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Surplus sharing in Cournot oligopoly

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We characterize equilibria of oligopolistic markets where identical firms with constant marginal cost compete à la Cournot. For given maximal willingness to pay and maximal total demand, we first identify all combinations of equilibrium consumer surplus and industry profit that can arise from arbitrary demand functions. Then, as a further restriction, we fix the average willingness to pay above marginal cost (i.e., first-best surplus) and identify all possible triples of consumer surplus, industry profit, and deadweight loss.

Keywords. Cournot, monopoly.

JEL classification. D42, D43.

1. Introduction

Antoine Augustin Cournot's pioneering mathematical analysis of monopoly and oligopoly, published in his *Recherches sur les Principes Mathematiques de la Theorie des Richesses* (1838), has had an enormous influence in economics.[1] Cournot's model has been a building block for a large number of seminal works in a variety of fields, including international trade (e.g., Brander and Krugman (1983), Atkeson and Burstein (2008)), the study of market power in macroeconomics (Hart (1982)), and in industrial organization (Bresnahan and Reiss (1990), Berry (1992)) and antitrust merger policy (Farrell and Shapiro (1990)). After nearly 200 years, countless papers have explored and extended Cournot's work, which remains a benchmark for theories of price formation in the absence of perfect competition (Vives (1989)).

In this paper, we advance and systematize some of the existing literature by characterizing all Cournot equilibrium outcomes as consumer demand varies. More precisely, we consider an oligopolistic market where a fixed number of firms compete à la Cournot and have the same constant marginal cost of production. Our main objective is to identify the set of those surplus divisions between firms and a mass of consumers that can

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[1]Treatment of those subjects remains almost unchanged to this day, to the point that contemporary economics students would hardly notice if excerpts from the *Recherches* were to appear in textbooks.

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arise under *some* demand function. To this end, we identify those triples of consumer surplus, industry profit, and deadweight loss that can arise in an equilibrium outcome under arbitrary demand functions with a given level of first-best surplus.

Let us explain our characterization result for the case when there is a unit mass of consumers and the common marginal cost of production is zero. In this case, the first-best surplus, s, associated with a market demand is simply the average willingness to pay of consumers. We represent market outcomes as points on the positive quadrant of a Cartesian plane, with industry profit on the x axis and consumer surplus on the y axis; see Figure 1 for illustration. A priori, the only constraint that restricts surplus-sharing is that the sum of industry profit and consumer surplus cannot exceed s; that is, any equilibrium payoff profile must be below the $y = s - x$ line. We show that the set of implementable market outcomes is characterized by a triangular shape, described by the points $(\Pi_s, s - \Pi_s)$, $(\Pi^s, 0)$, $(s, 0)$ and represented by the dotted area on Figure 1. The value Π_s is the minimum industry profit that can arise and this profit can only occur in an efficient equilibrium, so the corresponding consumer surplus is $s - \Pi_s$. If the industry profit, Π, is between Π^s $(\geq \Pi_s)$ and s, any value of consumer surplus between 0 and $s - \Pi$ can be induced by an appropriately chosen demand curve. In contrast, when Π is between Π_s and Π^s, the equilibrium consumer surplus cannot be arbitrarily small. We characterize the *curve* connecting $(\Pi_s, s - \Pi_s)$ and $(\Pi^s, 0)$, which identifies the minimum level of consumer surplus for each industry profit on this domain.

Remarkably, among all equilibria of all possible demand functions, the one that maximizes consumer surplus is efficient and also minimizes industry profit. Moreover, if $n \to \infty$, then $\Pi_s \to 0$ and $\Pi^s \to s$. Hence, the set of implementable outcomes converges

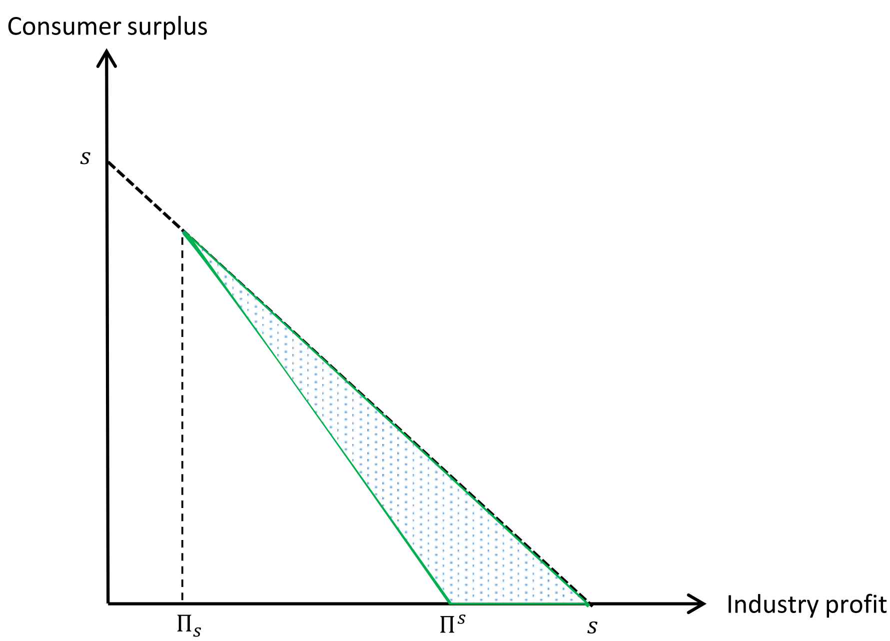

FIGURE 1. Implementable couples of consumer surplus and industry profit (dotted area).

to the *entire* first-best Pareto frontier. To paraphrase this using a common jargon of the literature, Cournot markets with constant marginal cost are quasi-competitive: all consumers with value above the marginal cost will be served in the limit as the number of firms increases. However, convergence to a competitive equilibrium is not guaranteed, because price may not approach the marginal cost of production and industry profit may remain positive.[2]

The "if" parts of our proofs are constructive. For each achievable market outcome (i.e., triple of consumer surplus, industry profit, and deadweight loss), we present an (inverse) demand function and a symmetric oligopoly equilibrium quantity that attains it. Our construction relies on a set of demand functions that, in equilibrium, induce a (common) *residual* demand that is unit-elastic with respect to profit, leaving firms indifferent between playing equilibrium and producing alternative quantities. Intuitively, this property is crucial for our construction because, at any given profit level and equilibrium price, demand can be raised to increase consumer surplus without altering firms' incentives up until firms become indifferent between the current price and a higher price they may induce by lowering the quantity produced.

Three papers are most closely related. First, Condorelli and Szentes (2020) identify the highest level of consumer surplus attainable in a monopolistic market, assuming inverse demand generates a given mean consumer value. The maximum consumer surplus is attained when the demand is unit-elastic (with respect to profit) and the price is such that all consumers are served. Second, as shown in Neeman (2003) and Kremer and Snyder (2018), it turns out that unit-elastic demand also generates the minimum monopoly profit. Taken together, these results fully characterize the combinations of producer and consumer surpluses achievable in a monopolistic market for some demand with a given average consumer value. In our paper, we offer a characterization that applies to an arbitrary number of firms competing à la Cournot.[3]

The papers mentioned above show that in the monopoly case, a *single* demand function can be used to span all achievable levels of consumer surplus for a given profit level. To explain this, we note that since this demand is unit-elastic, the seller is indifferent between setting prices on a large range. When the largest of these prices is set, consumer surplus is zero. At the lowest price, each consumer is served, so the allocation is efficient and consumer surplus is maximized. As discussed, unit-elasticity also plays a role in our analysis. In particular, we demonstrate that, holding industry profit fixed, consumer surplus is maximized by an inverse demand curve that induces unit-elastic residual demand. However, the problem of identifying all achievable levels of consumer surplus is a more subtle problem in the case of oligopoly. The inverse demand that makes *residual* demands unit-elastic has, typically, a unique equilibrium quantity. Hence, for any given total surplus, a different demand must be found to implement *each* achievable combination of consumer and producer surplus. In fact, in the oligopoly case, it is not possible

[2]As will be discussed in Section 4, the existing literature identifies convergence to a competitive equilibrium in the case of strictly decreasing inverse demand functions where the minimal consumer valuation is below the marginal cost.

[3]Kremer and Snyder (2018) also compute tight bounds on deadweight loss for a market with homogeneous firms engaging in Cournot competition. In terms of our characterization, they identify the point that generates no consumer surplus and minimizes profit (i.e., $(\Pi^s, 0)$ in Figure 1).

to perfectly trade-off profit and consumer surplus at *all* profit levels without introducing deadweight loss. In particular, following the illustration in Figure 1, we have $\Pi_s < \Pi^s$ and, therefore, the set of implementable industry profit–consumer surplus pairs is not a right triangle as in the monopoly case where $\Pi_s = \Pi^s$.

