Theoretical Economics 17 (2022), 929–942 1555-7561/20220929

Correction to "Incentive-compatible voting rules with positively correlated beliefs"

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Theorem 1 in Bhargava, Majumdar, and Sen (2015) provides a necessary condition for a social choice function to be locally robust ordinal Bayesian incentive compatible with respect to a belief system satisfying top-set correlation. In this paper, we provide a counterexample to that theorem and consequently provide a new necessary condition for the same in terms of sequential ordinal nondomination.

Keywords. Ordinal Bayesian incentive compatibility, correlated beliefs, sequential ordinal nondomination property..

JEL classification. D71, D82.

1. Introduction

A social choice function (SCF) selects an alternative at every collection of preferences of the agents in a society. An SCF is called ordinal Bayesian incentive compatible (OBIC) with respect to a belief if, by misreporting his sincere preference, no agent can increase his expected utility according to his belief for any utility function representing his sincere preference. An SCF is called locally robust OBIC (LOBIC) with respect to a belief if it is OBIC with respect to all beliefs lying in a small neighborhood of the original belief. LOBIC ensures that agents are incentivized to reveal their sincere preferences even if the designer is *slightly* unsure about their beliefs.

Theorem 1 in Bhargava, Majumdar, and Sen (2015) says that a unanimous SCF is LOBIC with respect to a top-set- (TS-) correlated belief if and only if it satisfies a property called ordinal nondomination (OND). We show that the "only if" part of this theorem is not correct, and consequently we provide a necessary condition for LOBIC in terms of sequential ordinal nondomination (sequential OND).[1]

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We are grateful to Arunava Sen for suggesting the problem to us and for helpful discussions on the paper.

[1]In the proof of Theorem 1 in Bhargava, Majumdar, and Sen (2015), the authors consider two cases. However, to our understanding, there is a third case that the authors have missed.

 https://doi.org/10.3982/TE4495

It is worth emphasizing that we do not put any restriction on the beliefs of the agents.[2] In particular, beliefs are not required to be independent or even common.[3]

2. A counterexample to Theorem 1 in Bhargava, Majumdar, and Sen (2015)

Theorem 1 in Bhargava, Majumdar, and Sen (2015) says that every TS-LOBIC SCF satisfies OND. Later, they remark that the statement holds for every LOBIC SCF. In what follows, we provide a counterexample to this statement. We consider beliefs satisfying TS correlation to clarify the fact that the result does not hold for TS-LOBIC SCFs also. Before proceeding to the counterexample, let us recall the following definitions from Bhargava, Majumdar, and Sen (2015).

Let A be a finite set of alternatives and let $N = \{1, \dots, n\}$ be a set of n agents. We denote by $\mathbb{P}$ the set of all (strict) preferences on A. An SCF is a mapping $f : \mathbb{P}^n \to A$. A belief μ_i for agent i is a probability distribution on the set $\mathbb{P}^n$, i.e., it is a map $\mu_i : \mathbb{P}^n \to [0, 1]$ such that $\sum_{P \in \mathbb{P}^n} \mu_i(P) = 1$. The utility function $u : A \to \mathbb{R}$ represents $P_i \in \mathbb{P}$ if and only if for all $a, b \in A$, we have $aP_ib \iff u(a) > u(b)$.

Definition 2.1. A belief for an agent i, μ_i, is TS correlated if for all $P_i \in \mathbb{P}$, all $k = 1, \dots, m-1$, and all $D \subset A$ such that $D \neq B_k(P_i)$ [4] and $|D| = k$, we have

$$\sum_{P_{-i} | B_k(P_j) = B_k(P_i)\ \forall\ j \neq i} \mu_i(P_{-i}|P_i) > \sum_{P_{-i} | B_k(P_j) = D\ \forall\ j \neq i} \mu_i(P_{-i}|P_i). \tag{1}$$

Definition 2.2. An SCF $f : \mathbb{P}^n \to A$ is OBIC with respect to belief system $\mu_N = (\mu_1, \dots, \mu_n)$ if for all $i \in N$, for all integers $k = 1, \dots, m$, and for all $P_i, P_i' \in \mathbb{P}$, we have

$$\sum_{P_{-i} | f(P_i, P_{-i}) \in B_k(P_i)} \mu_i(P_{-i} \mid P_i) \geq \sum_{P_{-i} | f(P_i', P_{-i}) \in B_k(P_i)} \mu_i(P_{-i} \mid P_i). \tag{2}$$

Definition 2.3. An SCF $f : \mathbb{P}^n \to A$ is LOBIC with respect to the belief system μ_N if there exists $\epsilon > 0$ such that f is OBIC with respect to all μ_N' such that $\mu_N' \in B_\epsilon(\mu_N)$. [5]

Definition 2.4. An SCF $f : \mathbb{P}^n \to A$ is TS-LOBIC with respect to a belief system μ_N if

(i) μ_i satisfy TS correlation for all $i \in N$

(ii) f is LOBIC with respect to μ_N.

