\widehat{\sigma}^2_{\varepsilon_2}}{1+\rho}\right)^{-1}.$$

Moreover, $a_1 = \widetilde{N}(y,\rho)/\lambda_1$ and $a_2 = \widetilde{D}(y,\rho)/\lambda_2$. Hence,

$$\frac{\partial a_1}{\partial\rho} = \frac{\big(\partial\widetilde{N}(y,\rho)/\partial y\big)(\partial y/\partial\rho) + \partial\widetilde{N}(y,\rho)/\partial\rho}{\lambda_1}.$$

Thus, to show $\partial a_1/\partial\rho < 0$, it suffices to prove that

$$\frac{\partial\widetilde{N}(y,\rho)}{\partial y}\frac{\partial y}{\partial\rho} + \frac{\partial\widetilde{N}(y,\rho)}{\partial\rho} < 0. \tag{33}$$

Direct computation yields $\partial\widetilde{N}(y,\rho)/\partial y < 0$. Then (33) is equivalent to

$$\frac{\partial y}{\partial\rho} > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}. \tag{34}$$

Moreover, recall that y in equilibrium is the unique positive value that satisfies (32). Thus, applying the implicit function theorem, it follows that

$$\frac{\partial y}{\partial\rho} = -\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)/\partial\rho}{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)/\partial y}.$$

Then (34) can be rewritten as

$$-\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)/\partial\rho}{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)/\partial y} > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y},$$

or using the fact that in equilibrium $\partial(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho))/\partial y > 0$, (34) is satisfied if and only if

$$-\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)}{\partial\rho} > -\left(\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}\right)\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)}{\partial y}. \tag{35}$$

Notice that

$$-\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)}{\partial\rho} = -\frac{\big(\partial\widetilde{N}(y,\rho)/\partial\rho\big)\widetilde{D}(y,\rho) - \widetilde{N}(y,\rho)\big(\partial\widetilde{D}(y,\rho)/\partial\rho\big)}{\widetilde{D}^2(y,\rho)},$$

or using (31),

$$-\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)}{\partial\rho} = -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho - \lambda_1 y\big(\partial\widetilde{D}(y,\rho)/\partial\rho\big)/\lambda_2}{\widetilde{D}(y,\rho)}.$$

Analogously,

$$\frac{\partial\big(\lambda_1 y/\lambda_2 - \widetilde{N}(y,\rho)/\widetilde{D}(y,\rho)\big)}{\partial y} = \frac{\lambda_1}{\lambda_2} - \frac{\partial\widetilde{N}(y,\rho)/\partial y - \lambda_1 y\big(\partial\widetilde{D}(y,\rho)/\partial y\big)/\lambda_2}{\widetilde{D}(y,\rho)}.$$

Therefore, (35) is equivalent to

$$\begin{aligned}&\frac{\partial\widetilde{N}(y,\rho)/\partial\rho - \lambda_1 y\big(\partial\widetilde{D}(y,\rho)/\partial\rho\big)/\lambda_2}{\widetilde{D}(y,\rho)} \\ &\quad > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}\left(\frac{\lambda_1}{\lambda_2} - \frac{\partial\widetilde{N}(y,\rho)/\partial y - \lambda_1 y\big(\partial\widetilde{D}(y,\rho)/\partial y\big)/\lambda_2}{\widetilde{D}(y,\rho)}\right)\end{aligned}$$

or

$$-\frac{y\big(\partial\widetilde{D}(y,\rho)/\partial\rho\big)}{\widetilde{D}(y,\rho)} > -\frac{\partial\widetilde{N}(y,\rho)/\partial\rho}{\partial\widetilde{N}(y,\rho)/\partial y}\left(1 + \frac{y\big(\partial\widetilde{D}(y,\rho)/\partial y\big)}{\widetilde{D}(y,\rho)}\right). \tag{36}$$

Moreover, recall that $a_2 = \widetilde{D}(y,\rho)/\lambda_2$. The positiveness of a_2 tells us that $\widetilde{D}(y,\rho) > 0$. After some algebra, we have that $\partial\widetilde{D}(y,\rho)/\partial\rho < 0$, $\partial\widetilde{N}(y,\rho)/\partial\rho < 0$, and $\partial\widetilde{D}(y,\rho)/\partial y > 0$. Hence, we conclude that the left-hand side of (36) is positive, whereas the right-hand side of (36) is negative since $\partial\widetilde{N}(y,\rho)/\partial y < 0$. Consequently, the fact that (36) is satisfied allows us to conclude that $\partial a_1/\partial\rho < 0$.