There is a small literature that seeks to identify bounds on market outcomes in Cournot oligopoly, based on specific properties of demand functions.[4] Anderson and Renault (2003) derive bounds on the ratios of deadweight loss and consumer surplus to producer surplus based on the degree of curvature of the inverse demand function. They show that the "more concave" is the demand, the larger is the share of producer surplus to overall surplus and the smaller is the consumer surplus relative to producer surplus. Johari and Tsitsiklis (2005) establish a lower bound of $2/3$ on the ratio between the sum of consumer and producer surplus and first-best surplus, when the (inverse) demand function is affine and firms are heterogeneous, with their cost function convex. Tsitsiklis and Xu (2014) extend the previous paper by providing smaller lower bounds for general convex (inverse) demand. Moreover, they show that arbitrary high efficiency losses are possible if demand is allowed to be concave. In contrast to these papers, our bounds do not rely on knowledge about the curvature of the demand function. Also, we obtain a complete characterization of all consumer and producer surplus couples for any given first-best surplus. However, we focus only on the case where firms are symmetric and their cost function is linear.

Finally, there is a large literature on Cournot oligopoly that focuses on issues of existence, uniqueness, and stability of equilibria, and on comparative statics. We are unable to survey the major contributions of this literature here, but we refer to Vives (2001).

The paper is organized as follows. After introducing the model, we study the case where demand functions are bounded but there is no restriction on first-best surplus. In Section 4, which contains the main results of this paper, we impose the additional restriction on the first-best surplus.

2. Model

A market is populated by a mass $b > 0$ of consumers and $n \in \mathbb{N}$ firms, all supplying a homogeneous good at common marginal cost $c \in (0, +\infty)$. Consumers have unit demand and heterogeneous willingness to pay for the good. Firms compete à la Cournot: each firm i decides the quantity $q_i \in [0, +\infty)$ that it brings to the market and a nonnegative price is determined by the market-clearing condition. The maximal willingness-to-pay among consumers is u $(> c)$. Let $\mathcal{P}$ be the set of all admissible inverse demand functions. That is, $\mathcal{P}$ consists of those functions $P : [0, \infty) \to [0, u]$ that are left-continuous, non-increasing, and such that $P(x) = 0$ for $x > b$. Then, if the inverse demand function is P and $Q = \sum_i q_i$ is the total supply, the market price is $P(Q)$, firm i's profit is $(P(Q) - c)q_i$ and consumer surplus is $\int_0^Q P(x)\,dx - QP(Q)$.

[4] A related problem, explored in Carvajal, Deb, Fenske, and Quah (2013), consists of identifying revealed preference tests for Cournot equilibrium.

Without loss of generality, we focus on symmetric equilibria, where all firms produce the same quantity.[5] We say that $(q, \dots, q)$, or simply q, is a Cournot equilibrium of P if

$$q = \arg\max_{x \geq 0} \big[P\big((n-1)q + x\big)x - cx \big],$$

and in this case we write $q \in \mathcal{E}(P)$.[6]

In this market, efficiency requires that a consumer is served if and only if his willingness-to-pay is larger than the marginal cost of production. Therefore, for each inverse demand curve $P \in \mathcal{P}$, we define the first-best surplus as

$$\mathrm{FB}(P) = \int_0^b \max\big\{0, P(x) - c\big\}\, dx.$$

Then let $\mathrm{CS}(P, q)$ and $\Pi(P, q)$ denote consumer surplus and the profit of a firm, respectively, if the inverse demand curve is P and each firm produces q. Formally,

$$\mathrm{CS}(P, q) = \int_0^{nq} P(x)\, dx - nqP(nq) \quad \text{and} \quad \Pi(P, q) = q\big(P(nq) - c\big).$$

Finally, we define deadweight loss under P if each firm produces q as

$$\mathrm{DWL}(P, q) = \mathrm{FB}(P) - \mathrm{CS}(P, q) - n\Pi(P, q).$$

3. Surplus sharing with arbitrary demand

Our first goal is to characterize those combinations of consumer surplus and industry profit that can arise in an equilibrium for some inverse demand in $\mathcal{P}$. Preliminarily, we identify with $\overline{\pi}$ the maximum *feasible* profit level for inverse demand functions in $\mathcal{P}$. That is, $\overline{\pi} = b(u - c)/n$, because the maximum symmetric profit level is achieved if all consumers, a measure b, are served and the market price is the maximal willingness-to-pay, u. This section is devoted to establishing the following result.

Proposition 1. *There exist $P \in \mathcal{P}$ and $q \in \mathcal{E}(P)$ such that $\Pi(P, q) = \pi$ and $\mathrm{CS}(P, q) = v$ if and only if $\pi \in (0, \overline{\pi}]$ and*

$$v \in \left[0, (n-1)(\overline{\pi} - \pi) - \pi \log\left(\frac{\pi}{\overline{\pi}}\right)\right].$$

Figure 2 illustrates the statement of Proposition 1 in three cases where $b = u = 1$ and $c = 0$ (i.e., $\lim_{c \downarrow 0}$), so $\overline{\pi} = 1/n$. The three curves plot the Pareto frontier of feasible payoff profiles, consumer surplus, and total industry profit for $n = 1$, 2, and 5. By

[5] We show in the Appendix that for any asymmetric equilibrium, there exists a symmetric one where the same total quantity is produced. Hence, consumer surplus and industry profit are the same in the two equilibria.

[6] Following McManus (1964), a symmetric equilibrium exists under the stated assumptions. Equilibria may exist for unbounded demand. However, no bound can be placed on market outcomes if demand is unbounded and no further restriction is imposed, as we shall show toward the end of next section.

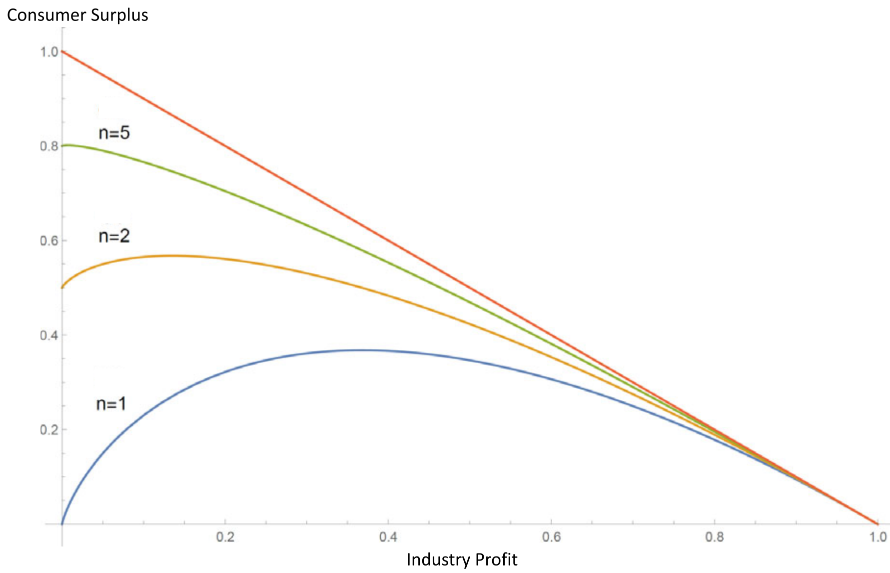

FIGURE 2. Achievable profit and surplus couples in $\mathcal{P}$ ($b = u = 1$, $c = 0$).

the proposition, any payoff profile weakly below these curves can arise in an equilibrium. As is apparent from the picture, the set of feasible producer–consumer surplus combinations expands as n grows and the market becomes more competitive. Without additional hypotheses on the set of demand functions, equilibrium alone imposes very few restrictions on how the surplus is shared in an oligopolistic market populated by a *large* number of identical firms with constant marginal cost.

Let us explain the main steps of the proof of this proposition. The key to this characterization result is to consider a parametric class of inverse demand functions and to show that these functions induce any possible equilibrium payoff profiles. This set is parameterized by the couple (π, q) and each generic element, denoted $P_{(\pi,q)}$, exhibits the properties that q constitutes a symmetric equilibrium under $P_{(\pi,q)}$ and that the corresponding profit of each firm is π. Our analysis consists of three steps. First, we demonstrate that this parametric set of demand functions spans the Pareto frontier of payoff profiles in the following sense. Fix any arbitrary inverse demand function in $\mathcal{P}$ and its symmetric equilibrium in which the individual quantity is q while the profit of each firm is π. Then the symmetric equilibrium where each firms produces q under $P_{(\pi,q)}$ generates larger consumer surplus than the former equilibrium does, while industry profit remains the same. Second, confining attention to the parameterized set of demand functions $\{P_{(\pi,q)}\}_{q \leq b/n}$, we can easily obtain an upper bound to consumer surplus for each possible profit level. This upper bound is achieved under demand function $P_{(\pi,b/n)}$. Third, we show that for any given profit π, every consumer surplus level, from 0 to the upper bound, can be obtained as the symmetric equilibrium q of some $P_{(\pi,q)}$, as q ranges between the minimal level compatible with individual profit π, $\pi/(u - c)$ and the maximal, b/n.

