Theoretical Economics 17 (2022), 841–881 1555-7561/20220841

A common-value auction with state-dependent participation

Stephan Lauermann
Department of Economics, University of Bonn

Asher Wolinsky
Department of Economics, Northwestern University

This paper analyzes a common-value, first-price auction with state-dependent participation. The number of bidders, which is unobservable to them, depends on the true value. For participation patterns with many bidders in each state, the bidding equilibrium may be of a "pooling" type—with high probability, the winning bid is the same across states and is below the ex ante expected value—or of a "partially revealing" type—with no significant atoms in the winning bid distribution and an expected winning bid increasing in the true value. Which of these forms will arise is determined by the likelihood ratio at the top of the signal distribution and the participation across states. We fully characterize this relation and show how the participation pattern determines the extent of information aggregation by the price.

Keywords. Auctions, search, information aggregation.

JEL classification. D44, D82.

1. Introduction

In various auctions and similar trading scenarios, participation is state-dependent—its extent may be correlated with information relevant for the bidding. This might be the case when the decisions on the costly recruitment of participants are made by an informed seller or when the participants are induced to participate by the value of correlated outside options. Strategic participants take this dependence into account and it affects their behavior. A situation of this sort arises, for example, when a privately informed borrower chooses how many lenders to contact to obtain a loan. The main objective of this paper is to shed light on how such state-dependent participation affects prices and the aggregation of information by the market.

Price formation with state-dependent participation can take different forms. This paper explores it by studying auctions in which the number of bidders varies across

Stephan Lauermann: s.lauermann@uni-bonn.de
Asher Wolinsky: a-wolinsky@northwestern.edu
We gratefully acknowledge support from the National Science Foundation under Grants SES-1123595 and SES-1061831 for early versions of this paper. This work was also supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2126/1-390838866 and through the CRC TR 224 (Project B04). Deniz Kattwinkel, Guannan Luo, Au Pak, Justus Preusser, Andre Speit, Matthew Thomas, Quitze Valenzuela-Stookey, and Qinggong Wu provided excellent research assistance and comments.

 https://doi.org/10.3982/TE3733

states and bidders can learn about the state from their own participation. We view the auction model as a convenient abstraction of a free-form price-formation process in a decentralized market environment rather than a formal mechanism. The specific auction format and some of the other features are selected to facilitate the clear exposition of the insights concerning the strategic effects of state-dependent participation rather than tailored to fit a specific application.

Specifically, we analyze a first-price auction for a single good with two value-states, ℓ and h, such that the common value of the good, v_ω, $\omega = \ell, h$, satisfies $v_h > v_\ell$. In state ω, there are n_ω bidders. Bidders do not observe ω or n_ω but get private, conditionally independent signals that are drawn from a distribution G_ω with support $[\underline{x}, \bar{x}]$ and density g_ω. The likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is increasing, so higher signals are relatively more likely in state h.[1] In this world, bidders know that different states may be associated with different participation, and they draw some inference about the overall participation from their own presence at the auction. This augments their private signal information, and the compound posterior likelihood ratio of the states depends both on the signal likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ (as it would in a standard auction environment) and on the participation ratio $\frac{n_h}{n_\ell}$. The objective of this paper is to explore the implications of this feature.

Our main characterization result (Theorem 1) concerns the forms of the bidding equilibria when n_ℓ and n_h are large. Specifically, the key magnitude is the "compound" posterior likelihood ratio, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$, and the form of the equilibrium varies dramatically according to whether this ratio is below or above 1. If this compound ratio is below 1, then any bidding equilibrium is necessarily of a pooling type: there is some bid b below the ex ante expected value such that, with probability close to 1, the winning bid is equal to b in both states. In fact, in this case, any bid b from an interval below the ex ante expected value can be supported as the outcome of such a pooling equilibrium. If this compound ratio is above 1, then any bidding equilibrium is of a partially revealing type: there are no significant atoms in the winning bid distribution, and the expected winning bid is higher in state h than in state ℓ. In being partially revealing, the equilibrium in this case resembles the equilibrium of an ordinary common-value auction. However, we will see that the degree of revelation is affected by the state-dependent participation, which may either dampen or enhance separation, depending on $\frac{n_h}{n_\ell}$.

These results regarding the two basic types of equilibria are explained by the form of the "winner's inference," that is, $\frac{\Pr(\text{all other bids} \leq b|h)}{\Pr(\text{all other bids} \leq b|\ell)}$, given a common bidding strategy β. When there are many bidders, for a strictly increasing bidding strategy β to be an equilibrium, the expected value conditional on winning must be increasing in the bid. But this is the case only if this winner's inference is increasing in b. The analysis will show that, given a common, strictly increasing bidding strategy β, and large n_ℓ and n_h, the relationship between the ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ and 1 determines whether the winner's inference of the relevant bidders (those with x near $\bar{x}$) is increasing or decreasing in x.

[1] This is the same basic model as in Lauermann and Wolinsky (2017), discussed below.

The extent of information aggregation by the price can be thought of casually as reflected by the closeness of the price to the true value and, more formally, as how informative the price is as a signal of the true state. It depends on the form of the equilibrium and on the ratio $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$. The price aggregates no information in the pooling equilibrium and aggregates some information in the partially revealing equilibria (the distribution of the winning bid in state h stochastically dominates that of state ℓ). The extent of information aggregation in the partially revealing equilibria increases in $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$; that is, the expected price is closer to the true value, and, more generally, the price is a more informative signal of the state.[2]

Our discussion of information aggregation continues the discussion of this question by Milgrom (1979) and Wilson (1977) in the context of an ordinary common-value auction. Translated to the two-state model considered here, Milgrom's (1979) result is that the winning bid in an ordinary common-value auction approaches the true value as the number of bidders grows if and only if the likelihood ratio of the two states is unbounded over the support of the signal distribution. Our analysis recognizes the additional information due to the state-dependent participation and points out that this may dampen or enhance information revelation. Specifically, in an ordinary large common-value auction without state-dependent participation, the price aggregates only the bidders' information, and the extent of information aggregation depends on the informativeness of the private signals as captured by $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$. With state-dependent participation, additional information can be injected into the price via $\frac{n_h}{n_\ell}$. For a given value of $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$, the larger $\frac{n_h}{n_\ell}$ is, the more information is incorporated into the price. In particular, the price aggregates information better than it does in a large ordinary auction with the same signal structure whenever $\frac{n_h}{n_\ell}$ is larger than 1, and it is worse at aggregating the information when $\frac{n_h}{n_\ell}$ is smaller than 1.

In our model, the participation is deterministic conditional on the state. We sketch an extension to random participation in Section 5.2. There, we also discuss some related contributions on auctions with uncertain bidder numbers.

State-dependent participation may arise for a variety of reasons. In Lauermann and Wolinsky (2017), it arises via the recruitment decision of a seller. The seller knows ω and solicits n_ω bidders at a constant cost per bidder. The subsequent interaction is the same as in the model of the current paper. Their main result is that, with binary signals, there exists an equilibrium in which the endogenous participation pattern gives rise to an atom in the bid distribution. Lauermann and Wolinsky (2021) show that an equilibrium of the partially revealing type always exists and the solicitation uniquely pins down the ratio $\frac{n_h}{n_\ell}$ in such an equilibrium (when the solicitation cost is small). These results regarding the equilibrium solicitation build on the characterization of the bidding behavior in the present paper.

State-dependent participation in a common-value setting may also arise from bidders' entry decisions, especially when the entry costs are correlated with the state (but not only in that case). We discuss bidder entry in Section 5.3.2, where we also discuss

[2] More precisely, the limit price distribution for large n_h and n_ℓ is equivalent to a distribution over posteriors. This distribution is Blackwell more informative about the state as the limit of $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}\frac{n_h}{n_\ell}$ increases.

related work on entry into interdependent value auctions by Murto and Välimäki (2019) and Atakan and Ekmekci (2020).

Our analysis may add insights to many scenarios in which auctions are used to study informal price competition in markets. For example, consider the competition among incompletely informed banks over the business of a potential borrower. Broecker (1990) and Riordan (1993) model this scenario as an ordinary common-value auction—the borrower contacts all of a fixed number of banks for quotes. This and our companion papers recognize that such competition may be significantly affected when the borrower chooses how many banks to contact based on its private information. For example, the inevitability of atoms established in the present paper implies that certain state-dependent contact patterns of a broad class will result in banks pooling on a unique quote, which resembles a collusive outcome.

Finally, information aggregation problems with state-dependent participation have been considered in other settings as well. In Lauermann and Wolinsky (2016), an informed seller contacts buyers sequentially in a random search model. The expected number of contacted bidders depends on the state via the sampling behavior. It is shown that the equilibrium outcome is necessarily pooling when signals are boundedly informative and the search costs are small. The fact that partially separating equilibria do not exist is related to the absence of price competition in the sequential search setting.

There is also some relationship between information aggregation in auctions and elections; see Feddersen and Pesendorfer (1997). Ekmekci and Lauermann (2020, 2021) consider the effects of a potentially state-dependent number of voters on information aggregation in elections, showing that information aggregation can fail when the number of voters is correlated with the state.

2. Setup and preliminary characterization

Basics

This is a single-good, common-value, first-price auction environment with two underlying states, h and ℓ. There are N potential bidders (buyers). The common values of the good for all potential bidders in the two states are v_ℓ and v_h, respectively, with $0 \leq v_\ell < v_h$.

Nature draws a state $\omega \in \{\ell, h\}$ with prior probabilities $\rho_\ell > 0$ and $\rho_h > 0$, $\rho_\ell + \rho_h = 1$, and in state ω randomly draws n_ω bidders from the pool, $1 \leq n_\omega \leq N$. A bold $\mathbf{n}$ denotes the vector (n_ℓ, n_h).[3]

Each of the n_ω bidders observes a private signal $x \in [\underline{x}, \bar{x}]$. Conditional on the state $\omega \in \{\ell, h\}$, signals are independently and identically distributed according to a cumulative distribution function (c.d.f.) G_ω. A bidder does not observe ω or n_ω, but she believes that her probability of being invited to the auction in state ω is $\frac{n_\omega}{N}$.

[3]The participation is exogenous here, but, as mentioned before, it can be endogenized in several ways.

The set of feasible bids, P_Δ, is a grid with step size $\Delta \geq 0$

$$P_\Delta \triangleq \begin{cases} [0, v_\ell] \cup \{v_\ell + \Delta, v_\ell + 2\Delta, \dots, v_h - \Delta, v_h\} & \text{for } \Delta > 0, \\ [0, v_h] & \text{for } \Delta = 0. \end{cases}$$

Notice that even for the case of $\Delta > 0$, the set P_Δ contains the continuum of prices on $[0, v_\ell]$.[4] The grid is introduced to deal with later existence issues.

The n_ω bidders simultaneously submit bids $b \in P_\Delta$. The highest bid wins, and ties are broken randomly with equal probabilities. If the winning bid is p in state $\omega \in \{h, \ell\}$, then the payoffs are $v_\omega - p$ for the winning bidder and 0 for all others.

We call this the "bidding game" and denote it by $\Gamma_0(\mathbf{n}, N, \Delta)$. The ordinary common-value auction is a special case with $n_\ell = n_h$.

The signal

The signal distributions G_ω, $\omega \in \{\ell, h\}$, have no atoms and strictly positive densities g_ω on an identical support, $[\underline{x}, \bar{x}]$. The likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$ is nondecreasing and right continuous, with $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \lim_{x \to \bar{x}} \frac{g_h(x)}{g_\ell(x)}$. This is the (weak) monotone likelihood ratio property (MLRP): larger values of x indicate a (weakly) higher likelihood of the higher state. The signals are nontrivial and boundedly informative, that is,

$$0 < \frac{g_h(\underline{x})}{g_\ell(\underline{x})} < 1 < \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty.$$

A bidder's posterior probability of ω, conditional on being solicited and receiving signal x, is

$$\Pr[\omega | x, \text{sol}; \mathbf{n}] \triangleq \frac{\rho_\omega g_\omega(x) \dfrac{n_\omega}{N}}{\rho_\ell g_\ell(x) \dfrac{n_\ell}{N} + \rho_h g_h(x) \dfrac{n_h}{N}},$$

where ρ_ω, $g_\omega(x)$, and $\frac{n_\omega}{N}$, respectively, reflect the information contained in the prior belief, in the signal x, and in the bidder being invited. We use "sol" to denote the event that the bidder was solicited. Notice that N cancels out, and hence, it does not play any role in the analysis.

Bidding

A bidding strategy β prescribes a bid as a function of the signal realization,

$$\beta : [\underline{x}, \bar{x}] \to P_\Delta.$$

We study strategies that are symmetric and pure.

Given a bidding strategy β employed by n other bidders, the probability of winning with a bid b in state ω is $\pi_\omega(b; \beta, n)$. From here on, $(\beta, \mathbf{n})$ and (β, n) will typically be

[4] This avoids some irrelevant distinctions between the case in which the bottom equilibrium bid is v_ℓ and the case in which it is $v_\ell - \Delta$.

suppressed from the arguments, and we write expressions such as $\Pr[\omega|x, \text{sol}]$ and $\pi_\omega(b)$ with the understanding that they depend on a specific profile $(\beta, \mathbf{n})$.

Expected payoff

Given the bidding strategy β and the participation $\mathbf{n} = (n_\ell, n_h)$, the interim expected payoff to a bidder who bids b, conditional on participating and observing the signal x, is

$$U(b|x, \text{sol}) = \Pr[\text{win at } b|x, \text{sol}]\big(\mathbb{E}[v|x, \text{sol}, \text{win at } b] - b\big), \tag{1}$$

where

$$\Pr[\text{win at } b|x,\text{sol}] = \frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b) + \rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h}, \tag{2}$$

and

$$\mathbb{E}[v|x,\text{sol},\text{win at } b] = \frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b) v_\ell + \rho_h g_h(x) n_h \pi_h(b) v_h}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b) + \rho_h g_h(x) n_h \pi_h(b)}, \tag{3}$$

where $(\beta, \mathbf{n})$ is suppressed from the arguments of $\mathbb{E}[v| \dots]$ and $\Pr[\text{win at } b| \dots]$, according to the convention adopted above.

Bidding equilibrium

A *bidding equilibrium* of $\Gamma_0(\mathbf{n}, N, \Delta)$ is a bidding strategy β such that $b = \beta(x)$ maximizes $U(\cdot|x, \text{sol})$ for all x.

3. Equilibrium monotonicity

With state-dependent participation, monotonicity is not immediate because the signals also inform bidders about the number of competitors rather than just about the value. If fewer bidders are solicited when $\omega = h$, a higher signal implies both, a higher value and less competition. The following example illustrates this consideration.

Example of a nonmonotone bidding equilibrium

Let $[\underline{x}, \bar{x}] = [0, 1]$, with $g_h(x) = 2x$ and $g_\ell(x) = 2 - 2x$. Therefore, the signal likelihood ratios are ∞ at $x = 1$ and 0 at $x = 0$, and so these signals reveal the state to be h and ℓ, respectively.[5] Further, suppose that $v_\ell > 0$, $n_h = 1$, and $n_\ell = 100$. It follows that $\pi_h(b; \beta, 1) = 1$ for all $b \geq 0$ in state h because there is no competition. Hence, given that $x = 1$ reveals that the state is h, in every bidding equilibrium it must be that $\beta(1) = 0$. So, if β were weakly increasing, then $\beta(x) = 0$ for all x. However, this strategy cannot be an equilibrium. At $x = 0$, the expected payoff from bidding $b = 0$ is $\frac{1}{100} v_\ell$, whereas the expected payoff from bidding $b' = \varepsilon$ is $v_\ell - \varepsilon$. Because $v_\ell > 0$, a deviation to b' is

[5]The example violates the bounded likelihood-ratio assumption. This simplifies the argument but is not essential.

profitable for small ε. Therefore, in this example, there is no weakly increasing bidding equilibrium.

However, when either at least two bidders participate in the auction in both states or $v_\ell = 0$ (unlike in the example), a bidding equilibrium strategy β is monotonic in the sense that, for any bidding equilibrium there is an equivalent monotone bidding equilibrium. A bidding equilibrium $\widetilde{\beta}$ is said to be *equivalent* to a bidding equilibrium β if the implied joint distributions over bids and states are identical.

Proposition 1 (Monotonicity of bidding equilibrium). *Suppose that either $v_\ell = 0$ or $n_\omega \geq 2$, $\omega = \ell, h$ and β is a bidding equilibrium.*

(i) If $x' > x$, then $U(\beta(x')|x', \text{sol}) \geq U(\beta(x)|x, \text{sol})$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

(ii) There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.

Therefore, if the likelihood ratio $\frac{g_h}{g_\ell}$ is strictly increasing everywhere, then a bidding equilibrium β is necessarily monotonic; if $\frac{g_h}{g_\ell}$ is constant over some interval, then β need not be monotonic over it, since all those signals contain the same information. However, in this case, there is an equivalent monotone bidding equilibrium that is obtained by reordering the bids over such intervals.[6]

This proposition is not proved separately since it is a special case of a more general version, called Proposition 4, which will be stated and proved in Appendix A.3.2.

The main observation in the proof is that, for $b \geq v_\ell$, $U(b|x, \text{sol}; \beta, \mathbf{n})$ satisfies single crossing with respect to b and x for any β (monotone or not). Therefore, above v_ℓ, best responses are monotone, and so are equilibrium bids.

