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Robust group strategy-proofness

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Strategy-proofness (SP) is a sought-after property in social choice functions because it ensures that agents have no incentive to misrepresent their private information at both the interim and ex post stages. Group strategy-proofness (GSP), however, is a notion that is applied to the ex post stage but not to the interim stage. Thus, we propose a new notion of GSP, coined *robust group strategy-proofness* (RGSP), which ensures that no group benefits by deviating from truth telling at the interim stage. We show for the provision of a public good that the *Minimum Demand* rule (Serizawa (1999)) satisfies RGSP when the production possibilities set satisfies a particular topological property. In the problem of allocating indivisible objects, an acyclicity condition on the priorities is both necessary and sufficient for the *Deferred Acceptance* rule to satisfy RGSP, but is only necessary for the *Top Trading Cycles* rule. For the allocation of divisible private goods among agents with single-peaked preferences (Sprumont (1991)), only free disposal, group replacement monotonic rules within the class of sequential allotment rules satisfy RGSP.

Keywords. Robust group strategy-proofness, minimum demand rule, top trading cycles, deferred acceptance, acyclic priorities, free disposal, group replacement monotonicity.

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1. Introduction

Strategy-proofness (SP) in social choice functions is a highly desirable property. It guarantees that the participating agents never (strictly) benefit by misrepresenting their private information in any *realized state*. This definition is equivalent to requiring that truth telling is a weakly dominant strategy in the direct revelation game associated with any strategy-proof rule. Although SP is demanding, many prominent rules satisfy it. These include, for instance, the *Deferred Acceptance* (DA) or *Top Trading Cycles* (TTC) rules in the problem of allocating indivisible objects, the *Minimum Demand* rule in the provision of public goods, and the *Uniform Rule* in allotment economies with single-peaked preferences.

For strategy-proof rules, truth telling is an optimal strategy for each individual when there is asymmetric information. In this sense, SP is a notion that is not only applicable to the *ex post* stage but also to the *interim* stage. Given the prevalence of asymmetric information in socioeconomic situations, the importance of strategy-proof rules in practice is deservedly paramount.

Despite its desiderata, SP does not take into account the possibility of group deviations. The notion of group strategy-proofness (GSP) corrects this oversight. Specifically, GSP requires that no group benefits by misreporting its private information in any *realized state*.[1] Although this definition is a straightforward adaptation of SP to the possibility of group deviations, GSP, unlike its individual version, is not an interim-stage notion. In other words, depending on the informational structure, a group may find it profitable to misreport its private information in the direct revelation games associated with some group strategy-proof rules. The following example illustrates how a group strategy-proof rule can be manipulated in the presence of asymmetric information.

Example 1.1 (Motivating Example). Let us consider a Scarf–Shapley economy with three agents and three indivisible objects. A(nn), B(eth), and C(arol) own objects a, b, and c, respectively. The agents have strict preferences, and these objects are allocated according to the celebrated Top Trading Cycles rule, f^{TTC}, which is known to satisfy both SP and GSP.[2] Hence, at any state (or, equivalently, preference profile), no agent or group of agents can profit by misrepresenting their preferences. We now introduce some informational asymmetry. Suppose that the agents know that Ann and Beth have preferences R_A and R_B, respectively, represented by the following utility functions:

$$u_A(c) = 10, \qquad u_A(b) = 6, \qquad u_A(a) = 1,$$
$$u_B(c) = 10, \qquad u_B(a) = 6, \qquad u_B(b) = 1.$$

[1] See Barberá et al. (2016) and Barberá et al. (2010), which establish sufficient conditions for the equivalence of strategy-proofness and group strategy-proofness.

[2] In Section 3, we give the formal definition of the Top Trading Cycles rule with respect to a given priority function. To use the rule in this example, set the priority function so that each object gives the highest priority to its owner.

Carol's preferences, on the other hand, could be either R_C or $\tilde{R}_C$, which are represented by the following utility functions:

$$u_C(a) = 10, \qquad u_C(b) = 6, \qquad u_C(c) = 1,$$
$$\tilde{u}_C(b) = 10, \qquad \tilde{u}_C(a) = 6, \qquad \tilde{u}_C(c) = 1.$$

Carol knows her own preferences, but Ann and Beth believe Carol's preferences are either R_C or $\tilde{R}_C$ with equal probability. Would the agents reveal their preferences truthfully in this case?

Because f^{TTC} is strategy-proof, truth telling is a dominant strategy for each agent. Hence, in the absence of communication, the agents would reveal their preferences truthfully. Consequently, depending on Carol's preferences, their allocation would be

$$f^{\mathrm{TTC}}(R_A, R_B, R_C) = (c, b, a) \quad \text{and} \quad f^{\mathrm{TTC}}(R_A, R_B, \tilde{R}_C) = (a, c, b).$$

The expected utilities of Ann and Beth are 5.5.

However, what happens if Ann and Beth can coordinate? In this case, they could misreport their preferences as $\tilde{R}_A$ and $\tilde{R}_B$, which are represented by the following utility functions, respectively:

$$\tilde{u}_A(b) = 10, \qquad \tilde{u}_A(c) = 6, \qquad \tilde{u}_A(a) = 1,$$
$$\tilde{u}_B(a) = 10, \qquad \tilde{u}_B(c) = 6, \qquad \tilde{u}_B(b) = 1.$$

Then the TTC rule would make the following assignment:

$$f^{\mathrm{TTC}}(\tilde{R}_A, \tilde{R}_B, R_C) = f^{\mathrm{TTC}}(\tilde{R}_A, \tilde{R}_B, \tilde{R}_C) = (b, a, c).$$

As a result of the misreport, both Ann and Beth get the expected utility of 6, which is an improvement over truth telling. Consequently, whether a group deviates from truth telling could depend on the information (or belief), even for rules, such as TTC, that satisfy GSP.[3] ◊

The example above illustrates an important point: in the presence of asymmetric information, rules satisfying SP remain nonmanipulable by individuals, whereas rules satisfying GSP may become manipulable by some groups. In this particular example, some group of agents who face uncertainty about those who are not in the group are able to insure against the risk by jointly misreporting their own preferences. This means that something is lost in the standard generalization from SP to GSP. We believe that the nonrobustness of GSP to asymmetric information—a realistic feature in many economic environments—is a rather serious drawback.

To address this shortcoming, we propose a new notion of GSP, coined *robust group strategy-proofness* (RGSP). We follow the Wilson doctrine, which refers to the vision, articulated in Wilson (1987), that economic theory should not rely heavily on the common

[3]We note here that our notion of GSP depends only on the ordinal (not cardinal) utility rankings over outcomes.

knowledge assumption. As in Bergemann and Morris (2005), we allow agents to have any belief regarding others' types. The key requirement for a blocking coalition is that its members must be able to rationalize their collusive agreement. Specifically, any coalition member should have a type who improves under the new arrangement for some belief of hers. In our terminology, such a type provides one tier of reasoning for the collusion. Furthermore, this type should be able to justify her belief. This means that the support of the belief should include only those who can provide one tier of reasoning for the collusion. In this case, we say the type provides two tiers of reasoning for the collusion. Continuing with the same logic, a type provides $k+1$ tiers of reasoning if the type improves by colluding over truth telling for some belief whose support is over those who can provide k tiers of reasoning. If each coalition member is able to provide infinite tiers of reasoning for the collusion, we say the coalition rationalizes the collusion. A rule satisfies RGSP if no coalition can rationalize any collusion. Thus, our notion ensures that no group agrees to deviate from truth telling regardless of beliefs.

Unlike GSP, our notion does not require that the members of any blocking coalition have degenerate and identical beliefs about others' types. Consequently, RGSP is much more stringent than GSP, which in turn is more demanding than SP. This raises a question: How common are robust group strategy-proof rules? To provide an answer, we consider some classic settings with well-known rules satisfying GSP.

We first investigate the provision of a single public good, for which agents must split the cost. Here, we find that the Minimum Demand rule with equal cost-sharing and convex costs (Serizawa (1999)) satisfies RGSP when the production set of public goods satisfies a technical condition. Specifically, the production range should be completely closed below: All subsets must contain their infimum. For instance, this requirement is satisfied if the range of public good production is finite.

Next, we turn to the allocation of private goods and consider the problem of allocating indivisible objects among agents with strict preferences. The DA rule satisfies RGSP if and only if objects' priorities are acyclic: Each object gives its highest priorities to a fixed set of agents (although the exact order could vary from object to object) and its lowest priorities to the remaining agents in a common order. For the other celebrated rule in this setting—Top Trading Cycles (TTC)—the same condition is necessary but not sufficient. Consequently, the DA rule outperforms the TTC rule in terms of RGSP.

Finally, we examine the allocation of a divisible resource on the domain of single-peaked preferences. We find the classic uniform rule of Sprumont (1991) violates RGSP. Thus, we widen our search for robust group strategy-proof rules to the larger class of rules, studied by Barberá et al. (1997), that are characterized by efficiency, SP, and replacement monotonicity. Every rule in this class satisfies GSP, but only those that satisfy free disposability[4] and group replacement monotonicity are robust group strategy-proof.

To the best of our knowledge, RGSP is the first notion of GSP from the interim perspective. Wilson's seminal paper (Wilson (1978)) started the literature on core under

[4]Free disposal rules designate an individual who is allocated the leftover resource after satisfying the others. See Section 3.3 for the formal definition.

asymmetric information that also focuses on the interim stage. Although both GSP and the core concern coalitional deviations, they are different. Deviating coalitions in the former manipulate the revelation game through joint misreports, but they do not have any power to change game rules. In the latter, however, any deviating coalition opts out of the allocation process altogether and is free to allocate the coalition's resources in any way. On the technical side, the literature on core with asymmetric information fixes the informational structure and the agents have common beliefs, whereas our model does not have the common knowledge assumption.

The paper proceeds as follows: The next section describes our framework and defines RGSP. Section 3 studies three different well-known settings and identifies some rules satisfying RGSP. Section 4 discusses possible modifications of RGSP and concludes. The Appendix contains proofs.

2. Setup

Let $N = \{1, \dots, n\}$ be a finite set of *agents* where $n \geq 2$. For each $i \in N$, let A_i be *agent i's set of alternatives*, and $A = A_1 \times \cdots \times A_n$ is the *set of (social) alternatives*. For each $i \in N$, let $\mathcal{A}_i$ be a σ-algebra on A_i and $\triangle A_i$ be *the set of probability measures* on $(A_i, \mathcal{A}_i)$. Every agent $i \in N$ has a preference relation R_i over $\triangle A_i$, and P_i and I_i stand for the strict and the indifference part of R_i, respectively. The *set of preference relations* for agent i is $\mathcal{R}_i$. We assume that each $R_i \in \mathcal{R}_i$ is represented by an expected utility function. That is, for each $R_i \in \mathcal{R}_i$, there exists a Bernoulli utility function $u_i^{R_i} : A_i \to \mathbb{R}$ such that for all $\alpha_i, \alpha_i' \in \triangle A_i$,

$$\alpha_i \, R_i \, \alpha_i' \quad \Longleftrightarrow \quad \int_{A_i} u_i^{R_i} \, d\alpha_i \geq \int_{A_i} u_i^{R_i} \, d\alpha_i'.^{5}$$

The set of preference profiles is $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$, and we refer to the typical element $R = (R_i)_{i \in N} \in \mathcal{R}$ as a *state*. In addition, we use the term *(payoff) type* R_i for agent i whose preference relation is R_i. For each $R \in \mathcal{R}$, $i \in N$, and $S \subseteq N$, we use the following conventional notation: $R_{-i} \equiv (R_j)_{j \neq i}$, $R \equiv (R_i, R_{-i})$, $R_S \equiv (R_i)_{i \in S}$, $R_{-S} \equiv (R_i)_{i \notin S}$, $R \equiv (R_S, R_{-S})$, $\mathcal{R}_{-i} \equiv \times_{j \neq i} \mathcal{R}_j$, $\mathcal{R} \equiv \mathcal{R}_i \times \mathcal{R}_{-i}$, $\mathcal{R}_S \equiv \times_{i \in S} \mathcal{R}_i$, $\mathcal{R}_{-S} \equiv \times_{i \notin S} \mathcal{R}_i$, and $\mathcal{R} \equiv \mathcal{R}_S \times \mathcal{R}_{-S}$. We sometimes write $R_{i_1 i_2 \cdots i_m}$ for $R_{\{i_1, i_2, \dots, i_m\}}$, $\mathcal{R}_{i_1 i_2 \cdots i_m}$ for $\mathcal{R}_{\{i_1, i_2, \dots, i_m\}}$, $R_{-i_1 i_2 \cdots i_m}$ for $R_{-\{i_1, i_2, \dots, i_m\}}$, and $\mathcal{R}_{-i_1 i_2 \cdots i_m}$ for $\mathcal{R}_{-\{i_1, i_2, \dots, i_m\}}$.

As we indicated in the Introduction, each agent knows her own type but not necessarily the other agents' types. In other words, we are interested in the interim stage analysis in which each type R_i can have any subjective belief regarding other agents' types. To formalize, for each i, fix a σ-algebra $\mathcal{G}_i$ on $\mathcal{R}_{-i}$, and let $\mathcal{B}_i$ be *the set of all probability measures* on $(\mathcal{R}_{-i}, \mathcal{G}_i)$. Type R_i's belief β_i is a probability measure in $\mathcal{B}_i$. We assume that $\mathcal{G}_i$ includes the singleton set $\{R_{-i}\}$ for each $R_{-i} \in \mathcal{R}_{-i}$, implying that all the degenerate probability measures are in $\mathcal{B}_i$. The notation $\mathrm{Supp}(\beta_i)$ stands for the support of β_i.

Let $X \subseteq A$ be the set of *feasible outcomes*—the social alternatives that are allowed in the model—and $x = (x_1, \dots, x_n) \in X$ be a *feasible outcome*. If each $x \in X$ is constant

[5]Here, the expectation is a Lebesgue integral, and $u_i^{R_i} : A_i \to \mathbb{R}$ is a random variable, i.e., a Borel-measurable function on $(A_i, \mathcal{A}_i)$.

across agents (i.e., if $x_i = x_j$ for all $i, j \in N$), then the model features *pure public goods*. We assume that there are no externalities, i.e., for each $x, y \in X$ we have $x\ R_i\ y$ if and only if $x_i\ R_i\ y_i$.[6]

A rule (or *social choice function*) f is a function that maps each state $R \in \mathcal{R}$ to a feasible outcome, i.e., $f : \mathcal{R} \to X$. The notation $f_i(R)$ indicates the alternative agent i obtains in state R under rule f.

The planner's goal is to implement f, but she does not know the realized state. In this paper, we concentrate only on direct mechanisms, i.e., the planner collects type reports from agents and determines the outcome according to f based on these reports. We now consider the agents' expected utilities in the interim stage depending on type reports. For each $S \subseteq N$, $i \in S$, and $\tilde{R}_S \in \mathcal{R}_S$, we first define the function $f_i^{\tilde{R}_S} : \mathcal{R}_{-i} \to A_i$ such that $f_i^{\tilde{R}_S}(R_{-i}) \equiv f_i(\tilde{R}_S, R_{-S})$ for all $R_{-i} \in \mathcal{R}_{-i}$. If a coalition S were to report $\tilde{R}_S$ (regardless of its members' types) while the others report their preferences truthfully, then the expected utility of $i \in S$ with type R_i and belief β_i is

$$\int_{\mathcal{R}_{-i}} u_i^{R_i} \circ f_i^{\tilde{R}_S}\, d\beta_i$$

where $u^{R_i} \circ f_i^{\tilde{R}_S}(R_{-i}) \equiv u_i^{R_i}(f_i^{\tilde{R}_S}(R_{-i}))$ for all $R_{-i} \in \mathcal{R}_{-i}$.[7]

Before we move on, let us note that only the ordinal utility information is relevant for the new notion we propose in the next section. We clarify this issue in Remark 2.9. Thus, all of the examples we consider from here on will only specify preference rankings and not utility functions.

2.1 *Robust group strategy-proofness*

Strategy-proofness (SP) has long been a cornerstone of mechanism design. In the direct revelation game associated with a strategy-proof rule, truth telling is a weakly dominant strategy.

Definition 2.1 (SP: Definition I). A rule is *strategy-proof* if for every $R, \tilde{R} \in \mathcal{R}$ and $i \in N$,

$$f(R_i, \tilde{R}_{-i})\ R_i\ f(\tilde{R}).$$

For strategy-proof rules, regardless of one's belief regarding the others' types and regardless of the others' strategies, one weakly prefers truth telling over any other report. Thus, strategy-proof rules provide very strong incentives to report truthfully in the direct revelation game associated with these rules.[8] In the literature, the following technically equivalent definition is widely used.

[6]This property is sometimes called *selfishness*.

[7]For the expected utility to be well-defined, it is sufficient to assume that $f_i^{\tilde{R}_S} : \mathcal{R}_{-i} \to A_i$ is measurable from $(\mathcal{R}_{-i}, \mathcal{G}_i)$ to $(A_i, \mathcal{A}_i)$. Then because $u_i^{R_i}$ is a random variable, the function $u_i^{R_i} \circ f_i^{\tilde{R}_S}$ is a random variable on $(\mathcal{R}_{-i}, \mathcal{G}_i)$.

[8]There has been some recent work arguing that some strategy-proof rules work better in practice than others. See, for instance, Saijo et al. (2007), Li (2017), and Bochet and Tumennasan (2017).

DEFINITION 2.2 (SP: Definition II). An agent $i \in N$ *manipulates* f at state $R \in \mathcal{R}$ if there exists $\tilde{R}_i \in \mathcal{R}_i$ such that

$$f(\tilde{R}_i, R_{-i}) \, P_i \, f(R).$$

A rule f is *strategy-proof* if no agent can manipulate it at any state.

