Theoretical Economics 17 (2022), 291–329 1555-7561/20220291

Statistical sunspots

William A. Branch
Department of Economics, University of California, Irvine

Bruce McGough
Department of Economics, University of Oregon

Mei Zhu
Key Laboratory of Mathematical Economics, Shanghai University of Finance and Economics (SUFE)

This paper shows that belief-driven economic fluctuations are a general feature of many determinate macroeconomic models. In environments with hidden state variables, forecast-model misspecification can break the link between indeterminacy and sunspots by establishing the existence of "statistical sunspots" in models that have a unique rational expectations equilibrium. To form expectations, agents regress on a set of observables that can include serially correlated nonfundamental factors (e.g., sunspots, judgment, expectations shocks, etc.). In equilibrium, agents attribute, in a self-fulfilling way, some of the serial correlation observed in data to extrinsic noise, i.e., statistical sunspots. This leads to sunspot equilibria in models with a unique rational expectations equilibrium. Unlike many rational sunspots, these equilibria are found to be generically stable under learning. Applications are developed in the context of a New Keynesian and an asset-pricing model.

Keywords. Adaptive learning, animal spirits, business cycles, optimal monetary policy.

JEL classification. D82, D83, E40, E50.

1. Introduction

There is a long and venerable history in macroeconomics of proposing theories of exogenous movements in expectations as an independent driving force amid multiple equilibria.[1] For example, many forward-looking macroeconomic models can generate

William A. Branch: wbranch@uci.edu
Bruce McGough: bmcgough@uoregon.edu
Mei Zhu: zhu.mei@mail.shufe.edu.cn
We thank the referees for helpful comments and suggestions. This paper has benefited from discussions with John Duffy, Stefano Eusepi, Cars Hommes, Blake LeBaron, Guillaume Rocheteau, and John Williams. We also thank seminar participants at U.C. Irvine, the 2017 Workshop on Expectations in Dynamic Macroeconomics at the St. Louis Fed, the 2017 Society of Computational Economics Meetings, and the Institute of Mathematical Behavioral Science. Mei Zhu acknowledges financial support from NSFC funding (Grant 11401365, 71850002) and China Scholarship Council (file 201506485009).

[1]For instance, Blanchard (2011) writes, "…the world economy is pregnant with multiple equilibria—self-fulfilling outcomes of pessimism or optimism, with major macroeconomic implications."

Available at https://econtheory.org. https://doi.org/10.3982/TE3752

endogenous volatility through sunspot equilibria that exhibit self-fulfilling dependence on extrinsic variables, i.e., "animal spirits." Yet, most research directs its focus to models that exhibit *determinacy*, that is, (local) uniqueness of equilibrium. Even with indeterminate models, sunspot equilibria are generally not stable when rational expectations are replaced with reasonable adaptive learning rules. As a result, according to a wide literature that argues in favor of stability under learning as an important consistency and equilibrium selection mechanism, sunspot equilibria are of limited interest to applied economists.[2] This paper's contribution is to introduce a novel equilibrium phenomenon that we call *statistical sunspots*. Statistical sunspots are endogenous fluctuations that arise in equilibrium even in models that feature a unique rational expectations equilibrium. What is more, they are often stable under adaptive learning.

We consider economic environments where some exogenous variables are hidden or unobserved, following a line of research that begins with Marcet and Sargent (1989). Unobservable shocks are a staple of any economy whose equilibrium path is driven by many exogenous forces, only a subset of which is measured and observable by agents. Examples that we have in mind include "natural rate" shocks to output and interest rates, monetary policy shocks, noise traders, and asset float in asset markets, among others. We adopt a *restricted perceptions* viewpoint that unobserved exogenous variables may lead agents to use misspecified forecasting models, though we will impose a consistency to this misspecification.[3] Specifically, in a *restricted perceptions equilibrium*, beliefs coincide with the projection of the endogenous state vector onto the individuals' restricted set of observables, which results in a set of cross-equation restrictions analogous to those that feature prominently in rational expectations models.[4]

The insight of this paper is that this set of observables can include serially correlated extrinsic shocks, statistical sunspots, which can be interpreted as judgment, sentiment, expectations shocks, sunspots, etc. Because agents do not observe the full state vector, they attribute, in a self-fulfilling way, some of the serial correlation observed in the data to these extrinsic factors. Because the existence of statistical sunspots does not hinge on indeterminacy, statistical sunspot equilibria are often stable under adaptive learning, a sharp contrast from rational sunspots.

To make our results applicable across a broad range of economic environments, we adopt as our laboratory a general linear, univariate, forward-looking model that depends on a serially correlated exogenous process, unobservable to agents. Even though the fundamental shock is hidden, if the agents were able to estimate an econometric forecasting model using an infinitely long history of the endogenous variable, then the economy would replicate the full-information rational expectations equilibrium. However, there is a long tradition for forecasters, who often face data limitations and/or degrees-of-freedom limitations, to formulate and estimate parsimonious, vector autoregressive (VAR) models with a finite set of variables. In a restricted perceptions equilibrium (RPE),

[2]See, for instance, Sargent (1993, 2008), Evans and Honkapohja (2001), and Woodford (2013).

[3]The introduction to White (1994) states explicitly, "...an economic or probability model is...a crude approximation to...the 'true' relationships.... Consequently, it is necessary to view models as misspecified."

[4]Piazzesi (2016): "Cross-equation restrictions constrain the parameters associated with agents' expectations to be consistent with the parameters from the equilibrium probability distribution."

though, these misspecified beliefs are optimal in the sense that the forecasting model coefficients reflect the least-squares projection of the true data generating process onto the space spanned by the limited number of regressors. We show existence of a *fundamentals RPE* where the endogenous state variable is driven only by the fundamental exogenous shock. More interestingly, we also provide necessary and sufficient conditions for the existence of multiple RPE that depend on a serially correlated extrinsic variable—a statistical sunspot. These equilibria exist even though we restrict attention to models that have a unique rational expectations equilibrium. Rather than a continuum of sunspot equilibria, we show that at most three RPE exist: the fundamental RPE and two symmetric RPEs that are driven by sunspots. The existence of these sunspot RPEs depends on the strength of expectational feedback, the serial correlation of the hidden fundamental shock, the serial correlation of the sunspot shock, and the signal to noise ratio of the shocks' innovations.

A particularly striking feature of sunspot RPE is their potential to be stable under adaptive learning. When the model's expectational feedback is not too strong, the fundamentals RPE is the model's unique RPE, and it is stable under adaptive learning. However, when sunspot RPE do exist they often are stable under learning while the fundamentals RPE is unstable. A common finding in models of adaptive learning is that determinacy implies stability under learning.[5] It is, therefore, not completely surprising to find a similar arrangement for statistical sunspots. As the expectational feedback in the model strengthens, the learning dynamics appear to bifurcate, destabilizing the fundamentals RPE and transferring stability to the newly emerged sunspot RPE.

We show that statistical sunspots introduce excess volatility to an economy. The economic volatility of sunspot RPE are bounded below by rational expectations and above by the fundamentals RPE. We use these results to illustrate the practical and empirical relevance of the theoretical results via two applications. The first explores the empirical implications of statistical sunspots for the excess volatility puzzle in stock prices. We use a calibrated asset-pricing model to explore the empirical implications of statistical sunspots by identifying the RPE sunspot process that can generate the excess volatility observed in data. The second application reconsiders a theme from studies into the design of monetary policy rules that, under learning, policymakers face an improved stabilization trade-off via a systematic, aggressive response whenever inflation deviates from its target rate.[6] We specify an optimal monetary policy problem in a New Keynesian environment with unobserved exogenous factors. We show that an optimal policy instrument rule responds less aggressively to inflation innovations than under perfect information. In a related matter, optimal monetary policy will coordinate the economy on a (learnable) sunspot RPE.

The paper proceeds as follows. Section 2 develops the theory in a simple environment. Section 3 presents the main results. Section 4 examines robustness of the main results. Section 5 presents the applications, while Section 6 concludes. All proofs are contained in the Appendix.

[5]Although common, there are counterexamples to the tight relationship between determinacy and expectational stability.

[6]See, for example, Eusepi and Preston (forthcoming).

1.1 *Related literature*

While this paper takes a bounded rationality perspective, there is an existing literature that identifies sunspot-like fluctuations that emerge within a unique rational expectations equilibrium. Angeletos and La'O (2013) introduce random matching frictions into a Lucas islands framework: firms have an incentive to trade with the other islands but make production decisions before being randomly matched with another firm, which may have firm-specific productivity and beliefs. Higher-order beliefs emerge from a timing protocol where each firm forecasts the terms of trade before the identity of their trading partner or the beliefs of other firms is revealed. This structure formalizes imperfect communication as (heterogeneous) beliefs are not observed until firms physically meet. An extension of their model shows how an information cascade can converge to the sentiment equilibrium as information is exchanged when firms trade bilaterally.

Similarly, Benhabib, Wang, and Wen (2015) show the existence of rational expectations equilibria driven by (potentially) serially correlated sentiment shocks. In their centralized environment, firms make decisions based on expected demand while households base decisions on expected income, which depend in part on sentiments about aggregate income. While households observe sentiments, firms receive a noisy signal about their demand. These firms are rational and solve their signal-extraction problem. When firms cannot distinguish between idiosyncratic demand shocks and aggregate sentiment shocks a strategic complementarity can arise so that there is a rational expectations equilibrium that features dependence on stochastic sentiment shocks, with the variance of those shocks also pinned down by the equilibrium. There is, however, also a rational expectations equilibrium that exists without dependence on the sentiment shock. While the model structure is essentially static, Benhabib et al. (2015) develop an adaptive learning formulation where the agents learn the sentiment-shock variance in real time and find that the sentiment-shock equilibrium is stable.

In contrast, our paper takes a bounded rationality perspective. The presence of hidden random variables and econometric considerations motivate agents to formulate expectations by fitting finite-order autoregressive models used to forecast payoff-relevant aggregate variables. Given those beliefs, the agents satisfy their respective optimization conditions and (temporary) market equilibrium prevails. Like Angeletos and La'O (2013), the statistical sunspot equilibrium arises in economies that are determinate. And, similar to Benhabib et al. (2015), it is the statistical sunspot equilibrium, and not the fundamental one, that is stable under learning. One potential advantage to our restricted perceptions approach is simplicity. By taking a bounded rationality perspective, statistical sunspot equilibria arise via the parsimony in their forecasting equations, a simple and intuitive structure for expectation formation. In fact, by developing our results within the context of a general expectational difference equation, robust statistical sunspots are easily incorporated across a range of economic environments. In particular, we develop applications in an overlapping generations with money economy, an asset-pricing model, and a New Keynesian model. The way in which statistical sunspots introduce additional volatility into the economy is another distinguishing feature from previous studies. In Angeletos and La'O (2013) and Benhabib et al. (2015), the sentiment-driven equilibria are more volatile than the fundamental equilibrium. Statistical sunspot equilibria, while more volatile than what would occur under rational expectations, are less volatile than the fundamental equilibrium. We explore the policy implications of this result and show that a central bank may wish to coordinate the economy on the sunspot equilibrium, with economic volatility that is lower than the fundamental equilibrium.

This paper is also related to a literature that studies the equilibrium implications of econometric model misspecification. Marcet and Sargent (1989) first introduce the idea that in an environment with private information, an adaptive learning process will converge to a limited information rational expectations equilibrium. Subsequently, Sargent (1999) and Evans and Honkapohja (2001) extend the idea to where the misspecification never vanishes. Sargent (1999) introduces the self-confirming equilibrium (SCE) concept into macroeconomic models. In an SCE, the coefficients of an agents' forecasting model satisfy a least-squares orthogonality condition, as in a restricted perceptions equilibrium, but in equilibrium these coefficients take the same values as if it were a rational expectations equilibrium: beliefs are correct along the equilibrium path and misspecified along paths that are never reached. An SCE is a refinement of a restricted perceptions equilibrium. Branch and Evans (2006) illustrate how the misspecification can arise endogenously. Cho and Kasa (forthcoming) provide a justification for restricted perceptions as a sort of Gresham's law of Bayesian model averaging that leads to correctly specified models being driven out of the agents' forecast model set. We are very much in the spirit of Sargent's (2008) essay on small deviations from the rational expectations hypothesis that preserve beliefs being pinned down by cross-equation restrictions and, yet, deliver an independent role for beliefs in economic fluctuations.

The results here also relate to a very large literature on sunspot equilibria in rational expectations models, e.g., Shell (1977), Cass and Shell (1983), Azariadis (1981), Azariadis and Guesnerie (1986), Guesnerie (1986), and Guesnerie and Woodford (1992). In the same spirit as this paper, Eusepi (2009) studies the connection between expectations-driven fluctuations and indeterminacy in one- and two-sector business cycle models. Woodford (1990) is the first to show that sunspot equilibria could be stable under learning in overlapping generations models. Evans and McGough (2005a) and Duffy and Xiao (2007) show that sunspot equilibria in applied business cycle models like Benhabib and Farmer (1994) and Farmer and Guo (1994) are unstable under learning. Evans and McGough (2005b) show that an alternative representation of sunspot equilibria can be stable in a New Keynesian model with an expectations-based interest rate rule.

Finally, this paper is most closely related to Sargent (1991) and Bullard, Evans, and Honkapohja (2008). We discuss our model in the context of Sargent (1991) throughout, and devote the end of Section 4 to an extensive discussion of the connection between the theory here and Bullard et al. (2008).

2. Restricted perceptions equilibria

A restricted perceptions equilibrium (RPE) is a solution concept that relaxes the rational expectations hypothesis while preserving its salient features of optimality and internal consistency. This section formalizes the concept of an RPE, developing the results within a simple univariate model with unobserved first-order autoregressive (AR(1)) shocks, and introduces the *statistical sunspots* concept.

2.1 *The laboratory model*

We take, as our laboratory, an ad hoc univariate model given by the pair of equations

$$y_t = \alpha \hat{E}_t y_{t+1} + \gamma z_t, \tag{1}$$

$$z_t = \rho z_{t-1} + \varepsilon_t. \tag{2}$$

Equation (1) is the expectational difference equation that determines the endogenous variable y_t as a linear function of expectations of y_{t+1} and a serially correlated *fundamental* shock z_t. Our view is that underlying the ad hoc model (1) is a carefully specified general equilibrium (GE) model, with $\hat{E}_t y_{t+1}$ capturing the agents' homogeneous time-t subjective forecast of y_{t+1}: see Section 2.2 for an example of an economy giving rise to our laboratory model. With this interpretation, (1) determines the (temporary) equilibrium value for aggregate y_t derived from an environment where individuals solve their respective optimization problem by as given their subjective expectations over payoff-relevant variables whose determination is treated as exogenous by each of the agents. We assume $0 < \rho < 1$ and that ε_t is white noise with variance σ_ε^2. Throughout, assume that $0 < \alpha < 1$ so that, under rationality, the model is determinate and features positive expectational feedback.