The two conditions in the proposition ensure that equilibrium bids are necessarily above v_ℓ. First, if $v_\ell = 0$, then this simply follows from the restriction of bids to be positive. Second, if there are at least two bidders, then a "Bertrand" argument implies that bids must be at least v_ℓ. For an intuition, note that it is common knowledge that the value is at least v_ℓ. As in the standard Bertrand game with complete information, 2 bidders are already sufficient. The assumption that $[0, v_\ell] \subset P_\Delta$ is used in this part of the proof.[7]

The single-crossing property implies that the proof does not have to distinguish between the cases of $\Delta > 0$ and $\Delta = 0$ above v_l. Moreover, the single-crossing property implies that our restriction to pure strategies is without loss of generality.

In light of Proposition 1, from now on, whenever $n_\omega \geq 2$, $\omega = \ell, h$, attention is confined to nondecreasing bidding equilibria.

[6]Although strict MLRP evidently simplifies the argument, we chose to require only weak MLRP beause this admits discrete signals as a special case, which is useful for some examples and results.

[7]Murto and Välimäki (2019) also show that nonmonotone bidding may occur in a common-value auction with random participation when there is a chance that there is only a single bidder.

4. Bidding equilibria with many bidders

This section characterizes bidding equilibria when there are many bidders in each state. From a substantive point of view, the many bidders case is the relevant case for the questions of competitiveness and information aggregation in markets. From an analytical point of view, this case makes it easier to get clean characterization results and to identify the underlying economic mechanism.

4.1 *Preliminaries*

We look at a sequence of bidding games $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ such that $\Delta^k \geq 0$, $\lim \Delta^k = 0$,

$$\lim_{k\to\infty} n_\omega^k = \infty \quad \text{for } \omega = \ell, h,$$

and

$$\lim_{k\to\infty} \frac{n_h^k}{n_\ell^k} = r \in [0, \infty],$$

and at a corresponding sequence of bidding equilibria β^k. We are interested in the limits of equilibrium magnitudes as $k \to \infty$.[8]

With many bidders, only bids associated with signals that are sufficiently close to $\bar{x}$ have a significant probability of winning. Therefore, the object of interest is the equilibrium distribution of the *winning* bid in state ω, namely,

$$F_\omega(p|\beta, n) \triangleq \big(G_\omega\big(\{x : \beta(x) \leq p\}\big)\big)^n,$$

and its pointwise limit, rather than the distribution of all the bids.

The notation's density is reduced as follows. First, when we discuss a fixed sequence $\{(\beta^k, \mathbf{n}^k)\}_{k=1}^\infty$, then magnitudes induced by $(\beta^k, \mathbf{n}^k)$ are typically written as $U^k(b|x, \text{sol})$, $F_\omega^k(p)$, etc. (rather than as $U(b|x, \text{sol}; \beta^k, \mathbf{n}^k)$, $F_\omega(p|\beta^k, n_\omega^k)$, etc.). Second, since nearly all limits are with respect to k, we generally omit the delimiter $k \to \infty$. Finally, we sometimes use the abbreviations

$$\bar{g} \triangleq \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \quad \text{and} \quad \rho \triangleq \frac{\rho_h}{\rho_\ell}.$$

4.2 *Winning bid distribution: Pooling versus partially revealing*

Our main characterization result shows that, for large k, the form of $F_\omega^k(p)$ is determined by $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} \lim \frac{n_h^k}{n_\ell^k} = \bar{g}r$. It exhibits a large atom at the top if $\bar{g}r < 1$, and it is essentially free of atoms if the reverse inequality holds.

Let $\mathbb{E}[v]$ denote the expected ex-ante value of the good, $\mathbb{E}[v] = \rho_\ell v_\ell + \rho_h v_h$, and let

$$\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] \triangleq \lim \mathbb{E}^k[v|\bar{x}, \text{sol}] \equiv \frac{v_\ell + \rho\bar{g}r v_h}{1 + \rho\bar{g}r}, \tag{4}$$

[8]By assumption, $N^k \geq n_\omega^k$ for $\omega = \ell, h$, and so $N^k \to \infty$.

be the limit posterior conditional on the highest signal $\bar{x}$ and being solicited. Note that $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] > \mathbb{E}[v]$ if $\bar{g}r > 1$ and $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] < \mathbb{E}[v]$ if $\bar{g}r < 1$. Thus, if $\bar{g}r < 1$, then just being included in the auction already involves a "participation curse" that depresses the value estimate held by any bidder below the prior.

Intuitively, $\bar{g}r \gtrless 1$ determines whether the expected number of relevant bidders (those with signals close to $\bar{x}$) is higher in state h or ℓ. This is because the expected number of bidders having signals in an ε-neighborhood of $\bar{x}$ is $n_\omega^k(1 - G_\omega(\bar{x} - \varepsilon)) \approx n_\omega^k g_\omega(\bar{x})\varepsilon$.

Theorem 1. *For every sequence of bidding games $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ with $\Delta^k > 0$ for all k, $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$, and $\lim \frac{n_h^k}{n_\ell^k} = r$, there exists a sequence of bidding equilibria β^k.*

(i) If $\bar{g}r > 1$, then for any such sequence,

$$\lim F_\omega^k(p) = \Phi_\omega(p|r),$$

where $\Phi_\omega(\cdot|r)$ is an atomless distribution that is uniquely determined by r with support $[v_\ell, \overline{\mathbb{E}}[v|\bar{x}, \text{sol}]]$.

(ii) If $\bar{g}r < 1$, then

(a) for any such sequence, there is a sequence of bids $\hat{b}^k$ such that

$$\lim\left[F_\omega^k\left(\hat{b}^k + \Delta^k\right) - F_\omega^k\left(\hat{b}^k - \Delta^k\right)\right] = 1,$$

with $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] \le \liminf \hat{b}^k$ and $\limsup \hat{b}^k \le \mathbb{E}[v]$.

(b) for any $\hat{b}$ with $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] < \hat{b} < \mathbb{E}[v]$, there is a sequence of equilibria β^k such that

$$\lim\left[F_\omega^k(\hat{b}) - F_\omega^k\left(\hat{b} - \Delta^k\right)\right] = 1.$$

(iii) If $\bar{g}r = 1$, then for any such sequence, $\lim F_\omega^k(p)$ has mass 1 on $\mathbb{E}[v]$.

Note that the theorem speaks about the (limit of the) distribution of the *winning* bid rather than the distribution of the submitted bids. Thus, Part 2 does not mean that, for large k, most equilibrium bids are $\hat{b}^k$ or $\hat{b}^k + \Delta^k$ but rather that the winning bid is very likely to be either $\hat{b}^k$ or $\hat{b}^k + \Delta^k$.

The proof shows that the distributions Φ_ω mentioned in Part 1 are

$$\Phi_\ell(p|r) \triangleq \begin{cases} 1 & \text{if } p \ge \overline{\mathbb{E}}[v|\bar{x}, \text{sol}] \\ \left(\dfrac{1}{\rho\bar{g}r}\dfrac{p - v_\ell}{v_h - p}\right)^{\frac{1}{\bar{g}r-1}} & \text{if } v_\ell < p \le \overline{\mathbb{E}}[v|\bar{x}, \text{sol}] \\ 0 & \text{if } p \le v_\ell \end{cases} \tag{5}$$

and

$$\Phi_h(\cdot|r) \triangleq \left(\Phi_\ell(\cdot|r)\right)^{\overline{g}r}, \tag{6}$$

and thus are uniquely determined by r, as claimed.

For the special case of $n_\ell = n_h = n$ (i.e., $r = 1$), the explicit characterization of the winning bid distribution is essentially implied by the analysis of Murto and Välimäki (2015).

Part 3 of the theorem implies that the limit equilibrium outcome is continuous in $\bar{g}r$ at $\bar{g}r = 1$. As $\bar{g}r \to 1$ from above, the distributions Φ_ω converge to a mass point at $\mathbb{E}[v]$, and, as $\bar{g}r \to 1$ from below, $\overline{\mathbb{E}}[v|\bar{x}, \mathrm{sol}]$ converges to $\mathbb{E}[v]$, and so the interval of outcomes in Part 2 collapses.

The assumption that $\Delta^k > 0$ along the sequence is only used to show the existence of equilibrium.[9] The characterization results concerning the forms of the equilibria hold with $\Delta = 0$ as well.

Proposition 2. *Consider any sequence of bidding games $\Gamma_0(\boldsymbol{\eta}^k, N^k, \Delta)$ with $\Delta = 0$, such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim \frac{n_h^k}{n_\ell^k} = r$. Then the characterization results of Theorem 1 (i.e., items 1, 2a, and 3) hold for any corresponding sequence of bidding equilibria β^k.*

The proof in Appendix A.1 shows Proposition 2 first, before allowing for a grid and proving Theorem 1.

4.3 *Key ideas from the proof of Theorem 1*

The following two observations highlight the key intuition of the theorem. First, if bidders with signals close to $\bar{x}$ are tied at a common bid, it must be that $\bar{g}r < 1$. Second, if bidders with signals close to $\bar{x}$ use a strictly increasing bidding strategy, it must be that $\bar{g}r > 1$.

Pooling on a common bid Suppose the equilibrium bidding strategies β^k are such that bidders with signals close to the top are tied at a common bid, that is, for all large k and some $\hat{b}$ and x^k,

$$\beta^k(x) = \hat{b} \quad \text{for all } x \in \left[x^k, \bar{x}\right], \tag{7}$$

and suppose that the winning bid is equal to $\hat{b}$ with probability 1 in the limit,

$$\lim\left[G_h\left(x^k\right)\right]^{n_h^k - 1} = \lim\left[G_\ell\left(x^k\right)\right]^{n_\ell^k - 1} = 0,$$

that is, x^k is not too close to $\bar{x}$.

Since the auction ends with a winning bid of $\hat{b}$ in both states when k is large, the bidders' ex ante rationality requires

$$\mathbb{E}[v] \geq \hat{b}.$$

[9]Existence without a grid is discussed in Section 5.1.

When bidding $\hat{b}$, the winning probability, and hence, the expected payoffs vanish to zero. A bidder who ε-overbids $\hat{b}$, however, wins with probability 1 in both states, and the expected value conditional on winning is $\mathbb{E}[v|\bar{x}, \text{sol}]$ in the limit (since winning contains no further information). Therefore, for a bidder with a signal $\bar{x}$ not to overbid $\hat{b}$, it must be that

$$\hat{b} \geq \mathbb{E}[v|\bar{x}, \text{sol}].$$

For both of the above inequalities to hold simultaneously, it must be that $\mathbb{E}[v|\bar{x}, \text{sol}] \leq \mathbb{E}[v]$, which holds if and only if $\bar{g}r \leq 1$; see (4). Therefore, $\bar{g}r \leq 1$ is necessary for an atom of the form (7).

Strictly increasing bids Suppose the bidding strategy β^k is strictly increasing near the top, in the following sense: for some $c \in (0, 1)$, one can choose x^k such that, for each k, the strategy β^k is strictly increasing on $[x^k, \bar{x}]$ and

$$\left[G_\ell\left(x^k\right)\right]^{n_\ell^k - 1} = c;$$

so, the winning probability at $\beta^k(x^k)$ in state ℓ is constant at c for all k. Of course, $x^k \to \bar{x}$ as $k \to \infty$.

Since there is an increasingly large number of bidders, the bidders' expected equilibrium profits are zero in the limit. However, given our construction, bidders with signals x^k and $\bar{x}$ win with a strictly positive, nonvanishing probability at $\beta^k(x^k)$ and $\beta^k(\bar{x})$, respectively, even for $k \to \infty$ (the bid $\beta^k(\bar{x})$ wins with probability 1). For their profits to go to zero, it must therefore be that

$$\beta^k(\bar{x}) \approx \mathbb{E}^k\left[v|\bar{x}, \text{sol, win at } \beta^k(\bar{x})\right],$$

and

$$\beta^k\left(x^k\right) \approx \mathbb{E}^k\left[v|x^k, \text{sol,win at } \beta^k\left(x^k\right)\right]. \tag{8}$$

Given $\beta^k(\bar{x}) > \beta^k(x^k)$, it must be that

$$\lim \mathbb{E}^k\left[v|\bar{x}, \text{sol, win at } \beta^k(\bar{x})\right] \geq \lim \mathbb{E}^k\left[v|x^k, \text{sol, win at } \beta^k\left(x^k\right)\right]. \tag{9}$$

Since $x^k \to \bar{x}$, whether inequality (9) holds depends on the "winner's inference" from winning at $\beta^k(\bar{x})$ versus $\beta^k(x^k)$. In the following, we show that (9) requires $\bar{g}r \geq 1$.

Obviously, the probability of winning is 1 in both states at $\beta^k(\bar{x})$, and so the winner's inference is

$$\frac{\pi^k\left(\beta^k(\bar{x})|h\right)}{\pi^k\left(\beta^k(\bar{x})|\ell\right)} = 1,$$

for all k. At $\beta^k(x^k)$, we have

$$\frac{\pi^k\left(\beta^k\left(x^k\right)|h\right)}{\pi^k\left(\beta^k\left(x^k\right)|\ell\right)} = \frac{\left[G_h\left(x^k\right)\right]^{n_h^k - 1}}{\left[G_\ell\left(x^k\right)\right]^{n_\ell^k - 1}}.$$

Simple algebra shows that, when $\lim[G_\ell(x^k)]^{n_\ell^k-1} = q \in (0,1)$, then $\lim[G_h(x^k)]^{n_h^k-1} = q^{\bar{g}r}$.[10] Therefore,

$$\lim \frac{\pi^k(\beta^k(x^k)|h)}{\pi^k(\beta^k(x^k)|\ell)} = q^{\bar{g}r-1}.$$

The expected value conditional on winning is

$$\lim \mathbb{E}^k[v|x^k, \text{sol, win at } \beta^k(x^k)] = \frac{v_\ell + \rho \lim \dfrac{g_h(x^k)}{g_\ell(x^k)} \dfrac{\pi^k(\beta^k(x^k)|h)}{\pi^k(\beta^k(x^k)|\ell)} v_h}{1 + \rho \lim \dfrac{g_h(x^k)}{g_\ell(x^k)} \dfrac{\pi^k(\beta^k(x^k)|h)}{\pi^k(\beta^k(x^k)|\ell)}},$$

and using the above we have

$$\lim \mathbb{E}^k[v|x^k, \text{sol, win at } \beta^k(x^k)] = \frac{v_\ell + \rho\bar{g}rq^{\bar{g}r-1}v_h}{1+\rho\bar{g}rq^{\bar{g}r-1}}. \tag{10}$$

So, for (9) to hold it must be that (10) is increasing in q, which is the case if and only if $\bar{g}r \geq 1$; thus, $\bar{g}r \geq 1$ is necessary for β^k to be strictly increasing at the top.[11]

4.4 *Revenue and information aggregation in large auctions*

Theorem 1 has implications for how the parameters $\bar{g}(\equiv \frac{g_h(\bar{x})}{g_\ell(\bar{x})})$ and $r(\equiv \lim \frac{n_h^k}{n_\ell^k})$ affect the expected equilibrium revenue and the extent of information aggregation in the limit and for large k.

The interim expected revenue is $\mathbb{E}^k[p|\omega] \triangleq \mathbb{E}[p|\omega; \beta^k, n_\omega^k]$. In the partially revealing case of $\bar{g}r > 1$, the revenue converges to a unique limit, $\overline{\mathbb{E}}[p|\omega] = \lim \mathbb{E}^k[p|\omega]$, the ex ante revenue $\rho_\ell\overline{\mathbb{E}}[p|\ell] + \rho_h\overline{\mathbb{E}}[p|h]$ is equal to the ex ante value $\mathbb{E}[v]$, and $\overline{\mathbb{E}}[p|h] > \mathbb{E}[v] > \overline{\mathbb{E}}[p|\ell]$.

In the pooling case of $\bar{g}r < 1$, the revenue is approximately equal to the atom for large k, $\mathbb{E}^k[p|\omega] \approx \hat{b}^k$, and so it is independent of the state (i.e., $\lim[\mathbb{E}^k[p|h] - \mathbb{E}^k[p|\ell]] = 0$). The atom, and hence, the revenue may vary along the sequence but is bounded, $\limsup \mathbb{E}^k[p|\omega] \leq \mathbb{E}[v]$, with a strict inequality for some sequences of equilibria.

Corollary 1. *Consider a sequence of bidding games* $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ *such that* $\Delta^k \geq 0$, $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\lim \frac{n_h^k}{n_\ell^k} = r$, *and a corresponding sequence of bidding equilibria* β^k, *with* $\overline{\mathbb{E}}[p|\omega] = \lim \mathbb{E}^k[p|\omega]$ *(when it exists).*

---

[10] With $Q_\ell \equiv -\lim n_\ell^k(1 - G_\ell(x^k))$, we have $\lim[G_\ell(x^k)]^{n_\ell^k-1} = e^{Q_\ell} = q$. By l'Hospital's rule, $Q_h \equiv -\lim n_h^k(1 - G_h(x^k)) = \bar{g}rQ_\ell$, and so $\lim[G_h(x^k)]^{n_h^k-1} = e^{Q_h} = e^{Q_\ell\bar{g}r} = q^{\bar{g}r}$. Intuitively, the number of bidders with signals $\geq x^k$ is approximately Poisson distributed with means $n_\ell^k(1-G_\ell(x^k))$ and $n_h^k(1-G_h(x^k))$, respectively.

