Theoretical Economics 16 (2021), 1139–1194 1555-7561/20211139

Mechanism design with financially constrained agents and costly verification

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A principal distributes an indivisible good to budget-constrained agents when both valuation and budget are agents' private information. The principal can verify an agent's budget at a cost. The welfare-maximizing mechanism can be implemented via a two-stage scheme. First, agents report their budgets, receive cash transfers, and decide whether to enter a lottery over the good. Second, recipients of the good can sell it on a resale market but must pay a sales tax. Low-budget agents receive a higher cash transfer, pay a lower price to enter the lottery, and face a higher sales tax. They are also randomly inspected.

Keywords. Mechanism design, budget constraints, efficiency, costly verification.

JEL classification. D45, D61, D82, H42.

1. Introduction

Governments worldwide allocate a variety of valuable resources to agents who are financially constrained. In Singapore, for example, 80% of the population's housing needs are met by the Housing and Development Board (HDB), a government agency founded in 1960 to provide affordable housing. Similar public housing programs prevail in many other countries. In China, several cities limit the supply of vehicle licenses to curb growth in the number of private vehicles, and various cities have implemented different mechanisms. For example, Shanghai allocates vehicle licenses through an auction-like mechanism, while Beijing uses a vehicle license lottery (see Rong et al. 2019). The evaluation of existing mechanisms has attracted attention from researchers and policymakers.[1]

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This paper is based on a chapter of my dissertation at the University of Pennsylvania and was supported in part by an SAS dissertation completion fellowship. I would like to express my gratitude to Rakesh Vohra for his encouraging support from the very beginning of the project. I am grateful to the two anonymous referees for their insightful comments and suggestions, which greatly improved the paper. For their valuable discussions, I am grateful to (in no particular order) Arjada Bardhi, Aislinn Bohren, Hanming Fang, Qing Gong, Daniel Hauser, Ilwoo Hwang, Nick Janetos, Shinsuke Kambe, George Mailath, Steven Matthews, Mariann Ollar, Mallesh Pai, Andrew Postlewaite, Debraj Ray, Nishant Ravi, Michael Richter, Phillip Strack, and the participants in seminars at University of Pennsylvania, City University of Hong Kong, Tsinghua University, Hong Kong Baptist University, The University of Hong Kong, Peking University, University of Groningen, University of Bonn, National University of Singapore, the 11th Japan-Taiwan-Hong Kong Contract Theory Conference, the 18th ACM Conference on Economics and Computation, the 2016 NASM of the Econometric Society, the 27th International Conference on Game Theory, and EconCon 2016. All remaining errors are my sole responsibility.

[1] For more examples, see the discussion of Che et al. (2013).

 https://doi.org/10.3982/TE3907

One justification for governments assuming this role is that a competitive market outcome will not maximize welfare if households are financially constrained. Financial constraints mean that in a competitive market, some households with high valuations will not obtain resources, while households with low valuations but access to cash will. A question then naturally arises: What is the welfare-maximizing (or optimal) mechanism in circumstances when both valuations and financial constraints are households' private information?

The mechanism design literature concerning this question has focused on mechanisms with only monetary transfers and has ignored the possibility of governments (or the principal) verifying the private information supplied by households (or agents). Indeed, in many instances, governments rely on household reports of their ability to pay when determining allocations, and governments can verify this information and punish a household that makes a false statement. For example, applicants for HDB flats in Singapore are subject to a set of eligibility conditions concerning age, family nucleus, monthly income, etc. Verification presumably allows governments to more accurately target those in need of help. However, the verification process is often costly, as governments may need to verify an applicant's employment history over the past few months or years or examine the business records of self-employed applicants. Even if the verification cost for one individual is low, the total cost can be substantial for a large population. For example, in 2018, the HDB of Singapore received over 47,000 applications, which accounts for 0.8% of Singapore's population.

Hence, it is important to explore how the option of costly verification affects the optimal mechanism. Verification allows the principal to better target financially constrained agents, but it is costly and reduces the amount of money available for subsidies. The principal faces a tradeoff between allocative efficiency and verification cost.

To study these questions, I consider a mechanism design problem in which there is a unit mass of a continuum of agents and a limited supply of indivisible goods. Each agent has two-dimensional private information—his valuation $v \in [\underline{v}, \overline{v}]$ and his budget b. Agents cannot be compelled to pay more than their budget. For simplicity, I assume that there are only two possible budgets, $b_2 > b_1$. The principal can verify an agent's budget at a cost and impose a penalty if the agent is found to have lied about his budget. The principal is also subject to a budget balance (BB) constraint requiring that the revenue from selling the good must exceed the total verification cost. This constraint rules out the possibility that the principal can offer an unlimited subsidy and relieve all budget constraints. I characterize the optimal mechanism that maximizes utilitarian efficiency.

In the optimal direct mechanism, there exist three cutoffs $v_1^* \leq v_2^* \leq v_2^{**}$, as illustrated in Figure 1. High-budget agents whose valuations exceed v_2^{**} (those in A) receive the good at the full price. High-budget agents whose valuations lie in the range $[v_2^*, v_2^{**}]$ and low-budget agents whose valuations exceed v_1^* (those in $B + C$) receive the good with probability $a^* \in (0, 1]$ at a discount price, with low-budget agents receiving a deeper discount. The remaining low-valuation agents receive no good but do receive a cash transfer, with low-budget agents receiving a higher cash transfer. Finally, only low-budget

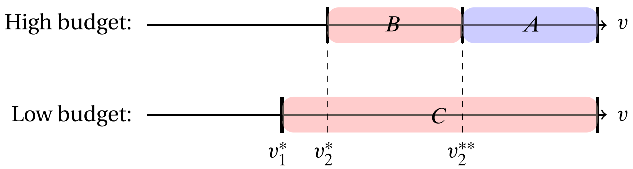


FIGURE 1. Illustration of the optimal mechanism.

agents are inspected randomly, and the inspection probability increases with the probability of assignment.

This paper is closely related to Che et al. (2013) (hereafter CGK), who also consider the problem of assigning resources to budget-constrained agents but do not consider the possibility that the principal can verify agent budget at a cost. Their optimal mechanism can be viewed as a special case of the above optimal mechanism in which the cost of verification precludes the principal from ever using it. In this case, all agents receive the same cash transfer, and agents in $B + C$ in Figure 1 receive the same price discount regardless of their budget (which implies that $v_1^* = v_2^*$). Thus, verification affects the optimal mechanism in a simple way: low-budget agents are favored in terms of *higher cash transfers or lower payments.* A rough intuition is that high-budget agents prefer to receive the good with the same probability as low-budget agents but at a higher price rather than receiving the good with a lower probability but at the same price. Thus, favoring low-budget agents in terms of the payment saves verification cost.[2]

Similar to CGK, the optimal mechanism here involves both cash and in-kind transfers (the provision of goods at discount prices). Verification allows the principal to better target low-budget agents and to provide them more transfers. If verification is cheap, the principal subsidizes low-budget agents mainly by offering them more cash transfers, which do not distort allocation. As verification becomes costlier, the difference in cash transfers declines, but that in in-kind transfers increases. This is because in-kind transfers are cheaper in terms of verification costs, as they are attractive only to high-valuation agents. Eventually, when verification becomes sufficiently costly, agents of both budgets receive the same amount of cash and in-kind transfers, and the optimal mechanism collapses to that in CGK.

The above findings thus have useful implications for the optimal method of subsidizing financially constrained people. First, it provides an explanation for the widespread observation that in many transfer programs, governments give benefits in kind, but they also expend considerable resources verifying income (Currie and Gahvari 2008). That is, an in-kind transfer may be an important aspect of transfer programs when verifying an agent's financial information is costly. Second, if we interpret verification cost as a measure of institutional quality (such as bureaucratic efficiency and

[2]CGK characterize the optimal mechanism in a simple 2×2 model, in which each agent has two possible valuations and two possible budgets. Therefore, by assumption, region B does not exist in their model, and only low-budget agents are induced to consume a random amount. They also consider a different form of budget constraint, which, as will become clear later, does not qualitatively affect the optimal mechanism.

information availability), the above findings suggest that countries with better institutions should rely more on cash transfers, while those with worse institutions should rely more on in-kind transfers.

In a direct mechanism, agents report their budgets and valuations, which is unrealistic because households are rarely asked to report their valuations in practice. As the second main result of this paper, I show that the optimal direct mechanism can be implemented via *random assignment with regulated resale and cash subsidy*, which is based on a similar scheme in CGK. The implementation consists of two stages. In the first stage, agents report only their budgets. The principal then provides them with cash transfers and the opportunity to participate in a lottery. Those who report low budgets receive more cash transfers and pay a lower price to enter the lottery. The principal then assigns the good at random (with uniform probability) among all lottery participants. In the second stage, a resale market opens, but the principal regulates the market by imposing a sales tax. Sellers who report low budgets in the first stage are subject to a higher sales tax. Finally, only agents who report low budgets are inspected randomly.

An important feature of this implementation is that low-budget agents receive a higher subsidy for their initial purchase and face more restrictions in the resale market. Intuitively, low-valuation agents may also want to participate in the lottery for the opportunity to sell the good later, which lowers the probability that low-budget high-valuation agents obtain the good. To reduce arbitrage, the principal imposes restrictions on resale. Since low-budget agents pay a lower price to enter the lottery and have a stronger incentive to engage in arbitrage, they face more restrictions in the resale market. As I will discuss in Section 4, this feature is present in the affordable housing program in Singapore. This feature also distinguishes this paper from CGK, in whose implementation agents of both budgets receive the same subsidy and face the same restriction. Therefore, CGK cannot explain the observed positive correlation between subsidies and restrictions in the Singapore program.

Introducing costly verification is technically challenging because it is no longer sufficient to consider "local" incentive compatibility (IC) constraints. Since the IC constraints between distant types can also bind, one cannot anticipate a priori the set of binding IC constraints. Specifically, if only valuations are private information, it is sufficient to consider adjacent IC constraints; if budgets are also private information, but the principal cannot verify budgets, it is sufficient to consider two one-dimensional deviations. These, however, no longer apply when both valuations and budgets are private information and the principal can verify budgets at a cost. In this case, in addition to adjacent IC constraints of underreporting values, one must consider deviations in which an agent misreports both dimensions of his private information. As a result, the local approach commonly used does not work here.

To solve this problem, I develop a novel method. First, I restrict attention to a class of allocation rules that have sufficient structure to help me track binding IC constraints and that are also rich enough to approximate any general allocation rule well. Specifically, I approximate the allocation rule of each budget type using step functions. When restricting attention to step functions, the binding IC constraints corresponding to the underreporting of budgets are between different budget types whose valuations are the jump discontinuity points of their allocation rules. This structure allows me to write the optimal verification rule as a function of the possible values and the jump discontinuity points of the allocation rule. I then solve a modified version of the principal's problem in which the allocation rule of low-budget types is restricted to take at most M distinct values. Finally, because step functions can approximate the optimal allocation rule arbitrarily well when M is sufficiently large, I can obtain a characterization of the optimal mechanism in the limit.

The remainder of the paper is organized as follows. Section 1.1 discusses related work. Section 2 presents the model. Section 3 characterizes the direct optimal mechanism. Section 4 provides a simple implementation. Section 5 studies the per-unit price constraint considered in CGK. All proofs are relegated to the Appendix.

1.1 *Related literature*

Financially constrained agents This paper is closely related to the literature on mechanism design problems with financially constrained agents. There is much work analyzing the revenue or efficiency of standard auctions when agents are financially constrained. Che and Gale (1998, 2006) study the revenue and efficiency of first-price, second-price, and all-pay auctions. Benoît and Krishna (2001) study multiple-object auctions and compare sequential and simultaneous ascending auctions. Brusco and Lopomo (2008) study strategic demand reduction in simultaneous ascending auctions and show that inefficiencies can arise even if the probability of bidders being budget constrained goes to zero.

Relatively fewer works study the design of an optimal mechanism. Laffont and Robert (1996) and Maskin (2000) study the revenue-maximizing and efficiency-maximizing mechanisms, respectively, when agents have the same commonly known budget. Malakhov and Vohra (2008) study the revenue-maximizing mechanism when one agent has a commonly known budget constraint, and the other agent is not budget constrained. Che and Gale (2000) characterize the revenue-maximizing mechanism when selling a good to a single buyer with a privately known budget constraint. Pai and Vohra (2014) generalize Che and Gale (2000) to the case of multiple buyers when a buyer's valuation and budget are drawn independently. Richter (2019), similar to this paper, considers a setting in which there is a unit mass of a continuum of agents and a limited supply of goods. In Richter (2019), agents have linear preferences for an unlimited supply of goods. He finds that the surplus-maximizing mechanism features a linear price for goods with a uniform cash subsidy to all agents.

This paper contributes to the literature by considering the possibility that the principal can verify an agent's budget at a cost.

Costly state verification The paper is related to the costly state verification literature. The first significant contribution to this series is from Townsend (1979), who studies a model with a principal and a single agent. In Townsend (1979), verification is deterministic. Border and Sobel (1987) and Mookherjee and Png (1989) generalize it by allowing random verification. Gale and Hellwig (1985) consider the effects of costly verification in the context of credit markets. In a recent contribution, Ben-Porath et al. (2014) study the allocation problem in a costly state verification framework when there are multiple agents and monetary transfer is not possible. Li (2020) extends Ben-Porath et al. (2014) to environments in which the principal's ability to punish an agent is sufficiently limited. Erlanson and Kleiner (2020) consider a model with costly verification in which a principal chooses between implementing a new policy and maintaining the status quo. In the above papers, each agent has only one-dimensional private information. By contrast, in this paper, both valuation and budget are private information, and the principal can verify only budget.

Costless or ex post verification This paper is also related to the literature on costless or ex post verification. Glazer and Rubinstein (2004) can be interpreted as a model of a principal and one agent with limited but costless verification and no monetary transfers. Mylovanov and Zapechelnyuk (2017) study a model of multiple agents with costless verification but sufficiently limited punishments. This paper differs from these earlier studies in that verification is costly and there are monetary transfers.

2. Model

There is a unit mass of a continuum of agents and a mass $S \in (0, 1)$ of indivisible goods. Each agent has two-dimensional private information, a private valuation (v) of the good and a privately known budget (b). His private valuation can take a continuum of possible values, that is, $v \in [\underline{v}, \overline{v}] \subset \mathbb{R}_+$. For tractability, there are only two possible budgets, that is, $b \in \{b_1, b_2\}$. For ease of exposition, assume $b_2 > \overline{v} > b_1 > \underline{v}$, so that a high-budget agent is never budget constrained in an individually rational mechanism. The type of agent is a pair consisting of his valuation and his budget, $t := (v, b)$, and the type space is $T := [\underline{v}, \overline{v}] \times \{b_1, b_2\}$. v and b are independent. Each agent has a high budget with probability π and a low budget with probability $1 - \pi$. The valuation v is distributed with cumulative distribution function F and strictly positive density f.

The principal can verify an agent's budget at a cost $k \geq 0$ and can impose a nontransferable penalty $c > 0$. Verification perfectly reveals an agent's budget. The penalty c is large enough that an agent never finds it optimal to misreport his budget if he is certain that he will be inspected. For later use, let $\rho := k/c$. As will become clear, ρ measures the "effective" verification cost to the principal. The cost to an agent of having his report verified is zero. This assumption is reasonable if the goods are valuable to agents and disclosure costs are negligible.

The usual version of the revelation principle (see, e.g., Myerson 1979 and Harris and Townsend 1981) does not apply to models with verification. However, it is not difficult to extend the argument to this type of environment.[3] Specifically, I show in Appendix A

[3]See Townsend (1988) and Ben-Porath et al. (2014) for further discussion and the extension of the revelation principle to various verification models, not including the environment considered in this paper. This result is also related to Green and Laffont (1986), who study implementation with evidence. In their paper, agents, rather than the principal, bear the burden of proof.

that attention can be restricted to direct mechanisms without loss of generality. Furthermore, I assume that the principal can only punish an agent who is inspected and found to have lied about his budget in a direct mechanism. This assumption, however, is not without loss of generality. Roughly, if we relax this assumption, in an optimal mechanism, the principal will sometimes punish a low-budget agent without verifying his budget. In this case, punishment plays a role similar to that of "red tape" in Banerjee (1997) and can be used to screen agents with different valuations when their valuation exceeds their ability to pay. I abstract away from this role of punishment in the paper.

A direct mechanism is thus a triple (a, p, q), where $a : T \to [0, 1]$ is the allocation rule, $p : T \to \mathbb{R}$ is the payment rule, and $q : T \to [0, 1]$ is the verification rule. Specifically, for each reported type $t \in T$, $a(t)$ denotes the probability that an agent obtains the good, $p(t)$ denotes the payment that an agent must make, and $q(t)$ denotes the probability of verification. In this definition, I implicitly assume that payment rules are deterministic. This assumption is without loss of generality based on an argument similar to that in Pai and Vohra (2014). Note further that the mechanisms considered here allow for the possibility of cross-subsidization.

The utility of an agent who has type $t = (v, b)$ and reports $\hat{t}$ is

$$u(\hat{t}, t) := \begin{cases} a(\hat{t})v - p(\hat{t}) & \text{if } \hat{b} = b \text{ and } p(\hat{t}) \leq b, \\ a(\hat{t})v - q(\hat{t})c - p(\hat{t}) & \text{if } \hat{b} \neq b \text{ and } p(\hat{t}) \leq b, \\ -\infty & \text{if } p(\hat{t}) > b. \end{cases}$$

That is, an agent has a standard quasilinear utility up to his budget constraint and cannot pay more than his budget. If an agent who lies about his budget is inspected, he will receive a penalty (c).

With transferable utilities, the welfare criterion I use is simply utilitarian efficiency.[4] Given quasilinear preferences, the total value realized minus the total verification cost is an equivalent criterion. The principal's problem is

$$\max_{a,p,q} \mathbb{E}_t\big[a(t)v - q(t)k\big], \tag{$\mathcal{P}$}$$

subject to

$$u(t) := u(t, t) \geq 0, \quad \forall t \in T, \tag{IR}$$

$$u(t) \geq u(\hat{t}, t), \quad \forall t \in T, \hat{t} \in \big\{\hat{t} \in T \,|\, p(\hat{t}) \leq b\big\}, \tag{IC}$$

$$p(t) \leq b, \quad \forall t \in T, \tag{BC}$$

$$\mathbb{E}_t\big[p(t) - q(t)k\big] \geq 0, \tag{BB}$$

$$\mathbb{E}_t\big[a(t)\big] \leq S. \tag{S}$$

[4]For why utilitarian efficiency is a reasonable welfare criterion, see Vickrey (1945) and Harsanyi (1955).

The individual rationality (IR) constraint requires that each agent receives a nonnegative expected payoff from participating in the mechanism. The (IC) constraint requires that it is weakly better for an agent to report his true type than any other type whose payment he can afford. The budget constraint (BC) states that an agent cannot be asked to make a payment larger than his budget b. Note that (BC) follows from (IR). This budget constraint is the same as that found in Che and Gale (2000) and Pai and Vohra (2014) but different from that in CGK, who adopt a per-unit price constraint. I discuss the differences between the two frameworks in Section 5. The principal's budget balance (BB) constraint requires that the revenue raised from selling the good must exceed the verification cost. (BB) rules out the possibility that the principal can subsidize without limit and relieve all budget constraints. Finally, the limited supply (S) constraint requires that the amount of the good assigned cannot exceed the supply. A mechanism (a, p, q) is *feasible* if it satisfies (IR), (IC), (BC), (BB), and (S).

I impose the following two assumptions throughout the paper.

Assumption 1. $\frac{1-F}{f}$ *is nonincreasing.*

Assumption 2. f *is nonincreasing.*

Assumption 1 is the standard monotone hazard rate condition, which is often adopted in the mechanism design literature. This assumption ensures that allocating more goods to agents with higher valuations rather than to those with lower valuations yields higher revenues for the principal. Assumption 2 states that agents are less likely to have higher valuations than to have lower valuations. As will become clear later, these two regularity assumptions rule out complicated pooling regions in the optimal mechanism. These two assumptions are also imposed in Richter (2019) and Pai and Vohra (2014) and are satisfied by some commonly used distributions, such as uniform distributions, exponential distributions, and left truncations of a normal distribution.

I briefly discuss some other assumptions of the model here. In the model, I assume that the penalty is nontransferable for ease of exposition. If the penalty is transferable, then (BC) also requires that an agent must be able to afford the payment and the penalty. As will become clear, in the optimal mechanism, the penalty is only used as an incentive to prevent high-budget agents from under-reporting their budgets. Therefore, the above constraint will not be binding, and the analysis in this paper still applies. I also assume that verification is perfect. If verification is imperfect in the sense that the principal cannot detect a lie with some probability, then it can be viewed as the case where c is the expected penalty (i.e., the actual penalty times the probability of detection). Finally, I assume that the principal's budget and the supply of the good is fixed. In practice, a government can increase the budget of a transfer program or increase the supply of the good (e.g., building more flats or providing more hospital beds) at some cost. The model can be easily modified to accommodate these possibilities. Allowing for the principal to raise money or increase the supply at some social cost will not qualitatively change the optimal mechanism.

2.1 *Competitive market*

Before proceeding to solve the optimal mechanism, I first illustrate the potential inefficiency caused by the budget constraint by considering the competitive market as an assignment method. I also allow for cash transfers from the principal subject to the (BB) constraint.

In a competitive market, the supply is S at any nonnegative price. The demand at price $p \geq 0$ is given by the mass of agents willing and able to pay p after receiving the cash transfer from the principal. The equilibrium price is the maximum price at which demand exceeds supply.

Let v^{FB} denote the critical value such that $1 - F(v^{FB}) = S$. If all agents with valuations above v^{FB} consume the good, the total value is maximized. Clearly, this first-best allocation will arise as the competitive market outcome with market-clearing price v^{FB} if supply is abundant or agents have ample budgets so that low-budget agents are able to pay v^{FB}.

Proposition 1. *The first-best can be attained if and only if* $b_1 + v^{FB}(1 - F(v^{FB})) \geq v^{FB}$, *where* v^{FB} *is the solution to* $1 - F(v^{FB}) = S$ *and strictly decreasing in* S.

Throughout the remainder of the paper, I maintain Assumption 3 so that the first-best is not attainable. In particular, a competitive market combined with cash transfers will not maximize welfare.

Assumption 3. $b_1 + v^{FB}(1 - F(v^{FB})) < v^{FB}$.

To illustrate why the competitive market outcome may be inefficient, consider Figure 2, where the supply is assumed to be sufficiently limited so that only high-budget agents are able to pay the competitive market price. As a result, only high-budget agents with valuations above v^{CM} consume the good, where $\pi[1 - F(v^{CM})] = S$ and $b_1 < v^{CM} < b_2$ for S sufficiently small. Clearly, $v^{CM} < v^{FB}$. In Figure 2, the first-best allocation would give the good to agents in region $B + C$, whereas the market assigns it to those in $A + B$. Compared with the first-best allocation, in the competitive market, some low-budget agents with high valuations will not consume the good, while some high-budget agents with lower valuations will.

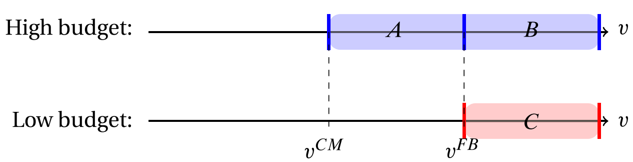


Figure 2. Competitive market fails to achieve efficiency.