The focus of our paper is the equilibrium interaction between subjective expectations and forecast-model misspecification. We assume agents are identical in their expectations formation, and we adopt the following *behavioral primitives*.

- B.1. Agents form expectations using linear forecasting models.
- B.2. Agents do *not* condition their forecasting models on current or lagged values of the fundamental shock.
- B.3. Agents do *not* condition their *expectations* on current values of the endogenous variable.
- B.4. Agents form subjective expectations over payoff-relevant *aggregate variables* by conditioning on *aggregate variables*.

We discuss each of these assumptions in turn.

Assumption B.1 is natural since the underlying model is linear.[7] Assumption B.2 captures that z_t is "unobserved": we assume that agents do not condition their forecasts on z at any lag. Two separate, but related, scenarios motivate Assumption B.2. Most obviously, if agents are simply unaware of its existence then they would not include it in their forecasting equation. More broadly, we have in mind a bounded rationality environment where z_t is unobserved and agents, while possibly aware of its potential impact, do

[7]Though in the adaptive learning literature it is standard to assume agents use linear forecasting models even when the underlying economy is nonlinear.

not have the data or sophistication to solve the associated signal extraction problem. In practice, degree-of-freedom limitations may lead agents to favor parsimonious models. The details of any particular justification for Assumption B.2 will depend on the precise modeling environment, and could include, for example, drifts in a central bank's inflation target, aggregate markup shocks, natural rate shocks, the presence of noise traders, or asset float. In Section 2.2, we develop an application where agent-specific productivity is driven by idiosyncratic fundamentals and an unobservable aggregate productivity shock. Section 5.1 is an application to asset-pricing that does feature homogeneous agents and unobservable variations in the outside supply of a risky asset.[8]

Assumption B.3 is, within models of adaptive learning, the *conventional timing* protocol. It predicates that the agent cannot condition current expectations on current realizations of endogenous variables. This assumption is standard, though not ubiquitous, in the learning literature; see Evans and Honkapohja (2001), Chapter 8). Assumption B.3 has the advantage of avoiding the simultaneous determination of expectations and contemporaneous endogenous variables. The existence of statistical sunspots is robust to the alternative *contemporaneous timing* assumption. However, we gain tractability in the stability analysis by imposing Assumption B.3.

Finally, Assumption B.4 represents the behavioral assumption behind how individuals take actions when they hold subjective (nonrational) beliefs (e.g., Woodford (2013)). Agents make decisions given their subjective expectations over aggregate variables that are treated as exogenous by the individual agent. Assumption B.4 maintains that when forecasting those aggregate variables, agents only incorporate aggregate variables into their predictions. Boundedly rational agents will incorporate those aggregate variables that are observable and correlated with the payoff-relevant variables that they seek to forecast. Statistical sunspots can matter precisely because of expectational feedback.

The theory of expectation formation proposed in this paper is based on the "cognitive consistency" principle. Even though our agents are boundedly rational, they will forecast like a good economist who specifies and estimates an econometric model for the aggregate payoff-relevant variables. To fix ideas, we respect attention to a prespecified collection of forecasting models of the AR(p) form that also conditions on some other extrinsic (sunspot) process. In this environment, the natural equilibrium concept is a restricted perceptions equilibrium where the forecast model used by agents is optimal among those under consideration, where the objective minimized is mean-square error.

The consequences of Assumptions B.1–B.4 for individual behavior are best understood in the context of a particular model, which is the subject of the next section.

[8] Some readers may question whether any variable is truly hidden as opposed to being obscured by an observable a noisy signal. We exploit the noisy signal interpretation in our example environment in the next section, where agents' productivity is driven by an unobservable aggregate shock and idiosyncratic shocks. While the idea that a large economy, or market, might be driven by hidden and unmeasured forces should be uncontroversial, we briefly remark that statistical sunspots can still arise in a more general environment with noisy signals. In a companion paper, we demonstrate that if the noisy signal, and the shock z_t, are sufficiently correlated then there exist multiple equilibria, including sunspot RPE's. We focus on the case of an unobservable z_t because we find it a compelling description of many environments and it delivers precise results that illustrate our main ideas.

2.2 *An overlapping generations model with unobserved shocks*

There is a continuum of agents born at each time t, indexed by $\omega_t \in \Omega$.[9] Each agent lives for two periods, and the population is constant and normalized to 1. Agents are yeoman farmers, each owning a production technology that is linear in labor and produces a common consumption good. Production is subject to a shock that has both aggregate and idiosyncratic components. The young work and the old consume. There is no storage technology other than fiat currency. Workers trade goods for money in competitive markets and then, when old, they trade their money for goods.

Each young agent chooses their labor supply $n_t(\omega_t)$ and money-holdings $M_t(\omega_t)$ to maximize their objective $u(c^e_{t+1}(\omega_t)) - \nu(n_t)$, subject to the budget constraints $z_t(\omega_t)n_t(\omega_t) = q_t M_t(\omega_t)$ and $c_{t+1}(\omega_t) = q_{t+1}M_t(\omega_t)$. Let $u(c)$ be constant relative risk aversion (CRRA), with risk aversion σ and let $v'(n) = 1$. Here, q_t is the goods price of money and $c_{t+1}(\omega_t)$ is consumption when old, with the superscript e referencing an expectation. For simplicity, we impose that agents use point expectations and form expectations of future consumption based on expectations of future prices: $c^e_{t+1}(\omega_t) = q^e_{t+1}(\omega_t)M_t(\omega_t)$. Finally, the variable $z_t(\omega_t)$ captures the multiplicative productivity shock so that $z_t(\omega_t)n_t(\omega_t)$ is the amount of goods produced by agent ω_t.

Agent ω_t's money demand is

$$M_t(\omega_t) = z_t(\omega_t)^{\frac{1}{\sigma}} q_t^{-\frac{1}{\sigma}} \big(q^e_{t+1}(\omega_t)\big)^{\frac{1-\sigma}{\sigma}}.$$

Market-clearing requires that $\int_\Omega M_t(\omega_t)\, d\omega_t = 1$. Equilibrium price is determined as $q_t^{1/\sigma} = \int_\Omega z_t(\omega_t)^{1/\sigma} (q^e_{t+1}(\omega_t))^{1-\sigma/\sigma}\, d\omega_t$. Given subjective beliefs about payoff-relevant aggregate prices, each agent ω_t formulates optimal money demand, markets clear, and equilibrium price is a function of (aggregate) expected prices.

Now assume $z_t(\omega_t)$ includes both an idiosyncratic and an aggregate component according to

$$\log\big(z_t(\omega_t)\big) = \log(z_t) + \log\big(\zeta_t(\omega_t)\big) \quad \text{and} \quad \log(z_t) = \rho \log(z_{t-1}) + \varepsilon_t,$$

where the $\log(\zeta_t(\omega_t))$ are independent and identically distributed (iid) mean zero and independent across agents, ε_t is iid zero mean with small support, and $\log(\zeta_t(\omega_t))$ and ε_t are independent processes.

Having surveyed the economic environment we discuss the connection to the behavioral assumptions in the preceding section.

- B.1. By restricting attention to linear forecast models, it is natural to assume that agents make decisions using their point forecasts.
- B.2. Imposing this behavioral primitive, we assume that agents only observe their own productivity, $z_t(\omega_t)$. If they were fully rational agents, and aware of aggregate z_t and its impact on their own productivity, then they would also understand that

[9] The overlapping generations (OLG) model discussed here was developed in, and borrows heavily from, Evans and McGough (2018).

their optimal forecasts of aggregate prices depend on aggregate productivity. These rational agents would use signal extraction—say, via a regression on an infinite history of prices—to acquire the information about z_t available in $z_t(\omega_t)$ and use it to forecast z_{t+1}. However, we assume that the agents are *not fully rational*. We assume they either are not aware of the aggregate component to productivity, or they find it too costly or econometrically infeasible, to consider forecast methods that incorporate $z_t(\omega_t)$.

- B.3. We assume agents do not use period-t price to forecast q_{t+1}. The timing of events is as follows: at the beginning of the period agents update their beliefs and then make optimal decisions conditional on those beliefs. This timing structure builds on a tradition where forecasting is a statistical decision separate from the economic decision.

- B.4. Operationally, each agent forecasts aggregate prices using only data on aggregate prices (and possibly sunspot processes). In a related matter, the agents do not observe, or forecast with, the expectations of the other agents. Without common knowledge of beliefs, an individual agent would not be able to estimate the hidden shock as the residual between the aggregate state and expectations.

With homogeneous expectations the (temporary) equilibrium price is $q_t = \vartheta z_t(q^e_{t+1})^{1-\sigma}$, with $\vartheta^{\frac{1}{\sigma}} = \int_\Omega \zeta_t(\omega_t)^{\frac{1}{\sigma}}\, d\omega_t$. Taking logs and centering at the nonstochastic steady state yields the linear reduced-form system $\hat{q}_t = (1-\sigma)\hat{q}^e_{t+1} + \hat{z}_t$, where $\hat{\star}$ is $\log(\star)$ in deviation from steady-state form, for $\star \in \{q, z\}$. Thus the model's equilibrium dynamics take the form of our generic model (1)–(2), in which agents' subjective expectations satisfy B.1–B.4.

2.3 *Rational expectations equilibria*

It is useful to review the rational expectations case. A process y_t is a rational expectations equilibrium (REE) of the model (1)–(2) if it is uniformly bounded a.s. and satisfies (1) when $\hat{E}_t$ is replaced with E_t, i.e., the conditional expectations operator. Of course, this loose definition neglects to specify the information upon which the rational agent conditions. We consider several cases.

If z_t is observable at time t, then the model has a unique rational expectations equilibrium of the form $y_t = (1-\alpha\rho)^{-1}\gamma z_t$. Now suppose y_t is assumed observable at time t, but z_t, as in Assumption B.2, is assumed "unobservable" (the reason for the quotes will become apparent). We may posit an REE of the form $y_t = by_{t-1} + \text{noise}_t$. Then $E_t y_{t+1} = bE_t y_t = by_t$. Using (1) we find

$$y_t = \alpha b y_t + \gamma z_t \quad \Longrightarrow \quad y_t = \frac{\gamma}{1-\alpha b}(1-\rho L)^{-1}\varepsilon_t \quad \Longrightarrow \quad y_t = \rho y_{t-1} + \frac{\gamma}{1-\alpha b}\varepsilon_t. \tag{3}$$

When $b = \rho$, the process generated by (3) is an REE and the same process as when z_t is observed. Provided that y_t is observable to rational agents, the full information REE obtains.

What happens when rational agents cannot condition on y_t and z_t is assumed unobserved, i.e., Assumptions B.2 and B.3? First write the rational model as $y_t = \alpha E_{t-1} y_{t+1} + \gamma z_t$ to emphasize the timing protocol B.3. The equilibrium in this case is an AR(∞), as any finite-order autoregression fit to the data will be underparameterized. To see this point, first made by Marcet and Sargent (1989), suppose agents use a forecasting model, also called a *perceived law of motion* (PLM), of the AR(p) form $y_t = b_1 y_{t-1} + \cdots + b_p y_{t-p} + \epsilon_t$. We may compute

$$\begin{aligned} E_{t-1} y_{t+1} &= b_1 E_{t-1} y_t + b_2 y_{t-1} + \cdots + b_p y_{t-p+1} \\ &= b_1 (b_1 y_{t-1} + \cdots + b_p y_{t-p}) + b_2 y_{t-1} + \cdots + b_p y_{t-p+1}. \end{aligned}$$

Given the perceived law of motion, the corresponding data generating process, called the *actual law of motion* (ALM), can be found by plugging expectations into the model and rearranging,

$$(1-\rho L)\left(1 - \alpha \sum_{j=1}^{p} (b_1 b_j + b_{j+1}) L^j \right) y_t = \gamma \varepsilon_t,$$

where b_{p+1} is set to zero. If the agents use an AR(p) perceived law of motion, then the actual law of motion is an AR($p+1$). A rational expectations equilibrium, however, requires the consistency of the perceived and actual laws of motion. The only possible PLM consistent with the implied ALM is an AR(∞). In this case, recovering the hidden shocks requires an infinitely long history of the endogenous variables.

2.4 *Restricted perceptions equilibria*

Incorporating all of the behavioral primitives B.1–B.4, we now make our key *bounded rationality* assumption: agents' forecast models include only a finite number of regressors. We view this final assumption as reasonable: in practice, forecasters have finite data, face degree-of-freedom limitations, and generally favor parsimonious models. To fix ideas, we focus on the case of PLMs that include only one lag of the endogenous variable, as well an extrinsic process, that is, an AR(1) plus sunspot. The contribution of this paper is to show that these restricted perceptions give scope for equilibrium dependence on this process, i.e., on statistical sunspots, and further, to show that these sunspot RPE are often stable under adaptive learning.

For most of the remainder of the paper, agents will form forecasts while using the PLM

$$\left.\begin{aligned} y_t &= b y_{t-1} + d\eta_t + \epsilon_t \\ \eta_t &= \phi \eta_{t-1} + \nu_t \end{aligned}\right\} \quad \Rightarrow \quad \hat{E}_t y_{t+1} = b^2 y_{t-1} + d(b+\phi)\eta_t, \tag{4}$$

where η_t is the extrinsic noise term, i.e., the statistical sunspot, and $0 < \phi < 1$. Note that, consistent with our expectations timing protocol, agents do observe and condition on the exogenous process η_t when forming forecasts.

For simplicity, we assume that η_t is orthogonal to the fundamentals shocks: ν_t and ε_s are independent for all t, s.[10] The extrinsic noise, η_t, can be thought of as a statistical sunspot variable that proxies for waves of optimism/pessimism, sentiment shocks, judgment or add factors, political shocks, etc. We call them statistical sunspots to distinguish them from rational sunspots that are typically martingale difference sequences. A statistical sunspot, alternatively, is a generically serially correlated exogenous process that will impact agent beliefs only if there is a statistical relationship between the state y and η. Importantly, whether and when agents use η in their forecast model will arise as an equilibrium property: d is pinned down via cross-equation restrictions.

Hommes and Zhu (2014), working in a similar framework, assign to agents an AR(1) perceived law of motion,

$$y_t = by_{t-1} + \epsilon_t \quad \Rightarrow \quad \hat{E}_t y_{t+1} = b^2 y_{t-1}, \tag{5}$$

where ϵ_t is a (perceived) white noise process.[11] In the AR(1) perceived law of motion, the coefficient b corresponds to the first-order autocorrelation coefficient. Hommes and Zhu (2014) define a behavioral learning equilibrium as a stochastic process for y_t satisfying (1), given that expectations are formed from (5), and with b equal to the first-order autocorrelation coefficient of y_t. The novelty in this paper is to expand the agents' set of observable variables to include extrinsic random variables.