Concerning the effect of ρ on c_1, recall that $c_1 = a_1/\Delta_1 = (1 + \rho + \widehat{\sigma}^2_{\varepsilon_1})a_1/(1+\rho)$. This expression tells us that c_1 is the product of two decreasing positive functions in ρ. Therefore, $\partial c_1/\partial\rho < 0$.

Next we study how a_1 and c_1 vary with a change in $\sigma^2_{\varepsilon_i}$ and $\sigma^2_{\varepsilon_j}$. To do that, first we analyze the effect of $\sigma^2_{\varepsilon_i}$ on d_1 and d_2. From Proposition 1, we know that $a_i = \Delta_i c_i > 0$,

$i = 1, 2$. Therefore, (2) implies that $d_i = ((n_i - 1)\Delta_i^{-1}a_i + n_j\Delta_j^{-1}a_j)^{-1}$. Substituting the expressions of (17) and the expression for Δ_i given in Lemma A2, it follows that

$$d_i = \left(\frac{(n_i - 1)n_j}{\Omega_i} + \frac{n_j n_i}{\Omega_j}\right)^{-1},$$

where $\Omega_i = n_j \Upsilon_i(d_i + \lambda_i) + n_i(\Upsilon_i - 1)(d_j + \lambda_j)$ and $\Omega_j = n_i \Upsilon_j(d_j + \lambda_j) + n_j(\Upsilon_j - 1)(d_i + \lambda_i)$, with $\Upsilon_i = \Xi_j/(\Xi_j - \Psi_i) = (1 - \rho^2 + \widehat{\sigma}^2_{\varepsilon_i})/((1-\rho)(1 + \rho + \widehat{\sigma}^2_{\varepsilon_i})) > 1$. Therefore, we derive the following equations that are satisfied in equilibrium, i.e., $F_i(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2) = 0$, $i = 1, 2$, where

$$F_i(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2) = \frac{(n_i - 1)n_j d_i}{\Omega_i} + \frac{n_i n_j d_i}{\Omega_j} - 1.$$

Let $DF_{d_1,d_2}(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)$ denote the matrix

$$\begin{pmatrix} \partial F_1(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial d_1 & \partial F_1(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial d_2 \\ \partial F_2(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial d_1 & \partial F_2(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial d_2 \end{pmatrix}.$$

After some tedious algebra, it can be shown that the determinant of $DF_{d_1,d_2}(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)$ is strictly positive. In particular, it is not null and, therefore, this matrix is invertible. Hence, we can apply the implicit function theorem and we have

$$\begin{aligned} &\begin{pmatrix} \partial d_1/\partial\sigma^2_{\varepsilon_1} & \partial d_1/\partial\sigma^2_{\varepsilon_2} \\ \partial d_2/\partial\sigma^2_{\varepsilon_1} & \partial d_2/\partial\sigma^2_{\varepsilon_1} \end{pmatrix} \\ &\quad = -\left(DF_{d_1,d_2}(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)\right)^{-1} \\ &\qquad \times \begin{pmatrix} \partial F_1(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial\sigma^2_{\varepsilon_1} & \partial F_1(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial\sigma^2_{\varepsilon_2} \\ \partial F_2(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial\sigma^2_{\varepsilon_1} & \partial F_2(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial\sigma^2_{\varepsilon_2} \end{pmatrix}. \end{aligned} \tag{37}$$

It is easy to see that all the elements of $(DF_{d_1,d_2}(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2))^{-1}$ are positive. Moreover, $\partial F_i(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial\sigma^2_{\varepsilon_i} < 0$ and $\partial F_i(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)/\partial\sigma^2_{\varepsilon_j} < 0$. Hence, (37) implies that $\partial d_i/\partial\sigma^2_{\varepsilon_i} > 0$ and $\partial d_i/\partial\sigma^2_{\varepsilon_j} > 0$.

Next we study the comparative statics of c_1 and c_2 with respect to $\sigma^2_{\varepsilon_1}$. Recall that $c_i = n_j/\Omega_i$. Using the fact that Υ_1, d_1, and d_2 are increasing in $\sigma^2_{\varepsilon_1}$ and that Υ_2 is independent of $\sigma^2_{\varepsilon_1}$, we have that Ω_1 and Ω_2 are increasing in $\sigma^2_{\varepsilon_1}$, which allows us to conclude that c_1 and c_2 are decreasing in $\sigma^2_{\varepsilon_1}$. Combining these results with the fact that Δ_1 is decreasing in $\sigma^2_{\varepsilon_1}$ and Δ_2 is independent of $\sigma^2_{\varepsilon_1}$, it follows that a_1 and a_2 are decreasing in $\sigma^2_{\varepsilon_1}$, since $a_1 = \Delta_1 c_1$ and $a_2 = \Delta_2 c_2$.