Observe that the definition above takes the ex-post approach. Specifically, once the state is realized each agent contemplates deviating from truth telling under the assumption that the others reveal their types truthfully. Our concern is the interim stage where each agent knows one's own type but not that of the others. Thus, we introduce another definition, which is also technically equivalent to Definition I.

DEFINITION 2.3 (SP: Definition III). An agent $i \in N$ *subjectively manipulates* f if there exists $R_i, \tilde{R}_i \in \mathcal{R}_i$, and a belief $\beta_i \in \mathcal{B}_i$ such that

$$\int_{\mathcal{R}_{-i}} u_i^{R_i} \circ f_i^{\tilde{R}_i} \, d\beta_i > \int_{\mathcal{R}_{-i}} u_i^{R_i} \circ f_i^{R_i} \, d\beta_i.$$

A rule f is *strategy-proof* if no agent can subjectively manipulate it.

Here, each agent contemplates deviating from truth telling after observing her type. As in Definition II, the deviating agent assumes that the other agents reveal their types truthfully. We emphasize that there is no restriction on the belief one can have regarding the others' types. In this sense, SP is a belief-free notion.

SP does not consider the possibility of group deviations. The group version of SP seeks to correct this oversight. The standard notion of group strategy-proofness (GSP) is a generalization of Definition II of SP, and its roots are traced to the notions of *core* and *strong Nash equilibrium* (Aumann (1959)). Below we state the formal definition.[9]

DEFINITION 2.4 (Group Strategy-Proofness). A coalition $S \subseteq N$ *manipulates* f at state $R \in \mathcal{R}$ if there exists $\tilde{R}_S \in \mathcal{R}_S$ such that $f(\tilde{R}_S, R_{-S}) \, P_i \, f(R)$ for each $i \in S$. A rule f is *group strategy-proof* if it is not manipulated by any coalition at any state.

This definition states that in the direct revelation game associated with any rule satisfying GSP, no coalition improves its members by deviating from truth telling in any realized state.[10] Because GSP is a generalization of Definition II of SP, it is an ex post notion. However, as we demonstrated in Example 1.1, GSP is not an interim notion: some groups may improve over truth telling in the interim stage for rules satisfying GSP. To remedy this problem, we propose a notion of GSP in the spirit of Definition III of SP.[11]

[9] See, for instance, Barberá et al. (2010) or Barberá et al. (2016).

[10] Notice that we only consider deviating coalitions in which each member is strictly better off. The literature sometimes calls this version weak GSP. On the other hand, a rule f satisfies "strong" GSP if there does not exist S, $R \in \mathcal{R}$, and $\tilde{R}_S$ such that $f(\tilde{R}_S, R_{-S}) \, R_i \, f(R)$ for each $i \in S$ and $f(\tilde{R}_S, R_{-S}) \, P_j \, f(R)$ for some $j \in S$.

[11] To the best of our knowledge, the group equivalent of dominant strategy has not been defined. Hence, we do not pursue a generalization of Definition I.

In our proposed notion, agents contemplate forming coalitions to collectively deviate from truth telling as in GSP. Thus, the agents assume that everyone's status-quo behavior is truth telling. We justify this assumption as follows: our notion is more demanding than SP, which means that each agent realizes that truth telling is an optimal strategy from an individual's perspective in the absence of collusion. Therefore, when an agent is not in a deviating coalition, she acts rationally and reports her preferences truthfully.

We first define the analog of subjective manipulation in Definition III of SP for coalitional deviations. In addition, this notion needs to be belief-free. To achieve this goal, we follow the approach used in epistemic game theory as well as in the robust mechanism design literature.[12,13] To be concrete, suppose that some coalition S, after deliberating, agrees to some report $\tilde{R}_S$. This means that the members of S are able to justify their agreement to $\tilde{R}_S$. For this, we require that the coalition members are able to provide infinite tiers of reasoning for the collusion:

- Type R_i of agent $i \in S$ is able to provide *one tier of reasoning* for the collusion if S reporting $\tilde{R}_S$ improves R_i over truth telling for some belief of hers.
- Type R_i is able to provide *two tiers of reasoning* for the collusion if S reporting $\tilde{R}_S$ improves R_i over truth telling for some belief of hers whose support is over those who can provide one tier of reasoning.
- Type R_i is able to provide k *(≥ 2) tiers of reasoning* for the collusion if S reporting $\tilde{R}_S$ improves R_i over truth telling for some belief of hers whose support is over those who can provide $k-1$ tiers of reasoning.

If S were to successfully commit to $\tilde{R}_S$, then each member of S would need to provide infinite tiers of reasoning for the collusion. We supplement this discussion with the following two examples.

Example 2.5. The set of agents is $\{1,2\}$ and the set of (public) alternatives is $\{a,b,c\}$. Each agent $i \in \{1,2\}$ can have three preferences, R_i, $\bar{R}_i$, and $\tilde{R}_i$. The rule f and the preferences are as follows:

f

	R_2	$\bar{R}_2$	$\tilde{R}_2$
R_1	a	a	a
$\bar{R}_1$	a	b	b
$\tilde{R}_1$	a	b	c

R_1	$\bar{R}_1$	$\tilde{R}_1$
a	b	c
	c	b
	a	a

R_2	$\bar{R}_2$	$\tilde{R}_2$
a	b	c
	c	b
	a	a

Observe here that rule f satisfies both SP and GSP. Suppose now that agents 1 and 2 agree to report $(\tilde{R}_1, \tilde{R}_2)$. Clearly, type R_1 or type R_2 would never agree to $(\tilde{R}_1, \tilde{R}_2)$ because

[12] For instance, see Dekel and Siniscalchi (2015) and Bergemann and Morris (2005).

[13] One key difference is that beliefs in our case are only about the payoff types while in other literatures they include strategies. The justification for this is that in our setting, the agents stick with truth telling due to SP.

they get their most preferred alternative, namely a, by reporting their type truthfully. In other words, neither R_1 nor R_2 provides one tier of reasoning for $(\tilde{R}_1, \tilde{R}_2)$. Type $\bar{R}_1$, on the other hand, improves over truth telling if she believes agent 2's type is R_2. However, because agent R_2 cannot provide one tier of reasoning, $\bar{R}_1$ cannot provide two tiers of reasoning for the collusion. Similar arguments show that $\bar{R}_2$ cannot provide two tiers of reasoning. Let us now concentrate on $\tilde{R}_1$, who provides one tier of reasoning for the collusion if she has a belief assigning a nonzero probability to the event that agent 2's type is either R_2 or $\bar{R}_2$. Unfortunately, neither R_2 nor $\bar{R}_2$ can provide two tiers of reasoning for the collusion, as noted above. Thus, $\tilde{R}_1$ is unable to provide three tiers of reasoning for her commitment to $(\tilde{R}_1, \tilde{R}_2)$. The same analysis is true for type $\tilde{R}_2$. Therefore, $(\tilde{R}_1, \tilde{R}_2)$ cannot be justified. ◊

The next example shows how a group can justify a misreport.

EXAMPLE 2.6. The set of agents is $\{1, 2, 3\}$ and the set of (public) alternatives is $\{a, b, c, d, e\}$. Each agent $i \in \{1, 2, 3\}$ can have two preferences, R_i and $\tilde{R}_i$. The rule f and the preferences are as follows:

f

	R_3		$\tilde{R}_3$	
	R_2	$\tilde{R}_2$	R_2	$\tilde{R}_2$
R_1	a	b	c	b
$\tilde{R}_1$	d	e	d	e

R_1	$\tilde{R}_1$	R_2	$\tilde{R}_2$	R_3	$\tilde{R}_3$
b	d	d	b	a	c
c	e	a	e		
e		e			
a		b			
d		c			

Suppose that agents 1 and 2 agree to report $(\tilde{R}_1, \tilde{R}_2)$. Observe that neither $\tilde{R}_1$ nor $\tilde{R}_2$ is able to provide one tier of reasoning for $(\tilde{R}_1, \tilde{R}_2)$. Type R_1, however, can provide one tier of reasoning if she believes that agents 2 and 3 have types (R_2, R_3). Similarly, R_2 provides one tier of reasoning with his belief that agents 1 and 3 have types $(R_1, \tilde{R}_3)$. Then, of course, R_1 provides two tiers of reasoning for her commitment to $(\tilde{R}_1, \tilde{R}_2)$ with the same belief she uses in her one tier of reasoning. The same applies to R_2. This way, (R_1, R_2) can provide infinite tiers of reasoning for their commitment to $(\tilde{R}_1, \tilde{R}_2)$. Observe here that the two agents learn each other's type already after two tiers of reasoning once they agree to $(\tilde{R}_1, \tilde{R}_2)$. However, they never agree on agent 3's type. ◊

Let us formalize the discussion above. As before, suppose that coalition S agrees to $\tilde{R}_S$. Fix any $\hat{\mathcal{R}} = \times_{i \in N} \hat{\mathcal{R}}_i \subseteq \mathcal{R}$ with $\hat{\mathcal{R}}_i = \mathcal{R}_i$ for all $i \notin S$. Think of $\hat{\mathcal{R}}$ as the set of types that can provide some tiers of reasoning for the collusion. We are looking for the set of types that can provide the next tier of reasoning. Of course, a certain type of member of S can provide the next tier of reasoning only if the collusion brings a better payoff than truth telling for some belief of hers whose support is in $\hat{\mathcal{R}}$. For technical convenience, we assume all the types of those not in S can provide the next tier of reasoning. To eliminate the types that cannot benefit by colluding, we define an operator $\xi[\tilde{R}_S] : 2^{\mathcal{R}} \to 2^{\mathcal{R}}$ such that $\xi[\tilde{R}_S](\hat{\mathcal{R}}) = \times_{i \in N} \xi_i[\tilde{R}_S](\hat{\mathcal{R}})$ where for each $i \notin S$,

$$\xi_i[\tilde{R}_S](\hat{\mathcal{R}}) = \mathcal{R}_i$$

and for each $i \in S$,

$$\xi_i[\tilde{R}_S](\hat{\mathcal{R}}) = \left\{ R_i \in \hat{\mathcal{R}}_i \;\middle|\; \begin{array}{l} \int_{\mathcal{R}_{-i}} u_i^{R_i} \circ f_i^{\tilde{R}_S}\, d\beta_i > \int_{\mathcal{R}_{-i}} u_i^{R_i} \circ f_i^{R_i}\, d\beta_i \\ \text{for some } \beta_i \text{ with } \mathrm{Supp}(\beta_i) \subseteq \hat{\mathcal{R}}_{-i} \end{array} \right\}.$$

Because each agent has an expected utility function, $\xi[\tilde{R}_S](\hat{\mathcal{R}})$ can be written for each $i \in S$ as follows:

$$\xi_i[\tilde{R}_S](\hat{\mathcal{R}}) = \left\{ R_i \in \hat{\mathcal{R}}_i \mid u_i^{R_i}\big(f_i(\tilde{R}_S, R_{-S})\big) > u_i^{R_i}\big(f_i(R_i, R_{-i})\big) \text{ for some } R_{-i} \in \hat{\mathcal{R}}_{-i} \right\}.$$

As a result, the operator ξ depends only on ordinal utility rankings over alternatives.

Let us now discuss when S justifies $\tilde{R}_S$. If a type can provide infinite tiers of reasoning, then the collusion must be preferable over truth telling for some belief of hers that has a support over those who can provide infinite tiers of reasoning. Thus, the set of types that can provide infinite tiers of reasoning is necessarily a fixed point of $\xi[\tilde{R}_S]$. At the same time, a type of S's member who is a part of any nonempty fixed point of $\xi[\tilde{R}_S]$ can provide infinite tiers of reasoning for $\tilde{R}_S$ with some belief whose support is over the fixed point. Thus, $\xi[\tilde{R}_S]$ having a nonempty fixed point is both necessary and sufficient for S to provide infinite tiers of reasoning for the collusion. As previously noted, our approach is an adaptation of the robust mechanism design literature to our setting. Thus, we will borrow the term "rationalization" from this point on.

Definition 2.7 (Rationalization). A coalition S *rationalizes* a report $\tilde{R}_S$ if there exists a nonempty set $\mathcal{R}^{\tilde{R}_S} \in \mathcal{T}$ such that

$$\xi[\tilde{R}_S]\big(\mathcal{R}^{\tilde{R}_S}\big) = \mathcal{R}^{\tilde{R}_S}.$$

The operator $\xi[\tilde{R}_S]$ does not restrict the types of those not in S. Thus, we sometimes use abusive language and say that $\mathcal{R}_S^{\tilde{R}_S}$ is a fixed point of $\xi[\tilde{R}_S]$.

Let us remark that it is possible for S to collude at $\tilde{R}_S$ even when their type is $\tilde{R}_S$, i.e., $\tilde{R}_S \in \mathcal{R}_S^{\tilde{R}_S}$ where $\mathcal{R}^{\tilde{R}_S}$ is a nonempty fixed point of $\xi[\tilde{R}_S]$. Here, everyone reports her type truthfully, but each must believe that at least one other agent in S is misreporting her type. Otherwise, we would have $\{\tilde{R}_S\} = \mathcal{R}_S^{\tilde{R}_S}$, but by the definition of ξ, $\xi[\tilde{R}_S](\mathcal{R}^{\tilde{R}_S}) = \xi[\tilde{R}_S](\{\tilde{R}_S\}) = \emptyset$, i.e., $\{\tilde{R}_S\}$ is not a fixed point of $\xi[\tilde{R}_S]$.

We are now ready to introduce our notion of GSP.

Definition 2.8 (Robust Group Strategy-Proofness). A coalition $S \subseteq N$ *subjectively manipulates* f if there exists $\tilde{R}_S \in \mathcal{R}_S$ such that S rationalizes $\tilde{R}_S$. A rule $f : \mathcal{R} \to X$ satisfies *robust group strategy-proofness* (RGSP) if there exists no coalition $S \subseteq N$ that can subjectively manipulate f.

Remark 2.9. RGSP is dependent only on ordinal (not cardinal) utility rankings over alternatives. This is simply because the operator ξ is dependent only on ordinal utility rankings. Therefore, one can think of the reports the agents submit in the direct revelation mechanism as simply ordinal rankings of alternatives.

Let us now investigate the relationship between our notion and GSP. Consider any rule f, which violates GSP. This means that there exists a state R, a coalition S, and a report $\tilde{R}_S$ with $u_i^{R_i}(f_i(\tilde{R}_S, R_{-S})) > u_i^{R_i}(f_i(R))$ for all $i \in S$. Then $\{R_S\}$ is a fixed point of $\xi[\tilde{R}_S]$ because each member of S prefers $\tilde{R}_S$ to truth telling for her belief that assigns probability 1 to R_{-i}. Hence, RGSP implies GSP. The opposite is not true. To demonstrate this, let us revisit the motivating example in the Introduction. Consider $(\tilde{R}_A, \tilde{R}_B)$. Observe that $\{(R_A, R_B)\}$ is a fixed point of $\xi[(\tilde{R}_A, \tilde{R}_B)]$ because both women would agree to report $(\tilde{R}_A, \tilde{R}_B)$ if they believe that the state is either (R_A, R_B, R_C) or $(R_A, R_B, \tilde{R}_C)$ with equal probability. Consequently, RGSP is more demanding than GSP.

Next, we remark that if the deviations are restricted to individuals (not to coalitions), then our definition is equivalent to SP. Each type of agent i weakly prefers truth telling to any other report regardless of beliefs in the direct revelation game associated with any strategy-proof rule. This means that for any $\tilde{R}_i$, $\xi[\tilde{R}_i](\mathcal{R}) = \emptyset$. On the other hand, if a rule is not strategy-proof, then there exists at least one state R in which some type R_i finds it profitable to deviate to another report $\tilde{R}_i$. Consequently, $\{R_i\}$ is a fixed point of $\xi[\tilde{R}_i]$. Hence, the rule fails RGSP. This discussion confirms our expectation that if one restricts the size of coalitions to one, then our notion is equivalent to SP.

RGSP does not rely on the agents' knowledge of the others' beliefs. Thus, we are following the Wilson doctrine, which questions the reliance on the common knowledge assumption. As a result, robust group strategy-proof rules are highly desirable. A potential pitfall of RGSP is that the set of rules satisfying it may be very limited. Clearly, the dictatorship rules in the case of public good economies and the serial dictatorship rules in the case of private good economies satisfy RGSP.[14] In the next section, we consider three settings and investigate whether some well-known rules satisfy RGSP.

3. Robust group strategy-proof rules

Before we consider how stringent RGSP is in different applications, let us introduce some concepts that will be needed later. The first one is efficiency, which requires that no agent can be made better off without hurting others.

Definition 3.1 (Efficiency). A rule f is efficient if for all R, there exists no $y \in X$ such that

$$y \, P_i \, f(R) \quad \text{for some } i \in N$$
$$y \, R_j \, f(R) \quad \forall j \in N.$$

The next property is nonbossiness (Satterthwaite and Sonnenschein (1981)), which states that one cannot change the alternatives the others obtain under a rule without affecting her own.

[14] A rule in a public good setting is a dictatorship if one specific agent's top choice is always selected. Such an agent would not be part of any colluding coalition, and so neither would any other agent. Hence, a dictatorship satisfies RGSP. In the case of private good settings, a rule is a serial dictatorship if there is an ordering of agents such that they choose their most preferred alternative sequentially based on availability. Of course, the highest-ranked agent has no incentive to be a part of any blocking coalition. This then prevents the next highest-ranked agent from colluding, and so on.