Is it reasonable that individuals would be able to observe η_t and not fundamental variables such as z_t? In our view, the answer is yes. The process η_t could be any collection of information that agents think is informative about the state of the market or the economy that does not have a direct, payoff-relevant effect except through agents' beliefs. The motivation for models with restricted perceptions is that agents do not know the structural model that generates data. A good econometrician would include all observable variables that help predict the state. The restricted perceptions, or bounded rationality, assumption is that individuals formulate models with only a finite number of lags. We show that when RPE exist that include dependence on η_t, these equilibria are stable under learning so that eventually agents would come to believe that these nonfundamental variables drive, in part, the endogenous state variable y_t. Thus, these are self-fulfilling equilibria. Surprisingly, RPE with dependence on η_t arise across a broad spectrum of determinate economic models, are stable under learning, and do not require heterogeneous information or beliefs.

Given the perceived law of motion (4), the ALM is

$$y_t = \alpha b^2 y_{t-1} + \alpha d(b + \phi)\eta_t + \gamma z_t. \tag{6}$$

[10] Relaxing this restriction leads, more generally, to stable sunspot equilibria.

[11] In Hommes and Zhu (2014), the PLM model in fact also includes a constant term. Since the analysis of the mean is relatively trivial and is not the main point of this work, here we assume that the means are zero and known, without loss of generality. We also assume that the agents know the stochastic process determining η_t. This does not impact the learning stability analysis since η is exogenous, with a sufficiently long history of ηs, the agents would precisely estimate ϕ and σ_ν. Unlike η, y is determined via a self-referential system.

Importantly, the PLM is underparameterized: the ALM (6) depends on y_{t-1}, η_t, and z_t, while the PLM depends only on y_{t-1} and η_t.[12]

The economy is in a restricted perceptions equilibrium when agents' forecast model is optimal among those under consideration. For the model at hand, this means that the agents' beliefs, as summarized by the coefficients (b, d), coincide with the coefficients obtained by projecting the ALM onto the span of y_{t-1} and η_t. That is, beliefs satisfy the least-squares orthogonality condition

$$EX_t\big(y_t - X_t'\Theta\big) = 0, \tag{7}$$

where $X_t' = (y_{t-1}, \eta_t)$, $\Theta_t' = (b, d)$, and where the expectation is taken with respect to the asymptotic distribution implied by the ALM under beliefs Θ. In an RPE, agents are unable to detect their misspecification within the context of their forecasting model. A sufficiently long history of data will reveal the misspecification to agents, so an RPE is appropriate for settings where data are slow to reflect the serial correlation in the residuals of the regression equations.

The set of restricted perceptions equilibria is characterized by studying the *T-map*, which captures the projection of the ALM onto the span of y_{t-1} and η_t, and for given beliefs Θ, is written $T(\Theta) = (EX_tX_t')^{-1}EX_ty_t$. An RPE is a fixed point of the T-map. Straightforward calculations produce

$$T(\Theta) = \begin{pmatrix} 1 & -d\phi\dfrac{\sigma_\eta}{\sigma_y} \\ 0 & (1 - b\phi)\dfrac{\sigma_y}{\sigma_\eta} \end{pmatrix} \begin{pmatrix} \operatorname{corr}(y_{t-1}, y_t) \\ \operatorname{corr}(y_t, \eta_t) \end{pmatrix}, \tag{8}$$

where $\operatorname{corr}(x, w)$ is the correlation coefficient between the variables x and w. The equilibrium coefficients, (b, d), depend on the correlation between the endogenous state variable y_t and the lag variable y_{t-1} as well as between y_t and the sunspot η_t. These correlation coefficients, in turn, depend on the belief coefficients (b, d). It is this self-referential feature of the model that makes the set of RPE interesting to characterize.

2.5 *Stability under adaptive learning*

The equilibrium selection mechanism in this paper is stability under adaptive learning. Here, we provide a brief outline of the methods that we use to conduct stability analysis.

Adaptive learning agents update their forecasting model coefficients each period by using recursive least squares (RLS) or other closely related econometric procedures. Denote by Θ_t the estimated coefficients of the PLM based on data $\{X_s\}_{s=0}^t$. The RLS algorithm can be written

$$\begin{aligned} R_t &= R_{t-1} + \kappa_t\big(X_tX_t' - R_{t-1}\big), \\ \Theta_t &= \Theta_{t-1} + \kappa_t R_t^{-1} X_t\big(y_t - \Theta_{t-1}'X_t\big), \end{aligned} \tag{9}$$

[12] Alternatively, one can write (6) as

$$y_t = \big(\alpha b^2 + \rho\big)y_{t-1} - \alpha b^2\rho y_{t-2} + \alpha d(b + \phi)(1 - \rho L)\eta_t + \gamma\varepsilon_t.$$

It is straightforward to see that if the forecast model was extended to be an AR(p), then the actual law of motion is AR$(p + 1)$.

where y_t is determined via the ALM (6) using beliefs Θ_{t-1}, and R_t is the estimated second-moment matrix of the regressors. The gain, κ_t, measures how much weight new data receives. If $\kappa_t = t^{-1}$, then RLS reproduces ordinary least squares; another common choice is constant gain, $\kappa_t = \kappa > 0$ (typically small), which results in a version of discounted least squares with the geometric discounting of past data at rate $1-\kappa$.

A restricted perceptions equilibrium is characterized by a beliefs vector Θ^*. If the algorithm (9) results in convergence to Θ^*, when initialized near it, then the RPE is said to be stable under adaptive learning. It is well established that, in a broad class of models, stability under reasonable learning algorithms, such as RLS, is governed by the *E-stability principle*, which holds that the stability of Θ^* under adaptive learning is implied by its Lyapunov stability under the following system of ordinary differential equations, referred to as the *E-stability ode*:[13]

$$\dot{\Theta} = T(\Theta) - \Theta. \tag{10}$$

A restricted perceptions equilibrium, Θ^*, is E-stable if it is a Lyapunov stable rest point to (10).

That the E-stability principle governs stability of an equilibrium is intuitive since (10) dictates that the estimated coefficients Θ are adjusted in the direction of the best linear projection of the data generated by the estimated coefficients onto the class of statistical models defined by the PLM (4). Stability of (10) thus answers the question of whether following a perturbation to the perceived coefficients, Θ will tend to return to the restricted perceptions equilibrium value.

A final comment is warranted. The equilibrium and stability concepts identified in this section are not specific to the regressors $X_t = (y_{t-1}, \eta_t)'$. Indeed, the set of regressors could be expanded to include any finite number of lags of y, as well as current and lagged values of other stationary extrinsic processes.

3. Existence and stability of RPE

In this section we present results on existence and stability of restricted perceptions equilibria. When agents restrict their regressors to include only fundamentals, then there is a unique RPE and it is always E-stable. When agents also include the sunspot η_t among their regressors, then the existence and stability results for RPE are parameter dependent.

3.1 *Existence*

A *fundamentals restricted perceptions equilibrium* is an RPE in which $b \neq 0$, $d = 0$, since there is no dependence on the extrinsic variable. Conversely, if $b, d \neq 0$, then the equilibrium is a sunspot RPE that features endogenous fluctuations, i.e., a statistical sunspot equilibrium. This section establishes existence and characterizes the set of RPE.

[13] Evans and Honkapohja (2001) present a number of theorems that cover "stability" under learning. Most often employed are theorems on convergence to an equilibrium with probability 1, with initial conditions in a suitable neighborhood of the equilibrium. However, there are also theorems that cover local convergence in probability and global convergence, as well as weak convergence. The E-stability conditions typically play an important role across all stability concepts.

The T-map for the fundamentals RPE results from setting $d = 0$ in (8):

$$b \to \frac{\alpha b^2 + \rho}{1 + \alpha b^2 \rho}. \tag{11}$$

A fundamentals RPE is a fixed point, $\hat{b}$, of (11), and, it should be noted, is equivalent to the behavioral learning equilibrium in Hommes and Zhu (2014).

The component of the T-map corresponding to d is given by

$$d \to \frac{d\alpha(b + \phi)(1 - b\phi)}{1 - \alpha b^2 \phi}. \tag{12}$$

Evidently, $d = 0$ is a fixed point of this mapping, corresponding to the fundamentals RPE. When $d^* \neq 0$ is a fixed point, we see from (12) that $b^* = (\alpha(1 - \phi^2))^{-1}(1 - \alpha\phi)$.[14] Given that value of $b = b^*$, the remainder of the T-map can be solved for d,

$$(d^*)^2 = \xi(b^*, \alpha, \rho, \phi)\left(\frac{\sigma_\varepsilon^2}{\sigma_\nu^2}\right), \tag{13}$$

where

$$\xi(b, \alpha, \rho, \phi) = \frac{\gamma^2\{\rho - b[1 - \alpha b(1 - b\rho)]\}(1 - \alpha b^2 \phi)(1 - \phi^2)}{\alpha(1 - \alpha b^2 \rho)(1 - \rho^2)(b + \phi)\phi(1 - \alpha\phi)}.$$

Note that, by (13), sunspot RPE, when they exist, come in pairs. The following result establishes existence of fundamental and sunspot RPE.

Theorem 1 (Existence of RPE). *Let $\phi \in (0, 1)$ be fixed. Then the following statements hold:*

(i) There exists a unique fundamentals RPE $(b, d) = (\hat{b}, 0)$, where $\hat{b}$ is a fixed point to (11).

(ii) There exists $0 < \tilde{\rho}(\alpha, \phi) < 1$ so that sunspot RPE $(b, d) = (b^*, \pm d^*)$ exists if and only if $\tilde{\alpha} \equiv (1 + \phi(1 - \phi))^{-1} < \alpha < 1$ and $\tilde{\rho}(\alpha, \phi) < \rho < 1$.

Corollary 1. *If $4/5 < \alpha < 1$, then sunspot RPE exist for sufficiently large ρ.*

Theorem 1 provides necessary and sufficient conditions under which a given sunspot, parameterized by ϕ, is supported as a restricted perceptions equilibrium. Corollary 1 shows that for a given structural parameter $\alpha > 4/5$, there will exist sunspot RPE provided the serial correlation of the fundamental shock is sufficiently strong. The existence of statistical sunspot equilibria requires that α is sufficiently large, i.e., there is strong expectational feedback in the model.

[14] In fact, relaxing Assumption B.3 delivers the same value for b^*. It is easy to verify that relaxing B.3, $E_t y_{t+1} = by_t + d\phi\eta_t$ and the T_d component is $d \to d\alpha\phi(1 - b\phi)/(1 - b\alpha)$, which yields the same fixed point b^* with $d^* \neq 0$.

Figure 1 illustrates Theorem 1 by plotting the fixed points of the T-map.[15] The four panels comprise comparative static experiments discussed in the next paragraph. For the descriptive purposes of this paragraph, focus attention on the dashed (as opposed to solid or dotted) lines and curves in Figure 1(a), as well as the solid line $d = 0$, which is common to all panels. The fixed points of the T-map's b component has a parabolic shape . The fixed points of the T-map's d component comprise the horizontal line at $d = 0$ and the vertical line at $b = b^*$. The intersection of these contours are restricted perceptions equilibria, as identified by the large (black) dots.

Figure 1 also illustrates comparative statics. Consider again Figure 1(a), which illustrates the comparative static effect from increasing α: solid curves and lines correspond to $\alpha = 0.85$, dashed to $\alpha = 0.90$, and dotted to $\alpha = 0.93$. For $\alpha = 0.85$ we see that $b^* > 1$, indicating that no sunspot RPE exist. As α increases, b^* falls *and* the parabolic contour shifts to the right. Already for $\alpha = 0.90$, both fundamentals and sunspot RPE exist. Panels 1(b) and 1(d) show the comparative statics from variation in ϕ and σ_v^2, and, as a consequence, do not change the fundamental RPE.[16]

The 1(a) panel anticipates the stability results to come in the next subsection. As α increases, the vertical $b = b^*$ line shifts to the left and the fundamentals RPE $\hat{b}$ shifts to the right. When $b^* = \hat{b}$, the system appears to undergo a "pitchfork-like" bifurcation, resulting in the genesis of two new fixed points—the sunspot RPE—that inherit the stability of the fundamentals RPE (which hence destabilizes). Unfortunately, the algebraic complexity of the T-map precludes conducting formal bifurcation analysis.

Notice in Figure 1 that the autoregressive coefficient b in the fundamentals RPE is greater than the same coefficient in the sunspot RPE. In fact, this result can be made formal.

Proposition 1. *If $\rho > \tilde{\rho}$ and $\alpha > \tilde{\alpha}$, then $b^* < \hat{b}$.*

The intuition is as follows: when $d \neq 0$, the agents' model tracks the serial correlation in the model—arising from the hidden shock z and the self-fulfilling serial correlation from agents' beliefs—through *both* the lagged endogenous variable and the extrinsic noise.

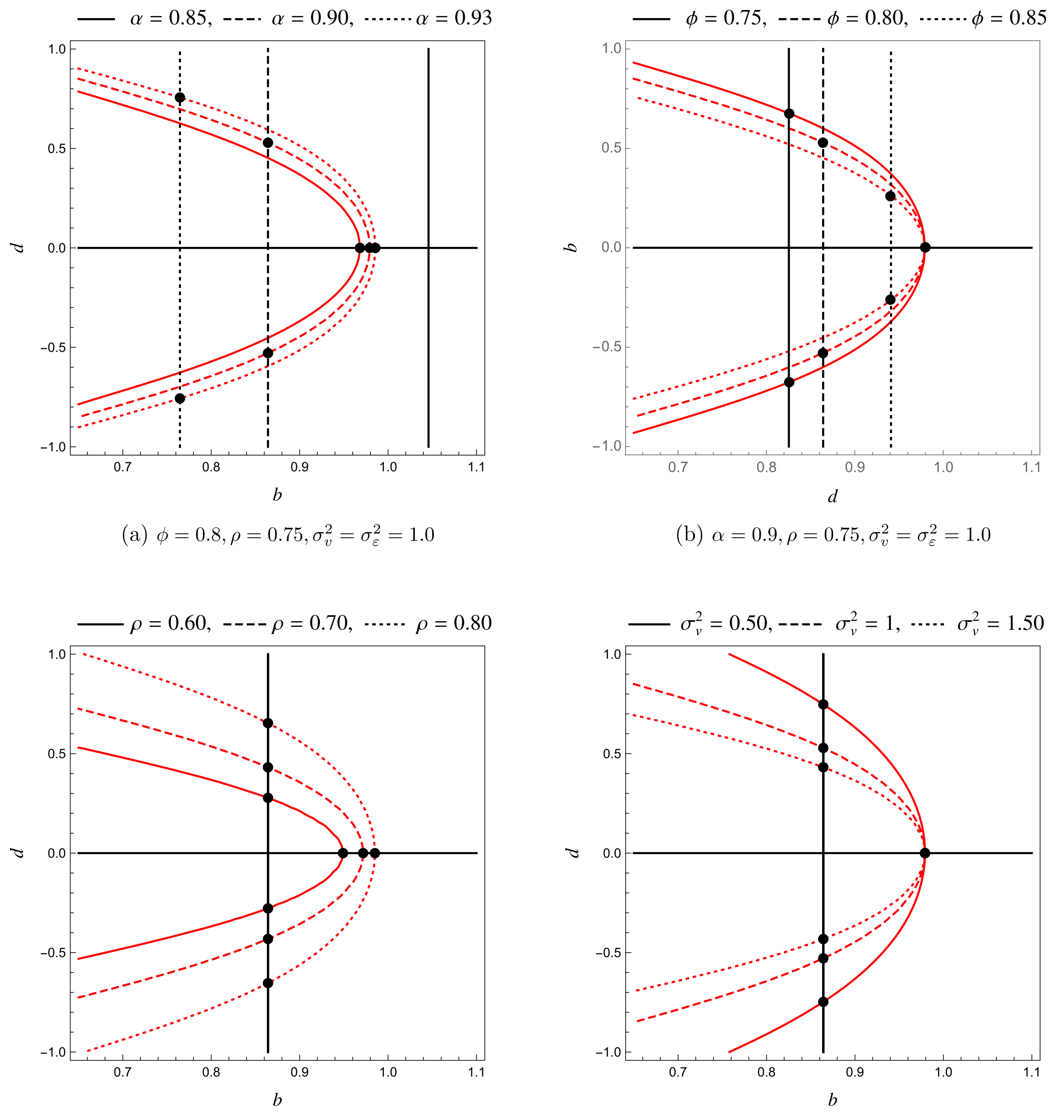
This proposition has a somewhat unexpected consequence: it is not at all obvious whether agents coordinating on the statistical sunspot equilibria will make the resulting process for y_t more or less volatile than the equilibrium where they condition on lagged y alone. Results on this question are presented below.