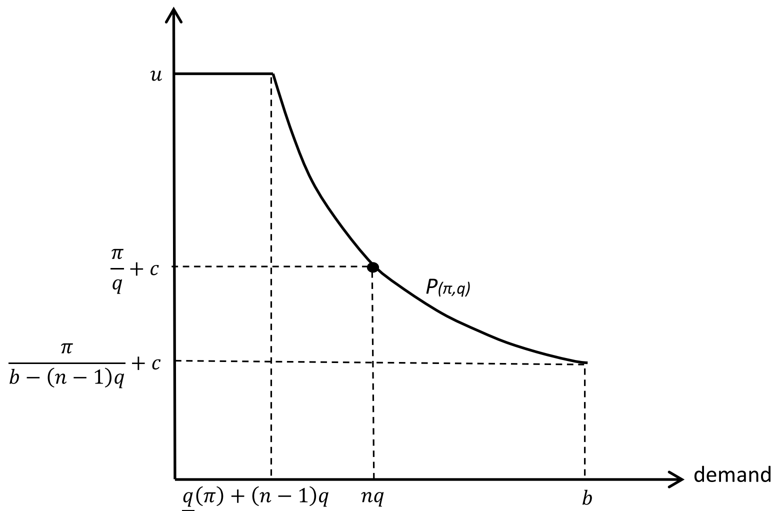

FIGURE 3. Inverse demand function $P_{(\pi,q)}$.

To begin with, we now define the aforementioned class of inverse demand functions. Denote with $\underline{q}(\pi)$ the minimal quantity that can generate profit π, that is, $\underline{q}(\pi) = \pi/(u - c)$. Then, for each $\pi \in [0, \bar{\pi}]$ and $q \in [\underline{q}(\pi), b/n]$, let

$$P_{(\pi,q)}(Q) = \begin{cases} u & \text{if } Q \in \big[0, \underline{q}(\pi) + (n-1)q\big] \\ \dfrac{\pi}{Q - (n-1)q} + c & \text{if } Q \in (\underline{q}(\pi) + (n-1)q, b] \\ 0 & \text{if } Q > b. \end{cases}$$

Figure 3 provides a graphical representation of a demand function in this class.

It is immediate to verify that $P_{(\pi,q)}(Q)$ is increasing in both π and q. We state this useful property in the next lemma, without providing an explicit proof.

LEMMA 1. *$P_{(\pi',q)} \geq P_{(\pi,q)}$ if $\bar{\pi} \geq \pi' \geq \pi \geq 0$ and $P_{(\pi,q')} \geq P_{(\pi,q)}$ if $b/n \geq q' \geq q \geq \underline{q}(\pi)$.*

The following lemma identifies an important property of our class of inverse demand functions. It states that any feasible profit level π and quantity level q arises as an equilibrium of the inverse demand curve $P_{(\pi,q)}$.

LEMMA 2. *For all $\pi \in [0, \bar{\pi}]$ and $q \in [\underline{q}(\pi), b/n]$, $q \in \mathcal{E}(P_{(\pi,q)})$ and $\Pi(P_{(\pi,q)}, q) = \pi$.*

As will be shown in the proof, the demand function $P_{(\pi,q)}$ exhibits unit-elasticity of the *residual* demand with respect to profit for quantities in $(\underline{q}(\pi), b - (n-1)q)$, when all other firms supply q. In particular, for each individual firm, producing any quantity in that interval is a best reply to the other firms producing q and it generates profit π.

PROOF OF LEMMA 2. Fix a $\pi \in [0, \bar{\pi}]$ and a $q \in [\underline{q}(\pi), b/n]$. Note that $P_{(\pi,q)}(nq) = \pi/q + c$, so $\Pi(P_{(\pi,q)}, q) = \pi$. It remains to show that $q \in \mathcal{E}(P_{(\pi,q)})$. To this end, consider the *residual* demand faced by i under demand $P_{(\pi,q)}$ when all other firms are producing q. This is

$$P_{(\pi,q)}\big(q_i+(n-1)q\big)=\begin{cases} u & \text{if } q_i\in\big[0,\underline{q}(\pi)\big] \\ \dfrac{\pi}{q_i}+c & \text{if } q_i\in(\underline{q}(\pi),b-(n-1)q] \\ 0 & \text{if } q_i>b-(n-1)q. \end{cases}$$

Observe that any quantity in the interval $[\underline{q}(\pi),b-(n-1)q]$ generates profit π and any quantity outside of this interval induces profit less than π. To conclude the proof, we argue that if $q\in[\underline{q}(\pi),b/n]$, then $q\in[\underline{q}(\pi),b-(n-1)q]$. This immediately follows because $q\le b/n$ implies $q\le b-(n-1)q$. □

Let us point out here a crucial difference between this result for monopoly versus oligopoly. When $n>1$ and $q<b/n$, we have $P_{(\pi,q)}\neq P_{(\pi,b/n)}$. On the other hand, if $n=1$, then $P_{(\pi,q)}=P_{(\pi,b/n)}$ for all q. So, in the case of monopoly, any $q\in[\underline{q}(\pi),b/n]$ is an equilibrium under $P_{(\pi,b/n)}$ and this inverse demand curve induces any consumer surplus that is consistent with π.

Next we provide a key building block of this paper. We show that the family of inverse demand functions defined above Pareto-dominates other demand functions in the following sense. Suppose that q is a symmetric equilibrium when demand is determined by P and the profit of an individual firm is π. Then the consumer surplus in this equilibrium is smaller than in the symmetric equilibrium of $P_{(\pi,q)}$, where every firm produces q and obtains profit π. When $n=1$, this result is analogous to Lemma 1 in Condorelli and Szentes (2020).

Lemma 3. *If $P\in\mathcal{P}$, $q\in\mathcal{E}(P)$, and $\Pi(P,q)=\pi$, then $P\le P_{(\pi,q)}$ and $\mathrm{CS}(P_{(\pi,q)},q)\ge\mathrm{CS}(P,q)$.*

Let us explain the intuition behind the statement of this lemma. As mentioned earlier, the identifying feature of the inverse demand curve $P_{(\pi,q)}$ is that the residual demand curve faced by a firm, provided that every other firm produces q, is unit-elastic. In particular, each firm is indifferent between producing a large range of quantities. This means that when demand is determined by $P_{(\pi,q)}$, the consumers' willingness-to-pay is set to be as high as possible while still providing the firms with just enough incentive to produce quantity q.

Proof of Lemma 3. First, observe that if q is an equilibrium in P and each firm's profit is π, then setting any other quantity x against $(n-1)q$ induces a payoff less than π, that is, for any $x\ge 0$,

$$\pi\ge x\big[P\big(x+(n-1)q\big)-c\big].$$

By denoting $Q=x+(n-1)q$ and rearranging, it follows that, for $Q\in[(n-1)q,+\infty)$,

$$P(Q)\le\frac{\pi}{Q-(n-1)q}+c. \tag{1}$$

Next we show that $P(Q) \leq P_{(\pi,q)}(Q)$ for $Q \geq 0$. This immediately follows from inequality (1) and the definition of $P_{(\pi,q)}$ for $Q \geq (n-1)q$. For $Q < (n-1)q \leq (n-1)q + \pi/(u-c)$, the inequality follows from the fact that $P_{(\pi,q)}(Q) = u$, while $P(Q) \leq u$ because the largest willingness-to-pay is u.

We now establish that $P_{(\pi,q)}$ generates (weakly) larger consumer surplus than P. In particular,

$$\mathrm{CS}(P_{(\pi,q)}, q) - \mathrm{CS}(P, q) = \int_0^{nq} \bigl(P_{(\pi,q)}(x) - P(x)\bigr)\, dx \geq 0,$$

because $P_{(\pi,q)} \geq P$, as argued above. $\square$

Building on the previous results, the following lemma identifies an upper bound on consumer surplus for each profit level π. It establishes that there exists no symmetric equilibrium under any demand function generating individual firm profit π that attains a consumer surplus higher than the equilibrium of $P_{(\pi,b/n)}$, where all firms produce b/n.

Lemma 4. *For any $P \in \mathcal{P}$ and $q \in \mathcal{E}(P)$ with $\Pi(P,q) = \pi$, we have $\mathrm{CS}(P_{(\pi,b/n)}, b/n) \geq \mathrm{CS}(P, q)$.*

Proof. Lemma 2 establishes that b/n is an equilibrium of $P_{(\pi,b/n)}$ and q is an equilibrium of $P_{(\pi,q)}$. Lemma 3 establishes that $\mathrm{CS}(P_{(\pi,q)}, q) \geq \mathrm{CS}(P, q)$. To see that $\mathrm{CS}(P_{(\pi,b/n)}, b/n) \geq \mathrm{CS}(P_{(\pi,q)}, q)$, recall the last displayed equation in the proof of Lemma 3 and observe that $b/n \geq q$ and, for each $\pi \in [0, \bar{\pi}]$ and q, q' such that $\pi/(u-c) \leq q' \leq q \leq b/n$, we have $P_{(\pi,q')}(Q) \leq P_{(\pi,q)}(Q)$ for $Q \in [0, \infty)$, by Lemma 1. $\square$

We are now ready to prove Proposition 1. To explain the remaining part of the argument, recall that $\overline{\pi} = b(u-c)/n$ is the largest feasible profit. Furthermore, for each $\pi \in (0, \overline{\pi})$, the inverse demand curve $P_{(\pi,b/n)}$ maximizes consumer surplus across all inverse demand functions that also generate profit π. But then the set of inverse demand curves $\{P_{(\pi,q)}\}_{q=\underline{q}(\pi)}^{b/n}$ spans the whole range of consumer surpluses that are consistent with profit π. The reason is that when the symmetric equilibrium quantity is set to the smallest level that is consistent with profit π, that is, $\underline{q}(\pi)$, the consumer surplus is 0 and the consumer surplus generated by $P_{(\pi,q)}$ is continuous in q.

