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Informative tests in signaling environments

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We study a receiver's learning problem of choosing an informative test in a signaling environment. Each test induces a signaling subgame. Thus, in addition to its direct effect on the receiver's information, a test has an indirect effect through the sender's signaling strategy. We show that the informativeness of signaling in the equilibrium that a test induces depends on the relative informativeness of the test's high and low grades. Consequently, we find that the receiver's preference relation over tests needs not comply with Blackwell's (1951) order. Our findings may shed light on phenomena such as grade inflation and information coarsening.

Keywords. Signaling games, information design, strategic learning, strategic information transmission.

JEL classification. D82, D83, C72.

1. Introduction

When decision-makers gather information, they need to decide what kind of information to learn, i.e., which test to choose from their set of available tests.[1] Often, the decision-maker (henceforth the *receiver*) makes the learning decision in a signaling environment where an informed agent (henceforth the *sender*) can take observable costly actions. For example, when public certifiers, such as safety and environmental organizations, test how firms perform in a particular area, firms can signal by spending money on unproductive channels such as advertising or donations. When job market recruiters test potential candidates to evaluate their competence for the job, candidates can signal by indicating they would accept a low wage offer for their initial employment period. When academic institutions test students to identify their qualities, students can signal by participating in extracurricular activities. In this paper, we analyze the receiver's preferences over tests in such a signaling environment.

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[1] Our notion of a test is identical to the notion of an experiment in Blackwell (1951).

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We consider the following model. There is a sender who is fully informed about the state of the world, which is either low or high. There is a receiver who is initially uninformed about the state. The receiver wants to choose an action that matches the state. In the first stage of the game, the receiver chooses an informative test from a given set of feasible tests. The second stage of the game is a signaling stage à la Spence (1973), except for the following two features. First, the sender's cost of signaling does not depend on the state;[2] we thus simply say that the sender chooses a signaling cost. Second, the sender observes the test choice of the receiver but not the realization of the test (henceforth grade) before choosing his signaling cost. In the last stage of the game, after observing the test's grade and the sender's signaling cost, the receiver chooses an action. The sender's payoff is equal to the receiver's action minus his signaling cost.

In our model, since the sender's cost of signaling does not depend on the state, if the receiver does not observe a grade of an informative test about the state, then in the unique perfect sequential equilibrium Grossman and Perry (1986), the sender chooses not to signal, and the receiver does not learn any information.[3] However, if the receiver privately observes a grade of an informative test about the sender's type, then the different types of the sender face different subjective probabilities regarding the test's grades. Therefore, although the sender's payoff is independent of his type, his expected payoff depends on his type. In this case, there exists a unique equilibrium that satisfies the D1 criterion (Bank and Sobel (1987), and Cho and Kreps (1987)); see Daley and Green (2014). In this equilibrium, the low and high types of the sender may choose different signaling costs;[4] i.e., the equilibrium signaling strategy of the sender discloses information about the state. Therefore, in our model, the receiver's test choice affects the information that the receiver obtains in equilibrium in two ways. The first is through the test's intrinsic informativeness level. The second is through the informativeness level of the signaling strategy in the equilibrium that the test induces. We analyze how these two effects influence the receiver's preference relation over tests.

We start by characterizing the unique equilibrium that satisfies the D1 criterion for a given test. Depending on the prior on the state, two kinds of equilibria can emerge: one is the fully pooling equilibrium, where both types of the sender choose not to signal; the other is a semi-pooling equilibrium in which the high type selects one signaling cost while the low type mixes between selecting this cost and not signaling at all. Additionally, we find that in the latter equilibrium, the belief that the receiver develops after observing the pooling signaling cost is the belief that utilizes the test optimally in the sense that it maximizes the difference between the expected posteriors of the high type

[2]If the sender's signaling costs depend on the state and satisfy the single-crossing condition, then the case where the receiver is uninformed corresponds to Spence's (1973) signaling model. In Spence's model the fully separating equilibrium, in which the state is perfectly revealed, is the unique equilibrium that satisfies stability-based refinements. Therefore, the receiver's optimal learning strategy is not to learn information independently.