DEFINITION 3.2 (Nonbossiness). A rule f is nonbossy if whenever $f_i(\tilde{R}_i, R_{-i}) = f_i(R)$ for some $i \in N$, $\tilde{R}_i \in \mathcal{R}_i$ and $R \in \mathcal{R}$ we have that $f(\tilde{R}_i, R_{-i}) = f(R)$.

Notice that nonbossiness is vacuously satisfied for public good environments.

3.1 *Provision of public goods*

In this subsection, we consider the public good cost-sharing problem of Serizawa (1999). In this environment, n agents submit their preferences over the level of public good provision, which has production range Y. We assume that $Y \subseteq \mathbb{R}_+$ (nonnegative reals), $0 \in Y$, and Y is compact. The public good $y \in Y$ is produced at cost $C(y)$. We assume that $C(y)$ is (strictly) increasing, convex, and $C(0) = 0$. For any convex subset $Y' \subseteq Y$, we assume that $C(y)$ is differentiable and has strictly positive derivative at each $y \in Y'$.

Each agent i contributes $t_i \geq 0$ (transfer) to the provision of the public good. A typical alternative for agent i is (t_i, y), and $A_i = \mathbb{R}_+ \times Y$ is the set of alternatives. The (Bernoulli) utility function for agent i with preference R_i is $u_i^{R_i} : \mathbb{R}_+ \times Y \to \mathbb{R}_+$. We assume that $u_i^{R_i}(\cdot, \cdot)$ is continuous, strictly quasi-concave, and strictly monotonic in the sense that it is decreasing in the transfer and increasing in the provision of public good.

The set of feasible outcomes is $X = \{(t_1, t_2, \ldots, t_n, y) \mid y \in Y \ \& \ \sum_{i=1}^n t_i = C(y)\}$. A rule $f = (f_1, \ldots, f_n, f_y)$ is a function mapping $\mathcal{R}$ to X. We will concentrate on a prominent rule in the literature: *the minimum demand rule with respect to the equal cost-sharing scheme*, denoted by f^{ME}. For this rule, the amount each agent pays is determined by the equal cost-sharing scheme $\tau : Y \to \mathbb{R}_+^n$ with $\tau_i(y) = C(y)/n$ for all $i \in N$ and $y \in Y$. Observe here that, corresponding to the equal cost-sharing scheme, each type R_i's utility depends only on the provision of public good. That is, if y units of public good are provided then R_i's utility is $v_i^{R_i}(y) \equiv u_i^{R_i}(C(y)/n, y)$. Let $p(R_i) = \{y \in Y \mid v_i^{R_i}(y) \geq v_i^{R_i}(y'), \forall y' \in Y\}$ be the *peak correspondence*. Furthermore, let $\underline{p}(R_i) = \inf p(R_i)$ and $\bar{p}(R_i) = \sup p(R_i)$. For the minimum demand rule, $f_y^{\mathrm{ME}}(R) = \min_{i \in N}\{\bar{p}_i(R_i)\}$. In summary, the minimum demand rule with respect to the equal cost-sharing scheme is $f^{\mathrm{ME}}(R) = (C(f_y^{\mathrm{ME}}(R))/n, \ldots, C(f_y^{\mathrm{ME}}(R))/n, f_y^{\mathrm{ME}}(R))$ for all $R \in \mathcal{R}$.

Given the equal cost-sharing scheme, $v_i^{R_i}(y)$ represents an induced preference over Y for type R_i. We abuse notation so that $y' R_i y$ if and only if $v_i^{R_i}(y') \geq v_i^{R_i}(y)$. We define the following notions of single-peakedness.

DEFINITION 3.3. Preferences R_i are *weakly single-peaked* if they satisfy the following:

- $p(R_i)$ has at most two elements and $Y \cap (\underline{p}(R_i), \bar{p}(R_i)) = \emptyset$.
- For all $y, y' \in Y$, $y' < y \leq \underline{p}(R_i)$ implies $y \, P_i \, y'$.
- For all $y, y' \in Y$, $\bar{p}(R_i) \leq y < y'$ implies $y \, P_i \, y'$.

Preferences R_i are *single-peaked* on Y if they are weakly single-peaked and $|p(R_i)| = 1$.

We denote single-peaked preferences and weakly single-peaked preferences (on Y) as $\mathcal{R}_i^s$ and $\mathcal{R}_i^w$, respectively. The analysis is greatly simplified if $\mathcal{R} = \mathcal{R}^w$. The following lemma states that this is the case if $u_i^{R_i}(\cdot, \cdot)$ and $C(\cdot)$ satisfy certain properties.

LEMMA 3.4. *If $u_i^{R_i}(y, t_i)$ is continuous, strictly quasi-concave, increasing in y, and decreasing in t_i, and $C(y)$ is convex on Y, then R_i is weakly single-peaked on Y. In addition, if Y is convex, then R_i is single-peaked.*

PROOF. For a proof, see the Appendix. □

Lemma 3.4 identifies conditions that guarantee the weak single-peakedness of preferences. We thus focus on weakly single-peaked preferences. While the minimum demand rule satisfies GSP, for this rule to satisfy RGSP one needs an additional restriction, which we introduce below.

DEFINITION 3.5. The set Y is *completely closed below (CCB)* if $\inf Y' \in Y'$ for all $Y' \subseteq Y$.

When Y is CCB, there is no infinite, (strictly) decreasing sequence in Y. Any finite subset of the nonnegative real numbers is CCB. While some infinite sets (such as the nonnegative integers) are CCB, others (such as $\mathbb{R}_+$) are not CCB. The following result characterizes the RGSP of the minimum demand rule.

PROPOSITION 3.6. *For $\mathcal{R} = \mathcal{R}^w$, the minimum demand rule with respect to the equal cost-sharing scheme satisfies RGSP if and only if Y is CCB.*

PROOF. For any $Y' \subseteq Y$ and $i \in N$, define $\mathcal{R}_i(Y') = \{R_i \in \mathcal{R}_i \mid p(R_i) \cap Y' \neq \emptyset, Y' \subseteq Y\}$ and $\mathcal{R}(Y') = \times_i \mathcal{R}_i(Y')$. In words, this is the set of weakly-single peaked preferences that have a peak in Y'.

Necessity: Suppose Y is not CCB. Set $S = \{1, 2\}$ and fix an arbitrary $Y^* \subseteq Y$ with $\inf Y^* \notin Y^*$. Then, for any $R_i \in \mathcal{R}_i(Y^*) \cap \mathcal{R}_i^s$ and $\epsilon > 0$, there exists $y \in (\inf Y^*, \inf Y^* + \epsilon)$. Furthermore, fix $\tilde{R}_S$ such that $\tilde{R}_i \in \mathcal{R}_i(Y^*) \cap \mathcal{R}_i^s$ for $i = 1, 2$ and $\bar{p}(\tilde{R}_1) = \bar{p}(\tilde{R}_2)$. For each $i \in S$, set

$$\mathcal{R}_i^* = \left\{R_i \in \mathcal{R}_i\left(Y^*\right) \cap \mathcal{R}_i^s \mid \bar{p}(R_i) < \bar{p}(\tilde{R}_i) \;\&\; \forall \epsilon > 0, \exists y \in \left(\inf Y^*, \inf Y^* + \epsilon\right) \text{ with } \bar{p}(\tilde{R}_i)\, P_i\, y\right\}$$

and for each $i \notin S$, set

$$\mathcal{R}_i^* = \left\{R_i \in \mathcal{R}_i\left(Y^*\right) \cap \mathcal{R}_i^s \mid \bar{p}(R_i) \geq \bar{p}(\tilde{R}_1)\right\}.$$

In addition, $\mathcal{R}^* = \times_{i \in N} \mathcal{R}_i^*$. By construction of $\mathcal{R}^*$, for any $R \in \mathcal{R}^*$, we have that $f_y^{\mathrm{ME}}(\tilde{R}_S, R_{-S}) = \bar{p}(\tilde{R}_1) = \bar{p}(\tilde{R}_2)$. Let us now consider $\xi[\tilde{R}_S](\mathcal{R}_S^* \times \mathcal{R}_{-S})$. Fix $i \in S$ and $R_i^* \in \mathcal{R}_i^*$. By construction, there exists $y^* \in (\inf Y^*, \bar{p}(R_i^*))$ such that $\bar{p}(\tilde{R}_i) P_i^* y^*$. For $j \in S \setminus \{i\}$, let $R_j^* \in \mathcal{R}_j$ have $\bar{p}(R_j^*) = y^*$. Fix any $R_{-S}^* \in \mathcal{R}_{-S}^*$. Observe that $\bar{p}(R_j^*) < \bar{p}(R_i^*) < \bar{p}(R_\ell^*)$ for all $\ell \notin S$. Thus, $f_y^{\mathrm{ME}}(R^*) = \bar{p}(R_j^*) = y^*$ and $f_y^{\mathrm{ME}}(\tilde{R}_S, R_{-S}^*) = \bar{p}(\tilde{R}_i) P_i^* f_y^{\mathrm{ME}}(R^*)$. In other words, if i with R_i^* has a belief that assigns probability 1 to the event that the others' type is R_{-i}^* then i agreeing to $\tilde{R}_S$ is profitable. Thus, any $R_i^* \in \mathcal{R}_i^*$ is in $\xi_i[\tilde{R}_S](\mathcal{R}_S^* \times \mathcal{R}_{-S})$. Consequently, $\xi_S[\tilde{R}_S](\mathcal{R}_S^* \times \mathcal{R}_{-S}) = \mathcal{R}_S^*$. Thus, f^{ME} does not satisfy RGSP.

Sufficiency: Consider arbitrary set S, $\hat{\mathcal{R}}_S \neq \emptyset$ and $\tilde{R}_S \in \mathcal{R}_S$. We will show that $\xi[\tilde{R}_S](\hat{\mathcal{R}}_S \times \mathcal{R}_{-S}) \neq \hat{\mathcal{R}}_S \times \mathcal{R}_{-S}$. Let

$$\hat{Y} = \left\{y \in Y \mid y = \bar{p}(R_i) \text{ for some } i \in S \;\&\; R_i \in \hat{\mathcal{R}}_i\right\}.$$

Because Y is CCB, $\inf \hat{Y} \in \hat{Y}$. This further implies that there exists $i \in S$ and $R_i^* \in \hat{\mathcal{R}}_i$ with $\bar{p}(R_i^*) = \inf \hat{Y}$. We show that $R_i^* \notin \xi_i[\tilde{R}_S](\hat{\mathcal{R}}_S \times \mathcal{R}_{-S})$. Suppose otherwise. Then there exists $R_{-i} \in \hat{\mathcal{R}}_{S\setminus\{i\}} \times \mathcal{R}_{-S}$ such that $f_y^{\mathrm{ME}}(\tilde{R}_S, R_{-S})\ P_i^*\ f_y^{\mathrm{ME}}(R_i^*, R_{-i})$. By construction, $f_y^{\mathrm{ME}}(R_i^*, R_{-i}) \leq \inf \hat{Y} = \bar{p}(R_i^*)$. By weak single-peakedness, we can dispose of the equality case immediately. The case $f_y^{\mathrm{ME}}(R_i^*, R_{-i}) < \inf \hat{Y}$ implies that there is an agent $j \notin S$ with $\bar{p}(R_j) < \inf \hat{Y}$. Then $f_y^{\mathrm{ME}}(\tilde{R}_S, R_{-S}) \leq \bar{p}(R_j)$. By weak single-peakedness, $f_y^{\mathrm{ME}}(R_i^*, R_{-i})\ R_i^*\ f_y^{\mathrm{ME}}(\tilde{R}_S, R_{-S})$, a contradiction. Consequently, $\hat{\mathcal{R}}_S$ is not a fixed point of $\xi[\tilde{R}_S]$. Given that S, $\tilde{R}_S$ and $\hat{\mathcal{R}}_S$ are selected arbitrarily, the minimum rule satisfies RGSP. □

To understand how Y being CCB guarantees the RGSP of the minimum demand rule, consider a deviating coalition S with proposed misreport $\tilde{R}_S$. For simplicity, let us suppose that the agents have single-peaked preferences. To see that $\xi[\tilde{R}_S]$ has no fixed point, fix an arbitrary set $\hat{\mathcal{R}}$. Let us collect the peaks of those in S under preferences $\hat{\mathcal{R}}_S$. The set of peaks must contain its infimum because Y is CCB. This means that under some preference in $\hat{\mathcal{R}}_S$, someone in S must have a preference peak equal to the infimum. But then she has no incentive to collude: if she believes someone outside of S has a lower peak than she has, then under truth telling the minimum demand rule implements a lower level of public good than her peak. Of course, the collusion cannot increase the level of public good implemented by the minimum demand rule because of the outsider's peak. On the other hand, if she believes that no one outside of S has a lower peak than she has then truth telling leads to the implementation of her peak. Thus, she will not improve by colluding. Hence, $\xi[\tilde{R}_S](\hat{\mathcal{R}}) \neq \hat{\mathcal{R}}$. Given that $\hat{\mathcal{R}}$ is an arbitrary set, $\xi[\tilde{R}_S]$ has no fixed point outside of the empty set.

3.2 *Allocation of indivisible goods in strict preference domains*

In this subsection, we focus on the allocation of indivisible objects. The set of alternatives for each agent is the same and consists of $m \geq 2$ indivisible objects, i.e., $A_i = O = \{0, o_1, \ldots, o_m\}$, where 0 is the null object. Each object $o \in O$ can be allocated to up to $q_o \in \mathbb{Z}_{++}$ agents. For technical convenience, we set $q_0 = \infty$. We refer to q_o as object o's quota and the collection of quotas $q = (q_o)_{o\in O}$ as the quota. An outcome x is feasible if $|\{i \in N \mid x_i = o\}| \leq q_o$ for all $o \in O$. Agent i's set of preferences, $\mathcal{R}_i$, is the set of all possible strict preference relations over A_i. We examine two prominent rules: *Gale's Top Trading Cycles* (TTC) and *Deferred Acceptance* (DA). To introduce these rules formally, we need to define the priority functions. For each object $o \in O$, a priority function π_o is a bijection from N to $\{1, \ldots, n\}$. We say agent i has a higher priority than j at object o if $\pi_o(i) < \pi_o(j)$. Let $\pi \equiv (\pi_o)_{o\in O}$. The TTC rule is efficient while the DA rule is stable.[15]

Top trading cycles mechanism: A rule f^{TTC} is a TTC rule with respect to priority π and quota q if the outcome for a given profile R is found according the following algorithm. In round 1, the set of active agents is N and the set of active objects is O.

[15]A rule f is stable if for each R, (i) $f_i(R)\ R_i\ 0$ for each $i \in N$, and (ii) there does not exist i and $o \in O \setminus \{0\}$ such that $o\ P_i\ f_i(R)$, $\pi_o(i) < \pi_o(j)$ and $f_j(R) = o$.

Round k: Each active agent points to her most preferred active object under R, and each active object points to the active agent who has the highest priority among the active agents at the object under π. Every agent who points to the null object is matched to it. Among the unmatched active agents and active objects we look for trading cycles where a cycle is an ordered set $\{i^1, o^1, i^2, o^2, \ldots, i^k, o^k\}$ such that for each $l \in \{1, \ldots, k\}$, i^l points to object o^l while o^l points to i^{l+1} where $i^{k+1} = i^1$. Each agent in any cycle is matched to the object to which she pointed. The set of active agents at round $k+1$ is modified by eliminating the agents who are matched in round k. The set of active objects in round $k+1$ consists of the objects that have not filled their quotas.

The algorithm stops when each agent is matched to an object (possibly the null object).

Deferred acceptance mechanism: A rule f^{DA} is a DA rule with respect to priority π if the outcome for a given profile R is found by the following algorithm.

Round 1: Each agent "applies" to her most preferred object. Each object o "holds" up to a maximum of q_o applicants with the highest priorities (if there are any) and rejects all others.

Round k: Each agent whose application was rejected in the previous round applies to her most preferred object which has not yet rejected her. Each object o considers the pool of applicants composed of the current applicants and the agents whom o has been holding from the previous round (if there are any). Each object o "holds" up to a maximum of q_o agents in the pool who have the highest priority and rejects all others.

The algorithm stops when no applicant is rejected, and each object is assigned to the agents whom it held at the final round.

It is well known that the two rules discussed above satisfy GSP.[16] Here, we investigate their performance in terms of RGSP. First, we point out that neither rule satisfies RGSP.

EXAMPLE 3.7. Let $N = \{1, 2, 3\}$ and suppose that $O = \{a, b, 0\}$. The quota for each object other than the null object is 1. Let the object priorities and the agent preferences be given as follows:

$$\begin{aligned} \pi_a(1) < \pi_a(2) < \pi_a(3), \qquad & \pi_b(1) < \pi_b(3) < \pi_b(2), \\ a\ P_1\ b\ P_1\ 0, \qquad & b\ \tilde{P}_1\ a\ \tilde{P}_1\ 0, \\ b\ P_2\ a\ P_2\ 0, \qquad & b\ \tilde{P}_2\ 0\ \tilde{P}_2\ a, \\ a\ P_3\ b\ P_3\ 0, \qquad & a\ \tilde{P}_3\ 0\ \tilde{P}_3\ b. \end{aligned}$$

If agents 2 and 3 have the respective preferences of R_2 and R_3, then depending on agent 1's preferences $f \in \{f^{\mathrm{DA}}, f^{\mathrm{TTC}}\}$ returns the following outcome:

$$f(R_1, R_2, R_3) = (a, 0, b) \quad \text{and} \quad f(\tilde{R}_1, R_2, R_3) = (b, a, 0).$$

[16]The DA rule does not satisfy strong GSP as defined in footnote 11, but the TTC does.