3.2 *Expectational stability*

We now turn to the assessment of stability under adaptive learning. As an aside, note that the stability of the fundamentals RPE may depend on whether the agents include

[15] In our simulations for the general model, we assume $\gamma = 1$ without loss of generality.

[16] The comparative statics for σ_ε^2 is qualitatively the same as for σ_ν^2.



(a) $\phi = 0.8, \rho = 0.75, \sigma_v^2 = \sigma_\varepsilon^2 = 1.0$

(b) $\alpha = 0.9, \rho = 0.75, \sigma_v^2 = \sigma_\varepsilon^2 = 1.0$

(c) $\alpha = 0.9, \phi = 0.8, \sigma_v^2 = \sigma_\varepsilon^2 = 1.0$

(d) $\alpha = 0.9, \phi = 0.8, \rho = 0.75, \sigma_\varepsilon^2 = 1.0$

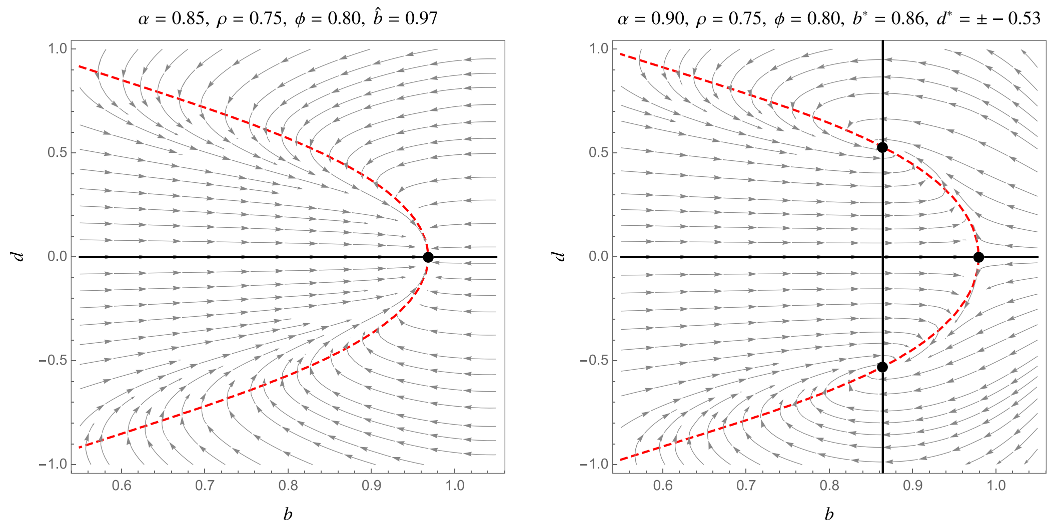
FIGURE 1. Existence and comparative statics.

sunspots among their regressors. Unless otherwise stated, we assume that agents always regress on sunspots.[17]

THEOREM 2 (Stability of RPE). *Let $\phi \in (0, 1)$ be fixed, and let $\tilde{\rho}$ and $\tilde{\alpha}$ be as in Theorem 1.*

(i) If agents do not regress on the sunspot process, the fundamentals RPE is E-stable.

[17]There are notions of *weak* and *strong* stability that can be applied to address these nuances, but are ultimately distracting from the primary analysis.



(a) Unique E-stable fundamentals RPE (b) Multiple E-stable sunspot RPE

FIGURE 2. E-stability dynamics.

(ii) If $0 < \alpha < \tilde{\alpha}$ *or if* $0 < \rho < \tilde{\rho}$, *then the fundamentals RPE is E-stable.*

(iii) There exists $\tilde{\rho} < \breve{\rho} < 1$ *so that if* $\tilde{\alpha} < \alpha < 1$ *and* $\breve{\rho} < \rho < 1$, *then the fundamentals RPE is not E-stable and the two sunspot RPE are E-stable.*

Item (iii) of this theorem is particularly striking: for a region of parameter space *only* the sunspot RPE are E-stable. This point will be examined numerically in Section 3.3.

As an illustration, consider Figure 2, which plots the vector field of the E-stability ode for two calibrations of the model. The left panel provides an example with $\alpha < \tilde{\alpha}$: there is a unique fixed point to the T-map and, as in part (ii) of the theorem, the vector field indicates stability. In the right panel, $\alpha > \tilde{\alpha}$ and sunspot RPE exist. The vector field indicates both the stability of the sunspot RPE and the instability of the fundamentals RPE. Though it is a bit hard to see in the figure, if d is restricted to be 0, then the vector field also indicates stability of the fundamentals RPE, corresponding to part (i) of Theorem 2.

Figure 3 provides a graphical illustration of the existence and stability conditions established by Theorem 2. In the figure are the sets of (α, ϕ) such that sunspot RPE exist (shaded regions) and are E-stable (cross-hatched regions) for various values of ρ. The relationships between parameters and existence/stability are subtle. For mid-values of ρ and high values of expectational feedback α, sunspot RPE exist for almost all ϕ and they are always stable. Lower values of ρ tend to preclude existence of sunspot RPE, and higher values of ρ introduce the possibility of unstable sunspot RPE. The broad conclusion: the existence of E-stable sunspot equilibria depends, nonlinearly, on the serial correlation properties of the omitted variable and the sunspot as well as the strength of expectational feedback in the model.

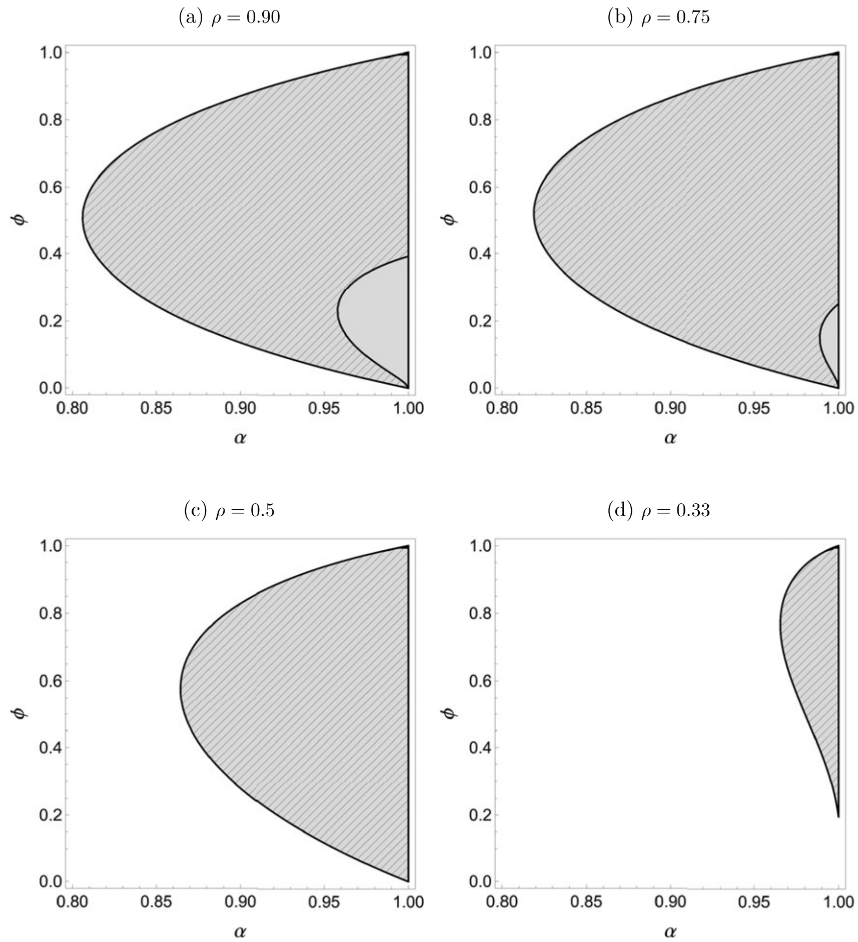


FIGURE 3. E-stability regions.

3.3 *Basins of attraction and mean dynamics*

The E-stability dynamics are suggestive of robust stability under adaptive learning. However, the formal connection between E-stability and stability under real-time adaptive learning via RLS is typically local in nature: we cannot infer the basins of attraction of the RLS algorithm from Figures 2 and 3. Alternatively, these basins can be examined via simulation and through analysis of the RLS algorithm's *mean dynamics*. In this section, we first discuss the concept of mean dynamics and then use it to assess the robustness of our sunspot RPE's stability to variation in initial conditions.

Recursive least squares, as well as other related learning rules, fit under a class of *stochastic recursive algorithms* (SRAs)

$$\begin{aligned}\Phi_t &= \Phi_{t-1} + \kappa_t H(\Phi_{t-1}, W_t),\\ W_{t+1} &= A(\Phi_t)W_t + B(\Phi_t)\zeta_{t+1}.\end{aligned} \tag{14}$$

Think of Φ_t as the estimate that is being updated as new data W_t arrive. In this light, H is the correction to the previous estimate, usually determined by some measure of forecast error, and κ_t is the gain, which, as we discussed earlier, measures how much weight to place on the correction. See Evans and Honkapohja (2001) for a complete discussion.

Now fix a value of Φ and define $h(\Phi) = \lim_{t\to\infty} EH(\Phi, W_t)$, where the expectation is taken against the distribution of W_t implied by the fixed value of Φ. The theory of SRAs tells us that the behavior of the algorithm (14) is closely related to the behavior of the ode $\dot{\Phi} = h(\Phi)$. If, for example, $\gamma_t = t^{-1}$ and Φ^* is a Lyapunov stable fixed point of the ode, then the algorithm (14) will induce convergence to Φ^* when appropriately initialized. Indeed, this result is the basis of the E-stability principle.

The ode $\dot{\Phi} = h(\Phi)$ also speaks to the behavior of (14) under small constant gain $\gamma_t \equiv \gamma > 0$: the trajectory of the ode associated with a given initial condition provides a good approximation to the expected path of the learning algorithm (14), i.e., the *mean dynamics*. In this way, the mean dynamics provide useful information about the transitional learning dynamics even far away from an equilibrium.

Turning to the case at hand, recall the notation $X_t' = (y_{t-1}, \eta_t)$ and $\Theta_t' = (b_t, d_t)$. Let $\Omega(\Theta) = \lim_{t\to\infty} EH(X_t X_t')$, where the expectation is taken against the distribution of X_t induced by the fixed beliefs Θ and the ALM (6). The mean dynamics ode is computed to be

$$\begin{aligned} \dot{\Theta} &= R^{-1}\Omega(\Theta)\big(T(\Theta) - \Theta\big), \\ \dot{R} &= \Omega(\Theta) - R. \end{aligned} \tag{15}$$

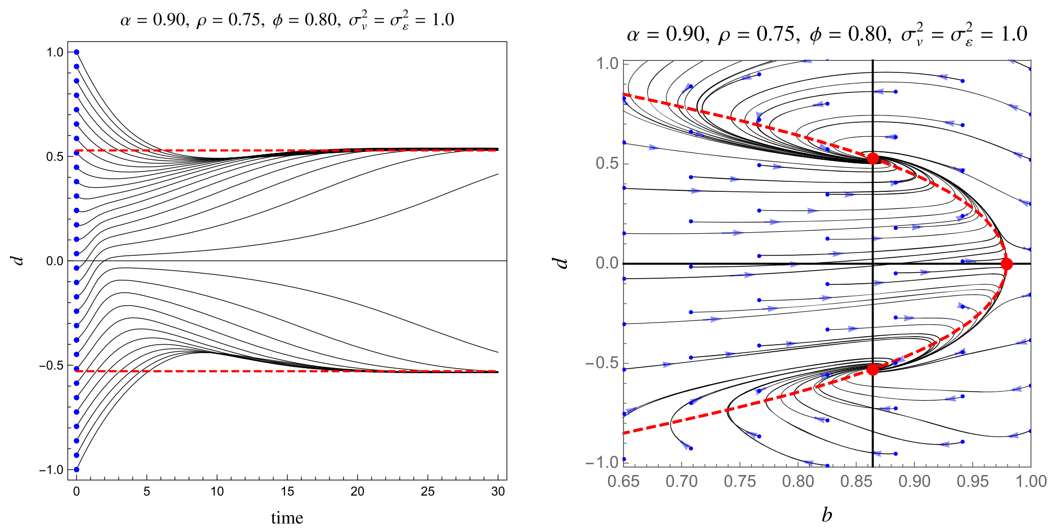
Here R is a 2×2 matrix corresponding to the second-moment matrix of the learning model's regressors. Note the important role played by R: its evolution distinguishes the mean dynamics from those induced by the E-stability ode.

Figure 4 provides results from numerical assessment of the system (15) using the indicated parameterization. Panel 4(a) provides the mean dynamics time paths of the sunspot coefficient d for various initial conditions d_0, as represented by the dots rising vertically above $t = 0$. The horizontal dashed lines identify the upper and lower sunspot RPE values of d. For each time path, $b_0 = \hat{b}$, that is, we start the agent off with beliefs coinciding with the fundamentals RPE.[18] In each case, agents learn to believe in the sunspot RPE and which of the two symmetric sunspot RPEs depends on their initial beliefs.