3. Optimal mechanism

In this section, I solve for the optimal mechanism. First, the (IC) constraints can be separated into two categories:

$$a(v,b)v - p(v,b) \geq a(\hat{v},b)v - p(\hat{v},b), \quad \forall v,\hat{v},b, \tag{IC-v}$$

$$a(v,b)v - p(v,b) \geq a(\hat{v},\hat{b})v - q(\hat{v},\hat{b})c - p(\hat{v},\hat{b}),$$
$$\forall v,b,(\hat{v},\hat{b}) \in \big\{(\hat{v},\hat{b}) | p(\hat{v},\hat{b}) \leq b\big\}, \tag{1}$$

(IC-v) corresponds to a misreport only of value, and (1) corresponds to a misreport of both value and budget. By the standard argument, (IC-v) holds if and only if, for all b, $a(v,b)$ is nondecreasing in v and the envelope condition holds: $p(v,b) = va(v,b) - \int_{\underline{v}}^{v} a(\nu,b)\,\mathrm{d}\nu - u(\underline{v},b)$ for all v. The difficulty arises from (1), which allows for the possibility to underreport or overreport budgets. In what follows, I first consider a relaxed problem by replacing (1) with the constraint corresponding to the underreporting of budgets:

$$a(v,b_2)v - p(v,b_2) \geq a(\hat{v},b_1)v - q(\hat{v},b_1)c - p(\hat{v},b_1), \quad \forall v,\hat{v}. \tag{IC-b}$$

This relaxation formalizes the intuition that the principal's main concern is preventing high-budget agents from falsely claiming to be low-budget agents. Clearly, in the optimal solution to the relaxed problem, agents reporting high budget will never be inspected (i.e., $q(v,b_2) = 0$ for all v). Later, I verify that the optimal mechanism of the relaxed problem automatically satisfies the IC constraints corresponding to the overreporting of budgets. In other words, a solution to the relaxed problem also solves the original problem.

To summarize, the principal's relaxed problem is

$$\max_{a,p,a} \mathbb{E}_t\big[a(t)v - q(t)k\big], \tag{$\mathcal{P}'$}$$

subject to (IR), (IC-v), (IC-b), (BC), (BB), and (S).

3.1 *No verification*

Before solving the general model, I first consider the benchmark case in which the principal does not verify agent budgets (i.e., $q = 0$), which is optimal if the verification cost is sufficiently high. In this case, as will become clear in the discussion below, it is sufficient to consider two *one-dimensional deviations*, which greatly simplifies the analysis. Although some of the results may be familiar, it helps to highlight the technical challenge arising from introducing costly verification. Denote the principal's problem in this case by $\mathcal{P}_{NV}$ and the corresponding relaxed problem ignoring the overreporting of budgets by $\mathcal{P}'_{NV}$.

Observe first that in this case, (IC-b) holds if and only if (IC-v) holds and a high-budget agent has no incentives to misreport *only* his budget:

$$a(v,b_2)v - p(v,b_2) \geq a(v,b_1)v - p(v,b_1), \quad \forall v. \tag{2}$$

To see this, note that

$$a(v, b_2)v - p(v, b_2) \geq a(v, b_1)v - p(v, b_1) \geq a(\hat{v}, b_1)v - p(\hat{v}, b_1),$$

where the first inequality follows from (2) and the second follows from (IC-v). Thus, it is sufficient to consider the two one-dimensional deviations: misreport only the value and misreport only the budget. The above inequality states that if a type-(v, b_2) agent has no incentive to report (v, b_1), then he has no incentive to report $(\hat{v}, b_1)$. This argument does not hold when there is verification because it is possible that types (v, b_1) and $(\hat{v}, b_1)$ are inspected with different probabilities. Instead, to determine the set of binding (IC-b) constraints, one must identify for each low-budget type $(\hat{v}, b_1)$, the high-budget type who benefits most from mimicking $(\hat{v}, b_1)$, which depends on the allocation rule. As a result, one cannot anticipate, a priori, which (IC-b) constraint binds. I will discuss this in more detail in Section 3.2.

Theorem 1 characterizes the optimal direct mechanism in the absence of verification.

Theorem 1. *Suppose the principal does not verify agent budgets. The optimal mechanism of $\mathcal{P}_{NV}$ satisfies the following properties: There exist two thresholds v^* and v^{**} with $v^* \leq v^{**}$.*

(i) Low-valuation agents of both budgets do not receive the good but do receive positive cash transfers:

$$a(v, b) = 0 \quad \textit{and} \quad p(v, b) = u^* > 0 \quad \forall v < v^*.$$

(ii) High-budget intermediate-valuation agents and low-budget agents with intermediate or high valuations receive the good with probability $a^* \in (0, 1]$ at a discount price:

$$\begin{cases} a(v, b_2) = a^* & \textit{and} \quad p(v, b_2) = b_1 \quad \forall v \in [v^*, v^{**}) \\ a(v, b_1) = a^* & \textit{and} \quad p(v, b_1) = b_1 \quad \forall v \in \left[v^*, \overline{v}\right]. \end{cases}$$

(iii) High-budget high-valuation agents receive the good at the full price:

$$a(v, b_2) = 1 \quad \textit{and} \quad p(v, b_2) = b_1 + \left(1 - a^*\right)v^{**} \quad \forall v \geq v^{**}.$$

The optimal mechanism obtained here shares features similar to those found in CGK and Pai and Vohra (2014). First, similar to CGK, low-valuation agents receive the same positive cash transfer regardless of their budgets. Second, similar to Pai and Vohra (2014), high-budget agents whose valuations are in $[v^*, v^{**}]$ are pooled with low-budget agents whose valuations are at least v^*. They are both offered a random assignment at a discount price. To understand this pooling result, consider two agents with the same valuation v but different budgets $b_2 > b_1$. Then, (IC-b) implies that as long as agent (v, b_2)'s payment is less than b_1, he must receive the good with the same probability as (v, b_1). By a similar argument, they must make the same payment as well. Their consumption is distorted downward to ensure IC for high-budget high-valuation agents who pay a higher price to finance the cash transfer.

3.2 *The general case*

I now turn to the general model with costly verification. Using the envelope condition, (IC-b) becomes the following: for all v and $\hat{v}$,

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v} a(\nu, b_2)\,\mathrm{d}\nu \geq u(\underline{v}, b_1) + a(\hat{v}, b_1)(v - \hat{v}) - q(\hat{v}, b_1)c + \int_{\underline{v}}^{\hat{v}} a(\nu, b_1)\,\mathrm{d}\nu. \tag{IC-b}$$

First, for each $\hat{v}$, I identify the type of high-budget agents whose gain from falsely claiming to be type-$(\hat{v}, b_1)$ is the largest. (IC-b) holds if and only if for each $\hat{v}$, $q(\hat{v}, b_1)c \geq \sup_v \Delta(v, \hat{v})$, where

$$\Delta(v, \hat{v}) := u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v} a(\nu, b_2)\,\mathrm{d}\nu + a(\hat{v}, b_1)(v - \hat{v}) + \int_{\underline{v}}^{\hat{v}} a(\nu, b_1)\,\mathrm{d}\nu$$

is what type-(v, b_2) expects to gain by reporting $(\hat{v}, b_1)$ in the absence of punishment. Since $\partial\Delta(v, \hat{v})/\partial v = -a(v, b_2) + a(\hat{v}, b_1)$ is nonincreasing in v, $\Delta(v, \hat{v})$ is concave in v and achieves its maximum at $v = v^d(\hat{v})$, where

$$v^d(\hat{v}) := \inf\{v | a(v, b_2) \geq a(\hat{v}, b_1)\}.$$

If the allocation rules for both budgets are continuous in v, the high-budget agents who benefit most from mimicking $(\hat{v}, b_1)$ are those who obtain the goods with the same probability as type-$(\hat{v}, b_1)$. This point is illustrated by Figure 3, which plots an allocation rule for high-budget agents, $a(\cdot, b_2)$, and an allocation rule for low-budget agents, $a(\cdot, b_1)$, as a function of their valuations v. Consider type-(v, b_2) who receives the good with probability $a(v, b_2) < a(\hat{v}, b_1)$. Then his expected gain from reporting $(\hat{v}, b_1)$ can be rewritten as

$$\Delta(v, \hat{v}) = \Delta(\hat{v}) - \int_{\hat{v}}^{v} a(\nu, b_2)\,\mathrm{d}\nu + a(\hat{v}, b_1)(v - \hat{v}), \tag{3}$$

where $\Delta(\hat{v})$ is a constant depending only on $\hat{v}$. In (3), the second term is the information rent that type-(v, b_2) loses (region A), and the last term is his gain due to the increased probability of receiving the good (region $A + B$). Thus, the expected gain (region B) increases as v increases as long as $a(v, b_2) < a(\hat{v}, b_1)$ and is maximized when $a(v, b_2) = a(\hat{v}, b_1)$.

Since the principal's objective function is strictly decreasing in q, the optimal verification rule satisfies

$$q(\hat{v}, b_1) = \frac{1}{c}\max\{0, \Delta(v^d(\hat{v}), \hat{v})\}. \tag{4}$$

Note that from the principal's perspective, the "effective" cost of inspecting type $(\hat{v}, b_1)$ is $\rho = k/c$ since $kq(\hat{v}, b_1) = \rho\max\{0, \Delta(v^d(\hat{v}), \hat{v})\}$. Note also that $v^d(\cdot)$ depends on the allocation rule. As a result, one cannot anticipate, a priori, which (IC-b) constraint binds.

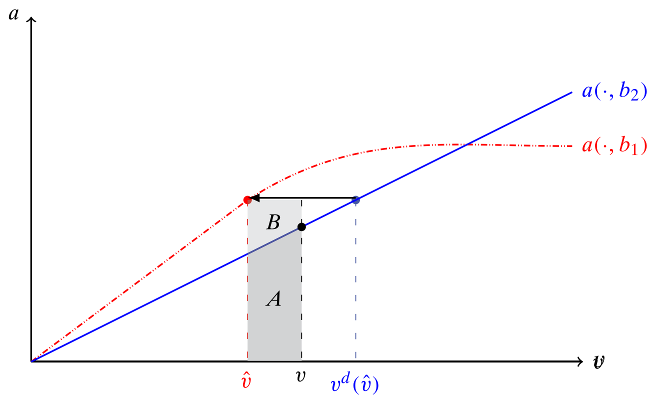


Figure 3. The set of binding (IC-b) constraints.

Furthermore, it is not only among local types that (IC-b) constraints are frequently binding. These difficulties are inherent in all multidimensional problems, and as a result, the existing approaches in the mechanism design literature do not apply to this problem.[5]

To track the binding (IC-b) constraints, I solve the principal's problem by first approximating the allocation rule using step functions. An allocation rule is a *step allocation rule* if for each b, $a(\cdot, b)$ is a step function. I can further narrow down the focus to a special class of step allocation rules. Let $M \geq 2$ be an integer. A step allocation rule a is an M-*step allocation rule* if it has the following structure: Let $\{v_1^m\}_{m=0}^{M+1}$ with $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $\{v_2^m\}_{m=0}^{M+1}$ with $\underline{v} \leq v_2^0 \leq v_2^1 \leq \cdots \leq v_2^M \leq \overline{v}$ be two partitions of the interval $[\underline{v}, \overline{v}]$, respectively. For $i = 1, 2$ and $m = 1, \ldots, M$,

$$a(v, b_i) = a^m \text{ for } v \in [v_i^{m-1}, v_i^m),$$

where $0 \leq a^1 < a^2 < \cdots < a^M \leq 1$, $v_1^0 = \underline{v}$, $v_1^M = \overline{v}$, $a(v, b_2) = 0$ for $v < v_2^0$, and $a(v, b_2) = 1$ for $v \geq v_2^M$. Roughly speaking, if a is an M-step allocation rule, $a(\cdot, b_1)$ and $a(\cdot, b_2)$ take the same set of M possible values, and in addition, $a(\cdot, b_2)$ can take the values of 0 or 1.

It is useful to focus on M-step allocation rules since they make it easy to track the set of binding (IC-b) constraints. To see this, consider $v \in [v_1^{m-1}, v_1^m)$; the high-budget agents who benefit most from misreporting type (v, b_1) are those with valuation $v^d(v) = v_2^{m-1}$ (i.e., those who also obtain the good with probability a^m). Therefore, the optimal verification rule satisfies $q(v, b_1) = q^m$ for all $v \in [v_1^{m-1}, v_1^m)$ and all $m = 1, \ldots, M$, where

$$q^m = \frac{1}{c} \max\left\{0, u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} \left(a^j - a^{j-1}\right)\left(v_2^{j-1} - v_1^{j-1}\right)\right\}, \tag{5}$$

where $a^0 = 0$ and $a^{M+1} = 1$. In other words, it is sufficient to consider (IC-b) between (v_1^{m-1}, b_1) and (v_2^{m-1}, b_2) for $m = 1, \ldots, M$. This set of (IC-b) constraints can be tracked by tracking the jump discontinuity points of the allocation rule, making the problem tractable.

[5]See Rochet and Stole (2003) for a survey on the multidimensional mechanism design problem.

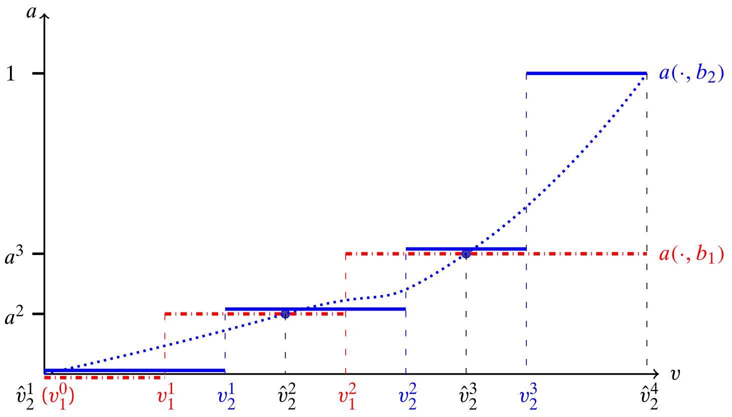


FIGURE 4. Proof sketch of Lemma 1.

Lemma 1 shows that it is without loss of generality to focus on M-step allocation rules among all step allocation rules.

LEMMA 1. *If a is an optimal step allocation rule of $\mathcal{P}'$, then a is an M-step allocation rule for some integer $M \geq 2$.*

To prove Lemma 1, I prove something stronger. Consider a feasible allocation rule where $a(\cdot, b_1)$ is a nondecreasing step function and $a(\cdot, b_2)$ is any nondecreasing function. Suppose that there exists a payment rule and a verification rule to be used in conjunction with the allocation rule such that the resulting mechanism is feasible. Then we can construct another feasible mechanism whose allocation rule is an M-step allocation rule and strictly improves welfare. The construction follows a weight-shifting argument and is illustrated in Figure 4. In Figure 4, $a(\cdot, b_1)$ (the dash–dotted line) takes three distinct values $\{a^1, a^2, a^3\}$ with $0 = a^1 < a^2 < a^3 < 1$, and $a(\cdot, b_2)$ is a continuous increasing function (the dotted curve). Let $\hat{v}_2^m$ be such that $a(\hat{v}_2^m, b_2) = a^m$ for $m = 1, 2, 3$. For each m, reduce the probability that high-budget agents in $[\hat{v}_2^m, v_2^m]$ receive the good to a^m and increase the probability that those in $[v_2^m, \hat{v}_2^{m+1}]$ receive the good to a^{m+1}. The choice of v_2^m is uniquely determined such that the supply to high-budget agents in $[\hat{v}_2^m, \hat{v}_2^{m+1}]$ remains unchanged. The resulting new $a(\cdot, b_2)$ (the solid line) is a step function, taking four distinct values $\{a^1, a^2, a^3, 1\}$. As a result, the new allocation rule is a 3-step allocation rule. Clearly, this change strictly improves welfare. Redefine the payment rule using the envelope condition, and let the verification rule remain unchanged. It can be verified that the new mechanism is feasible if f satisfies the two regularity assumptions. In particular, Assumption 1 ensures that the revenue goes up so that (BB) holds. Under both mechanisms, the binding (IC-b) constraints are from $(\hat{v}_2^m, b_2)$ to (v, b_1) for $v \in [v_1^{m-1}, v_1^m)$. Assumption 2 ensures that the information rent accrued to $(\hat{v}_2^m, b_2)$ goes up so that (IC-b) holds.

Consider the principal's problem ($\mathcal{P}'$) with two modifications:

$$\max_{a,p,q} \mathbb{E}_t\big[a(t)v - q(t)k\big], \qquad (\mathcal{P}'(M, d))$$

subject to (IR), (IC-v), (IC-b), (BC), (S),

$$a \text{ is an } M'\text{-step allocation rule for some } M' \le M,$$

$$\mathbb{E}\big[p(t) - q(t)k\big] \ge -d. \tag{BB-d}$$

The first modification restricts attention to M-step allocation rules for which the number of steps is bounded from above. The second modification is to relax the BB constraint by $d \ge 0$ and is made for technical reasons. As will become clear below, an optimal mechanism of $\mathcal{P}'$ can be approximated arbitrarily well by a feasible mechanism of $\mathcal{P}'(M, d)$ for M sufficiently large and d sufficiently small. In what follows, I first solve $\mathcal{P}'(M, d)$ for all $M \ge 2$ and $d > 0$ and then take $M \to \infty$ and $d \to 0$.

3.2.1 *Solve* $\mathcal{P}'(M, d)$ In this section, I solve $\mathcal{P}'(M, d)$. Let $V(M, d)$ denote the value of $\mathcal{P}'(M, d)$. The main result of this section is showing that $V(M, d) = V(2, d)$ for all $M \ge 2$ and $d \ge 0$. In other words, for all $M \ge 2$, the optimal allocation rule of $\mathcal{P}'(M, d)$ is a 2-step allocation rule. Readers who are not interested in the technical details can skip this section with little loss of continuity.

First, remember that the optimal verification rule of $\mathcal{P}'(M, d)$ is given by (5). The optimization problem is nonlinear due to the presence of the maximum operator. To make the problem linear, I show the following.

Lemma 2. *Under an optimal mechanism of* $\mathcal{P}'(M, d)$*, for any* $v \in [\underline{v}, \overline{v}]$*, some high-budget type exists that weakly prefers to falsely claim to be the low-budget type* (v, b_1) *in the absence of verification.*

Hence, the optimal verification rule satisfies $q(v, b_1) = q^m$ for all $v \in [v_1^{m-1}, v_1^m)$ and all $m = 1, \dots, M$, where

$$q^m = \frac{1}{c}\left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}\big(a^j - a^{j-1}\big)\big(v_2^{j-1} - v_1^{j-1}\big)\right]. \tag{6}$$

Using (6), $\mathcal{P}'(M, d)$ can be written as a linear problem in terms of $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$, and the allocation rule a.

Next, I show that the optimal verification probability is nondecreasing in a low-budget agent's reported value.

Lemma 3. *In an optimal mechanism of* $\mathcal{P}'(M, d)$, $v_2^1 - v_1^1 \ge 0$. *If* $V(M, d) > V(M-1, d)$ *for* $M \ge 3$*, then*

$$v_2^{M-1} - v_1^{M-1} > \cdots > v_2^1 - v_1^1 \ge 0.$$

As a result, the verification probability in an optimal mechanism of $\mathcal{P}'(M, d)$ *is nondecreasing in the reported value, that is,* $q^M > \cdots > q^2 \ge q^1 \ge 0$.

To understand this monotonicity result, consider a low-budget agent and a high-budget agent, both receiving the good with probability a^m. Let p_1^m and p_2^m denote their

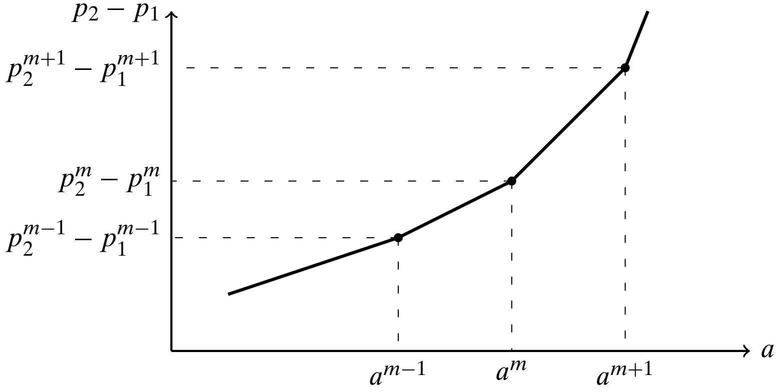


FIGURE 5. Illustration of Lemma 3.

payments, respectively. The difference in their payments, to which the verification probability is proportional, is

$$p_2^m - p_1^m = u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}).$$

Figure 5 graphs the payment difference against the assignment probability. The three bold-faced points correspond to the assignment probabilities a^{m-1}, a^m, and a^{m+1}, respectively. Note that the difference in marginal prices $([(p_2^{m+1} - p_2^m) - (p_1^{m+1} - p_1^m)]/(a^{m+1} - a^m) = v_2^m - v_1^m)$ is represented by the slope of the line connecting two adjacent points. By Lemma 3, this difference increases with the probability of assignment. Hence, the linear envelope of the three points is convex. Intuitively, low-budget agents with higher valuations are more likely to receive the good and are also more likely to be budget constrained. The optimal mechanism subsidizes them by prescribing a larger difference in marginal prices.

Using the monotonicity result, I can further simplify the principal's problem. For fixed jump discontinuity points v_i^ms, $\mathcal{P}'(M, d)$ is linear in $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$, and a^ms. Hence, an optimal solution can be obtained at an extreme point of the feasible region. The monotonicity of the verification probability implies that in addition to the monotonicity constraints on the a^ms, there are only finitely many other constraints binding. As a result, for an M sufficiently large, there are finitely many distinct a^ms in an optimal mechanism. More formally, $V(M, d) = V(M - 1, d)$ for M sufficiently large. Under Assumptions 1 and 2, I can further prove that the optimal allocation rule of $\mathcal{P}'(M, d)$ is a 2-step allocation rule, that is, $V(M, d) = V(M - 1, d)$ for $M \geq 3$. To obtain a rough intuition for this result, consider an M-step allocation rule. We can reduce the number of steps by reducing the assignment probability for agents with relatively low values and increasing that for agents with relatively high values. This change clearly improves the total value. It also raises the revenue by Assumption 1 and relaxes the budget constraint by Assumption 2. However, this change can make the underreporting of budgets more attractive and increase the total verification cost. The two regularity assumptions ensure that the benefit exceeds the cost.

LEMMA 4. *For all $M \geq 2$ and $d \geq 0$, the optimal allocation rule of $\mathcal{P}'(M, d)$ is a 2-step allocation rule, that is, $V(M, d) = V(2, d)$.*

3.2.2 *Optimal mechanism* In this section, I characterize the optimal mechanism of the original problem ($\mathcal{P}$) by proving that an optimal mechanism of $\mathcal{P}'(2, 0)$ is also an optimal mechanism of $\mathcal{P}$. In other words, the optimal allocation rule is a 2-step allocation rule.