[3]In this degenerate environment, all the sender's types share the same preferences over pairs of receiver's beliefs and actions. Therefore, multiple equilibria satisfy the D1 criterion; in each of them, the sender's types are indifferent between the equilibrium actions. These equilibria are ranked in terms of efficiency. Grossman and Perry's refinement selects the efficient fully pooling equilibrium.

[4]The high (low) type corresponds to the realization of the high (low) state.

and the low type. This belief determines the informativeness level of the sender's signaling strategy in equilibrium: the higher this belief is, the more separation there is in equilibrium, and the more informative the sender's equilibrium signaling strategy is.

We identify the main property that determines the informativeness level of the sender's signaling strategy in the equilibrium that a test induces. Call a high (low) grade a grade that the high (low) type receives with a higher probability than the low (high) type. Our novel insight is that what matters is not the informativeness level of the test per se, but rather the relative informativeness of the test's high and low grades. Specifically, we argue that the more informative the test's low grades are relative to its high grades, the higher is the belief that the signaling cost induces, and the more informative the sender's signaling is. The intuition for this insight is that a test whose low grades are more informative than its high grades is essentially better at identifying the low type. Intuitively, such a test is used optimally when the interim belief is high, i.e., when the belief assigns a low probability to the event that the sender's type is low, as it manifests the test's relative advantage.

We present a series of formal results that establish this insight and study its implications for the receiver's preference relation over tests. In Lemma 4, we identify conditions under which a test that is derived from another test by increasing the informativeness of its low (high) grades induces more (less) informative signaling in equilibrium. A direct implication of this lemma is that a more informative test in the sense of Blackwell (1951) does not necessarily induce a more informative signaling strategy than a less informative test.[5] We present a sufficient condition for a test to be both intrinsically more informative and more informative in terms of the signaling strategy it induces than another test, and we show that the receiver prefers the former test to the latter independently of the prior. We then show that because more informative tests do not necessarily induce more informative signaling strategies, the receiver's preference relation over tests does not comply with Blackwell's partial order. Specifically, there are cases where the receiver prefers a less informative test to a more informative test independently of the prior.

We next analyze the receiver's preference relation over pairs of symmetric tests,[6] one whose low grades are more informative than its high grades and one whose high grades are more informative than its low grades. We show that in the absence of signaling, the receiver's preferences over these tests depend on the prior. However, when the sender can signal, we show that since the former test induces a more informative equilibrium signaling strategy than the latter, the receiver always prefers the former test to the latter, independently of the prior.

Our model corresponds to a learning problem in signaling environments with two key features. First, the sender's signaling costs are independent of the property that is

[5] In the paper, we say that a test/signal is more informative than another test/signal if and only if it dominates it in the sense of Blackwell (1951).

[6] Two tests are symmetric if, for every grade, the probability of the low (high) type to observe the grade under the first test is equal to the probability of the high (low) type to observe this grade under the second test.

the subject of interest. Second, the receiver can commit to the test.[7] The applications we mention at the beginning of this section seem to include these features, and our results may help shed light on several phenomena in the context of these applications.[8] For example, grade inflation in schools, where the vast majority of the students receive grades A and B, and only a small fraction of the students receive lower marks, is criticized for not providing sufficient information to the market about the students' qualities.[9] However, our results show that because a grade inflation policy may include low grades that are more informative than high grades, such a policy may provide more information to the market than other grading policies that are intrinsically more informative. Our results may also help explain why public certifiers deliberately coarsen the information they reveal to the public Harbaugh and Rasmusen (2018) or why employers delegate their hiring decisions to human resources companies which may be relatively less well equipped to assess the quality of job candidates.[10] The explanation we offer is that the receiver may intentionally choose a less informative test because it induces a more informative signaling strategy.