If agents 2 and 3 misreport their preferences as $\tilde{R}_2$ and $\tilde{R}_3$, respectively, then depending on agent 1's preferences the outcome is as follows:

$$f(R_1, \tilde{R}_2, \tilde{R}_3) = (a, b, 0) \quad \text{and} \quad f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = (b, 0, a).$$

Here, agent 3 prefers $f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3)$ to $f(\tilde{R}_1, R_2, R_3)$ while agent 2 prefers $f(R_1, \tilde{R}_2, \tilde{R}_3)$ to $f(R)$. In other words, $\{(R_2, R_3)\}$ is a fixed point of $\xi[(\tilde{R}_2, \tilde{R}_3)]$. Thus, both the TTC and DA rules with respect to priority π do not satisfy RGSP. ◊

The preceding example leads to our next question: what conditions guarantee that the DA and TTC rules satisfy RGSP? Given that priorities are designed by the planner, we identify a restriction on π. We need one more definition to introduce this restriction formally: for each agent i and object o, let $U_o(i)$ denote the set of agents who have higher priorities than i at o, i.e., $U_o(i) \equiv \{j \in N \mid \pi_o(j) < \pi_o(i)\}$.

Definition 3.8 (Acyclicity). A priority function and quota pair (π, q) is *acyclic* if there do not exist objects a, b and agents i, j, k satisfying the following two conditions:

(C) $\pi_a(i) < \pi_a(j) < \pi_a(k)$, and $\pi_b(k) < \pi_b(i)$ or $\pi_b(k) < \pi_b(j)$.

(S) There exist disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U_a(j)$, $N_b \subseteq U_b(i)$, $|N_a| = q_a - 1$, and $|N_b| = q_b - 1$.

Similar acyclicity conditions have been introduced in the literature before. Ergin acyclicity (2002), which is both necessary and sufficient for the efficiency of the DA rule, is obtained by replacing Condition (C) in Definition 3.8 with the following condition.

(EC) $\pi_a(i) < \pi_a(j) < \pi_a(k)$, and $\pi_b(k) < \pi_b(i)$.

Condition (C) is weaker than (EC). Thus, acyclicity is more stringent than Ergin acyclicity. Kesten (2006) studied the stability of the TTC rule and identified the following acyclicity condition as both necessary and sufficient.

Definition 3.9 (Kesten Acyclicity). A priority function and quota pair (π, q) is *Kesten acyclic* if there do not exist objects a, b and agents i, j, k satisfying the following two conditions:

(KC) $\pi_a(i) < \pi_a(j) < \pi_a(k)$, $\pi_b(k) < \pi_b(i)$, and $\pi_b(k) < \pi_b(j)$.

(KS) There exists a set $N_a \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U_a(i) \cup (U_a(j) \setminus U_b(k))$ and $|N_a| = q_a - 1$.

Acyclicity is neither weaker nor stronger than Kesten acyclicity. Clearly, Condition (C) is weaker than (KC). When each object has the quota of 1, both conditions (S) and (KS) are vacuously satisfied. Hence, in such cases, acyclicity is more demanding than Kesten acyclicity. However, when the quota is not 1 for some objects, acyclicity is sometimes satisfied whereas Kesten acyclicity is not.

EXAMPLE 3.10. Consider three objects $\{a, b, c, 0\}$ with $q_a = q_b = 1$ and $q_c = 2$, and three agents $\{1, 2, 3\}$. The priority structures are the following:

$$\pi_a(1) < \pi_a(2) < \pi_a(3),$$
$$\pi_b(2) < \pi_b(1) < \pi_b(3),$$
$$\pi_c(3) < \pi_c(1) < \pi_c(2).$$

Notice that (π, q) is acyclic but not Kesten acyclic. ◊

Lemma 3.11 below identifies an intuitive structure that acyclic priorities satisfy. Specifically, for any pair of objects a and b, an acyclic priority structure divides agents into two classes. High-ranked (low-ranked) agents, specifically those with a priority weakly lower (strictly higher) than $q_a + q_b$, must be the same for objects a and b. Furthermore, the low-ranked agents must have identical ranking across goods a and b.

LEMMA 3.11. *The following two statements are equivalent:*

(i) A pair (π, q) is acyclic.

(ii) A pair (π, q) satisfies that for any $a, b \in O$, and agent $i \in N$,

$$\pi_a(i) \leq q_a + q_b \iff \pi_b(i) \leq q_a + q_b$$

and

$$\pi_a(i) > q_a + q_b \implies \pi_a(i) = \pi_b(i).$$

PROOF. For a proof, see the Appendix. □

Our main results of this section relate the DA and TTC rules to acyclicity. We next show that acyclicity is necessary for the DA and TTC rules to satisfy RGSP.

THEOREM 3.12. *Let f be the TTC or DA rule with respect to (π, q). If f satisfies RGSP then (π, q) is acyclic.*

PROOF. Pick $f \in \{f^{\mathrm{DA}}, f^{\mathrm{TTC}}\}$ and suppose that (π, q) is cyclic. By Lemma 3.11, there exist a and b such that either (a) $\pi_a(\ell) \leq q_a + q_b \iff \pi_b(\ell) \leq q_a + q_b$ for any ℓ but $q_a + q_b < \pi_a(\ell) \neq \pi_b(\ell)$ for some ℓ or (b) there exist ℓ with $\pi_a(\ell) \leq q_a + q_b < \pi_b(\ell)$. Without loss of generality, assume that $q_a \leq q_b$.

Case (a). In this case, we can find j and k with $q_a + q_b < \pi_a(j) < \pi_a(k)$ and $q_a + q_b < \pi_b(k) < \pi_b(j)$. Let i be the agent for whom $\pi_a(i) = q_a$. By assumption, $\pi_b(i) \leq q_a + q_b$. Let $N_a = \{\ell \in N \mid \pi_a(\ell) < q_a\}$. The assumptions of this case allow the construction of N_b such that $N_a \cap N_b = \emptyset$, $|N_b| = q_b - 1$ and for each $\ell \in N_b$, $\pi_a(\ell) \leq q_a + q_b \geq \pi_b(\ell)$.

Let R be such that all agents prefer 0 to any object in $O \setminus \{a, b, 0\}$. Let R_{-ijk} be such that a is the most preferred object for each $\ell \in N_a$, b for each $\ell \in N_b$ and 0 for each $\ell \notin N_a \cup N_b \cup \{i, j, k\}$.

We now consider R_{ijk} and $\tilde{R}_{ijk}$ such that

$$a\ P_i\ b\ P_i\ 0 \quad \text{and} \quad 0\ \tilde{P}_i\ a\ \tilde{P}_i\ b$$
$$a\ P_j\ b\ P_j\ 0 \quad \text{and} \quad b\ \tilde{P}_j\ 0\ \tilde{P}_j\ a$$
$$a\ P_k\ b\ P_k\ 0 \quad \text{and} \quad a\ \tilde{P}_k\ 0\ \tilde{P}_k\ b.$$

Consequently, we have that

$$f_{ijk}(R) = (a, 0, b) \qquad f_{ijk}(\tilde{R}_{jk}, R_{-jk}) = (a, b, 0)$$
$$f_{ijk}(\tilde{R}_i, R_{-i}) = (0, a, b) \qquad f_{ijk}(\tilde{R}_{ijk}, R_{-ijk}) = (0, b, a).$$

Observe that j prefers $f(\tilde{R}_{jk}, R_{-jk})$ to $f(R)$ under R_j. At the same time, k prefers $f(\tilde{R}_{ijk}, R_{-ijk})$ to $f(\tilde{R}_i, R_{-i})$ under R_k. Thus, $\{R_{jk}\}$ is a fixed point of $\xi[\tilde{R}_{jk}]$. Consequently, in this case, f does not satisfy RGSP, a contradiction.

Case (b). There must exist j with $\pi_a(j) \leq q_a + q_b < \pi_b(j)$ and k with $\pi_a(k) > q_a + q_b \geq \pi_b(k)$. We claim that $\pi_a(j) \leq q_a$. Suppose otherwise. Let i be the agent for whom $\pi_a(i) = q_a$. By following the exact same steps as in case (a), we reach a contradiction. Hence, $\pi_a(j) \leq q_a$. By symmetry, $\pi_b(k) \leq q_b$. Let $N^1 = \{\ell \in N \mid \pi_a(\ell) \leq q_a \ \&\ \pi_b(\ell) > q_a + q_b\}$. We know that $j \in N^1$. Let $n^1 = |N^1|$. Pick any ℓ with $q_a < \pi_a(\ell) \leq q_a + q_b$. We now claim that $\pi_b(\ell) \leq q_b + q_a - n^1$. Suppose otherwise. This means that $q_b \leq q_b + q_a - n^1 < \pi_b(\ell) \leq q_a + q_b$. Fix an agent $i \in N^1$. Then we must have that $\pi_a(i) \leq q_a < \pi_a(\ell) \leq q_a + q_b < \pi_a(k)$ and $\pi_b(k) \leq q_b < q_a + q_b - n^1 < \pi_b(\ell) \leq q_a + q_b < \pi_b(i)$. Let $N_a = \{i' \neq i \mid \pi_a(i') \leq q_a\}$. By construction, $|N_a| = q_a - 1$. Clearly, $|U_b(\ell)| \geq q_a + q_b - n^1$. In addition, $N^1 \cap U_b(\ell) = \emptyset$. Note here that at most $q_a - n^1$ agents in N_a can have higher priorities than ℓ at b. Consequently, $|U_b(\ell) \setminus N_a| \geq q_b + q_a - n^1 - (q_a - n^1) = q_b$. Thus, we can construct N_b such that $|N_b| = q_b - 1$, $k \notin N_b$, and $N_b \subset U_b(\ell) \setminus N_a$. By relabeling j by ℓ in the proof of part (a), we reach a contradiction. Consequently, any i' with $q_b + q_a - n^1 < \pi_b(i') \leq q_a + q_b$ must have $\pi_a(i') \leq q_a$. Set now $N^2 = N^1 \cup \{i' \in N \mid q_b + q_a - n^1 < \pi_b(i') \leq q_a + q_b\}$. Observe that $|N^2| = 2n^1$. By using the same logic as before, we can show that for each ℓ with $q_a < \pi_a(\ell) \leq q_a + q_b$ must have $\pi_b(\ell) \leq q_b + q_a - 2n^1$. In turn, any i' with $q_b + q_a - 2n^1 < \pi_b(i') \leq q_a + q_b$ must have $\pi_a(i') \leq q_a$. Continuing with the same logic, we eventually find that any ℓ with $q_a < \pi_a(\ell) \leq q_a + q_b$ must have $\pi_b(\ell) \leq q_b$. However, there are q_b such agents. In addition, as pointed out earlier, $\pi_b(k) \leq q_b$. Thus, there are at least $q_b + 1$ agents with the top q_b priorities at b, a contradiction. □

The result above highlights the importance of acyclicity for truth telling under DA and TTC: it is impossible for either rule to satisfy RGSP when (π, q) is cyclic. We now turn our attention to the result that the DA rule satisfies RGSP when priorities are acyclic. First, we present the following two lemmas that are instrumental in the proof.

LEMMA 3.13. *Suppose (π, q) is acyclic and let $\underline{a}$ and $\bar{a}$ be nonnull objects with the lowest and highest quotas, respectively. For any a and agent i,*

$$\pi_{\underline{a}}(i) \leq q_{\underline{a}} + q_{\bar{a}} \quad \Longrightarrow \quad \pi_a(i) \leq q_{\underline{a}} + q_{\bar{a}}$$

and

$$\pi_{\underline{a}}(i) > q_{\underline{a}} + q_{\bar{a}} \quad \Longrightarrow \quad \pi_a(i) = \pi_{\underline{a}}(i).$$

PROOF. For a proof, see the Appendix. □

LEMMA 3.14. *Let $N^* \equiv \{i \in N \mid \pi_{\underline{a}}(i) \leq q_{\underline{a}} + q_{\bar{a}}\}$ and (π, q) be acyclic. Furthermore, let a be the most preferred object of i under some preference profile R. If $\pi_a(i) \leq q_a + q_{\underline{a}}$ then:*

(a) $f_i^{\mathrm{DA}}(R)$ is at worst the second most preferred object of i under R.

(b) $f_i^{\mathrm{DA}}(R) = a$ whenever the set

$$N_a(R, i) \equiv \left\{ j \in N^* \mid \pi_a(j) < \pi_a(i) \ \& \ a \, R_j \, b, \forall b \in O \right\}$$

has no more than $q_a - 1$ agents.

PROOF. For a proof, see the Appendix. □

We are now ready to present the sufficiency of acyclicity for the RGSP of the DA rule.

THEOREM 3.15. *If (π, q) is acyclic, then the DA rule satisfies RGSP.*

PROOF. Suppose the DA rule with respect to (π, q) does not satisfy RGSP. Then there must exist S and $\tilde{R}_S$ such that S rationalizes $\tilde{R}_S$.

Clearly, $|S| \neq 1$; otherwise, we obtain a contradiction with the SP of the DA. We need several steps to complete the proof.

CLAIM 1. *If S can rationalize $\tilde{R}_S$, then $S \cap N^*$ can rationalize $\tilde{R}_{N^* \cap S}$.*

PROOF OF CLAIM 1. Let $\mathcal{R}^{\tilde{R}_S}$ be a fixed point of $\xi[R_S]$. Let $S \cap N^* \neq \emptyset$. For each $i \in S$, there exists $R^i \in \mathcal{R}^{\tilde{R}_S}$ such that

$$f^{\mathrm{DA}}\big(\tilde{R}_S, R^i_{-S}\big) \, P_i^i \, f^{\mathrm{DA}}\big(R^i\big).$$

By Lemma 3.13, for each $i \in N^*$, $j \notin N^*$ and $b \in O$,

$$\pi_b(i) < \pi_b(j).$$

It is easy to see that for any R and $\bar{R}_{-N^*}$,

$$f^{\mathrm{DA}}_{N^*}(R) = f^{\mathrm{DA}}_{N^*}(R_{N^*}, \bar{R}_{-N^*}).$$

Consequently, for each $i \in N^* \cap S$ and R_{-N^*},

$$f^{\mathrm{DA}}_{N^*}\big(\tilde{R}_{S \cap N^*}, R^i_{N^* \setminus S}, R_{-N^*}\big) = f^{\mathrm{DA}}_{N^*}\big(\tilde{R}_S, R^i_{-S}\big) \ \& \ f^{\mathrm{DA}}_{N^*}\big(R^i\big) = f^{\mathrm{DA}}_{N^*}\big(R^i_{N^*}, R_{-N^*}\big).$$

Hence, for each $i \in N^* \cap S$ and R_{-N^*}, we have that

$$f^{\mathrm{DA}}\big(\tilde{R}_{S \cap N^*}, R^i_{N^* \setminus S}, R_{-N^*}\big) \, P_i^i \, f^{\mathrm{DA}}\big(R^i_{N^*}, R_{-N^*}\big).$$

Consequently, $S \cap N^*$ can rationalize $\tilde{R}_{N^* \cap S}$.

Due to Claim 1, we can assume without loss of generality that $S \subseteq N^*$. In addition, fix any $\mathcal{R}^{\tilde{R}_S}$ which is a fixed point of $\xi[\tilde{R}_S]$.

Claim 2. *Pick any agent $i \in S$ and R_i. If there exists a such that $\pi_a(i) \leq q_a + q_{\underline{a}}$ and aR_ib for all b then $R_i \notin \mathcal{R}_i^{\tilde{R}_S}$.*

Proof of the Claim 2. We prove the claim by induction. Assume that for any $j \in S$, R_j and b with $\pi_b(j) \leq \kappa$ where $\kappa < q_b + q_{\underline{a}}$ and bR_jc for all c, $R_j \notin \mathcal{R}_j^{\tilde{R}_S}$. Pick $i \in S$, R_i and a with $\pi_a(i) = \kappa + 1$ and aR_ic for all c. Suppose $R_i \in \mathcal{R}_i^{\tilde{R}_S}$. Then there exists $R \in \mathcal{R}^{\tilde{R}_S}$ such that

$$f^{\mathrm{DA}}(\tilde{R}_S, R_{-S}) \; P_i \; f^{\mathrm{DA}}(R).$$

If $\kappa \leq q_a - 1$, then $f_i^{\mathrm{DA}}(R)$ is i's most preferred object, contradicting the relation above. Hence, let $\kappa \geq q_a$. Let $a_i = f_i^{\mathrm{DA}}(R)$ and, by Lemma 3.14, a_i is the second most preferred object under R_i. Thus, $f_i^{\mathrm{DA}}(\tilde{R}_S, R_{-S}) = a$. Consider $N_a(R, i)$, which is defined in Lemma 3.14(b). The same lemma yields that $|N_a(R, i)| \geq q_a$. By the induction assumption, $N_a(R, i) \cap S = \emptyset$. Then by Lemma 3.14(a), $f_i^{\mathrm{DA}}(\tilde{R}_S, R_{-S}) \neq a$, a contradiction.

By extending the same arguments as above, we obtain the following claim.

Claim 3. *Let $\underline{a}^+ \in O \setminus \{\underline{a}\}$ be an object with $q_{\underline{a}^+} \leq q_b$ for all $b \in O \setminus \{\underline{a}\}$. Pick any agent $i \in S$ and R_i. If $\pi_a(i) \leq q_{\underline{a}^+} + q_{\underline{a}}$ and $\underline{a}R_ib$ for all b, then $R_i \notin \mathcal{R}_i^{\tilde{R}_S}$.*

Claim 4. $S = \emptyset$.