Intuitively, an allocation rule can be well approximated by some step allocation rule. We know from Lemma 1 that an optimal step allocation rule of $\mathcal{P}'$ must be an M-step allocation rule for some integer M. Thus, an optimal allocation rule of $\mathcal{P}'$ can be well approximated by some M-step allocation rule. Let V denote the value of $\mathcal{P}'$. Then, for any $d > 0$ there exists $\overline{M}(d) > 0$ such that for all $M > \overline{M}(d)$,

$$V - V(2, d) = V - V(M, d) \leq (1 - \pi)(1 + \rho)\frac{\mathbb{E}[v]}{M},$$

where the equality holds by Lemma 4. By the standard argument, one can show that $V = V(2, 0)$ by first letting M go to infinity and then letting d go to zero. That is, an optimal mechanism of $\mathcal{P}'(2, 0)$ also solves $\mathcal{P}'$. It is easy to verify that an optimal solution to $\mathcal{P}'(2, 0)$ satisfies the (IC) constraints corresponding to the over-reporting of budgets and, therefore, solves $\mathcal{P}$.

Finally, the optimal mechanism is *unique*. To understand this result, suppose on the contrary, that there are two optimal mechanisms. Since $\mathcal{P}'$ is linear in (a, p, q), the convex combination of these two optimal mechanisms would also be optimal. However, the convex combination of two 2-step allocation rules is not a 2-step allocation rule in general, so it cannot be optimal. Hence, there exists a unique optimal mechanism.

THEOREM 2. *The unique optimal mechanism of $\mathcal{P}$ satisfies the following properties: There exist three thresholds $v_1^*(\rho)$, $v_2^*(\rho)$, and $v_2^{**}(\rho)$ with $v_1^*(\rho) \leq v_2^*(\rho) \leq v_2^{**}(\rho)$.*

(i) *Low-valuation agents of both budgets do not receive the good but do receive cash transfers:*

$$\begin{cases} a(v, b_2) = 0 \quad \text{and} \quad p(v, b_2) = -u_2^*(\rho) \quad \forall v < v_2^*(\rho) \\ a(v, b_1) = 0 \quad \text{and} \quad p(v, b_1) = -u_1^*(\rho) \quad \forall v < v_1^*(\rho) \end{cases},$$

where low-budget agents receive a higher cash transfer, that is, $u_1^*(\rho) > u_2^*(\rho) \geq 0$.

(ii) *High-budget intermediate-valuation agents and low-budget agents with intermediate or high valuations receive the good with probability $a^*(\rho) \in (0, 1]$ at a discount price:*

$$\begin{cases} a(v, b_2) = a^*(\rho) \quad \text{and} \quad p(v, b_2) = -u_2^*(\rho) + a^*(\rho)v_2^*(\rho) > b_1 \\ \quad \forall v \in [v_2^*(\rho), v_2^{**}(\rho)) \\ a(v, b_1) = a^*(\rho) \quad \text{and} \quad p(v, b_1) = -u_1^*(\rho) + a^*(\rho)v_1^*(\rho) = b_1 \\ \quad \forall v \in \left[v_1^*(\rho), \overline{v}\right] \end{cases},$$

where low-budget agents receive a deeper discount.

(iii) High-budget high-valuation agents receive the good at the full price:

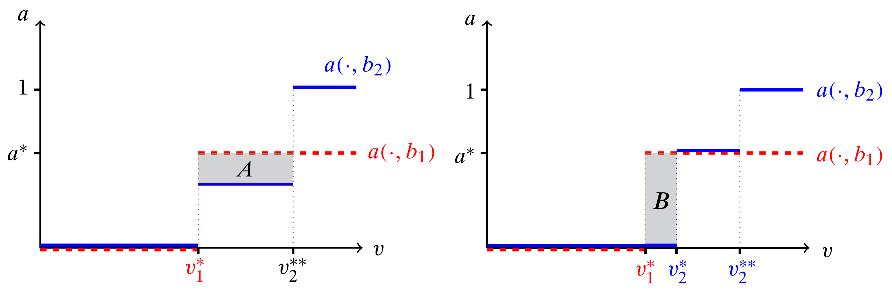
$$a(v,b_2)=1 \quad \textit{and} \quad p(v,b_2)=-u_2^*(\rho)+a^*(\rho)v_2^*(\rho)+\big(1-a^*(\rho)\big)v_2^{**}(\rho) \quad \forall v\geq v^{**}(\rho).$$

Only low-budget agents are inspected randomly, and the inspection probability increases with the probability of assignment:

$$q(v,b_1)=\begin{cases}\frac{1}{c}\big[u_1^*(\rho)-u_2^*(\rho)\big] & \textit{if } v<v_1^*(\rho)\\ \frac{1}{c}\big[u_1^*(\rho)-u_2^*(\rho)+a^*(\rho)\big(v_2^*(\rho)-v_1^*(\rho)\big)\big] & \textit{if } v\geq v_1^*(\rho)\end{cases}.$$

Comparing the optimal mechanism with that in Theorem 1, verification affects the optimal mechanism in a simple way: low-budget agents are favored in terms of *higher cash transfers or lower payments*. This result can be illustrated via Figure 6. In each graph of Figure 6, the solid line represents the allocation rule for high-budget agents and the dashed line represents the allocation rule for low-budget agents. The mechanism in Figure 6a favors low-budget agents in terms of the probability of assignment. We can increase the probability of assignment and the payment for high-budget intermediate-valuation agents as in Figure 6b so that low-budget agents are favored in terms of the payment. Consider type (v,b_1) with $v\geq v_1^*$. In Figure 6a, type (v_2^{**},b_2) has the strongest incentive to report (v,b_1), and its gain is equal to the difference in cash transfers plus the difference in information rents (area A). In Figure 6b, type (v_2^*,b_2) has the strongest incentive to report (v,b_1), and its gain is equal to the difference in cash transfers plus area B. Assumption 2 ensures that area A is larger than area B as high-budget agents receive more information rents in the latter case. Hence, favoring low-budget agents in terms of the payment improves allocative efficiency while saving verification cost.

Similar to the case without verification, the optimal mechanism involves both cash transfers and in-kind transfers (the provision of goods at discount prices). Measure



(a) Favor low-budget agents in terms of the probability of assignment

(b) The optimal mechanism favors low-budget agents in terms of payment

FIGURE 6. Favoring low-budget agents in terms of payment improves allocative efficiency while saving verification cost. (a) Favor low-budget agents in terms of the probability of assignment. (b) The optimal mechanism favors low-budget agents in terms of payment.

the value of an in-kind transfer by the additional price paid by high-budget high-valuation agents to receive the good (i.e., $p(\overline{v}, b_2) - p(\underline{v}, b_2) = a^* v_2^* + (1 - a^*) v_2^{**}$).[6] Then, the in-kind transfer received by high-budget intermediate-valuation agents is $a^*[a^* v_2^* + (1 - a^*) v_2^{**}] - a^* v_2^*$, and that received by low-budget agents with intermediate or high valuations is $a^*[a^* v_2^* + (1 - a^*) v_2^{**}] - a^* v_1^*$. Thus, low-budget agents receive more in-kind transfers as well as more cash transfers. This is in contrast to the case without verification, where agents receive the same amount of cash transfers and the same amount of in-kind transfers regardless of their budgets.

To understand the difference, consider how the change in the effective verification cost (ρ) would affect the optimal mechanism. High-budget agents are tempted to underreport their budgets precisely because of the differences in cash and in-kind transfers. To sustain IC, all agents reporting low budgets are inspected with some probability. Intuitively, as verification becomes more costly (i.e., ρ increases), the principal wants to inspect agents less frequently to save verification cost. To maintain IC, the principal must reduce the differences in cash and in-kind transfers. Proposition 2 shows that for sufficiently large ρ, agents of both budget types receive the same amount of cash and in-kind transfers, no one is inspected, and Theorem 2 reduces to Theorem 1.

PROPOSITION 2. *(i) If $\rho \geq \pi/(1-\pi)$, then agents of both budgets receive the same amount of cash transfers, that is, $u_1^* = u_2^*$. (ii) If $\rho \geq \pi/[S(1-\pi)]$, then agents of both budgets also receive the same amount of in-kind transfers, that is, $v_1^* = v_2^*$. In this case, no one is inspected.*

Figure 7 plots the impact of an increase in ρ on the difference in cash transfers ($u_2^* - u_1^*$) and the difference in in-kind transfers ($a^*(v_2^* - v_1^*)$) for a numerical example.

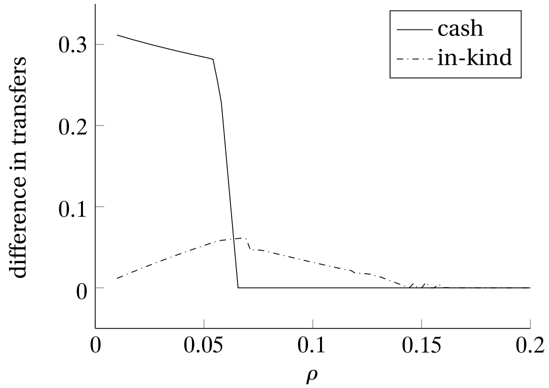


FIGURE 7. The impact of an increase in the effective verification cost (ρ) on the differences in cash and in-kind transfers. In this numerical example, v is uniformly distributed on $[0, 1]$, $S = 0.4$, $b_1 = 0.2$, $\pi = 0.5$, and $\rho \in [0, 0.2]$.

[6]In the literature, the value of an in-kind transfer is often measured by its market value. Since the paper does not explicitly model the private market, I use the additional price paid by high-budget high-valuation agents as a proxy.

Interestingly, the difference in in-kind transfers does not change monotonically. As ρ increases, the difference in cash transfers declines, but the difference in in-kind transfers first increases and then decreases. This is because although cash transfer is more efficient in the sense that it does not introduce any distortion in allocation, it is more expensive in terms of the verification cost, as it is attractive to everyone regardless of their valuations. By contrast, an in-kind transfer is attractive only to agents whose valuations are sufficiently high. Therefore, the principal optimally subsidizes low-budget agents by offering them more in-kind transfers instead of more cash transfers as ρ increases. The differences in cash and in-kind transfers reflect the amount of benefits accrued to low-budget agents. Thus, as discussed in the Introduction, the above findings have useful implications for the optimal method of subsidizing financially constrained people. First, in-kind transfers may be employed when verifying the financial information of an agent is costly. Second, countries with higher verification costs or worse institutions should rely more on in-kind transfers.

4. Implementation

The optimal direct mechanism has a simple implementation, which exhibits some of the features of the affordable housing program in Singapore.

Consider the following *random assignment with regulated resale and cash subsidy* (RwRRC) scheme, which is based on the RwRRC scheme in CGK. The scheme consists of two stages.

1. In the first stage, agents report their budgets. Those who report a low budget are inspected with probability $(u_1^* - u_2^*)/c$. The principal offers cash transfers u_1^* to low-budget agents and u_2^* to high-budget agents. The principal also offers agents the choice of participating in a lottery over the good. To enter the lottery, low-budget agents pay a price of $p_1^* := a^* v_1^*$ and high-budget agents pay a price of $p_2^* := a^* v_2^*$. The principal distributes the good at random with a uniform probability among all participants. Each participant receives the good with a probability of no more than a^*.

2. In the second stage, a resale market opens, in which agents can purchase goods from one another (and from the principal if not all goods are distributed in the first stage). The sales taxes are $\tau_1^* := v_2^{**} - v_1^*$ for low-budget sellers and $\tau_2^* := v_2^{**} - v_2^*$ for high-budget sellers. Agents who report low budgets in the first stage and choose not to sell the good they received in the second stage are inspected with probability $(v_2^* - v_1^*)/c$.

Under the RwRRC scheme, lottery participants expect to receive the good with probability a^* in equilibrium, and the unique equilibrium price in the resale market is $p^s = v_2^{**}$. Then, in equilibrium, an agent makes zero profit by participating in the lottery and selling the good he received. Let $\hat{v}$ be such that $a^*[1 - F(\hat{v})] = S$. It is an equilibrium in which in the first stage, all agents report their budgets truthfully and opt into the lottery when their valuations exceed $\hat{v}$. In the second stage, all low-budget recipients with valuations below v_1^* and all high-budget recipients with valuations below v_2^* will sell the good. Resale demand comprises those high-budget agents who did not receive the good initially but who are willing to pay $p^s = v_2^{**}$. The equilibrium outcome coincides with that of the optimal direct mechanism.

Proposition 3. *The optimal direct mechanism is implemented by RwRRC with u_1^*, u_2^*, v_1^*, v_2^*, v_2^{**}, and a^* given by Theorem 2.*

The RwRRC scheme exhibits some of the features of Singapore's affordable housing program. In Singapore, the affordable housing program is administered by the HDB, which develops and sells new flats to eligible buyers and regulates an open market in which buyers can directly purchase resale flats from existing owners.[7] To purchase an HDB flat, a household needs to meet several eligibility conditions. The HDB also offers financial aid to buyers in the form of housing grants. For example, for a 3-room flat, the monthly household income ceiling for an eligible buyer is 12,000 SGD, and a first-time buyer whose monthly household income is no more than 8,500 SGD can receive a housing grant of up to 80,000 SGD. Applicants for HDB flats are required to submit documents to verify their eligibility. Any person who provides false information is liable on conviction to a fine not exceeding 5,000 SGD or to imprisonment for a term not exceeding 6 months or to both.[8]

Similar to the RwRRC scheme, the allocation of HDB flats consists of two stages. The HDB periodically launches sales of new flats, and a lottery is held to decide the allocation of flats among the applicants. Once sold, HDB resale flats can be purchased on the open market at any time, and these flats are usually more expensive. This resale market is regulated. Unlike the RwRRC scheme, owners of HDB flats do not pay sales taxes, but they must have resided in their flats for some time, referred to as the minimum occupation period (MOP), before they are eligible to resell or sublet their flats.

If we interpret the sales tax in the RwRRC scheme as a form of restriction on resale, low-budget agents who receive more financial aid in their initial purchases are subject to more severe restrictions on resale. This feature is also present in the Singapore program. In the Singapore program, the length of MOP positively depends on the financial aid received. If a flat is purchased with a housing grant, the owner is required to reside in their flat for at least 5 years before they can resell or sublet. By contrast, a flat purchased without a housing grant is subject to no MOP requirement or a shorter one.[9,10]

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[7]In Singapore, 90% of HDB flats are owned by their residents. The remainder are rental flats for people who cannot afford to purchase the cheapest form of HDB flats despite financial aid.

[8]The HDB of Singapore. (2019, May 10). [Resale procedures of HDB flats]. Retrieved from https://www.hdb.gov.sg/cs/infoweb/e-resale/resale-procedures

[9]The HDB of Singapore. (2018, Jan 1). [Eligibility conditions for flat owners to sell their HDB flats]. Retrieved from http://www.hdb.gov.sg/cs/infoweb/residential/selling-a-flat/eligibility

[10]The HDB of Singapore. (2018, Jan 8). [Eligibility conditions for flat owners to rent out their HDB flats]. Retrieved from http://www.hdb.gov.sg/cs/infoweb/residential/renting-out-a-flat-bedroom/renting-out-your-flat/eligibility

Another difference between the Singapore program and the RwRRC scheme is that in the Singapore program, an applicant pays only if they win the lottery and purchase a flat. By contrast, agents in the RwRRC scheme pay for the lottery ticket regardless of whether they receive the good, which some may consider unrealistic. I discuss this issue in Section 5.

5. Per-unit price constraint

In the optimal direct mechanism, agents make payments to the principal regardless of whether they receive the good, which some may consider unrealistic.[11] The question, then, is whether this direct mechanism can be implemented by a mechanism in which agents pay if and only if they receive the good and the payment is within their budgets. Such an implementation is impossible if $a^* < 1$. To guarantee that such an implementation always exists, we can replace (BC) with the following *per-unit price constraint* considered by CGK:

$$p(t) \leq a(t)b, \quad \forall t = (v, b). \tag{PC}$$

The optimal mechanisms in these two settings share qualitatively similar features.

The key observation is that for an mechanism to be incentive compatible, the per-unit price paid by an agent must be nondecreasing in his reported value. Thus, similar to (BC), (PC) holds if and only if it holds for low-budget agents with the highest possible valuation. Using this observation, the results of Theorem 1 extend and characterize the optimal mechanism when the principal does not verify agent budgets. This result extends the results in Section 3 of CGK by allowing for an agent's valuation to take a continuum of possible values.

If the principal can verify an agent's budget at a cost, then by an approximation argument similar to that in Section 3.2, I can prove that the optimal allocation rule is an M-step allocation rule for some integer $2 \leq M \leq 5$. However, it may not be a 2-step allocation rule even if we impose the regularity conditions. Intuitively, with the more stringent (PC) constraint, more benefits accrue to low-budget agents, especially those with low valuations, making it more attractive to underreport budgets and making verification more costly. Thus, the principal may benefit from increasing the number of steps, which saves verification cost. To restore the optimality of a 2-step allocation rule, I need an additional assumption ruling out in-kind transfers to agents with sufficiently low valuations, that is, $a(v, b) = 0$ for all $v < b_1$. Note that this assumption is satisfied if either (BC) holds or the principal does not verify agent budgets.[12]

This optimal mechanism can be implemented by a modified RwRRC scheme, in which a lottery participant in the first stage pays *only if* he receives the good. If verification is sufficiently costly or the principal does not verify agent budgets, then this implementation is identical to the RwRRC scheme in CGK.

[11] Such an "all-pay" feature is common in optimal mechanisms with financially constrained agents. For example, in Laffont and Robert (1996), Maskin (2000), and Pai and Vohra (2014), the optimal mechanism can be implemented by an all-pay or a modified all-pay auction.

[12] A more detailed analysis can be found in the working paper version of this paper (Li 2017).

Appendix A: The revelation principle

Consider a general mechanism that consists of a message space $\mathcal{M}$ and a quadruplet (a, p, q, θ), where $a : \mathcal{M} \to [0, 1]$ maps a message to the probability that an agent obtains the good, $p : \mathcal{M} \to \mathbb{R}$ maps a message to the payment an agent must make, $q : \mathcal{M} \to [0, 1]$ maps a message to the probability of verification and $\theta : \mathcal{M} \times \{n, b_1, b_2\} \to [0, 1]$ denotes the punishment rule. In particular, $\theta(m, n)$ denotes the probability that an agent is punished if his budget is not inspected and $\theta(m, b)$ denotes the probability that an agent is punished if his budget is inspected and revealed to be b.

Given a mechanism, an agent with type $t = (v, b)$ chooses $m \in \mathcal{M}$ to maximize his expected payoff:

$$a(m)v - p(m) - \big(1 - q(m)\big)\theta(m, n)c - q(m)\theta(m, b)c$$

subject to the constraint that $p(m) \leq b$. Let $m^*(t)$ denote the solution to the agent's payoff maximization problem. For ease of exposition, I assume that $m^*(t)$ is deterministic, but it is easy to accommodate mixed strategies. If the agent's problem has multiple solutions, then some deterministic selection rule is used. Consider a new mechanism with message space T. Let $a^*(t) = a(m^*(t))$, $p^*(t) = p(m^*(t))$, $q^*(t) = a(m^*(t))$, and $\theta^*(t, \cdot) = \theta(m^*(t), \cdot)$. Then the new mechanism is incentive compatible. Clearly, an agent has no incentive to report $\hat{t}$ such that $p^*(\hat{t}) > b$. For $\hat{t}$ such that $p^*(\hat{t}) \leq b$, we have

$$\begin{aligned} &a\big(m^*(t)\big)v - p\big(m^*(t)\big) - \big(1 - q\big(m^*(t)\big)\big)\theta\big(m^*(t), n\big)c - q\big(m^*(t)\big)\theta\big(m^*(t), b\big)c \\ &\quad \geq a\big(m^*(\hat{t})\big)v - p\big(m^*(\hat{t})\big) - \big(1 - q\big(m^*(\hat{t})\big)\big)\theta\big(m^*(\hat{t}), n\big)c - q\big(m^*(\hat{t})\big)\theta\big(m^*(\hat{t}), b\big)c. \end{aligned}$$

The inequality simply follows from the fact that $m^*(t)$ maximizes type t's payoff in the original mechanism. Furthermore, the principal's payoff in the truth telling equilibrium is as same as that in the original mechanism.

Hence, it is without loss of generality to focus on direct mechanisms in which $\mathcal{M} = T$. In the main body of the paper, I assume that the principal can only punish an agent who is inspected and found to have lied about his budget. That is, $\theta(t, n) = 0$, $\theta(t, \hat{b}) = 1$ if $\hat{b} \neq b$, and $\theta(t, \hat{b}) = 0$ if $\hat{b} = b$.

Appendix B: Omitted proofs

This section is organized as follows. Appendix B.1 proves Proposition 1. Appendix B.2 consists of proofs in Section 3.1. Appendix B.3 consists of proofs in Section 3.2. Appendix B.4 proves Proposition 3.

B.1 *Proof of Proposition 1*

Let v^{FB} denote the critical value such that $1 - F(v^{FB}) = S$. If $b_1 + v^{FB}(1 - F(v^{FB})) \geq v^{FB}$, then the first-best outcome will arise as the competitive market outcome with market-clearing price v^{FB} and a uniform cash transfer of $v^{FB} - b_1$ to all agents.

Next, we prove the "only if" part. If the first-best is achieved, the allocation rule satisfies: $a(v, b) = 1$ if $v \geq v^{FB}$ and $a(v, b) = 0$ otherwise. By the standard argument, (IC) implies that $p(v, b) = v^{FB} - u(\underline{v}, b)$ if $v \geq v^{FB}$ and $p(v, b) = -u(\underline{v}, b)$ otherwise. Since $p(\overline{v}, b_1) \leq b_1$ by (BC), and the total verification cost is zero in the first-best, $u(\underline{v}, b_2) = u(\underline{v}, b_1) \geq v^{FB} - b_1$. Then (BB) holds only if

$$b_1 + v^{FB}\big(1 - F\big(v^{FB}\big)\big) \geq v^{FB}.$$

B.2 *No verification*

To prove Theorem 1, I first prove Lemmas 5 and 6. Lemma 5 says that in an optimal mechanism, agents receive the same amount of cash transfers regardless of their budgets. One implication of Lemma 5 is that in an optimal mechanism agents receive positive cash transfers regardless of their budgets.

Lemma 5. *Suppose the principal does not verify agent budgets. In an optimal mechanism of $\mathcal{P}'_{NV}$, $u(\underline{v}, b_1) = u(\underline{v}, b_2)$.*

Proof. Let (a, p) be an optimal mechanism of $\mathcal{P}'_{NV}$. Using the envelope condition, (2) can be rewritten as

$$u(\underline{v}, b_2) + \int_{\underline{v}}^{v} a(\nu, b_2)\,\mathrm{d}\nu \geq u(\underline{v}, b_1) + \int_{\underline{v}}^{v} a(\nu, b_1)\,\mathrm{d}\nu, \quad \forall v. \tag{7}$$

If $v = \underline{v}$, (7) reduces to $u(\underline{v}, b_2) \geq u(\underline{v}, b_1)$. Suppose, on the contrary, that $u(\underline{v}, b_2) > u(\underline{v}, b_1)$. Then we can construct another feasible mechanism (a^*, p^*) strictly improving welfare.