Related literature

In Spence's (1973) seminal paper about signaling, as in many subsequent signaling papers, the sender's signaling costs satisfy a single-crossing condition, such that the unique equilibrium under stability-based refinements is fully separating where the receiver gains full information. Daley and Green (2014) and Frankel and Kartik (2019) consider signaling environments with information loss. Daley and Green (2014) study a signaling environment à la Spence, where the receiver observes an exogenous test. They show that if the receiver's test is sufficiently informative, then the unique equilibrium that satisfies the D1 criterion involves some pooling.[11] Frankel and Kartik (2019) consider a signaling environment where a sender's type is two-dimensional, such that

[7]Additionally, if the realization of the grade of the test were public, then the signaling of the sender should take place before the realization of the grade. Note that in our model in which the receiver privately observes the grade, there is no restriction on the order of the signaling action and the realization of the grade.

[8]These applications include a certifier and a firm, schools and students, and an employer and a potential candidate. In these applications, it seems natural that the receiver can commit to the test but not to her action. Moreover, in many cases, it is natural to presume that the signaling cost of the sender is independent of the property that is the subject of interest. That is, that the monetary loss of a firm on advertising or donation does not depend on its quality, that a candidate's disutility from a lower salary is not related to his competence level, and that the cost of participation in extracurricular activities is independent of a student's academic level.

[9]The phenomenon of grade inflation is well documented. Several papers present models that predict grade inflation as the outcome of strategic interaction between competing universities; see, e.g., Chan, Hao, and Suen (2007), Popov and Bernhardt (2013), and Boleslavsky and Cotton (2015).

[10]We thank an anonymous referee for suggesting this example.

[11]Alós-Ferrer and Prat (2012) study a similar question. They find that in the presence of a test, pooling equilibria can satisfy the intuitive criterion Cho and Kreps (1987). Other papers analyze signaling environments that are different from Spence (1973) where the receiver is exposed to additional information; see, e.g., Weiss (1983) and Feltovich, Harbaugh, and To (2002).

one dimension corresponds to the intrinsic value of the sender and the other dimension corresponds to his ability to signal. They show that, generically, there is some loss of information in equilibrium.

Ball (2021) studies an information design problem of choosing a scoring rule, a mapping from the sender's action to a distribution of scores, in a multi-feature extension of Frankel and Kartik's (2019) model.[12] Specifically, he shows that a less informative scoring rule may induce a more informative sender's signaling strategy and so the receiver can obtain more information in equilibrium by coarsening the scoring rule. Bonatti and Cisternas (2020) study a scoring rule design problem in a dynamic monopolistic screening setting and find a qualitatively similar result. In this paper, we study the information design problem of choosing an informative test in a variation of Daley and Green's (2014) model and find a related qualitative result: a less informative test may induce a more informative signaling strategy and, thus, may ultimately lead to more information revelation in equilibrium. However, the mechanisms by which scoring rules and tests affect the sender's signaling strategy are different. A scoring rule affects the sender's signaling strategy by altering the way the receiver observes the chosen signaling cost, whereas a test affects the sender's signaling strategy by modifying the sender's types preferences over the receiver's interim belief, i.e., her belief following the signaling action. Specifically, in tests, unlike in scoring, different sender's types face different expected posterior beliefs for the same signaling cost. This property enables meaningful signaling in environments where the sender's payoff is state-independent.

Our work joins other papers that deal with environments where a receiver's test choice affects the strategic behavior of an informed sender. Rosar (2017) and Harbaugh and Rasmusen (2018) both study a receiver's optimal test choice in environments where the sender can decide on whether or not to participate in the test. The information that the receiver obtains from a test also depends on the way it affects the sender's participation strategy. Rosar (2017) considers an environment with a risk-averse sender who is partially informed about whether his quality is high or low. Harbaugh and Rasmusen (2018) consider an environment where the sender is fully informed about his quality and incurs an exogenous fixed cost from participating in the test. Both papers find that the receiver's optimal test uses coarse grading to increase participation. Boleslavsky and Kim (2021) study the problem of Bayesian persuasion, where a sender chooses a test to affect a receiver's behavior, in environments where the underlying state is generated by another agent's unobservable effort. The sender's test affects the receiver's information in equilibrium further by affecting the strategic behavior of the agent. Hence, the sender's optimal test also depends on how the test incentivizes the agent's effort in the equilibrium it induces.