[11] The explicit solution in (10) and $\beta^k(x^k) \approx \mathbb{E}[v|x^k, \text{sol, win at } \beta^k(x^k)]$ are used in the proof to derive the closed form of Φ_ω (the winning bid distribution in the limit).

(i) If $\bar{g}r > 1$, *then*

$$\overline{\mathbb{E}}[p|\ell] < \mathbb{E}[v] < \overline{\mathbb{E}}[p|h], \tag{11}$$

and

$$\rho_\ell \overline{\mathbb{E}}[p|\ell] + \rho_h \overline{\mathbb{E}}[p|h] = \mathbb{E}[v]. \tag{12}$$

(ii) If $\bar{g}r < 1$, *then*

$$\lim\left[\mathbb{E}^k[p|h] - \mathbb{E}^k[p|\ell]\right] = 0,$$

and $\limsup \mathbb{E}^k[p|\omega] \leq \mathbb{E}[v]$.

Proof. The equality in Part 2 of the result is immediately implied by Part 2 of Theorem 1.

For Part 1, (12) follows from direct calculation using the explicit form of the winning bid distribution Φ_ω given by (5).[12] Then (11) follows from (12) and the fact that Φ_h first-order stochastically dominates Φ_ℓ. □

Recall from Theorem 1 that Part 2 of the corollary applies not only to $\mathbb{E}^k[p|\omega]$ but also to the realized price.

Information aggregation by the orice We use the term information aggregation to describe the information conveyed by the price about the state. We will examine how it depends on the parameters $\bar{g}$ and r first informally and then more formally. When $\bar{g}r < 1$, the price fails to aggregate the information since exactly the same price prevails in both states with high probability.

In the partially revealing case of $\bar{g}r > 1$, the extent of aggregation can be evaluated by comparing the limit distributions of the winning bid Φ_ω in the two states. Inspection of (5)–(6) reveals the following facts. First, when $\bar{g}r$ is near 1, then Φ_h and Φ_ℓ are nearly identical. Second, when $\bar{g}r$ is large, then Φ_ω is concentrated near v_ω in both states and actually approaches a mass point on v_ω as $\bar{g}r \to \infty$. Thus, a price observation is not a very informative signal of the state if $\bar{g}r$ is near 1, but it is so if $\bar{g}r$ is very large.

More formally, we claim that $\bar{g}r$ determines the informativeness of the price about the state in the sense of Blackwell's criterion. Recall an information structure is a set of signals S and a conditional distribution $H(s|\omega)$ over S, for every state $\omega \in \{\ell, h\}$. In the auction environment at hand, $S = [\underline{x}, \bar{x}]$, $s \in S$ is the first-order statistic of the individual signals of the participating bidders, and hence, $H(s|\omega) = (G_\omega(s))^{n_\omega}$. For any prior likelihood ratio ρ, this information structure induces a distribution $\Psi_\omega(\rho)$ over a set of posteriors in each state ω. The functions Ψ_ω are an equivalent representation of the underlying information structure. In the case of a monotone bidding equilibrium, the distributions of the winning bid, viewed as functions of ρ, are equivalent to the Ψ_ωs because there is a one-to-one relationship between the bid and the posterior. In the limit

[12]The calculation is simplified by changing the integration variable to $y = \frac{1}{\rho\bar{g}r}\frac{p-v_\ell}{v_h-p}$ in the integral $\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] = \rho_\ell \int p\, d\Phi_\ell(p) + \rho_h \int p\, d\Phi_\ell(p)$. Alternatively, it follows from (8) and the law of iterated expectations.

of a sequence of equilibria such that $\lim \frac{n_h^k}{n_\ell^k} = r$, these Ψ_ω's are equivalent to the limits of the winning bid distributions $\Phi_\omega(\cdot|\rho, \bar{g}, r)$. As we just said, although the elements in the support of $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ are expected values, they are in a one-to-one relationship with the posteriors. Therefore, when we say below that $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$, $\omega = \ell, h$, the statement is about the underlying information structure in which the decision maker's signal is the winning bid. Now, we can inquire formally how the parameters $\bar{g}$ and r affect the informativeness of the equilibrium price.

Corollary 2. *(i) If $\bar{g}r > \bar{g}'r' > 1$, then $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$ according to Blackwell's criterion.*

(ii) $\overline{\mathbb{E}}[p|\ell]$ is decreasing in $\bar{g}r$ and $\overline{\mathbb{E}}[p|h]$ is increasing.

(iii) $\overline{\mathbb{E}}[p|\omega] \to v_\omega$ as $\bar{g}r \to \infty$ $\omega = \ell, h$.

The proof is in Appendix A.2. Notice the asymmetry between the cases of $\bar{g}r > 1$ and $\bar{g}r < 1$. For the case with $\bar{g}r > 1$, the informativeness of the price varies monotonically with $\bar{g}r$. For $\bar{g}r < 1$, recall from Theorem 1 that the limit distribution is state-independent and contains no information, for all values of $\bar{g}r$.

Finally, the case of $n_\ell = n_h$ (i.e., $r = 1$) is just the ordinary common value auction. Milgrom (1979) shows that information aggregation in such a large auction is nearly perfect—in the sense of the winning bid approaching the true value—iff $\bar{g} = \infty$. Adapting Milgrom's analysis to the case of finite $\bar{g}$, it is intuitive that the winning bid gets closer to the true value as $\bar{g}$ grows. The corollary verifies this and also shows that the price becomes more informative in the more general sense of Blackwell's criterion.[13]

4.5 *Failure of affiliation of beliefs*

Another way to describe the role of $\bar{g}r$ in determining the equilibrium outcome is in terms of the affiliation between the value and the highest signal. Let $y_{[\mathbf{n}]}$ denote the highest signal realization given participation $\mathbf{n} = (n_\ell, n_h)$. The c.d.f. of $y_{[\mathbf{n}]}$ conditional on ω is $(G_\omega(x))^{n_\omega - 1}$. Therefore, the likelihood ratio of the states at $y_{[\mathbf{n}]} = x$ is

$$\frac{n_h}{n_\ell} \frac{g_h(x)}{g_\ell(x)} \frac{\big(G_h(x)\big)^{n_h - 1}}{\big(G_\ell(x)\big)^{n_\ell - 1}}. \tag{13}$$

In ordinary auctions with $n_h = n_\ell = n$, this likelihood ratio is increasing in x, which means that $y_{[\mathbf{n}]}$ is affiliated with the value. In contrast, with state-dependent participation, the likelihood ratio (13) need not be increasing; in fact, it is *decreasing* for x sufficiently close to $\bar{x}$ if $\frac{n_h}{n_\ell} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} < 1$. Therefore, $y_{[\mathbf{n}]}$ might not be positively affiliated with the value.

[13]This seems to be a somewhat novel observation for ordinary common value auctions as well.

5. Discussion

5.1 *Existence without grid*

The difficulty in establishing existence directly in the model with a continuum of bids is due to the possible presence of atoms in the bid distribution. Therefore, the bidders' *equilibrium* payoffs might not be continuous in their bids, and this precludes the application of "off-the-shelf" existence results. This is why we look instead at the limit of a sequence of equilibria for a vanishingly small grid (that way, existence is guaranteed by established results, e.g., Athey (2001)).

One issue with this approach is that such a limit is not necessarily an equilibrium of the continuum case, as the limit might exhibit atoms that are absent in the sequence. To see this, consider a sequence of games with grid P_{Δ^k}, with $\beta^k(x) = b$ for $x < \hat{x}$ and $\beta^k(x) = b + \Delta^k$ for $x \geq \hat{x}$. In the limit as $\Delta^k \to 0$, $\lim \beta^k(x) = b$ for all x. Therefore, the winning probability in the limit is strictly higher than the limit of winning probabilities for bidders with $x < \hat{x}$ and it is strictly lower for bidders with $x \geq \hat{x}$. Such merging of atoms may imply that the limit strategy does not need to be an equilibrium of the game with a continuum of bids, even if the elements of the sequence are.

This issue may be resolved by simply defining equilibrium to be the limit outcome as the grid's step goes to zero, or by using the related approach of Jackson et al. (2002). Roughly speaking, bidders submit two numbers, their actual bid and their "eagerness to trade"; the winning bidder is selected from among those who are tied for the "most eager" designation within the group of those who are tied at the highest bid. In the example of the previous paragraph, the limit strategy will have all bidders bid b, but those with $x \geq \hat{x}$ (who bid $b + \Delta^k$ along the sequence) express eagerness $\overline{e}$, while those with $x < \hat{x}$ (who bid b along the sequence) express eagerness $\underline{e} < \overline{e}$. In case of a tie at b, the winner is chosen randomly from among those with $\overline{e}$ if such exists and otherwise from those with $\underline{e}$ (b bidders who announce anything else have even lower priority). With this approach, the winning probabilities and payoffs with a strategy that is the limit of a convergent sequence of bidding strategies are the limit of the winning probabilities and payoffs along the sequence. Therefore, the limit of a convergent sequence of equilibrium bidding strategies, for a vanishingly small grid, is an equilibrium of the continuum limit (of the modified game) itself.[14]

5.2 *Random state-dependent participation*

In the model considered so far, participation $\mathbf{n} = (n_\ell, n_h)$ is deterministic. In many cases of interest, however, participation is random. Let $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ denote participation distributions, where $\eta_\omega(n)$ is the probability with which $n = 1, \ldots, N$ bidders are invited in state ω. The expected payoff $U(b|x; \beta, \boldsymbol{\eta})$ and the probability of winning $\pi_\omega(b|\beta, \boldsymbol{\eta})$ are now functions of $\boldsymbol{\eta}$. The bidding game given $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$ is $\Gamma_0(\boldsymbol{\eta}, N, \Delta)$, and a bidding equilibrium is defined as before.

[14] For a detailed discussion of these existence problems in a related model with an uncertain number of bidders, see Lauermann and Speit (2020).

Appendix A.3.1 presents the explicit expressions of U and π_ω for this case. It also presents the proof that any bidding equilibrium is monotone using the single crossing property of the buyers' preferences; see Proposition 4. In addition, for certain forms of random state-dependent participation, the characterization of the bidding equilibria of large auctions in Theorem 1 holds. In one such form, the support of η^k_ω is contained in $\{n^k_\omega, \dots, n^k_\omega + m\}$ for some fixed integer $m > 0$.

Proposition 3. *Consider any sequence of bidding games $\Gamma_0(\boldsymbol{\eta}^k, N^k, \Delta^k)$ such that for every k the support of η^k_ω is contained in $\{n^k_\omega, \dots, n^k_\omega + m\}$ for some fixed integer $m > 0$ and $\Delta^k \to 0$, $\min\{n^k_\ell, n^k_h\} \to \infty$, and $\lim \frac{n^k_h}{n^k_\ell} = r$. Then the conclusions of Theorem 1 hold.*

This observation is not surprising because in this case the randomness becomes relatively negligible as $\min\{n^k_\ell, n^k_h\} \to \infty$. and $n^k_\omega \to \infty$. The proof is in Appendix A.3.3.

If the uncertainty over the number of bidders has full support, the extension is not straightforward. We know from Murto and Välimäki (2019) and Lauermann and Speit (2020) that even when the distribution is independent of the state, the equilibrium strategies might be nonmonotone and existence is not guaranteed.[15]

Nevertheless, based on the results in Lauermann and Speit (2020), we conjecture that if the state-dependent participation is Poisson distributed with means given by μ_h and μ_ℓ, then our characterization results would extend with μ_h/μ_ℓ replacing n_h/n_ℓ. However, this would require further analysis.

5.3 *Endogenizing state-dependent participation*

In the main body of this paper, $\mathbf{n} = (n_\ell, n_h)$ is exogenously given. We now discuss ways to endogenize this feature.

5.3.1 *Seller solicitation* In Lauermann and Wolinsky (2017, 2021), an informed seller solicits bidders optimally at a cost s per invited bidder. This analysis builds on the bidding behavior of the present paper and explores the first-stage equilibrium solicitation. It confirms that the different forms of state-dependent participation considered in the present paper may arise endogeneously in the equilibria of a larger model. The analysis focuses on the limit equilibrium outcomes of a sequence of auctions obtained as the solicitation cost s vanishes. Lauermann and Wolinsky (2021) establishes the existence of a sequence of equilibria whose limit outcome is of the partially revealing type and shows that there is a unique such outcome (with respect to the limit ratio $\frac{n_h}{n_\ell}$ and the bid distributions). There is an accompanying characterization of how this unique limit ratio $\frac{n_h}{n_\ell}$ varies with the informativeness of the signal, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})}$, and the prior belief. Lauermann and Wolinsky (2017) shows that, for a binary signal, there also exist multiple (limit) equilibrium outcomes of the pooling type.

[15]See also Harstad et al. (2008) for the effect of an uncertain number of bidders in the large double-auction setting of Pesendorfer and Swinkels (1997).

5.3.2 *Costly entry* Bidders' entry decisions are another potential source of state-dependent participation. Suppose that there is a fixed population N of potential bidders and that each bidder i has to incur a cost c_i to enter the auction. The distribution of these costs depends on the state ω. In equilibrium, bidders will enter if their cost is lower than their expected payoff of participating. This will result in state-dependent participation. One can think of c_i as being determined by the value of an alternative option, the availability of which depends on the state. The bidding stage of this interaction is provided by our model. The equilibrium participation will be determined by a condition on the optimality of bidders' entry decisions.

In a simple binary example of this sort, the costs $c_i \in \{\underline{c}, \overline{c}\}$ with $\underline{c} < \overline{c}$ and only those with cost $\underline{c}$ choose to enter. In state ω, there are n_ω bidders with cost $\underline{c}$. If n_ω is stochastic (e.g., if bidders' costs are drawn independently from $\{\underline{c}, \overline{c}\}$ with a state-dependent probability), this would give rise to a random number of bidders conditional on the state, as discussed in Section 5.2.

Consider next an alternative scenario in which bidders' entry decisions are made in the interim stage after they learned their signal. Suppose that all bidders have a state-dependent outside option that they would prefer to the participation in the auction if the state is h with sufficiently high probability. In particular, suppose that there is an interior $\hat{x}$ such that bidders enter if and only if $x \leq \hat{x}$. This scenario may seem different from what we consider in this paper since bidders learn only their signal and not the extra information of being selected into the auction. However, as shown next, it can be transformed into an equivalent form that is an instance of the stochastic version of our model.

If the bidders' signals are independent draws out of G_ω, the number of entering bidders in state ω is distributed binomially with N draws and success probability $G_\omega(\hat{x})$; hence, the expected participation is $\bar{n}_\omega = G_\omega(\hat{x})N$. Then the model with $\bar{n}_\omega$ and the distribution $\widehat{G}_\omega(x) = \frac{G_\omega(x)}{G_\omega(\hat{x})}$ is an instance of the stochastic version of our state-dependent participation model discussed in Section 5.2.

These descriptions are equivalent since the bidders' information turns out to be the same in both: In the original scenario, a bidder observes only the signal, x; hence, the information that it gets beyond the prior is summarized by the likelihood ratio $\frac{g_h(x)}{g_\ell(x)}$. In the transformed model that fits our framework, a bidder's information is both the signal x and being in the auction, which is summarized by the likelihood ratio $\frac{\bar{n}_h}{\bar{n}_\ell}\frac{\widehat{g}_h(\hat{x})}{\widehat{g}_\ell(\hat{x})}$. Observe that these two likelihood ratios are indeed equal,

$$\frac{\bar{n}_h}{\bar{n}_\ell}\frac{\widehat{g}_h(\hat{x})}{\widehat{g}_\ell(\hat{x})} = \frac{G_h(\hat{x})N}{G_\ell(\hat{x})N}\frac{\dfrac{g_h(x)}{G_h(\hat{x})}}{\dfrac{g_\ell(x)}{G_\ell(\hat{x})}} = \frac{g_h(x)}{g_\ell(x)}.$$

Costly entry into common value auctions has been studied in Murto and Välimäki (2019). They show that the resulting uncertainty about the bidder number may give rise to nonmonotone bidding strategies and a failure of the linkage principle. Atakan and Ekmekci (2020) study the bidders' choice among two competing auctions and the extent of arbitrage. Their model is quite different from ours. The main relationship to our work is in the state-dependent participation that in their model is implied by the endogenous state-dependent opportunity cost of entry.

Yet another model of state-dependent participation is in the spirit of rational inattention. Bidders become aware of an auction with a probability that depends on the state, and they may choose optimally the state-dependent probability of becoming aware of the auction in the ex ante stage.

5.4 *Broader class of environments*

We analyze large, first-price auctions in a binary state world and, strictly speaking, the results pertain to that environment. However, this model is just a means to illustrate the main insights concerning the effects of state-dependent participation that are likely to be relevant for a broader set of environments. The previous subsection already presents scenarios to which this analysis applies (random state-dependent participation and endogenous participation). We now discuss further scenarios to illustrate the potential broader scope.

Other auction formats Although we have not performed the full analysis, it seems that the qualitative results continue to hold for a second-price auction as well. In this case, the functional forms of the limit price distribution will be different, but the main insights would not change.

Two states The qualitative insights of the strategic inference from the state-dependent participation do not seem to depend on the two-state assumption. We use this assumption to establish the monotonicity of the equilibrium bidding strategy. If monotonicity can be established for the multiple state case, perhaps by resorting to stronger assumptions, then the extension to a world of multiple states would probably be quite straightforward.