Definition 2.5. An SCF $f : \mathbb{P}^n \to A$ satisfies OND if for all $i \in N$, for all $P_i, P_i' \in \mathbb{P}$, and all $P_{-i} \in \mathbb{P}^{n-1}$ such that $f(P_i', P_{-i}) P_i f(P_i, P_{-i})$, there exists $P_{-i}' \in \mathbb{P}^{n-1}$ such that the following statements hold:

[2] Although the statement of Theorem 1 in Bhargava, Majumdar, and Sen (2015) involves beliefs that satisfy TS correlation, they have remarked that the "only if" part of the theorem is more general as it holds for arbitrary beliefs. Since we deal with the "only if" part of this theorem, we present our result for arbitrary beliefs.

[3] See Majumdar and Sen (2004) and Mishra (2016) for details on LOBIC SCFs under independent priors.

[4] Recall that $B_k(P_i)$ denotes top k alternatives in the ordering P_i.

[5] The function $B_\epsilon(\mu_i)$ denotes the open ball of radius ϵ centered at μ_i.

TABLE 1. Example of an SCF that does not satisfy OND but satisfies LOBIC.

1 \ 2	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*c*	*a*	*b*	*b*
acb	*a*	*a*	*a*	*b*	*c*	*a*
bac	*b*	*a*	*b*	*b*	*a*	*c*
bca	*c*	*b*	*b*	*b*	*a*	*c*
cab	*a*	*a*	*b*	*c*	*c*	*c*
cba	*a*	*c*	*b*	*a*	*c*	*c*

(i) Either $f(P_i, P'_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P'_{-i}) P_i f(P'_i, P_{-i})$.

(ii) Either $f(P_i, P_{-i}) = f(P'_i, P'_{-i})$ or $f(P_i, P_{-i}) P_i f(P'_i, P'_{-i})$.

EXAMPLE 2.1. Suppose that there are two agents $\{1, 2\}$ and three alternatives $\{a, b, c\}$. Consider the domain $\mathbb{P}$ containing the set of all preferences over $\{a, b, c\}$. We denote by *abc* the preference where a, b, and c are the top-ranked, second-ranked, and third-ranked alternatives, respectively. In Table 1, we present an SCF, say $\hat{f}$, and in Tables 2 and 3, we present the conditional beliefs μ_1 and μ_2 of agent 1 and agent 2, respectively. These tables are self-explanatory.

We claim the following facts about the SCF $\hat{f}$ and the prior beliefs μ_1 and μ_2. Claims 2.1 and 2.2 establish that the SCF $\hat{f}$ is TS-LOBIC with respect to (μ_1, μ_2), while Claim 2.3 says that $\hat{f}$ does not satisfy the OND property. This contradicts Theorem 1 in Bhargava, Majumdar, and Sen (2015). ◊

CLAIM 2.1. *The conditional beliefs μ_1 and μ_2 satisfy TS correlation.*

PROOF. Recall that a belief system μ_N is TS correlated if for all $i \in N$, all $P_i \in \mathbb{P}$, all $k = 1, \dots m-1$, and all $D \subset A$ such that $D \neq B_k(P_i)$ and $|D| = k$, we have

$$\sum_{P_{-i} | B_k(P_j) = B_k(P_i)\ \forall\ j \neq i} \mu_i(P_{-i}|P_i) > \sum_{P_{-i} | B_k(P_j) = D\ \forall\ j \neq i} \mu_i(P_{-i}|P_i). \tag{3}$$

Observe from Tables 2 and 3 that for all $i = \{1, 2\}$ and all $P_i \in \mathbb{P}$, $P_{-i} = P_i$ implies $\mu_i(P_{-i}|P_i) = 0.51$. Since preferences P_{-i} such that $P_{-i} = P_i$ will always appear only in

TABLE 2. Conditional belief of agent 1.

1 \ 2	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	0.51	0.02	0.04	0.17	0.20	0.06
acb	0.02	0.51	0.20	0.01	0.09	0.17
bac	0.17	0.15	0.51	0.02	0.14	0.01
bca	0.15	0.17	0.02	0.51	0.01	0.14
cab	0.15	0.14	0.01	0.17	0.51	0.02
cba	0.01	0.21	0.23	0.01	0.03	0.51

TABLE 3. Conditional belief of agent 2.

$1 \setminus 2$	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	0.51	0.02	0.01	0.01	0.01	0.09
acb	0.02	0.51	0.09	0.25	0.25	0.01
bac	0.09	0.17	0.51	0.02	0.15	0.17
bca	0.01	0.01	0.02	0.51	0.06	0.20
cab	0.17	0.20	0.17	0.20	0.51	0.02
cba	0.20	0.09	0.20	0.01	0.02	0.51

the left hand side of inequality (3) and the corresponding probability is more than 0.5, it follows that inequality (3) will always be satisfied by the belief system (μ_1, μ_2). This shows that (μ_1, μ_2) satisfy TS correlation. □

CLAIM 2.2. *The SCF $\hat{f}$ is TS-LOBIC with respect to $\mu_N = (\mu_1, \mu_2)$.*

The proof of this claim is relegated to Appendix A.