Proof of Claim 4. Suppose otherwise. Let $i \in S$ be the agent with the best priority at $\underline{a}$ among those who are in S. Let $N^- \equiv \{j \in N^* \mid \pi_{\underline{a}}(j) < \pi_{\underline{a}}(i)\}$ and $N^+ \equiv \{j \in N^* \mid \pi_{\underline{a}}(j) > \pi_{\underline{a}}(i)\}$. By construction, $N^- \cap S = \emptyset$. By Lemma 3.11, if $\pi_{\underline{a}}(i) \leq q_{\underline{a}} + q_{\underline{a}^+}$ then $\pi_a(i) \leq q_a + q_{\underline{a}}$ for each $a \in O$. This and Claims 2 and 3 imply $i \notin S$. Thus,

$$\pi_{\underline{a}}(i) > q_{\underline{a}} + q_{\underline{a}^+}.$$

Because $i \in S$, there exists $R \in \mathcal{R}^{\tilde{R}_S}$ such that

$$f^{\mathrm{DA}}(\tilde{R}_S, R_{-S}) \; P_i \; f^{\mathrm{DA}}(R).$$

Let b^* be the most preferred object of i under R. Denote $a^* \equiv f_i^{\mathrm{DA}}(\tilde{R}_S, R_{-S})$. Under the DA algorithm at profile R, a^* rejects i.

Let us partition O into $O^- \equiv \{a \in O \mid \pi_a(i) > q_a + q_{\underline{a}}\}$ and $O^+ \equiv \{a \in O \mid \pi_a(i) \leq q_a + q_{\underline{a}}\}$. By Claim 2, $b^* \in O^-$.

Pick $a \in O^-$. By Lemma 3.11, $\pi_a(i) = \pi_{\underline{a}}(i)$, $\pi_a(j) < \pi_a(i)$ for $j \in N^-$ and $\pi_a(j) > \pi_a(i)$ for $j \in N^+$. Thus, $a \in O^-$ rejects i only in favor of those in N^-.

We now show that in the first step of the DA algorithm at R, at least one object $a \in O^-$ receives q_a applicants in N^-. Suppose otherwise. We know that eventually a^* receives

q_{a^*} applicants in N^-. Thus, at some point, someone in N^- must be rejected by some object. Consider the very first step of the DA in which some agent $j \in N^-$ is rejected by some object b which cannot be in O^-. By then, i is still held by her most preferred object b^*. This means b holds at least q_b agents who have priorities better than $q_b + q_{\underline{a}}$. Consequently, b^* will not reject any applicants who hold its first $q_{b^*} + q_b$ priorities. Because $b \notin O^-$, $\pi_{\underline{a}}(i) \leq q_b + q_{\underline{a}} \leq q_{b^*} + q_a$. Given that $\pi_{\underline{a}}(i) = \pi_{b^*}(i)$ (recall $b^* \in O^-$), we obtain that i gets b^* under profile R, a contradiction.

Pick any $b \in O^+$. By Lemma 3.11, each $j \in N^*$ must have $\pi_b(j) \leq q_b + q_{\underline{a}}$. Suppose that someone in N^* applies to b in the DA algorithm at a report in which the preferences of N^- are R_{N^-}. We know that at least one object $a \in O^-$ receives q_a applicants in $N^- \subset N^*$. Everyone in N^- has one of b's top $q_b + q_{\underline{a}} \leq q_b + q_a$ priorities. Thus, b does not reject any applicant in N^* at any report in which the preferences of N^- are R_{N^-}.

Consider now R and $(\tilde{R}_S, R_{-S})$ in which N^- reports the same preferences. Thus, those objects in O^- which receive more applicants from N^- than their quotas are the same in step 1 of the DA at the two profiles. Thus, any agent in $N^* \supset N^-$ who applies to $b \in O^+$ is not rejected. Thus, the set of agents in N^- who are rejected in the first step of the DA is the same at the two profiles. In fact, this is true at any step of the DA. Hence,

$$f^{\mathrm{DA}}_{N^-}(R) = f^{\mathrm{DA}}_{N^-}(\tilde{R}_S, R_{-S}).$$

Clearly, there cannot be any object $a \in O^-$ such that aP_ib^* and (strictly) less than q_a agents in N^- get a under R. Consequently, if i was assigned to any object $a \in O^-$, it is because b^*R_ia. Hence, $f^{\mathrm{DA}}_i(R)\ P_i\ f^{\mathrm{DA}}_i(\tilde{R}_S, R_{-S})$, a contradiction. □

Notice that acyclicity is both sufficient and necessary for DA to satisfy RGSP. However, acyclicity is only a necessary condition for TTC to satisfy RGSP. In the following example, we present a case in which the TTC rule violates RGSP under acyclic priorities.

Example 3.16. Recall Example 3.10. We have already pointed out that (π, q) is acyclic. We will now show that f^{TTC} does not satisfy RGSP in this example. We will consider a coalition $S = \{1, 3\}$ with preferences:

$$b\ R_1\ a\ R_1\ c\ R_1\ 0$$

$$b\ R_3\ a\ R_3\ c\ R_3\ 0.$$

Agent 2's preferences are one of the following:

$$c\ R_2\ b\ R_2\ a\ R_2\ 0$$

$$b\ \tilde{R}_2\ c\ \tilde{R}_2\ a\ \tilde{R}_2\ 0.$$

Under the TTC, $f^{\mathrm{TTC}}_1(R_1, R_2, R_3) = a$, $f^{\mathrm{TTC}}_3(R_1, R_2, R_3) = b$, $f^{\mathrm{TTC}}_1(R_1, \tilde{R}_2, R_3) = a$ and $f^{\mathrm{TTC}}_3(R_1, \tilde{R}_2, R_3) = c$. Now consider the report:

$$b\ \tilde{R}_1\ c\ \tilde{R}_1\ a\ \tilde{R}_1\ 0$$

$$a\ \tilde{R}_3\ c\ \tilde{R}_3\ b\ \tilde{R}_3\ 0.$$

The outcomes now change. Specifically, $f_1^{\mathrm{TTC}}(\tilde{R}_1, R_2, \tilde{R}_3) = b$, $f_3^{\mathrm{TTC}}(\tilde{R}_1, R_2, \tilde{R}_3) = a$, $f_1^{\mathrm{TTC}}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = c$, and $f_3^{\mathrm{TTC}}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = a$.

Observe that agent 1 prefers $f_1^{\mathrm{TTC}}(\tilde{R}_1, R_2, \tilde{R}_3) = b$ to $f_1^{\mathrm{TTC}}(R_1, R_2, R_3) = a$, while agent 3 prefers $f_3^{\mathrm{TTC}}(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = a$ to $f_3^{\mathrm{TTC}}(R_1, \tilde{R}_2, R_3) = c$. Therefore, $\{1, 3\}$ rationalizes $\tilde{R}_{\{1,3\}}$. ◊

The example above along with Theorems 3.12 and 3.15 demonstrate that the DA rule outperforms the TTC rule in terms of RGSP: both rules do not satisfy RGSP if priorities are cyclic. However, acyclicity is sufficient only for DA to satisfy RGSP. Consequently, DA satisfies RGSP whenever TTC does, but the opposite is not true.

Remark 3.17. It is well known that strong GSP implies nonbossiness in this setting.[17] One may wonder if RGSP also implies nonbossiness. This is not the case. To see this, consider an example with 2 agents and 3 objects. The rule matches agent 1 to her most preferred object but agent 2 to agent 1's second most preferred object. This rule is bossy, but it satisfies RGSP because agent 1 has no incentive to be in any colluding coalition.

3.3 *Division of finite, divisible resources in single-peaked preference domains*

In this subsection, we focus on the setting first studied in Sprumont (1991). Here, the planner allocates a divisible, finite stock of a resource among agents with single-peaked preferences. Specifically, $\Omega > 0$ is the stock, and the set of feasible outcomes is $X = \{x \in \mathbb{R}_+^n \mid \sum_{i \in N} x_i = \Omega\}$. A preference relation R_i is *single-peaked* over $[0, \Omega]$ if there exists $p(R_i) \in [0, \Omega]$ such that for each $x_i, y_i \in [0, \Omega]$, the condition $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$ implies $x_i \, P_i \, y_i$. For each agent $i \in N$, $\mathcal{R}_i$ is the set of single-peaked preferences over $[0, \Omega]$.

A rule that is central in this model is the so-called *Uniform rule*, denoted f^u and defined for each $R \in \mathcal{R}$ and each $i \in N$ as

$$f_i^u(R) = \begin{cases} \min\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \geq \Omega \\ \max\{p(R_i), \lambda\} & \text{if } \sum_{i \in N} p(R_i) \leq \Omega \end{cases}$$

where λ solves $\sum_{i \in N} f_i^u(R) = \Omega$.

Sprumont (1991) shows that the only rule that satisfies efficiency, SP, and equal treatment of equals is the uniform rule.[18] Furthermore, this rule is group strategy-proof. However, we show below that the uniform rule fails RGSP.

Example 3.18. Let $N = \{1, 2, 3\}$ and $\Omega = 13$. Let us now show that f^u fails RGSP. Let $p(R_1) = 2$ and $p(R_2) = 7$. If $p(R_3) = 3$, then $f^u(R_1, R_2, R_3) = (3, 7, 3)$. When $p(\tilde{R}_3) = 5$, we have $f^u(R_1, R_2, \tilde{R}_3) = (2, 6, 5)$. Suppose that agents 1 and 2 report $\tilde{R}_1$ and $\tilde{R}_2$, respectively, so that $p(\tilde{R}_1) = 1.5$ and $p(\tilde{R}_2) = 7.5$. Then $f^u(\tilde{R}_1, \tilde{R}_2, R_3) = (2.5, 7.5, 3)$ and

[17] See Pápai (2000).

[18] A rule f satisfies the equal treatment of equals if whenever $R_i = R_j$ for any $i, j \in N$, and R, $f_i(R) = f_j(R)$.

$f^u(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = (1.5, 6.5, 5)$. Clearly, agent 1 prefers $f^u(\tilde{R}_1, \tilde{R}_2, R_3) = (2.5, 7.5, 3)$ to $f^u(R_1, R_2, R_3) = (3, 7, 3)$, and agent 2 $f^u(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) = (1.5, 6.5, 5)$ to $f^u(R_1, R_2, \tilde{R}_3) = (2, 6, 5)$. Thus, $\{(R_1, R_2)\}$ is a fixed point of $\xi[(\tilde{R}_1, \tilde{R}_2)]$. Hence, f^u violates RGSP. ◊

Due to the negative result above, we search for robust group strategy-proof rules within a larger class of rules that are known to satisfy GSP. Barberá et al. (1997) study the class of rules characterized by SP, efficiency, and replacement monotonicity.[19]

Definition 3.19 (Replacement Monotonicity (Barberá et al. (1997))). A rule f satisfies replacement monotonicity if whenever $f_i(\tilde{R}_i, R_{-i}) \geq f_i(R)$ for some i, R, and $\tilde{R}_i$,

$$f_j(\tilde{R}_i, R_{-i}) \leq f_j(R), \quad \forall j \neq i.$$

As Barberá et al. (1997) demonstrate, replacement monotonicity implies nonbossiness. In addition, it is well known that any rule satisfying all of efficiency, SP, and replacement monotonicity is group strategy-proof. Because the uniform rule satisfies replacement monotonicity, Example 3.18 shows that efficiency, SP, and replacement monotonicity together do not imply RGSP. To understand why this is the case, let us investigate Example 3.18 closely. There, agent 3's preference peak determines whether the sum of the agents' peaks strictly exceeds Ω (overdemanded case) or is strictly less than Ω (underdemanded case). This leads to a possibility in which a joint misreport of agents 1 and 2 helps agent 1 in an underdemanded case and agent 2 in an overdemanded case. This suggests that either the overdemanded or under-demanded cases need to be ruled out. In a literal sense, we cannot achieve this, but one can manufacture a similar situation by appointing one agent who is allocated the "leftover" resource after satiating the remaining agents. This way the agents other than the unlucky one never get (strictly) more resource than their peak. We call such rules *free disposal rules*.

Definition 3.20 (Free Disposal Rule). A rule f is a free disposal rule if there exists $i^* \in N$ such that

$$f_{i^*}(R) = \max\left\{0, \Omega - \sum_{i \neq i^*} p(R_i)\right\}.$$

We now show that any rule satisfying RGSP, efficiency, and replacement monotonicity is a free disposal rule.

Theorem 3.21. *If some rule f satisfies RGSP, efficiency, and replacement monotonicity then f is a free disposal rule.*

Proof. For a proof, see the Appendix. □

We next demonstrate that free disposability along with efficiency, SP, and replacement monotonicity does not lead to RGSP. The following example illustrates this point.

[19] This is the class of sequential allotment rules. See Barberá et al. (1997) for the formal definition.

EXAMPLE 3.22. $N = \{1, 2, \ldots, 5\}$ and $\Omega = 20$.

Case 1: If $\sum_{i=1}^{4} p(R_i) < 20$, then

$$f_i(R) = \begin{cases} p(R_i) & \text{if } i = \{1, \ldots, 4\} \\ 20 - \sum_{j=1}^{4} p(R_j) & \text{if } i = 5. \end{cases}$$

Case 2: If $\sum_{i=1}^{4} p(R_i) \geq 20$, then

$$\begin{aligned} f_1(R) &= \min\big\{p(R_1), 7 + \max\big\{0, 3 - p(R_3)\big\} + \max\big\{0, 10 - p(R_2) - p(R_4)\big\}\big\} \\ f_2(R) &= \min\big\{p(R_2), 7 + \max\big\{0, 3 - p(R_4)\big\} + \max\big\{0, 10 - p(R_1) - p(R_3)\big\}\big\} \\ f_3(R) &= \min\big\{p(R_3), \max\big\{3, 20 - \min\big\{10,\ p(R_2) + p(R_4)\big\} - f_1(R)\big\}\big\} \\ f_4(R) &= \min\big\{p(R_4), \max\big\{3, 20 - \min\big\{10,\ p(R_1) + p(R_3)\big\} - f_2(R)\big\}\big\} \\ f_5(R) &= 0. \end{aligned}$$

The rule above can be interpreted as follows: 7 units of resource are earmarked for each of agents 1 and 2, and 3 units for each of agents 3 and 4. If either agent 1 or 3 demands less than her earmarked amount (i.e., the agent's preference peak is less the earmarked amount), then the other's earmarked amount increases by the difference. The same applies to agents 2 and 4. If the total demand of agents 1 and 3 is less than 10 units, first increase agent 2's earmarked amount by the difference and then agent 4's if 2 demands less than her new earmarked amount. If agents 2 and 4 demand less than 10, then a similar scenario unfolds starting with agent 1's and then with 3's earmarked amount. If the total demand of the first four agents is less than 20, then agent 5 is allocated the excess supply.

Clearly, f is a free disposal rule, but it does not satisfy RGSP, which we show below.

Suppose that $p(R_1) = 7$, $p(R_2) = 5$, $p(R_3) = p(R_4) = 5$, $p(R_5) = 0$, $p(\tilde{R}_1) = 5$ and $p(\tilde{R}_2) = 7$. Then

$$f(R_1, R_2, R_3, R_4, R_5) = (7, 5, 3, 5, 0) \quad \text{and} \quad f(\tilde{R}_1, \tilde{R}_2, R_3, R_4, R_5) = (5, 7, 5, 3, 0).$$

If agents 3 and 4 report their preferences as $\tilde{R}_3$ and $\tilde{R}_4$ where $p(\tilde{R}_3) = p(\tilde{R}_4) = 4$, then

$$f(R_1, R_2, \tilde{R}_3, \tilde{R}_4, R_5) = (7, 5, 4, 4, 0) \quad \text{and} \quad f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4, R_5) = (5, 7, 4, 4, 0).$$

Clearly,

$$\begin{aligned} &f(R_1, R_2, \tilde{R}_3, \tilde{R}_4, R_5)\ P_3\ f(R_1, R_2, R_3, R_4, R_5) \\ &f(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4, R_5)\ P_4\ f(\tilde{R}_1, \tilde{R}_2, R_3, R_4, R_5). \end{aligned}$$

Thus, $\{(R_3, R_4)\}$ is a fixed point of $\xi[(\tilde{R}_3, \tilde{R}_4)]$ yielding that f fails RGSP. ◊

The rule considered in the example above fails RGSP for the following reason: the total allocation to agents 1, 2, and 5 is the same for profiles $(R_1, R_2, R_3, R_4, R_5)$ and $(\tilde{R}_1, \tilde{R}_2, R_3, R_4, R_5)$, but the allocations to agents 3 and 4 differ for the above mentioned two profiles even though their preferences remain unchanged. One of them is better off while the other is worse off under one preference profile, but the situation reverses under the other profile. Thus, agents 3 and 4 want to avoid such risks, which is accomplished through a joint misreport. As we will see next, if a rule is to satisfy RGSP then the scenario we have just described cannot occur. In fact, whenever some group's total allocation increases with a preference change of the group, each remaining agent should receive less of the resource.

Definition 3.23 (Group Replacement Monotonicity). A rule f is group replacement monotonic if whenever $\sum_{i\in S} f_i(\tilde{R}_S, R_{-S}) \geq \sum_{i\in S} f_i(R)$ for some $S \subseteq N$, R, and $\tilde{R}_S$ we have that $f_j(\tilde{R}_S, R_{-S}) \leq f_j(R)$, $\forall j \in N \setminus S$.

Clearly, group replacement monotonicity is more demanding than replacement monotonicity. We next show that RGSP along with efficiency, SP, and replacement monotonicity implies group replacement monotonicity.

Theorem 3.24. *Any rule f that satisfies RGSP, efficiency, and replacement monotonicity must satisfy group replacement monotonicity.*

Proof. For a proof, see the Appendix. □

We are finally ready to show that efficiency, SP, group replacement monotonicity, and free disposal are sufficient for RGSP.