Proof of Proposition 1. To prove the "only if" part, recall that we have already argued that $\bar{\pi}$ is the largest feasible equilibrium profit, so π must indeed be in the interval $(0, \bar{\pi}]$. Moreover, by Lemma 4, $\mathrm{CS}(P, q) \leq \mathrm{CS}(P_{(\pi,b/n)}, b/n)$, that is, v must be in the interval $[0, \mathrm{CS}(P_{(\pi,b/n)}, b/n)]$. Finally, a straightforward computation yields that

$$\mathrm{CS}(P_{(\pi,b/n)}, b/n) = (n-1)\left[\frac{b(u-c)}{n} - \pi\right] - \pi \log\left(\frac{n\pi}{b(u-c)}\right).$$

To argue the "if" part, for each $\pi \in (0, \bar{\pi})$ and $q \in [0, b/n]$, consider $P_{(\pi,q)}$ and recall that q is an equilibrium in $P_{(\pi,q)}$ by Lemma 2. Furthermore, $\mathrm{CS}(P_{(\pi,q)}, q)$ is continuous and strictly increasing in q, and $\mathrm{CS}(P_{(\pi,\underline{q}(\pi))}, \underline{q}(\pi)) = 0$. $\square$

Our previous results allow us to easily identify the maximum consumer surplus for an equilibrium under an inverse demand function in $\mathcal{P}$. In light of Lemma 4, finding an inverse demand function in $\mathcal{P}$ and an equilibrium that maximizes consumer surplus is equivalent to maximizing $\mathrm{CS}(P_{(\pi,b/n)}, b/n)$ in $\pi \in (0, \bar{\pi})$. The proof is omitted because it involves a straightforward maximization problem.

Corollary 1. *Let* $\pi^* = n\bar{\pi}/e^n$. *Then* $\mathrm{CS}(P_{(\pi^*,b/n)}, b/n) = n(n-1)\bar{\pi} + n\bar{\pi}/e^n \geq \mathrm{CS}(P, q)$ *for any* $P \in \mathcal{P}$, $q \in \mathcal{E}(P)$.

To conclude this section, let us discuss two implications of Proposition 1. First, by providing an explicit expression for $\mathrm{CS}(P_{(\pi,b/n)}, b/n)$, the proof of Proposition 1 shows that no meaningful bound can be placed on the ratio of consumer to producer surplus. In fact, $\lim_{\pi\to 0} \mathrm{CS}(P_{(\pi,b/n)}, b/n)/(n\pi) = \infty$ and $\lim_{\pi\to\bar{\pi}} \mathrm{CS}(P_{(\pi,b/n)}, b/n)/(n\pi) = 0$. Second, we observe that the bounds described by Proposition 1 can be used to identify restrictions on the set of price–quantity pairs that can arise in any equilibrium, given the number of firms operating in the market. To see this, use the bound we obtained above and the fact that consumer surplus cannot exceed $(u - P)Q$ in any equilibrium, where P and Q are the equilibrium price and total quantity. For instance, consider the case of $n = 1$ depicted in Figure 2, where $u = b = 1$ and $c = 0$. In this case, $1/e$ is the maximum achievable consumer surplus, which implies $Q(1 - P) \leq 1/e$. This provides a restriction on the attainable price–quantity pairs. An analogous exercise can be performed for $n > 1$. However, as Figure 2 illustrates, when the number of firms grows large, the maximum consumer surplus goes to 1, which makes the condition on consumer surplus vacuous.

4. Surplus sharing for given first-best surplus

The aim of this section is to provide a complete characterization of all possible triples of consumer surplus, industry profit, and deadweight loss that can arise in an equilibrium for some inverse demand curve for a given first-best surplus. We accomplish this goal by considering each feasible level of first-best surplus, s, and focusing attention on the set of those demands that are consistent with that level of surplus. Then we characterize those pairs of consumer surplus and industry profit that can occur in an equilibrium under some inverse demand curve with first-best surplus equal to s. Of course, the deadweight loss in each equilibrium can be computed as the difference between s and the sum of consumer surplus and industry profit.

As a preliminary step, note that the first-best surplus must be weakly positive and can never exceed $b(u - c)(= n\overline{\pi})$. This latter surplus can be achieved only if the willingness-to-pay of each consumer is u. That is, $0 \leq \mathrm{FB}(P) \leq b(u - c)$ for each $P \in \mathcal{P}$. For each $s \in (0, b(u - c)]$, let $\mathcal{P}_s$ denote the set of those inverse demand curves that generate surplus s, that is, $\mathcal{P}_s = \{P \in \mathcal{P} : \mathrm{FB}(P) = s\}$.

As mentioned above, we characterize those combinations of consumer and producer surplus that can arise in an equilibrium under some inverse demand function in $\mathcal{P}_s$. Let us now explain the main steps of our arguments leading to this result and il-

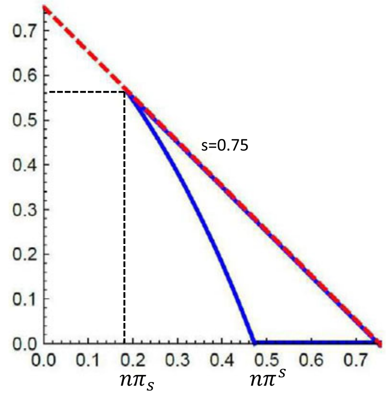

FIGURE 4. Achievable payoffs for $n = 2$, $b = u = 1$, $c = 0$, and $s = 3/4$.

lustrate them on Figure 4 for the case of $n = 2$, $u = b = 1$, $c = 0$, and $s = 0.75$. First, we pin down the smallest possible equilibrium profit for an individual firm, π_s, under $\mathcal{P}_s$. In the example of Figure 4, we have $2\pi_{0.75} \approx 0.18$. Then we identify the Pareto frontier of the set of equilibrium payoff profiles under $\mathcal{P}_s$. More precisely, for each $\pi \in [\pi_s, s/n]$, we construct an inverse demand curve in $\mathcal{P}_s$ and an equilibrium such that industry profit is $n\pi$ and consumer surplus is $s - n\pi$, that is, the equilibrium is efficient. On Figure 4, the set of efficient equilibrium payoffs corresponds to the straight line connecting point $(2\pi_{0.75}, 0.75 - 2\pi_{0.75})$ to point $(0.75, 0)$. The remaining piece of our characterization result is to figure out how small the equilibrium consumer surplus can be for each $\pi \in [\pi_s, s/n]$. Indeed, the final step is to show that for each surplus s, there is a threshold value of individual profit, π^s, such that if π is above this value, or $\pi \in [\pi^s, s/n]$, then any consumer surplus can arise in equilibrium between 0 and $s - n\pi$. In the numerical example, $2\pi^{0.75} \approx 0.47$. In contrast, if π is smaller than the threshold, $\pi \in [\pi_s, \pi^s]$, then the consumer surplus is bounded away from 0. Indeed, our last proposition pins down the smallest equilibrium value of consumer surplus for each such profit level. On Figure 4, the curved line describes the smallest possible consumer surplus that can arise in equilibrium for a given industry profit. Note that this curve is continuous, decreasing, and concave, and reaches 0 at $2\pi^{0.75}$, a point where deadweight loss is maximal.

To begin our analysis, we now turn our attention to identifying the minimal equilibrium profit that can arise if the first-best surplus is s. We define a profit level, π_s, for each $s \in (0, n\overline{\pi}]$ and then show that the equilibrium profit of a firm is at least π_s if the inverse demand curve is in $\mathcal{P}_s$. Let $\pi_s = 0$ if $s \in (0, (n-1)\bar{\pi}]$ and let π_s be the solution of $s = \mathrm{FB}(P_{(\pi,b/n)})$ in π, that is, $s = \mathrm{FB}(P_{(\pi_s,b/n)})$ if $s \in ((n-1)\bar{\pi}, n\bar{\pi}]$.