[12] Frankel and Kartik (2021) consider the case where the receiver can commit to her action following each of the sender's signaling costs in Frankel and Kartik's (2019) setting. They show that the receiver finds it optimal to commit to taking a more moderate action than her best response following each of the sender's signaling costs. Whitmeyer (2021) compares the receiver's payoff from the optimal scoring rule and her maximal payoff when she can commit to her actions and characterizes conditions under which the optimal scoring rule yields the same payoff for the receiver as in the commitment case.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we consider the possible equilibria in the sequential subgame that a test induces. In Section 4, we show how the properties of a test affect the informativeness level of the sender's signaling strategy in equilibrium and study the implications of this result on the receiver's preferences over tests. Section 5 is devoted to a discussion. Section 6 concludes. Most of the proofs are relegated to the Appendix.

2. The model

2.1 *The environment*

There is a sender (he) who either has a low value (henceforth the *low type*) or a high value (henceforth the *high type*). The sender's type is high with a prior probability of $\mu_0 \in (0, 1)$. For simplicity, we assume that the values of the low type and the high type are 0 and 1, respectively, and identify the sender's type with its value.[13] The sender knows his type, $\omega \in \{0, 1\}$, and can signal by taking a costly observable action $c \in \mathbb{R}_+$ (henceforth *signaling cost*). There is a receiver (she) who chooses a test $\pi : \{0, 1\} \to \Delta S$ from a finite set of feasible tests $\Pi^R \subseteq \Pi$, and an action, $a \in \mathbb{R}$.[14] We assume that Π is the set of all partially informative tests; i.e., the set of all tests that are informative but not fully informative. The receiver's payoff as a function of the sender's type and her action is $U^R(\omega, a) = -(\omega - a)^2$. The sender's payoff is the receiver's action minus his signaling cost, $U^S(\omega, a, c) = a - c$. Note that the sender's payoff does not depend directly on his type. The receiver and both of the sender's types act to maximize their expected payoffs.

2.2 *Time line*

The game consists of three periods. In the first period, the receiver chooses a test $\pi \in \Pi^R$. In the second period, the sender observes the receiver's test choice and decides on a signaling cost $c \in \mathbb{R}_+$. The receiver observes the sender's signaling cost and forms an interim belief about the sender's type, denoted by $\mu_\pi(c) \in [0, 1]$. In the third period, the receiver observes the test's grade $s \in S$, forms a posterior belief about the sender's type, and takes an action $a \in \mathbb{R}$.

2.3 *Notations*

To simplify the exposition, we introduce a series of notations. Since the type space is binary, we identify a belief $\mu \in \Delta\{0, 1\}$ with a number $\mu \in [0, 1]$ that corresponds to the probability that the belief μ assigns to the event that the sender's type is 1, which is also the expected value given the belief. We denote by $\text{pos}_\pi(s, \mu) \in [0, 1]$ the posterior belief given a test $\pi \in \Pi$, an initial belief $\mu \in [0, 1]$, and a grade $s \in S$ that is derived using Bayes's

[13] All our results hold for arbitrary values of the high type and the low type h and l with $h > l$.

[14] We assume without loss of generality that all tests have the same grade set S, as we can always define S to be the union over all the possible grades in the set of feasible tests. We assume that Π^R is finite to ensure the existence of an optimal test.

rule whenever possible,[15] i.e.,

$$\text{pos}_\pi(s, \mu) := \frac{\mu \cdot \pi(s|1)}{(1-\mu) \cdot \pi(s|0) + \mu \cdot \pi(s|1)}.$$