Theorem 3.25. *Suppose that f satisfies all of efficiency, strategy-proofness, group replacement monotonicity, and free disposal. Then f satisfies RGSP.*

Proof. Let i^* be the agent with $f_{i^*}(R) = \max\{0, \Omega - \sum_{i\neq i^*} p(R_i)\}$ for all R. Let us denote $N \setminus \{i^*\}$ by N^*. Suppose that f fails RGSP.

Claim 1. *There exists S and $\tilde{R}_S$ such that $|S| \geq 2$, $S \subset N^*$ and $\mathcal{R}^{\tilde{R}_S} \neq \emptyset$, which is a fixed point of $\xi[\tilde{R}_S]$.*

Proof of Claim 1. Because f is not robust group strategy-proof, there must exist $\bar{S}$ and $\bar{R}_{\bar{S}}$ such that $\mathcal{R}^{\bar{R}_{\bar{S}}} \neq \emptyset$ is a fixed point of $\xi[\bar{R}_{\bar{S}}]$. If $i^* \notin \bar{S}$, then we are done. Suppose $i^* \in \bar{S}$. Because f is strategy-proof, $|\bar{S}| \geq 2$. For each $i \in \bar{S}$, fix any $R_i \in \mathcal{R}_i^{\bar{R}_{\bar{S}}}$. Then there exists $R^i_{-i} \in \mathcal{R}_{-i}^{\bar{R}_{\bar{S}}}$ such that

$$f\big(\bar{R}_{\bar{S}}, R^i_{-\bar{S}}\big) \, P_i \, f\big(R_i, R^i_{-i}\big). \tag{1}$$

Fix $R^*_{i^*}$ with $p(R^*_{i^*}) = 0$. Because f is a free disposal rule, $f(R_i, R^i_{-i}) = f(R_i, R^*_{i^*}, R^i_{N^*\setminus\{i\}})$ and $f(\bar{R}_{\bar{S}}, R^i_{-\bar{S}}) = f(\bar{R}_{\bar{S}\setminus\{i^*\}}, R^*_{i^*}, R^i_{-\bar{S}\cup\{i^*\}})$. Thus, for agent i of type R_i,

$$f\big(\bar{R}_{S\setminus\{i^*\}}, R^*_{i^*}, R^i_{-\bar{S}\cup\{i^*\}}\big)\, P_i\, f\big(R_i, R^*_{i^*}, R^i_{N^*\setminus\{i\}}\big).$$

The proof is complete once we set $S = \bar{S}\setminus\{i\}$ and $\tilde{R}_S = \bar{R}_{\bar{S}\setminus\{i^*\}}$.

Claim 2. *Fix R^* with $p(R^*_i) = \Omega$ for all i. There exists $S \subset N^*$ and $\tilde{R}_S$ such that $R^*_S \in \mathcal{R}^{\tilde{R}_S}_S$ where $\mathcal{R}^{\tilde{R}_S}_S$ is a fixed point of $\xi[\tilde{R}_S]$. In addition, for each $i \in S$, there exists R^i_{-S} with*

$$f\big(\tilde{R}_S, R^i_{-S}\big)\, P_i\, f\big(R^*_S, R^i_{-S}\big).$$

Proof of Claim 2. By Claim 1, there exist $S \subset N^*$, $\tilde{R}_S$, and $\mathcal{R}^{\tilde{R}_S} \neq \emptyset$ such that $\mathcal{R}^{\tilde{R}_S}$ is a fixed point of $\xi[\tilde{R}_S]$. Fix any $i \in S$ and $R_i \in \mathcal{R}^{\tilde{R}_S}_i$. There exists $R^i_{-i} \in \mathcal{R}^{\tilde{R}_S}_{-i}$ such that

$$f\big(\tilde{R}_S, R^i_{-S}\big)\, P_i\, f\big(R_i, R^i_{-i}\big). \tag{2}$$

Because f is a free disposal, efficient rule, for (2) to hold, it must be that $p(R_i) + \sum_{j\in N^*\setminus\{i\}} p(R^i_j) > \Omega$ and $p(R_i) > f_i(R_i, R^i_{-i})$. In addition, (2) and the single-peakedness of preferences yield $f_i(\tilde{R}_S, R^i_{-S}) > f(R_i, R^i_{-i})$. Furthermore, because $p(R^*_i) \geq p(R_i) > f_i(R_i, R^i_{-i})$, $f(R^*_i, R^i_{-i}) = f(R_i, R^i_{-i})$ (Lemma A.1(b)). As a result,

$$f\big(\tilde{R}_S, R^i_{-S}\big)\, P^*_i\, f\big(R^*_i, R^i_{-i}\big).$$

This proves that $R^*_S \in \mathcal{R}^{\tilde{R}_S}_S$. Consider $f(R_i, R^*_{S\setminus\{i\}}, R^i_{-S})$. By replacing R^i_j by R^*_j sequentially for each $j \in S\setminus\{i\}$ and by using replacement-monotonicity and Lemma A.1(c), we obtain that

$$f_i\big(R_i, R^i_{-i}\big) \geq f_i\big(R_i, R^*_{S\setminus\{i\}}, R^i_{-S}\big).$$

We know that $p(R^*_i) \geq p(R_i) > f_i(R_i, R^i_{-i})$. By Lemma A.1(b),

$$f\big(R^*_S, R^i_{-S}\big) = f\big(R_i, R^*_{S\setminus\{i\}}, R^i_{-S}\big).$$

Consequently,

$$f\big(\tilde{R}_S, R^i_{-S}\big)\, P^*_i\, f\big(R^*_S, R^i_{-i}\big).$$

This completes the proof. We are finally ready to prove the theorem.

Thanks to Claim 2, fix $S \subset N^*$, $\tilde{R}_S$ and R^i_{-S} such that for all $i \in S$,

$$f\big(\tilde{R}_S, R^i_{-S}\big)\, P_i\, f\big(R^*_S, R^i_{-S}\big). \tag{3}$$

Consider $\sum_{j\in N\setminus S} f_j(R^*_S, R^i_{-S})$. Let j^* be the agent in S such that

$$\sum_{j\in N\setminus S} f_j\big(R^*_S, R^{j^*}_{-S}\big) \geq \sum_{j\in N\setminus S} f_j\big(R^*_S, R^i_{-S}\big), \quad \forall i \in S.$$

By group replacement monotonicity, it must be that

$$f_j\big(R_S^*, R_{-S}^{j^*}\big) \leq f_j\big(R_S^*, R_{-S}^{i}\big), \quad \forall j, i \in S. \tag{4}$$

Furthermore, we know that

$$p(\tilde{R}_i) \leq p\big(R_i^*\big), \quad \forall i \in S$$

and

$$f_{j^*}\big(\tilde{R}_S, R_{-S}^{j^*}\big) > f_{j^*}\big(R_S^*, R_{-S}^{j^*}\big).$$

Clearly, $(R_S^*, R_{-S}^{j^*})$ is an overdemanded case. However, for the last inequality to hold, there must exist some $\hat{j} \in S$ ($\hat{j} \neq j^*$) with

$$p(\tilde{R}_{\hat{j}}) < f_{\hat{j}}\big(R_S^*, R_{-S}^{j^*}\big).$$

By combining this with (4), we have that

$$p(\tilde{R}_{\hat{j}}) < f_{\hat{j}}\big(R_S^*, R_{-S}^{j^*}\big) \leq f_{\hat{j}}\big(R_S^*, R_{-S}^{\hat{j}}\big).$$

Efficiency (if there is an overdemand) or the fact that $\hat{j} \neq i^*$ imply that

$$f_{\hat{j}}\big(\tilde{R}_S, R_{-S}^{\hat{j}}\big) \leq p(\tilde{R}_{\hat{j}}) < f_{\hat{j}}\big(R_S^*, R_{-S}^{\hat{j}}\big) \leq p\big(R_{\hat{j}}^*\big).$$

By single-peakedness, we have

$$f\big(R_S^*, R_{-S}^{\hat{j}}\big)\, P_{\hat{j}}^*\, f\big(\tilde{R}_S^*, R_{-S}^{\hat{j}}\big)$$

which contradicts (3). Thus, f must satisfy RGSP. □

The theorem above shows that any robust group strategy-proof rule in the class of rules satisfying SP, efficiency, and replacement monotonicity must satisfy the additional requirements of group replacement monotonicity and free disposability. The new requirements are restrictive but many rules satisfy them (along with efficiency and SP). For instance, consider the following rule: first appoint an agent who is allocated 0 or the leftover stock depending on whether the other agents can be fully satiated. If the others demand more than the total stock, then use the uniform rule to determine the allocation among these lucky agents.[20]

4. Discussion and conclusion

We have proposed a new notion of GSP, RGSP, that takes into account asymmetric information about types. While our notion is belief-free, one could think of various notions of GSP in the interim stage by placing requirements on the beliefs of the members of a blocking coalition. A natural restriction is to assume that the agents have common

[20]Here, the uniform rule can be replaced by any fixed path rule of Moulin (1999).

beliefs, in which case the new definition of interim GSP would require that no coalition profit by deviating from truth telling for any common belief. However, this approach has its own problem: because expected utilities are used, the Bernoulli utilities become very important. To be specific, recall that Ann and Beth were able to collude in Example 1.1. However, now change the example by altering the Bernoulli utilities to $u_A(b) = u_B(a) = 4$. Then, as long as Ann and Beth have a common belief, they cannot collude at $(\tilde{R}_A, \tilde{R}_B)$ and improve. Thus, the new notion of GSP would depend on cardinal utilities, which is somewhat unsatisfactory. This discussion demonstrates potential challenges of defining interim notions of GSP under the common knowledge assumption.

Another natural way to modify RGSP is to consider the information blocking coalitions share. In our notion, the deviating coalition members rationalize their cooperation but do not explicitly share any information. One possible requirement is that the blocking coalition members reveal their types to each other truthfully.[21] Clearly, such a modification should enlarge the set of rules satisfying the new version of RGSP. However, this does not happen in some of the settings and rules we consider in this paper.[22]

One open question is whether the rules satisfying RGSP and GSP coincide under certain conditions. Specifically, it is desirable to identify sufficient conditions on the rules or the preference domain that guarantee the equivalence of robust group strategy-proof and group strategy-proof rules in general environments as Barberá et al. (2016) do for the equivalence of SP and GSP. We leave this question for future research.

Appendix

A.1 *Proofs*

Proof of Lemma 3.4. To prove this lemma, it suffices to show that $v_i^{R_i}(\cdot)$ is strictly quasi-concave on Y. Fix any $\lambda \in (0, 1)$ and $y, y' \in Y$ such that $\lambda y + (1-\lambda)y' \in Y$. Because u_i is strictly quasi-concave, we have

$$u_i^{R_i}\big(\lambda C(y)/n + (1-\lambda)C\big(y'\big)/n,\, \lambda y + (1-\lambda)y'\big) > \min\big\{u_i^{R_i}\big(C(y)/n,\, y\big),\, u_i^{R_i}\big(C\big(y'\big)/n,\, y'\big)\big\}.$$

The convexity of $C(y)$ and the assumption that u_i is strictly decreasing t_i imply that

$$u_i^{R_i}\big(C\big(\lambda y + (1-\lambda)y'\big)/n,\, \lambda y + (1-\lambda)y'\big) \geq u_i^{R_i}\big(\lambda C(y)/n + (1-\lambda)C\big(y'\big)/n,\, \lambda y + (1-\lambda)y'\big).$$

The two inequalities above along with the definition of $v_i^{R_i}(\cdot)$ imply that $v_i^{R_i}$ is strictly quasi-concave. □

[21] In this case, the credibility issue will arise.

[22] Formally, a rule f satisfies *fine robust group strategy-proofness (FRGSP)* if there is no S, R_S, $\tilde{R}_S$ and $(R^i_{-S})_{i\in S}$ such that $u_i(f(\tilde{R}_S, R^i_{-S}), R_i) > u_i(f(R_S, R^i_{-S}), R_i)$ for all $i \in S$. Acyclicity remains the necessary and sufficient condition for DA to satisfy FRGSP while it is only necessary for TTC. Within the class of efficient, strategy-proof, and replacement monotonic rules, only the group replacement monotonic, free disposal rules satisfy FRGSP. These results can be provided upon request.

PROOF OF LEMMA 3.11. (i) $\Longrightarrow$ (ii): Fix any $a, b \in O \setminus 0$ and pick i with $\pi_a(i) \leq q_a + q_b$. In contrast to the lemma, suppose $\pi_b(i) > q_a + q_b$. Then there exists k with $\pi_a(k) > q_a + q_b$ but $\pi_b(k) \leq q_a + q_b$. First let us show that $\pi_a(i) = q_a + q_b$ and $\pi_a(k) = q_a + q_b + 1$. In contrast assume that $\pi_a(i) < q_a + q_b$ or $\pi_a(k) > q_a + q_b + 1$. Then let j be the agent with $\pi_a(j) = \max\{\pi_a(i) + 1, q_a\}$. Observe that $j \neq k$. Pick now a set $N_a \subset U_a(j) \setminus \{i, j, k\}$ with $|N_a| = q_a - 1$. Consider the set $U_b(i)$ which satisfies $|U_b(i)| \geq q_a + q_b$. As a result,

$$\left|U_b(i) \setminus \left(\{i, j, k\} \cup N_a\right)\right| \geq q_a + q_b - 2 - (q_a - 1) = q_b - 1.$$

Thus, we can find $N_b \subseteq U_b(i) \setminus \{i, j, k\}$ such that $N_b \cap N_a = \emptyset$ and $|N_b| = q_b - 1$. Consequently, i, j, k and a, b satisfy both (C) and (S). This contradicts (π, q) is acyclic. Thus, $\pi_a(i) = q_a + q_b$ and $\pi_a(k) = q_a + q_b + 1$. Also, the same proof implies that $\pi_b(k) = q_a + q_b$ and $\pi_b(i) = q_a + q_b + 1$. Furthermore, for any i', $\pi_a(i') < q_a + q_b$ if and only if $\pi_b(i') < q_a + q_b$. Pick k^* with $\pi_b(k^*) = q_a + q_b - 1$. Observe that $\pi_a(k^*) < q_a + q_b$. Thus, $\pi_a(k^*) \leq q_a + q_b - 1 < \pi_a(i) < \pi_a(k)$ and $\pi_b(k^*) = q_a + q_b - 1 < \pi_b(k) < \pi_b(i)$. Furthermore, observe that $|U_a(i) \setminus \{i, k, k^*\}| = |U_b(k^*) \setminus \{i, k, k^*\}| = q_a + q_b - 2$. We can thus find disjoint sets $\bar{N}_a \subset U_a(i)$ and $\bar{N}_b \subset U_b(k^*)$ such that $|\bar{N}_a| = q_a - 1$ and $|\bar{N}_b| = q_b - 1$. Then i, k, k^*, $\bar{N}_a$ and $\bar{N}_b$ satisfy both (C) and (S). This contradicts the acyclicity of (π, q).

We now show that $\pi_a(i) > q_a + q_b$ implies $\pi_a(i) = \pi_b(i)$. Suppose otherwise. Because of the first part of this lemma, there must exist j and k with $q_a + q_b < \pi_a(j) < \pi_a(k)$ and $q_a + q_b < \pi_b(k) < \pi_b(j)$. Let i be the agent with $\pi_b(i) = q_a + q_b$. By the first part, $\pi_a(i) \leq q_a + q_b$. Consider i, j, k, a and b, and note that condition (C) is satisfied because $\pi_a(i) < \pi_a(j) < \pi_a(k)$ and $\pi_b(k) < \pi_b(j)$. In addition, $\pi_a(k) > \pi_a(j) > q_a + q_b \geq \pi_a(i)$ implies $|U_a(j)| \geq q_a + q_b - 1$. At the same time, $\pi_b(j) > \pi_b(k) > q_a + q_b = \pi_b(i)$ implies that $|U_b(i)| = q_a + q_b - 1$. Thus, we can find disjoint sets $N_a \subset U_a(j)$ and $N_b \subset U_b(i)$ such that $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. Consequently, Condition (S) is satisfied contradicting that (π, q) is acyclic.

(ii) $\Longrightarrow$ (i): Suppose that (π, q) is cyclic. Fix $i, j, k \in N$ and $a, b \in O$ satisfying conditions (C) and (S). If $\pi_a(k) > q_a + q_b$ then $\pi_b(k) = \pi_a(k)$ as required by (ii). In addition, (ii) gives that either $\pi_a(i) \leq q_a + q_b$ or $\pi_a(i) = \pi_b(i)$. In both cases, $\pi_b(i) < \pi_b(k)$. The same argument yields that $\pi_b(j) < \pi_b(k)$. This contradicts Condition (C). Consequently, $\pi_a(k) \leq q_a + q_b$. Then $\pi_a(j) \leq q_a + q_b$ and $\pi_a(i) \leq q_a + q_b$, which, together with (ii), implies that $\pi_b(i) \leq q_a + q_b$. Thus, $|U_a(j)| \leq q_a + q_b$ and $|U_b(i)| \leq q_a + q_b$. In addition, by (ii), the first $q_a + q_b$ priorities in both objects must be given to the same agents. Thus, $|U_a(j) \cup U_b(i) \setminus \{i, j, k\}| \leq q_a + q_b - 3$. Then there cannot exist two disjoints sets $N_a \subseteq U_a(j) \setminus \{i, j, k\}$ and $N_b \subseteq U_b(i) \setminus \{i, j, k\}$ with $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. Consequently, Condition (S) is violated. □

PROOF OF LEMMA 3.13. Pick any $a \in O$ and any i with $\pi_{\underline{a}}(i) \leq q_{\underline{a}} + q_{\bar{a}}$. Because $q_a \leq q_{\bar{a}}$, Lemma 3.11 yields that

$$\pi_{\underline{a}}(i) \leq q_{\underline{a}} + q_a \quad \Longrightarrow \quad \pi_a(i) \leq q_{\underline{a}} + q_a \leq q_{\underline{a}} + q_{\bar{a}}$$

and

$$\pi_{\underline{a}}(i) > q_{\underline{a}} + q_a \quad \Longrightarrow \quad \pi_a(i) = \pi_{\underline{a}}(i) \leq q_{\underline{a}} + q_{\bar{a}}.$$

On the other hand, if $\pi_{\underline{a}}(i) > q_{\underline{a}} + q_{\bar{a}}$, by Lemma 3.11, $\pi_a(i) = \pi_{\underline{a}}(i) > q_{\underline{a}} + q_{\bar{a}}$. □

Proof of Lemma 3.14. (a) If $f_i^{\mathrm{DA}}(R) \neq a$, then a must reject i at some point. Afterwards, a holds q_a agents who have better priorities than i at a. Let i apply to her second most preferred object b. By Lemma 3.11, the first $q_a + q_b$ priorities at both a and b belong to the same group of agents. Given that q_a of these are held by a, b does not reject any agent who has one of its first $q_a + q_b$ priorities. Hence, i is assigned to its second most preferred object.