Note that the full phase space, including the variables Θ and R, is six dimensional. Panel 4(b) provides the projection on the b-d plane of trajectories corresponding to different initial conditions, which now are on a lattice over the b-d plane. The larger dots locate the RPE sunspots, and the dashed curve and the solid lines are the same as in Figure 2. Again, we observe convergence to the sunspot RPE regardless of initial beliefs. For smaller values of b_0, some trajectories converge to the upper sunspot even though $d_0 < 0$.[19] This type of behavior is not predicted by the E-stability dynamics.

[18] For all trajectories in both panels the matrix R was initialized at $\Omega(\Theta)$.

[19] No such behavior is evident in panel 4(a) because $b_0 = \hat{b}$ and $\hat{b}$ has a relatively large value.



(a) Mean dynamics of d with $b_0 = \hat{b}$

(b) Mean dynamics in b-d plane

FIGURE 4. Mean dynamics.

Theorem 2 and Figure 3 show the possibility of unstable sunspot RPE. What happens to the learning dynamics in this case? Numerical examinations indicate that as the sunspot RPE destabilize, a limit cycle emerges, and further, this limit cycle appears to be quite robust. Figure 5 gives an example where the mean dynamics predict a stable limit cycle local to a sunspot RPE (large dot) even when beliefs are initialized (smaller dots) far away.

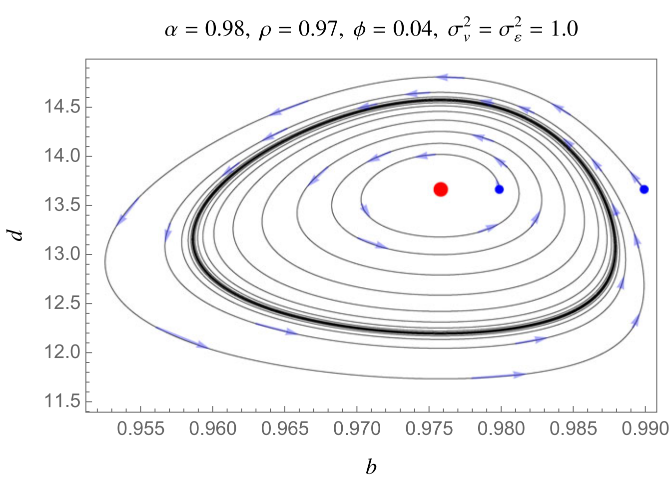
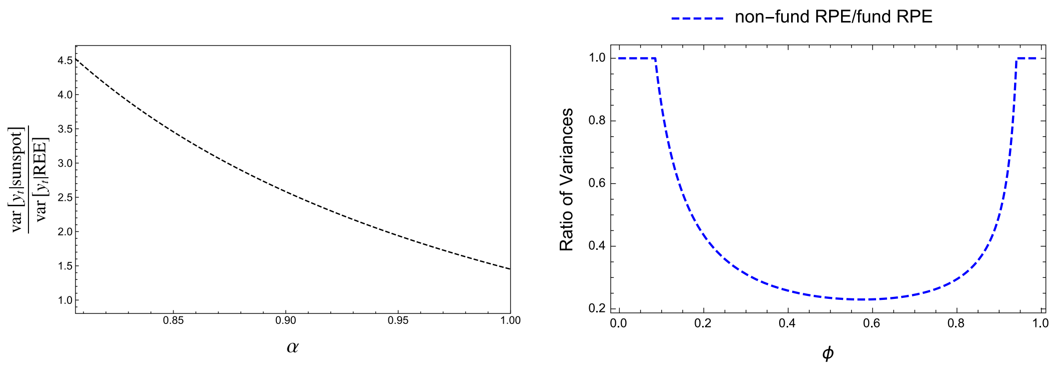


FIGURE 5. Limit cycle.



(a) Relative variance of sunspot RPE to REE. (b) Relative variance of sunspot to fund. RPE.

Figure 6. Volatility in a sunspot RPE.

3.4 *Endogenous fluctuations and economic volatility*

Conventional models of sunspot equilibria are viewed as inefficient since they introduce serial correlation and volatility that would not exist without coordination on the self-fulfilling equilibria. Do the endogenous fluctuations that arise in a sunspot restricted perceptions equilibrium lead to more or less economic volatility? We establish two results. First, the statistical sunspot process, under fairly general conditions, exhibits greater volatility than the equilibrium process under rational expectations. Second, the sunspot RPE exhibits *lower* volatility than the fundamentals RPE. Excess volatility introduced by sunspots is bounded below by the rational expectations equilibrium and above by the fundamentals RPE.

This section explores these issues, first, by establishing the following result on the additional volatility introduced into a determinate economy by sunspot RPE.

Proposition 2. *Let $\alpha > \tilde{\alpha}$ and $\rho > \tilde{\rho}$ so that multiple restricted perceptions equilibria exist. There exists a $\hat{\rho}(\alpha, \phi)$ such that* $\mathrm{var}(y_t|$ *sunspot RPE* $) > \mathrm{var}(y_t|$ *REE* $)$ *for all $\tilde{\rho} < \rho < \hat{\rho}$.*

A graphical illustration of the necessary and sufficient conditions is provided in Figure 6(a) for the numerical example $\alpha = 0.9$, $\rho = 0.7$, and $\phi = 0.4$. The statistical sunspot RPE brings additional volatility into the economy above and beyond rational expectations, for any sunspot innovation process, provided that the serial correlation of the fundamental shock z_t is within an interval with an upper bound. When might a rational expectations equilibrium be more volatile than the statistical sunspot? In the extreme cases of α close to the indeterminacy region and ρ close to 1.

The next proposition presents the surprising and yet intuitive result that the fundamentals RPE is more volatile than statistical sunspots—and rational expectations equilibria.

Proposition 3. *Let $\alpha > \tilde{\alpha}$ and $\rho > \tilde{\rho}$ so that multiple restricted perceptions equilibria exist. For ρ sufficiently large, the fundamentals RPE is more volatile than the sunspot RPE.*

Numerical analysis suggests that the result holds for a broad set of parameters. The intuition for the finding is as follows. Imagine an increase in ϕ, i.e., making the statistical sunspot more volatile. This has offsetting effects for relatively large ϕ.[20] On the one hand, it increases b^*, which tends to lift the variance of the sunspot RPE. On the other hand, d^* decreases and less weight is placed on the extrinsic shock η whose serial correlation properties no longer track the fundamental shock z as well. This, in turn, tends to push down the sunspot RPE's variance. As ρ becomes large enough, the weight on the sunspot is sufficiently small so that the fundamentals RPE is more volatile. In general, the relative variances of the RPE depend on the elasticities of the RPE coefficients b and d. Numerical explorations suggest that the effect of a lower self-fulfilling serial correlation via the b coefficient outweighs the impact on the d coefficient, as seen in Figure 6(b). This figure plots the excess volatility of the two RPEs, i.e., the ratio of the RPE variance to the variance in the rational expectations equilibrium, as a function of ϕ.[21] The figure clearly demonstrates the non-monotonic effect of ϕ on the variance of the sunspot RPE. From a relatively high value of ϕ, the comparative static effect of decreasing ϕ is to decrease b^* and increase d^*, as the sunspot better tracks the serial correlation in the model. However, for some lower value of ϕ, the comparative static effect of decreasing ϕ then increases b^* and decreases d^* as the RPE becomes closer to the fundamental RPE. In between these critical values there is a non-monotonic effect from ϕ on economic volatility in the nonfundamentals RPE.

The results presented in this section have important empirical implications that we address in Section 5.

4. Robustness and discussion

Because the fundamental shock z is hidden, and the forecasting model used by agents within a restricted perceptions equilibrium is misspecified, it is reasonable to question the robustness of statistical sunspots as an equilibrium concept. There are, however, several ways in which the sunspot RPE may be robust despite this misspecification.

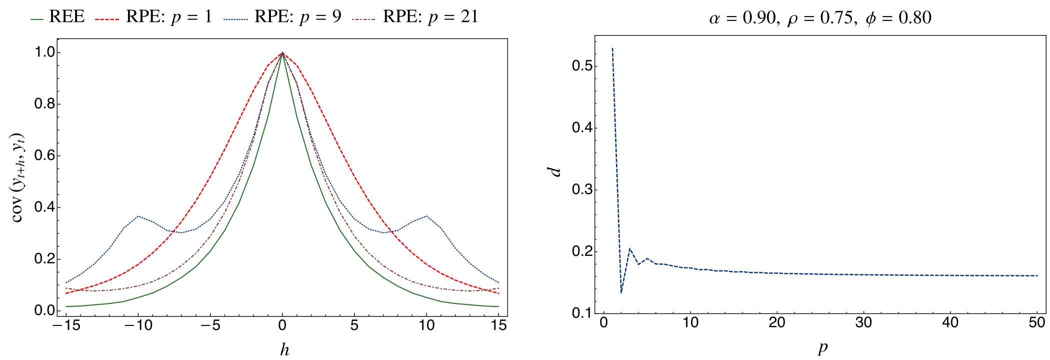
(i) *Robustness of sunspot RPE to AR(p) perceived laws of motion.* The focus, so far, has been on the case where the PLM is an AR(1) plus sunspot. This is the simplest case that facilitates precise statements on existence and E-stability. However, Section 2.3 showed that if the agents hold an AR(∞) perceived law of motion, without a sunspot term, then the agent could recover the full information rational expectations equilibrium. A natural question, then, is do sunspot RPE exist for AR(p) plus sunspot PLMs and $p > 1$? General results are not available so we continue with our numerical example: $\alpha = 0.90$, $\rho = 0.75$, $\phi = 0.8$, and $\sigma_\varepsilon = \sigma_v = 1$. Now assume a perceived law of motion of the form

$$y_t = b_1 y_{t-1} + \cdots + b_p y_{t-p} + d\eta_t.$$

Figure 7 illustrates the existence of sunspot RPE and compares the autocorrelation properties of the RPE to the REE.

[20] In fact, the effect of ϕ on the b^* is first decreasing and then increasing.

[21] The parameter values here are $\alpha = 0.95$, $\rho = 0.6$, $\gamma = 1$, and $\sigma_\varepsilon^2 = \sigma_\nu^2 = 1$.



(a) Autocovariogram.

(b) AR(p) sunspot RPE: $p = 1, 2, \ldots, 50$.

FIGURE 7. AR(p) statistical sunspots.

Figure 7(b) plots the RPE value for the sunspot coefficient d as a function of the lag length p in the PLM. For this parameterization, a sunspot RPE exists for PLMs with lag lengths up to at least 50. For PLMs with $p = 10$ or more, there is a very modest comparative static effect on d from lengthening the lags in the forecasting equation. Figure 7(a) plots the autocovariogram for the REE and the RPE across $p = 1, 9, 21$. The solid line corresponds to the full information REE or what would arise if agents were to forecast with an AR(∞) and not condition on a sunspot. The sunspot RPE are more autocorrelated than the REE, though as the lag length p increases, the RPE covariograms reflect a similar autocorrelation structure as the REE, especially for lower-order autocorrelations.

From Figure 7 one can see how sunspot RPE could be robust to global (mis-) specification tests. In most real-world settings, forecasters face data and/or degrees-of-freedom limitations that lead them to forecast with parsimonious, in p, models. That AR(p) model, though, is misspecified, and as more and more data accumulate, the agent may detect the misspecification and consider an alternative lag length based on an econometric procedure like the Akaike information criterion (AIC). Figure 7 shows that, at least for some parameterizations of the model, a sunspot RPE continues to exist under the alternative PLM specification. A specification-testing process like this could continue until the agent arrives at the AR(∞) forecasting equation. It is not obvious whether sunspot RPE would continue to exist for all p, but Figure 7(b) suggests the possibility.[22] Regardless, over the medium-term, sunspot RPE would remain robust to this particular form of global misspecification testing.

(ii) *The sunspot RPE appear to be robustly learnable and the fundamental RPE is not.* The E-stability results showed that, for a region of the parameter space where sunspot RPE exist, the statistical sunspot equilibrium is the only E-stable RPE. Through numerical analysis of the mean dynamics, it was indicated that if agents

[22]In ongoing research, we explore the stability of statistical sunspots where the PLM also nests the REE by letting $p \to \infty$ or relaxing B.3.

place even the smallest prior on the sunspot, i.e., $d \approx 0$, then they will learn to believe in sunspots and the economy will converge to the statistical sunspot equilibrium. Even outside that region of the parameter space, Figure 5 provided an example where the sunspot RPE is not E-stable and the learning dynamics converge to a limit cycle that depends explicitly on the statistical sunspot with time-varying estimates for (b, d). In this sense, statistical sunspots are robust to adaptive learning.

(iii) *The sunspot RPE is robust to local econometric (mis-)specification tests.* A local econometric (mis-)specification test would recursively apply, for instance, a likelihood ratio test against the null that their AR(p) econometric model of the economy is a good descriptor of the data. Employing that idea in the present environment, the equilibrium itself is defined so that if agents within the statistical sunspot RPE were to test their model against a (local) alternative, they would fail to reject the null. This is because the likelihood ratio test statistics are constructed based on sample estimates of the least-squares "orthogonality" term $X_t(y_t - X_t'\Theta)$, which is equal to zero, on average, within the RPE with a fixed Θ at its RPE value. It would be an interesting exercise for future research to study whether learning can "escape" from a sunspot RPE by employing the methodology developed by Cho and Kasa (2015).

(iv) *Incentives to deviate from a sunspot RPE.* We now ask whether individual agents face an incentive to deviate from a sunspot RPE. Without loss of generality, assume that the choice of forecasting model is a statistical one and agents assess their forecasting success based on mean-squared forecast errors. Within a sunspot RPE, each agent can calculate his/her mean-square forecast error (MSE) with the RPE as

$$\mathrm{MSE}^1 = E\big(y_{t+1} - E_t^1 y_{t+1}\big)^2.$$

Now imagine that a single agent with zero mass were able to costlessly recover the full-information REE and now contemplates forecasting with the fundamentals model

$$E_t^2 y_{t+1} = c\rho z_t,$$

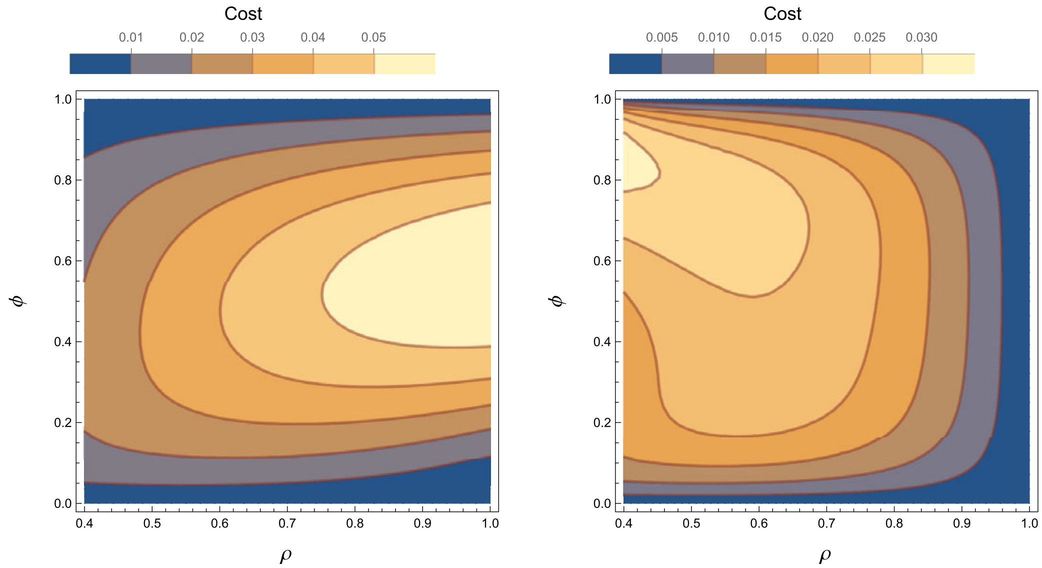
where $c = Ey_t z_t / Ez_t^2$.[23] This forecast model produces mean forecast errors

$$\mathrm{MSE}^2 = E\big(y_{t+1} - E_t^2 y_{t+1}\big)^2.$$

We will say that a sunspot RPE is robust if $\mathrm{MSE}^1 < \mathrm{MSE}^2$. We have the following result on whether a sunspot RPE is robust, given that it exists.