In relation to (f) and (g), note that Lemma A2 shows that the only equilibrium coefficient affected by the quantity offered in the auction (Q) and by the prior mean of the valuations ($\overline{\theta}_i$ and $\overline{\theta}_j$) is b_i. Using (2), we get that d_i is independent of these parameters.

Finally, concerning (h), notice that by similar reasoning as before, we derive the following equations that are satisfied in equilibrium, i.e., $F_i(n_1, n_2, d_1, d_2) = 0$, $i = 1, 2$, where $F_i(n_1, n_2, d_1, d_2) = F_i(\sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_2}, d_1, d_2)$. Hence,

$$\begin{pmatrix} \partial d_1/\partial n_1 & \partial d_1/\partial n_2 \\ \partial d_2/\partial n_1 & \partial d_2/\partial n_2 \end{pmatrix}$$
$$= -\big(DF_{d_1,d_2}(n_1, n_2, d_1, d_2)\big)^{-1} \begin{pmatrix} \partial F_1(n_1, n_2, d_1, d_2)/\partial n_1 & \partial F_1(n_1, n_2, d_1, d_2)/\partial n_2 \\ \partial F_2(n_1, n_2, d_1, d_2)/\partial n_1 & \partial F_2(n_1, n_2, d_1, d_2)/\partial n_2 \end{pmatrix}.$$

Taking into account that all the elements of the previous two matrices are positive, we conclude that $\partial d_i/\partial n_i < 0$ and $\partial d_i/\partial n_j < 0$. □

Proof of Corollary 3. Suppose that $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$. Using the expressions of Ξ_i and Δ_i, it is easy to see that in this case, $\Xi_2\Delta_2^{-1} > \Xi_1\Delta_1^{-1}$. Next, we distinguish two cases.

Case 1: $(n_1 + n_2 - 2)n_1/((n_1 + n_2)(n_1 + n_2 - 1)) \geq 1 - \Xi_2\Delta_2^{-1}$. Evaluating the polynomial $G(\zeta)$, stated in the proof of Lemma A2, at $\zeta = 1$, we have that in this case, $G(1) \geq 0$. This implies that $z \leq 1$ and, therefore, $c_1 \leq c_2$. In addition, using the expressions of d_1 and d_2, we get $\text{sgn}\{d_1 - d_2\} = \text{sgn}\{c_1 - c_2\}$, which implies $d_1 \leq d_2$. Finally, notice that $\Delta_1 \leq \Delta_2$ whenever $\sigma^2_{\varepsilon_1} \geq \sigma^2_{\varepsilon_2}$. Hence, $a_1/a_2 = z\Delta_1/\Delta_2 \leq 1$.

Case 2: $(n_1 + n_2 - 2)n_1/((n_1 + n_2)(n_1 + n_2 - 1)) < 1 - \Xi_2\Delta_2^{-1}$. Notice that

$$\frac{(n_1 + n_2 - 2)n_2}{(n_1 + n_2)(n_1 + n_2 - 1)} - \big(1 - \Xi_1\Delta_1^{-1}\big) \leq \frac{(n_1 + n_2 - 2)n_1}{(n_1 + n_2)(n_1 + n_2 - 1)} - \big(1 - \Xi_2\Delta_2^{-1}\big),$$

since $\Xi_2\Delta_2^{-1} > \Xi_1\Delta_1^{-1}$ and $n_1 \geq n_2$. Thus, in this case, we have that $H_N(1) < 0$ and $H_D(1) < 0$. Taking into account the shape of these polynomials, the previous two inequalities imply that $\overline{z}_D > 1 > \overline{z}_N$. However, Proposition A1 indicates that in this case there is no equilibrium. □

Appendix B: Proofs of results in Section 4

Proof of Proposition 4. Without any loss of generality suppose that $\overline{\theta}_i \geq \overline{\theta}_j$. Using the expression of the optimal demand function, bidders of type j are expected to be buyers when $\overline{\theta}_j > \mathbb{E}[p]$. From the expression of $\mathbb{E}[p]$, the previous inequality is satisfied provided that

$$\frac{n_i(\overline{\theta}_i - \overline{\theta}_j)}{d_i + \lambda_i} < Q. \tag{38}$$