(b) Suppose that $|N_a(R, i)| \leq q_a - 1$. Clearly, i cannot be rejected by a in Step 1 of the DA algorithm. Pick any $j \notin N_a(R, i)$ such that $\pi_a(j) < \pi_a(i)$ and j prefers some object b to a. Let $\hat{N} \equiv \{k \in N \mid \pi_a(k) \leq q_a + q_b\}$. Clearly, $\pi_a(j) < \pi_a(i) \leq q_a + q_b$, and by Lemma 3.11, N^* which includes i and j holds the first $q_a + q_b$ priorities at both a and b. Suppose j is rejected from b and then applies to a. In the DA algorithm, j is rejected only in favor of q_b agents who in this case are in $\hat{N}$. However, $|\{k \in N \mid \pi_a(k) < \pi_a(i)\}| \leq q_a + q_b - 1$. Given that q_b of these agents are held by b, there can be only $q_a - 1$ applicants whose priorities are better than i's at a. Hence, i cannot be rejected by a when j applies to a. Given that j is picked arbitrarily, the proof is complete. □

The following well-known results are used in some of the proofs in Section 3.3.

Lemma A.1. *Consider the Sprumont setting. If some rule f is strategy-proof, efficient and replacement monotonic, then f satisfies the following conditions.*

(a) f is peak-only, i.e., for any two profiles R and $\tilde{R}$ with $p(R_i) = p(\tilde{R}_i)$ for all $i \in N$, $f(R) = f(\tilde{R})$.

(b) If $p(R_i) \leq f_i(R)$ and $p(\tilde{R}_i) \leq f_i(R)$ for some i, R and $\tilde{R}_i$ then $f(R) = f(\tilde{R}_i, R_{-i})$. Similarly, if $p(R_i) \geq f_i(R)$ and $p(\tilde{R}_i) \geq f_i(R)$ for some i, R and $\tilde{R}_i$ then $f(R) = f(\tilde{R}_i, R_{-i})$.

(c) If $p(\tilde{R}_i) \geq f_i(R) \geq p(R_i)$ for some i, R and $\tilde{R}_i$, then $p(R_i) \leq f_i(\tilde{R}_i, R_{-i}) \leq p(\tilde{R}_i)$. Similarly, if $p(\tilde{R}_i) \leq f_i(R) \leq p(R_i)$ for some i, R and $\tilde{R}_i$, then $p(R_i) \geq f_i(\tilde{R}_i, R_{-i}) \geq p(\tilde{R}_i)$.

Proof of Theorem 3.21. We will prove the theorem in several steps.

Claim 1. *If there exist R_i, R_j, R^i_{-ij} and R^j_{-ij} with $f_i(R_i, R_j, R^i_{-ij}) > p(R_i)$ and $f_j(R_i, R_j, R^j_{-ij}) < p(R_j)$, then $f_i(R_i, R_j, R^j_{-ij}) = 0$.*

Proof of Claim 1. Suppose otherwise. Because $f_i(R_i, R_j, R^i_{-ij}) > p(R_i) \geq 0$ it must be that $f_j(R_i, R_j, R^i_{-ij}) < \Omega$. By efficiency,

$$f_i\big(R_i, R_j, R^j_{-ij}\big) \leq p(R_i) < f_i\big(R_i, R_j, R^i_{-ij}\big)$$

$$f_j\big(R_i, R_j, R^j_{-ij}\big) < p(R_j) \leq f_j\big(R_i, R_j, R^i_{-ij}\big)$$

and for any $k \in N \setminus \{i, j\}$,

$$p\big(R^i_k\big) \leq f_k\big(R_i, R_j, R^i_{-ij}\big)$$
$$f_k\big(R_i, R_j, R^j_{-ij}\big) < p\big(R^j_k\big).$$

Let $\tilde{R}^i_{-ij}$ and $\tilde{R}^j_{-ij}$ be preferences such that $p(\tilde{R}^i_k) = f_k(R_i, R_j, R^i_{-ij})$ and $p(\tilde{R}^j_k) = f_k(R_i, R_j, R^j_{-ij})$ for each $k \in N \setminus \{i, j\}$. By Lemma A.1(b), it must be that $f(R_i, R_j, R^i_{-ij}) = f(R_i, R_j, \tilde{R}^i_{-ij})$ and $f(R_i, R_j, R^j_{-ij}) = f(R_i, R_j, \tilde{R}^j_{-ij})$. Fix $\epsilon > 0$ such that $p(R_i) < f_i(R_i, R_j, R^i_{-ij}) - \epsilon$, $0 < f_i(R_i, R_j, R^j_{-ij}) - \epsilon$, $f_j(R_i, R_j, R^j_{-ij}) + \epsilon < p(R_j)$ and $f_j(R_i, R_j, R^i_{-ij}) + \epsilon < \Omega$. Fix $\tilde{R}_i$ and $\tilde{R}_j$ with $p(\tilde{R}_i) = f_i(R_i, R_j, R^j_{-ij}) - \epsilon$ and $p(\tilde{R}_j) = f_j(R_i, R_j, R^i_{-ij}) + \epsilon$. Consider now $f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij})$. By construction of ϵ and $\tilde{R}_j$, we must have that $p(R_i) + p(\tilde{R}_j) + \sum_{k \neq i,j} p(\tilde{R}^i_k) < \Omega$. By Lemma A.1(c), efficiency and replacement monotonicity, we find that $f_i(R_i, \tilde{R}_j, \tilde{R}^i_{-ij}) = f_i(R_i, R_j, \tilde{R}^i_{-ij}) - \epsilon > p(R_i)$, $f_j(R_i, \tilde{R}_i, \tilde{R}^i_{-ij}) = p(\tilde{R}_j)$ and $f_k(R_i, \tilde{R}_i, \tilde{R}^i_{-ij}) = p(\tilde{R}^i_k)$ for all $k \neq i, j$. Because the preferences are single-peaked, $f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij})\ P_i\ f(R_i, R_j, \tilde{R}^i_{-ij})$. Finally, let us consider $(\tilde{R}_i, \tilde{R}_j, \tilde{R}^i_{-ij})$ which differs from $(R_i, \tilde{R}_j, \tilde{R}^i_{-ij})$ only in the preferences of i. Because $p(\tilde{R}_i) < p(R_i) < f_i(R_i, \tilde{R}_i, \tilde{R}^i_{-ij})$, by Lemma A.1(b), it must be that $f(\tilde{R}_i, \tilde{R}_j, \tilde{R}^i_{-ij}) = f(R_i, \tilde{R}_j, \tilde{R}^i_{-ij})$. Consequently, we find $f(\tilde{R}_i, \tilde{R}_j, \tilde{R}^i_{-ij})\ P_i\ f(R_i, R_j, \tilde{R}^i_{-ij})$. By using the mirror image arguments, we find that $f(\tilde{R}_i, \tilde{R}_j, \tilde{R}^j_{-ij})\ P_j\ f(R_i, R_j, \tilde{R}^j_{-ij})$. The last two findings mean that $\{(R_i, R_j)\}$ is a fixed point of $\xi[\tilde{R}_{ij}]$ contradicting the RGSP of f.

Claim 2. *For each R with $\sum_{j \in N} p(R_j) < \Omega$ there exists a unique agent i^R with $f_{i^R}(R) > p(R_{i^R})$.*

Proof of Claim 2. Because $\sum_{i \in N} f_i(R) = \Omega$, efficiency implies the existence of an agent j with $p(R_j) < f_j(R)$. In contrast to the claim suppose that there are more than two such agents. Select any one of them randomly and denote the selected agent by i^*. Fix $\epsilon > 0$ so that $\min_{i \in \{j | p(R_j) < f_j(R)\}} \{f_i(R)\} - 4\epsilon > 0$. Let R^1 be a preference profile such that

$$p\big(R^1_{i^*}\big) = f_{i^*}(R) - 3\epsilon$$
$$p\big(R^1_i\big) = f_i(R) \quad \forall i \neq i^*.$$

Observe here that

$$\sum_{i \in N} p\big(R^1_i\big) = \Omega - 3\epsilon.$$

In addition, Lemma A.1(b) implies that

$$f\big(R^1\big) = f(R).$$

Consequently,

$$f_{i^*}(R^1) = f_{i^*}(R) = p(R^1_{i^*}) + 3\epsilon$$
$$f_i(R^1) = f_i(R) = p(R^1_i) \quad \forall i \neq i^*.$$

Pick any $j^* \neq i^*$ with $f_{j^*}(R) > p(R_{j^*})$ which is feasible because of the supposition. Fix a preference profile R^2 such that

$$p(R^2_{j^*}) = f_{j^*}(R^1) + 2\epsilon$$
$$p(R^2_i) = f_i(R^1) \quad \forall i \neq j^*.$$

Observe that R^2 and R^1 differ only in agent j^*'s peak. In addition, $\sum_{i\in N} p(R^2_i) = \Omega - \epsilon < \Omega$. Now Lemma A.1(c), efficiency and replacement monotonicity imply that

$$f_{i^*}(R^2) = f_{i^*}(R^1) - 2\epsilon = f_{i^*}(R) - 2\epsilon = p(R^2_{i^*}) + \epsilon$$
$$f_{j^*}(R^2) = f_{j^*}(R^1) + 2\epsilon = f_{j^*}(R) + 2\epsilon = p(R^2_{j^*})$$
$$f_i(R^2) = f_i(R^1) = f_i(R) = p(R^2_i) \quad \forall i \neq i^*, j^*.$$

Pick any agent $k^* \neq i^*, j^*$. Fix a preference profile R^3 such that

$$p(R^3_{k^*}) = f_{k^*}(R^2) + 2\epsilon$$
$$p(R^3_i) = f_i(R^2) \quad \forall i \neq k^*.$$

Observe that R^3 and R^2 differ only in agent k^*'s peak. In addition, $\sum_{i\in N} p(R^3_i) = \Omega + \epsilon > \Omega$. By Lemma A.1(c), we know that

$$f_{k^*}(R^3) \geq f_{k^*}(R^2).$$

By replacement monotonicity we know that

$$f_i(R^3) \leq f_i(R^2) \quad \forall i \neq k^*.$$

Given that k^*'s allocation at most increases by 2ϵ, i^*'s decreases by 2ϵ at most. Thus, $f_{i^*}(R^3) \geq f_{i^*}(R^2) - 2\epsilon = f_{i^*}(R) - 4\epsilon > 0$ (by construction of ϵ). If $f_i(R^3) < f_i(R^2) = p(R^3_i)$ for some $i \neq k^*$ then we reach contradiction to Claim 1 by comparing R^2 and R^3. Thus, we must have that

$$f_i(R^3) = f_i(R^2) = p(R^3_i) \quad \forall i \neq k^*, i^*. \tag{5}$$

We could have reached R^3 from R^1 by changing k^*'s preferences first and then j^*'s. This would give us

$$f_i(R^3) = p(R^3_i) \quad \forall i \neq j^*, i^*.$$

By combining this with (5), we find that

$$f_i(R^3) = p(R_i^3) = f_i(R) \quad \forall i \neq i^*, j^*, k^*$$
$$f_{j^*}(R^3) = p(R_{j^*}^3) = f_{j^*}(R) + 2\epsilon$$
$$f_{k^*}(R^3) = p(R_{k^*}^3) = f_{k^*}(R) + 2\epsilon$$
$$f_{i^*}(R^3) = p(R_{i^*}^3) - \epsilon = f_{i^*}(R) - 4\epsilon.$$

Consider a preference profile R^4 which satisfies

$$p(R_{i^*}^4) = f_{i^*}(R) + 2\epsilon > f_{i^*}(R^3)$$
$$p(R_i^4) = f_i(R^3) \quad \forall i \neq k^*.$$

The profiles R^3 and R^4 differ in i^*'s peak. By Lemma A.1, we know that

$$f(R^4) = f(R^3).$$

Consequently, $f_{i^*}(R^4) < f_{i^*}(R)$ and $f_{j^*}(R^4) > f_{j^*}(R)$. These inequalities will reverse if we switch the places of i^* and j^* in the sequence of preference changes that reach R^4 from R. This is the contradiction we are looking for.

Claim 3. *There exists unique agent i^* such that whenever $\sum_{i\in N} p(R_i) < \Omega$, $f_{i^*}(R) = \Omega - \sum_{i\neq i^*} p(R_i)$.*

Proof of Claim 3. By Claim 2 and efficiency, we know that for each R, there exists i^R, such that $f_{i^R}(R) = \Omega - \sum_{i\neq i^R} p(R_i)$. To prove the current claim, it suffices to show that for any two profiles R and $\tilde{R}$ with $\sum_{i\in N} p(R_i) < \Omega$ and $\sum_{i\in N} p(\tilde{R}_i) < \Omega$ we have $i^R = i^{\tilde{R}}$. Let us partition $N \setminus \{i^R\}$ into two sets: $N^- = \{i \in N \setminus \{i^R\} \mid p(\tilde{R}_i) < p(R_i)\}$ and $N^+ = \{i \in N \setminus \{i^R\} \mid p(\tilde{R}_i) \geq p(R_i)\}$. If $N^- \neq \emptyset$, then fix a random agent $i \in N^-$. Let us now consider $f(\tilde{R}_i, R_{-i})$. Because i's preference peak decreased there is still an underdemand. Since $p(\tilde{R}_i) < p(R_i) = f_i(R)$, Lemma A.1(c) implies $f_i(\tilde{R}_i, R_{-i}) \leq f_i(R)$. Then by replacement monotonicity $f_{i^R}(\tilde{R}_i, R_{-i}) \geq f_{i^R}(R) > p(R_{i^R})$. This immediately implies $i^{(\tilde{R}_i, R_{-i})} = i^R$. By changing the preferences of those in N^- sequentially and using the same arguments we find that

$$i^{(\tilde{R}_{N^-}, R_{-N^-})} = i^R.$$

We now change R_{-N^+} to $\tilde{R}_{-N^+}$ sequentially one agent's preferences at a time. Observe that along these changes we will always have an underdemand because $\sum_{i\in N} p(\tilde{R}_i) < \Omega$. Pick any agent $i \in N^+$. If $p(R_i) = p(\tilde{R}_i)$ then by Lemma A.1(a) and nonbossiness,

$$f(\tilde{R}_{\{i\}\cup N^-}, R_{-\{i\}\cup N^-}) = f(\tilde{R}_{N^-}, R_{-\cup N^-}) \;\&\; i^{(\tilde{R}_{\{i\}\cup N^-}, R_{-\{i\}\cup N^-})} = i^R.$$

If $p(R_i) < p(\tilde{R}_i)$, by (b) and (c) of Lemma A.1, we find

$$p(R_i) = f_i(R) = f_i(\tilde{R}_{N^-}, R_{-\cup N^-}) \leq f_i(\tilde{R}_{\{i\}\cup N^-}, R_{-\{i\}\cup N^-}) \leq p(\tilde{R}_i).$$

Because there is an underdemand, we must have

$$f_i(\tilde{R}_{\{i\}\cup N^-}, R_{-\{i\}\cup N^-}) = p(\tilde{R}_i).$$

By replacement monotonicity and efficiency, we find that

$$f_j(\tilde{R}_{\{i\}\cup N^-}, R_{-\{i\}\cup N^-}) = p(\tilde{R}_j), \quad \forall j \neq i^R.$$

Consequently,

$$i^{(\tilde{R}_{\{i\}\cup N^-}, R_{-\{i\}\cup N^-})} = i^R.$$

By using the same arguments sequentially, we find that

$$i^{(R_{i^R}, \tilde{R}_{-i^R})} = i^R.$$

Finally, because $p(\tilde{R}_j) = f_j(R_{i^R}, \tilde{R}_{-i^R})$ for all $j \neq i^R$, we cannot have the case in which $p(\tilde{R}_{i^R}) > f_{i^R}(R_{i^R}, \tilde{R}_{-i^R})$ because $\sum_{j\in N} p(\tilde{R}_j) < \Omega$. Then by Lemma A.1(b), we have that

$$f(\tilde{R}) = f(R_{i^R}, \tilde{R}_{-i^R}) \ \& \ i^{\tilde{R}} = i^R.$$

Let i^* be the agent for whom $f_{i^*}(R) > p(R_{i^*})$ for all R with $\sum_{i\in N} p(R_i) < \Omega$.