Proposition 4. *Let $\tilde{\alpha} < \alpha < 1$ and $\tilde{\rho} < \rho < 1$, so that multiple RPE exist. The sunspot RPE are robust.*

[23] We also found robustness if the zero-mass agent were to set c equal to its rational expectations equilibrium value.



(a) Isocost: $\alpha \to 1$.

(b) Isocost: $\alpha \to (1 + \phi - \phi^2)^{-1}$.

FIGURE 8. Incentive to deviate to rational expectations. Each isocost line represents, given α, the $\bar{C}$ such that a sunspot RPE is robust for any $C \geq \bar{C}$. Cost is normalized by $\mathrm{var}(y_t)$.

Proposition 4 shows that, within a sunspot RPE, an individual agent would not have an incentive to deviate to a PLM that depends only on the hidden shock. This result is not surprising because when the other agents forecast with the AR(1) plus sunspot model, then the zero-mass agent's model is also misspecified since it does not include the sunspot shock.

Assume instead that the zero-mass agent is able to recover the actual law of motion, via rational expectations, though possibly at some cost C. That is, suppose that an individual agent holds the PLM,

$$y_t = \delta_0 y_{t-1} + \delta_1 \eta_t + \delta_2 z_t,$$

where $\delta_0 = \alpha b^2$, $\delta_1 = \alpha d(b + \phi)$, and $\delta_2 = \gamma$. The mean-squared forecast error in this case is $\delta_1^2 \sigma_v^2 + \delta_2^2 \sigma_\varepsilon^2$. If recovering the actual law of motion is costless, then agents will obviously have an incentive to use the alternative model and the sunspot RPE is not robust to rational expectations. We now ask, "What would the cost to acquiring rational expectations have to be for the sunspot RPE to be robust to rational expectations?"

Figure 8 illustrates the critical cost threshold $\bar{C}(\alpha, \rho, \phi)$ such that the sunspot RPE are robust to rational expectations for all $C \geq \bar{C}$. The figure plots two slices of the plane holding the feedback parameter α at its smallest and largest values required for sunspot RPE to exist. We have normalized $\bar{C}$ by the equilibrium variance of y_t. Thus, a $\bar{C} = 0.02$ carries the interpretation that the sunspot RPE are robust against rational expectations as long as the cost of acquiring rational expectations is 2 percent or greater of the equilibrium variance. The isocost curves in Figure 8 show that a cost of 5% or greater will lead to a robust sunspot RPE across all alternative parameterizations of α, ρ, and ϕ. The cost can be as low as 1% if the sunspot is sufficiently serially correlated.

This sense in which the sunspot RPE are robust if a single agent with zero mass contemplated deviating from the equilibrium is similar to a condition required within the "exuberance equilibrium" identified by Bullard et al. (2008). The "near-rational" expectations developed by Bullard et al. (2008) assume that the agents in the economy form expectations by first fitting an autoregressive moving average (ARMA) model to historical data and then adjusting their econometric forecasts by incorporating judgment, modeled as a purely extrinsic shock like what we explore in this paper. An exuberance equilibrium arises when the ARMA model used by agents is the best-fitting model, in a least-squares sense, and a zero-mass agent would prefer to incorporate the judgment/sunspot given that all other agents are incorporating it. Finally, an exuberance equilibrium must also lead to ARMA coefficients that would be recoverable using a least-squares learning procedure. The robustness contemplated by Bullard et al. concerns whether including the judgment variable in the PLM is a best response. In this section, we considered robustness in the sense that a zero-mass agent would prefer a different PLM.

There are, evidently, many similarities behind the ideas in this paper and those developed in Bullard et al. (2008). We close this section by briefly comparing the two approaches. In short, the Bullard et al. paper has a distinct motivation and interpretation of the sunspot variable, whereas we have a sharper equilibrium selection and we are able to demonstrate a stronger robustness.

The exuberance equilibrium is motivated by a setting where a decisionmaker turns over forecasting to two distinct agents, one who generates a purely statistical forecast from an ARMA model and the other who adjusts the statistical forecast with qualitative judgments, modeled as a purely extrinsic noise term; hence, the similarity. The exuberance equilibrium requires that (1) the autocovariance generating function of the ARMA model matches the same object in the true data generating process, (2) that those ARMA coefficients are recoverable from a reasonable learning algorithm, and (3) that including the qualitative judgment is robust in the sense that a zero-mass agent would want to incorporate the judgment given that everyone else is. There are several differences from our equilibrium. First, our agents obey the "cognitive consistency" principle by formulating a purely econometric model to generate forecasts of aggregate variables beyond their control. We simply allow them to place a prior that the sunspot variable—which is purely extrinsic and potentially serially correlated—improves their forecasts. Second, we show that when the sunspot equilibrium exists and the agents condition on the sunspot in their PLM, the sunspot equilibrium is stable under learning and the fundamental equilibrium is not stable. The exuberance equilibrium is stable with or without the judgment in their New Keynesian application. Thus, we have a strong equilibrium selection result. Third, Bullard, Evans, and Honkapohja (2008) point out that the exuberance equilibrium would not be robust to a decision-maker who optimally weights the purely statistical forecast relative to the judgment component. The weight on the statistical sunspot in the restricted perceptions equilibrium, alternatively, does reflect this optimal weighting as its coefficient in the forecast rule comes from the projection of the observable variable onto the sunspot.

The existence of statistical sunspots is self-fulfilling in the sense that the agents' econometric model is capturing, in part, the serial correlation observed in the data that would not exist if the agents did not believe it was there. That is, by forecasting with the AR(1) plus sunspot model the agents are introducing additional serial correlation into the model that is self-confirming due to the self-referential properties of the model. This is the intuition for why a large feedback effect is required to establish existence. As we show in the next section, the restrictions that the equilibrium places on the statistical sunspot process lead to distinct policy implications since the fundamental equilibrium is more volatile than the sunspot RPE. These distinctions do not diminish the contribution made by Bullard, Evans, and Honkapohja. The two approaches are clearly related and their papers were highly influential on the development of our theory.

5. Applications

We turn now to two economic applications that illustrate the practical and empirical relevance of statistical sunspots. The first application addresses the excess volatility puzzle in rational expectations models of stock prices (Shiller (1981)). The second application is a New Keynesian model where the central bank commits to an optimal instrument rule and revisits the proposition that nonrational expectations imply a more "hawkish" monetary policy rule (cf. Orphanides and Williams (2005a) and Eusepi and Preston (forthcoming)). We show that across a broad range of potential policymaker preferences for relative price stability, the optimal rule is consistent with the existence of a sunspot restricted perceptions equilibrium. We also show that the optimal rule within the class considered is less aggressive against inflation than it would be in a perfect information environment.

5.1 *Excess volatility in asset prices*

Shiller (1981) estimates that stock prices are roughly five times more volatile than one would expect if prices reflected the expected present value of future dividends, i.e., under rational expectations. The previous section established that in environments with hidden variables, a statistical sunspot equilibrium exhibits volatility bounded below by rational expectations and above by the fundamental restricted perceptions equilibrium. Here we adopt the calibrated asset-pricing model in Branch and Evans (2010) and pin down the serial correlation in the sunspot process to generate excess price volatility in line with Shiller's estimate.

The environment is a mean–variance linear asset-pricing model similar to De Long, Shleifer, Summers, and Waldmann (1990). Investors are mean–variance maximizers who allocate wealth across two assets.[24] There is a risk-free asset that pays a gross rate

[24] Branch and Evans (2011) derive the mean–variance asset-pricing equations from an OLG model with a stochastically fluctuating population of young agents. Given beliefs about next period's price and dividend, young agents solve a portfolio optimization problem with CARA preferences. Market equilibrium is perturbed by fluctuations in the per capita outside supply of the asset, an unobservable variable. We refer the reader to Branch and Evans (2011) for details.

of return $R = \beta^{-1} > 1$, where β is the discount factor, and a risky asset that yields a (stochastic) dividend stream $\{q_t\}$ and trades at *ex dividend* price p_t. In this framework, demand for the risky asset is

$$z_{dt} = \frac{\hat{E}_t(p_{t+1} + q_{t+1}) - \beta^{-1} p_t}{a\sigma^2},$$

where $\hat{E}_t(p_{t+1} + q_{t+1})$ is the subjective expectation of the one-period-ahead payoff, a is the coefficient of risk aversion, and σ^2 is the variance of excess returns $p_{t+1} + q_{t+1} - \beta^{-1} p_t$. The equilibrium price p_t is determined by market-clearing, i.e., $z_{dt} = z_t$, where z_t is the (stochastic) supply of the risky asset. The term z_t captures fluctuations in outside share supply, also called asset float, and proxies for variations in the availability of publicly tradable shares. There is empirical evidence that fluctuations in asset float can be an important determinant of asset prices (see Baker and Wurgler (2000)). Here we assume that dividends, q_t, are observable and asset float, z_t, is unobserved by traders.

The equilibrium price equation is

$$p_t = \hat{E}_t(p_{t+1} + q_{t+1}) - a\sigma^2 z_t,$$

which takes the same form as (1), extended to include two exogenous shocks. We assume that dividends and share supply are determined by a pair of (uncorrelated) stationary AR(1) processes

$$q_t = \delta q_{t-1} + u_t,$$
$$z_t = \rho z_{t-1} + \varepsilon_t.$$

After imposing the behavioral primitives, traders' beliefs will come from a linear forecasting rule for stock prices that depends on the observables, lagged stock prices, dividends, and any other information in the form of the statistical sunspot η. The perceived law of motion, therefore, is

$$p_t = bp_{t-1} + cq_t + d\eta_t \quad \Rightarrow \quad \hat{E}_t p_{t+1} = b^2 p_{t-1} + c(b + \delta)q_t + d(b + \rho)\eta_t.$$

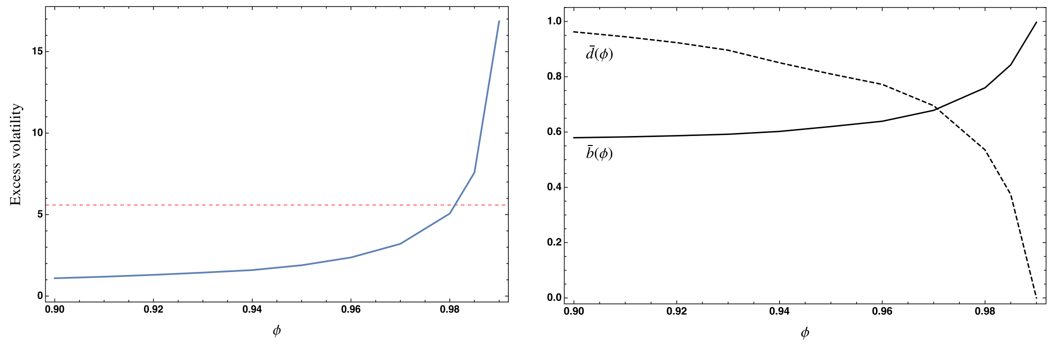
As before, the statistical sunspot follows:

$$\eta_t = \phi\eta_{t-1} + v_t.$$

The values for the coefficients (b, c, d) are pinned down in a restricted perceptions equilibrium, following the same steps from earlier in the paper.[25]

We now show that the model can capture the magnitude of excess stock price volatility estimated by Shiller (1981). We begin by adopting the calibration in Branch and Evans (2010). For this exercise, the key parameters are the autoregressive parameters and innovation variances for the dividend and share supply processes. Data on real dividends are provided by Shiller and the share supply series is from Baker and Wurgler (2000). Our

[25]The presence of multiple fundamental shocks raises the possibility of multiple fundamental RPE. In the calibration, though, there is a unique fundamental RPE.



(a) Excess price volatility (dash=target). (b) RPE belief coefficients as function of ϕ.

FIGURE 9. Excess price volatility in calibrated asset-pricing model.

resulting estimates lead us to set $\delta = 0.95$, $\rho = 0.8837$, $\sigma_u = 0.0022$, and $\sigma_\varepsilon = 1.7$. We set $\beta = 0.9975$, based on the one-month risk-free rate, and the risk parameter, $\sigma^2 = 0.1233$, is based on the excess return volatilities in Guidolin and Timmermann (2007). Finally, we set the coefficient of risk aversion $a = 2$.

Figure 9 presents the results from the quantitative exercise. Figure 9(a) plots stock price volatility in a restricted perceptions equilibrium, relative to volatility in the rational expectations equilibrium, as a function of ϕ, the autoregressive coefficient in the statistical sunspot variable. That is, the plot measures excess volatility. The figure sets $\sigma_v^2 = 0.1$ and focuses on the empirically relevant range for ϕ. A more serially correlated sunspot increases excess volatility. The dashed line is the Shiller target for excess volatility. A value of $\phi \approx 0.98$ delivers an excess volatility in equilibrium consistent with the data.[26]

Panel 9(b) gives further insight into the excess volatility by plotting the E-stable RPE values for b, d as a function of ϕ. As the sunspot becomes more serially correlated, the equilibrium weight, d, placed on the sunspot in the forecasting equation decreases and the autoregressive coefficient b increases. As the sunspot approaches a driftless random walk, the sunspot equilibrium no longer exists and the fundamental RPE exhibits an excess volatility several times larger than what is observed in the data.

5.2 *Optimal monetary policy*

A seminal result by Orphanides and Williams (2005a) is that for economies with nonrational agents who update their forecasting models using an adaptive learning rule, the optimal monetary policy rule involves a more aggressive response to inflation deviations from target.[27] The intuition behind this well known result is that with nonrational expectations, the central bank seeks to minimize inflation volatility in order to help anchor

[26] In this exercise, the identified value of ϕ depends on the normalized value of σ_v. A precise value for ϕ would require another moment to match, which is beyond the scope of this application.