We denote by $V_\pi^R(\mu)$ the expected payoff of the receiver from a test $\pi \in \Pi$ and an initial belief μ when she chooses her action to be equal to the expected value of the sender given her posterior belief, i.e.,

$$V_\pi^R(\mu) := -\left(\mu \cdot \sum_{s \in S} \pi(s|1)\big(1 - \text{pos}_\pi(s, \mu)\big)^2 + (1-\mu) \cdot \sum_{s \in S} \pi(s|0)\big(-\text{pos}_\pi(s, \mu)\big)^2\right).$$

We denote by $V_\pi^\omega(\mu)$ the expected value of the receiver's posterior belief from the perspective of the sender of type ω, $\omega \in \{0, 1\}$, from a test $\pi \in \Pi$ and an initial belief μ, i.e.,

$$V_\pi^\omega(\mu) := \sum_{s \in S} \pi(s|\omega) \cdot \text{pos}_\pi(s, \mu).$$

We denote by $U_\pi^\omega(\mu, c)$ the expected payoff of the sender of type ω, $\omega \in \{0, 1\}$, from a test $\pi \in \Pi$, an initial belief μ, and a signaling cost c when the receiver chooses her action to be equal to the expected value of the sender given her posterior belief, i.e.,

$$U_\pi^\omega(\mu, c) \equiv V_\pi^\omega(\mu) - c.$$

2.4 *Strategies and equilibrium*

We now define the players' strategies and the equilibrium concept we use.

Definition 1. A *strategy for the sender* is a mapping $\sigma : \{0, 1\} \times \Pi^R \to \Delta\mathbb{R}_+$ that assigns to each pair of type $\omega \in \{0, 1\}$ and test $\pi \in \Pi^R$ a probability distribution over the possible signaling costs.

We let $\sigma_\pi(c|\omega)$ denote the probability that $\sigma(\pi, \omega)$ assigns to the signaling cost c, and $\text{supp}(\sigma(\pi)) \equiv \text{supp}(\sigma(\pi, 0)) \cup \text{supp}(\sigma(\pi, 1))$.

Definition 2. The *first-period strategy of the receiver* is a choice of test $\pi \in \Pi^R$. The *third-period strategy of the receiver* is a mapping $a : \Pi^R \times S \times \mathbb{R}_+ \to \Delta\mathbb{R}$ that assigns to each tuple of test $\pi \in \Pi^R$, grade $s \in S$, and signaling cost $c \in \mathbb{R}_+$ a probability distribution over the receiver's possible actions. A *strategy for the receiver* is a pair $\{\pi, a(\cdot, \cdot, \cdot)\}$.

Every test $\pi \in \Pi^R$ induces a signaling subgame. Our equilibrium concept requires that the sender's strategy, the receiver's third-period strategy, and the receiver's interim

[15] The only cases where $\text{pos}_\pi(s, \mu_\pi(c))$ cannot be obtained using Bayes' rule is when $\pi(s|1) = 0$ ($\pi(s|0) = 0$), $\pi(s|0) > 0$ ($\pi(s|1) > 0$), and $\mu = 1$ ($\mu = 0$). That is, cases where the receiver observes a grade that only one of the types can get when the receiver's initial belief is that the sender's type is the other type with certainty. In this case, the only reasonable conclusion is that the initial belief is incorrect, i.e., $\text{pos}_\pi(s, 1) = 0$ ($\text{pos}_\pi(s, 0) = 1$).

beliefs must form a perfect Bayesian equilibrium (henceforth PBE) that satisfies the D1 criterion (Bank and Sobel (1987), Cho and Kreps (1987)) in every subgame that is induced by any test $\pi \in \Pi^R$. Specifically, since the receiver's loss function is quadratic, then, in equilibrium, the unique action that the receiver selects is equal to the sender's expected value given her posterior belief from observing the signaling cost and the test's realized grade, which is derived using Bayes' rule. The receiver's first-period strategy must choose a test $\pi^* \in \Pi^R$ that induces a subgame whose equilibrium outcome maximizes the receiver's expected payoff.