We construct the new mechanism by giving low-budget agents more cash transfers and fewer goods. Specifically, let $u^*(\underline{v}, b_1) = u^*(\underline{v}, b_2) = (1 - \pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2)$. Let v^- and v^+ be such that

$$v^- = \sup\left\{v \Big| \int_{\underline{v}}^{v} a(\nu, b_1)\,\mathrm{d}\nu + u(\underline{v}, b_1) - (1 - \pi)u(\underline{v}, b_1) - \pi u(\underline{v}, b_2) \leq 0\right\}, \quad \text{and}$$

$$v^+ = \sup\left\{v \geq v^- \Big| (1 - \pi)\int_{\underline{v}}^{v^-} a(\nu, b_1)f(\nu)\,\mathrm{d}\nu - \pi\int_{v^-}^{v}\big[a(v, b_2) - a(\nu, b_2)\big]f(\nu)\,\mathrm{d}\nu \geq 0\right\}.$$

Assume without loss of generality that

$$(1 - \pi)\int_{\underline{v}}^{v^-} a(\nu, b_1)f(\nu)\,\mathrm{d}\nu - \pi\int_{v^-}^{v^+}\big[a\big(v^+, b_2\big) - a(\nu, b_2)\big]f(\nu)\,\mathrm{d}\nu = 0. \tag{8}$$

We construct the new allocation rule by reducing the assignment probability for low-budget agents whose valuations are below v^- and increasing that for high-budget agents whose valuations lie in $[v^-, v^+]$. Let

$$a^*(v, b_1) = \begin{cases} 0 & \text{if } v \leq v^- \\ a(v, b_1) & \text{if } v > v^- \end{cases}, \quad \text{and}$$

$$a^*(v, b_2) = \begin{cases} a(v, b_2) & \text{if } v \le v^- \\ a(v^+, b_2) & \text{if } v^- < v \le v^+ \\ a(v, b_2) & \text{if } v > v^+ \end{cases}.$$

Clearly, $a^*(v, b)$ is nondecreasing in v for both b. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u^*(\underline{v}, b)$ for all v and b. By construction, the new mechanism (a^*, p^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare. In what follows, I verify that (a^*, p^*) satisfies (BC), (BB), and (IC-b) in turn.

By the definition of v^-,

$$\begin{aligned} p^*(\overline{v}, b_1) &= \overline{v}a^*(\overline{v}, b_1) - \int_{\underline{v}}^{\overline{v}} a^*(\nu, b_1)\,\mathrm{d}\nu - u^*(\underline{v}, b_1) \\ &\le \overline{v}a(\overline{v}, b_1) - \int_{v^-}^{\overline{v}} a(\nu, b_1)\,\mathrm{d}\nu - \int_{\underline{v}}^{v^-} a(\nu, b_1)\,\mathrm{d}\nu - u(\underline{v}, b_1) = p(\overline{v}, b_1) \le b_1. \end{aligned}$$

Hence, the new mechanism (a^*, p^*) satisfies (BC).

Using the envelope condition,

$$\begin{aligned} \mathbb{E}\big[p^*(v, b)\big] &= -(1-\pi)u^*(\underline{v}, b_1) - \pi u^*(\underline{v}, b_2) + \mathbb{E}\left[\left(v - \frac{1 - F(v)}{f(v)}\right)a^*(v, b)\right] \\ &\ge -(1-\pi)u(\underline{v}, b_1) - \pi u(\underline{v}, b_2) + \mathbb{E}\left[\left(v - \frac{1 - F(v)}{f(v)}\right)a(v, b)\right] = \mathbb{E}\big[p(v, b)\big], \end{aligned}$$

where the inequality holds by Assumption 1. Therefore, (a^*, p^*) satisfies (BB).

Define

$$\Delta(v) := u^*(\underline{v}, b_2) + \int_{\underline{v}}^{v} a^*(\nu, b_2)\,\mathrm{d}\nu - u^*(\underline{v}, b_1) - \int_{\underline{v}}^{v} a^*(\nu, b_1)\,\mathrm{d}\nu, \quad \forall v.$$

To show that (IC-b) holds, it suffices to show that (a^*, p^*) satisfies (7), that is, $\Delta(v) \ge 0$ for all v. For all $v \le v^-$, since $u^*(\underline{v}, b_2) = u^*(\underline{v}, b_1)$ and $a^*(\nu, b_2) \ge 0 = a^*(\nu, b_1)$ for all $\nu \le v$, $\Delta(v) \ge 0$. For $v = v^+$,

$$\begin{aligned} \Delta(v^+) &= u^*(\underline{v}, b_2) + \int_{\underline{v}}^{v^+} a^*(\nu, b_2)\,\mathrm{d}\nu - u^*(\underline{v}, b_1) - \int_{\underline{v}}^{v^+} a^*(\nu, b_1)\,\mathrm{d}\nu \\ &\ge u(\underline{v}, b_2) + \int_{\underline{v}}^{v^-} a(\nu, b_2)\,\mathrm{d}\nu + \int_{v^-}^{v^+} a(v^+, b_2)\,\mathrm{d}\nu - u(\underline{v}, b_1) \\ &\quad - \int_{v^-}^{v^+} a(\nu, b_1)\,\mathrm{d}\nu - \frac{1}{\pi}\int_{v^-}^{v^+} a(\nu, b_1)\,\mathrm{d}\nu \\ &\ge u(\underline{v}, b_2) + \int_{\underline{v}}^{v^-} a(\nu, b_2)\,\mathrm{d}\nu + \int_{v^-}^{v^+} a(v^+, b_2)\,\mathrm{d}\nu - u(\underline{v}, b_1) \\ &\quad - \int_{v^-}^{v^+} a(\nu, b_1)\,\mathrm{d}\nu - \int_{v^-}^{v^+} a(\nu, b_1)\,\mathrm{d}\nu - \int_{v^-}^{v^+} \big[a(v^+, b_2) - a(v, b_2)\big]\,\mathrm{d}v \end{aligned}$$

$$= u(\underline{v}, b_2) + \int_{\underline{v}}^{v^+} a(\nu, b_2)\,\mathrm{d}\nu - u(\underline{v}, b_1) - \int_{\underline{v}}^{v^+} a(v, b_1)\,\mathrm{d}\nu \geq 0,$$

where the first inequality holds by the definition of v^-. To see that the second inequality holds, observe that

$$\begin{aligned}
&\pi \int_{v^-}^{v^+} \big[a\big(v^+, b_2\big) - a(v, b_2)\big]\,\mathrm{d}v \\
&\quad\geq \pi \int_{v^-}^{v^+} \big[a\big(v^+, b_2\big) - a(v, b_2)\big] f(v) \frac{1}{f\big(v^-\big)}\,\mathrm{d}v \\
&\quad= (1-\pi) \int_{\underline{v}}^{v^-} a(v, b_1) f(v) \frac{1}{f\big(v^-\big)}\,\mathrm{d}v \\
&\quad\geq (1-\pi) \int_{\underline{v}}^{v^-} a(v, b_1)\,\mathrm{d}v,
\end{aligned}$$

where the first and the third lines hold by Assumption 2 and the second line holds by (8). Since $a^*(v, b) = a(v, b)$ for all $v \geq v^+$ and b, $\Delta(v) \geq 0$ for all $v \geq v^+$. Finally, $\Delta'(v) = a(v^+, b_2) - a(v, b_1)$ for $v \in (v^-, v^+)$, which is nonincreasing. Hence, $\Delta(v)$ is concave over (v^-, v^+). Since $\Delta(v^-) \geq 0$ and $\Delta(v^+) \geq 0$, we have $\Delta(v) \geq 0$ for all $v \in (v^-, v^+)$. Hence, (IC-b) holds.

Thus, (a^*, p^*) is a feasible mechanism and strictly improves welfare, which contradicts the optimality of (a, p). Hence, it must be that $u(\underline{v}, b_2) = u(\underline{v}, b_1)$. □

Lemma 6 says that for any given v, an optimal mechanism on average allocates more resources to high-budget agents whose valuations are below v than to low-budget agents whose valuations are below v.

Lemma 6. *Suppose the principal does not verify agent budgets. In an optimal mechanism of $\mathcal{P}'_{NV}$, the allocation rule satisfies*

$$\int_{\underline{v}}^{v} a(\nu, b_2) f(\nu)\,\mathrm{d}\nu \geq \int_{\underline{v}}^{v} a(\nu, b_1) f(\nu)\,\mathrm{d}\nu, \quad \forall v. \tag{9}$$

Proof. Given Lemma 5, (7) becomes

$$\int_{\underline{v}}^{v} a(\nu, b_2)\,\mathrm{d}\nu \geq \int_{\underline{v}}^{v} a(\nu, b_1)\,\mathrm{d}\nu, \quad \forall v. \tag{10}$$

For each $b \in \{b_1, b_2\}$, we have

$$\int_{\underline{v}}^{v} a(\nu, b) f(\nu)\,\mathrm{d}\nu = f(v) \int_{\underline{v}}^{v} a\big(\nu', b\big)\,\mathrm{d}\nu' - \int_{\underline{v}}^{v} \bigg[\int_{\underline{v}}^{\nu} a\big(\nu', b\big)\,\mathrm{d}\nu'\bigg] f'(\nu)\,\mathrm{d}\nu.$$

Since $f \geq 0$ and $-f' \geq 0$, (9) follows from (10). □

Proof of Theorem 1. We first solve the optimal mechanism of $\mathcal{P}'_{NV}$ and then verify that the optimal mechanism satisfies the (IC) constraints corresponding to the overreporting of budgets.

Let (a, p) be a feasible mechanism that satisfies the conditions in Lemmas 5 and 6. Consider another mechanism (a^*, p^*) constructed as follows.

Let $\hat{v} = \inf\{v | a(v, b_2) \geq a(\overline{v}, b_1)\}$. Note that $\hat{v} = \overline{v}$ if $a(\overline{v}, b_1) > a(\overline{v}, b_2)$ and $\hat{v} = \underline{v}$ if $a(\overline{v}, b_1) \leq a(\underline{v}, b_2)$. Let a^* be as follows:

$$a^*(v, b_1) = \begin{cases} a(\overline{v}, b_1) & \text{if } v \geq v_1^*, \\ 0 & \text{otherwise}, \end{cases}$$

where v_1^* satisfies $a(\overline{v}, b_1)[1 - F(v_1^*)] = \int_{\underline{v}}^{\overline{v}} a(v, b_1) f(v)\,\mathrm{d}v$, and

$$a^*(v, b_2) = \begin{cases} 1 & \text{if } v \geq v_2^{**}, \\ a(\overline{v}, b_1) & \text{if } v_2^* \leq v < v_2^{**}, \\ 0 & \text{otherwise}, \end{cases}$$

where $v_2^* \leq \hat{v}$ satisfies $a(\overline{v}, b_1)[F(\hat{v}) - F(v_2^*)] = \int_{\underline{v}}^{\hat{v}} a(v, b_2) f(v)\,\mathrm{d}v$ and $v_2^{**} \geq \hat{v}$ satisfies $1 - F(v_2^{**}) + a(\overline{v}, b_1)[F(v_2^{**}) - F(\hat{v})] = \int_{\hat{v}}^{\overline{v}} a(v, b_2) f(v)\,\mathrm{d}v$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b. By construction, the new mechanism (a^*, p^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare unless $a^* = a$ almost surely. By Assumption 1, (a^*, p^*) satisfies (BB). By Assumption 2, (a^*, p^*) satisfies (BC):

$$\begin{aligned}
&p(\overline{v}, b_1) - p^*(\overline{v}, b_1) \\
&\quad = \int_{\underline{v}}^{v_1^*} \big[a(\overline{v}, b_1) - a(v, b_1)\big]\,\mathrm{d}v + \int_{v_1^*}^{\overline{v}} \big[0 - a(v, b_1)\big]\,\mathrm{d}v \\
&\quad \geq \int_{\underline{v}}^{v_1^*} \big[a(\overline{v}, b_1) - a(v, b_1)\big] f(v) \frac{1}{f(v_1^*)}\,\mathrm{d}v + \int_{v_1^*}^{\overline{v}} \big[0 - a(v, b_1)\big] f(v) \frac{1}{f(v_1^*)}\,\mathrm{d}v \\
&\quad = \frac{1}{f(v_1^*)} \left\{ a(\overline{v}, b_1)\big[1 - F(v_1^*)\big] - \int_{\underline{v}}^{\overline{v}} a(v, b_1) f(v)\,\mathrm{d}v \right\} = 0.
\end{aligned}$$

Next, we verify that (a^*, p^*) also satisfies (IC-b). It suffices to show that $v_1^* \geq v_2^*$. If $v_1^* \geq \hat{v}$, then $v_1^* \geq v_2^*$. If $v_1^* < \hat{v}$, then

$$\begin{aligned}
a(\overline{v}, b_1)\big[F(\hat{v}) - F(v_1^*)\big] &= \int_{\underline{v}}^{\hat{v}} a(v, b_1) f(v)\,\mathrm{d}v + \int_{\hat{v}}^{\overline{v}} \big[a(v, b_1) - a(\overline{v}, b_1)\big] f(v)\,\mathrm{d}v \\
&\leq \int_{\underline{v}}^{\hat{v}} a(v, b_1) f(v)\,\mathrm{d}v \\
&\leq \int_{\underline{v}}^{\hat{v}} a(v, b_2) f(v)\,\mathrm{d}v = a(\overline{v}, b_1)\big[F(\hat{v}) - F(v_2^*)\big],
\end{aligned}$$

where the second inequality holds by Lemma 6. In this case, it must be that $a(\overline{v}, b_1) > 0$ since otherwise $a(\overline{v}, b_1) = 0 \le a(\underline{v}, b_2)$, which implies $\hat{v} = \underline{v} \le v_1^*$. Hence, $v_2^* \le v_1^*$. Thus, (a^*, p^*) satisfies (IC-b).

Therefore, (a^*, p^*) is a feasible mechanism and strictly improves welfare unless $a^* = a$ almost surely. Suppose $v_2^* < v_1^*$, it is welfare improving to increase v_2^* and reduce v_1^* without affecting any constraint. Hence, it is optimal to set $v_1^* = v_2^* = v^*$. Let $v^{**} = v_2^{**} \ge v^*$ and $u^* = u(\underline{v}, b_1) = u(\underline{v}, b_2)$. Then, the optimal allocation rule must satisfy $a(v, b_1) = \chi_{\{v \ge v^*\}} \min\{\frac{u^*+b_1}{v^*}, 1\}$ and $a(v, b_2) = \chi_{\{v \ge v^*\}} \min\{\frac{u^*+b_1}{v^*}, 1\} + \chi_{\{v \ge v^{**}\}}(1 - \min\{\frac{u^*+b_1}{v^*}, 1\})$, where χ_V is the indicator function of set V. Let $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\,\mathrm{d}\nu - u^*$ for all v and b. This completes the characterization of the optimal mechanism of $\mathcal{P}'_{NV}$. Finally, it is easy to see that low-budget agents have no incentive to overreport their budget. □

B.3 *The general case*

In this section, I first prove Lemmas 1 and 2. Appendix B.3.1 proves Lemma 3, Appendix B.3.2 proves Lemma 4, and Appendix B.3.3 contains the proof of Theorem 2. Appendix B.3.4 contains the proof of Proposition 2.

Proof of Lemma 1. Let (a, p, q) be a feasible mechanism of $(\mathcal{P}')$. Suppose $a(\cdot, b_1)$ is a non-decreasing step function. That is, there exist an integer $M \ge 2$, $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $0 \le a^1 < a^2 < \cdots < a^M \le 1$ such that $a(v, b_1) = a^m$ if $v \in [v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Consider another mechanism (a^*, p^*, q^*) constructed as follows.

Let $\hat{v}_2^m = \inf\{v | a(v, b_2) \ge a^m\}$ for $m = 1, \dots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. For each $m = 1, \dots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2) f(v)\,\mathrm{d}v = a^{m-1}\bigl[F\bigl(v_2^{m-1}\bigr) - F\bigl(\hat{v}_2^{m-1}\bigr)\bigr] + a^m\bigl[F\bigl(\hat{v}_2^m\bigr) - F\bigl(v_2^{m-1}\bigr)\bigr], \tag{11}$$

where $a^0 = 0$ and $a^{M+1} = 1$. Let $a^*(v, b_2) = a^m$ for $v \in [v_2^{m-1}, v_2^m)$ and $m = 1, \dots, M$, $a^*(v, b_2) = 0$ for $v < v_2^0$, and $a^*(v, b_2) = 1$ for $v \ge v_2^M$. Note that if $a^1 = 0$, then $v_2^0 = \underline{v}$. If $a^M = 1$, then any $v_2^M \in [\hat{v}_2^M, \overline{v}]$ satisfies (11), and in this case let $v_2^M = v_2^{M-1}$. Let $a^*(\cdot, b_1) = a(\cdot, b_1)$, $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b, and $q^*(\cdot, b_1) = q(\cdot, b_1)$. By construction, (a^*, p^*, q^*) satisfies (IR), (IC-v), (BC), and (S), and strictly improves welfare unless $a^* = a$ almost surely. By Assumption 1, (a^*, p^*, q^*) satisfies (BB).

Next, we verify that (a^*, p^*, q^*) satisfies (IC-b). Since (a, p, q) satisfies (IC-b), $q(v, b_1) \ge q^m$ for $v \in (v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$, where

$$q^m = \frac{1}{c}\max\biggl\{0, u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\,\mathrm{d}\nu$$
$$+ a^m\bigl(\hat{v}_2^m - v_1^{m-1}\bigr) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\,\mathrm{d}\nu\biggr\}.$$

Hence, it suffices to show that, for $m = 1, \dots, M$,

$$q^m c \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\,\mathrm{d}\nu + a^m\big(v_2^{m-1} - v_1^{m-1}\big) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\,\mathrm{d}\nu. \tag{12}$$

By Assumption 2,

$$\begin{aligned}
\int_{v_2^{m-1}}^{\hat{v}_2^m} \big[a^*(\nu, b_2) - a(\nu, b_2)\big]\,\mathrm{d}\nu &\geq \int_{v_2^{m-1}}^{\hat{v}_2^m} \big[a^*(\nu, b_2) - a(\nu, b_2)\big] f(\nu) \frac{1}{f\big(v_2^{m-1}\big)}\,\mathrm{d}\nu \\
&= \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} \big[a(\nu, b_2) - a^*(\nu, b_2)\big] f(\nu) \frac{1}{f\big(v_2^{m-1}\big)}\,\mathrm{d}\nu \\
&\geq \int_{\hat{v}_2^{m-1}}^{v_2^{m-1}} \big[a(\nu, b_2) - a^*(\nu, b_2)\big]\,\mathrm{d}\nu.
\end{aligned}$$

Therefore, for $m = 1, \dots, M$,

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} \big[a^*(\nu, b_2) - a(\nu, b_2)\big]\,\mathrm{d}\nu \geq 0. \tag{13}$$

Then we have

$$\begin{aligned}
&\text{RHS of (12)} \\
&= u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a^*(\nu, b_2)\,\mathrm{d}\nu + a^m\big(\hat{v}_2^m - v_1^{m-1}\big) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\,\mathrm{d}\nu \\
&\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\,\mathrm{d}\nu + a^m\big(\hat{v}_2^m - v_1^{m-1}\big) + \int_{\underline{v}}^{v_1^{m-1}} a(\nu, b_1)\,\mathrm{d}\nu \\
&\leq q^m c,
\end{aligned}$$

where the equality holds since $a^*(v, b_2) = a^m$ for $v \in (v_2^{m-1}, \hat{v}_2^m)$, and the first inequality holds by (13).

Thus, (a^*, p^*, q^*) is a feasible mechanism, and strictly improves welfare unless $a^* = a$ almost surely. Observe that a^* is an M-step allocation rule. □

Proof of Lemma 2. To prove Lemma 2, I first prove that under an optimal mechanism of $\mathcal{P}'(M, d)$, either no high-budget type has incentives to mimic low budget type, or for any low-budget type, some high-budget type exists that weakly prefers to mimic the low-budget type in the absence of verification.

Lemma 7. *An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies one of the following two conditions:*

(C1) For all $m = 1, \dots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \geq 0. \tag{14}$$

(C2) For all $m = 1, \dots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \leq 0.$$

Clearly, if (C2) holds, the optimal verification rule is $q = 0$. In this case, the optimal mechanism of ($\mathcal{P}'$) given in Section 3.1 is also a feasible mechanism of $\mathcal{P}'(M, d)$ and satisfies (C1) with equality. Thus, an optimal mechanism of $\mathcal{P}'(M, d)$ satisfies (C1). □

Proof of Lemma 7. The proof is by contradiction. Let (a, p, q) be an optimal mechanism of $\mathcal{P}'(M, d)$. Assume without loss that a is an M-step allocation rule, and q is given by (5). Suppose (a, p, q) satisfies neither (C1) nor (C2). Then we show that one can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. This contradicts the optimality of (a, p, q). Therefore, (a, p, q) must satisfy either (C1) or (C2).

We break the proof into three cases.

Case 1. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) < 0$. Let $m > 1$ be such that $v_2^{m'-1} - v_1^{m'-1} \leq 0$ for all $m' < m$ and $v_2^{m-1} - v_1^{m-1} > 0$. If there is no such m, then (a, p, q) satisfies (C2). Define $\hat{v} \in [v_1^{m-1}, v_2^{m-1}]$ as follows. If $F(v_1^m) > \pi F(v_2^{m-1}) + (1-\pi)F(v_1^{m-1})$, let $\hat{v}$ be such that $F(\hat{v}) = \pi F(v_2^{m-1}) + (1-\pi)F(v_1^{m-1})$; otherwise, let $\hat{v} = v_1^m$. Clearly, $\hat{v} \leq v_1^m$. Consider two different cases in turn: (i) $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) \geq \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1(v_2^0 - v_1^0)]$ and (ii) $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) < \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1(v_2^0 - v_1^0)]$. In each of these two cases, we construct a new mechanism by increasing the cash transfers to low-budget agents while reducing the amount of goods allocated to them.

(i) Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) \geq \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1(v_2^0 - v_1^0)]$. Let $\tilde{v}_1^{m-1} \in [v_1^{m-1}, \hat{v}]$ be such that

$$(a^m - a^{m-1})(\tilde{v}_1^{m-1} - v_1^{m-1}) = \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) - a^1(v_2^0 - v_1^0)].$$

Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$ and $m' \neq m-1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in [\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = (1-\pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) - \pi a^1(v_2^0 - v_1^0)$ and $u^*(\underline{v}, b_2) = (1-\pi)u(\underline{v}, b_1) + \pi u(\underline{v}, b_2) + (1-\pi)a^1(v_2^0 - v_1^0)$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u^*(\underline{v}, b)$ for all v and b. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. By construction, (a^*, p^*, q^*) also satisfies (IR), (IC-v), and (S) and strictly improves welfare. By Assumption 1, (BB) holds.