[27] For a general discussion of this robust finding, see the excellent survey by Eusepi and Preston (forthcoming), in particular, Result 5a. See, also, Stiglitz (2019).

private-sector expectations. This section revisits Orphanides and Williams (2005a) with our theory of restricted perceptions and endogenous volatility. We find that in a New Keynesian economy, with hidden variables and homogeneous expectations, a statistical sunspot equilibrium will exist under optimal policy (within the class of rules considered) and the optimal policy response to inflation innovations is less aggressive under restricted perceptions than rational expectations.

5.2.1 *A new Keynesian model with hidden variables* We adapt Orphanides and Williams (2005a) to the present environment,

$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \kappa y_t + u_t, \tag{16}$$

$$y_t = x_t + z_t, \tag{17}$$

where π_t is the inflation rate, y_t is the output gap, and u_t and z_t are aggregate supply and aggregate demand shocks, respectively. Equation (16) is a standard New Keynesian Phillips curve that comes from the aggregate supply block. Without loss of generality, assume that $\sigma_u^2 \to 0$. Equation (17) is the aggregate demand equation and it relates the output gap to the central bank's policy variable, x_t, up to noise z_t. As before, we assume that $z_t = \rho z_{t-1} + \varepsilon_t$. The central bank is able to perfectly control aggregate demand up to a serially correlated exogenous noise. The central bank treats z_t as exogenous to their policy. We assume that this control error is unobservable to private-sector agents.

The central bank has an optimal instrument rule of the form

$$x_t = -\theta(\pi_t - \bar{\pi}), \quad \theta \geq 0,$$

where $\bar{\pi}$ is the long-run inflation target, which we set to $\bar{\pi} = 0$. The form of this policy rule is what Orphanides and Williams (2005b) refer to as an "outcome-based" rule that adjusts the output gap (aggregate demand) whenever inflation deviates from target. The central bank chooses the coefficient in its policy rule in order to minimize a quadratic loss function

$$\mathcal{L} = (1 - \lambda)\breve{E} y_t^2 + \lambda \breve{E} \pi_t^2,$$

where $\breve{E}$ is the central bank's unconditional expectations operator.

Given that z_t is unobservable, the agents in the economy formulate their expectations from the forecasting rule

$$\pi_t = b\pi_{t-1} + d\eta_t \quad \Rightarrow \quad \hat{E}_t \pi_{t+1} = b^2 \pi_{t-1} + d(b + \phi)\eta_t,$$

where, again, $\eta_t = \phi\eta_{t-1} + v_t$. Plugging expectations, the policy rule, and the aggregate demand equation into (16) produces the actual law of motion for inflation,

$$\pi_t = \frac{\beta b^2}{1 + \kappa\theta}\pi_{t-1} + \frac{\beta d(b + \phi)}{1 + \kappa\theta}\eta_t + \frac{\kappa}{1 + \kappa\theta} z_t,$$

which is the same form as (6) with $\alpha = \beta/(1 + \kappa\theta)$ and $\gamma = \kappa/(1 + \kappa\theta)$. Notice that there is a unique rational expectations equilibrium for all θ. Given a value for θ, the belief coefficients (b, d) are pinned down in a restricted perceptions equilibrium. We denote the RPE inflation process as $\pi_t(\theta)$.

5.2.2 *Optimal policy* Under optimal policy, there is a symmetric Nash equilibrium where the central bank's policy rule and the beliefs of agents are determined simultaneously. Then the optimal policy rule is determined according to

$$\theta^*(\lambda) = \arg\min_\theta \big[(1-\lambda)\theta^2 + \lambda\big] E\big[\pi_t(\theta)\big]^2 + t.i.p.$$

and $E[\pi_t(\theta)]^2 = a_1 E z_t^2 + a_2 E \eta_t^2$, where a_1 and a_2 are complicated expressions of the model parameters θ, β, κ, ρ, and ϕ, and E is now the unconditional mathematical expectation taken with respect to the distributions of z_t and η_t.[28] The resulting optimal monetary policy is, therefore, also a complicated function of these parameters and a closed-form solution is unavailable.

We also compute the equilibrium outcomes when the policy rule is derived under perfect information, that is, rational expectations with z_t observable to agents. This alternative scenario facilitates a comparison of the optimal θ under restricted perceptions to rational expectations and also allows us to consider an experiment where the central bank mistakenly assumes that the private sector has perfect information about aggregate demand. Under perfect-information rational expectations, the optimal policy rule solves

$$\theta^R(\lambda) = \arg\min_\theta \big[(1-\lambda)\theta^2 + \lambda\big] \left(\frac{\kappa}{1+\kappa\theta-\beta\rho}\right)^2 \frac{\sigma_\varepsilon^2}{1-\rho^2}.$$

From the associated first-order condition, it is straightforward to show that

$$\theta^R(\lambda) = \frac{\kappa\lambda}{(1-\lambda)(1-\beta\rho)}$$

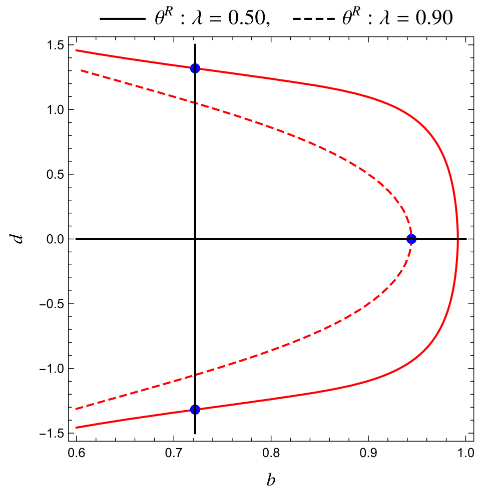
and the optimal inflation response is increasing in λ.[29]

Since $\theta^*(\lambda)$ does not have a closed-form solution, we proceed numerically. We set $\beta = 0.99$ and $\kappa = 0.10$, both within the empirically relevant range. As a benchmark, we set $\rho = 0.5$, $\phi = 0.75$, and $\sigma_\varepsilon = \sigma_v = 1$, and focus on how λ impacts the existence of statistical sunspot RPE and the optimal policy rule under restricted perceptions and rational expectations. Figure 10 plots the T-maps for $\lambda = 0.5, 0.9$.

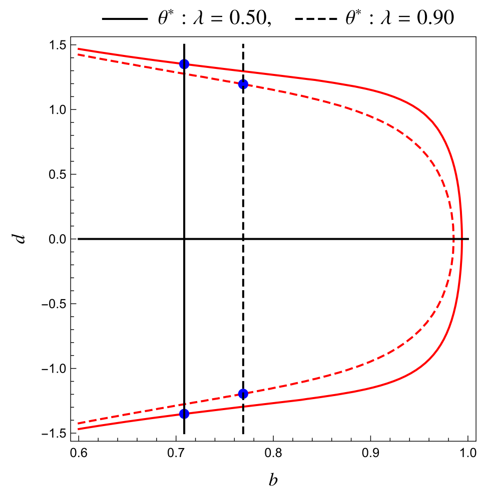
Figure 10 provides examples of how sunspot RPE might emerge in a New Keynesian model where monetary policy is set according to a rule that guarantees existence of a unique rational expectations equilibrium. In each panel, the solid (dashed) lines correspond to $\lambda = 0.5$ ($\lambda = 0.9$), the relative weight on inflation stabilization in the policymaker's loss function. Panel 10(a) plots the T-maps associated to two different values

[28] We make the assumption on the central bank's beliefs, under $\check{E}$, that the policymaker treats the control error as exogenous to his/her policy instrument so that $\check{E}(x_t z_t) = 0$. This assumption does not impact the qualitative finding regarding optimal policy under restricted perceptions. Instead, the assumption is necessary for the optimal policy under the rational expectations problem to have a bounded solution in order to facilitate a comparison to Orphanides–Williams. Relaxing this restriction so that $\check{E}(x_t z_t) \neq 0$ leads to a finite θ under restricted perceptions but not rational expectations.

[29] This optimal value θ^R is the direct effect of z on π, in an REE, times the central bank's preference for inflation stabilization relative to output gap stabilization.



(a) θ^R



(b) θ^*

FIGURE 10. T-maps for policy rules θ^R and θ^*, and $\lambda = 0.5, 0.9$.

for λ when policy is set according to the rule with θ^R, the policy rule that is optimal under rational expectations, but private-sector agents have restricted perceptions. For $\lambda = 0.5$ (more generally, for λ not too large), the optimal policy rule delivers a restricted perceptions equilibrium driven by statistical sunspots. If the policymaker places a high weight on inflation stabilization (dashed line in Figure 10(a)), then there is a unique fundamentals RPE. Panel 10(b) plots the T-maps for the same values of λ, but where policy optimally responds to minimize the loss in a restricted perceptions equilibrium. In these cases, optimal policy is consistent with a sunspot RPE. Notice that as λ increases, $\hat{b}$, d^* decreases and b^* increases.

To provide more general results for the optimal policy θ^*, we are able to establish the following analytic result.

PROPOSITION 5. *Let $\beta \to 1$. If λ is sufficiently small, then sunspot RPE exist in the New Keynesian model with optimal policy and restricted perceptions.*

To gain greater insights into these existence results and the policy implications, Figure 11 plots θ^R and θ^* as a function of λ. Both policy rules exhibit optimal reaction coefficients that are increasing in the relative preference for inflation stabilization: more hawkish preferences lead to more aggressive policy reactions to inflation innovations. However, it is apparent that the optimal policy under restricted perceptions features less active policy than what is optimal under rational expectations. This result is surprising in light of the findings in Orphanides–Williams and a large number of other papers that nonrational expectations implies monetary policy should react more strongly in response to inflation deviations from target.

Figure 12 provides some insight into optimal policy under statistical sunspots, and the intuition underlying Figure 11. Panels 12(a) and 12(b) plot the sunspot RPE policymaker losses and inflation variance as a function of the policy coefficient θ. The

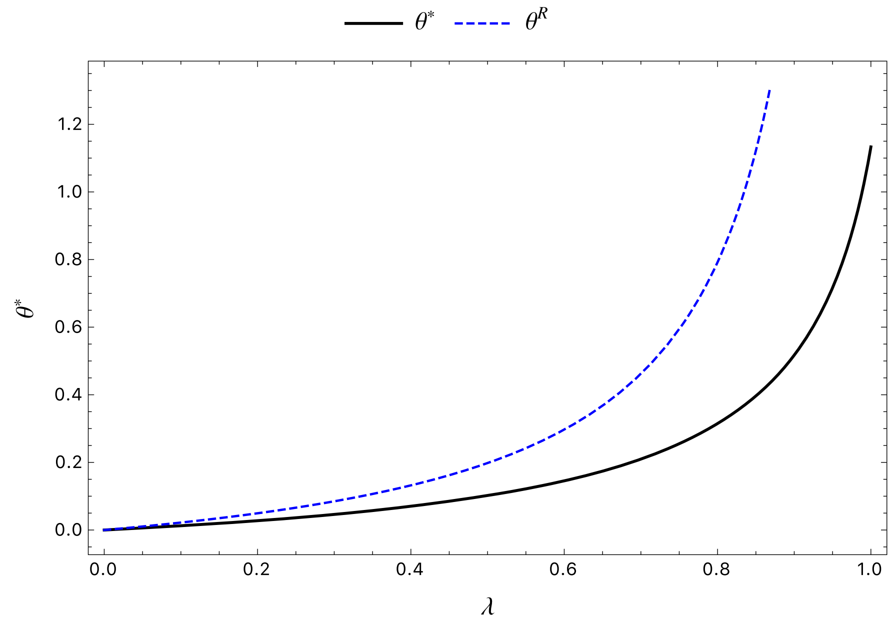
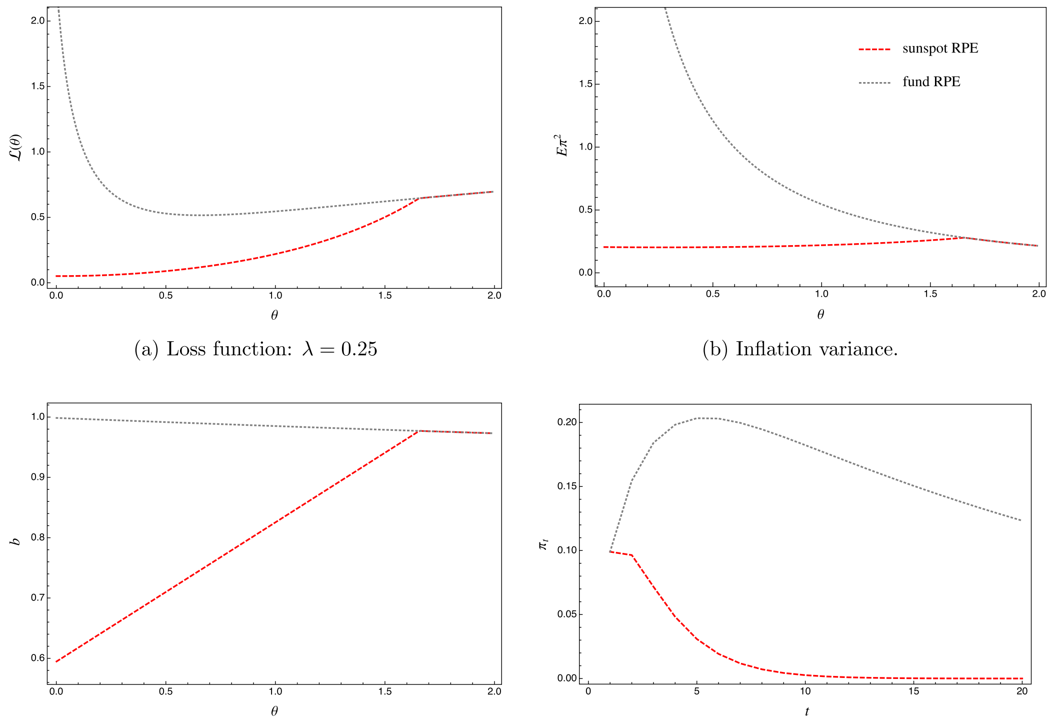


Figure 11. Optimal policy rules: θ^R versus θ^*.

dashed line corresponds to the sunspot RPE and the dotted line is the fundamental RPE. For θ large enough the sunspot RPE will not exist: this is equivalent to the low α case in Section 3. For smaller values of α, though, inflation variance is lower in the sunspot RPE than the fundamentals RPE, as we saw earlier. This translates, as well, into lower losses. Thus the optimal policy will choose a lower policy coefficient and, through the learning stability, the economy will converge to the lower volatility sunspot equilibrium. Figure 12(c) confirms the intuition that the sunspot RPE leads to a lower perceived autocorrelation and, consequently, a lower economic volatility. Figure 12(d) plots the impulse response to a positive shock to the hidden fundamental variable z. With the much higher perceived autocorrelation, the fundamental RPE impulse response delivers a strong and persistent impact on inflation. Conversely, the sunspot RPE displays a geometric decay and very little persistence. Of course, in the sunspot RPE, the policymaker has to balance that stronger response from the fundamental shock against the sunspot shock. However, we have shown that there is less economic volatility in a sunspot versus fundamental RPE.