Definition 3. We say that a pair of strategies $\{\sigma^*(\cdot,\cdot), \{\pi^*, a^*(\cdot,\cdot,\cdot)\}\}$ and a system of interim beliefs $\{\mu_\pi(c)\}_{\pi\in\Pi^R, c\in\mathbb{R}_+}$ form an *equilibrium* if and only if the following conditions hold:

(i) If $c \in \operatorname{supp}(\sigma^*(\pi))$, then the receiver's interim belief $\mu_\pi(c)$ is obtained from μ_0 using Bayes' rule, i.e.,

$$\mu_\pi(c) = \frac{\mu_0 \cdot \sigma_\pi(c \mid 1)}{\mu_0 \cdot \sigma_\pi(c \mid 1) + (1-\mu_0)\cdot \sigma_\pi(c \mid 0)}.$$

If $c \notin \operatorname{supp}(\sigma^*(\pi))$, then the interim belief $\mu_\pi(c)$ must satisfy the D1 criterion, as defined below.

(ii) For every $\pi \in \Pi$, $s \in S$, and $c \in \mathbb{R}_+$ we have that

$$a^*(\pi, s, c) = \operatorname{pos}_\pi\big(s, \mu_\pi(c)\big).$$

(iii) If $c \in \operatorname{supp}(\sigma^*(\pi, \omega))$, then

$$c \in \operatorname*{argmax}_{c'\in\mathbb{R}_+} U_\pi^\omega\big(\mu_\pi(c'), c'\big).$$

(iv) The test π^* satisfies

$$\pi^* \in \operatorname*{argmax}_{\pi\in\Pi^R} \sum_{c\in\operatorname{supp}(\sigma^*(\pi))} \big(\mu_0\cdot\sigma_\pi^*(c|1) + (1-\mu_0)\cdot\sigma_\pi^*(c|0)\big)\cdot V_\pi^R\big(\mu_\pi(c)\big).$$

In condition (iv) of the definition we implicitly assume that $|\operatorname{supp}(\sigma^*(\pi))| < \infty$. This assumption is valid because, as we show in the following section, for every $\pi \in \Pi$ in any D1 equilibrium, it holds that $|\operatorname{supp}(\sigma^*(\pi))| < \infty$. The D1 criterion that is mentioned in condition (i) of the definition is defined as follows.

Definition 4. Consider a PBE of the subgame that is induced by some $\pi \in \Pi$ with equilibrium strategies $\{\sigma^*(\pi,\cdot), a^*(\pi,\cdot\cdot)\}$. For each $\omega \in \{0,1\}$, consider some $c_\omega \in \operatorname{supp}(\sigma^*(\pi,\omega))$. For every $c \notin \operatorname{supp}(\sigma^*(\pi))$ and $\omega \in \{0,1\}$, define $B^\omega(c) := \{\mu | U_\pi^\omega(\mu, c) > U_\pi^\omega(\mu_\pi(c_\omega), c_\omega)\}$. The D1 criterion requires that if $B^{\omega'}(c) \subset B^\omega(c)$, then $\mu_\pi(c) = \omega$.

The intuition behind the widely used D1 criterion is the following. Given an equilibrium, once the receiver observes a signaling cost that is not an element of the support of each of the types' equilibrium strategies, she cannot use Bayes' rule to infer her belief. Instead, the receiver develops her belief as follows. For each type, the receiver computes the set of beliefs that would keep this type's expected payoff given the cost higher than its equilibrium expected payoff. If some type's set is contained in another type's set, then the D1 criterion requires that this type would not be an element in the belief's support. The economic interpretation is that the posterior belief after observing the cost excludes types that are dominated by other types in terms of their motivation to deviate to this cost.

3. The second-period subgame

In this section, we analyze the equilibria in the subgame that is induced by a test $\pi \in \Pi$. We start with a preliminary stage in which we show that for every test $\pi \in \Pi$, there exists a unique initial belief that maximizes the difference between the expected values of the receiver's posterior beliefs from the perspective of the sender's high type and low type.