Next we study the relationship between $\mathbb{E}[p]$ and n_i. Note that rewriting the expression of the expected equilibrium price, it follows that

$$\mathbb{E}[p] = \overline{\theta}_j + \frac{n_i(d_j + \lambda_j)(\overline{\theta}_i - \overline{\theta}_j)}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)} - \frac{Q}{\dfrac{n_i}{d_i + \lambda_i} + \dfrac{n_j}{d_j + \lambda_j}}.$$

Differentiating this expression with respect to n_i, we have

$$\frac{\partial}{\partial n_i}\mathbb{E}[p]=\frac{\partial}{\partial n_i}\left(\frac{n_i(d_j+\lambda_j)}{n_i(d_j+\lambda_j)+n_j(d_i+\lambda_i)}\right)(\overline{\theta}_i-\overline{\theta}_j)-\frac{\partial}{\partial n_i}\left(\frac{1}{\dfrac{n_i}{d_i+\lambda_i}+\dfrac{n_j}{d_j+\lambda_j}}\right)Q.$$

Using Proposition 3(ii), it follows that $\frac{\partial}{\partial n_i}(\frac{1}{\frac{n_i}{d_i+\lambda_i}+\frac{n_j}{d_j+\lambda_j}})<0$. Hence, from (38), it follows that

$$\begin{aligned}\frac{\partial}{\partial n_i}\mathbb{E}[p]>{}&\frac{\partial}{\partial n_i}\left(\frac{n_i(d_j+\lambda_j)}{n_i(d_j+\lambda_j)+n_j(d_i+\lambda_i)}\right)(\overline{\theta}_i-\overline{\theta}_j)\\&-\frac{\partial}{\partial n_i}\left(\frac{1}{\dfrac{n_i}{d_i+\lambda_i}+\dfrac{n_j}{d_j+\lambda_j}}\right)\frac{n_i(\overline{\theta}_i-\overline{\theta}_j)}{d_i+\lambda_i}\end{aligned}$$

and, from direct computations,

$$\frac{\partial}{\partial n_i}\mathbb{E}[p]>\left(d_j+\lambda_j-\frac{n_i(d_j+\lambda_j)\dfrac{\partial}{\partial n_i}d_i}{d_i+\lambda_i}\right)\frac{\overline{\theta}_i-\overline{\theta}_j}{n_i(d_j+\lambda_j)+n_j(d_i+\lambda_i)}.$$

From Proposition 3(ii), we know that $\frac{\partial}{\partial n_i}d_i<0$. Hence, we can conclude that $\mathbb{E}[p]$ increases with n_i. Given that the proofs of how the other underlying parameters affect the expected equilibrium price are similar to the previous one, they are omitted.

Now suppose that $|\overline{\theta}_i-\overline{\theta}_j|$ is high enough or $Q=0$. The results we have just derived may not hold. For example, let us focus on the relationship between the expected price and n_1 when $|\overline{\theta}_i-\overline{\theta}_j|$ is high enough. To study this relationship, we first show that $n_2(d_1+\lambda_1)/(n_1(d_2+\lambda_2))$ decreases with n_1. Recall that $d_2=((n_2-1)n_1/\Omega_2+n_1n_2/\Omega_1)^{-1}$. Using the expressions of Ω_i, we have

$$1=\left(\frac{n_2-1}{\Upsilon_2+(\Upsilon_2-1)\dfrac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}}+\frac{n_2}{\Upsilon_1\dfrac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}+\Upsilon_1-1}\right)^{-1}+\frac{\lambda_2}{d_2+\lambda_2}.$$

The fact that d_2 decreases with n_1 implies that $\lambda_2/(d_2+\lambda_2)$ increases with n_1. Then the previous inequality tells us that $\frac{n_2-1}{\Upsilon_2+(\Upsilon_2-1)\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}}+\frac{n_2}{\Upsilon_1\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}+\Upsilon_1-1}$ increases with n_1. For this to be possible, $\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}$ needs to be decreasing in n_1. Given that the expected price satisfies

$$\begin{aligned}\mathbb{E}[p]={}&\left(1+\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}\right)^{-1}\overline{\theta}_1+\left(1-\left(1+\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}\right)^{-1}\right)\overline{\theta}_2\\&-\left(\frac{n_1}{d_1+\lambda_1}+\frac{n_2}{d_2+\lambda_2}\right)^{-1}Q,\end{aligned}$$

we have that the relationship between the expected price and n_1 is ambiguous. For instance, if $\overline{\theta}_2$ is low enough, then the fact that d_1, d_2, and $n_2(d_1+\lambda_1)/(n_1(d_2+\lambda_2))$ are decreasing in n_1 allows us to conclude that the expected price increases with n_1. However, if $\overline{\theta}_2$ is large, and $\overline{\theta}_1$ and Q are low enough, then the expected price decreases with n_1. □

Lemma B1. *The equilibrium quantities solve the distorted benefit maximization program*

$$\max_{x_1,x_2} \mathbb{E}\big[n_1\big(\theta_1 x_1 - (d_1+\lambda_1)x_1^2/2\big) + n_2\big(\theta_2 x_2 - (d_2+\lambda_2)x_2^2/2\big)|t\big]$$
$$s.t.\ n_1 x_1 + n_2 x_2 = Q,$$

taking as given the equilibrium parameters d_1 and d_2.