CLAIM 2.3. *The SCF $\hat{f}$ does not satisfy the OND property.*

PROOF. Consider $P_1 = abc$, $P_1' = acb$, and $P_2 = bac$. We have $\hat{f}(P_1', P_2)P_1\hat{f}(P_1, P_2)$. However, there is no P_2' such that $\hat{f}(P_1, P_2)P_1\hat{f}(P_1', P_2')$ or $\hat{f}(P_1, P_2) = \hat{f}(P_1', P_2')$ and $\hat{f}(P_1, P_2')P_1\hat{f}(P_1', P_2)$ or $\hat{f}(P_1, P_2') = \hat{f}(P_1', P_2)$. Therefore, $\hat{f}$ does not satisfy the OND property. □

This completes the verification that Example 2.1 is indeed a counterexample to Theorem 1 in Bhargava, Majumdar, and Sen (2015).

3. A NECESSARY CONDITION FOR LOBIC WITH RESPECT TO A (ANY) CORRELATED BELIEF

In this section, we provide a necessary condition for an SCF to be LOBIC with respect to an arbitrary correlated belief. Our necessary condition uses the notion of sequential OND. In contrast to the OND property where a gain of agent i by manipulation can be paid back at *exactly one* preference profile P'_{-i}, in case of sequential OND the same can happen through a sequence of preference profiles $(P^1_{-i}, \dots, P^k_{-i})$ for some $k \geq 1$. Note that OND is a special case of sequential OND where the length of the sequence is 1. We use the following notation to facilitate the formal definition of sequential OND: for a preference P, we use the notation R to denote the weak part of it, that is, aRb implies either aPb or $a = b$.

DEFINITION 3.1. For an SCF $f : \mathbb{P}^n \to A$ and a pair of distinct preferences (P_i, P_i') in $\mathbb{P}$, a sequence $(P^1_{-i}, \dots, P^k_{-i})$ of elements of $\mathbb{P}^{n-1}$ is called an *OND sequence* for f with respect to (P_i, P_i') if for all $l = 1, \dots, k-1$, we have $f(P_i', P^l_{-i})P_i f(P_i', P^{l+1}_{-i})$ and $f(P_i, P^{l+1}_{-i})R_i f(P_i', P^l_{-i})$.

Definition 3.2. An SCF $f : \mathbb{P}^n \to A$ satisfies *sequential OND* property if for all $i \in N$, all $P_i, P_i' \in \mathbb{P}$, and all $P_{-i} \in \mathbb{P}^{n-1}$ with $f(P_i', P_{-i})P_i f(P_i, P_{-i})$, there exists an OND sequence $(P_{-i}^1, \dots, P_{-i}^k)$ for f with respect to (P_i, P_i') such that $f(P_i, P_{-i})R_i f(P_i', P_{-i}^k)$, $f(P_i, P_{-i}^1)R_i f(P_i', P_{-i})$, and $f(P_i', P_{-i})P_i f(P_i', P_{-i}^1)$.

In what follows, we argue that the SCF $\hat{f}$ in Table 1 satisfies the sequential OND property.

Claim 3.1. *The SCF $\hat{f}$ satisfies the OND property (and, hence, the sequential OND property) for all situations except the one where $P_1 = abc$, $P_1' = acb$, and $P_2 = bac$.*

The proof of this claim is relegated to Appendix B.

For the case where $P_1 = abc$, $P_1' = acb$, and $P_2 = bac$, consider the sequence $(P_2^1 = bca, P_2^2 = cab)$ of preferences of agent 2. Note that (i) $\hat{f}(P_1, P_2^1)R_1\hat{f}(P_1', P_2)$, (ii) $\hat{f}(P_1, P_2)R_1\hat{f}(P_1', P_2^2)$, (iii) $\hat{f}(P_1, P_2^2)R_1\hat{f}(P_1', P_2^1)$, and (iv) $\hat{f}(P_1', P_2)P_1\hat{f}(P_1', P_2^1) \times P_1\hat{f}(P_1', P_2^2)$. Thus, $\hat{f}$ satisfies the sequential OND property.

Theorem 3.1. *An SCF is LOBIC with respect to some belief system only if it satisfies the sequential OND property.*

Proof. Suppose an SCF $f : \mathbb{P}^n \to A$ is LOBIC with respect to some belief system μ_N. Since f is LOBIC, we assume that $\mu_i(P_{-i} \mid P_i) > 0$ for all $P_i \in \mathbb{P}$, all $P_{-i} \in \mathbb{P}^{n-1}$, and all $i \in N$. We show that f satisfies the sequential OND property, that is, for all $i \in N$, all $P_i, P_i' \in \mathbb{P}$, and all $P_{-i} \in \mathbb{P}^{n-1}$ with $f(P_i', P_{-i})P_i f(P_i, P_{-i})$, there exists an OND sequence $(P_{-i}^1, \dots, P_{-i}^k)$ for f with respect to (P_i, P_i') such that $f(P_i, P_{-i})R_i f(P_i', P_{-i}^k)$, $f(P_i, P_{-i}^1)R_i f(P_i', P_{-i})$, and $f(P_i', P_{-i})P_i f(P_i', P_{-i}^1)$.