Unboundedly informative signals It has been assumed throughout that the signals are boundedly informative, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} < \infty$. While this assumption was used in the analysis, some of the results extend to a setting with an unboundedly informative signal. In particular, when the signal is unboundedly informative, $\frac{g_h(\bar{x})}{g_\ell(\bar{x})} = \infty$, then the shape of the equilibrium will depend on the speed of divergence of $\frac{n_\ell}{n_h} \to \infty$.[16] In an extension of Lauermann and Wolinsky (2017) with seller solicitation, we give an example with unboundedly informative signals that shows that a pooling equilibrium may arise.

Large auctions The focus on large auctions is natural for discussing information aggregation. Nevertheless, the strategic effects of state-dependent participation are just as relevant for trading scenarios with few participants. Still, we focus on large auctions because the analysis is simpler. For example, in the partially revealing case, large numbers guarantee that bids are near the expected values, and thus simplify the argument. However, such proximity may already hold for fairly low numbers, and perhaps other arguments utilizing more directly the structure of the equilibrium might be used.

[16]Specifically, given a sequence (n_h^k, n_ℓ^k) and some $q \in (0, 1)$, let x_q^k solve $(1 - G_h(x_q^k))^{n_h^k} = q$ for all k. Then what will matter is whether $\lim_{k\to\infty} \frac{(1-G_h(x_q^k))^{n_h^k}}{(1-G_\ell(x_q^k))^{n_\ell^k}} > 1$ for all q.

Outlook There are a number of open questions left for future research. We already noted that it may be worth exploring certain entry models more systematically; see Section 5.3.2. In addition, exploring equilibrium bidding with a small, state-dependent bidder number may be interesting, as this may open up further questions such as the revenue comparison across auction formats, optimal information design, and the information acquisition incentives of bidders. For these questions, the results for the pooling equilibria arising here may be different from those implied by the standard equilibria. Finally, extending the analysis from the pure common-values case to the general interpendent-values case may be worthwhile in order to explore the implications of pooling for the allocative performance of the auction.

Appendix

Auxiliary result: Winning probability at atoms The following lemma is restated from Lauermann and Wolinsky (2017). It derives an expression for the winning probability in the case of a tie. Define

$$x_-(b) \triangleq \inf\{x \in [\underline{x}, \bar{x}] | \beta(x) \geq \bar{b}\}$$

and

$$x_+ \triangleq \sup\{x \in [\underline{x}, \bar{x}] | \beta(x) \leq \bar{b}\}.$$

Lemma 1 (Lauermann and Wolinsky (2017)). *Suppose β is nondecreasing and, for some $\bar{b}$, $x_- = x_-(\bar{b}) < x_+(\bar{b}) = x_+$. Then*

$$\pi_\omega(\bar{b}) = \frac{G_\omega(x_+)^n - G_\omega(x_-)^n}{n\big(G_\omega(x_+) - G_\omega(x_-)\big)} = \int_{x_-}^{x_+} \frac{\big(G_\omega(x)\big)^{n-1} g_\omega(x)\, dx}{G_\omega(x_+) - G_\omega(x_-)}. \tag{14}$$

Observe that the last expression is the expected probability of a randomly drawn signal from $[x_+, x_-]$ to be the highest. Thus, $\pi_\omega(\bar{b})$ "averages" what would be the winning probabilities of the types in $[x_+, x_-]$ if β were strictly increasing.

A.1 *Proof of Proposition 2 and Theorem 1 (large bidding equilibria)*

Here, and in the rest of the Appendix, we often use the abbreviation

$$\lambda \triangleq \overline{g} \lim \frac{n_h^k}{n_\ell^k}.$$

A.1.1 *Preliminary comments* The finite grid ($\Delta^k > 0$) is needed only for the existence claims but not for the characterization results. We therefore proceed as follows. First, we show the characterization results for the no-grid case of $\Delta^k = 0$ because this case is less cluttered, proving Proposition 2. Second, we resurrect the finite grid with $\Delta^k > 0$ to explain the adaptations of the proof that it requires, proving the characterization parts of Theorem 1. Finally, we establish the existence of equilibria, especially those described in Part 2 of Theorem 1.

We prepare the proof with a number of auxiliary lemmas that hold for $\Delta^k \geq 0$.

A.1.2 *Auxiliary lemmas* The next lemma formalizes the idea that the number of bidders with signals close to $\bar{x}$ is Poisson distributed.

Lemma 2 (Poisson-approximation). *Consider some sequence* $(x^k, \mathbf{n}^k)$ *with* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\lim \frac{n_h^k}{n_\ell^k} = r < \infty$. *If*

$$\lim \bigl(G_\ell (x^k)\bigr)^{n_\ell^k} = q$$

for some $q \in [0, 1]$, *then*

$$\lim \bigl(G_h (x^k)\bigr)^{n_h^k} = q^{\bar{g} r}.$$

Proof of Lemma 2. Let $Q_\omega \triangleq \lim(1 - G_\omega(x^k)) n_\omega^k \in [0, \infty) \cup \infty$. Observe that

$$\lim \bigl(G_\omega (x^k)\bigr)^{n_\omega^k} = \lim \biggl(1 - \frac{1 - G_\omega(x^k)}{n_\omega^k} n_\omega^k \biggr)^{n_\omega^k} = e^{-Q_\omega}.$$

The lemma clearly holds with $q = 0$ if $\lim x^k < \bar{x}$. So, suppose $\lim x^k = \bar{x}$. Then $\lim \frac{1-G_h(x^k)}{1-G_\ell(x^k)} = \bar{g}$, and so we have $Q_h = Q_\ell \bar{g} \lim(n_h^k / n_\ell^k)$. Therefore, $q = e^{-Q_\ell}$ implies

$$\lim \bigl(G_h (x^k)\bigr)^{n_h^k} = e^{-Q_h} = e^{Q_\ell \bar{g} \lim(n_h^k / n_\ell^k)} = q^{\bar{g} r}. \qquad \square$$

Recall that

$$U(b|x, \text{sol}; \beta, \mathbf{n}) = \frac{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b; \beta, n_\ell)(v_\ell - b) + \rho_h g_h(x) n_h \pi_h(b; \beta, n_h)(v_h - b)}{\rho_\ell g_\ell(x) n_\ell + \rho_h g_h(x) n_h}. \tag{15}$$

Lemma 3 ("Zero profit"). *For any* $\varepsilon > 0$, *there is an* $M(\varepsilon)$ *such that, if* $n_\omega > M(\varepsilon)$, $\omega = \ell, h$, *then* $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) < \varepsilon$ *for all* x *in every bidding equilibrium* β.

Remark. We do not suppress here β, $\mathbf{n}$ from the arguments of U since the claim concerns a range of $\mathbf{n}$ and all corresponding equilibria β.

Proof of Lemma 3. By (15) and the right continuity of $\frac{g_h}{g_\ell}$, $(U(b|\cdot, \text{sol}; \beta, \mathbf{n}))_{b,\beta,\mathbf{n}}$ is a family of functions that is uniformly (right) equicontinuous: For every $\varepsilon > 0$ and x, there is some $z_\varepsilon > 0$ such that

$$\bigl|U\bigl(b|x', \text{sol}; \beta, \mathbf{n}\bigr) - U(b|x, \text{sol}; \beta, \mathbf{n})\bigr| \le \frac{\varepsilon}{2},$$

for all b, all $(\beta, \mathbf{n})$ and all x' such that $0 \le x' - x \le z_\varepsilon$; similarly at $\bar{x}$ for all x' s.t. $\bar{x} - x' \le z_\varepsilon$.[17]

[17]The monotonicity of $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n})$ in x, which is established in Lemma 8, implies that it would be sufficient to argue the result for $\bar{x}$.

Suppose $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) = \varepsilon > 0$ for some $x < \bar{x}$ (the case $x = \bar{x}$ is analogous and omitted). From β being a bidding equilibrium, for all $x' > x$ s.t. $x' - x \leq z_\varepsilon$,

$$\left|U\big(\beta(x)|x, \text{sol}; \beta, \mathbf{n}\big) - U\big(\beta(x')|x', \text{sol}; \beta, \mathbf{n}\big)\right| \leq \frac{\varepsilon}{2}. \tag{16}$$

Therefore,

$$\begin{aligned}
&U\big(\beta(x)|x, \text{sol}; \beta, \mathbf{n}\big) - \frac{\varepsilon}{2} \\
&\quad \leq \inf_{x' \in [x, x+z_\varepsilon]} U\big(\beta(x')|x', \text{sol}; \beta, \mathbf{n}\big) \\
&\quad \leq \sum_{\omega=\ell,h} \rho_\omega \frac{\int_x^{x+z_\varepsilon} \big[v_\omega - \beta(x')\big]\pi_\omega\big(\beta(x'); \beta, n_\omega\big)\, dG_\omega(x')}{G_\omega(x+z_\varepsilon) - G_\omega(x)} \\
&\quad \leq \sum_{\omega=\ell,h} \rho_\omega \frac{v_\omega \int_{\underline{x}}^{\bar{x}} \pi_\omega\big(\beta(x'); \beta, n_\omega\big) dG_\omega(x')}{G_\omega(x+z_\varepsilon) - G_\omega(x)} \\
&\quad = \sum_{\omega=\ell,h} \frac{\rho_\omega v_\omega}{n_\omega\big(G_\omega(x+z_\varepsilon) - G_\omega(x)\big)} \\
&\quad \leq \frac{\mathbb{E}[v]}{\min_{\omega\in\{\ell,h\}} \big(n_\omega\big(G_\omega(x+z_\varepsilon) - G_\omega(x)\big)\big)},
\end{aligned}$$

where the first inequality follows from (16), the second follows from the definition of U, the third owes to increasing the term in the numerator, and the fourth from the fact that the expected probability of winning over all signals is $1/n_\omega$. Now, let $M(\varepsilon)$ be large enough so that, for $n_\omega \geq M(\varepsilon)$, the RHS is smaller than $\frac{\varepsilon}{2}$. Therefore, for any $\mathbf{n}$ such that $n_\omega \geq M(\varepsilon)$, $U(\beta(x)|x, \text{sol}; \beta, \mathbf{n}) < \varepsilon$. □

Corollary 3. *Let* $(\mathbf{n}^k)_{k=1}^\infty$ *be such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $(\beta^k)_{k=1}^\infty$ *be a corresponding sequence of bidding equilibria.*

(i)

$$\lim \sup_{x\in[\underline{x},\bar{x}]} U^k\big(\beta^k(x)|x, \text{sol}\big) = 0. \tag{17}$$

(ii) If, for some sequence $(b^k)_{k=1}^\infty$ *of bids and some* ω, $\lim \pi_\omega^k(b^k) > 0$, *then for any sequence* $(x^k)_{k=1}^\infty$,

$$\lim \mathbb{E}^k\big[v|x^k, \text{sol, win } \textit{at } b^k\big] \leq \lim b^k. \tag{18}$$

(iii) If $\lim \pi_\omega^k(\beta^k(x^k)) > 0$ *for some* ω *and sequence* $(x^k)_{k=1}^\infty$, *then*

$$\lim \beta^k\big(x^k\big) = \lim \mathbb{E}^k\big[v|x^k, \text{sol, win } \textit{at } \beta^k\big(x^k\big)\big]. \tag{19}$$

Proof of Corollary 3. From Lemma 2, $\lim \pi_h^k(\beta^k(x^k)) > 0 \Leftrightarrow \lim \pi_\ell^k(\beta^k(x^k)) > 0$. Therefore, $\lim \pi_\omega^k(b^k) > 0$ for some ω is sufficient for $\lim \pi_\omega^k(b^k) > 0$ for all ω.

Parts (i) and (ii) follow immediately from Lemma 3 that would be contradicted if (17) or (18) did not hold. Part (iii) is immediate from (18) and the individual rationality condition,

$$\beta^k(x^k) \leq \mathbb{E}^k[v|x^k, \text{sol, win at } \beta^k(x^k)]. \qquad \square$$

Recall that $\overline{g} \triangleq \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$.

Lemma 4. *Let* $\mathbf{n}^k$ *be such that* $\min\{n_\ell^k, n_h^k\} \to \infty$ *and* $\overline{g} \lim \frac{n_h^k}{n_\ell^k} < 1$, *and let* $(\beta^k)_{k=1}^\infty$ *be a corresponding sequence of (nondecreasing) bidding strategies. If* $(b^k)_{k=1}^\infty$ *is a sequence of bids such that* $b^k < \beta^k(\bar{x})$ *for all* k *and* $\lim \pi_\ell^k(b^k) \in (0, 1)$, *then*

$$\lim \mathbb{E}^k[v|\bar{x}, \text{sol, win } at\ b^k] > \lim \mathbb{E}^k[v|\bar{x}, \text{sol, win } at\ \beta^k(\bar{x})].$$

Proof of Lemma 4. Divide throughthe numerator and denominator of (3) by $\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)$ to express it in terms of the compound likelihood ratio $\frac{\rho_h}{\rho_\ell} \frac{g_h(x)}{g_\ell(x)} \frac{n_h}{n_\ell} \frac{\pi_h(b)}{\pi_\ell(b)}$ as

$$\mathbb{E}[v|x\text{,sol,win at } b] = \frac{v_\ell + \dfrac{\rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)} v_h}{1 + \dfrac{\rho_h g_h(x) n_h \pi_h(b)}{\rho_\ell g_\ell(x) n_\ell \pi_\ell(b)}}. \tag{20}$$

Hence, we have to show that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} > \lim \frac{\pi_h^k(\beta^k(\bar{x}))}{\pi_\ell^k(\beta^k(\bar{x}))}. \tag{21}$$

Let

$$\hat{q} \triangleq \lim\left(G_\ell\left(x_+^k(b^k)\right)\right)^{n_\ell^k - 1},$$
$$\hat{q}_- \triangleq \lim\left(G_\ell\left(x_-^k(b^k)\right)\right)^{n_\ell^k - 1},$$

with $1 \geq \hat{q} \geq \hat{q}_- > 0$ by $\lim \pi_\ell^k(b^k) \in (0, 1)$. Recall $\lambda \triangleq \overline{g} \lim \frac{n_h^k}{n_\ell^k}$. We first show the following:

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \hat{q}^{\lambda - 1} > 1 \quad \text{if } \hat{q}_- = \hat{q} \tag{22}$$

and

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q} - \hat{q}_-)} > \hat{q}^{\lambda - 1} \geq 1 \quad \text{if } \hat{q}_- < \hat{q}. \tag{23}$$

To derive (22), note that[18]

$$\left(G_\omega(x_-^k)\right)^{n_\omega^k-1} \le \pi_\omega(b^k|\beta^k, n_\omega^k) \le \left(G_\omega(x_+^k)\right)^{n_\omega^k-1}.$$

Hence, whenever $\lim(G_\ell(x_-^k))^{n_\ell^k-1} = q_- = \hat{q} = \lim(G_\ell(x_+^k))^{n_\ell^k-1}$, Lemma 2 implies

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{\hat{q}^\lambda}{\hat{q}} = \hat{q}^{\lambda-1}.$$

To derive (23), recall from Lemma 1 that

$$\pi_\omega^k(b^k) = \frac{\left(G_\omega(x_+^k)\right)^{n_\omega^k} - \left(G_\omega(x_-^k)\right)^{n_\omega^k}}{n_\omega^k\left[G_\omega(x_+^k) - G_\omega(x_-^k)\right]}, \tag{24}$$

and hence, using Lemma 2,

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \frac{n_\ell^k}{n_h^k} \frac{G_\ell(x_+^k) - G_\ell(x_-^k)}{G_h(x_+^k) - G_h(x_-^k)} \frac{G_h(x_+^k)^{n_h^k} - G_h(x_-^k)^{n_h^k}}{G_\ell(x_+^k)^{n_\ell^k} - G_\ell(x_-^k)^{n_\ell^k}} = \frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q} - \hat{q}_-)}.$$

To show the inequality $\frac{(\hat{q})^\lambda - (\hat{q}_-)^\lambda}{\lambda(\hat{q}-\hat{q}_-)} > \hat{q}^{\lambda-1}$, let $Q \triangleq \frac{\hat{q}_-}{\hat{q}} < 1$. Then the inequality is equivalent to $Q^\lambda - \lambda Q + \lambda < 1$. Since $\lambda < 1$, the LHS is increasing in Q over $[0, 1)$ and is equal to 1 at $Q = 1$, so the inequality holds.

Let

$$\bar{x}_-^k \triangleq x_-^k\left(\beta^k(\bar{x})\right) \quad \text{and} \quad q \triangleq \lim\left(G_\ell(\bar{x}_-^k)\right)^{n_\ell^k}.$$

Since, by the hypothesis, $b^k < \beta^k(\bar{x})$ for all k, we have $q \ge \hat{q}$.

Case 1. Suppose that $q = 1$. Since

$$\pi_\omega^k\left(\beta^k(\bar{x})\right) \ge \left(G_\omega(\bar{x}_-^k)\right)^{n_\omega^k-1},$$

we have $\lim \pi_\ell^k(\beta^k(\bar{x})) = q(=1)$. By Lemma 2, $\lim(G_h(\bar{x}_-^k))^{n_h^k} = q^\lambda = 1$ as well. So, $\lim \frac{\pi_h^k(\beta^k(\bar{x}))}{\pi_\ell^k(\beta^k(\bar{x}))} = 1$. This, (22), and (23) imply (21).