Proof. The Lagrangian function of the maximization program is given by

$$\mathcal{L}(x_1, x_2, \mu) = n_1\big(t_1 x_1 - (d_1+\lambda_1)x_1^2/2\big) + n_2\big(t_2 x_2 - (d_2+\lambda_2)x_2^2/2\big) - \mu(n_1 x_1 + n_2 x_2 - Q),$$

where μ denotes the Lagrange multiplier. Differentiating, we obtain the FOCs

$$n_1\big(t_1 - (d_1+\lambda_1)x_1\big) - \mu n_1 = 0 \tag{39}$$
$$n_2\big(t_2 - (d_2+\lambda_2)x_2\big) - \mu n_2 = 0 \tag{40}$$
$$n_1 x_1 + n_2 x_2 = Q. \tag{41}$$

From (39) and (40), it follows that $x_i = (t_i - \mu)/(d_i + \lambda_i)$, $i = 1, 2$. Substituting these expressions in (41) and operating, we have $\mu = (\frac{n_1 t_1}{d_1+\lambda_1} + \frac{n_2 t_2}{d_2+\lambda_2} - Q)(\frac{n_1}{d_1+\lambda_1} + \frac{n_2}{d_2+\lambda_2})^{-1}$. Then plugging this expression into (39) and (40), we get the expressions of the equilibrium quantities given in (8). In addition, since the objective function is concave and the constraint is a linear equation, we conclude that the critical point is a global maximum. Hence, the equilibrium quantities are the solutions of the optimization problem stated in Lemma B1. □

Proposition B1. *Let $Q = (n_1+n_2)q$ and let $\mu_i = n_i/(n_1+n_2)$. Then there exists a unique price-taking equilibrium, and the equilibrium coefficients of the demand function for a type-i bidder are given by*

$$b_i^o = \frac{\widehat{\sigma}_{\varepsilon_i}^2\big(\mu_j(\overline{\theta}_i - \rho\overline{\theta}_j) + \rho\lambda_j q\big)}{\mu_i\rho\widehat{\sigma}_{\varepsilon_i}^2\lambda_j + \mu_j\big(1-\rho^2+\widehat{\sigma}_{\varepsilon_i}^2\big)\lambda_i}\quad a_i^o = \frac{\mu_j\big(1-\rho^2\big)}{\mu_i\rho\widehat{\sigma}_{\varepsilon_i}^2\lambda_j + \mu_j\big(1-\rho^2+\widehat{\sigma}_{\varepsilon_i}^2\big)\lambda_i}$$
$$c_i^o = \frac{\mu_j(1-\rho)\big(1+\rho+\widehat{\sigma}_{\varepsilon_i}^2\big)}{\mu_i\rho\widehat{\sigma}_{\varepsilon_i}^2\lambda_j + \mu_j\big(1-\rho^2+\widehat{\sigma}_{\varepsilon_i}^2\big)\lambda_i}.$$

Proof. In the competitive setup, the FOC of the optimization problem for a type-i bidder is given by $\mathbb{E}[\theta_i|s_i, p] - p - \lambda_i x_i = 0$. Doing computations similar to the proof of

Lemma A1, we derive the system of equations[29]

$$b_i = \left((1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j + \Psi_i\left(\frac{q-\mu_i b_i - \mu_j b_j}{\mu_j a_j}\right)\right)\Big/\lambda_i \tag{42}$$

$$a_i = \left(\Xi_i - \frac{\mu_i}{\mu_j}\frac{a_i}{a_j}\Psi_i\right)\Big/\lambda_i \tag{43}$$

$$c_i = \left(1 - \Psi_i\left(\frac{\mu_i c_i + \mu_j c_j}{\mu_j a_j}\right)\right)\Big/\lambda_i, \quad i, j = 1, 2, j \neq i. \tag{44}$$