Since f is LOBIC, for all agents $i \in N$, all preferences P_i of agent i, all misreported preferences P_i', and all $l = 1, \dots, m$, we have

$$\sum_{P_{-i} \mid f(P_i, P_{-i}) \in B_l(P_i)} \mu(P_{-i} \mid P_i) \geq \sum_{P_{-i} \mid f(P_i', P_{-i}) \in B_l(P_i)} \mu(P_{-i} \mid P_i). \tag{4}$$

Consider an agent $i \in N$, two preferences $\bar{P}_i, \bar{P}_i' \in \mathbb{P}$, and a preference profile $\bar{P}_{-i} \in \mathbb{P}^{n-1}$ of the other agents such that $f(\bar{P}_i', \bar{P}_{-i})\bar{P}_i f(\bar{P}_i, \bar{P}_{-i})$. If there does not exist any such instance, then f satisfies sequential OND vacuously. Let $f(\bar{P}_i, \bar{P}_{-i}) = a$ and $f(\bar{P}_i', \bar{P}_{-i}) = b$. We proceed to show that there is an OND sequence $(P_{-i}^1, \dots, P_{-i}^k)$ for f with respect to $(\bar{P}_i, \bar{P}_i')$ such that $f(\bar{P}_i, \bar{P}_{-i})\bar{R}_i f(\bar{P}_i', P_{-i}^k)$, $f(\bar{P}_i, P_{-i}^1)\bar{R}_i f(\bar{P}_i', \bar{P}_{-i})$, and $f(\bar{P}_i', \bar{P}_{-i})\bar{P}_i f(\bar{P}_i', P_{-i}^1)$.

Consider the upper contour set $B(b, \bar{P}_i)$ of b at $\bar{P}_i$. Because $b\bar{P}_i a$, we have $a \notin B(b, \bar{P}_i)$. Applying (4) to the upper contour set $B(b, \bar{P}_i)$, we have

$$\sum_{P_{-i} \mid f(\bar{P}_i, P_{-i}) \in B(b, \bar{P}_i)} \mu(P_{-i} \mid \bar{P}_i) \geq \sum_{P_{-i} \mid f(\bar{P}_i', P_{-i}) \in B(b, \bar{P}_i)} \mu(P_{-i} \mid \bar{P}_i). \tag{5}$$

Because $f(\bar{P}_i, \bar{P}_{-i}) = a$, $a \notin B(b, \bar{P}_i)$, and $\mu_i(\bar{P}_{-i} \mid \bar{P}_i) > 0$, by (5) there must exist $\hat{P}_{-i}$ such that $f(\bar{P}_i, \hat{P}_{-i}) \in B(b, \bar{P}_i)$ and $f(\bar{P}'_i, \hat{P}_{-i}) \notin B(b, \bar{P}_i)$. Let us denote this $\hat{P}_{-i}$ by P^1_{-i}.

Since $f(\bar{P}_i, P^1_{-i})\bar{R}_i f(\bar{P}'_i, \bar{P}_{-i})$ and $f(\bar{P}'_i, \bar{P}_{-i})\bar{P}_i f(\bar{P}'_i, P^1_{-i})$, if $f(\bar{P}_i, \bar{P}_{-i})\bar{R}_i f(\bar{P}_i{}', P^1_{-i})$, then the sequence (P^1_{-i}) is an OND sequence for f with respect to $(\bar{P}_i, \bar{P}'_i)$ satisfying the requirement of Definition 3.2 for the current instance. Suppose instead $f(\bar{P}_i{}', P^1_{-i})\bar{P}_i f(\bar{P}_i, \bar{P}_{-i})$. Let $f(\bar{P}_i{}', P^1_{-i}) = c$.

Consider the upper contour set $B(c, \bar{P}_i)$. Applying (4) to $B(c, \bar{P}_i)$, we have

$$\sum_{P_{-i} \mid f(\bar{P}_i, P_{-i}) \in B(c, \bar{P}_i)} \mu(P_{-i} \mid \bar{P}_i) \geq \sum_{P_{-i} \mid f(\bar{P}'_i, P_{-i}) \in B(c, \bar{P}_i)} \mu(P_{-i} \mid \bar{P}_i). \tag{6}$$