Case 2. Suppose that $q < 1$. So, there is an atom at $\beta^k(\bar{x})$. First, consider $\lambda \in (0, 1)$. As before, using Lemmas 2 and 1, we have

$$\lim \frac{\pi_h^k\left(\beta^k(\bar{x})\right)}{\pi_\ell^k\left(\beta^k(\bar{x})\right)} = \frac{1 - q^\lambda}{\lambda(1-q)} < q^{\lambda-1}, \tag{25}$$

[18]This can be verified using Lemma 1. For example, expanding the formula for π_ω gives

$$\pi_\omega(b^k|\beta^k, n_\omega^k) = \frac{1}{n_\omega^k}\left[G_\omega(x_+^k)^{n_\omega^k-1} + G_\omega(x_+^k)^{n_\omega^k-2} G_\omega(x_-^k) + \cdots + G_\omega(x_-^k)^{n_\omega^k-1}\right] \ge \frac{n_\omega^k G_\omega(x_+^k)^{n_\omega^k-1}}{n_\omega^k}.$$

where the last inequality follows from $\lambda \in (0,1)$, $q \in (0,1)$, and straightforward algebraic manipulation.[19]

Since $q \geq \hat{q} > 0$ and $\lambda < 1$, we have $\hat{q}^{\lambda-1} \geq q^{\lambda-1}$. Now, this together with (22), (23), and (25) imply (21).

If $\lambda = 0$, by Lemma 2, $\lim G_h(x_-^k(b^k))^{n_h^k} = 1$, and hence, $\lim \pi_h^k(\beta^k(\bar{x})) = 1$. Thus, (21) follows from $\lim \pi_\ell^k(b^k) < \lim \pi_\ell^k(\beta^k(\bar{x}))$. □

A.1.3 *Proof of Proposition 2 (characterization for the case of no grid)* We use the following lemma in the proof.

Lemma 5. *Suppose $\Delta^k = 0$ for all k. Let $\mathbf{n}^k$ be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\overline{g} \lim \frac{n_h^k}{n_\ell^k} > 1$. Let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of equilibrium bidding strategies. If $(\beta^k)_{k=1}^\infty$ contains a sequence of nonvanishing atoms $(b^k)_{k=1}^\infty$, that is, $\lim(G_\ell(x_+^k(b^k)))^{n_\ell^k} > \lim(G_\ell(x_-^k(b^k)))^{n_\ell^k}$, then*

$$\lim_{k\to\infty} b^k < \lim_{k\to\infty} \lim_{\varepsilon\to 0} \mathbb{E}^k[v|x_+^k, \text{sol}, \text{win } at\ b^k + \varepsilon].$$

Proof of Lemma 5. By bidders' individual rationality, $\mathbb{E}^k[v|x_-^k, \text{sol}, \text{win at } b^k] \geq b^k$. Therefore, the claim will follow from $\lim \lim_{\varepsilon\to 0} \mathbb{E}^k[v|x_+^k, \text{sol}, \text{win at } b^k + \varepsilon] > \lim \mathbb{E}^k[v|x_-^k, \text{sol}, \text{win at } b^k]$, which in turn will follow from

$$\lim \frac{g_h(x_-^k)}{g_\ell(x_-^k)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h(x_+^k)}{g_\ell(x_+^k)} \frac{(G_h(x_+^k(b^k)))^{n_h^k}}{(G_\ell(x_+^k(b^k)))^{n_\ell^k}}. \tag{26}$$

Let $q_- = \lim G_\ell(x_-^k)^{n_\ell^k}$ and $q_+ = \lim G_\ell(x_+^k)^{n_\ell^k}$. Note that $G_\omega(x_+^k)^{n_\omega^k} \approx G_\omega(x_+^k)^{n_\omega^k - 1}$ for large k. By the hypothesis of the lemma, $q_+ > 0$. By Lemma 2, $\lim G_h(x_-^k)^{n_h^k} = (q_-)^\lambda$ and $\lim G_h(x_+^k)^{n_h^k} = (q_+)^\lambda$. Recall from Lemma 1 that

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \frac{n_\ell^k}{n_h^k} \frac{G_\ell(x_+^k) - G_\ell(x_-^k)}{G_h(x_+^k) - G_h(x_-^k)} \frac{G_h(x_+^k)^{n_h^k} - G_h(x_-^k)^{n_h^k}}{G_\ell(x_+^k)^{n_\ell^k} - G_\ell(x_-^k)^{n_\ell^k}}. \tag{27}$$

Using this and the above observations,

$$\lim \frac{g_h(x_-^k)}{g_\ell(x_-^k)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \lim \left(\frac{g_h(x_-^k)}{g_\ell(x_-^k)} \frac{G_\ell(x_+^k) - G_\ell(x_-^k)}{G_h(x_+^k) - G_h(x_-^k)} \right) \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \frac{(q_+)^\lambda - (q_-)^\lambda}{\lambda(q_+ - q_-)}. \tag{28}$$

Now, $\lim \frac{g_h(x_+^k)}{g_\ell(x_+^k)} = \frac{g_h(\bar{x})}{g_\ell(\bar{x})}$ and by MLRP

$$\frac{g_h(x_-^k)}{g_\ell(x_-^k)} \frac{G_\ell(x_+^k) - G_\ell(x_-^k)}{G_h(x_+^k) - G_h(x_-^k)} \leq 1.$$

[19] With $Q = \frac{1}{q}$, the inequality is equivalent to $(Q)^\lambda - \lambda Q + \lambda < 1$. The right-hand side equals 1 if $Q = 1$ and is increasing in Q on $(0,1)$ by $\lambda \in (0,1)$; hence, the inequality holds.

Therefore, we may establish (26) by showing that

$$\frac{(q_+)^\lambda - (q_-)^\lambda}{\lambda(q_+ - q_-)} < (q_+)^{\lambda-1}. \tag{29}$$

Letting $Q = \frac{q_-}{q_+} < 1$, (29) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is decreasing in Q over $[0, 1)$ and is equal to 1 at $Q = 1$. Therefore, (29) holds and so does (26). □

We now prove proposition 2. By Proposition 1, we may assume that each bidding strategy β^k is monotone.

Case 1: Suppose $\overline{g}r < 1$. Given any $\varepsilon \in (0, 1)$, let (x^k) be such that $\lim(G_h(x^k))^{n_h^k} = \varepsilon$ for all k. We show that

$$\lim\bigl(G_h\bigl(x_+^k\bigl(\beta^k\bigl(x^k\bigr)\bigr)\bigr)\bigr)^{n_h^k} = 1,$$

with $x_+^k(b) = \sup\{x|\beta^k(x) = b\}$. This implies

$$\lim\bigl(G_h\bigl(x_+^k\bigl(b^k\bigr)\bigr)\bigr)^{n_h^k} - \bigl(G_h\bigl(x_-^k\bigl(b^k\bigr)\bigr)\bigr)^{n_h^k} \geq 1 - \varepsilon.$$

Then by Lemma 2 and $\bar{g}r < 1$, this inequality holds for $\omega = \ell$ as well. Since we can choose ε arbitrarily small, this establishes the claim.

Let $y_+^k \equiv x_+^k(b)$, and suppose to the contrary that

$$\lim\bigl(G_h\bigl(y_+^k\bigr)\bigr)^{n_h^k} < 1. \tag{30}$$

Since $\beta^k(x^k) < \beta^k(\bar{x})$, (30) implies that there exists b^k with $\beta^k(x^k) < b^k < \beta^k(\bar{x})$ and

$$\lim \pi_\ell^k\bigl(b^k\bigr) \in (0, 1). \tag{31}$$

Hence, the zero-profit condition (18) from Corollary 3 requires that

$$\lim b^k \geq \lim \mathbb{E}^k\bigl[v|\bar{x}, \text{sol}, \text{win at } b^k\bigr]. \tag{32}$$

Given (31) and (32), Lemma 4 implies that

$$\lim \mathbb{E}^k\bigl[v|\bar{x}, \text{sol}, \text{win at } b^k\bigr] > \lim \mathbb{E}^k\bigl[v|\bar{x}, \text{sol}, \text{win at } \beta^k(\bar{x})\bigr]. \tag{33}$$

Individual rationality requires that

$$\lim \mathbb{E}^k\bigl[v|\bar{x}, \text{sol}, \text{win at } \beta^k(\bar{x})\bigr] \geq \lim \beta^k(\bar{x}). \tag{34}$$

Hence, (32)–(34) together imply a contradiction to $b^k < \beta^k(\bar{x})$. Thus, (30) cannot hold, which proves the claim.

Case 2a: Suppose $\overline{g}r > 1$ **and** $r \neq \infty$. Let us establish first that there are no atoms in the limit. Suppose to the contrary that $\beta^k(x) = b^k$ for all $x \in (x_-^k, x_+^k)$ and $\lim(G_\ell(x_+^k))^{n_\ell^k} > \lim(G_\ell(x_-^k))^{n_\ell^k} \geq 0$. Thus,

$$\lim_{k\to\infty} \lim_{\varepsilon\to 0} \pi_\ell^k\bigl(b^k + \varepsilon\bigr) = \lim_{k\to\infty} \bigl(G_\ell\bigl(x_+^k\bigr)\bigr)^{n_\ell^k} > 0. \tag{35}$$

This and Lemma 5 implies that

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0} U^k\big(b^k+\varepsilon|x_+^k,\text{sol}\big)>0. \tag{36}$$

contradicting the zero-profit condition (17). Thus, there can be no atom.

Next, let us derive the functional form. Take any $\alpha\in(0,1)$. Let $(x^k)_{k=1}^\infty$ be such that $(G_\ell(x^k))^{n_\ell^k-1}=\alpha$ for all k. By the absence of atoms (just established above), $\lim \pi_\omega^k(\beta^k(x^k))=\lim(G_\omega(x^k))^{n_\omega^k-1}=\lim F_\omega^k(\beta^k(x^k))$. By Corollary (3)(iii),

$$\lim \beta^k\big(x^k\big)=\lim \mathbb{E}^k\big[v|x^k,\text{sol, win at }\beta^k\big(x^k\big)\big]$$

Therefore, expressing $\mathbb{E}^k[v|x^k,\text{sol, win at }\beta^k(x^k)]$ in terms of the compound likelihood ratio as in (20) and using $\lim \pi_\omega^k(\beta^k(x^k))=\lim(G_\omega(x^k))^{n_\omega^k-1}$,

$$\begin{aligned}\lim \beta^k\big(x^k\big)&=\lim\frac{v_\ell+\dfrac{\rho_h}{\rho_\ell}\dfrac{g_h(x^k)}{g_\ell(x^k)}\dfrac{n_h^k}{n_\ell^k}\dfrac{\pi_h^k(\beta^k(x^k))}{\pi_\ell^k(\beta^k(x^k))}v_h}{1+\dfrac{\rho_h}{\rho_\ell}\dfrac{g_h(x^k)}{g_\ell(x^k)}\dfrac{n_h^k}{n_\ell^k}\dfrac{\pi_h^k(\beta^k(x^k))}{\pi_\ell^k(\beta^k(x^k))}}\\&=\lim\frac{v_\ell+\dfrac{\rho_h}{\rho_\ell}\dfrac{g_h(x^k)}{g_\ell(x^k)}\dfrac{n_h^k}{n_\ell^k}\dfrac{\big(G_h(x^k)\big)^{n_h^k-1}}{\big(G_\ell(x^k)\big)^{n_\ell^k-1}}v_h}{1+\dfrac{\rho_h}{\rho_\ell}\dfrac{g_h(x^k)}{g_\ell(x^k)}\dfrac{n_h^k}{n_\ell^k}\dfrac{\big(G_h(x^k)\big)^{n_h^k-1}}{\big(G_\ell(x^k)\big)^{n_\ell^k-1}}}.\end{aligned} \tag{37}$$

From $\lim(G_\omega(x^k))^{n_\omega^k-1}>0$, we have $x^k\to\bar{x}$. This, Lemma 2, and $\lim(G_\omega(x^k))^{n_\omega^k-1}=\lim F_\omega^k(\beta^k(x^k))$ imply

$$\lim\frac{\big(G_h(x^k)\big)^{n_h^k-1}}{\big(G_\ell(x^k)\big)^{n_\ell^k-1}}=\lim\frac{\big(G_h(x^k)\big)^{n_h^k}}{\big(G_\ell(x^k)\big)^{n_\ell^k}}=\big[\lim\big(G_\ell(x^k)\big)^{n_\ell^k}\big]^{\lambda-1}=\alpha^{\lambda-1},$$

where, as before, $\lambda=\overline{g}\lim\frac{n_h^k}{n_\ell^k}$. Using this observation and letting $\lim\beta^k(x^k)=p$, we can rewrite (37) as

$$p=\frac{v_\ell+\rho\lambda\alpha^{\lambda-1}v_h}{1+\rho\lambda\alpha^{\lambda-1}}. \tag{38}$$

Thus, for every $\alpha\in(0,1)$, we can find the unique p such that $\lim F_\ell^k(p)=\alpha$. This gives a function $\hat{p}(\alpha)$ that is continuous and strictly increasing on $(0,1)$. The limit distribution $\Phi_\ell(p)$ is simply the inverse of $\hat{p}$, meaning, the α solution of (38) for given p. Finally, from Lemma 2, $\lim F_h^k(p)=\Phi_h(p)$.

Case 2b: Suppose $r=\infty$. In this case, $\Phi_\omega(\cdot|r)$ is degenerate with probability mass 1 on v_ω. Given bidders' individual rationality constraint, it is sufficient to show that $\Phi_h(\cdot|r)$ is degenerate with probability mass 1 on v_h. But this follows directly from the zero profit condition and the observation that, given $r = \infty$, if $\lim F_h^k(p) > 0$ for some $p < v_h$, then $\lim \mathbb{E}^k[v|x^k, \text{sol, win at } p] = v_h$.

Case 3: Suppose $\overline{g}r = 1$. From bidders' individual rationality,

$$\rho_\ell \lim \mathbb{E}^k[p|\ell] + \rho_h \lim \mathbb{E}^k[p|h] \leq \mathbb{E}[v]. \tag{39}$$

We show that, for any $p < \mathbb{E}[v]$, $\lim F_\omega^k(p) = 0$. This together with (39) implies the proposition, since if $\lim F_\omega^k(p) < 1$ for some $p > \mathbb{E}[v]$, (39) would be violated.

Suppose to the contrary that, for some $p < \mathbb{E}[v]$, $\lim F_\omega^k(p) > 0$. Therefore, given that bids are from the continuum, there is $p' < \mathbb{E}[v]$, such that $q \triangleq \lim \pi_\ell^k(p') > 0$. Then there is a sequence $(b^k)_{k=1}^\infty$ such that β^k has no atom at b^k for any k, $b^k \geq p'$, and $\lim b^k = p'$. Letting $\hat{q} \triangleq \lim \pi_\ell^k(b^k)$, Lemma 2 and $\lambda = \overline{g} \lim \frac{n_h^k}{n_\ell^k} = 1$ imply

$$\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{\hat{q}^\lambda}{\hat{q}} = 1.$$

Thus, from (3), $\lim \mathbb{E}^k[v|\bar{x}, \text{sol, win at } b^k] = \mathbb{E}[v] > \lim b^k$. Since also $\lim \pi_\omega^k(b^k) > 0$ from $b^k > p'$ and $\lim \pi_\ell^k(p') > 0$, we have

$$\lim U^k(b^k|\bar{x}, \text{sol}) > 0,$$

contradicting the zero-profit condition (17). Thus, such $(b^k)_{k=1}^\infty$ cannot exist. Therefore, $\lim \pi_\omega^k(p) = 0$ for all $p < \mathbb{E}[v]$, as needed.

This shows the characterization results for the no-grid case, proving Proposition 2.

A.1.4 *Proving the characterization results of Theorem 1* We now consider the finite grid $(\Delta^k > 0)$. Most of the above proof goes through with no change. We will therefore only present the arguments that have to be adjusted, rather than reproduce the entire proof. These are in the instances where a "slight undercutting" argument is used, and the adjusted arguments ensure that, for a sufficiently fine grid, the above proof goes through.

Case 1: $\bar{g}r < 1$. Given any $\varepsilon \in (0, 1)$, let x^k be such that $(G_h(x^k))^{n_h^k} = \varepsilon$ for all k. Let $b^k = \beta^k(x^k)$. As before, the result holds if

$$\lim \bigl(G_h\bigl(x_+^k\bigl(b^k + \Delta^k\bigr)\bigr)\bigr)^{n_h^k} = 1. \tag{40}$$

Suppose to the contrary that (40) fails and $\lim(G_h(x_+^k(b^k + \Delta^k)))^{n_h^k} < 1$. Then

$$b^k + \Delta^k < \beta^k(\bar{x}).$$

Moreover,

$$\lim \pi_h^k\bigl(b^k + \Delta^k\bigr) \leq \lim \bigl(G_h\bigl(x_+^k\bigl(b^k + \Delta^k\bigr)\bigr)\bigr)^{n_h^k} < 1$$

and

$$\lim \pi_h^k\bigl(b^k + \Delta^k\bigr) \geq \lim \bigl(G_h\bigl(x_-^k\bigl(b^k + \Delta^k\bigr)\bigr)\bigr)^{n_h^k} \geq \lim \bigl(G_h\bigl(x^k\bigr)\bigr)^{n_h^k} = \varepsilon > 0.$$

Hence, the zero-profit condition (17) requires that

$$\lim\big(b^k + \Delta^k\big) \geq \lim \mathbb{E}^k\big[v|\bar{x}, \text{sol, win at } b^k + \Delta^k\big].$$

Now, $\lim \pi_h^k(b^k + \Delta^k) \in (0, 1)$ and $b^k + \Delta^k < \beta^k(\bar{x})$ for all k implies via Lemma 4 (β^ks that have support only on the grid are a special case considered in that lemma) that

$$\lim \mathbb{E}^k\big[v|\bar{x}, \text{sol, win at } b^k + \Delta^k\big] > \lim \mathbb{E}^k\big[v|\bar{x}, \text{sol, win at } \beta^k(\bar{x})\big].$$

The bidders' individual rationality requires that

$$\lim \mathbb{E}^k\big[v|\bar{x}, \text{sol, win at } \beta^k(\bar{x})\big] \geq \lim \beta^k(\bar{x}).$$

Together, the last three displayed inequalities contradict $b^k + \Delta^k < \beta^k(\bar{x})$.