Next, we verify that (a^*, p^*, q^*) satisfies (IC-b). For $\hat{v} \in [\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = 1, \ldots, m-1$, (IC-b) holds since

$$u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \le 0 \le q^*(\hat{v}, b_1)c.$$

For $\hat{v} \in [\tilde{v}_1^{m'-1}, \tilde{v}_1^{m'})$, $m' = m, \ldots, M$, we have $q^*(\hat{v}, b_1) = q^m$. Then (IC-b) holds since

$$\begin{aligned} & u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ &= \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\tilde{v}_2^{m-1} - \tilde{v}_1^{m-1} - v_2^{m-1} + v_1^{m-1}) \\ &\quad - a^1(v_2^0 - v_1^0) \\ &\le \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \frac{(a^m - a^{m-1})(v_1^{m-1} - \tilde{v}_1^{m-1})}{\pi} - a^1(v_2^0 - v_1^0) \\ &= \sum_{j=1}^{m'} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + u(\underline{v}, b_1) - u(\underline{v}, b_2) = q^{m'} c, \end{aligned}$$

where the inequality holds since by Assumption 2,

$$\begin{aligned} v_2^{m-1} - \tilde{v}_2^{m-1} &\ge \frac{1}{f(\tilde{v}_2^{m-1})}[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] \\ &\ge \frac{1-\pi}{\pi} \frac{1}{f(\tilde{v}_1^{m-1})}[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})] \ge \frac{1-\pi}{\pi}(\tilde{v}_1^{m-1} - v_1^{m-1}). \end{aligned}$$

Thus, (a^*, p^*, q^*) is feasible and strictly improves welfare.

(ii) Suppose $(a^m - a^{m-1})(\hat{v} - v_1^{m-1}) < \pi[u(\underline{v}, b_2) - u(\underline{v}, b_1) + a^1(v_2^0 - v_1^0)]$**.** Let $\tilde{v}_1^{m-1} = \hat{v}$. Let $\tilde{v}_2^{m-1} \in [\hat{v}, v_2^{m-1}]$ be such that $\pi[F(v_2^{m-1}) - F(\tilde{v}_2^{m-1})] = (1-\pi)[F(\tilde{v}_1^{m-1}) - F(v_1^{m-1})]$. Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $i = 1, 2$, and $m' \ne m-1$. Let $a^*(v, b_1) = a^{m-1}$ if $v \in (v_1^{m-1}, \tilde{v}_1^{m-1})$, and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^m$ if $v \in [\tilde{v}_2^{m-1}, v_2^{m-1})$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1})$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1-\pi)(a^m - a^{m-1})(\hat{v} - v_1^{m-1})/\pi$. Then $u^*(\underline{v}, b_2) > u^*(\underline{v}, b_1) + a^1(v_2^0 - v_1^0) \ge 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u^*(\underline{v}, b)$ for all v and b. Let $q^*(v, b_1) = q(v, b_1)$ for all v. By a similar argument to that of case (i), (a^*, p^*, q^*) is feasible and strictly improves welfare.

Case 2. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) \ge 0$. Let $m \ge 2$ be such that (14) holds for all $m' < m$ and

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0.$$

If there is no such m, then (a, p, q) satisfies (C1). It must be the case that $v_2^{m-1} < v_1^{m-1}$. Suppose $v_2^{m-1} < v_2^M$ in Case 2. We discuss the case of $v_2^M = v_2^{m-1} < v_1^{m-1}$ in Case 3.

Let $m^* \geq m$ be the smallest m' such that $v_2^{m'} > v_2^{m-1}$. That is, $v_2^{m^*} > v_2^{m-1}$ and $v_2^{m'} = v_2^{m-1}$ for $m' = m, \dots, m^* - 1$. Let $\hat{v} \in [v_2^{m-1}, v_1^{m-1}]$ be such that

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m-1} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + (a^m - a^{m-1})(\hat{v} - v_1^{m-1}) = 0.$$

Consider two different cases in turn: (i) $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] \leq (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$ and (ii) $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] > (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$.

(i) Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] \leq (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$**.** Let $\tilde{v}_2^{m^*} \in [\hat{v}, v_2^{m^*})$ be such that

$$(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] = (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\tilde{v}_2^{m^*})]. \tag{15}$$

Let $\tilde{v}_2^{m'} = \hat{v}$ for $m' = m-1, \dots, m^*-1$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in [\tilde{v}_2^{m^*}, v_2^{m^*})$, and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\, d\nu - u(\underline{v}, b)$ for all v and b. Clearly, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. By construction, (a^*, p^*, q^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare. By Assumption 1, (BB) holds.

Next, we show that (a^*, p^*, q^*) satisfies (IC-b). That is, for $m' = 1, \dots, M$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'} c.$$

This is trivial for $m' \leq m$. For $m' = m+1, \dots, m^*$, we have $\tilde{v}_2^{m'-1} = \tilde{v}_2^{m-1} \leq v_1^{m-1} < v_1^{m'-1}$. Hence,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \sum_{j=m+1}^{m'} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - v_1^{j-1}) < 0 \leq q^{m'} c.$$

Finally, consider $m' \geq m^* + 1$. It suffices to show that

$$\begin{aligned} &u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m^*+1} (a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ &\quad \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m^*+1} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}), \end{aligned}$$

which holds if and only if

$$(a^{m^*} - a^{m-1})(\tilde{v}_2^{m-1} - v_2^{m-1}) \leq (a^{m^*+1} - a^{m^*})(v_2^{m^*} - \tilde{v}_2^{m^*}).$$

The above inequality holds by (15) and Assumption 2.

Thus, (a^*, p^*, q^*) is feasible and strictly increases welfare.

(ii) Suppose $(a^{m^*} - a^{m-1})[F(\hat{v}) - F(v_2^{m-1})] > (a^{m^*+1} - a^{m^*})[F(v_2^{m^*}) - F(\hat{v})]$**.** Let $\tilde{v}_2^{m-1}$ be such that

$$(a^{m^*} - a^{m-1})\big[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})\big] = (a^{m^*+1} - a^{m^*})\big[F(v_2^{m^*}) - F(\tilde{v}_2^{m-1})\big].$$

Let $\tilde{v}_2^{m'} = \tilde{v}_2^{m-1}$ for $m' = m, \dots, m^*$ and $\tilde{v}_2^{m'} = v_2^{m'}$ if $m' < m-1$ or $m' > m^*$. Let $\tilde{v}_1^{m'} = v_1^{m'}$ for all m'. Let $a^*(v, b_1) = a(v, b_1)$. Let $a^*(\cdot, b_2)$ such that $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_2) = a^{m^*+1}$ if $v \in [\tilde{v}_2^{m-1}, v_2^{m^*})$, and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b. Let $q^*(v, b_1) = q(v, b_1)$ all v. By a similar argument to that of case (ii), (a^*, p^*, q^*) is feasible and strictly improves welfare.

Case 3. Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) \geq 0$ and $v_2^M = v_2^{m-1} < v_1^{m-1}$, where $m \geq 2$ is such that (14) holds for all $m' < m$ and $u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) < 0$.

Let $\tilde{v}_1^{m-1} = v_1^{m-1} - \varepsilon$ for some $\varepsilon > 0$ and $\tilde{v}_2^{m'} = v_2^{m-1} + \delta$ for $m' = m-1, \dots, M$, where $\delta > 0$ is such that

$$(1-\pi)(a^m - a^{m-1})\big[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})\big] = \pi(1 - a^{m-1})\big[F(\tilde{v}_2^{m-1}) - F(v_2^{m-1})\big].$$

Let $\tilde{v}_i^{m'} = v_i^{m'}$ for $m' \neq m-1$ and $i = 1, 2$. Let $\varepsilon > 0$ be such that

$$\min\left\{\tilde{v}_1^{m-1} - v_1^{m-2}, u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1})\right\} = 0.$$

Since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m-1}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \geq 0$, we have $\tilde{v}_2^{m'} \leq \tilde{v}_1^{m'}$ for all $m' \geq m-1$. Let $a^*(v, b_2) = a^{m-1}$ if $v \in (v_2^{m-1}, \tilde{v}_2^{m-1})$, $a^*(v, b_1) = a^m$ if $v \in [\tilde{v}_1^{m-1}, v_1^{m-1})$ and $a^*(v, b) = a(v, b)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b. Since $a^*(\overline{v}, b_1) = a(\overline{v}, b_1)$ and $a^*(v, b_1) \geq a(v, b_1)$ for all v, we have $p^*(\overline{v}, b_1) \leq p(\overline{v}, b_1) \leq b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q^m$ if $v \in [\tilde{v}_1^{m-1}, \tilde{v}_1^m)$ for $m = 1, \dots, M$. Then the change of the verification cost is

$$k(q^m - q^{m-1})\big[F(v_1^{m-1}) - F(\tilde{v}_1^{m-1})\big].$$

Since $q^m = 0 \leq q^{m-1}$, the verification cost is reduced. Furthermore, by Assumption 1, the revenue increases. Hence, (BB) holds. By construction, (a^*, p^*, q^*) also satisfies (IR), (IC-v), and (S) and strictly improves welfare.

Next, we show that (IC-b) is satisfied. That is, for $m' = 1, \dots, M$,

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq q^{m'}c.$$

This is trivial for $m' < m$. For $m' \geq m$, this holds since

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m'}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \leq 0 = q^{m'}c.$$

Thus, (a^*, p^*, q^*) is feasible and strictly increases welfare. □

B.3.1 *Proof of Lemma 3* We prove Lemma 3 by proving Lemmas 8 and 9.

LEMMA 8. *An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies that $v_2^1 \geq v_1^1$.*

LEMMA 9. *Suppose $V(M, d) > V(M-1, d)$ for some $M \geq 3$. An optimal mechanism of $\mathcal{P}'(M, d)$ satisfies that $v_2^{M-1} - v_1^{M-1} > \cdots > v_2^1 - v_1^1$.*

PROOF OF LEMMA 8. Let (a, p, q) be an optimal mechanism of $\mathcal{P}'(M, d)$. Assume without loss that a is an M-step allocation rule, and q is given by (6). Suppose, on the contrary, that $v_2^1 < v_1^1$. Since (14) holds for $m = 2$, it must be that $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$. Hence, it is either (i) $u(\underline{v}, b_1) > u(\underline{v}, b_2) \geq 0$, or (ii) $a^1 > 0$ and $v_2^0 > v_1^0$. In what follows, we consider these two cases in turn. In each case, we show that one can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. This contradicts the optimality of (a, p, q). Therefore, $v_2^1 \geq v_1^1$.

(i) Suppose $u(\underline{v}, b_1) > u(\underline{v}, b_2) \geq 0$. Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^1 = v_1^1 - \pi\varepsilon/(1-\pi)$ and $\tilde{v}_2^1 > v_2^1$ be such that $(1-\pi)[F(v_1^1) - F(\tilde{v}_1^1)] = \pi[F(\tilde{v}_2^1) - F(v_2^1)]$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 \leq \tilde{v}_1^1$. By Assumption 2,

$$\begin{aligned}
\tilde{v}_2^1 - v_2^1 &\leq \left[F\left(\tilde{v}_2^1\right) - F\left(v_2^1\right)\right]\frac{1}{f\left(\tilde{v}_2^1\right)} \\
&\leq \frac{1-\pi}{\pi}\left[F\left(v_1^1\right) - F\left(\tilde{v}_1^1\right)\right]\frac{1}{f\left(\tilde{v}_1^1\right)} \\
&\leq \frac{1-\pi}{\pi}\left(v_1^1 - \tilde{v}_1^1\right) = \varepsilon.
\end{aligned}$$

Let $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) + (a^2 - a^1)\varepsilon$ and $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) - \pi(a^2 - a^1)\varepsilon/(1-\pi)$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_1) \geq u^*(\underline{v}, b_2) > 0$. Let $a^*(v, b_1) = a^2$ if $v \in [\tilde{v}_1^1, v_1^1)$, $a^*(v, b_2) = a^1$ if $v \in (v_2^1, \tilde{v}_2^1)$ and $a^*(v, b) = a(v, b)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u^*(\underline{v}, b)$ for all v and b. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1)$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. By construction, (a^*, p^*, q^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare. By Assumption 1, (a^*, p^*, q^*) satisfies (BB).

Finally, we show that (a^*, p^*, q^*) satisfies (IC-b). Let $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$, and $m \neq 1$. For $\hat{v} < \tilde{v}_1^1$, $q^*(\hat{v}, b_1) = q^1$ and we have

$$\begin{aligned}
u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1\left(\tilde{v}_2^0 - \tilde{v}_1^0\right) &= u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) - \frac{(a^2 - a^1)\varepsilon}{1-\pi} \\
&\leq q^1 c.
\end{aligned}$$

For $\hat{v} \in (\tilde{v}_1^1, v_1^2)$, we have

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1\left(\tilde{v}_2^0 - \tilde{v}_1^0\right) + \left(a^2 - a^1\right)\left(\tilde{v}_2^1 - \tilde{v}_1^1\right) \\
&\quad = u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) + \left(a^2 - a^1\right)\left(v_2^1 - v_1^1\right) \\
&\qquad - \frac{(a^2 - a^1)\varepsilon}{1-\pi} + \left(a^2 - a^1\right)\left(\tilde{v}_2^1 - v_2^1 + v_1^1 - \tilde{v}_1^1\right)
\end{aligned}$$

$$\begin{aligned}
&\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
&\quad - \frac{(a^2 - a^1)\varepsilon}{1-\pi} + (a^2 - a^1)\left(\varepsilon + \frac{\pi\varepsilon}{1-\pi}\right) \\
&= u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\
&\leq \min\{q^2 c, q^1 c\} = q^*(\hat{v}, b_1)c,
\end{aligned}$$

where the first inequality holds since $\tilde{v}_2^1 - v_2^1 \leq \varepsilon$ and the last inequality holds since $v_2^1 < v_1^1$. For $m \geq 3$, $q^*(\hat{v}, b_1) = q^m$ for $\hat{v} \in [v_1^{m-1}, v_1^m)$. Since $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$, and $m \geq 3$, we have

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1(\tilde{v}_2^0 - \tilde{v}_1^0) + (a^2 - a^1)(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\
&\quad \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\
&\quad \leq q^m c.
\end{aligned}$$

Hence, (IC-b) is satisfied. Thus, (a^*, p^*, q^*) is feasible and strictly improves welfare.

(ii) Suppose $a^1 > 0$ and $v_2^0 > v_1^0$. Let $\varepsilon \in (0, a^1]$ be sufficiently small. Let $\tilde{v}_1^0 = v_1^0 = \underline{v}$, and

$$\tilde{v}_1^1 = \frac{(a^2 - a^1)v_1^1 + \varepsilon v_1^0}{a^2 - a^1 + \varepsilon} < v_1^1.$$

By Assumption 2, we have

$$\begin{aligned}
(a^2 - a^1)\left[F(v_1^1) - F(\tilde{v}_1^1)\right] &\leq (a^2 - a^1)(v_1^1 - \tilde{v}_1^1)f(\tilde{v}_1^1) \\
&= \varepsilon(\tilde{v}_1^1 - v_1^0)f(\tilde{v}_1^1) \leq \varepsilon\left[F(\tilde{v}_1^1) - F(v_1^0)\right].
\end{aligned}$$

Let $\Delta = \varepsilon[F(\tilde{v}_1^1) - F(v_1^0)] - (a^2 - a^1)[F(v_1^1) - F(\tilde{v}_1^1)] \geq 0$. If $v_2^1 > v_2^0$, then let $\tilde{v}_2^0 = v_2^0$ and $\tilde{v}_2^1$ be such that

$$\pi(a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = \pi\varepsilon\left[F(\tilde{v}_2^1) - F(v_2^0)\right] + (1-\pi)\Delta.$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 \geq \tilde{v}_2^0 \geq v_1^0$. If $v_2^1 = v_2^0$, then let $\tilde{v}_2^1 = \tilde{v}_2^0$ be such that

$$\pi(a^2 - a^1)\left[F(v_2^1) - F(\tilde{v}_2^1)\right] = (1-\pi)\Delta.$$

For $\varepsilon > 0$ sufficiently small, $\tilde{v}_2^1 = \tilde{v}_2^0 \geq v_1^0$. Let $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$, and $m \geq 2$. For $i = 1, 2$, let $a^*(v, b_i) = a^1 - \varepsilon$ if $v \in [\tilde{v}_i^0, \tilde{v}_i^1)$, $a^*(v, b_i) = a^2$ if $v \in [\tilde{v}_i^1, v_i^1)$, and $a^*(v, b_i) = a(v, b_i)$ otherwise. Let $u^*(\underline{v}, b) = u(\underline{v}, b)$ and $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u^*(\underline{v}, b)$ for all v and b. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1)$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. By construction, (a^*, p^*, q^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare. By Assumption 1, (a^*, p^*, q^*) satisfies (BB).

Finally, we show that (a^*, p^*, q^*) satisfies (IC-b). Suppose $v_2^1 > v_2^0$. For $\hat{v} < \tilde{v}_1^1$,

$$\begin{aligned} u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) &< u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) \\ &= q^1 c = q^*(\hat{v}, b_1)c. \end{aligned}$$

For $\hat{v} \in [\tilde{v}_1^1, \tilde{v}_1^2)$,

$$\begin{aligned} &u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\ &= u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\ &\quad + (a^2 - a^1 + \varepsilon)\tilde{v}_2^1 - \varepsilon v_2^0 - (a^2 - a^1)v_2^1 \\ &\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\ &\leq \min\{q^1 c, q^2 c\} \leq q^*(\hat{v}, b_1)c, \end{aligned}$$

where the first inequality holds by Assumption 2, and the second inequality holds since $v_2^1 < v_1^1$. For $m \geq 2$ and $\hat{v} \in [v_1^{m-1}, v_1^m)$, since $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$, and $m \geq 2$,

$$\begin{aligned} &u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(v_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\ &\quad + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \\ &\leq q^m c = q^*(\hat{v}, b_1)c. \end{aligned}$$

Hence, (IC-b) is satisfied.

Suppose $v_2^0 = v_2^1$. For $v < \tilde{v}_1^1$,

$$\begin{aligned} u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) &< u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) \\ &= q^1 c = q^*(\hat{v}, b_1)c. \end{aligned}$$

For $v \in [\tilde{v}_1^1, \tilde{v}_1^2)$,

$$\begin{aligned} &u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\ &= u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) + a^2(\tilde{v}_2^1 - v_2^1) \\ &\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + (a^2 - a^1)(v_2^1 - v_1^1) \\ &\leq \min\{q^1 c, q^2 c\} \leq q^*(\hat{v}, b_1)c, \end{aligned}$$

where the first inequality holds since $\tilde{v}_2^1 \leq v_2^1$ and the second inequality holds since $v_2^1 < v_1^1$. For $m \geq 2$ and $\hat{v} \in [v_1^{m-1}, v_1^m)$, since $\tilde{v}_i^m = v_i^m$ for $i = 1, 2$, and $m \geq 2$,

$$\begin{aligned} &u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1 - \varepsilon)(\tilde{v}_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(\tilde{v}_2^1 - \tilde{v}_1^1) \\ &\quad + \sum_{j=3}^{m}(a^j - a^{j-1})(\tilde{v}_2^{j-1} - \tilde{v}_1^{j-1}) \end{aligned}$$

$$\le q^m c = q^*(\hat{v}, b_1)c.$$

Hence, (IC-b) is satisfied.

Thus, (a^*, p^*, q^*) is feasible and strictly improves welfare. □

Before proving Lemma 9, we first write down the first-order conditions of $\mathcal{P}'(M, d)$ for later use. By Lemma 2, $\mathcal{P}'(M, d)$ can be written as follows, where $\varphi(v) = v - [1 - F(v)]/f(v)$ denotes the virtual value function and the Greek letters in parentheses denote the corresponding Lagrangian multipliers:

$$\max_{\substack{u(\underline{v},b_1),u(\underline{v},b_2),\\ \{a^m\}_{m=1}^M,\{v_1^m\}_{m=1}^{M-1},\{v_2^m\}_{m=0}^M}} \pi \sum_{m=1}^{M+1} \int_{v_2^{m-1}}^{v_2^m} a^m v f(v)\,\mathrm{d}v + (1-\pi)\sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} a^m v f(v)\,\mathrm{d}v$$

$$-(1-\pi)\rho \sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} \left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \right] f(v)\,\mathrm{d}v,$$

subject to

$$\pi \sum_{m=1}^{M+1} a^m \big[F(v_2^m) - F(v_2^{m-1})\big] + (1-\pi)\sum_{m=1}^{M} a^m \big[F(v_1^m) - F(v_1^{m-1})\big] \le S, \tag{β}$$

$$a^M v_1^{M-1} - \sum_{j=1}^{M-1} a^j (v_1^j - v_1^{j-1}) - u(\underline{v}, b_1) \le b_1, \tag{η}$$

$$-(1-\pi)u(\underline{v}, b_1) + (1-\pi)\sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} a^m \varphi(v) f(v)\,\mathrm{d}v$$

$$-(1-\pi)\rho \sum_{m=1}^{M} \int_{v_1^{m-1}}^{v_1^m} \left[u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \right] f(v)\,\mathrm{d}v$$

$$-\pi u(\underline{v}, b_2) + \pi \sum_{m=1}^{M+1} \int_{v_2^{m-1}}^{v_2^m} a^m \varphi(v) f(v)\,\mathrm{d}v \ge -d, \tag{λ}$$

$$u(\underline{v}, b_1) \ge 0, \; u(\underline{v}, b_2) \ge 0, \tag{ξ_1, ξ_2}$$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m} (a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \ge 0, \quad m = 1, \dots, M, \tag{μ^m}$$

$$0 = a^0 \le a^1 \le a^2 \le \cdots \le a^M \le a^{M+1} = 1, \tag{$\alpha^1, \dots, \alpha^{M+1}$}$$

$$\underline{v} = v_1^0 \le v_1^1 \le \cdots \le v_1^M = \overline{v}, \tag{$\gamma_1^1, \dots, \gamma_1^M$}$$

$$\underline{v} \le v_2^0 \le v_2^1 \le \cdots \le v_2^M \le \overline{v}. \tag{$\gamma_2^0, \dots, \gamma_2^{M+1}$}$$

Let $M \ge 3$ be an integer. We note that if a mechanism is a feasible solution to $\mathcal{P}'(M - 1, d)$, it is also a feasible solution to $\mathcal{P}'(M, d)$. Clearly, $V(M-1, d) \le V(M, d)$. Suppose

$V(M-1,d) < V(M,d)$, then in an optimal solution to $\mathcal{P}'(M,d)$ the allocation rule must be an M-step allocation rule, that is,

$$0 = a^0 \le a^1 < a^2 < \cdots < a^M \le a^{M+1} = 1,$$
$$\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}.$$