Because $f(\bar{P}_i{}', P^1_{-i})\bar{P}_i f(\bar{P}_i, \bar{P}_{-i})$, we have that $f(\bar{P}_i, \bar{P}_{-i}) \notin B(c, \bar{P}_i)$. Hence, by (6) there must exist P^*_{-i} such that $f(\bar{P}_i, P^*_{-i}) \in B(c, \bar{P}_i)$ and $f(\bar{P}'_i, P^*_{-i}) \notin B(c, \bar{P}_i)$. As before, let us denote that P^*_{-i} by P^2_{-i}. By the definition of P^2_{-i}, we have $f(\bar{P}'_i, P^2_{-i}) \notin B(c, \bar{P}_i)$. This, together with the fact that $f(\bar{P}'_i, P^1_{-i}) = c$, implies $f(\bar{P}'_i, P^1_{-i})\bar{P}_i f(\bar{P}'_i, P^2_{-i})$, and, hence, $P^1_{-i} \neq P^2_{-i}$. Since $f(\bar{P}_i, P^1_{-i})\bar{R}_i f(\bar{P}'_i, \bar{P}_{-i})$ and $f(\bar{P}'_i, \bar{P}_{-i})\bar{P}_i f(\bar{P}'_i, P^1_{-i})$, if $f(\bar{P}_i, \bar{P}_{-i})\bar{R}_i f(\bar{P}_i{}', P^2_{-i})$, then (P^1_{-i}, P^2_{-i}) is an OND sequence for f with respect to $(\bar{P}_i, \bar{P}'_i)$ satisfying the requirements of Definition 3.2 for the current instance. If not, then we proceed to the next step.

Continuing in this manner we can construct an OND sequence $(P^1_{-i}, P^2_{-i}, \ldots, P^k_{-i})$ for f with respect to $(\bar{P}_i, \bar{P}'_i)$ such that $f(\bar{P}_i, \bar{P}_{-i})\bar{R}_i f(\bar{P}'_i, P^k_{-i})$, $f(\bar{P}_i, P^1_{-i})\bar{R}_i f(\bar{P}'_i, \bar{P}_{-i})$, and $f(\bar{P}'_i, \bar{P}_{-i})\bar{P}_i f(\bar{P}'_i, P^1_{-i})$. The termination of the process is guaranteed by the fact that $P^1_{-i}, P^2_{-i}, \ldots, P^k_{-i}$ are all distinct. To see why they are distinct, note that, in a similar way as we have shown $f(\bar{P}'_i, P^1_{-i})\bar{P}_i f(\bar{P}'_i, P^2_{-i})$ in the preceding paragraph, we can show $f(\bar{P}'_i, P^1_{-i})\bar{P}_i f(\bar{P}'_i, P^2_{-i})\bar{P}_i, \ldots, \bar{P}_i f(\bar{P}'_i, P^k_{-i})$. This in particular means $P^1_{-i}, P^2_{-i}, \ldots, P^k_{-i}$ are all distinct. □

Appendix A: Proof of claim 2.2

We have shown in Claim 2.1 that both μ_1 and μ_2 satisfy TS correlation. So we need to show that $\hat{f}$ is LOBIC with respect to (μ_1, μ_2). Let $\hat{f}^{\mu_N}_B(P'_i|P_i) = \sum_{P_{-i}|\hat{f}(P'_i, P_{-i}) \in B} \mu_i(P_{-i} \mid P_i)$ denote the aggregate probability induced by $\hat{f}$ according to μ_N on an upper contour set B of P_i when his sincere preference is P_i and he (mis)reports it as P'_i. Therefore, to show that $\hat{f}$ is OBIC with respect to a TS correlated μ_N, we need to show that for each agent $i \in \{1, 2\}$, each sincere preference P_i, each misreport P'_i of agent i, and each upper contour set B of P_i, $\sum_{P_{-i}|\hat{f}(P_i, P_{-i}) \in B} \mu_i(P_{-i} \mid P_i) \geq \sum_{P_{-i}|\hat{f}(P'_i, P_{-i}) \in B} \mu_i(P_{-i} \mid P_i)$, i.e., $\hat{f}^{\mu_N}_B(P_i|P_i) \geq \hat{f}^{\mu_N}_B(P'_i|P_i)$. Suppose that the sincere preference of agent 1 is $P_1 = abc$. Suppose further that he considers a misreport as $P'_1 = acb$. Note that his conditional belief $\mu_1(\cdot \mid abc)$ at $P_1 = abc$ is given in the first row of Table 2. Further note that the (nontrivial) upper contour sets of the preference abc are $\{a\}$ and $\{a, b\}$. The believed (through $\mu_1(\cdot \mid abc)$) probability that the outcome lies in the upper contour set $\{a\}$ (that is, the outcome is a) when 1 reports abc is $\hat{f}^{\mu_N}_{\{a\}}(abc|abc) =$

TABLE 4. Upper contour probabilities when $P_1 = abc$.

P_1'	$\hat{f}^{\mu_N}_{B_1}(P_1' \mid P_1)$	$\hat{f}^{\mu_N}_{B_2}(P_1' \mid P_1)$
abc*	0.70*	0.96*
acb	0.63	0.80
bac	0.22	0.94
bca	0.20	0.43
cab	0.53	0.57
cba	0.68	0.72

$\mu_1(abc \mid abc) + \mu_1(acb \mid abc) + \mu_1(bca \mid abc) = 0.51 + 0.02 + 0.17 = 0.70$. Similarly, the believed (through $\mu_1(\cdot \mid abc)$ and not through $\mu_1(\cdot \mid acb)$) probability that the outcome is a when 1 misreports his preference as acb is $\hat{f}^{\mu_N}_{\{a\}}(acb|abc) = \mu_1(abc \mid abc) + \mu_1(acb \mid abc) + \mu_1(bac \mid abc) + \mu_1(cba \mid abc) = 0.51 + 0.02 + 0.04 + 0.06 = 0.63$. Since $\hat{f}^{\mu_N}_{\{a\}}(abc|abc) \geq \hat{f}^{\mu_N}_{\{a\}}(acb|abc)$, we have that the requirement of OBIC is satisfied for this instance.