To see that π_s is well defined on $((n-1)\bar{\pi}, n\bar{\pi}]$, observe that for each $\pi \in [0, \bar{\pi}]$,

$$\begin{aligned}\mathrm{FB}(P_{(\pi,b/n)}) &= \mathrm{CS}(P_{(\pi,b/n)}, b/n) + n\Pi(P_{(\pi,b/n)}, b/n)\\ &= \pi\left[1 - \log\left(\frac{\pi}{\bar{\pi}}\right)\right] + (n-1)\bar{\pi}.\end{aligned}$$

Note that the right-hand side is continuous, and strictly increasing in π, and its value is $(n-1)\overline{\pi}$ at 0 and $n\overline{\pi}$ at $\overline{\pi}$. Then, by the intermediate value theorem, the equation $s = \text{FB}(P_{(\pi_s, b/n)})$ has a unique solution.[7]

In what follows, for each $s \in (0, n\overline{\pi}]$, we construct an inverse demand curve P_s and show that under P_s, there is an equilibrium in which the profit of each firm is π_s. Moreover, the equilibrium profit under any demand in the set $\mathcal{P}_s$ is weakly larger than π_s. To this end, for each $s \in ((n-1)\bar{\pi}, n\bar{\pi}]$, let $P_s = P_{(\pi_s, b/n)}$, and for $s \in [0, (n-1)\bar{\pi}]$, let P_s be defined as

$$P_s(Q) = \begin{cases} \dfrac{sn}{b(n-1)} + c & \text{if } Q \in \big[0, b(n-1)/n\big] \\ c & \text{if } Q \in (b(n-1)/n, b] \\ 0 & \text{if } Q > b. \end{cases}$$

Next we argue that $\text{FB}(P_s) = s$ for all $s \in (0, n\overline{\pi}]$, that is, $P_s \in \mathcal{P}_s$. This is obvious if $s \in ((n-1)\bar{\pi}, n\bar{\pi}]$ because, on this domain, $P_s = P_{(\pi_s, b/n)}$ and $s = \text{FB}(P_{(\pi_s, b/n)})$ by the definition of π_s. If $s \in [0, (n-1)\bar{\pi}]$, then

$$\text{FB}(P_s) = \frac{b(n-1)}{n} \frac{sn}{b(n-1)} s = s.$$

We also note that b/n is an equilibrium under P_s and $\Pi(P_s, b/n) = \pi_s$.

We are now ready to prove that π_s is the minimal equilibrium profit under any inverse demand curve in $\mathcal{P}_s$. In fact, the next lemma states that the equilibrium b/n under P_s does not only induce the smallest profit, but also no deadweight loss and, therefore, the largest consumer surplus across all equilibria under $\mathcal{P}_s$.

Lemma 5. *For each $P \in \mathcal{P}_s$ and $q \in \mathcal{E}(P)$,*

(i) $\pi_s = \Pi(P_s, b/n) \leq \Pi(P, q)$

(ii) $s - n\pi_s = \text{CS}(P_s, b/n) \geq \text{CS}(P, q)$.

Proof. Consider first part (i). This statement is obvious for $s \in (0, (n-1)\bar{\pi}]$ because $\Pi(P_s, b/n) = \pi_s = 0$. Therefore, we focus on $s \in ((n-1)\bar{\pi}, n\bar{\pi}]$. By way of contradiction, suppose that q is an equilibrium under some $P \in \mathcal{P}_s$ and $\Pi(P, q) = \pi' < \pi_s$. Consider now the (efficient) equilibrium b/n under $P_{(\pi', b/n)}$. By Lemmas 1 and 3,

$$P_{(\pi', b/n)} \geq P_{(\pi', q)} \geq P$$

and, hence, $\text{FB}(P_{(\pi', b/n)}) \geq \text{FB}(P)$. Since $\text{FB}(P_{(\pi, b/n)})$ is strictly increasing in π, $\pi_s > \pi'$ implies $s = \text{FB}(P_s) = \text{FB}(P_{(\pi_s, b/n)}) > \text{FB}(P_{(\pi', b/n)}) \geq \text{FB}(P) = s$, a contradiction.

To see part (ii), note that

$$s - n\pi_s = \text{CS}(P_s, b/n) \geq s - n\Pi(P, q) \geq \text{CS}(P, q),$$

[7] If $b = u = 1$ and $c = 0$, $\pi_s = \frac{n-1-ns}{nW_{-1}(\frac{n-1-ns}{e})}$ for $s > \frac{n-1}{n}$. Here W_{-1} is the lower branch of the Lambert W function. While it cannot be expressed in terms of elementary functions, it is defined by $W_{-1}(xe^x) = x$ for $x \leq -1$.

where the first inequality follows from part (i) and the second inequality holds because the sum of the industry profit and the consumer surplus cannot exceed the first-best surplus. $\square$

Our next goal is to characterize the set of efficient equilibrium payoff profiles for each s. To state our objective more precisely, note first that the equilibrium profit of a firm, $\Pi(P, q)$, for any equilibrium q of any $P \in \mathcal{P}_s$ cannot exceed s/n and is weakly larger than π_s (by part (i) of Lemma 5). In what follows, we demonstrate that for each $\pi \in [\pi_s, s/n]$, there exists an inverse demand curve in $\mathcal{P}_s$ and an efficient equilibrium under it such that the profit of each firm is π and the consumer surplus is $s - n\pi$. To this end, we next introduce a new parametric class of demands and show that such inverse demand curves lie in $\mathcal{P}_s$.

For each $\pi \in (0, \bar{\pi}]$, $q \in [\underline{q}(\pi), b/n]$, and $k \in [\underline{q}(\pi) + (n-1)q, nq]$, let

$$P^k_{\pi,q}(Q) = \begin{cases} P_{(\pi,q)}(k) & \text{if } Q \in [0, k] \\ P_{(\pi,q)}(Q) & \text{if } Q \in (k, b] \\ 0 & \text{if } Q > b. \end{cases}$$

Observe that $P^k_{\pi,q}$ is a truncated version of $P_{(\pi,q)}$ because $P^k_{\pi,q}(Q) = P_{(\pi,q)}(Q)$ for $Q \geq k$ and, below k, the value of $P^k_{\pi,q}$ is defined to be $P_{(\pi,q)}(k)$. Intuitively, $P^k_{\pi,q}$ is derived from $P_{(\pi,q)}$ by setting equal to $P_{(\pi,q)}(k)$ the willingness-to-pay to of each consumer whose valuation is higher than that. From the proof of Lemma 2, it follows that producing q is an equilibrium under $P^k_{\pi,q}$ and it generates individual profit π.

Figure 5 depicts an example of a demand function $P^k_{\pi,q}$ for $b = u = 1$ and $c = 0$ and the division of the first-best surplus between consumer surplus (CS), industry profit $(n\pi)$, and deadweight loss (DWL) in the q equilibrium.

We are now ready to state the following proposition.

Proposition 2. *For each $s \in (0, n\bar{\pi}]$ and $\pi \in [\pi_s, s/n]$, there exist $P \in \mathcal{P}_s$ and $q \in \mathcal{E}(P)$ such that $\Pi(P, q) = \pi$ and $\mathrm{CS}(P, q) = s - n\pi$.*

Let us explain the proof of this proposition. Recall that producing b/n is an efficient equilibrium under $P_{(\pi,b/n)}$, that is, the consumer surplus under $P_{(\pi,b/n)}$ is just the difference between the first-best surplus and the industry profit. Of course, $P_{(\pi,b/n)}$ generates too much surplus, that is, $P_{(\pi,b/n)} \notin \mathcal{P}_s$. Hence, we modify it by truncating it and show that there exists a k such that $P^k_{\pi,b/n} \in \mathcal{P}_s$. Since b/n is an efficient equilibrium under $P^k_{\pi,b/n}$ that induces profit π to each firm, $P = P^k_{\pi,b/n}$ and $q = b/n$ satisfy the statement of the proposition.

Proof of Proposition 2. Lemma 5 established the statement for $\pi = \pi_s$. We now show that for each $\pi \in (\pi_s, s/n]$, there exists (a unique) k^0 such that $\mathrm{FB}(P^{k^0}_{\pi,b/n}) = s$ and, therefore, $P^{k^0}_{\pi,b/n} \in \mathcal{P}_s$. To show the existence of k^0, observe first that $P^{\pi/(u-c)+(n-1)b/n}_{\pi,b/n} = P_{(\pi,b/n)}$. Since $\pi > \pi_s$, we must have $\mathrm{FB}(P_{(\pi,b/n)}) > \mathrm{FB}(P_{(\pi_s,b/n)}) = \mathrm{FB}(P_s) = s$. Also

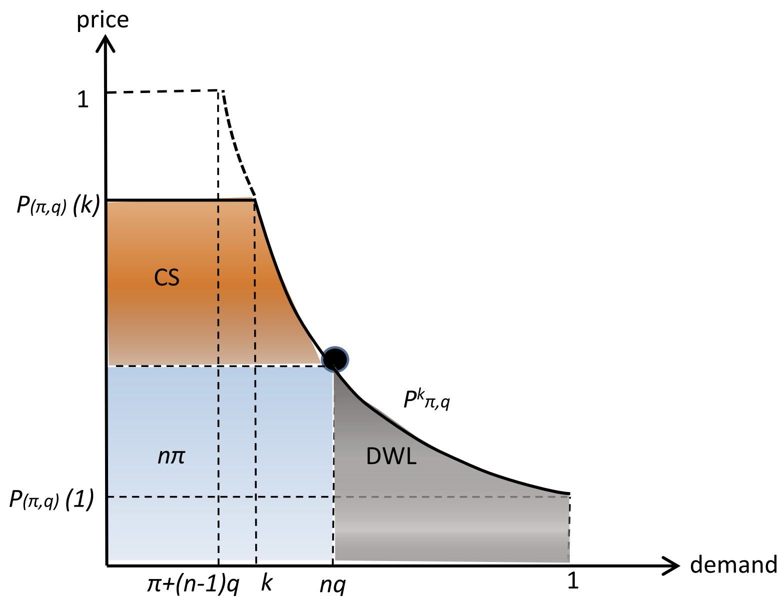

FIGURE 5. Example of demands $P^k_{\pi,q}$ for $b = u = 1$ and $c = 0$.

note that $\mathrm{FB}(P^b_{\pi,b/n}) = n\pi \le s$ because $\pi \le s/n$. Since $\mathrm{FB}(P^k_{\pi,b/n})$ is continuous and strictly decreasing in k, the intermediate value theorem implies the existence of k_0 at which $\mathrm{FB}(P^{k^0}_{\pi,b/n}) = s$. Finally, the proof is concluded by noting that in the equilibrium b/n under $P^{k^0}_{\pi,b/n}$, the profit of each firm is π and consumer surplus is equal to $\mathrm{FB}(P^{k^0}_{\pi,b/n}) - n\Pi(P^{k^0}_{\pi,b/n}, b/n) = s - n\pi$. □

In words, the result above says that as long as the profit of each firm is at least π_s, *any* division of the first-best surplus between consumers and producers is attained by some demand function without incurring any deadweight loss, irrespective of the number of firms in the market. Furthermore, when the number of firms is sufficiently large, $n > 1/(1-s)$, then $\pi_s = 0$ and, hence, any such divisions are attainable.