Hence, $\alpha^2 = \cdots = \alpha^M = 0$ and $\gamma_1^1 = \cdots = \gamma_1^M = 0$. Then the first-order conditions of $\mathcal{P}'(M,d)$ are

$$\begin{aligned}
&\pi\bigg[\int_{v_2^{m-1}}^{v_2^m}\big[v+\lambda\varphi(v)\big]\,\mathrm{d}v - \beta\big[F\big(v_2^m\big)-F\big(v_2^{m-1}\big)\big]\bigg]\\
&\quad + (1-\pi)\bigg[\int_{v_1^{m-1}}^{v_1^m}\big[v+\lambda\varphi(v)\big]f(v)\,\mathrm{d}v - \beta\big[F\big(v_1^m\big)-F\big(v_1^{m-1}\big)\big]\bigg]\\
&\quad - (1-\pi)(1+\lambda)\rho\big(v_2^{m-1}-v_1^{m-1}\big)\big[F\big(v_1^m\big)-F\big(v_1^{m-1}\big)\big]\\
&\quad + (1-\pi)(1+\lambda)\rho\big(v_2^m - v_1^m - v_2^{m-1} + v_1^{m-1}\big)\big[1-F\big(v_1^m\big)\big]\\
&\quad + \eta\big(v_1^m - v_1^{m-1}\big) + \mu^m\big(v_2^{m-1}-v_1^{m-1}\big)\\
&\quad - \big(v_2^m - v_1^m - v_2^{m-1} + v_1^{m-1}\big)\sum_{j=m+1}^{M}\mu^j\\
&\quad + \alpha^m - \alpha^{m+1} = 0, \qquad\qquad (a^m, 1\le m\le M-1)
\end{aligned}$$

$$\begin{aligned}
&\pi\bigg[\int_{v_2^{M-1}}^{v_2^M}\big[v+\lambda\varphi(v)\big]f(v)\,\mathrm{d}v - \beta\big[F\big(v_2^M\big)-F\big(v_2^{M-1}\big)\big]\bigg]\\
&\quad + (1-\pi)\bigg[\int_{v_1^{M-1}}^{v_1^M}\big[v+\lambda\varphi(v)\big]f(v)\,\mathrm{d}v - \beta\big[F\big(v_1^M\big)-F\big(v_1^{M-1}\big)\big]\bigg]\\
&\quad - (1-\pi)(1+\lambda)\rho\big(v_2^{M-1}-v_1^{M-1}\big)\big[F\big(v_1^M\big)-F\big(v_1^{M-1}\big)\big]\\
&\quad - \eta v_1^{M-1} + \mu^M\big(v_2^{M-1}-v_1^{M-1}\big) + \alpha^M - \alpha^{M+1} = 0, \qquad\qquad (a^M)
\end{aligned}$$

$$\begin{aligned}
&\big(a^{m+1}-a^m\big)\bigg\{(1-\pi)\big[\big(\beta-(1+\lambda)v_1^m\big)f\big(v_1^m\big) + (\lambda+\rho+\lambda\rho)\big(1-F\big(v_1^m\big)\big)\\
&\quad + (1+\lambda)\rho\big(v_2^m - v_1^m\big)f\big(v_1^m\big)\big] - \sum_{j=m+1}^{M}\mu^j - \eta\bigg\} = 0, \qquad (v_1^m, 1\le m\le M-1)
\end{aligned}$$

$$\begin{aligned}
&a^1\bigg\{\pi\big[\big(\beta-(1+\lambda)v_2^0\big)f\big(v_2^0\big) + \lambda\big[1-F\big(v_2^0\big)\big]\big] - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{M}\mu^j\bigg\}\\
&\quad + \gamma_2^0 - \gamma_2^1 = 0, \qquad\qquad (v_2^0)
\end{aligned}$$

$$\begin{aligned}&\left(a^{m+1}-a^{m}\right)\Bigg\{\pi\left[\left(\beta-(1+\lambda)v_2^m\right)f\left(v_2^m\right)+\lambda\left[1-F\left(v_2^m\right)\right]\right]\\&\quad-(1-\pi)(1+\lambda)\rho\left[1-F\left(v_1^m\right)\right]+\sum_{j=m+1}^{M}\mu^j\Bigg\}\\&\quad+\gamma_2^m-\gamma_2^{m+1}=0, && (v_2^m, 1\le m\le M-1)\\&\pi\left(a^{M+1}-a^{M}\right)\left[\left(\beta-(1+\lambda)v_2^M\right)f\left(v_2^M\right)+\lambda\left[1-F\left(v_2^M\right)\right]\right]\\&\quad+\gamma_2^M-\gamma_2^{M+1}=0, && (v_2^M)\\&\eta+\sum_{m=1}^{M}\mu^m-(1-\pi)(\lambda+\rho+\lambda\rho)+\xi_1=0, && (u(\underline{v},b_1))\\&-\sum_{m=1}^{M}\mu^m-\pi\lambda+(1-\pi)(1+\lambda)\rho+\xi_2=0. && (u(\underline{v},b_2))\end{aligned}$$

The variables in the parentheses denote the variables with respect to which the first-order conditions are taken.

Proof of Lemma 9. Let (a,p,q) be an optimal solution of $\mathcal{P}'(M,d)$. We show that $v_2^{m+1}-v_1^{m+1}>v_2^m-v_1^m$ for all $m=1,\dots,M-2$.

For $m=1,\dots,M-1$, since $a^{m+1}>a^m$, the first-order condition of v_1^m becomes

$$\begin{aligned}&(1-\pi)\left[\left(\beta-(1+\lambda)v_1^m\right)f\left(v_1^m\right)+(\lambda+\rho+\lambda\rho)\left(1-F\left(v_1^m\right)\right)+(1+\lambda)\rho\left(v_2^m-v_1^m\right)f\left(v_1^m\right)\right]\\&\quad-\sum_{j=m+1}^{M}\mu^j-\eta=0.\end{aligned}$$

Then, for $m=1,\dots,M-1$,

$$v_2^m-v_1^m=\frac{1}{\rho}v_1^m-\frac{\lambda+\rho+\lambda\rho}{(1+\lambda)\rho}\frac{1-F\left(v_1^m\right)}{f\left(v_1^m\right)}-\frac{\beta}{(1+\lambda)\rho}+\frac{\eta+\sum_{j=m+1}^{M}\mu^j}{(1-\pi)(1+\lambda)\rho f\left(v_1^m\right)},$$

where the right-hand side is strictly increasing in v_1^m by Assumptions 1 and 2. If $\mu^{m+1}=0$, then $v_2^{m+1}-v_1^{m+1}>v_2^m-v_1^m$ since $v_1^{m+1}>v_1^m$.

Suppose $\mu^{m+1}>0$. Then $v_2^{m+1}-v_1^{m+1}\ge 0\ge v_2^m-v_1^m$ since (14) holds for m and $m+2$ and (14) holds with equality for $m+1$. We show that $v_2^{m+1}-v_1^{m+1}>v_2^m-v_1^m$. Suppose, on the contrary, that $v_2^{m+1}-v_1^{m+1}=v_2^m-v_1^m=0$. Then we can construct another feasible mechanism (a^*,p^*,q^*), which strictly improves welfare. Let $\hat{v}\in(v_1^m,v_1^{m+1})$ be such that

$$\left(a^{m+2}-a^{m+1}\right)\left[F\left(v_1^{m+1}\right)-F(\hat{v})\right]=\left(a^{m+1}-a^m\right)\left[F(\hat{v})-F\left(v_1^m\right)\right].$$

For each b, let $a^*(v,b)=a^m$ if $v\in(v_1^m,\hat{v})$, $a^*(v,b)=a^{m+2}$ if $v\in[\hat{v},v_1^{m+1})$ and $a^*(v,b)=a(v,b)$ otherwise. Let $p^*(v,b)=va^*(v,b)-\int_{\underline{v}}^{v}a^*(\nu,b)\,\mathrm{d}\nu-u(\underline{v},b)$ for all v and b and

$q^*(\cdot, b_1) = q(\cdot, b_1)$. Clearly, (a^*, p^*, q^*) is feasible and strictly improves welfare, which contradicts the optimality of (a, p, q). □

B.3.2 *Proof of Lemma 4* Let $M \geq 3$ be an integer. We want to show that $V(M-1, d) = V(M, d)$. Suppose, on the contrary, that $V(M-1, d) < V(M, d)$ (i.e., $\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}$ and $a^1 < a^2 < \cdots < a^M$). An optimal solution to $\mathcal{P}'(M, d)$ satisfies the first-order conditions given in Appendix B.3.1. In what follows, we show that these first-order conditions imply that $M \leq 2$, which contradicts the assumption that $M \geq 3$. Hence, it must be the case that $V(M-1, d) = V(M, d)$ for all $M \geq 3$.

We first provide a proof sketch of Lemma 4. Assume, for ease of exposition, that

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0, \qquad (\mu^1)$$

$$\underline{v} = v_1^0 < v_1^1 < \cdots < v_1^M = \overline{v}, \qquad (\gamma_1^1, \ldots, \gamma_1^M)$$

$$0 \leq v_2^0 < v_2^1 < \cdots < v_2^M < \overline{v}. \qquad (\gamma_2^0, \ldots, \gamma_2^{M+1})$$

Then $\mu^1 = \cdots = \mu^M = 0$, $\gamma_1^1 = \cdots = \gamma_1^M = 0$ and $\gamma_2^1 = \cdots = \gamma_2^{M+1} = 0$. The first-order conditions for v_1^m and v_2^m $(m = 1, \ldots, M-1)$ are

$$(1-\pi)\left\{\begin{array}{c}[\beta - v_1^m - \lambda\varphi(v_1^m)]f(v_1^m) + (1+\lambda)\rho[1 - F(v_1^m)] \\ + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\end{array}\right\} - \eta = 0, \tag{16}$$

$$\pi(\beta - v_2^m - \lambda\varphi(v_2^m))f(v_2^m) - (1-\pi)(1+\lambda)\rho[1 - F(v_1^m)] = 0. \tag{17}$$

We show below that if f satisfies Assumptions 1 and 2, then the above system of equations has at most one solution. Hence, $V(M, d) = V(2, d)$.

We break the formal proof into several claims. In all claims, we assume, without explicitly repeating these, that $u(\underline{v}, b_1)$, $u(\underline{v}, b_2)$, $\{a^m\}_{m=1}^M$, $\{v_1^m\}_{m=1}^{M-1}$, and $\{v_2^m\}_{m=0}^M$ define an optimal mechanism of $\mathcal{P}'(M, d)$ and β, η, λ, ξ_1, ξ_2, $\{\mu^m\}_{m=1}^M$, $\{\alpha^m\}_{m=1}^{M+1}$, $\{\gamma_1^m\}_{m=1}^M$, and $\{\gamma_2^m\}_{m=0}^{M+1}$ are the associated Lagrangian multipliers. By Lemma 3, $\mu^m = 0$ for $m = 3, \ldots, M$.

For later use, we note here that the sum of the first-order conditions of $a^{m'}$ from $m+1$ to M $(m = 0, \ldots, M-1)$ is given by

$$\begin{aligned}&\pi\left[\int_{v_2^m}^{v_2^M}[v + \lambda\varphi(v)]f(v)\,\mathrm{d}v - \beta[F(v_2^M) - F(v_2^m)]\right] \\ &+ (1-\pi)\left[\int_{v_1^m}^{v_1^M}[v + \lambda\varphi(v)]f(v)\,\mathrm{d}v - (1+\lambda)\rho(v_2^m - v_1^m)[1 - F(v_1^m)] - \beta[1 - F(v_1^m)]\right] \\ &- \eta v_1^m + (v_2^m - v_1^m)\sum_{j=m+1}^{M}\mu^j + \alpha^{m+1} - \alpha^{M+1} = 0,\end{aligned} \tag{18}$$

where $\alpha^2 = \cdots = \alpha^M = 0$ since $a^1 < a^2 < \cdots < a^M$.

Claim 1. $v_2^m > v_2^{m-1}$ *and* $\gamma_2^m = 0$ *for* $m = 2, \ldots, M-1$.

Proof. For $m=1,\dots,M-1$, since $a^{m+1}>a^m$, the first-order condition of v_1^m becomes

$$(1-\pi)\big[\big(\beta-(1+\lambda)v_1^m\big)f\big(v_1^m\big)+(\lambda+\rho+\lambda\rho)\big[1-F\big(v_1^m\big)\big]+(1+\lambda)\rho\big(v_2^m-v_1^m\big)f\big(v_1^m\big)\big] - \sum_{j=m+1}^{M}\mu^j-\eta=0.$$

Then, for $m=1,\dots,M-1$,

$$v_2^m=\frac{1+\rho}{\rho}v_1^m-\frac{\lambda+\rho+\lambda\rho}{(1+\lambda)\rho}\frac{1-F\big(v_1^m\big)}{f\big(v_1^m\big)}-\frac{\beta}{(1+\lambda)\rho}+\frac{\eta+\sum_{j=m+1}^{M}\mu^j}{(1-\pi)(1+\lambda)\rho f\big(v_1^m\big)},$$

where the right-hand side is strictly increasing in v_1^m by Assumptions 1 and 2. Let $m\in\{1,\dots,M-2\}$. If $\mu^{m+1}=0$, then $v_2^{m+1}>v_2^m$ since $v_1^{m+1}>v_1^m$. If $\mu^{m+1}>0$, then $v_2^{m+1}\geq v_1^{m+1}>v_1^m\geq v_2^m$ since (14) holds for m and $m+2$ and (14) holds with equality for $m+1$. Hence, $\gamma_2^{m+1}=0$. □

Claim 2. $\overline{v}+\lambda\varphi(\overline{v})>\beta\geq v_2^{M-1}+\lambda\varphi(v_2^{M-1})$.

Proof. Since $\mu^M=0$, the first-order condition of v_2^{M-1} implies that $\beta\geq v_2^{M-1}+\lambda\varphi(v_2^{M-1})$. Since $v_2^{M-1}>v_1^{M-1}$ and $\mu^M=0$, the first-order condition of a^M implies that

$$\pi\int_{v_2^{M-1}}^{v_2^M}\big[v+\lambda\varphi(v)-\beta\big]f(v)\,\mathrm{d}v+(1-\pi)\int_{v_1^{M-1}}^{v_1^M}\big[v+\lambda\varphi(v)-\beta\big]f(v)\,\mathrm{d}v\geq 0.$$

Hence, $\beta<\overline{v}+\lambda\varphi(\overline{v})$. □

Claim 3. $\gamma_2^M=\gamma_2^{M+1}=0$ *and* $v_2^M+\lambda\varphi(v_2^M)\leq\beta$.

Proof. First, we show that $\gamma_2^M=0$. Suppose $v_2^M+\lambda\varphi(v_2^M)>\beta\geq v_2^{M-1}+\lambda\varphi(v_2^{M-1})$, then $v_2^M>v_2^{M-1}$ and, therefore, $\gamma_2^M=0$. Suppose $v_2^M+\lambda\varphi(v_2^M)\leq\beta<\overline{v}+\lambda\varphi(\overline{v})$, then $v_2^M<\overline{v}$ and, therefore, $\gamma_2^{M+1}=0$. Since $\gamma_2^{M+1}=0$ and $v_2^M+\lambda\varphi(v_2^M)\leq\beta$, the first-order condition of v_2^M implies that $\gamma_2^M=0$. Hence, $\gamma_2^M=0$.

Next, we show that $v_2^M+\lambda\varphi(v_2^M)\leq\beta$. If $a^{M+1}>a^M$, then the first-order condition of v_2^M implies that $\beta\geq v_2^M+\lambda\varphi(v_2^M)$. If $a^M=a^{M+1}=1$, then by construction $v_2^M=v_2^{M-1}$ and, therefore, $v_2^M+\lambda\varphi(v_2^M)\leq\beta$ by Claim 2. Hence, $v_2^M+\lambda\varphi(v_2^M)\leq\beta$.

Finally, since $v_2^M+\lambda\varphi(v_2^M)\leq\beta<\overline{v}+\lambda\varphi(\overline{v})$, $v_2^M<\overline{v}$ and, therefore, $\gamma_2^{M+1}=0$. □

In what follows, we consider two cases in turn: $u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)=0$ and $u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)>0$.

B.3.2.1 *Case 1:* $u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)=0$.

Claim 4. *Suppose* $u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)=0$, *then* $\gamma_2^1=0$.

Proof. Suppose $\gamma_2^0 > 0$, then $v_2^0 = \underline{v}$. Since (14) holds for $m = 2$, we have $v_2^1 \geq v_1^1 > \underline{v} = v_2^0$. Hence, $\gamma_2^1 = 0$. Suppose $\gamma_2^0 = 0$ and $a^1 = 0$, then the first-order condition of v_2^0 implies that $\gamma_2^1 = \gamma_2^0 = 0$.

Assume for the rest of the proof that $\gamma_2^0 = 0$ and $a^1 > 0$. Suppose, on the contrary, that $\gamma_2^1 > 0$, then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Since $\gamma_2^1 > 0$, we have $v_2^0 = v_2^1 \geq v_1^1$. We consider two different cases: (i) $v_2^0 = v_2^1 = v_1^1$ and (ii) $v_2^0 = v_2^1 > v_1^1$.

(i) Suppose $v_2^0 = v_2^1 = v_1^1$. Let $\tilde{v}_1^1$ be such that $a^2(v_1^1 - \tilde{v}_1^1) = a^1(v_1^1 - \underline{v})$. Then, by Assumption 2, we have

$$\begin{aligned} a^2\big[F(v_1^1) - F(\tilde{v}_1^1)\big] &= (a^2 - a^1 + a^1)\big[F(v_1^1) - F(\tilde{v}_1^1)\big] \\ &\leq a^1\big[F(v_1^1) - F(\tilde{v}_1^1)\big] + (a^2 - a^1)f(\tilde{v}_1^1)(v_1^1 - \tilde{v}_1^1) \\ &= a^1\big[F(v_1^1) - F(\tilde{v}_1^1)\big] + a^1 f(\tilde{v}_1^1)(\tilde{v}_1^1 - \underline{v}) \leq a^1 F(v_1^1). \end{aligned}$$

Let $\tilde{v}_2^0 = \underline{v}$ and $\tilde{v}_2^1$ be such that $\pi a^2[F(v_2^1) - F(\tilde{v}_2^1)] = (1-\pi)[a^1 F(v_1^1) - a^2[F(v_1^1) - F(\tilde{v}_1^1)]]$. Let $a(v, b_i) = a^2$ if $v \in [\tilde{v}_i^1, v_i^1)$ and $a^*(v, b_i) = 0$ if $v \in [\underline{v}, \tilde{v}_i^1)$ for $i = 1, 2$, and $a^*(v, b) = a(v, b)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b. Then, by construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. By Assumption 1, (a^*, p^*, q^*) satisfies (BB). Clearly, (a^*, p^*, q^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies (IC-b). Let $\tilde{v}_1^m = v_1^m$ and $\tilde{v}_2^m = v_2^m$ for all $m \geq 2$. For $\hat{v} \in [\underline{v}, \tilde{v}_1^1)$, we have

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) < u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c = q^*(\hat{v}, b_1)c.$$

For $\hat{v} \in [\tilde{v}_1^1, v_1^1)$, we have

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c = q^*(\hat{v}, b_1)c,$$

where the inequality holds since $a^2(\tilde{v}_2^1 - \tilde{v}_1^1) \leq a^2(v_1^1 - \tilde{v}_1^1) = a^1(v_1^1 - \underline{v}) = a^1(v_2^0 - v_1^0)$. For $m \geq 2$ and $\hat{v} \in [v_1^{m-1}, v_1^{m-2})$, since $v_2^1 = v_1^1$, we have

$$\begin{aligned} &u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2(\tilde{v}_2^1 - \tilde{v}_1^1) + \sum_{j=3}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\ &\quad \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c = q^*(\hat{v}, b_1)c, \end{aligned}$$

where the last inequality holds by construction. Hence, (IC-b) is satisfied. Thus, (a^*, p^*, q^*) is feasible and strictly improves welfare, a contradiction.

(ii) Suppose $v_2^0 = v_2^1 > v_1^1$. Let $\varepsilon > 0$ be sufficiently small. Let $a^*(v, b_1) = a^1 - \varepsilon$ if $v < v_1^1$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $\tilde{v}_2^0 < v_2^0$ be such that $\pi(a^1 - \varepsilon)[F(v_2^0) - F(\tilde{v}_2^0)] = (1-\pi)\varepsilon F(v_1^1)$. For $\varepsilon > 0$ sufficiently small, $v_1^1 < \tilde{v}_2^0$. Let $a^*(v, b_2) = a^1 - \varepsilon$ if $v \in [\tilde{v}_2^0, \tilde{v}_2^1)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + \varepsilon(v_1^1 - v_1^0)$ and $u^*(\underline{v}, b_2) =$

$u(\underline{v}, b_2) - (1-\pi)\varepsilon(v_1^1 - v_1^0)/\pi$. For $\varepsilon > 0$ sufficiently small, $u^*(\underline{v}, b_2) \geq u^*(\underline{v}, b_1) > 0$. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b. Then, by construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. Then (a^*, p^*, q^*) satisfies (BB) by Assumption 1. Clearly, (a^*, p^*, q^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare.

Finally, we show that (a^*, p^*, q^*) satisfies (IC-b). Note that, by Assumption 2, we have

$$\begin{aligned}(a^1-\varepsilon)(v_2^0 - \tilde{v}_2^0) &\geq (a^1-\varepsilon)\frac{1}{f(\tilde{v}_2^0)}\big[F(v_2^0) - F(\tilde{v}_2^0)\big] \\ &\geq \frac{1-\pi}{\pi}\varepsilon\frac{1}{f(v_1^1)}F(v_1^1) \geq \frac{1-\pi}{\pi}\varepsilon(v_1^1 - v_1^0).\end{aligned}$$

Then, for $\hat{v} < v_1^1$, we have

$$\begin{aligned}&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1-\varepsilon)(\tilde{v}_2^0 - v_1^0) \\ &= u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{\varepsilon(v_1^1 - v_1^0)}{\pi} + (a^1-\varepsilon)(\tilde{v}_2^0 - v_2^0) - \varepsilon v_2^0 - (a^1-\varepsilon)v_1^0 \\ &\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1 v_2^0 + \frac{\varepsilon(v_1^1 - v_1^0)}{\pi} - \frac{(1-\pi)\varepsilon(v_1^1 - v_1^0)}{\pi} - \varepsilon v_2^0 - (a^1-\varepsilon)v_1^0 \\ &\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) + \varepsilon(v_1^1 - v_2^0) \\ &< u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = q^1 c = q^*(\hat{v}, b_1)c.\end{aligned}$$

For $\hat{v} \in [v_1^{m-1}, v_1^m)$ for $m = 2, \ldots, M$, we have

$$\begin{aligned}&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + (a^1-\varepsilon)(\tilde{v}_2^0 - v_1^0) + (a^2 - a^1 + \varepsilon)(v_2^1 - v_1^1) \\ &\qquad + \sum_{j=3}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) \\ &\leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) + \varepsilon(v_1^1 - v_2^0) + \varepsilon(v_2^1 - v_1^1), \\ &= u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{m}(a^j - a^{j-1})(v_2^{j-1} - v_1^{j-1}) = q^m c = q^*(\hat{v}, b_1)c.\end{aligned}$$

Hence, (IC-b) is satisfied. Thus, (a^*, p^*, q^*) is feasible and strictly improves welfare, a contradiction.

Hence, $\gamma_2^1 = 0$. □

By Claims 1, 3, and 4, we have $\gamma_2^m = 0$ for $m = 1, \ldots, M+1$. Thus, for $m = 1, \ldots, M-1$, v_1^m and v_2^m satisfy

$$(1-\pi)\big[(\beta - v_1^m - \lambda\varphi(v_1^m))f(v_1^m) + (1+\lambda)\rho\big[1 - F(v_1^m)\big] + (1+\lambda)\rho(v_2^m - v_1^m)f(v_1^m)\big]$$

$$-\sum_{j=m+1}^{M}\mu^j-\eta=0, \tag{19}$$

$$\pi\big(\beta-v_2^m-\lambda\varphi(v_2^m)\big)f(v_2^m)-(1-\pi)(1+\lambda)\rho\big[1-F(v_1^m)\big]+\sum_{j=m+1}^{M}\mu^j=0, \tag{20}$$

where (19) and (20) are the first-order conditions of v_1^m and v_2^m, respectively.