Note that $a_i/a_j = ((\Xi_i - \frac{\mu_i}{\mu_j}\frac{a_i}{a_j}\Psi_i)/\lambda_i)/((\Xi_j - \frac{\mu_j}{\mu_i}\frac{a_j}{a_i}\Psi_j)/\lambda_j)$. Hence,

$$a_i/a_j = \mu_j(\Psi_j\lambda_i\mu_j + \Xi_i\lambda_j\mu_i)/\big(\mu_i(\Psi_i\lambda_j\mu_i + \lambda_i\Xi_j\mu_j)\big).$$

Then plugging the previous expression into (43), we get

$$a_i = \frac{\mu_j(\Xi_i\Xi_j - \Psi_i\Psi_j)}{\mu_j\Xi_j\lambda_i + \mu_i\Psi_i\lambda_j}. \tag{45}$$

Furthermore, using (42) and after some algebra, we have

$$\mu_i b_i + \mu_j b_j = \frac{\dfrac{\mu_i}{\lambda_i}\left((1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j + \dfrac{\Psi_i}{\mu_j a_j}q\right) + \dfrac{\mu_j}{\lambda_j}\left((1-\Xi_j)\overline{\theta}_j - \Psi_j\overline{\theta}_i + \dfrac{\Psi_j}{\mu_i a_i}q\right)}{\dfrac{\Psi_i}{\lambda_i}\dfrac{\mu_i}{\mu_j a_j} + \dfrac{\Psi_j}{\lambda_j}\dfrac{\mu_j}{\mu_i a_i} + 1}.$$

Substituting (45) and the last expression into (42), it follows that

$$b_i = a_i\left(\frac{\Xi_j\overline{\theta}_i - \Psi_i\overline{\theta}_j}{\Xi_i\Xi_j - \Psi_i\Psi_j} - \overline{\theta}_i\right) + \frac{\lambda_j\Psi_i}{\mu_j\Xi_j\lambda_i + \mu_i\Psi_i\lambda_j}q. \tag{46}$$

In addition, from (44) and after some algebra, we get

$$\mu_i c_i + c_j\mu_j = \left(\frac{\mu_i}{\lambda_i} + \frac{\mu_j}{\lambda_j}\right)\Big/\left(\frac{\Psi_i}{\lambda_i}\frac{\mu_i}{\mu_j a_j} + \frac{\Psi_j}{\lambda_j}\frac{\mu_j}{\mu_i a_i} + 1\right).$$

Using (45) and the last expression in (44), we have

$$c_i = \frac{\mu_j(\Xi_j - \Psi_i)}{\mu_j\Xi_j\lambda_i + \mu_i\Psi_i\lambda_j}. \tag{47}$$

Finally, substituting the expressions for Ξ_i, Ξ_j, Ψ_i, and Ψ_j in (45)–(47), we obtain the formulas stated in this proposition. □

[29]To ease the notation, the superscript o is omitted in this proof.

Proof of Proposition 5. Performing computations similar to the proof of Lemma A1, we obtain that the equilibrium coefficients with subsidies $\kappa_i = d_i(c_1^o, c_2^o)$ satisfy

$$b_i = \frac{(1-\Xi_i)\overline{\theta}_i - \Psi_i\overline{\theta}_j - \dfrac{\Psi_i(n_i b_i + n_j b_j - Q)}{n_j a_j}}{d_i + \lambda_i - d_i(c_1^o, c_2^o)} \quad a_i = \frac{\Xi_i - \dfrac{n_i a_i}{n_j a_j}\Psi_i}{d_i + \lambda_i - d_i(c_1^o, c_2^o)} > 0$$

$$c_i = \frac{1 - \dfrac{\Psi_i(n_i c_i + n_j c_j)}{n_j a_j}}{d_i + \lambda_i - d_i(c_1^o, c_2^o)}, \quad i, j = 1, 2, j \neq i.$$

Comparing this system of equations and those derived in the proof of Proposition B1, using $Q = (n_1 + n_2)q$ and $\mu_i = n_i/(n_1 + n_2)$, we obtain that the equilibrium coefficients of the price-taking equilibrium solves this system. Therefore, we conclude that the quadratic subsidies, $\kappa_i x_i^2/2$ with $\kappa_i = d_i(c_1^o, c_2^o)$, induce an efficient allocation. The closed-form expressions for the optimal subsidy rates are

$$\kappa_i = \frac{1}{n_j(1-\rho)}\left(\frac{(n_i - 1)(1 + \widehat{\sigma}_{\varepsilon_i}^2 + \rho)}{n_i\lambda_j\rho\widehat{\sigma}_{\varepsilon_i}^2 + n_j\lambda_i(1 - \rho^2 + \widehat{\sigma}_{\varepsilon_i}^2)} + \frac{n_i(1 + \widehat{\sigma}_{\varepsilon_j}^2 + \rho)}{n_i\lambda_j(1 - \rho^2 + \widehat{\sigma}_{\varepsilon_j}^2) + n_j\lambda_i\rho\widehat{\sigma}_{\varepsilon_j}^2}\right)^{-1}.$$