To complete our characterization, we now also turn our attention to inefficient equilibria. For each value of first-best surplus s, we define a threshold value of individual firm profit, π^s. Then we show that if π is larger than π^s, then consumer surplus can be anything between 0 and $s - n\pi$. In contrast, when π is smaller than π^s, consumer surplus is bounded away from 0 and our last proposition characterizes this bound.

For each $s \in (0, n\overline{\pi}]$, let $\pi^s \in [\pi_s, s/n]$ be the solution to $\mathrm{FB}(P_{(\pi^s,\underline{q}(\pi^s))}) = s$. Recall that $\underline{q}(\pi)$ denotes the minimal quantity that can generate profit π, that is, $\underline{q}(\pi) = \pi/(u-c)$. To see that π^s is well defined, note first that $\mathrm{FB}(P_{(\pi_s,b/n)}) = s$ and, therefore, $\mathrm{FB}(P_{(\pi_s,\underline{q}(\pi_s))}) \le s$ because, by Lemma 1, $P_{(\pi_s,\underline{q}(\pi_s))} \le P_{(\pi_s,b/n)}$ and $P_{(\pi_s,b/n)} \ge c$. Second, note that $\mathrm{FB}(P_{(s/n,\underline{q}(\pi_s)}) \ge s$ as profit under $P_{(s/n,\underline{q}(\pi_s))}$ is s/n and $\mathrm{FB}(P_{(s/n,\underline{q}(\pi_s))}) \ge ns/n = s$. Then, since $\mathrm{FB}(P_{(\pi_s,\underline{q}(\pi_s))})$ is continuous and strictly increasing, the intermediate value theorem implies that the equation $\mathrm{FB}(P_{(\pi,\underline{q}(\pi))}) = s$ has a unique solution in π.

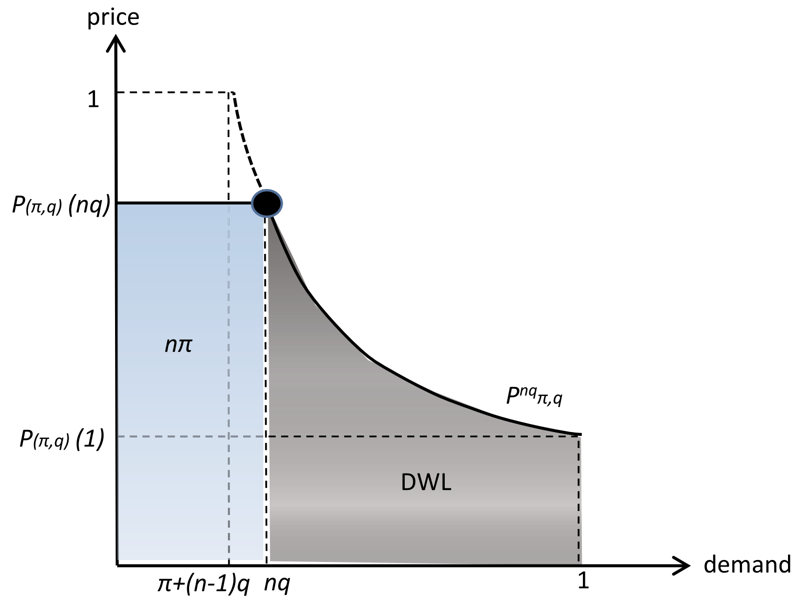

FIGURE 6. Example of demands $P^{nq}_{\pi,q}$ for $b = u = 1$ and $c = 0$.

PROPOSITION 3. *For every* $s \in (0, n\bar{\pi}]$, $\pi \in [\pi^s, s/n]$, *and* $v \in [0, s - n\pi]$ *there exists* $P \in \mathcal{P}_s$ *and* $q \in \mathcal{E}(P)$ *such that* $\Pi(P, q) = \pi$ *and* $\mathrm{CS}(P, q) = v$.

This proposition states that for given s and profit level π above π^s, any feasible combination of consumer surplus and deadweight loss is achievable. To prove it, we first show that zero consumer surplus can be achieved in equilibrium when $\pi \in [\pi^s, s/n]$ using inverse demand $P^{nq}_{\pi,q}$ for some q. Note that such demand induces zero consumer surplus as illustrated in Figure 6, where the equilibrium quantity is indicated with the black dot, the profit is the blue shaded (clearer) area, and deadweight loss (DWL) is the gray (darker) area. Finally, once again we appeal to the intermediate value theorem to argue that any intermediate level of consumer surplus is also achievable.

Before proceeding with the proof, we find it useful to establish some properties of the deadweight loss function, $\mathrm{DWL}(P, q)(= \mathrm{FB}(P) - \mathrm{CS}(P, q) - n\Pi(P, q))$ when restricted to the domain $\{P^k_{\pi,q}\}$.

LEMMA 6. $\mathrm{DWL}(P^k_{\pi,q}, q)$ *is continuous, strictly decreasing in* q, *and independent of* k.

PROOF. Since $k < nq$ for $q \in [\underline{q}(\pi), b/n]$ and $P^k_{\pi,q}(x) = P_{(\pi,q)}(x)$ for $x \geq nq \geq \pi/(u - c) + (n-1)q$, we have $\mathrm{DWL}(P^k_{\pi,q}, q) = \int_{nq}^{b}(P^k_{\pi,q}(x) - c)\,dx = \int_{nq}^{b}(P_{(\pi,q)}(x) - c)\,dx = \pi[-\log(q) + \log(b - (n-1)q)]$. See Figure 5 for a geometric intuition of the last part. □

PROOF OF PROPOSITION 3. As a first step, for each $s \in (0, (u - c)b]$ and $\pi \in [\pi^s, s/n]$, we determine q such that $\mathrm{FB}(P^{nq}_{\pi,q}) = s$. Clearly, if such q exists, then in the equilibrium q of this demand function, consumer surplus is 0. In fact, it is immediate to verify that

$\mathrm{CS}(P^{nq}_{\pi,q}, q) = 0$ because $P^{nq}_{\pi,q}$ is constant between 0 and the equilibrium total quantity nq (see Figure 6).

Hence, we show that for each s and π in the range identified by the statement, there exists $\underline{q}(\pi) \le q \le b/n$ and demand $P^{nq}_{\pi,q}$ such that $\mathrm{FB}(P^{nq}_{\pi,q}) = s$. To see this, observe that $P^{n\underline{q}(\pi)}_{\pi,\underline{q}(\pi)} = P_{(\pi,\underline{q}(\pi))}$ as $\pi/(u-c) + (n-1)\underline{q}(\pi) = n\underline{q}(\pi)$ since $\underline{q}(\pi) = \pi/(u-c)$. Then note that $\mathrm{FB}(P_{(\pi,\underline{q}(\pi))}) \ge \mathrm{FB}(P_{(\pi^s,\underline{q}(\pi))}) = s$, where the inequality follows from Lemma 1 observing that $\pi \ge \pi^s$ and $\underline{q}(\pi) \ge \underline{q}(\pi^s)$, while the equality follows from the definition of π^s. Second, consider that $\mathrm{FB}(P^b_{\pi,b/n}) = n\pi \le s$ because $P^b_{\pi,b/n}(Q) = P_{(\pi,b/n)}(b) = n\pi/b + c$ for $Q \in [0, b]$ and $\pi \le s/n$. The result that such a q exists, call it $q^0(\pi, s)$, follows from the intermediate value theorem by varying q in $\mathrm{FB}(P^{nq}_{\pi,q})$ between $\underline{q}(\pi)$ and b/n.