We show that the requirement of OBIC is satisfied for every instance by means of Tables 4–15. Each table stands for a sincere preference of an agent, for instance, the first table is for $P_1 = abc$. The possible misreports (and the sincere one) are listed in the first column and the corresponding total aggregate probability for different upper contour sets are mentioned in the next columns. Here, for $k = 1, 2$, by B_k we denote the upper contour set of the corresponding sincere preference containing k elements. For instance, in the first table, $B_1 = \{a\}$ and $B_2 = \{a, b\}$. Note that B_3 need not be considered as it contains all the elements a, b, c and, hence, its aggregate probability will always be 1. To help the reader, we have marked the row corresponding to the sincere preference with the symbol $*$ in each table. In each table, the fact that each probability in the first row is weakly bigger than those in the corresponding column establishes that $\hat{f}$ is OBIC.

TABLE 5. Upper contour probabilities when $P_1 = acb$.

P_1'	$\hat{f}^{\mu_N}_{B_1}(P_1' \mid P_1)$	$\hat{f}^{\mu_N}_{B_2}(P_1' \mid P_1)$
acb*	0.90*	0.99*
abc	0.54	0.74
bac	0.60	0.77
bca	0.09	0.28
cab	0.53	0.80
cba	0.03	0.80

Table 6. Upper contour probabilities when $P_1 = bac$.

P'_1	$\hat{f}^{\mu_N}_{B_1}(P'_1 \mid P_1)$	$\hat{f}^{\mu_N}_{B_2}(P'_1 \mid P_1)$
*bac**	0.70*	0.99*
abc	0.15	0.49
acb	0.02	0.86
bca	0.68	0.82
cab	0.51	0.83
cba	0.51	0.70

Table 7. Upper contour probabilities when $P_1 = bca$.

P'_1	$\hat{f}^{\mu_N}_{B_1}(P'_1 \mid P_1)$	$\hat{f}^{\mu_N}_{B_2}(P'_1 \mid P_1)$
*bca**	0.70*	0.99*
abc	0.15	0.17
acb	0.51	0.52
bac	0.68	0.82
cab	0.02	0.68
cba	0.02	0.34

Table 8. Upper contour probabilities when $P_1 = cab$.

P'_1	$\hat{f}^{\mu_N}_{B_1}(P'_1 \mid P_1)$	$\hat{f}^{\mu_N}_{B_2}(P'_1 \mid P_1)$
cab∗	0.70*	0.99*
abc	0.01	0.47
acb	0.51	0.83
bac	0.02	0.67
bca	0.17	0.68
cba	0.67	$0.99^{\dagger}$

Table 9. Upper contour probabilities when $P_1 = cba$.

P'_1	$\hat{f}^{\mu_N}_{B_1}(P'_1 \mid P_1)$	$\hat{f}^{\mu_N}_{B_2}(P'_1 \mid P_1)$
*cba**	0.75*	0.98*
abc	0.23	0.77
acb	0.03	0.04
bac	0.51	0.76
bca	0.52	0.97
cab	0.55	0.78

Table 10. Upper contour probabilities when $P_2 = abc$.

P'_2	$\hat{f}^{\mu_N}_{B_1}(P'_2 \mid P_2)$	$\hat{f}^{\mu_N}_{B_2}(P'_2 \mid P_2)$
*abc**	0.90*	0.99*
acb	0.79	0.80
bac	0.02	0.49
bca	0.71	0.83
cab	0.10	0.61
cba	0.02	0.53

Table 11. Upper contour probabilities when $P_2 = acb$.

P'_2	$\hat{f}^{\mu_N}_{B_1}(P'_2 \mid P_2)$	$\hat{f}^{\mu_N}_{B_2}(P'_2 \mid P_2)$
*acb**	0.90*	0.99*
abc	0.82	0.83
bac	0.51	0.53
bca	0.11	0.31
cab	0.18	0.98
cba	0.51	0.98

Table 12. Upper contour probabilities when $P_2 = bac$.

P'_2	$\hat{f}^{\mu_N}_{B_1}(P'_2 \mid P_2)$	$\hat{f}^{\mu_N}_{B_2}(P'_2 \mid P_2)$
*bac**	0.90*	0.99*
abc	0.51	0.98
acb	0.02	0.80
bca	0.62	0.83
cab	0.01	0.54
cba	0.01	0.10

Table 13. Upper contour probabilities when $P_2 = bca$.