6. Conclusion

The results in this paper show that sunspot equilibria can exist in models with a unique rational expectations equilibrium. When some state variables are unobserved by or hidden from agents who have restricted perceptions, they specify optimal parsimonious forecast models. The insight in this paper is that while certain fundamental, i.e., payoff relevant, variables may be hidden from agents, they may end up coordinating on an equilibrium that depends, in a self-fulfilling manner, on extrinsic variables that we



(a) Loss function: $\lambda = 0.25$ (b) Inflation variance.

(c) RPE AR(1) coeff. (d) Impulse response to z ($\theta = 0.1$).

Figure 12. Further intuition.

call statistical sunspots. These statistical sunspots overcome two limitations of sunspot theories based on rational expectations: they exist in the empirically relevant range of models, i.e., within the determinacy region, and the sunspot equilibria are often stable under learning.

This paper focuses on the theoretical properties of statistical sunspots with applications to excess volatility in stock prices and optimal monetary policy. The theory of statistical sunspots, though, has broad practical interest for dynamic stochastic general equilibrium (DSGE) models. Under appropriate conditions, statistical sunspots exist in standard formulations of real business cycle models; that is, sunspot equilibria can exist without relying on nonconvexities. Statistical sunspots can also exist in New Keynesian models with optimal monetary policy or Taylor-type rules that respect the "Taylor principle." In particular, policy advice to rule out expectations-driven cycles, i.e., unanchored expectations, is more subtle than what one would conclude under strict rational expectations.

Appendix

Proof of Theorem 1. The fundamental RPE solves $b = (\alpha b^2 + \rho)(1 + \alpha b^2 \rho)^{-1} \equiv g(b)$ with $|b| < 1$. A fundamentals RPE exists since $g(b)$ is continuous with $g(0) > 0$, $g(1) < 1$. Uniqueness is established in Appendix D of Hommes and Zhu (2014).

A sunspot RPE exists if and only if $|b^*| < 1$ and $d^{*2} > 0$. It is straightforward to see that $|b^*| < 1 \Leftrightarrow \alpha > 1/(1+\phi-\phi^2)$. Similarly, $d^{*2} > 0 \Leftrightarrow \rho > \tilde{\rho}$, where

$$\tilde{\rho} = \frac{\alpha^2\phi - \alpha\phi^2 - \alpha^3\phi^2 + \alpha\phi^4 + \alpha^3\phi^4 - \alpha^2\phi^5}{\alpha^2 + \alpha^3\phi^3 + 3\alpha^2\phi^4 - 6\alpha^2\phi^2 - \alpha^2\phi^6 - 1 + 3\alpha\phi}.$$

With the aid of a computer algebra system, we find that $\alpha > \tilde{\alpha} \implies \tilde{\rho} < 1$.[30] □

Proof of Proposition 1. We can prove this result by showing that $b^* < g(b^*) \equiv T_b(b^*, 0)$, which holds given $\tilde{\alpha} < \alpha < 1$ and $\tilde{\rho} < \rho < 1$. We have already shown that $\exists$ unique $\hat{b}$ such that $\hat{b} = g(\hat{b})$, $g(0) > 0$, $g(1) < 1$. If $b^* > \hat{b}$, then there would exist another fixed point in the interval $(b^*, 1)$—a contradiction. Therefore, $b^* < \hat{b}$. □

Proof of Theorem 2. Items are addressed in order.

(i) In case agents do not regress on the sunspot, the E-stability ode is given by

$$\dot{b} = \frac{\alpha b^2 + \rho}{1 + \alpha b^2 \rho} - b.$$

As shown in Hommes and Zhu (2014), the resting point $\hat{b}$ is Lyapunov stable.

(ii) If agents do regress on the sunspot, then the eigenvalues of DT evaluated at $d = 0$ are given by

$$\frac{2b\alpha(1-\rho^2)}{(1+b^2\alpha\rho)^2} \quad \text{and} \quad \frac{\alpha(b+\phi)(1-b\phi)}{1-b^2\alpha\phi}.$$

The first eigenvalue is real and less than 1, a result that follows from Hommes and Zhu (2014). It can be shown that the second eigenvalue is less than 1 when $\hat{b} < b^*$, which is equivalent to the conditions identified in statement 2 of the theorem.

(iii) Computer algebra shows that the determinant and trace of DT evaluated at (b^*, d^*) is independent of σ_ε^2 and σ_ν^2. Further, fixing ϕ and taking the limit as $\rho \to 1$ reveals that the determinant is positive and the trace is negative. The existence of the critical value $\check{\rho}$ follows from continuity. □

Proof of Propositions 2 and 3. Defining $\tilde{d} \equiv (d^*)^2\sigma_v^2/(\sigma_\varepsilon^2\gamma^2)$ and $\xi_1(b) \equiv \alpha b^2$,

$$\begin{aligned}\operatorname{var}(y_t|\text{sunspot RPE}) = {} & \frac{\gamma^2(1+\rho\xi_1(b^*))\sigma_\varepsilon^2}{(1-\rho\xi_1(b^*))(1-\xi_1^2(b^*))(1-\rho^2)} \\ & + \frac{\gamma^2\alpha^2(b^*+\phi)^2\tilde{d}(1+\phi\xi_1(b^*))\sigma_\varepsilon^2}{(1-\phi\xi_1(b^*))(1-\xi_1^2(b))(1-\phi^2)},\end{aligned}$$

[30]For this and other proofs, as well as for some of the formulae found in the body, we used the symbolic methods available in Mathematica. The code is available by request.

$$\operatorname{var}(y_t|\mathrm{REE}) = \frac{\gamma^2\sigma_\varepsilon^2}{(1-\alpha\rho)^2(1-\rho^2)},$$

$$\operatorname{var}(y_t|\text{fund RPE}) = \frac{\gamma^2(1+\rho\xi_1(\hat{b}))\sigma_\varepsilon^2}{(1-\rho\xi_1(\hat{b}))(1-\xi_1^2(\hat{b}))(1-\rho^2)}.$$

The ratio depends on α, ρ, and ϕ, and $\frac{\operatorname{var}(y_t|\text{sunspot RPE})}{\operatorname{var}(y_t|\mathrm{REE})} > 1$ follows from computer algebra.

Also, since $\hat{b} > b^*$, if ρ sufficiently large, we have

$$\frac{1+\rho\xi_1(\hat{b})}{(1-\rho\xi_1(\hat{b}))(1-\xi_1^2(\hat{b}))} > \frac{\alpha^2(b^*+\phi)^2\tilde{d}(1+\phi\xi_1(b^*))}{(1-\phi\xi_1(b^*))(1-\xi_1^2(b^*))}\frac{(1-\rho^2)}{(1-\phi^2)} + \frac{1+\rho\xi_1(b^*)}{(1-\rho\xi_1(b^*))(1-\xi_1^2(b^*))}. \tag{18}$$

□

Proof of Proposition 4. The relative forecast accuracy of the sunspot to the model conditioning on z_t is

$$F(\rho) = \frac{\gamma^2 P_3(\rho)}{\alpha^2(1-\rho^2)\phi(1-\alpha\phi)(1-\phi^2)^2\big[(1-\phi^2)^2-\rho(1-\alpha\phi)^2\big]^2}\sigma_\varepsilon^2,$$

where $P_3(\rho)$ is a cubic polynomial with $P_3(1) > 0$, $P_3(0) < 0$, $P_3(\tilde{\rho}) > 0$ whenever $\alpha > \tilde{\alpha}$. Furthermore, the symmetric axis of the differential $P_3'(\rho) > 1$. It follows that $F(\rho) > 0$ for all $\rho \in (\tilde{\rho}, 1)$. □

Proof of Proposition 5. We know from Theorem 1 that when a sunspot RPE exists, the value for θ^* will not be too large. A sufficient condition for this in the optimal policy solving the fixed point problem is $\lambda \to 0 \Rightarrow \theta^*_\lambda \to 0$. Then we have $(1+\phi(1-\phi))^{-1} \le \beta < 1$ ensures that $|b^*| < 1$. Straightforward algebra establishes that$(d^*)^2 > 0$ when $b = b^*$ and $\theta = \theta^*$ in the symmetric equilibrium. □

References

Angeletos, G.-M. and J. La'O (2013), "Sentiments." *Econometrica*, 81, 739–779. [294]

Azariadis, C. (1981), "Self-fulfilling prophecies." *Journal of Economic Theory*, 25, 380–396. [295]

Azariadis, C. and R. Guesnerie (1986), "Sunspots and cycles." *Review of Economic Studies*, 53, 725–737. [295]

Baker, M. and J. Wurgler (2000), "The equity share in new issues and aggregate stock returns." *Journal of Finance*, 55, 2219–2257. [318]

Benhabib, J. and R. A. Farmer (1994), "Indeterminacy and increasing returns." *Journal of Economic Theory*, 63, 19–41. [295]

Benhabib, J., P. Wang, and Y. Wen (2015), "Sentiments and aggregate demand fluctuations." *Econometrica*, 83, 549–585. [294]

Blanchard, O. J. (2011), "Four hard truths." I.M.F. Blog. [291]

Branch, W. A. and G. W. Evans (2006), "Intrinsic heterogeneity in expectation formation." *Journal of Economic Theory*, 127, 264–295. [295]

Branch, W. A. and G. W. Evans (2011), "Learning about risk and return: A simple model of bubbles and crashes." *American Economic Journal: Macroeconomics*, 3, 159–191. [317]

Branch, W. A. and G. W. Evans (2010), "Asset return dynamics and learning." *Review of Financial Studies*, 23, 1651–1680. [317, 318]

Bullard, J., G. W. Evans, and S. Honkapohja (2008), "Monetary policy, judgment and near-rational exuberance." *American Economic Review*, 98, 1163–1177. [295, 316]

Cass, D. and K. Shell (1983), "Do sunspots matter?" *Journal of Political Economy*, 91, 193–227. [295]

Cho, I.-K. and K. Kasa (2015), "Learning and model validation." *Review of Economic Studies*, 82, 45–82. [314]

Cho, I.-K. and K. Kasa (2017), "Gresham's law of model averaging." *American Economic Review*, 107, 11, 3589–3616. [295]

De Long, B., A. Shleifer, L. Summers, and R. Waldmann (1990), "Noise trader risk in financial markets." *Journal of Political Economy*, 98, 703–738. [317]

Duffy, J. and W. Xiao (2007), "Instability of sunspot equilibria in real business cycle models under adaptive learning." *Journal of Monetary Economics*, 54, 879–903. [295]

Eusepi, S. (2009), "A note on expectations-driven business cycles in economies with production externalities." *International Journal of Economic Theory*, 5, 9–23. [295]

Eusepi, S. and B. Preston (2018), "The science of monetary policy: An imperfect knowledge perspective." *Journal of Economic Literature*, 56, 1, 3–59. [293, 317, 319]

Evans, G. W. and S. Honkapohja (2001), *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey. [292, 295, 297, 303, 309]

Evans, G. W. and B. McGough (2005a), "Indeterminacy and the stability puzzle in non-convex economies." *The B.E. Journal of Macroeconomics (Contributions)*, 5, 8. [295]

Evans, G. W. and B. McGough (2005b), "Monetary policy, indeterminacy and learning." *Journal of Economic Dynamics and Control*, 29, 1809–1840. [295]

Evans, G. W. and B. McGough (2018), "Equilibrium selection, observability, and backward-stable solutions." *Journal of Monetary Economics*, 98, 1–10. [298]

Farmer, R. E. and J.-T. Guo (1994), "Real business cycles and the animal spirits hypothesis." *The Journal of Economic Theory*, 63, 42–72. [295]

Guesnerie, R. (1986), "Stationary sunspot equilibria in an N-commodity world." *Journal of Economic Theory*, 40, 103–128. [295]

Guesnerie, R. and M. Woodford (1992), "Endogenous fluctuations." In *Advances in Economic Theory*, 289–412, Cambridge University Press, Cambridge. [295]

Guidolin, M. and A. Timmermann (2007), "Asset allocation under multivariate regime switching." *Journal of Economic Dynamics and Control*, 31, 3503–3544. [319]

Hommes, C. H. and M. Zhu (2014), "Behavioral learning equilibria." *Journal of Economic Theory*, 150. [301, 304, 324, 325]

Marcet, A. and T. J. Sargent (1989), "Convergence of least-squares learning in environments with hidden state variables and private information." *Journal of Political Economy*, 97, 1306–1322. [292, 295, 300]

Orphanides, A. and J. C. Williams (2005a), "Imperfect knowledge, inflation expectations, and monetary policy." In *The Inflation-Targeting Debate* (B. Bernanke and M. Woodford, eds.), 201–234, University of Chicago Press. [317, 319, 320]

Orphanides, A. and J. C. Williams (2005b), "Inflation scares and forecast-based monetary policy." *Review of Economic Dynamics*, 8, 498–527. [320]

Piazzesi, M. (2016), "Estimation of rational expectations models." In *The New Palgrave Dictionary of Economics* (M. Vernengo, E. Perez Caldentey, and B. Rosser Jr., eds.). Elsevier. [292]

Sargent, T. J. (1991), "Equilibrium with signal extraction from endogenous variables." *Journal of Economic Dynamics and Control*, 15, 245–273. [295]

Sargent, T. J. (1993), *Bounded Rationality in Macroeconomics*. Oxford University Press, Oxford. [292]

Sargent, T. J. (1999), *The Conquest of American Inflation*. Princeton University Press, Princeton, NJ. [295]

Sargent, T. J. (2008), "Evolution and intelligent design." *American Economic Review*, 98, 5–37. [292]

Shell, K. (1977), "Monnaie et Allocation Intertemporelle." Working paper, CNRS Seminaire de E. Malinvaud, Paris. [295]

Shiller, R. J. (1981), "Do stock prices move too much to be justified by subsequent changes in dividends." *American Economic Review*, 71, 421–436. [317, 318]

Stiglitz, J. (2019), "An agenda for reforming economic theory." *Frontiers of Economics in China*, 14, 149–167. [319]

White, H. (1994), *Estimation, Inference and Specification Analysis*. Econometric Society Monograph, Cambridge University Press, Cambridge, UK. [292]

Woodford, M. (1990), "Learning to believe in sunspots." *Econometrica*, 58, 277–307. [295]

Woodford, M. (2013), "Macroeconomic analysis without the rational expectations hypothesis." *Annual Review of Economics*. [292, 297]

Co-editor Florian Scheuer handled this manuscript.

Manuscript received 7 May, 2019; final version accepted 13 March, 2021; available online 25 March, 2021.