If $\rho = 0$ (or, with full information, if $\sigma_{\varepsilon_i}^2 = 0$, $i = 1, 2$), then $\kappa_i^f = 1/((n_i - 1)\lambda_i^{-1} + n_j\lambda_j^{-1})$. □

Lemma B2. *The expected deadweight loss at an anonymous allocation* $(x_1(t), x_2(t))$ *satisfies*

$$\mathbb{E}[\mathrm{DWL}] = \frac{1}{2}\lambda_1 n_1 \mathbb{E}\big[\big(x_1(t) - x_1^o(t)\big)^2\big] + \frac{1}{2}\lambda_2 n_2 \mathbb{E}\big[\big(x_2(t) - x_2^o(t)\big)^2\big]. \tag{48}$$

Proof. Notice that $\mathbb{E}[\mathrm{TS}] = \mathbb{E}[\mathbb{E}[\mathrm{TS}|t]]$, where

$$\begin{aligned}\mathbb{E}[\mathrm{TS}|t] &= \mathbb{E}\big[n_1\big(\theta_1 x_1(t) - \lambda_1\big(x_1(t)\big)^2/2\big) + n_2\big(\theta_2 x_2(t) - \lambda_2\big(x_2(t)\big)^2/2\big)|t\big] \\ &= n_1\big(t_1 x_1(t) - \lambda_1\big(x_1(t)\big)^2/2\big) + n_2\big(t_2 x_2(t) - \lambda_2\big(x_2(t)\big)^2/2\big).\end{aligned}$$

A Taylor series expansion of $\mathbb{E}[\mathrm{TS}|t]$ around the price-taking equilibrium $(x_1^o(t), x_2^o(t))$, stopping at the second term due to the fact that $\mathbb{E}[\mathrm{TS}|t]$ is quadratic, yields

$$\begin{aligned}\mathbb{E}[\mathrm{TS}|t]\big(x(t)\big) &= \mathbb{E}[\mathrm{TS}|t]\big(x^o(t)\big) + \nabla\mathbb{E}[\mathrm{TS}|t]\big(x^o(t)\big)\big(x(t) - x^o(t)\big) \\ &\quad + \frac{1}{2}\big(x(t) - x^o(t)\big)' D^2\mathbb{E}[\mathrm{TS}|t]\big(x^o(t)\big)\big(x(t) - x^o(t)\big),\end{aligned}$$

where $\nabla\mathbb{E}[\mathrm{TS}|t](x^o(t))$ and $D^2\mathbb{E}[\mathrm{TS}|t](x^o(t))$ are, respectively, the gradient and the Hessian matrix of $\mathbb{E}[\mathrm{TS}|t]$ evaluated at $x^o(t)$. Because of optimality of $x^o(t)$,

$$\nabla\mathbb{E}[\mathrm{TS}|t]\big(x^o(t)\big) = (0, 0).$$

In addition, $D^2\mathbb{E}[\mathrm{TS}|t](x^o(t)) = \begin{pmatrix} -\lambda_1 n_1 & 0 \\ 0 & -\lambda_2 n_2 \end{pmatrix}$. Hence,

$$\mathbb{E}[\mathrm{TS}|t]\big(x(t)\big) - \mathbb{E}[\mathrm{TS}|t]\big(x^o(t)\big) = -\frac{1}{2}\lambda_1 n_1\big(x_1(t) - x_1^o(t)\big)^2 - \frac{1}{2}\lambda_2 n_2\big(x_2(t) - x_2^o(t)\big)^2$$

and, therefore, (48) is satisfied. □

Appendix C: Proofs of results in Section 5

Proof of Proposition 6. Using (28) and (29), it follows that $\lim_{n_1\to\infty} \overline{z}_N = \lim_{n_1\to\infty} \overline{z}_D = 0$. Furthermore, after some algebra, we have that the necessary and sufficient condition for the existence of an equilibrium (i.e., $\lim_{n_1\to\infty} \overline{z}_N/\overline{z}_D > 1$) is equivalent to $n_2 > \bar{n}_2(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2})$, where

$$\bar{n}_2\big(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2}\big) = \frac{\rho\big((2-\rho)\widehat{\sigma}^2_{\varepsilon_2} + 2\big(1-\rho^2\big)\big)\widehat{\sigma}^2_{\varepsilon_1}}{\big(1-\rho^2\big)\big(\big(1+\widehat{\sigma}^2_{\varepsilon_1}\big)\big(1+\widehat{\sigma}^2_{\varepsilon_2}\big) - \rho^2\big)}.$$