Case 2: $\bar{g}r > 1$. The critical lemma for this case was Lemma 5, which should be adapted as follows.

Lemma 6. *Suppose $\Delta^k > 0$ for all k. Let n^k be such that $\min\{n_\ell^k, n_h^k\} \to \infty$ and $\lim \frac{n_h^k}{n_\ell^k} \frac{g_h(\bar{x})}{g_\ell(\bar{x})} > 1$. Let $(\beta^k)_{k=1}^\infty$ be a corresponding sequence of bidding equilibria. If $(\beta^k)_{k=1}^\infty$ exhibits a sequence of nonvanishing atoms $(b^k)_{k=1}^\infty$, that is, $\lim(G_\ell(x_+^k(b^k)))^{n_\ell^k} > \lim(G_\ell(x_-^k(b^k)))^{n_\ell^k}$, then*

$$\lim b^k < \lim \mathbb{E}^k\big[v|x_+^k, \text{sol, win } at\ b^k + \Delta^k\big].$$

Proof of Lemma 6. If, in the limit, there is no atom at $b^k + \Delta^k$, that is, if $\lim(G_\ell(x_+^k(b^k + \Delta^k)))^{n_\ell^k} = \lim(G_\ell(x_-^k(b^k + \Delta^k)))^{n_\ell^k}$, then the original proof of the lemma works directly. If in the limit there is an atom at $b^k + \Delta^k$, then instead of (26) we have to establish

$$\lim \frac{g_h(x_-^k)}{g_\ell(x_-^k)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h(x_+^k)}{g_\ell(x_+^k)} \frac{\pi_h^k(b^k + \Delta^k)}{\pi_\ell^k(b^k + \Delta^k)}. \tag{41}$$

Let $x_{++}^k = x_+^k(b^k + \Delta^k)$ and note that $x_-^k(b^k + \Delta^k) = x_+^k(b^k) = x_+^k$. Also, recall $q_+ = \lim G_\ell(x_+^k)^{n_\ell^k}$ and let $q_{++} = \lim G_\ell(x_{++}^k)^{n_\ell^k}$. We already know from (26) that

$$\lim \frac{g_h(x_-^k)}{g_\ell(x_-^k)} \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} < \lim \frac{g_h(x_+^k)}{g_\ell(x_+^k)} \frac{\big(G_h(x_+^k(b^k))\big)^{n_h^k}}{\big(G_\ell(x_+^k(b^k))\big)^{n_\ell^k}} = \lim \frac{g_h(\bar{x})}{g_\ell(\bar{x})}(q_+)^{\lambda-1}$$

Analogous calculation to that of (27)–(28) in the proof yields

$$\begin{aligned}\lim \frac{g_h(x_+^k)}{g_\ell(x_+^k)} \frac{\pi_h^k(b^k + \Delta^k)}{\pi_\ell^k(b^k + \Delta^k)} &= \lim \frac{n_\ell^k}{n_h^k} \frac{g_h(x_+^k)}{g_\ell(x_+^k)} \frac{G_\ell(x_{++}^k) - G_\ell(x_+^k)}{G_h(x_{++}^k) - G_h(x_+^k)} \frac{G_h(x_{++}^k)^{n_h^k} - G_h(x_+^k)^{n_h^k}}{G_\ell(x_{++}^k)^{n_\ell^k} - G_\ell(x_+^k)^{n_\ell^k}} \\ &= \frac{g_h(\bar{x})}{g_\ell(\bar{x})} \frac{(q_{++})^\lambda - (q_+)^\lambda}{\lambda(q_{++} - q_+)}.\end{aligned}$$

Finally,

$$(q_+)^{\lambda-1} < \frac{(q_{++})^\lambda - (q_+)^\lambda}{\lambda(q_{++} - q_+)}, \tag{42}$$

since letting $Q = \frac{q_{++}}{q_+} > 1$, (42) is equivalent to $Q^\lambda - \lambda Q + \lambda > 1$. Since $\lambda > 1$, the LHS is increasing in Q over $[1, \infty)$ and is equal to 1 at $Q = 1$. Therefore, (42) and so does (41). This completes the adaptation of Lemma 5 for the case of finite price grid. □

We can now adapt the proof from Proposition 2. The proof uses a slight overbidding argument. The paragraph containing equations (35) and (36) should be modified as follows:

$$\lim_{k\to\infty} \lim_{\varepsilon^k\to 0} \pi_\ell^k\big(b^k + \Delta^k\big) \geq \lim_{k\to\infty} \big(G_\ell\big(x_+^k\big)\big)^{n_\ell^k} > 0, \tag{43}$$

where the first inequality is strict if in the limit there is an atom at $b^k + \Delta^k$, that is, if $\lim(G_\ell(x_+^k(b^k + \Delta^k)))^{n_\ell^k} < \lim(G_\ell(x_-^k(b^k + \Delta^k)))^{n_\ell^k}$. This and Lemma 6 implies that

$$\lim U^k\big(b^k + \Delta^k | x_+^k, \text{sol}\big) > 0. \tag{44}$$

Beyond that point, the proof from Proposition 2 continues unchanged.

Case 3: $\bar{g}r = 1$. The only necessary change required in the original proof of Proposition 2 is with respect to the choice of the sequence b^k. Note that given $\Delta^k \to 0$, under the stated hypothesis, there must still be a sequence b^k such that $b^k \geq p'$, $\lim b^k = p'$, and the probability of a tie at b^k is vanishing,

$$\lim \frac{\pi_\omega^k\big(b^k\big)}{\big(G_\omega\big(x_+^k\big(b^k\big)\big)\big)^{n_\omega^k}} = 1.$$

The remainder of the proof from Proposition 2 applies as before.

A.1.5 *Proving the existence claims of Theorem 1* Recall that $P_\Delta = [0, v_\ell) \cup \{v_\ell, v_\ell + \Delta, v_\ell + 2\Delta, \dots, v_h - \Delta, v_h\}$. Let $m = \| \{v_\ell, v_\ell + \Delta, \dots, v_h - \Delta, v_h\} \|$. Using the idea of Athey (2001), Σ_Δ is a set of vectors of dimension $m+1$ whose coordinates belong to $[\underline{x}, \bar{x}]$,

$$\Sigma_\Delta = \big\{\sigma = (\sigma_0, \sigma_1, \dots, \sigma_m) \in [\underline{x}, \bar{x}]^{m+1} \mid \underline{x} \triangleq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m \triangleq \bar{x}\big\},$$

where σ determines a monotone bidding strategy β_σ by $\beta_\sigma(x) = v_\ell + i\Delta$ if $x \in [\sigma_i, \sigma_{i+1})$, $i = 0, \dots, m-1$. Given $\varepsilon > 0$ and some $\hat{b} \in P_\Delta$, let $m(\hat{b}) = \| \{v_\ell, v_\ell + \Delta, \dots, \hat{b} - \Delta, \hat{b}\} \|$ and

$$\Sigma_\Delta(\hat{b}, \varepsilon) = \{\sigma \in \Sigma_\Delta | \sigma_{m(\hat{b})-1} = \bar{x} - \varepsilon, \sigma_{m(b)} = \bar{x}\};$$

that is, for $\sigma \in \Sigma_\Delta(\hat{b}, \varepsilon)$, the strategy $\beta_\sigma(x) = \hat{b}$ for all $x \in [\bar{x} - \varepsilon, \bar{x}]$.

Define the correspondence Ψ from $\Sigma_\Delta(\hat{b}, \varepsilon)$ into itself: For any $\sigma' \in \Sigma_\Delta(\hat{b}, \varepsilon)$, let

$$\Psi\big(\sigma'\big) = \Big\{\sigma \in \Sigma_\Delta(\hat{b}, \varepsilon) | \beta_\sigma(x) \in \arg\max_{b \leq \hat{b}} U(b|x, \text{sol}; \beta_\sigma, \mathbf{n}) \text{ for all } x \leq \bar{x} - \varepsilon\Big\},$$

that is, $\Psi(\sigma')$ is the best-response correspondence for $x \leq \bar{x} - \varepsilon$ when bidders are restricted to bid at most $\hat{b}$. The correspondence Ψ is nonempty, convex valued, and upper hemicontinuous. That Ψ is nonempty and convex valued follows immediately from the single-crossing property identified in Lemma 7, shown directly below, just as in Athey (2001). The upper hemicontinuity follows from the theorem of the maximum. Thus, by Kakutani's fixed-point theorem, there exists some $\sigma^*(\hat{b}, \varepsilon)$ such that $\sigma^* = \Psi(\sigma^*)$.

General existence claim If we choose $\varepsilon = 0$ and $\hat{b} = v_h$, then $\sigma^* = \Psi(\sigma^*)$ implies that σ^* is an equilibrium of the original game $\Gamma_0(\mathbf{n}, N, \Delta)$, proving the general existence claim at the start of Theorem 1.

Now, fix some sequence of bidding games $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ such that $\Delta^k > 0$, $\Delta^k \to 0$, $\min\{n_\ell^k, n_h^k\} \to \infty$, and $\lim \frac{n_h^k}{n_\ell^k} = r$, with $\bar{g}r < 1$.

Now, take any $q \in (0, 1)$ and let ε^k be such that $(G_\ell(\bar{x} - \varepsilon^k))^{n_\ell^k} = q$. Given some $\hat{b}$, let Ψ^k be the correspondence given $\hat{b}$, ε^k, Δ^k, $\mathbf{n}^k$, and let σ^k be one of its fixed points.

Claim 1. *For every $\hat{b}$ with $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] < \hat{b} < \mathbb{E}[v]$ and q small enough, the strategy $\beta^k = \beta_{\sigma^k}$ is a bidding equilibrium of $\Gamma_0(\mathbf{n}^k, N^k, \Delta^k)$ for k large enough.*

The claim implies the last remaining item from Theorem 1. To prove the claim, it is sufficient to show that $\bar{x}$ does not have an incentive to bid higher than $\hat{b}$ and $\bar{x} - \varepsilon$ has a strict incentive to bid $\bar{b}$, shown in Steps 2 and 3 below. This implies that β_{σ^k} is an optimal bid for all signals given the single-crossing property from Lemma 7 because the constraints are slack.

Step 1. For q small enough and every $\sigma^k \in \Sigma_k(\hat{b}, \varepsilon^k)$,

$$\lim \mathbb{E}^k\big[v|\bar{x} - \varepsilon^k, \text{sol, win at } \hat{b}\big] > \hat{b}.$$

By definition, $x_-^k(\hat{b}) = \sigma^k_{m^k(\hat{b})-1}$. Let $q_- = \lim(G_\ell(\sigma^k_{m^k(b)-1}))^{n_\ell^k}$, and note that $q_- \leq q$. From before, with $\lambda = \bar{g}r$, $\lim \frac{\pi_h^k(b^k)}{\pi_\ell^k(b^k)} = \frac{1-q_-^\lambda}{\lambda(1-q_-)}$ and so

$$\lim \frac{n_h}{n_\ell} \frac{g_h(\bar{x} - \varepsilon^k)}{g_\ell(\bar{x} - \varepsilon^k)} \frac{\pi_h^k(\hat{b})}{\pi_\ell^k(\hat{b})} = \frac{1 - q_-^\lambda}{1 - q_-},$$

which is arbitrarily close to 1 for q_- small enough. It follows that, for ever δ, there is some q small enough such that

$$\lim \mathbb{E}^k\big[v|\bar{x} - \varepsilon^k, \text{sol, win at } \hat{b}\big] \geq \mathbb{E}[v] - \delta.$$

Since $\hat{b} < \mathbb{E}[v]$, the claim follows.

Step 2. For q small enough and k large enough,

$$U\big(\hat{b}|\bar{x}, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k\big) > U\big(b'|\bar{x}, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k\big) \quad \text{for all } b' > \hat{b}.$$

From $U(b'|\bar{x}, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k) = \mathbb{E}^k[v|\bar{x}, \text{sol}] - b'$ and $\overline{\mathbb{E}}[v|\bar{x}, \text{sol}] < \hat{b} < b'$, we have $U(b'|\bar{x}, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k) < 0$.

From Step 1, $U(\hat{b}|\bar{x}, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k) > 0$ for q small enough and k large enough.

Step 3. For q small enough and k large enough,

$$U\big(\hat{b}|\bar{x}-\varepsilon^k, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k\big) > U\big(b'|\bar{x}-\varepsilon^k, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k\big) \quad \text{for all } b' < \hat{b}.$$

Note that

$$\frac{U\big(\hat{b}|\bar{x}-\varepsilon^k, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k\big)}{U\big(b'|\bar{x}-\varepsilon^k, \text{sol}; \beta_{\sigma^k}, \mathbf{n}^k\big)} = \frac{\overset{k}{\Pr}\big(\text{win at } \hat{b}|\bar{x}-\varepsilon^k\big)}{\overset{k}{\Pr}\big(\text{win at } b'|\bar{x}-\varepsilon^k\big)} \frac{\big(\mathbb{E}^k\big[v|\bar{x}-\varepsilon^k, \text{sol,win at } \hat{b}\big] - \hat{b}\big)}{\big(\mathbb{E}^k\big[v|\bar{x}-\varepsilon^k, \text{sol, win at } b'\big] - b'\big)}.$$

Also,

$$\mathbb{E}^k\big[v|\bar{x}-\varepsilon^k, \text{sol, win at } b'\big] - b' \geq v_h,$$

and

$$\lim \mathbb{E}^k\big[v|\bar{x}-\varepsilon^k, \text{sol, win at } \hat{b}\big] - \hat{b} > 0.$$

Therefore, it is sufficient to show that, for every $R > 1$ there is some q small enough such that

$$\lim \frac{\overset{k}{\Pr}\big(\text{win at } \hat{b}|\bar{x}-\varepsilon^k\big)}{\overset{k}{\Pr}\big(\text{win at } b'|\bar{x}-\varepsilon^k\big)} > R.$$

For this, in turn, it is sufficient to show that, for $\omega \in \{\ell, h\}$,

$$\lim \frac{\pi_\omega^k(\hat{b})}{\pi_\omega^k(b')} > R.$$

With $x_-^k = \sigma^k_{m^k(\hat{b})-1}$, we have $\beta^k(x) < \hat{b}$ iff $x \leq x_-^k$. Therefore, $\pi_\omega^k(b') \leq (G_\omega(x_-^k))^{n_\omega^k}$, and we have

$$\frac{\pi_\omega^k(\hat{b})}{\pi_\omega^k(b')} \geq \frac{1}{\big(G_\omega\big(x_-^k\big)\big)^{n_\omega^k}} \frac{1-\big(G_\omega\big(x_-^k\big)\big)^{n_\omega^k}}{n_\omega^k\big[1-G_\omega\big(x_-^k\big)\big]}.$$

If $(G_\omega(x_-^k))^{n_\omega^k} \to q_- \in (0,1)$, then

$$\lim \frac{1}{\big(G_\omega\big(x_-^k\big)\big)^{n_\omega^k}} \frac{1-\big(G_\omega\big(x_-^k\big)\big)^{n_\omega^k}}{n_\omega^k\big[1-G_\omega\big(x_-^k\big)\big]} = \frac{1-q_-}{-q_- \ln q_-}.$$

Now, the claim follows since $q_- \leq q$ and we can choose q small enough such that $\frac{1-q}{-q\ln q} < R$ (recall that $-q_- \ln q_- \to 0$ for $q_- \to 0$).

If $n_\omega^k[1-G_\omega(x_-^k)] \to \infty$, then the claim follows because $(G_\omega(x_-^k))^{n_\omega^k} n_\omega^k[1-G_\omega(x_-^k)]$ is increasing in $G_\omega(x_-^k)$ for $n_\omega^k[1-G_\omega(x_-^k)] \geq 1$, and hence, $(G_\omega(x_-^k))^{n_\omega^k} n_\omega^k[1-G_\omega(x_-^k)] \leq -q\ln q$ for q small enough. (To see it is increasing, write the expression as $\xi^n n[1-\xi]$ and note that $\frac{d}{d\xi}(\xi^n n[1-\xi]) = n\xi^{n-1}n[1-\xi] - \xi^n n = n\xi^{n-1}[n(1-\xi)-\xi] > 0$ for $n(1-\xi) > 1 > \xi$.)

This finishes the proof of Step 3. As noted before, Step 2 and Step 3 imply the claim.