Claim 4. *For each R with $\sum_{j\neq i^*} p(R_j) < \Omega$, $f_{i^*}(R) = \Omega - \sum_{j\neq i^*} p(R_j)$ and $f_i(R) = p(R_i)$ for all $i \neq i^*$.*

Proof of Claim 4. Claim 3 proves the current claim when $\sum_{i\in N} p(R_i) < \Omega$. In addition, efficiency yields the claim if $\sum_{i\in N} p(R_i) = \Omega$. Let us focus on the case in which $\sum_{i\in N} p_i(R_i) > \Omega$ but $\sum_{i\neq i^*} p(R_i) < \Omega$. If $f_{i^*}(R) < \Omega - \sum_{i\neq i^*} p(R_i)$ then by feasibility, there exists $i \neq i^*$ with $f_i(R) > p(R_i)$ which cannot happen when there is an overdemand. Hence, $f_{i^*}(R) \geq \Omega - \sum_{i\neq i^*} p(R_i)$. Set $N^* \equiv \{i \in N \setminus \{i^*\} \mid p(R_i) > 0\}$. If $N^* = \emptyset$, then $p(R_{i^*}) > \Omega$ and $p(R_i) = 0$ for all $i \neq i^*$. This case cannot happen in our model. Suppose $|N^*| \geq 2$. In contrast to the claim, suppose $f_{i^*}(R) > \Omega - \sum_{j\neq i^*} p(R_j)$. Fix any $\epsilon > 0$ such that $\epsilon < \min_{i\in N^*}\{p(R_i)\}$ and $\Omega - \sum_{j\neq i^*} p(R_j) + \epsilon < p(R_{i^*})$. Let $\tilde{R}_{i^*}$ be preferences of i^* such that $p(\tilde{R}_{i^*}) = \Omega - \sum_{j\neq i^*} p(R_j) + \epsilon < p(R_{i^*})$. Pick any arbitrary agent $j^* \in N^*$. Consider $\tilde{R}_{j^*}$ with $p(\tilde{R}_{j^*}) = 0$. We now investigate $(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*j^*})$ and observe that

$$p(\tilde{R}_{i^*}) + p(\tilde{R}_{j^*}) + \sum_{i\neq i^*, j^*} p(R_i) = p(\tilde{R}_{i^*}) + \sum_{i\neq i^*, j^*} p(R_i) = \Omega - \big(p(R_{j^*}) - \epsilon\big) < \Omega.$$

By Claim 3,

$$\begin{aligned}
f_{j^*}(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*j^*}) &= 0\\
f_i(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*j^*}) &= p(R_i) \quad \forall i \neq i^*, j^*\\
f_{i^*}(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*j^*}) &= \Omega - \sum_{j\neq i^*} f_j(\tilde{R}_{j^*}, \tilde{R}_{i^*}, R_{-i^*j^*}) > p(\tilde{R}_{i^*}).
\end{aligned}$$

Consider $f(\tilde{R}_{i^*}, R_{-i^*})$. By construction, $p(\tilde{R}_{i^*}) + \sum_{i \neq i^*} p(R_i) = \Omega + \epsilon$. If $f_{i^*}(\tilde{R}_{i^*}, R_{-i^*}) < \Omega - \sum_{i \neq i^*} p(R_i)$, then by feasibility, we will have that $p(R_j) < f_j(\tilde{R}_{i^*}, R_{-i^*})$ for some $j \neq i^*$ which cannot happen when there is an overdemand. If $f_{i^*}(\tilde{R}_{i^*}, R_{-i^*}) > \Omega - \sum_{i \neq i^*} p(R_i) > 0$, then for some $j \in N^*$, $p(R_j) > f_j(\tilde{R}_{i^*}, R_{-i^*})$. If $j \neq j^*, i^*$, there would be a contradiction with Claim 1 (compare $(\tilde{R}_j, \tilde{R}_i, R_{-i^*j^*})$ and $(R_j, \tilde{R}_i, R_{-i^*j^*})$. However, for all $k^* \neq j^*$, we must have that $p(R_{k^*}) = f_{k^*}(\tilde{R}_{i^*}, R_{-i^*})$. Recall that j^* is picked arbitrarily from N^* and $|N^*| \geq 2$. We could have chosen some other agent in N^*. This would give us that $p(R_{j^*}) = f_{j^*}(\tilde{R}_{i^*}, R_{-i^*})$. Consequently,

$$f_i(\tilde{R}_{i^*}, R_{-i^*}) = p(R_i) \quad \forall i \neq i^*$$

$$f_{i^*}(\tilde{R}_{i^*}, R_{-i^*}) = \Omega - \sum_{j \neq i^*} p(R_j) < p(\tilde{R}_{i^*}).$$

Finally, consider R which differs from $(\tilde{R}_{i^*}, R_{-i^*})$ only in i^*'s preferences. We know that $p(R_{i^*}) > p(\tilde{R}_{i^*}) > f(\tilde{R}_{i^*}, R_{-i^*})$. By Lemma A.1(c), we obtain that $f(R) = f(\tilde{R}_{i^*}, R_{-i^*})$, a contradiction. This proves the claim when $|N^*| \geq 2$. Lastly, consider $|N^*| = 1$. Denote the agent in N^* by j^*. Because $f_j(R) \leq p(R_j)$ for all j (thanks to efficiency), we have that $f_i(R) = 0$ for all $i \neq i^*, j^*$. If the current claim is not true, then it must be that $f_{i^*}(R) > \Omega - p(R_{j^*})$ and $f_{j^*}(R) < p(R_{j^*})$. Pick any agent $i \neq i^*, j^*$ and fix $\epsilon > 0$ such that $\epsilon < \Omega - p(R_{j^*})$. Fix preferences of i, $\tilde{R}_i$, such that $p(\tilde{R}_i) = \epsilon$. Consider $(\tilde{R}_i, R_{-i})$. By construction, $p(\tilde{R}_i) + \sum_{j \neq i^*, i} p(R_j) < \Omega$ but $p(\tilde{R}_i) + \sum_{j \neq i} p(R_j) > \Omega$. In addition, two agents, j^* and i, have peaks exceeding 0. Thus, as we showed for the $|N^*| \geq 2$ case, it must be that $p(\tilde{R}_i) = f_i(\tilde{R}_i, R_{-i}) = \epsilon > f_i(R)$ and $p(R_{j^*}) = f_{j^*}(\tilde{R}_i, R_{-i}) > f_{j^*}(R)$. These inequalities are incompatible with replacement monotonicity.

Claim 5. *If $\sum_{i \neq i^*} p(R_i) > \Omega$, then $f_{i^*}(R) = 0$.*

Proof of Claim 5. Suppose $f_{i^*}(R) > 0$. By efficiency, we know that $p(R_{i^*}) \geq f_{i^*}(R) > 0$. Construct $\tilde{R}$ so that $p(\tilde{R}_{i^*}) = p(R_{i^*})$, $p(\tilde{R}_i) \leq p(R_i)$ for all $i \neq i^*$, and $\sum_{i \neq i^*} p(\tilde{R}_i) = \Omega - \epsilon$ where $\epsilon < f_{i^*}(R) \leq p(R_{i^*})$. Observe that $\sum_{i \in N} p(\tilde{R}_i) > \Omega$. Thus, by Claim 4, $f_i(\tilde{R}) = p(\tilde{R}_i)$ for all $i \neq i^*$ and $f_{i^*}(\tilde{R}) = \epsilon$. Now let us change the preferences of those in $N \setminus \{i^*\}$ from $\tilde{R}_{-i^*}$ to R_{-i^*} sequentially. By construction, each agent's peak weakly increases. Thus, whenever we change some agent's preferences, by (b) and (c) of Lemma A.1, this agent's allocation weakly increases. This means that in each step, i^*'s allocation weakly decreases by replacement monotonicity. Consequently, $f_{i^*}(R) < \epsilon$ which is a contradiction. □

Proof of Theorem 3.24. By Theorem 3.21, there exists i^* such that $f_{i^*}(R) = \max\{0, \Omega - \sum_{j \neq i^*} p(R_j)\}$. Fix any S, R and $\tilde{R}_S$ satisfying $\sum_{j \in S} f_j(\tilde{R}_S, R_{-S}) \geq \sum_{j \in S} f_j(R)$.

Claim 1. *If $\sum_i p(R_i) \leq \Omega$ then $f_j(\tilde{R}_S, R_{-S}) \leq f_j(R)$, $\forall j \in N \setminus S$.*

Proof of Claim 1. Because f is a free disposal rule, efficiency implies that $f_i(R) = p(R_i)$ for all $i \neq i^*$ and $f_{i^*}(R) = \Omega - \sum_{i \neq i^*} f_i(R)$. If $\sum_{i \in S \setminus \{i^*\}} p(\tilde{R}_i) + \sum_{i \in -S \setminus \{i^*\}} p(R_i) \leq$

Ω, again we have $p(\tilde{R}_i) = f_i(\tilde{R}_S, R_{-S})$, for all $i \in S \setminus \{i^*\}$, $p(R_j) = f_j(\tilde{R}_S, R_{-S})$ for all $j \in N \setminus (S \cup \{i^*\})$, and $f_{i^*}(\tilde{R}_S, R_{-S}) = \Omega - \sum_{i \neq i^*} f_i(\tilde{R}_S, R_{-S})$. If $i^* \in S$, we have that $f_j(R) = f_j(\tilde{R}_S, R_{-S})$ for each $j \notin N \setminus S$ which is what we are looking for. If $i^* \notin S$, then each $j \in N \setminus S$ other than i^* gets their peak under $f(\tilde{R}_S, R_{-S})$. But because $f_j(R) = f_j(\tilde{R}_S, R_{-S})$ for each $j \in N \setminus S$ and $\sum_{i \in S} f_i(\tilde{R}_S, R_{-S}) \geq \sum_{i \in S} f_i(R)$, we have that $f_{i^*}(\tilde{R}_S, R_{-S}) \leq f_{i^*}(R)$. Consequently, we have shown that $f_j(\tilde{R}_S, R_{-S}) \leq f_j(R)$, $\forall j \in N \setminus S$ if $\sum_{i \in S \setminus \{i^*\}} p(\tilde{R}_i) + \sum_{i \in -S \setminus \{i^*\}} p(R_i) \leq \Omega$. Finally, let $\sum_{i \in S \setminus \{i^*\}} p(\tilde{R}_i) + \sum_{i \in -S \setminus \{i^*\}} p(R_i) > \Omega$. Then i^* gets $f_{i^*}(\tilde{R}_S, R_{-S}) = 0 \leq f_{i^*}(R)$. In addition, by efficiency $f_j(\tilde{R}_S, R_{-S}) \leq p(R_j) = f_j(R)$ for all $j \in N \setminus (S \cup \{i^*\})$. This is what we are looking for. □

Claim 2. *If $\sum_i p(R_i) \geq \Omega$ and $\sum_{i \in S} f_i(R) = \sum_{i \in S} f_i(\tilde{R}_S, R_{-S})$, then $f_j(R) = f_j(\tilde{R}_S, R_{-S})$ for each $j \in N \setminus S$.*

Proof of Claim 2. Set $S^* = \{j \in N \setminus S : f_j(R) \neq f_j(\tilde{R}_S, R_{-S})\}$. If $S^* = \emptyset$ then we are done. Suppose $S^* \neq \emptyset$ which means that the claim is false. Let $R^1_{N \setminus S^*}$ and $R^2_{N \setminus S^*}$ be such that

$$p\big(R^1_i\big) = f_i(R) \;\&\; p\big(R^2_i\big) = f_i(\tilde{R}_S, R_{-S}), \quad \forall i \in N \setminus S^*.$$

By repeatedly using Lemma A.1(b) we obtain that

$$f\big(R^1_{N \setminus S^*}, R_{S^*}\big) = f(R) \;\&\; f\big(R^2_{N \setminus S^*}, R_{S^*}\big) = f(\tilde{R}_S, R_{-S}). \tag{6}$$

By efficiency, $f_i(R) \leq p(R_i)$. If $f_j(R) = p(R_j)$ for all $j \in S^*$, then efficiency and feasibility imply $f_j(\tilde{R}_S, R_{-S}) = p(R_j) = f_j(R)$ for all $j \in S^*$ which contradicts that $S^* = \emptyset$. By feasibility, $\sum_{j \in S^*} f_j(R) = \sum_{j \in S^*} f_j(\tilde{R}_S, R_{-S})$. Consequently, it cannot be that the case in which $f_j(R) < f_j(\tilde{R}_S, R_{-S})$ for each $j \in S^*$. Thus, for some $j \in S^*$, $f_j(\tilde{R}_S, R_{-S}) < f_j(R) \leq p(R_j)$. By efficiency, we then have that $f_i(\tilde{R}_S, R_{-S}) \leq p(R_i)$ for all $i \in N$. Consequently, we have that

$$\min\big\{f_j(R), f_j(\tilde{R}_S, R_{-S})\big\} < \max\big\{f_j(R), f_j(\tilde{R}_S, R_{-S})\big\} \leq p(R_j) \quad \forall j \in S^*. \tag{7}$$

Let $\hat{R}_{S^*}$ be such that

$$p(\hat{R}_j) = \big(f_j(R) + f_j(\tilde{R}_S, R_{-S})\big)/2, \quad \forall j \in S^*.$$

Due to (7), we have

$$\min\big\{f_j(R), f_j(\tilde{R}_S, R_{-S})\big\} < p(\hat{R}_j) < \max\big\{f_j(R), f_j(\tilde{R}_S, R_{-S})\big\} \leq p(R_j) \quad \forall j \in S^*. \tag{8}$$

In addition,

$$\sum_{j \in S^*} p(\hat{R}_j) = \sum_{j \in S^*} f_j(R) = \sum_{j \in S^*} f_j(\tilde{R}_S, R_{-S}).$$

Consider now $(R^1_{N \setminus S^*}, \hat{R}_{S^*})$ and $(R^1_{N \setminus S^*}, \hat{R}_{S^*})$. By construction, $\sum_{i \in N \setminus S^*} p(R^1_i) + \sum_{i \in S^*} p(\hat{R}_i) = \Omega$ and $\sum_{i \in N \setminus S^*} p(R^2_i) + \sum_{i \in S^*} p(\hat{R}_i) = \Omega$. Thus, by efficiency,

$$f_j\big(R^1_{N \setminus S^*}, \hat{R}_{S^*}\big) = f_j\big(R^2_{N \setminus S^*}, \hat{R}_{S^*}\big) = p(\hat{R}_j), \quad \forall j \in S^*.$$

By combining the equation above, (6) and (8), we find that for each $j \in S^*$ with $f_j(R) > f_j(\tilde{R}_S, R_{-S})$,

$$f\big(R^2_{N\setminus S^*}, \hat{R}_{S^*}\big)\, P_j\, f\big(R^2_{N\setminus S^*}, R_{S^*}\big).$$

Similarly, for each j with $f_j(R) < f_j(\tilde{R}_S, R_{-S})$ then

$$f\big(R^1_{N\setminus S^*}, \hat{R}_{S^*}\big)\, P_j\, f\big(R^1_{N\setminus S^*}, R_{S^*}\big).$$

The two relations above mean that $\{R_{S^*}\}$ is a fixed point of $\xi[\hat{R}_{S^*}]$ contradicting that f is robust group strategy-proof.

Claim 3. *If $\sum_i p(R_i) \geq \Omega$ and $\sum_{i\in S} f_i(\tilde{R}_S, R_{-S}) > \sum_{i\in S} f_i(R)$ then $f_j(\tilde{R}_S, R_{-S}) \leq f_j(R)$ for all $j \in N \setminus S$.*

Proof of Claim 3. Suppose otherwise. By efficiency, we know that $f_i(\tilde{R}_S, R_{-S}) \leq p(\tilde{R}_i)$ for all $i \in S$. Fix R^1 be such that

$$p\big(R^1_i\big) = f_i(\tilde{R}_S, R_{-S}) \quad \forall i \in S$$
$$R^1_i = R_i \quad \forall i \in N \setminus S.$$

By repeatedly using Lemma A.1(b), we obtain that

$$f\big(R^1\big) = f(\tilde{R}_S, R_{-S}). \tag{9}$$

In addition, $p(R^1_i) = f_i(R^1)$ for any $i \in S$. Let R^2 such that

$$p\big(R^2_i\big) \leq p\big(R^1_i\big) \quad \forall i \in S$$
$$\sum_{i\in S} p\big(R^2_i\big) = \sum_{i\in S} f_i(R)$$
$$R^2_i = R_i \quad \forall i \in N \setminus S.$$

We now reach R^2 from R^1 by sequentially changing the preferences of those in S. Because $\sum_{i\in S} p(R^2_i) = \sum_{i\in S} f_i(R)$ and $f_i(R) \leq p(R_i)$ for all $i \in N$, at any step of this process, there will be an overdemand. Therefore, by efficiency and strategy-proofness, the allocation of the agent whose preference peak decreases must decrease to her new peak. Then by replacement monotonicity, the allocation of those in $N \setminus S$ must weakly increase. Hence, we find that

$$f_j(R) = f_j\big(R^2\big) \geq f_j\big(R^1\big) = f_j(\tilde{R}_S, R_{-S}) \quad \forall j \in N \setminus S \tag{10}$$

and

$$p\big(R^2_i\big) = f_i\big(R^2\big) \quad \forall i \in S.$$

By construction, $\sum_{i\in S} p(R^2_i) = \sum_{i\in S} f_i(R)$ and $R^2_{-S} = R_{-S}$. Thus, by Claim 2 we must have

$$f_j\big(R^2\big) = f_j(R), \quad \forall j \in N \setminus S.$$

By combining the equation above, (9) and (10), we complete the proof. □

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