Claim 5. *Suppose $u(\underline{v},b_1)-u(\underline{v},b_2)+a^1(v_2^0-v_1^0)=0$, then $M=2$.*

Proof. Let $\hat{m}=1$ if $\mu^2=0$ and $\hat{m}=2$ if $\mu^2>0$. For $m=\hat{m},\dots,M-1$, (19) and (20) become

$$\begin{aligned}&(1-\pi)\big\{\big(\beta-v_1^m-\lambda\varphi(v_1^m)\big)f(v_1^m)+(1+\lambda)\rho\big[1-F(v_1^m)\big]\\&\qquad+(1+\lambda)\rho\big(v_2^m-v_1^m\big)f(v_1^m)\big\}-\eta=0, \qquad (16)\\&\pi\big(\beta-v_2^m-\lambda\varphi(v_2^m)\big)f(v_2^m)-(1-\pi)(1+\lambda)\rho\big[1-F(v_1^m)\big]=0, \qquad (17)\end{aligned}$$

Let $m\in\{\hat{m},\dots,M-1\}$. Given β, η and λ, (16) and (17) define v_2^m as functions of v_1^m, denoted by g_1 and g_2, respectively. By Assumptions 1 and 2, $g_1'(v_1^m)>1$. By the implicit function theorem, we have

$$g_2'(v_1^m)=\frac{1-\pi}{\pi}\frac{(1+\lambda)\rho f(v_1^m)}{-\big(\beta-(1+\lambda)v_2^m\big)f'(v_2^m)+(1+2\lambda)f(v_2^m)}>0.$$

To see that the inequality holds, note that $(\beta-v-\lambda\varphi(v))f(v)$ is strictly decreasing in v for $v<v_2^M$. Taking the derivative with respect to v yields $(\beta-(1+\lambda)v)f'(v)-(1+2\lambda)f(v)<0$ for $v<v_2^M$.

Since $v_2^m\geq v_1^m\geq\underline{v}\geq 0$ by Lemma 3, $v+\lambda\varphi(v)<\beta$ for all $v<v_2^M$ by Claim 3, $\sum_{j=m+1}^{M}\mu^j=0$, $\eta\geq 0$, $\alpha^{m+1}=0$ and $\alpha^{M+1}\geq 0$, (18) implies that

$$\int_{v_1^m}^{\overline{v}}\big[v+\lambda\varphi(v)-\beta\big]f(v)\,\mathrm{d}v\geq 0,$$

which holds if and only if $v_1^m\geq\hat{v}(\beta)$, where

$$\hat{v}(\beta)=\inf\left\{\hat{v}\,\middle|\,\int_{\hat{v}}^{\overline{v}}\big[v+\lambda\varphi(v)-\beta\big]f(v)\,\mathrm{d}v\geq 0\right\}. \tag{21}$$

Next, we show that if $v_1^m\geq\hat{v}(\beta)$ and $v_2^m=g_2(v_1^m)\geq v_1^m$, then $g_2'(v_1^m)\leq 1$, where the inequality holds strictly if $v_1^m>\hat{v}(\beta)$. First, we have

$$\begin{aligned}&-\big(\beta-(1+\lambda)v_2^m\big)f'(v_2^m)+(1+2\lambda)f(v_2^m)\\&\quad=-\big(\beta-v_2^m-\lambda\varphi(v_2^m)\big)f'(v_2^m)+\lambda\left\{\frac{\big[1-F(v_2^m)\big]f'(v_2^m)}{f(v_2^m)}+f(v_2^m)\right\}+(1+\lambda)f(v_2^m)\\&\quad\geq(1+\lambda)f(v_2^m),\end{aligned}$$

where the last inequality holds since $\beta - v_2^m - \lambda\varphi(v_2^m) > 0$, $f' \leq 0$ by Assumption 2 and $[1-F(v_2^m)]f'(v_2^m) + f^2(v_2^m) \geq 0$ by Assumption 1. Then we have

$$g_2'(v_1^m) = \frac{1-\pi}{\pi}\frac{(1+\lambda)\rho f(v_1^m)}{-(\beta-(1+\lambda)v_2^m)f'(v_2^m) + (1+2\lambda)f(v_2^m)} \leq \frac{(1-\pi)\rho f(v_1^m)}{\pi f(v_2^m)}.$$

By Assumption 1,

$$f(v) \geq f(v_1^m)\frac{1-F(v)}{1-F(v_1^m)}, \quad \forall v \geq v_1^m. \tag{22}$$

Then, for $v_1^m \geq \hat{v}(\beta)$, we have

$$\begin{aligned} 1-F(v_1^m) &\geq \frac{f(v_1^m)}{1-F(v_1^m)}\int_{v_1^m}^{\bar{v}}\big(1-F(v)\big)\,\mathrm{d}v \\ &= \frac{f(v_1^m)}{1-F(v_1^m)}\bigg[(1+\lambda)\int_{v_1^m}^{\bar{v}}\big(1-F(v)\big)\,\mathrm{d}v - \lambda\int_{v_1^m}^{\bar{v}}\big(1-F(v)\big)\,\mathrm{d}v\bigg] \\ &= \frac{f(v_1^m)}{1-F(v_1^m)}\bigg[-(1+\lambda)v_1^m\big[1-F(v_1^m)\big] + \int_{v_1^m}^{\bar{v}}\big[v+\lambda\varphi(v)\big]f(v)\,\mathrm{d}v\bigg] \\ &\geq \big(\beta-(1+\lambda)v_1^m\big)f(v_1^m), \end{aligned}$$

where the first line holds by (22), the third line holds by integration by parts, and the last line holds since $v_1^m \geq \hat{v}(\beta)$. Combining this inequality and (17) yields

$$\begin{aligned} \big(\beta - v_2^m - \lambda\varphi(v_2^m)\big)f(v_2^m) &= \frac{1-\pi}{\pi}(1+\lambda)\rho\big[1-F(v_1^m)\big] \\ &= \frac{1-\pi}{\pi}\rho\big[\big[1-F(v_1^m)\big] + \lambda\big[1-F(v_1^m)\big]\big] \\ &\geq \frac{1-\pi}{\pi}\rho\big[\big(\beta-(1+\lambda)v_1^m\big)f(v_1^m) + \lambda\big[1-F(v_1^m)\big]\big] \\ &= \frac{1-\pi}{\pi}\rho\big[\beta - v_1^m - \lambda\varphi(v_1^m)\big]f(v_1^m). \end{aligned}$$

Hence,

$$g_2'(v_1^m) \leq \frac{1-\pi}{\pi}\frac{\rho f(v_1^m)}{f(v_2^m)} \leq \frac{\beta - v_2^m - \lambda\varphi(v_2^m)}{\beta - v_1^m - \lambda\varphi(v_1^m)} \leq 1.$$

Note that $g_2'(v_1^m) < 1$ if $v_1^m > \hat{v}(\beta)$ or $v_1^m < g_2(v_1^m)$.

Thus, there exists at most one $v_1^m \geq \hat{v}(\beta)$ such that $g_1(v_1^m) = g_2(v_1^m) \geq v_1^m$, that is, (16) and (17) have at most one solution such that $v_2^m \geq v_1^m \geq \hat{v}(\beta)$. Hence, $M \leq \hat{m}+1$.

If $\hat{m} = 1$, then $M \leq \hat{m}+1 \leq 2$. Assume for the rest of the proof that $\hat{m} = 2$ and $\mu^2 > 0$. In this case, $M \leq \hat{m}+1 \leq 3$. We want to show that $M = 2$. Suppose this is not true, then $M = 3$. Since $\mu^2 > 0$, $v_2^1 = v_1^1$. By the same argument proving that $v_1^m \geq \hat{v}(\beta)$ for $m \geq \hat{m}$, we have $v_1^1 \geq \hat{v}(\beta)$. Then we have $v_1^2 > v_1^1 \geq \hat{v}(\beta)$, and $g_2(v_1^2) = v_2^2 > v_1^2$. Since $g_2'(v) < 1$ if $v > \hat{v}(\beta)$ and $g_2(v) \geq v$, we have $g_2(v) > v$ for all $v_1^1 \leq v < v_1^2$. Hence, $v_2^1 = g_2(v_1^1) > v_1^1$, a contradiction to that $v_2^1 = v_1^1$. Hence, $M = 2$. □

B.3.2.2 *Case 2:* $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$. In this case, by Lemma 3, $\mu^1 = \mu^2 = 0$.

Claim 6. *Suppose* $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$, *then* $\gamma_2^1 = 0$.

Proof. Since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$, $\mu^1 = 0$. Suppose, on the contrary, that $\gamma_2^1 > 0$. Then $v_2^1 = v_2^0$. Suppose $\gamma_2^0 > 0$, then $v_2^1 = v_2^0 = \underline{v} = v_1^0 < v_1^1$, a contradiction to Lemma 3. Hence, $\gamma_2^0 = 0$.

Suppose $a^1 = 0$, then the first-order condition of v_2^0 implies that $\gamma_2^1 = 0$. Suppose $a^1 > 0$. Then the first-order conditions of v_2^0 and v_2^1 imply that

$$\begin{aligned}
&\pi\big(\beta - (1+\lambda)v_2^0\big)f\big(v_2^0\big) + \pi\lambda\big[1 - F\big(v_2^0\big)\big] \\
&\quad \geq (1-\pi)(1+\lambda)\rho \\
&\quad > (1-\pi)(1+\lambda)\rho\big[1 - F\big(v_1^1\big)\big] \\
&\quad \geq \pi\big(\beta - (1+\lambda)v_2^1\big)f\big(v_2^1\big) + \pi\lambda\big[1 - F\big(v_2^1\big)\big].
\end{aligned}$$

Since $(\beta - (1+\lambda)v)f(v) + \lambda[1 - F(v)]$ is strictly decreasing in v when $v + \lambda\varphi(v) < \beta$, we have $v_2^1 > v_2^0$ and, therefore, $\gamma_2^1 = 0$. □

By Claims 1, 3, and 6, we have $\gamma_2^m = 0$ for $m = 1, \ldots, M+1$. Thus, for $m = 1, \ldots, M-1$, v_1^m and v_2^m satisfies (16), (17), and (18). By an argument similar to that of Claim 5, we have the following.

Claim 7. *Suppose* $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$, *then* $M = 2$.

To summarize, we have shown in both cases that $M = 2$. However, this contradicts the assumption that $M \geq 3$. Hence, it must be that $V(M, d) = V(M-1, d)$ for all $M \geq 3$. This completes the proof of Lemma 4.

B.3.3 *Proof of Theorem 2* To prove Theorem 2, we first prove Lemmas 10 and 11.

Lemma 10. *In an optimal allocation rule of* $\mathcal{P}'(2, 0)$, $v_2^0 = \underline{v}$ *and* $a^1 = 0$.

Proof. First, we show that $v_2^0 = \underline{v}$. We consider two different cases: (i) $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = 0$ and (ii) $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$.

(i) Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) = 0$**.** Suppose, on the contrary, that $v_2^0 > \underline{v}$. Then we can construct another feasible mechanism (a^*, p^*, q^*), which strictly improves welfare. Since $v_2^0 > \underline{v} = v_1^0$, we have $u(\underline{v}, b_2) > u(\underline{v}, b_1)$, $a^1 > 0$ and $v_1^1 > \underline{v}$.

Let $\varepsilon > 0$ be sufficiently small. Let $\tilde{v}_1^0 = \underline{v} + \varepsilon$ and $\tilde{v}_2^0 < v_2^0$ be such that $\pi[F(v_2^0) - F(\tilde{v}_2^0)] = (1-\pi)F(\tilde{v}_1^0)$. For $\varepsilon > 0$ sufficiently small, $\tilde{v}_1^0 < \min\{v_1^1, \tilde{v}_2^0\}$. Let $u^*(\underline{v}, b_1) = u(\underline{v}, b_1) + a^1\varepsilon$ and $u^*(\underline{v}, b_2) = u(\underline{v}, b_2) - (1-\pi)a^1\varepsilon/\pi$. For $\varepsilon > 0$, sufficiently small, $u^*(\underline{v}, b_2) \geq u^*(\underline{v}, b_1) > 0$. Let $a^*(v, b_1) = 0$ if $v < \tilde{v}_1^0$ and $a^*(v, b_1) = a(v, b_1)$ otherwise. Let $a^*(v, b_2) = a^1$ if $v \in (\tilde{v}_2^0, v_2^0)$ and $a^*(v, b_2) = a(v, b_2)$ otherwise. Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u^*(\underline{v}, b)$ for all v and b. By construction, $p^*(\overline{v}, b_1) = p(\overline{v}, b_1) \leq b_1$. Hence, (BC) is satisfied. Let $q^*(v, b_1) = q(v, b_1)$ for all v. Clearly, (a^*, p^*, q^*) satisfies (IR), (IC-v), and (S), and strictly improves welfare. By Assumption 1, (a^*, p^*, q^*) satisfies (BB).

Finally, we show that (a^*, p^*, q^*) satisfies (IC-b). For $\hat{v} < \tilde{v}_1^0$, we have $u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) \leq 0 \leq q^*(\hat{v}, b_1)c$. For $\hat{v} \in [\tilde{v}_1^0, v_1^1)$,

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1\left(\tilde{v}_2^0 - \tilde{v}_1^0\right) \\
&\quad = u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) + \frac{a^1\varepsilon}{\pi} + a^1\left(\tilde{v}_2^0 - v_2^0\right) - a^1\varepsilon \\
&\quad \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1\left(v_2^0 - v_1^0\right) = q^1 c = q^*(\hat{v}, b_1)c,
\end{aligned}$$

where the inequality holds since by Assumption 2,

$$\begin{aligned}
v_2^0 - \tilde{v}_2^0 &\geq \frac{1}{f\left(\tilde{v}_2^0\right)}\left[F\left(v_2^0\right) - F\left(\tilde{v}_2^0\right)\right] \\
&\geq \frac{1-\pi}{\pi}\frac{F(\underline{v}+\varepsilon)}{f(\underline{v}+\varepsilon)} \geq \frac{1-\pi}{\pi}\varepsilon.
\end{aligned}$$

For $\hat{v} \in [v_1^1, \overline{v}]$,

$$\begin{aligned}
&u^*(\underline{v}, b_1) - u^*(\underline{v}, b_2) + a^1\left(\tilde{v}_2^0 - \tilde{v}_1^0\right) + \left(a^2 - a^1\right)\left(v_2^1 - v_1^1\right) \\
&\quad \leq u(\underline{v}, b_1) - u(\underline{v}, b_2) + \sum_{j=1}^{2}\left(a^j - a^{j-1}\right)\left(v_2^{j-1} - v_1^{j-1}\right) = q^2 c = q^*(\hat{v}, b_1)c.
\end{aligned}$$

Hence, (IC-b) is satisfied. Thus, (a^*, p^*, q^*) is feasible and strictly improves welfare, a contradiction. Hence, $v_2^0 = \underline{v}$.

(ii) Suppose $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$**.** Suppose, on the contrary, that $v_2^0 > \underline{v}$. In this case, $\gamma_2^0 = 0$. By construction, we have $a^1 > 0$ and $v_1^1 > \underline{v}$. Hence, $\alpha^1 = 0$. Since $u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^1(v_2^0 - v_1^0) > 0$ and $v_2^1 \geq v_1^1$, we have $\mu^1 = \mu^2 = 0$. Let

$$\begin{aligned}
\Delta(v_1, v_2) = \pi &\int_{v_2}^{v_2^M}\left[v + \lambda\varphi(v) - \beta\right]f(v)\,\mathrm{d}v - \eta v_1 - \alpha^{M+1} \\
&+ (1-\pi)\left[\int_{v_1}^{v_1^M}\left[v + \lambda\varphi(v) - \beta\right]f(v)\,\mathrm{d}v - (1+\lambda)\rho(v_2 - v_1)\left[1 - F(v_1)\right]\right].
\end{aligned}$$

Then $\Delta(v_1^0, v_2^0) = \Delta(v_1^1, v_2^1) = 0$ by (18). Since $v_2^0 > v_1^0$, and $v + \lambda\varphi(v) < \beta$ for $v < v_2^M$ by Claim 3 in the proof of Lemma 4, $\Delta(v_1^0, v_2^0) = 0$ implies that $\int_{v_1^0}^{v_1^M}[v + \lambda\varphi(v) - \beta]f(v)\,\mathrm{d}v \geq 0$, that is, $\hat{v}(\beta) = \underline{v}$, where $\hat{v}(\beta)$ is given by (21).

It follows from the proof of Lemma 4 that the first-order conditions of v_1^1 and v_2^1 can be reduced to (16) and (17). Given β, η, and λ, (16) and (17) define v_2^1 as functions of v_1^1, denoted by g_1 and g_2, respectively. By an argument similar to that in the proof of Claim 5 in the proof of Lemma 4, $g_1'(v) > 1$, and $g_2'(v) < 1$ if $v > \underline{v}$ and $g_2(v) \geq v$. Note that

$$\frac{\partial \Delta}{\partial v_1} = (1-\pi)\big[\big(\beta - v_1 - \lambda\varphi(v_1)\big)f(v_1) + (1+\lambda)\rho(v_2 - v_1)f(v_1) + (1+\lambda)\rho\big[1-F(v_1)\big]\big] - \eta,$$

$$\frac{\partial \Delta}{\partial v_2} = \pi\big(\beta - v_2 - \lambda\varphi(v_2)\big)f(v_2) - (1-\pi)(1+\lambda)\rho\big[1-F(v_1)\big].$$

Clearly, $\partial\Delta(v_1, g_1(v_1))/\partial v_1 = 0$ by (16) and $\partial\Delta(v_1, g_2(v_1))/\partial v_2 = 0$ by (17). Since $v_2^1 \geq v_1^1$, $g_2(v) > g_1(v)$ for all $v < v_1^1$. Then $\partial\Delta(v_1, g_2(v_1))/\partial v_1 > \Delta(v_1, g_1(v_1))/\partial v_1 = 0$ for all $v_1 < v_1^1$. Hence,

$$0 = \Delta\big(v_1^1, v_2^1\big) = \Delta\big(v_1^0, v_2^0\big) + \int_{v_1^0}^{v_1^1} \frac{\partial\Delta\big(v_1, g_2(v_1)\big)}{\partial v_1}\,\mathrm{d}v_1 > \Delta\big(v_1^0, v_2^0\big) = 0,$$

a contradiction. Hence, $v_2^0 = \underline{v}$.

Next, we show that $a^1 = 0$. Suppose, on the contrary, that $a^1 > 0$, then $\alpha^1 = 0$. Then v_2^0 satisfies

$$a^1\left\{\pi\big[\beta - v_2^0 - \lambda\varphi\big(v_2^0\big)\big]f\big(v_2^0\big) - (1-\pi)(1+\lambda)\rho + \sum_{j=1}^{2}\mu^j\right\} + \gamma_2^0 = 0, \tag{23}$$

$$\pi\int_{v_2^0}^{v_2^M}\big[v + \lambda\varphi(v) - \beta\big]f(v)\,\mathrm{d}v + (1-\pi)\int_{v_1^0}^{v_1^M}\big[v + \lambda\varphi(v) - \beta\big]f(v)\,\mathrm{d}v - \eta v_1^0 - \alpha^{M+1} = 0, \tag{24}$$

where (23) is the first-order condition of v_2^0, and (24) is (18) when $m = 0$. Since $v_2^0 > v_1^0$ and $v + \lambda\varphi(v) < \beta$ for $v < v_2^M$ by Claim 3 in the proof of Lemma 4, (24) implies that $\int_{v_1^0}^{v_1^M}[v + \lambda\varphi(v)]f(v)\,\mathrm{d}v - \beta \geq 0$, that is, $\hat{v}(\beta) = \underline{v}$. Since $g_2'(v) < 1$ if $v > \underline{v}$ and $g_2(v) \geq v$, and $g_2(v_1^1) = v_2^1 \geq v_1^1$, we have $g_2(v_1^0) > v_1^0 = \underline{v} = v_2^0$. However, by (23), $v_2^0 \geq g_2(v_1^0)$, a contradiction. Hence, $a^1 = 0$. □

Lemma 11. *For any $d > 0$, there exists $\overline{M}(d)$ such that for all $M > \overline{M}(d)$,*

$$V - V(M, d) \leq (1-\pi)(1+\rho)\frac{\mathbb{E}[v]}{M}.$$

Proof. Let (a, p, q) be an optimal mechanism of $(\mathcal{P}')$. Then $p(v, b) = va(v, b) - \int_{\underline{v}}^{v} a(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b, and q is defined by (4). Fix $M \geq 2$. Construct (a^*, p^*, q^*) as follows. Let $a^0 = 0$, $a^{M+1} = 1$ and $a^m = (m-1)a(\overline{v}, b_1)/M$ for $m = 1, \dots, M$. Let $v_1^0 = \underline{v}$, $v_1^M = \overline{v}$ and

$$v_1^m = \inf\big\{v | a(v, b_1) \geq a^{m+1}\big\} \quad \text{for } m = 0, \dots, M-1.$$

Then $\underline{v} = v_1^0 \leq v_1^1 \leq \dots \leq v_1^M = \overline{v}$ and $0 = a^0 \leq a^1 < a^2 < \dots < a^M \leq a^{M+1} = 1$. Let $a^*(v, b_1) = a^m$ if $v \in [v_1^{m-1}, v_1^m)$ for $m = 1, \dots, M$. Then $a(v, b_1) - 1/M \leq a^*(v, b_1) \leq$

$a(v, b_1)$ for all v. Let $\hat{v}_2^m = \inf\{v | a(v, b_2) \geq a^m\}$ for $m = 1, \ldots, M$, $\hat{v}_2^0 = 0$ and $\hat{v}_2^{M+1} = \overline{v}$. For each $m = 1, \ldots, M+1$, there exists $v_2^{m-1} \in [\hat{v}_2^{m-1}, \hat{v}_2^m]$ such that

$$\int_{\hat{v}_2^{m-1}}^{\hat{v}_2^m} a(v, b_2) f(v)\,\mathrm{d}v = a^{m-1}\big[F\big(v_2^{m-1}\big) - F\big(\hat{v}_2^{m-1}\big)\big] + a^m\big[F\big(\hat{v}_2^m\big) - F\big(v_2^{m-1}\big)\big].$$

Let $a^*(v, b_2) = a^m$ if $v \in [v_2^{m-1}, v_2^m)$ for $m = 1, \ldots, M$, $a^*(v, b_2) = 0$ if $v < v_2^0$ and $a^*(v, b_2) = 1$ if $v \geq v_2^M$. Clearly, a^* satisfies (S). Let $p^*(v, b) = va^*(v, b) - \int_{\underline{v}}^{v} a^*(\nu, b)\,\mathrm{d}\nu - u(\underline{v}, b)$ for all v and b. Let $q^*(v, b_1) = q(v, b_1) + v/(cM)$ for all v. Clearly, (a^*, p^*, q^*) satisfies (IR) and (IC-v).