To conclude the proof, we now show that for $\pi \in [\pi^s, s/n]$, all intermediate levels of consumer surplus between 0 and $s - n\pi$ can be achieved by some demand $P^k_{\pi,q} \in \mathcal{P}_s$. As a preliminary step observe two facts: (i) $\mathrm{DWL}(P_{(\pi,q^0(\pi,s))}, q^0(\pi, s)) = s - n\pi$ because by the first part of this proposition $\mathrm{DWL}(P^{nq^0(\pi,s)}_{\pi,q^0(\pi,s)}, q^0(\pi, s)) = s - n\pi$ but deadweight loss of $P^k_{\pi,q}$ does not depend on k (see Lemma 6); (ii) $\mathrm{DWL}(P_{(\pi,b/n)}, b/n) = 0$ by definition. Then, since $\mathrm{DWL}(P_{(\pi,q)}, q)$ is continuous and decreasing in q (see Lemma 6), then for any $x \in [0, s-n\pi]$ there exists $\hat{q}$ such that $\mathrm{DWL}(P_{(\pi,\hat{q})}, \hat{q}) = x$. Then, since $\mathrm{DWL}(P^k_{\pi,q}, q)$ does not depend on k (see Lemma 6) and since for any P, we have $\mathrm{DWL}(P, q) = \mathrm{FB}(P) - \mathrm{CS}(P, q) - n\Pi(P, q)$, we can establish our result if for all $q \in [q_0(\pi, s), b/n]$, we find k^q such that $\mathrm{FB}(P^{k^q}_{\pi,q}) = s$. Existence of k^q (where dependence on s and π is omitted to simplify notation) is demonstrated in the remainder of the proof.

To see that for all $q \ge q^0(\pi, s)$, there exists k^q such that $nq \ge k^q \ge nq^0(\pi, s)$ and $\mathrm{FB}(P^{k^q}_{\pi,q}) = s$, we can use again the intermediate value theorem after observing the following two things. First, $\mathrm{FB}(P^{nq^0(\pi,s)}_{\pi,q^0(\pi,s)}) = s \le \mathrm{FB}(P^{nq^0(\pi,s)}_{\pi,q})$ for $q \ge q^0(\pi, s)$, because, for given k, by Lemma 1, $P^k_{\pi,q} \ge P^k_{\pi,q^0}$ for all $q \ge q^0(\pi, s)$. Second, $\mathrm{FB}(P^{nq}_{\pi,q}) \le s$ for $q \ge q^0(\pi, s)$, because $\mathrm{FB}(P^{nq}_{\pi,q}) = n\pi + \mathrm{DWL}(P^{nq}_{\pi,q}, q) \le n\pi + \mathrm{DWL}(P^{nq^0(\pi,s)}_{\pi,q^0(\pi,s)}, q^0(\pi, s)) = \mathrm{FB}(P^{nq^0(\pi,s)}_{\pi,q^0(\pi,s)}) = s$, where the inequality follows since we have $\mathrm{DWL}(P^{nq}_{\pi,q}, q) \le \mathrm{DWL}(P^{nq^0(\pi,s)}_{\pi,q^0(\pi,s)}, q^0(\pi, s))$ due to $\mathrm{DWL}(P^k_{\pi,q}, q)$ being independent of k and decreasing in q for given π (see Lemma 6). □

Observe that if $n = 1$, then $\pi_s = \pi^s$ as $P_{(\pi_s,q)} = P_{(\pi_s,b/n)}$ for any $q \in [\pi, b]$. So, $\mathrm{FB}(P_{(\pi,\underline{q}(\pi))}) = \mathrm{FB}(P_{(\pi_s,b/n)}) = \mathrm{FB}(P_s)$. Hence, the above result completes the characterization for the monopoly. In this case, for any given s and $\pi \in [\pi_s, s]$, all surplus and deadweight loss combinations are achieved as different equilibria of the demand P_s. In particular, the highest consumer surplus $s - n\pi$ is achieved by the equilibrium b/n (see Condorelli and Szentes (2020)) and the lowest, equal to 0, by $\underline{q}(\pi) = \pi/(u-c)$ (see Kremer and Snyder (2018)), while all intermediate levels are achieved by equilibria where quantity ranges from $\underline{q}(\pi)$ to b/n. The achievable set is a *right triangle* defined by the following three vertexes in the monopoly profit–consumer surplus plane: $(n\pi_s, s - n\pi_s)$, $(s, 0)$, and $(n\pi_s, 0)$. See the first column of Figure 7 for an illustration.

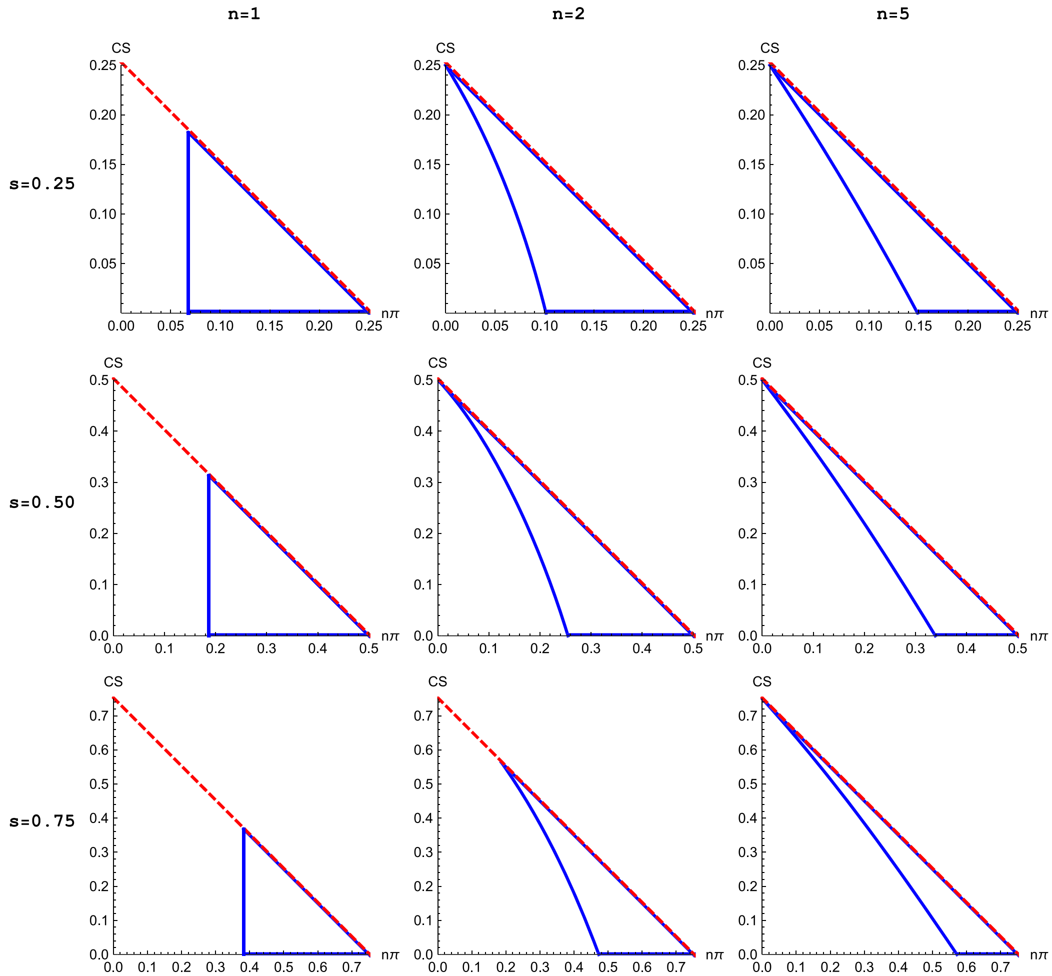

FIGURE 7. Achievable $(n\pi, \text{CS})$ couples in $\mathcal{P}_s$ within blue curves, $b = u = 1$ and $c = 0$.

To complete our analysis, we characterize next the possible levels of consumer surplus for each $\pi \in (\pi_s, \pi^s]$. To state our final result, we introduce one additional piece of notation. For $\pi \in [\pi_s, \pi^s]$, let $\hat{q}(\pi, s)$ solve $\text{FB}(P_{(\pi,\hat{q}(\pi,s))}) = s$. To see that $\hat{q}(\pi, s)$ is well defined, note first that $\text{FB}(P_{(\pi^s,\underline{q}(\pi^s))}) = s$. Then, by Lemma 1, for each $\pi \leq \pi^s$, $\text{FB}(P_{(\pi,\underline{q}(\pi))}) \leq s$ because $\underline{q}(\pi) \leq \underline{q}(\pi^s)$. Furthermore, $\text{FB}(P_{(\pi,b/n)}) \geq \text{FB}(P_s) = s$. Since $\text{FB}(P_{(\pi,q)})$ is continuous and strictly increasing in q, the intermediate value theorem implies the existence of the unique solution of $\text{FB}(P_{(\pi,\hat{q}(\pi,s))}) = s$.

PROPOSITION 4. *For $s \in (0, n\bar{\pi}]$ and $\pi \in [\pi_s, \pi^s]$, there exists $P \in \mathcal{P}_s$ and $q \in \mathcal{E}(P)$ such that $\Pi(P, q) = \pi$ and $\text{CS}(P, q) = v$ if and only if $v \in [\text{CS}(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s)), s - n\pi]$.*

PROOF. Assume by way of contradiction that $P \in \mathcal{P}_s$ and an equilibrium q of P exists such that $\Pi(P, q) = \pi \in [\pi_s, \pi^s]$ and $\text{CS}(P, q) < \text{CS}(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s))$.