P'_2	$\hat{f}^{\mu_N}_{B_1}(P'_2 \mid P_2)$	$\hat{f}^{\mu_N}_{B_2}(P'_2 \mid P_2)$
*bca**	0.78*	0.98*
abc	0.02	0.53
acb	0.51	0.52
bac	0.74	0.75
cab	0.01	0.47
cba	0.01	0.75

Table 14. Upper contour probabilities when $P_2 = cab$.

P_2'	$\hat{f}_{B_1}^{\mu_N}(P_2'\|P_2)$	$\hat{f}_{B_2}^{\mu_N}(P_2'\|P_2)$
cab^*	0.78*	0.99*
abc	0.06	0.85
acb	0.02	0.94
bac	0.01	0.26
bca	0.51	0.54
cba	0.74	$0.99^\dagger$

It remains to show that $\hat{f}$ is LOBIC with respect to μ_N, that is, there is a neighborhood of μ_N such that $\hat{f}$ is OBIC with respect to each $\hat{\mu}_N$ in the neighborhood. Using the continuity of the expectation and the fact that there are finitely many upper contour sets, we can always find a neighborhood of μ_N such that for all $\hat{\mu}_N$ in the neighborhood, all $i \in N$, all P_i and P_i', and all upper contour set B of P_i, $\hat{f}_B^{\mu_N}(P_i|P_i) > \hat{f}_B^{\mu_N}(P_i'|P_i)$ implies $\hat{f}_B^{\hat{\mu}_N}(P_i|P_i) > \hat{f}_B^{\hat{\mu}_N}(P_i'|P_i)$. However, this does not complete the proof, as we have $\hat{f}_B^{\mu_N}(P_i|P_i) = \hat{f}_B^{\mu_N}(P_i'|P_i)$ for some $i \in N$, some P_i, P_i', and some upper contour sets B of P_i. We have marked such instances in the tables with the symbol †. We need to argue that we can find some neighborhood of μ_N such that for each $\hat{\mu}_N$ in the neighborhood, $\hat{f}_B^{\hat{\mu}_N}(P_i|P_i) \geq \hat{f}_B^{\hat{\mu}_N}(P_i'|P_i)$ for each of these instances. Consider Table 8. Observe that $\hat{f}_{\{a,c\}}^{\mu_N}(cab|cab) = \hat{f}_{\{a,c\}}^{\mu_N}(cba|cba)$. Note in Table 1 that for each P_2, $\hat{f}_{\{a,c\}}(cab, P_2) = \hat{f}_{\{a,c\}}(cba, P_2)$. Here, by $\hat{f}_{\{a,c\}}(cab, P_2)$, we denote the probability that the outcome $\hat{f}(cab, P_2)$ belongs to the set $\{a, c\}$ (that is, $\hat{f}_{\{a,c\}}(cab, P_2) = 1$ if $\hat{f}(cab, P_2) \in \{a, c\}$, and $\hat{f}_{\{a,c\}}(cab, P_2) = 0$ otherwise). This fact implies that no matter what the prior belief $\hat{\mu}_N$ is, we will always have $\hat{f}_{\{a,c\}}^{\hat{\mu}_N}(cab|cab) = \hat{f}_{\{a,c\}}^{\hat{\mu}_N}(cba|cba)$. The same logic holds for other instances where $\hat{f}_B^{\mu_N}(P_i|P_i) = \hat{f}_B^{\mu_N}(P_i'|P_i)$ for some B. This proves that $\hat{f}$ is LOBIC with respect to μ_N.

Appendix B: Proof of Claim 3.1

For every preference P_i of agent $i \in \{1, 2\}$, we have a table in Tables 16–27. In the corresponding table, the first column presents preferences P_i' via which agent i can manipulate. Every other column presents a pair (P_{-i}, P_{-i}') (or $(\bar{P}_{-i}, \bar{P}_{-i}')$ or $(\hat{P}_{-i}, \hat{P}_{-i}')$) such that

Table 15. Upper contour probabilities when $P_2 = cba$.

P_2'	$\hat{f}_{B_1}^{\mu_N}(P_2'\|P_2)$	$\hat{f}_{B_2}^{\mu_N}(P_2'\|P_2)$
cba^*	0.90*	0.99*
abc	0.20	0.37
acb	0.51	0.71
bac	0.09	$0.99^\dagger$
bca	0.02	0.40
cab	0.54	0.63

TABLE 16. $P_1 = abc$.

P'_1	(P_{-1}, P'_{-1})	$(\bar{P}_{-1}, \bar{P}'_{-1})$
acb	$(bac, -)$	(cba, bca)
bac	(bac, cba)	(cab, bca)
bca	(bac, cba)	(cab, bca)
cab	(bac, cba)	
cba	(bac, cba)	

TABLE 17. $P_1 = acb$.

P'_1	(P_{-1}, P'_{-1})
abc	(bca, cba)
bac	(cab, cba)
bca	(cab, cba)
cab	(bca, bac)
cba	(bca, bac)

TABLE 18. $P_1 = bac$.