A.2 *Proof of Corollary 2*

Proof. 1. Observe first that $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative in the sense of Lehman (1988)'s criterion. To see this, consider a probability q of a Type I error (rejecting the hypothesis that $\omega = h$ when it is true), and let p_q and p'_q be the thresholds that achieve it, $q = \Phi_h(p_q|\rho, \bar{g}, r) = \Phi_h(p'_q|\rho, \bar{g}', r')$. The corresponding Type II errors satisfy $1 - \Phi_\ell(p_q|\rho, \bar{g}, r) = 1 - q^{1/\bar{g}r} < 1 - q^{1/\bar{g}'r'} = 1 - \Phi_\ell(p'_q|\rho, \bar{g}', r')$, which implies Lehman's ranking. Since in this two-state environment Lehman's ranking is equivalent to Blackwell's ranking (Jewitt (2007)), $\Phi_\omega(\cdot|\rho, \bar{g}, r)$, $\omega = \ell, h$, is more informative by that criterion as well.

2. Consider the following decision problem. A decision maker DM observes the winning bid p and has to select a value estimate $\widehat{v} \in [v_\ell, v_h]$. Its utility function is $u(\widehat{v}, \omega) = -(\widehat{v} - v_\omega)^2$. DM's posterior after observing p is $\Pr[\omega|\text{winning bid} = p]$. The optimal $\widehat{v}$ maximizes

$$U(\widehat{v}) = -\Pr[\ell|\text{ winning bid} = p]\mathbb{E}(\widehat{v} - v_\ell)^2 - \Pr[h|\text{ winning bid} = p]\mathbb{E}(\widehat{v} - v_h)^2,$$

and hence, it is $\Pr[\ell|\text{ winning bid} = p]v_\ell + \Pr[h|\text{ winning bid} = p]v_h = \mathbb{E}[v|\text{ win at } p]$. Since, as we observed before, at the (limit) equilibrium $p = \mathbb{E}[v|\text{ win at } p]$, the optimal $\widehat{v}$ is p itself.

Since $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ is Blackwell more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$, it has to yield higher optimal expected payoff for any payoff function and any prior. In particular, for any ρ,

$$\sum_\omega \rho_\omega \mathbb{E}\big[U(p)|\omega\big] \geq \sum_\omega \rho_\omega \mathbb{E}'\big[U(p)|\omega\big] \tag{45}$$

where $\mathbb{E}$ and $\mathbb{E}'$ are the expectations with respect to $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ and $\Phi_\omega(\cdot|\rho, \bar{g}', r')$, respectively. Now,

$$\begin{aligned}
&\sum_\omega \rho_\omega \mathbb{E}\big[U(p)|\omega\big] \\
&\quad = -\sum_\omega \rho_\omega \mathbb{E}\big[(p - v_\omega)^2|\omega\big] \\
&\quad = -\sum_\omega \rho_\omega \mathbb{E}\big(p^2|\omega\big) + 2(v_h - v_\ell)\rho_h \mathbb{E}(p|h) + 2v_\ell \rho_h \mathbb{E}(p|h) + 2v_\ell \rho_\ell \mathbb{E}(p|\ell) + C \\
&\quad = -\sum_\omega \rho_\omega \mathbb{E}\big(p^2|\omega\big) + 2(v_h - v_\ell)\rho_h \mathbb{E}(p|h) + 2v_\ell \mathbb{E}(v) + C,
\end{aligned}$$

where $C = -\rho_l v_\ell^2 - \rho_h v_h^2$ and we used $\mathbb{E}(v) = \rho_h \mathbb{E}(p|h) + \rho_\ell \mathbb{E}(p|\ell)$ from Claim 1. Analogously,

$$\begin{aligned}
&\sum_\omega \rho_\omega \mathbb{E}'\big[U(p)|\omega\big] \\
&\quad = -\sum_\omega \rho_\omega \mathbb{E}'\big(p^2|\omega\big) + 2(v_h - v_\ell)\rho_h \mathbb{E}'(p|h) + 2v_\ell \mathbb{E}(v) + C.
\end{aligned}$$

Therefore, (45) is equivalent to

$$-\sum_{\omega} \rho_\omega \mathbb{E}\big(p^2|\omega\big) + 2(v_h - v_\ell)\rho_h \mathbb{E}(p|h) \geq -\sum_{\omega} \rho_\omega \mathbb{E}'\big(p^2|\omega\big) + 2(v_h - v_\ell)\rho_h \mathbb{E}'(p|h). \quad (46)$$

Now, since $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ is Blackwell more informative than $\Phi_\omega(\cdot|\rho, \bar{g}', r')$ and so the posteriors are a mean-preserving spread of the latter, and since p^2 is a convex function of the posterior,

$$\sum_{\omega} \rho_\omega \mathbb{E}\big[p^2|\omega\big] \geq \sum_{\omega} \rho_\omega \mathbb{E}'\big[p^2|\omega\big].$$

Therefore, for (46) to hold we must have

$$\mathbb{E}\big[(p|h\big] \geq \mathbb{E}'[p|h].$$

Since $\mathbb{E}[v] = \rho_h \mathbb{E}(p|h) + \rho_\ell \mathbb{E}(p|\ell)$, the reverse inequality holds for $\mathbb{E}(p|\ell)$.

3. Since $\Phi_\omega(\cdot|\rho, \bar{g}, r)$ converges to a mass point on v_ω when $\bar{g}r \to \infty$, the result follows.[20] □

A.3 *Bidding equilibrium with random participation*

A.3.1 *Notation for random participation* Given participation distributions $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$, let

$$\overline{n}_\omega(\eta_\omega) \triangleq \sum_{n=1}^{N} n\eta_\omega(n), \quad \text{and} \quad \overline{\pi}_\omega(b; \beta, \eta_\omega) \triangleq \sum_{n=1}^{N} \eta_\omega(n) n \pi_\omega(b; \beta, n)/\overline{n}_\omega. \quad (47)$$

These are the expected number of bidders and the weighted average probability of winning in state ω. To make the expressions less dense, we omit here and later the argument of $\overline{n}_\omega(\eta_\omega)$ and write just $\overline{n}_\omega$ instead. Also, as before, when there is no danger of confusion, we will continue to omit the argument β and $\boldsymbol{\eta}$ from U, π_ω, E, etc. The counterpart of (15)—the expected payoff to a bidder who bids b given $\boldsymbol{\eta} = (\eta_\ell, \eta_h)$—is

$$U(b|x, \text{sol}) = \frac{\rho_\ell g_\ell(x)\overline{n}_\ell \overline{\pi}_\ell(b)(v_\ell - b) + \rho_h g_h(x)\overline{n}_h \overline{\pi}_h(b)(v_h - b)}{\rho_\ell g_\ell(x)\overline{n}_\ell + \rho_h g_h(x)\overline{n}_h}. \quad (48)$$

Expressions (1)–(2) can also be adapted to mixed strategies, with $\overline{n}_\omega$ and $\overline{\pi}_\omega$ just taking everywhere the place of n_ω and π_ω.

A.3.2 *Proof of monotonicity with random participation*

Proposition 4. *Suppose either $v_\ell = 0$ or η is such that $\eta_\ell(1) = \eta_h(1) = 0$, and β is a bidding equilibrium.*

(i) If $x' > x$, then $U(\beta(x')|x', \text{sol}) \geq U(\beta(x)|x, \text{sol})$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

[20]Recalling that $p \in [v_\ell, v_h]$, and hence, bounded.

(ii) There exists an equivalent bidding equilibrium $\widetilde{\beta}$, such that $\widetilde{\beta}$ is nondecreasing on $[\underline{x}, \bar{x}]$ and coincides with β over intervals over which $\frac{g_h}{g_\ell}$ is strictly increasing.

The proof of Proposition 4 relies on two lemmas.

LEMMA 7 (Single-crossing). *Given any bidding strategy β, any distribution $\boldsymbol{\eta}$ and any bids $b' > b \geq v_\ell$.*

(i) If $\overline{\pi}_\omega(b') > 0$ for some $\omega \in \{\ell, h\}$, then for all $x' > x$,

$$U\big(b'|x, \mathrm{sol}\big) \geq U(b|x, \mathrm{sol}) \quad \Rightarrow \quad U\big(b'|x', \mathrm{sol}\big) \geq U\big(b|x', \mathrm{sol}\big);$$

where the second inequality is strict if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

(ii) If $\overline{\pi}_\omega(b') = 0$ for some $\omega \in \{\ell, h\}$, then $\overline{\pi}_\omega(b) = 0$ for both ω, and $U(b'|x, \mathrm{sol}) = U(b|x, \mathrm{sol}) = 0$ for all x.

REMARK. The proof of Lemma 7 relies on the assumption that there are only two states. If bids are necessarily above v_ℓ (as is indeed implied by the next lemma), conditional on state ℓ, a higher bid is necessarily worse than a lower one. So, if two bids are optimal for some belief, the higher bid must be better if the state is h—implying that a higher belief must make the higher bid more attractive. This is the key role of that assumption.

The following lemma collects a number of additional properties of a bidding equilibrium β. One of them is a straightforward Bertrand property: when there are two or more bids in both states, then $\beta(x) \geq v_\ell$, for all x.

LEMMA 8 (Bertrand and other properties). *Suppose either $v_\ell = 0$ or $\eta_\ell(1) = \eta_h(1) = 0$ and β is a bidding equilibrium.*

(i) $\overline{\pi}_\omega(\beta(x)) > 0$ if $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$.

(ii) $\beta(x) \in [v_\ell, v_h)$ for almost all x.

(iii) $U(\beta(x')|x', \mathrm{sol}) \geq U(\beta(x)|x, \mathrm{sol})$ if $x' > x$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

The proof of the lemma utilizes that the set of feasible bids is dense below v_ℓ. If the price grid is finite below v_ℓ as well, equilibrium may involve bids just below v_ℓ —just like in the usual Bertrand pricing game with price grid—but such equilibria would not add anything important.

PROOF OF LEMMA 7. $b' > b \geq v_\ell$ implies $(v_\ell - b') < (v_\ell - b)$ and $\overline{\pi}_\ell(b') \geq \overline{\pi}_\ell(b)$. These together with the hypothesis $\overline{\pi}_\ell(b') > 0$ and $b' > b \geq v_\ell$ imply

$$\overline{\pi}_\ell\big(b'\big)\big(v_\ell - b'\big) < \overline{\pi}_\ell(b)(v_\ell - b). \tag{49}$$

Hence, $U(b'|x, \text{sol}) \geq U(b|x, \text{sol})$ requires

$$\overline{\pi}_h(b')(v_h - b') > \overline{\pi}_h(b)(v_h - b). \tag{50}$$

Rewriting $U(b'|x, \text{sol})$ yields

$$\frac{\rho_\ell g_\ell(x)\bar{n}_\ell}{\rho_\ell g_\ell(x)\bar{n}_\ell + \rho_h g_h(x)\bar{n}_h}\left[\bar{\pi}_\ell(b)(v_\ell - b) + \frac{\rho_h g_h(x)\bar{n}_h}{\rho_\ell g_\ell(x)\bar{n}_\ell}\bar{\pi}_h(b)(v_h - b)\right]. \tag{51}$$

It follows from $U(b'|x, \text{sol}) \geq U(b|x, \text{sol})$ and (49) that

$$\begin{aligned}&\frac{\rho_h g_h(x)\overline{n}_h}{\rho_\ell g_\ell(x)\overline{n}_\ell}\left[\overline{\pi}_h(b')(v_h - b') - \overline{\pi}_h(b)(v_h - b)\right]\\ &\quad\geq \overline{\pi}_\ell(b)(v_\ell - b) - \overline{\pi}_\ell(b')(v_\ell - b') > 0.\end{aligned}$$

Since $x' > x$ and $\frac{g_h(x)}{g_\ell(x)}$ is nondecreasing,

$$\begin{aligned}&\frac{\rho_h g_h(x')\overline{n}_h}{\rho_\ell g_\ell(x')\overline{n}_\ell}\left[\overline{\pi}_h(b')(v_h - b') - \overline{\pi}_h(b)(v_h - b)\right]\\ &\quad\geq \overline{\pi}_\ell(b)(v_\ell - b) - \overline{\pi}_\ell(b')(v_\ell - b') > 0.\end{aligned} \tag{52}$$

which implies

$$\begin{aligned}&U(b'|x', \text{sol})\\ &= \frac{\rho_\ell g_\ell(x')\overline{n}_\ell}{\rho_\ell g_\ell(x')\overline{n}_\ell + \rho_h g_h(x')\overline{n}_h}\left[\overline{\pi}_\ell(b')(v_\ell - b') + \frac{\rho_h g_h(x')\overline{n}_h}{\rho_\ell g_\ell(x')\overline{n}_\ell}\overline{\pi}_h(b')(v_h - b')\right]\\ &\geq \frac{\rho_\ell g_\ell(x')\overline{n}_\ell}{\rho_\ell g_\ell(x')\overline{n}_\ell + \rho_h g_h(x')\overline{n}_h}\left[\overline{\pi}_\ell(b)(v_\ell - b) + \frac{\rho_h g_h(x')\overline{n}_h}{\rho_\ell g_\ell(x')\overline{n}_\ell}\overline{\pi}_h(b)(v_h - b)\right]\\ &= U(b|x', \text{sol}).\end{aligned} \tag{53}$$

If $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$, then (52) and (53) hold with strict inequalities.

The last part of the lemma is immediate because G_h and G_ℓ are mutually absolutely continuous, so that $G_h(\{x|\beta(x) \leq b\}) = 0 \Leftrightarrow G_\ell(\{x|\beta(x) \leq b\}) = 0$. □

Proof of Lemma 8.

Step 0: If $\pi_\omega(b) > 0$ for some $n \geq 2$ and $\omega = \ell$ or h, then $\overline{\pi}_\omega(b) > 0$ for both ω and any η_ω.

Proof of Step 0. $\pi_\omega(b; \beta, n) > 0$ for some n and ω implies that $G_\omega(\{x|\beta(x) \leq b\}) > 0$. Since G_h and G_ℓ are mutually absolutely continuous, it follows that $G_{\omega'}(\{x|\beta(x) \leq b\}) > 0$ also for $\omega' \neq \omega$. Therefore, $\overline{\pi}_\omega(b) > 0$ for both ω and any η_ω. □

Step 1. $\beta(x) \geq v_\ell$ for almost all x.

Proof of Step 1. This is immediate if $v_\ell = 0$. So, suppose $\eta_\ell(1) = \eta_h(1) = 0$.

Let $\underline{b} \equiv \inf\{b | \pi_\omega(b) > 0$ for some n and $\omega\}$. Suppose $\underline{b} < v_\ell$. It may not be that β has an atom at $\underline{b}$ (i.e., $\int_{\{x:\beta(x)=\underline{b}\}} g_\omega(x)\,dx > 0$) since by a standard Bertrand argument $U(\underline{b} + \varepsilon | x, \text{sol}) > U(\underline{b} | x, \text{sol})$ for sufficiently small $\varepsilon \in (0, v_\ell - \underline{b})$. Therefore, there exists a sequence of x^k such that $\beta(x^k) \to \underline{b}$ and $\overline{\pi}_\omega(\beta(x^k)) \to 0$ (owing to $\eta_\omega(1) = 0$). Hence, equilibrium payoffs $U(\beta(x^k)|x^k, \text{sol}) \to 0$. However, by the definition of $\underline{b}$ and monotonicity of $\bar{\pi}_\omega$, $\overline{\pi}_\omega(b)$ is strictly positive for all $b \in (\underline{b}, v_\ell)$. Thus, for all $b \in (\underline{b}, v_\ell)$, the payoff $U(b|x, \text{sol}) > 0$. This contradicts the optimality of $\beta(x^k)$ for sufficiently large k, a standard Bertrand argument. Thus, $\underline{b} \geq v_\ell$. Finally, $\pi_\omega(b) = 0$ for all $b < v_\ell$ implies that $G_\omega(\{x | \beta(x) \geq v_\ell\}) = 1$, proving the step. □

Step 2. $\beta(x) < v_h$ for all x.

Proof of Step 2. It clearly cannot be that $G_\omega(\{x | \beta(x) > v_h\}) = 1$ for any ω, since this would imply that bidders have strictly negative payoffs in expectations. Suppose that $\beta(x') \geq v_h$ for some x'. From $G_\ell(\{x|\beta(x) > v_h\}) < 1$, $\beta(x') \geq v_h$ implies $\overline{\pi}(\beta(x')) > 0$ and $U(\beta(x')|x', \text{sol}) < 0$, a contradiction to the optimality of $\beta(x')$. □

Step 3. $\overline{\pi}_\omega(\beta(x)) > 0$ for almost all x for $\omega \in \{\ell, h\}$.

Proof of Step 3. Fix $\omega \in \{\ell, h\}$. Let $X = \{x | \overline{\pi}_\omega(\beta(x)) = 0\}$. The probability that in state ω all bidders are from that set is $\Sigma_n \eta_\omega(n)[G_\omega(X)]^n$. Since in that event some bidder has to win, we have $\Sigma_n \eta_\omega(n)[G_\omega(X)]^n \leq \Pr[\{\text{Winning bidder has signal } x \in X\}|\omega] \leq \overline{n}_\omega \int_{x\in X} \overline{\pi}_\omega(\beta(x)) g(x)\,dx = 0$. Hence, $G_\omega(X) = 0$. □

Step 4. For any $x' > x$, $U(\beta(x')|x', \text{sol}) \geq U(\beta(x)|x, \text{sol})$. The inequality is strict if and only if $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$. Thus, $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$ implies that $U(\beta(x')|x', \text{sol})$ is strictly positive.