There are three possibilities: either $q < \hat{q}(\pi, s)$ or $q = \hat{q}(\pi, s)$ or $q > \hat{q}(\pi, s)$. First, suppose $q < \hat{q}(\pi, s)$ and note that we must have $P \leq P_{(\pi,q)} < P_{(\pi,\hat{q}(\pi,s))}$, where the first inequality follows by Lemma 3 and the second from Lemma 1. Recalling the definition of FB, we must have $\text{FB}(P) \leq \text{FB}(P_{(\pi,q)}) < \text{FB}(P_{(\pi,\hat{q}(\pi,s))}) = s$, which contradicts $P \in \mathcal{P}_s$.

Second, if $q = \hat{q}(\pi, s)$, then $P_{(\pi,q)} = P_{(\pi,\hat{q}(\pi,s))}$. Hence, either $P = P_{(\pi,\hat{q}(\pi,s))}$ and, therefore, $\text{CS}(P, q) = \text{CS}(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s))$, a contradiction, or $P < P_{(\pi,\hat{q}(\pi,s))}$ in an interval with positive mass that gives $\text{FB}(P) < \text{FB}(P_{(\pi,\hat{q}(\pi,s))}) = s$, also a contradiction.

Third, suppose that $q > \hat{q}(\pi, s)$. Define $\hat{b}(P) = \max\{q : P(q) \geq c\}$ and note it is well defined because P is left-continuous and $u > c$. Observe that Lemma 3 implies $P \leq P_{(\pi,q)}$ and, therefore,

$$\begin{aligned}
\text{DWL}(P, q) &= \int_{nq}^{\hat{b}(P)} \big(P(x) - c\big)\, dx \\
&\leq \int_{nq}^{\hat{b}(P)} \big(P_{(\pi,q)}(x) - c\big)\, dx + \int_{\hat{b}(P)}^{b} \big(P_{(\pi,q)}(x) - c\big)\, dx \\
&= \text{DWL}(P_{(\pi,q)}, q),
\end{aligned}$$

where the second inequality follows because $P \leq P_{(\pi,q)}$ and $P_{(\pi,q)}(Q) \geq c$ for $Q \in [\hat{b}(P), b]$. Then recall from Lemma 6 that $\text{DWL}(P_{(\pi,x)}, x)$ is strictly decreasing in x and conclude that because $q > \hat{q}(\pi, s)$, we must have

$$\text{DWL}(P, q) \leq \text{DWL}(P_{(\pi,q)}, q) \leq \text{DWL}\big(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s)\big).$$

To find a contradiction with the hypothesis that equilibrium q of P generates lower consumer surplus, it is then sufficient to observe that

$$\begin{aligned}
\text{CS}\big(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s)\big) &= s - n\pi - \text{DWL}\big(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s)\big) \\
&\leq s - n\pi - \text{DWL}(P, q) = \text{CS}(P, q).
\end{aligned}$$

The proof that intermediate levels of consumer surplus can be attained is analogous to the one presented in the previous proposition. In particular, $\text{DWL}(P_{(\pi,q)}, q)$ is continuous, strictly decreasing (by Lemma 6), and goes from $s - n\pi - \text{CS}(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s))$ to 0 as x goes from $\hat{q}(\pi, s)$ to b/n. Hence, because $\text{DWL}(P^k_{\pi,q}, q) = \text{DWL}(P_{(\pi,q)}, q)$ for any k (also by Lemma 6) and $\text{CS}(P^k_{\pi,q}, q) = s - n\pi - \text{DWL}(P^k_{\pi,q}, q)$, we can conclude the proof if, for all $\pi \in [\pi_s, \pi^s]$, we can find k^q such that $\text{FB}(P^{k^q}_{\pi,q}) = s$ for all $q \in [\hat{q}(\pi, s), b/n]$. Details are omitted. □

Notably, it follows from Proposition 4 and the fact that $\text{CS}(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s))$ is concave in $\pi \in [\pi_s, \pi^s]$ that the maximal level of deadweight loss that can arise in any equilibrium is $s - n\pi^s$.

For a graphical illustration of Propositions 2, 3, and 4, Figure 7 plots the possible combinations of industry profit and consumer surplus for the case when $b = u = 1$, $c = 0$ for various values of s and n. Note that, as expected, $\hat{q}(\pi^s, s) = \pi^s/(u - c)$ and, therefore, $\mathrm{CS}(P_{(\pi^s,\hat{q}(\pi^s,s))}, \hat{q}(\pi^s, s)) = 0$. On the other hand, $\hat{q}(\pi_s, s) = b/n$ and, therefore, $\mathrm{CS}(P_{(\pi_s,\hat{q}(\pi_s,s))}, \hat{q}(\pi_s, s)) = \mathrm{CS}(P_{\pi_s}, b/n) = s - n\pi_s$. That is, as long as $\pi^s > \pi_s$ (see the second and third columns of Figure 7), there is a unique achievable level of consumer surplus at the minimal profit π_s and the equilibrium is efficient.

Inspection of Figure 7 suggests that as the number of firms increases, the maximal industry profit and the minimal consumer surplus stay constant, the *minimal* industry profit, $n\pi_s$ decreases in n, while the *maximum* consumer surplus $s - n\pi_s$ increases in n. Importantly, for each *fixed* level of industry profit, the minimum consumer surplus level achievable increases in n. These observations are stated formally below. We omit the proof.

Remark 1. $n\pi_s$ is decreasing in n and $\lim_{n\to\infty} n\pi_s = 0$, $n\pi^s$ is increasing in n and $\lim_{n\to\infty} n\pi^s = s$, and $\lim_{n\to\infty} \mathrm{CS}(P_{(\pi,\hat{q}(\pi,s))}, \hat{q}(\pi, s)) = s - n\pi$ for $0 \leq \pi < s/n$.

The remark states that inefficiency disappears as competition increases. The fact that inefficiency disappears in equilibrium implies that as the number of firms grows large, all consumers with value above marginal cost are served, no matter what the demand function is. The property that, for a given demand function, the quantity produced in equilibrium increases as the number of firm increases has been called *quasi-competitiveness*. The literature has provided a number of conditions on demand and cost function that imply quasi-competitiveness, but less has been said about convergence for large n. Notably, Amir and Lambson (2000) have shown that quasi-competitiveness holds quite generally with constant marginal cost and strictly decreasing demand functions. Our analysis contributes to this literature by showing that convergence to an efficient outcome will occur with constant marginal costs, regardless of demand.

The remark also points out that no matter how many firms produce on the market, there always exist demands and equilibria such that the industry profit remains large. The hypothesis that if the number of firms grows large, then the price goes to marginal cost and industry profit goes to zero was first put forward by Cournot himself in 1838. This property, called *convergence to a competitive equilibrium* in the literature, has been explored in a number of past papers (e.g., Frank (1965) and Ruffin (1971)). While it has been shown that increasing returns to scale may hinder convergence to competitive equilibrium, the property has been observed to hold with constant marginal cost when the demand is strictly decreasing and the lowest consumer valuation is equal to or below marginal cost. As our results emphasize, even with constant marginal costs, this property is not guaranteed to hold in general.

Appendix

We show that if there exists an asymmetric Cournot equilibrium, then there also exists a symmetric equilibrium where the total amount produced, and, therefore, industry profit, is the same. We illustrate this for the case of two firms, but the argument extends easily to multiple firms.

Suppose there exists equilibrium (q_1, q_2) with $Q = q_1 + q_2$. The following inequalities hold:

$$q_1\big[P(Q) - c\big] \geq q'\big[P\big(q' + q_2\big) - c\big] \quad \forall q'$$
$$q_2\big[P(Q) - c\big] \geq q''\big[P\big(q'' + q_1\big) - c\big] \quad \forall q''.$$

Now substitute $q' = Q/2 - q_2 + \hat{q}$ and $q'' = Q/2 - q_1 + \hat{\hat{q}}$. We can rewrite the above inequalities as

$$q_1\big[P(Q) - c\big] \geq (Q/2 - q_2 + \hat{q})\big[P(Q/2 + \hat{q}) - c\big] \quad \forall \hat{q}$$
$$q_2\big[P(Q) - c\big] \geq (Q/2 - q_1 + \hat{\hat{q}})\big[P(Q/2 + \hat{\hat{q}}) - c\big] \quad \forall \hat{\hat{q}}.$$

Summing up the two sets of inequalities we know the following must hold:

$$\begin{aligned} Q\big[P(Q) - c\big] \geq{}& (Q/2 - q_2 + \hat{q})\big[P(Q/2 + \hat{q}) - c\big] \\ &+ (Q/2 - q_1 + \hat{\hat{q}})\big[P(Q/2 + \hat{\hat{q}}) - c\big] \quad \forall \hat{q}, \hat{\hat{q}}. \end{aligned}$$

Since the above must hold for all $\hat{q}$, $\hat{\hat{q}}$, fix $\hat{q} = \hat{\hat{q}}$. The set of inequalities below must also hold:

$$Q\big[P(Q) - c\big] \geq (Q/2 - q_2 + \hat{q} + Q/2 - q_1 + \hat{q})\big[P(Q/2 + \hat{q}) - c\big] \quad \forall \hat{q}.$$

Finally, noting that $q_1 + q_2 = Q$ and dividing by 2, we get

$$Q/2\big[P(Q) - c\big] \geq \hat{q}\big[P(Q/2 + \hat{q}) - c\big] \quad \forall \hat{q},$$

which implies that there exists a symmetric equilibrium where both firms produce quantity $Q/2$.

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