Moreover, taking the limit in (25), it follows that $\lim_{n_1\to\infty} z = 0$ and

$$\lim_{n_1\to\infty} n_1 z = n_2 \Xi_1 \Delta_1^{-1}/\big(1 - \Xi_1\Delta_1^{-1}\big). \tag{49}$$

Using the expressions included in the statement of Lemma A2 and after some tedious algebra, we get $\lim_{n_1\to\infty} b_1 = q$, $\lim_{n_1\to\infty} a_1 = 0$, $\lim_{n_1\to\infty} c_1 = 0$, $\lim_{n_1\to\infty} a_2 = \Delta_2 \lim_{n_1\to\infty} c_2$,

$$\lim_{n_1\to\infty} b_2 = \frac{\widehat{\sigma}^2_{\varepsilon_2}\left(\dfrac{(n_2-1)\big(1-\rho^2\big)}{\big(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_2}\big)} + \dfrac{\big(1-\rho^2+\widehat{\sigma}^2_{\varepsilon_1}(1-2\rho)\big)}{\big(1+\widehat{\sigma}^2_{\varepsilon_1}\big)\big(1+\widehat{\sigma}^2_{\varepsilon_2}\big)-\rho^2}\right)(\overline{\theta}_2 - \rho\overline{\theta}_1 + q\rho\lambda_1)}{(1-\rho)\lambda_2\left(n_2(1+\rho) - \dfrac{\rho\widehat{\sigma}^2_{\varepsilon_1}\big(1+\rho+\widehat{\sigma}^2_{\varepsilon_2}\big)}{\big(1+\widehat{\sigma}^2_{\varepsilon_1}\big)\big(1+\widehat{\sigma}^2_{\varepsilon_2}\big)-\rho^2}\right)}$$

$$+ q\frac{\rho^2\widehat{\sigma}^2_{\varepsilon_2}\widehat{\sigma}^2_{\varepsilon_1}}{n_2\big(1-\rho^2\big)\big(\big(1+\widehat{\sigma}^2_{\varepsilon_1}\big)\big(1+\widehat{\sigma}^2_{\varepsilon_2}\big)-\rho^2\big)}$$

$$\lim_{n_1\to\infty} c_2 = \frac{n_2 - \bar{n}_2\big(\rho, \widehat{\sigma}^2_{\varepsilon_1}, \widehat{\sigma}^2_{\varepsilon_2}\big)}{\lambda_2\dfrac{1-\rho^2+\widehat{\sigma}^2_{\varepsilon_2}}{1-\rho}\left(\dfrac{n_2}{\big(1+\rho+\widehat{\sigma}^2_{\varepsilon_2}\big)} - \dfrac{\rho\widehat{\sigma}^2_{\varepsilon_1}}{(1+\rho)\big(\big(1+\widehat{\sigma}^2_{\varepsilon_1}\big)\big(1+\widehat{\sigma}^2_{\varepsilon_2}\big)-\rho^2\big)}\right)}. \tag{50}$$

Next, in relation to the expressions for d_1 and d_2, we have that

$$\lim_{n_1\to\infty} d_1 = \lim_{n_1\to\infty}\big((n_1-1)c_1 + n_2c_2\big)^{-1} = \left(\lim_{n_1\to\infty}\left(\frac{(n_1-1)}{n_1}n_1 z + n_2\right)\lim_{n_1\to\infty} c_2\right)^{-1} > 0.$$

The fact that $n_1 z$ and c_2 converge to a positive finite number, as shown in (49) and (50), implies that d_1 does not converge to zero (provided that $\rho\widehat{\sigma}^2_{\varepsilon_1} > 0$; if $\rho\widehat{\sigma}^2_{\varepsilon_1} = 0$, then it is easy to see that $\lim_{n_1\to\infty} n_1 z = \infty$). A similar result is obtained with the limit of d_2. In particular,

$$\lim_{n_1\to\infty} d_2 = \left(\left(\lim_{n_1\to\infty} n_1 z + n_2 - 1\right)\lim_{n_1\to\infty} c_2\right)^{-1} > \lim_{n_1\to\infty} d_1 > 0. \qquad \square$$

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Co-editor Simon Board handled this manuscript.

Manuscript received 14 March, 2019; final version accepted 26 October, 2020; available online 9 November, 2020.