P'_1	(P_{-1}, P'_{-1})	$(\bar{P}_{-1}, \bar{P}'_{-1})$
abc	(cab, bac)	(cba, bac)
acb	(cba, cab)	
bca	(acb, abc)	
cab		
cba		

TABLE 19. $P_1 = bca$.

P'_1	(P_{-1}, P'_{-1})	$(\bar{P}_{-1}, \bar{P}'_{-1})$
abc	(cba, acb)	(cab, acb)
acb	(cab, abc)	
bac	(abc, acb)	
cab	(cab, abc)	
cba	(cab, abc)	

TABLE 20. $P_1 = cab$.

P'_1	(P_{-1}, P'_{-1})
abc	(bac, cab)
acb	(bac, bca)
bac	
bca	(abc, cab)
cba	(acb, bca)

TABLE 21. $P_1 = cba$.

P'_1	(P_{-1}, P'_{-1})	$(\bar{P}_{-1}, \bar{P}'_{-1})$
abc	(bac, cab)	
acb	(bca, cba)	
bac	(abc, cab)	(bca, cab)
bca	(abc, cab)	(bca, cab)
cab	(bca, acb)	

TABLE 22. $P_2 = abc$.

P'_2	(P_{-2}, P'_{-2})	$(\bar{P}_{-2}, \bar{P}'_{-2})$
acb	(bac, cba)	(bca, cba)
bac	(bca, abc)	
bca	(bca, cab)	
cab	(bac, cba)	(bca, cba)
cba		

TABLE 23. $P_2 = acb$.

P'_2	(P_{-2}, P'_{-2})	$(\bar{P}_{-2}, \bar{P}'_{-2})$
abc	(bca, bac)	(cba, bac)
bac		
bca	(cba, cab)	
cab	(bca, abc)	
cba	(bca, abc)	

TABLE 24. $P_2 = bac$.

P'_2	(P_{-2}, P'_{-2})	$(\bar{P}_{-2}, \bar{P}'_{-2})$
abc	(abc, bca)	
acb	(abc, cba)	
bca	(abc, cab)	(acb, cba)
cab	(abc, cab)	
cba	(abc, bac)	

TABLE 25. $P_2 = bca$.

P'_2	(P_{-2}, P'_{-2})	$(\bar{P}_{-2}, \bar{P}'_{-2})$	$(\hat{P}_{-2}, \hat{P}'_{-2})$
abc			
acb	(cba, acb)		
bac	(abc, acb)	(cab, acb)	(cba, acb)
cab	(abc, bca)	(cba, bca)	
cba	(abc, acb)	(cba, acb)	

agent i manipulates at (P_i, P_{-i}) (or $(P_i, \bar{P}_{-i})$ or $(P_i, \hat{P}_{-i})$) via P'_i and P'_{-i} (or $\bar{P}'_{-i}$ or $\hat{P}'_{-i}$) satisfies the conditions in the definition of OND (Definition 2.5). For instance, Table 16 considers manipulation by agent 1 when his sincere preference is abc. The first element $P'_1 = acb$ and the second element $(P_{-1}, P'_{-1}) = (bac, -)$ in the first row indicates the fact that agent 1 manipulates at the preference profile (abc, bac) via acb, and there is no preference of agent 2 where the conditions in the Definition 2.5 are satisfied. The third element $(\bar{P}_{-1}, \bar{P}'_{-1}) = (cba, bca)$ of the first row indicates that agent 1 manipulates at the

TABLE 26. $P_2 = cab$.

P'_2	(P_{-2}, P'_{-2})	$(\bar{P}_{-2}, \bar{P}'_{-2})$
abc	(abc, bac)	(bca, cab)
acb	(abc, bca)	
bac	(abc, cab)	
bca	(abc, acb)	
cba	(bac, acb)	(bca, acb)

TABLE 27. $P_2 = cba$.

P'_2	(P_{-2}, P'_{-2})
abc	
acb	
bac	(abc, bac)
bca	(acb, abc)
cab	(acb, bca)

preference profile (abc, cba) via acb and the conditions in Definition 2.5 are satisfied at the preference bca of agent 2.

It follows from the Tables 16–27 that $\hat{f}$ satisfies the OND property for all cases except for the case when $P_1 = abc$, $P_1' = acb$, and $P_{-1} = bac$ (i.e., $P_2 = bac$).

References

Bhargava, Mohit, Dipjyoti Majumdar, and Arunava Sen (2015), "Incentive-compatible voting rules with positively correlated beliefs." *Theoretical Economics*, 10, 867–885. [929, 930, 931, 932]

Majumdar, Dipjyoti and Arunava Sen (2004), "Ordinally Bayesian incentive compatible voting rules." *Econometrica*, 72, 523–540. [930]

Mishra, Debasis (2016), "Ordinal Bayesian incentive compatibility in restricted domains." *Journal of Economic Theory*, 163, 925–954. [930]

Co-editor Federico Echenique handled this manuscript.

Manuscript received 31 August, 2020; final version accepted 1 July, 2021; available online 26 July, 2021.