We show that (IC-b) is satisfied, that is, for $m = 1, \ldots, M$ and $v \in [v_1^{m-1}, v_1^m)$,

$$cq^*(v, b_1) \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\,\mathrm{d}\nu + a^m\big(v_2^{m-1} - v\big) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\,\mathrm{d}\nu.$$

Let $m \in \{1, \ldots, M\}$. Recall that for $v \in [v_1^{m-1}, v_1^m)$, we have

$$cq(v, b_1) \geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\,\mathrm{d}\nu + a(v, b_1)\big(\hat{v}_2^m - v\big) + \int_{\underline{v}}^{v} a(\nu, b_1)\,\mathrm{d}\nu.$$

Then, for $v \in [v_1^{m-1}, v_1^m)$,

$$\begin{aligned}
cq^*(v, b_1) &= cq(v, b_1) + \frac{v}{M} \\
&\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\,\mathrm{d}\nu + a(v, b_1)\hat{v}_2^m - \left(a(v, b_1) - \frac{1}{M}\right)v \\
&\quad + \int_{\underline{v}}^{v} a(\nu, b_1)\,\mathrm{d}\nu \\
&\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a(\nu, b_2)\,\mathrm{d}\nu + a^m\big(\hat{v}_2^m - v\big) + \int_{\underline{v}}^{v} a(\nu, b_1)\,\mathrm{d}\nu \\
&\geq u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{\hat{v}_2^m} a^*(\nu, b_2)\,\mathrm{d}\nu + a^m\big(\hat{v}_2^m - v\big) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\,\mathrm{d}\nu \\
&= u(\underline{v}, b_1) - u(\underline{v}, b_2) - \int_{\underline{v}}^{v_2^{m-1}} a^*(\nu, b_2)\,\mathrm{d}\nu + a^m\big(v_2^{m-1} - v\big) + \int_{\underline{v}}^{v} a^*(\nu, b_1)\,\mathrm{d}\nu,
\end{aligned}$$

where the second inequality holds since $a(v, b) - 1/M \leq a^*(v, b) \leq a(v, b)$ and the third inequality holds by an argument similar to that in the proof of Lemma 1.

Finally, we show that (a^*, p^*, q^*) satisfies (BB-d) for M sufficiently large:

$$\begin{aligned}
&\mathbb{E}_t\big[p^*(t) - q^*(t)k\big] - \mathbb{E}_t\big[p(t) - q(t)k\big] \\
&\quad = \pi \int_{\underline{v}}^{\overline{v}} \varphi(v)\big[a^*(v, b_2) - a(v, b_2)\big]f(v)\,\mathrm{d}v
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\pi)\int_{\underline{v}}^{\overline{v}} \varphi(v)\big[a^*(v,b_1) - a(v,b_1)\big]f(v)\,\mathrm{d}v \\
&- (1-\pi)\int_{\underline{v}}^{\overline{v}} k\big[q^*(v,b_1) - q(v,b_1)\big]f(v)\,\mathrm{d}v \\
\geq& -\frac{\mathbb{E}[v]}{M} - (1-\pi)\frac{\mathbb{E}[v]}{M}\rho.
\end{aligned}
$$

For any $d > 0$, there exists $\overline{M}(d)$ such that for all $M > \overline{M}(d)$, we have

$$
\frac{\mathbb{E}[v]}{M} + (1-\pi)\frac{\mathbb{E}[v]}{M}\rho < 0.
$$

Thus, (a^*, p^*, q^*) is a feasible solution to $\mathcal{P}'(M, d)$ for $M > \overline{M}(d)$. Hence,

$$
\begin{aligned}
&V - V(M,d) \\
&\quad \leq (1-\pi)\bigg[\int_{\underline{v}}^{\overline{v}} v\big[a(v,b_1) - a^*(v,b_1)\big]f(v)\,\mathrm{d}v - \int_{\underline{v}}^{\overline{v}} \big[q(v,b_1) - q^*(v,b_1)\big]kf(v)\,\mathrm{d}v\bigg] \\
&\quad \leq (1-\pi)(1+\rho)\frac{\mathbb{E}[v]}{M}.
\end{aligned}
$$

□

PROOF OF THEOREM 2. By Lemmas 4 and 11, we have

$$
V - V(2,d) = V - V(M,d) \leq (1-\pi)(1+\rho)\frac{\mathbb{E}[v]}{M}.
$$

Let M go to infinity and we have $V(2,0) \leq V \leq V(2,d)$ for all $d > 0$. By the standard argument, $V(2,\cdot)$ is continuous. Let d go to zero and we have $V = V(2,0)$.

By Lemma 10, there exist $u(\underline{v}, b_1) \geq 0$, $u(\underline{v}, b_2) \geq 0$, $\underline{v} \leq v_1^1 \leq \overline{v}$, $\underline{v} \leq v_2^1 \leq v_2^2 \leq \overline{v}$ and $0 < a^2 \leq 1$ such that an optimal mechanism of $(\mathcal{P}')$ is given by

$$
\begin{aligned}
a(v,b_1) &= \chi_{\{v\geq v_1^1\}}a^2, \\
a(v,b_2) &= \chi_{\{v\geq v_2^1\}}a^2 + \chi_{\{v\geq v_2^2\}}\big(1-a^2\big), \\
p(v,b_1) &= -u(\underline{v},b_1) + \chi_{\{v\geq v_1^1\}}a^2 v_1^1, \\
p(v,b_2) &= -u(\underline{v},b_2) + \chi_{\{v\geq v_2^1\}}a^2 v_2^1 + \chi_{\{v\geq v_2^2\}}\big(1-a^2\big)v_2^2, \\
q(v,b_1) &= \frac{1}{c}\big[u(\underline{v},b_1) - u(\underline{v},b_2) + \chi_{\{v\geq v_1^1\}}a^2\big(v_2^1 - v_1^1\big)\big], \\
q(v,b_2) &= 0,
\end{aligned}
$$

where χ_V is the indicator function of set V. By Lemma 8, $v_2^1 \geq v_1^1$. It is easy to verify that the above mechanism satisfies the IC constraints corresponding to the overreporting of budgets and, therefore, solves $(\mathcal{P})$. Let $a^*(\rho) = a^2$, $v_1^*(\rho) = v_1^1$, $v_2^*(\rho) = v_2^1$, $v_2^{**}(\rho) = v_2^2$, $u_1^*(\rho) = u(\underline{v}, b_1)$, and $u_2^*(\rho) = u(\underline{v}, b_2)$.

Finally, we show the optimal mechanism is unique. Observe that in an optimal mechanism of ($\mathcal{P}$), (S), (BB), and (BC) hold with equality.

Suppose, on the contrary, that there are two different optimal mechanisms (a^*, p^*, q^*) and $(\hat{a}, \hat{p}, \hat{q})$. Then there exist $(u_1^*, u_2^*, a^*, v_1^*, v_2^*, v_2^{**})$ and $(\hat{u}_1^*, \hat{u}_2^*, \hat{a}^*, \hat{v}_1^*, \hat{v}_2^*, \hat{v}_2^{**})$ that characterize the two optimal mechanisms, respectively.

First, it is easy to verify that the convex combination of the two mechanisms $(\kappa a^* + (1-\kappa)\hat{a}, \kappa p^* + (1-\kappa)\hat{p}, \kappa q^* + (1-\kappa)\hat{q})$, where $\kappa \in (0,1)$, is also optimal. Denote this convex combination by (a, p, q). Second, $v_1^* = \hat{v}_1^*$ since otherwise a is a 3-step function, a contradiction to Lemma 4.

Third, we show that $v_2^* = \hat{v}_2^*$, $v_2^{**} = \hat{v}_2^{**}$ and $a^* = \hat{a}^*$. Suppose $a^* = \hat{a}^* = 1$. Since (S) holds with equality in an optimal mechanism, $v_2^* = v_2^{**} = \hat{v}_2^* = \hat{v}_2^{**}$.

Suppose $a^* < 1$ and $\hat{a}^* = 1$. Since (S) holds with equality in the two original mechanisms, $v_2^* < \hat{v}_2^*$. Then $a(v, b_1) = \chi_{\{v \geq v_1^*\}}[\kappa a^* + (1-\kappa)]$. If $v \in (v_2^*, \min\{v_2^{**}, \hat{v}_2^{**}\})$, then $a(v, b_2) = \kappa a^* < \kappa a^* + (1-\kappa)$, which is a contradiction to Lemma 1. Hence, it cannot be the case that $a^* < 1$ and $\hat{a}^* = 1$.

Suppose $a^* < 1$ and $\hat{a}^* < 1$. In this case, $a(v, b_1) = \chi_{\{v \geq v_1^*\}}[\kappa a^* + (1-\kappa)\hat{a}^*]$. Suppose, on the contrary, that $v_2^* < \hat{v}_2^*$. If $v \in (v_2^*, \min\{v_2^{**}, \hat{v}_2^*\})$, then $a(v, b_2) = \kappa a^* < \kappa a^* + (1-\kappa)\hat{a}^*$, which is a contradiction to Lemma 1. Hence, $v_2^* = \hat{v}_2^*$. Suppose, on the contrary, that $v_2^{**} < \hat{v}_2^{**}$. If $v \in (v_2^{**}, \hat{v}_2^{**})$, then $a(v, b_2) = \kappa + (1-\kappa)\hat{a}^* > \kappa a^* + (1-\kappa)\hat{a}^*$, a contradiction to Lemma 1. Hence, $v_2^{**} = \hat{v}_2^{**}$. Finally, since (S) holds with equality in the two original mechanisms, it must be the case $a^* = \hat{a}^*$.

Lastly, we show that $u_1^* = \hat{u}_1^*$ and $u_2^* = \hat{u}_2^*$. Since (BC) holds with equality in the two original mechanisms, $u_1^* = a^* v_1^* - b_1 = \hat{a}^* \hat{v}_1^* - b_1 = \hat{u}_1^*$. Note that in (BB) the coefficient in front of u_2^* is $(1-\pi)\rho - \pi$. If $\rho \neq \pi/(1-\pi)$, then $u_2^* = \hat{u}_2^*$ since (BB) holds with equality in the two original mechanisms. If $\rho = \pi/(1-\pi)$, then any $u_2^*, \hat{u}_2^* \in [0, u_1^*]$ satisfy (BB) in the two original mechanisms. Since the objective function is strictly increasing in u_2^* and $\hat{u}_2^*$, we have $u_2^* = u_1^* = \hat{u}_1^* = \hat{u}_2^*$. □

B.3.4 *Proof of Proposition 2* Theorem 2 also greatly simplifies the analysis of the optimal mechanism's properties. Now the principal's problem ($\mathcal{P}$) can be reduced to the following, where the Greek letters in parentheses denote the corresponding Lagrangian multipliers:

$$\max_{\substack{u(\underline{v},b_1),u(\underline{v},b_2),\\ a^2,v_1^1,v_2^1,v_2^2}} \pi\left[\int_{v_2^1}^{v_2^2} a^2 v f(v)\,\mathrm{d}v + \int_{v_2^2}^{\overline{v}} v f(v)\,\mathrm{d}v\right] + (1-\pi)\int_{v_1^1}^{\overline{v}} a^2 v f(v)\,\mathrm{d}v$$

$$- (1-\pi)\rho\big[u(\underline{v}, b_1) - u(\underline{v}, b_2)\big]F\big(v_1^1\big)$$

$$- (1-\pi)\rho\int_{v_1^1}^{\overline{v}}\big[u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2\big(v_2^1 - v_1^1\big)\big]f(v)\,\mathrm{d}v,$$

subject to

$$\pi a^2\big[F\big(v_2^2\big) - F\big(v_2^1\big)\big] + \pi\big[1 - F\big(v_2^2\big)\big] + (1-\pi)a^2\big[1 - F\big(v_1^1\big)\big] \leq S, \qquad (\beta)$$

$$a^2 v_1^1 - u(\underline{v}, b_1) \leq b_1, \qquad (\eta)$$

$$
\begin{aligned}
&- (1-\pi)u(\underline{v}, b_1) + (1-\pi)\int_{v_1^1}^{v_1^2} a^2\varphi(v)f(v)\,\mathrm{d}v \\
&\quad - (1-\pi)\rho\big[u(\underline{v}, b_1) - u(\underline{v}, b_2)\big]F\big(v_1^1\big) \\
&\quad - (1-\pi)\rho\int_{v_1^1}^{\overline{v}}\big[u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2\big(v_2^1 - v_1^1\big)\big]f(v)\,\mathrm{d}v - \pi u(\underline{v}, b_2) \\
&\quad + \pi\int_{v_2^1}^{v_2^2} a^2\varphi(v)f(v)\,\mathrm{d}v + \pi\int_{v_2^2}^{\overline{v}}\varphi(v)f(v)\,\mathrm{d}v \ge 0, \qquad (\lambda)
\end{aligned}
$$

$$u(\underline{v}, b_1) \ge 0, u(\underline{v}, b_2) \ge 0, \qquad (\xi_1, \xi_2)$$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) \ge 0, \qquad (\mu^1)$$

$$u(\underline{v}, b_1) - u(\underline{v}, b_2) + a^2\big(v_2^1 - v_1^1\big) \ge 0, \qquad (\mu^2)$$

$$0 \le a^2 \le a^3 = 1, \qquad (\alpha^2, \alpha^3)$$

$$\underline{v} \le v_1^1 \le \overline{v}, \qquad (\gamma_1^1, \gamma_1^2)$$

$$\underline{v} \le v_2^1 \le v_2^2 \le \overline{v}. \qquad (\gamma_2^1, \gamma_2^2, \gamma_2^3)$$

Let $a^* = a^2$, $v_1^* = v_1^1$, $v_2^* = v_2^1$, $v_2^{**} = v_2^2$, $u_1^* = u(\underline{v}, b_1)$, and $u_2^* = u(\underline{v}, b_2)$ denote an optimal solution of ($\mathcal{P}$). Let β, η, λ, μ^1, μ^2, α^3, ξ_1, and ξ_2 denote the corresponding Lagrangian multipliers. Then v_1^*, v_2^*, v_2^{**}, a^*, u_1^*, u_2^*, β, η, λ, μ^1, μ^2, α^3, ξ_1, and ξ_2 satisfy the following first-order conditions:

$$
\begin{aligned}
&(1-\pi)\big[\big(\beta - v_1^* - \lambda\varphi\big(v_1^*\big)\big)f\big(v_1^*\big) + (1+\lambda)\rho\big[1 - F\big(v_1^*\big)\big] + (1+\lambda)\rho\big(v_2^* - v_1^*\big)f\big(v_1^*\big)\big] \\
&\quad - \eta - \mu^2 = 0, \qquad (25)
\end{aligned}
$$

$$\pi\big(\beta - v_2^* - \lambda\varphi\big(v_2^*\big)\big)f\big(v_2^*\big) - (1-\pi)(1+\lambda)\rho\big[1 - F\big(v_1^*\big)\big] + \mu^2 = 0, \qquad (26)$$

$$\big(1 - a^*\big)\big(\beta - v_2^{**} - \lambda\varphi\big(v_2^{**}\big)\big)f\big(v_2^{**}\big) = 0,$$

$$
\begin{aligned}
&\pi\int_{v_2^*}^{v_2^{**}}\big[v + \lambda\varphi(v) - \beta\big]f(v)\,\mathrm{d}v \\
&\quad + (1-\pi)\bigg[\int_{v_1^*}^{\overline{v}}\big[v + \lambda\varphi(v) - \beta\big]f(v)\,\mathrm{d}v - (1+\lambda)\rho\big(v_2^* - v_1^*\big)\big[1 - F\big(v_1^*\big)\big]\bigg] \\
&\quad - \eta v_1^* + \mu^2\big(v_2^* - v_1^*\big) - \alpha^3 = 0,
\end{aligned}
$$

$$\eta + \mu^1 + \mu^2 - (1-\pi)(\lambda + \rho + \lambda\rho) + \xi_1 = 0, \qquad (27)$$

$$-\mu^1 - \mu^2 - \pi\lambda + (1-\pi)(1+\lambda)\rho + \xi_2 = 0. \qquad (28)$$

Furthermore, in the optimal mechanism, (S) and (BB) hold with equality:

$$(1-\pi)a^*\big[1 - F\big(v_1^*\big)\big] + \pi a^*\big[F\big(v_2^{**}\big) - F\big(v_2^*\big)\big] + \pi\big[1 - F\big(v_2^{**}\big)\big] = S,$$

$$- (1-\pi)u_1^* + (1-\pi)a^* v_1^*\big[1-F\big(v_1^*\big)\big] - \pi u_2^* + \pi a^* v_2^*\big[1-F\big(v_2^*\big)\big]$$
$$+ \pi\big(1-a^*\big)v_2^{**}\big[1-F\big(v_2^{**}\big)\big] - (1-\pi)\rho\big(u_1^*-u_2^*\big) - (1-\pi)\rho a^*\big(v_2^*-v_1^*\big)\big[1-F\big(v_1^*\big)\big] = 0.$$

Proof of Proposition 2. **Part (i):** Suppose, on the contrary, that $u_1^* > u_2^* \geq 0$. In this case, $\xi_1 = \mu^1 = \mu^2 = 0$. (28) implies that $\xi_2 = \pi\lambda - (1-\pi)(1+\lambda)\rho$. Since $\xi_2 \geq 0$, we have $\lambda[\pi - \rho(1-\pi)] \geq \rho(1-\pi) > 0$, which implies that $\rho < \pi/(1-\pi)$, a contradiction.

Part (ii): By Proposition 2 part (i), $u_1^* = u_2^*$. It suffices to show that $v_1^* = v_2^*$. Suppose, on the contrary, that $v_2^* > v_1^*$. In this case, $\mu^2 = 0$. Adding (27) and (28) yields $\eta - \lambda + \xi_1 + \xi_2 = 0$. Since $\xi_1, \xi_2 \geq 0$, we have $\eta \leq \lambda$. Divide (25) by $(1-\pi)f(v_1^*)$ and (26) by $\pi f(v_2^*)$, and take differences, we have

$$\big[1+(1+\lambda)\rho\big]\big(v_2^*-v_1^*\big) + \lambda\big[\varphi\big(v_2^*\big) - \varphi\big(v_1^*\big)\big] + (1+\lambda)\rho\frac{1-F\big(v_1^*\big)}{f\big(v_1^*\big)}$$
$$+ (1+\lambda)\rho\frac{1-\pi}{\pi}\frac{1-F\big(v_1^*\big)}{f\big(v_2^*\big)} - \frac{\eta}{(1-\pi)f\big(v_1^*\big)} = 0.$$

Since $v_2^* > v_1^*$, $f(v_2^*) \leq f(v_1^*)$, $\varphi(v_2^*) > \varphi(v_1^*)$ and $\eta \leq \lambda$, we have

$$0 \geq \big[1+(1+\lambda)\rho\big]\big(v_2^*-v_1^*\big) + \lambda\big[\varphi\big(v_2^*\big) - \varphi\big(v_1^*\big)\big] + (1+\lambda)\frac{\rho}{\pi}\frac{1-F\big(v_1^*\big)}{f\big(v_1^*\big)} - \frac{\lambda}{(1-\pi)f\big(v_1^*\big)}$$
$$> \frac{\rho}{\pi}\frac{1-F\big(v_1^*\big)}{f\big(v_1^*\big)} + \lambda\left[\frac{\rho}{\pi}\frac{1-F\big(v_1^*\big)}{f\big(v_1^*\big)} - \frac{1}{(1-\pi)f\big(v_1^*\big)}\right] \geq 0,$$

where the last inequality holds since $1-F(v_1^*) \geq S$ and $\rho \geq \pi/[S(1-\pi)]$, a contradiction. Hence, $v_1^* = v_2^*$. □

B.4 *Proof of Proposition 3*

Let a denote the probability that a lottery participant receives the good in the first stage and p^s denote the expected price that a buyer pays in the resale market. Assume without loss of generality that $p^s > b_1 + u_1^*$, so a low-budget agent cannot afford the good in the resale market. Consider a low-budget agent whose valuation is v and who reports his budget truthfully. His payoff is u_1^* if he does not enter the lottery. If he enters the lottery, there are two possibilities. If he retains the good he received, then his payoff is $u_1^* + av - p_1^* = u_1^* + av - a^* v_1^*$; otherwise, his payoff is $u_1^* - p_1^* + a(p^s - \tau_1^*) = u_1^* - a^* v_1^* + a(p^s - v_2^{**} + v_1^*)$. Clearly, it is optimal for him to retain the good if and only if $v \geq p^s - v_2^{**} + v_1^*$, and it is optimal for him to enter the lottery if and only if $a \max\{v, p^s - v_2^{**} + v_1^*\} \geq a^* v_1^*$.

Similarly, consider a high-budget agent whose valuation is v and who reports his budget truthfully. First, note that if it is optimal for him not to enter the lottery, then it is also optimal for him not to buy the good in the resale market, and his payoff is u_2^* in this case. Next, if it is optimal for him to sell the good in the resale market, then it is optimal for him not to buy the good in the resale market provided he did not receive it in the lottery. Then, if he enters the lottery, we need to consider three possibilities. If he retains the good when he receives it and buys it when he does not receive it, his payoff is $u_2^* - p_2^* + av + (1-a)(v - p^s) = u_2^* - a^* v_2^* + av + (1-a)(v - p^s)$; if he retains the good when he receives it and does not buy it when he does not receive it, his payoff is $u_2^* + av - p_2^* = u_2^* + av - a^* v_2^*$; if he sells the good, then his payoff is $u_2^* - p_2^* + a(p^s - \tau_2^*) = u_2^* - a^* v_2^* + a(p^s - v_2^{**} + v_2^*)$. Clearly, in the second stage, it is optimal for him to retain the good if and only if $v \geq p^s - v_2^{**} + v_2^*$ and buy the good if and only if $v \geq p^s$. In the first stage, it is optimal for him to enter the lottery if and only if $a \max\{v, p^s - v_2^{**} + v_2^*\} \geq a^* v_2^*$.

Hence, in the second stage, the demand at price p^s is equal to the measure of high-budget agents who did not receive the good in the lottery and whose value is above p^s,

$$\pi(1-a)\big[1 - F\big(p^s\big)\big],$$

and the supply is equal to the total supply S minus the measure of agents who retain the good they received in the lottery:

$$S - a(1-\pi)\left[1 - F\left(\max\left\{p^s - v_2^{**} + v_1^*, \frac{a^* v_1^*}{a}\right\}\right)\right]$$
$$- a\pi\left[1 - F\left(\max\left\{p^s - v_2^{**} + v_2^*, \frac{a^* v_2^*}{a}\right\}\right)\right].$$

For each a, there is a unique p^s such that demand is equal to supply. By construction, $a \leq a^*$. Suppose $a < a^*$, then the market-clearing condition implies that $p^s < v_2^{**}$. This implies that agents make a loss if they enter the lottery and sell the good they received. Hence, a low-budget agent enters the lottery only if $v > v_1^*$ and a high-budget agent buys the lottery only if $v > v_2^*$, which in turn implies that $a = a^*$, a contradiction. Thus, $a = a^*$ and $p^s = v_2^{**}$ in equilibrium.

In equilibrium, an agent makes zero profit by participating in the lottery and selling the good he received. All low-budget agents whose valuations are above v_1^* strictly prefer to participate in the lottery and retain the good they received. All high-budget agents whose valuations are above v_2^* strictly prefer to participate in the lottery and retain the good they received. In addition, all high-budget agents whose valuations are above v_2^{**} will buy the good in the second stage if they do not receive any via the lottery. Therefore, the RwRRC scheme implements the optimal direct mechanism.

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Co-editor Simon Board handled this manuscript.

Manuscript received 28 August, 2019; final version accepted 23 September, 2020; available online 25 September, 2020.