Proof of Step 4. From (48), it follows (after dividing the numerator and denominator by $g_\ell(x)$) that

$$U(b|x, \text{sol}) = \frac{\rho_\ell \overline{n}_\ell \overline{\pi}_\ell(b)(v_\ell - b) + \rho_h \dfrac{g_h(x)}{g_\ell(x)} \overline{n}_h \overline{\pi}_h(b)(v_h - b)}{\rho_\ell \overline{n}_\ell + \rho_h \dfrac{g_h(x)}{g_\ell(x)} \overline{n}_h}. \tag{54}$$

Therefore, for any $x' > x$,

$$U\big(\beta(x')|x', \text{sol}\big) \geq U\big(\beta(x)|x', \text{sol}\big) \geq U\big(\beta(x)|x, \text{sol}\big) \geq 0, \tag{55}$$

where the first and last inequalities are equilibrium conditions; the second inequality owes to $\frac{g_h(x')}{g_\ell(x')} \geq \frac{g_h(x)}{g_\ell(x)}$ and $\overline{\pi}_h(\beta(x))(v_h - \beta(x)) \geq 0 \geq \overline{\pi}_\ell(\beta(x))(v_\ell - \beta(x))$, which follows from Steps 1 and 2.

Suppose $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$. Now, either $\overline{\pi}_\omega(\beta(x)) > 0$, in which case $\overline{\pi}_h(\beta(x))(v_h - \beta(x)) > 0$, and it follows from (54) and $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$ that the second inequality in (55) is strict, or $\overline{\pi}_\omega(\beta(x)) = 0$, and hence, $U(\beta(x)|x, \text{sol}) = 0$. In the latter case, by Step 3, there is some $y \in (\underline{x}, x')$ such that $\overline{\pi}_\omega(\beta(y)) > 0$. We can choose y such that $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(y)}{g_\ell(y)}$ (recall that $\frac{g_h(\underline{x})}{g_\ell(\underline{x})} = \lim_{x\to\underline{x}} \frac{g_h(x)}{g_\ell(x)}$). By Step 2, $\overline{\pi}_h(\beta(y))(v_h - \beta(y)) > 0$. Since $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(y)}{g_\ell(y)}$, it follows from (54) and the fact that β is a bidding equilibrium that

$$U\big(\beta(x')|x', \text{sol}\big) \geq U\big(\beta(y)|x', \text{sol}\big) > U\big(\beta(y)|y, \text{sol}\big) \geq 0 = U\big(\beta(x)|x, \text{sol}\big).$$

Conversely, $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x)}{g_\ell(x)}$ implies

$$U\big(\beta(x')|x', \text{sol}\big) = U\big(\beta(x')|x, \text{sol}\big) \leq U\big(\beta(x)|x, \text{sol}\big) = U\big(\beta(x)|x', \text{sol}\big) \leq U\big(\beta(x')|x', \text{sol}\big),$$

where the inequalities are equilibrium conditions while the equalities owe to the fact that x and x' contain the same information. Therefore, $U(\beta(x')|x', \text{sol}) = U(\beta(x)|x, \text{sol})$. □

Step 5. The strict positivity of $U(\beta(x)|x, \text{sol})$ implies immediately that $\overline{\pi}_\omega(\beta(x)) > 0$ for any x for which $\frac{g_h(x)}{g_\ell(x)} > \frac{g_h(\underline{x})}{g_\ell(\underline{x})}$. (Step 3 established this only for almost all x). This proves Part 1 of the lemma.

This completes the proof of the lemma: Part 1 of the lemma is established in Step 5. Part 2 is established in Steps 1 and 2. Part 3 is established in Step 4. □

Proof of Proposition 4. Part 1: Proved by Lemma 8.

Part 2: Suppose that $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Since β is a bidding equilibrium, $U(\beta(x)|x, \text{sol}) \geq U(\beta(x')|x, \text{sol})$. By Lemma 8, $\overline{\pi}_\omega(\beta(x')) > 0$ and $\beta(x') \geq v_\ell$. Therefore, by Lemma 7, $U(\beta(x)|x', \text{sol}) > U(\beta(x')|x', \text{sol})$, contradicting the optimality of $\beta(x')$ for x'. Thus, the supposition $\beta(x') < \beta(x)$ is false. Hence, $\beta(x') \geq \beta(x)$ whenever $\frac{g_h(x')}{g_\ell(x')} > \frac{g_h(x)}{g_\ell(x)}$.

Next, suppose that $\frac{g_h(x')}{g_\ell(x')} = \frac{g_h(x)}{g_\ell(x)}$ for some $x, x' \in (\underline{x}, \bar{x}]$, but $\beta(x') < \beta(x)$. Then there is some interval containing x and x' over which $\frac{g_h(x)}{g_\ell(x)}$ is constant, say, C. Let $[x_-, x_+]$ be the closure of this interval. By the above argument, $\beta(x'') \leq \beta(x)$ whenever $x'' < x_- < x$ and $\beta(x) \leq \beta(x''')$ whenever $x < x_+ < x'''$. Define $\widetilde{\beta}_1(x)$ by

$$\widetilde{\beta}_1(x) = \inf\big\{b : G_h(x) \leq G_h\big(\{t|\beta(t) \leq b\}\big)\big\} \quad \text{if } x \in [x_-, x_+]$$

Thus, on $[x_-, x_+]$ the signals are essentially “reordered” to make $\widetilde{\beta}_1(x)$ monotone. Outside $[x_-, x_+]$, $\widetilde{\beta}_1(x)$ coincides with $\beta(x)$. Note that $\tilde{\beta}(x') \leq \tilde{\beta}(x) \leq \tilde{\beta}(x'')$ for all $x' < x_-$ and $x_+ < x''$. With this definition,

$$G_h\big(\{x|\widetilde{\beta}_1(x) \leq b\}\big) = G_h\big(\{x|\beta(x) \leq b\}\big),$$

for all b. That is, the distribution of bids induced by $\widetilde{\beta}_1$ is equal to the distribution of bids induced by β in state h. It is also the same in state ℓ because $\widetilde{\beta}_1 = \beta$ outside $[x_-, x_+]$ and because the distributions G_ℓ and G_h conditional on $x \in (x_-, x_+)$ are identical (owing to the constant $\frac{g_h(x)}{g_\ell(x)}$).

The equality of the distributions of bids under $\widetilde{\beta}_1$ and β implies that, for any $x \notin \{x_-, x_+\}$, $\tilde{\beta}_1(x)$ is optimal: for $x \notin [x_-, x_+]$ this follows immediately from $\widetilde{\beta}_1(x) = \beta(x)$;

for $x \in (x_-, x_+)$ this follows from $\tilde{\beta}_1(x) = \beta(y)$ where y is some value of the signal such that $\frac{g_h(y)}{g_\ell(y)} = \frac{g_h(x)}{g_\ell(x)}$. For $x \in \{x_-, x_+\}$, note that we can represent the distribution of signals by an equivalent pair of densities that is equal to the original densities almost everywhere, so that the resulting equilibrium still corresponds to the same distributional strategy. Here, $\tilde{\beta}_1$ can be rationalized at $\{x_-, x_+\}$ by changing the densities at the points $x \in \{x_-, x_+\}$. At x_-, if $\tilde{\beta}_1(x_-) = \tilde{\beta}_1(x_- + \varepsilon)$ for some ε (an atom), $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_\omega(x_-) = \lim_{\varepsilon\to 0} g_\omega(x_- + \varepsilon)$. Otherwise, $\tilde{\beta}_1(x_-)$ is rationalized by setting $g_\omega(x_-) = \lim_{\varepsilon\to 0} g_\omega(x_- - \varepsilon)$, similarly for x_+. It follows that $\tilde{\beta}_1$ is monotone on $[x_-, x_+]$ and that it is equivalent to β.

Repeating this construction for all intervals over which $\frac{g_h(x)}{g_\ell(x)}$ is constant, we get a sequence of bidding strategies (constructing the sequence by starting with the longest interval of signals on which $\frac{g_h(x)}{g_\ell(x)}$ is constant). Let $\tilde{\beta}$ be the pointwise limit of this sequence on $(\underline{x}, \bar{x}]$ and let $\tilde{\beta}(\underline{x}) = \lim_{\varepsilon\to 0} \beta(\underline{x} + \varepsilon)$. Then $\tilde{\beta}$ is an equivalent bidding equilibrium that is monotone on $[\underline{x}, \bar{x}]$, as claimed. □

A.3.3 *Proof of Proposition 3 for random participation* The following lemma shows that, for the purposes of this proof, $\boldsymbol{\eta}^k$ may be replaced by $\mathbf{n}^k$ without loss of generality. Once this is established, the proof of Theorem 1 applies and need not be repeated. Recall $\overline{n}_\omega(\eta_\omega)$ and $\overline{\pi}_\omega(b; \beta, \eta_\omega)$, $\omega = \ell, h$, from (47). Since we deal here explicitly with $\boldsymbol{\eta}$ and $\mathbf{n}$, we do not suppress them in the arguments of π and $\mathbb{E}[v|\ldots]$.

Lemma 9. *Consider a sequence of bidding games $\Gamma_0(N^k, \boldsymbol{\eta}^k, \Delta^k)$ such that the support of η^k_ω is contained in $\{n^k_\omega, \ldots, n^k_\omega + m\}$ for some fixed integer $m > 0$ and $\Delta^k \to 0$, $\min\{n^k_\ell, n^k_h\} \to \infty$ and $\lim_{k\to\infty} \frac{n^k_h}{n^k_\ell} = r$, and a corresponding sequence of bidding equilibria β^k.*

(i)

$$\lim \frac{\overline{n}^k_h}{\overline{n}^k_\ell} = \lim \frac{n^k_h}{n^k_\ell}, \tag{56}$$

(ii) For any (b^k) with $\lim (G_\omega(x_+(b^k)))^{n^k_\omega} > 0$,

$$\lim \frac{\overline{\pi}_h(b^k; \beta^k, \eta^k_h)}{\overline{\pi}_\ell(b^k; \beta^k, \eta^k_\ell)} = \lim \frac{\pi_h(b^k; \beta^k, n^k_h)}{\pi_\ell(b^k; \beta^k, n^k_\ell)}.$$

(iii) For any (b^k) with $\lim (G_\omega(x_+(b^k)))^{n^k_\omega} > 0$,

$$\lim \mathbb{E}[v|x^k, \text{sol, win } \textit{at } b^k; \beta^k, \boldsymbol{\eta}^k] = \lim \mathbb{E}[v|x^k, \text{sol, win } \textit{at } b^k; \beta^k, \mathbf{n}^k].$$

Remark. The condition $\lim (G_\omega(x_+(b^k)))^{n^k_\omega} > 0$ is needed for part (ii). For any fixed $x < \bar{x}$, if β^k is strictly increasing, it follows from $\pi_\omega(\beta^k(x); \beta^k, n^k_\omega) = (G_\omega(x))^{n^k_\omega - 1}$ that

$$\frac{\pi_h(\beta^k(x); \beta^k, n^k_h + 1)}{\pi_\ell(\beta^k(x); \beta^k, n^k_\ell)} = G_h(x) \frac{\pi_h(\beta^k(x); \beta^k, n^k_h)}{\pi_\ell(\beta^k(x); \beta^k, n^k_\ell)} < \frac{\pi_h(\beta^k(x); \beta^k, n^k_h)}{\pi_\ell(\beta^k(x); \beta^k, n^k_\ell)}.$$

Therefore, since $G_h(x) < 1$, the difference between these ratios is not vanishing as would be required for the result of the lemma to hold for this x. However, when $\lim(G_\omega(x_+(b^k)))^{n_\omega^k} > 0$, then $x_+(b^k) \to \bar{x}$, and hence, $G_\omega(x_+(b^k)) \to 1$. Fortunately, bids for which $\lim(G_\omega(x_+(b^k)))^{n_\omega^k} = 0$ can be neglected in the characterization proof (the winning bid is strictly higher than b^k with probability 1).

Proof of Lemma 9. Part (i) is immediate. Part (iii) follows from Parts (i) and (ii). So, we show Part (ii). For this, it is sufficient to show (shifting the counting integer by 1 to simplify the expressions below)

$$\lim \frac{\pi_\omega\left(b^k; \beta^k, n_\omega^k + 1\right)}{\pi_\omega\left(b^k; \beta^k, n_\omega^k + m + 1\right)} = 1.$$

From Lemma 1,

$$\frac{\pi_\omega\left(b^k; \beta^k, n_\omega^k + 1\right)}{\pi_\omega\left(b^k; \beta^k, n_\omega^k + m + 1\right)} = \frac{\displaystyle\int_{x_-^k}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k} g_\omega(x)\, dx}{\displaystyle\int_{x_-^k}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k + m} g_\omega(x)\, dx}.$$

The claim is now immediate if $x_-^k \to \bar{x}$ since

$$\frac{1}{G_\omega\left(x_+^k\right)^m} \le \frac{\displaystyle\int_{x_-^k}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k} g_\omega(x)\, dx}{\displaystyle\int_{x_-^k}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k + m} g_\omega(x)\, dx} \le \frac{1}{G_\omega\left(x_-^k\right)^m}, \tag{57}$$

and $G_\omega(x_+^k) \to 1$. Otherwise, we can choose some $\varepsilon > 0$ with $x_-^k < \bar{x} - \varepsilon$ for all k. Observe that

$$\lim \frac{\displaystyle\int_{\bar{x}-\varepsilon}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k} g_\omega(x)\, dx}{\displaystyle\int_{x_-^k}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k} g_\omega(x)\, dx} = 1.$$

The claim now follows using the previous bounds (57) because

$$\lim \frac{\pi_\omega\left(b^k; \beta^k, n_\omega^k + 1\right)}{\pi_\omega\left(b^k; \beta^k, n_\omega^k + m + 1\right)} = \lim \frac{\displaystyle\int_{\bar{x}-\varepsilon}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k} g_\omega(x)\, dx}{\displaystyle\int_{\bar{x}-\varepsilon}^{x_+^k} \left(G_\omega(x)\right)^{n_\omega^k + m} g_\omega(x)\, dx},$$

and because we can choose ε arbitrarily small such that $G_\omega(\bar{x} - \varepsilon) \cong 1$. □

Given Lemma 9, the proof of Proposition 3 is identical to the proof of Theorem 1.

References

Atakan, Alp and Mehmet Ekmekci (2020), "Market selection and information content of prices." Econometrica. (forthcoming). [844, 857]

Athey, Susan (2001), "Single crossing properties and the existence of pure strategy equilibria in games of incomplete information." *Econometrica*, 69, 861–889. [855, 869, 870]

Broecker, Thorsten (1990), "Credit-worthiness tests and interbank competition." *Econometrica*, 58, 429–452. [844]

Ekmekci, Mehmet and Stephan Lauermann (2020), "Manipulated electorates and information aggregation." *The Review of Economic Studies*, 87, 997–1033. [844]

Ekmekci, Mehmet and Stephan Lauermann (2021), "Information aggregation in Poisson-elections." Theoretical Economics. (forthcoming). [844]

Feddersen, Tom and Wolfgang Pesendorfer (1997), "Voting behavior and information aggregation in elections with private information." *Econometrica*, 65, 1029–1058. [844]

Harstad, Ron, Aleksandar Pekeč, and Ilia Tsetlin (2008), "Information aggregation in auctions with an unknown number of bidders." *Games and Economic Behavior*, 62, 476–508. [856]

Jackson, Matthew O., Leo K. Simon, Jeroen M. Swinkels, and William R. Zame (2002), "Communication and equilibrium in discontinuous games of incomplete information." *Econometrica*, 70, 1711–1740. [855]

Jewitt, Ian (2007), "Information order in decision and agency problems." Oxford. Report. [872]

Lauermann, Stephan and Andre Speit (2020), "Bidding in a common-value auctions with an uncertain number of competitors." Germany. CRC TR 224 Discussion Paper Series, available at IDEAS, University of Bonn and University of Mannheim. [855, 856]

Lauermann, Stephan and Asher Wolinsky (2016), "Search with adverse selection." *Econometrica*, 84, 243–315. [844]

Lauermann, Stephan and Asher Wolinsky (2017), "Bidder solicitation, adverse selection, and the failure of competition." *American Economic Review*, 107, 1399–1429. [842, 843, 856, 858, 859]

Lauermann, Stephan and Asher Wolinsky (2021), "Bidder solicitation in common-value auctions." Bonn. In preparation. [843, 856]

Lehman, Erich L. (1988), "Comparing location experiments." *Annals of Statistics*, 16, 521–533. [872]

Milgrom, Paul (1979), "A convergence theorem for competitive bidding with differential information." *Econometrica*, 47, 679–688. [843, 854]

Murto, Pauli and Juuso Välimäki (2015), "Large common value auctions with risk averse bidders." *Games and Economic Behavior*, 91, 60–74. [850]

Murto, Pauli and Juuso Välimäki (2019), "Informal affiliated auctions with costly entry." Helsinki. Report. [844, 847, 856, 857]

Pesendorfer, Wolfgang and Jeroen M. Swinkels (1997), "The loser's curse and information aggregation in common value auctions." *Econometrica*, 65, 1247–1281. [856]

Riordan, Michael (1993), "Competition and bank performance: A theoretical perspective." In *Capital Markets and Financial Intermediation* (Colin Mayer and Xavier Vives, eds.), 328–343, Cambridge University Press. [844]

Wilson, Robert (1977), "A bidding model of perfect competition." *Review of Economic Studies*, 44, 511–518. [843]

Co-editor Simon Board handled this manuscript.

Manuscript received 22 April, 2019; final version accepted 28 June, 2021; available online 12 July, 2021.