Lemma 1. *For every $\pi \in \Pi$, there exists a unique initial belief $\mu^*_\pi \in [0, 1]$ such that for every $\mu \in [0, 1]$ with $\mu \neq \mu^*_\pi$, it holds that $V^1_\pi(\mu^*_\pi) - V^0_\pi(\mu^*_\pi) > V^1_\pi(\mu) - V^0_\pi(\mu)$.*

We prove the lemma by exploiting general properties of Bayesian updating that allow us to show that the difference between the expected posteriors of the high type and the low type is strictly concave in the initial belief. Henceforth, we will refer to μ^*_π as the "dividing belief of test π," as it maximally separates the expected posteriors of the high type and low type. In Section 4, we analyze which properties of the test are linked to its dividing belief. In this section, we establish a connection between a test's dividing belief and the set of equilibria in the subgame it induces. First, we restate a result of Daley and Green (2014) that shows that, generically, for any informative test, there exists a unique PBE that satisfies the D1 criterion in the subgame that this test induces. This result allows us to identify each test $\pi \in \Pi$ with a unique equilibrium outcome.

Proposition 1 (Daley and Green (2014)). *Consider a test $\pi \in \Pi$. There exists a unique interim belief $\mu_\pi \in [0, 1]$ such that in the subgame that the test π induces, the following is true:*

(i) If $\mu_0 < \mu_\pi$, then there exists a unique equilibrium. In equilibrium, type 0 selects the signaling cost 0 with probability $\alpha := \frac{(\mu_\pi - \mu_0)}{(\mu_\pi(1-\mu_0))}$ and the signaling cost $c = V^0_\pi(\mu_\pi)$ with probability $1 - \alpha$, and type 1 selects the signaling cost $c = V^0_\pi(\mu_\pi)$ with probability 1.

(ii) If $\mu_0 > \mu_\pi$, then there exists a unique equilibrium. In equilibrium, both types pool and select the signaling cost 0.

(iii) If $\mu_0 = \mu_\pi$, then the set of equilibria are the fully pooling equilibria in which both types select the same signaling cost, and this cost is an element of the set $[0, V^0_\pi(\mu_\pi)]$.

As Proposition 1 shows, when $\mu_0 \neq \mu_\pi$, there exists a unique equilibrium in the subgame that the test π induces. When $\mu_0 = \mu_\pi$, there are multiple fully pooling equilibria, and the receiver's expected payoff is the same in all these equilibria. Since we are focusing on the test choice of the receiver, we identify an equilibrium outcome with the expected payoff it provides to the receiver. Accordingly, we say that a test π induces a subgame with a unique equilibrium outcome even in the case where $\mu_0 = \mu_\pi$. The next lemma shows that given a test $\pi \in \Pi$, there is a clear connection between the characterization of equilibria (Proposition 1) and the dividing belief μ^*_π.

Lemma 2. *For every $\pi \in \Pi$, it holds that $\mu_\pi = \mu^*_\pi$.*

We give here a short sketch of the proof to convey the mechanics of the model. Assume by way of contradiction that the dividing belief is feasible, i.e., that $\mu^*_\pi \geq \mu_0$, and that the high type selects with positive probability some signaling cost that induces, in equilibrium, an interim belief lower than the dividing belief. First, it must be that the low type also selects this signaling cost with positive probability, as otherwise, the belief given this signaling cost in equilibrium would be 1. Now consider a higher signaling cost such that given the dividing belief, the low type's payoff is the same as its payoff from selecting the original signaling cost. By the definition of the dividing belief, it must be that the high type's payoff given this cost and the dividing belief is higher than its payoff under the original signaling cost. It follows from the D1 criterion that the belief after such a deviation must be that the type that deviated is the high type. Clearly, this means that such a deviation is profitable (at least for the high type); thus, we get a contradiction. An analogous argument proves that no signaling cost that induces an interim belief that is higher than the dividing belief satisfies the D1 criterion.

Lemma 2 and Proposition 1 provide us with the following characterization of the equilibria in the subgame that a test $\pi \in \Pi$ induces: if $\mu^*_\pi \geq \mu_0$, then in the unique equilibrium, the high type selects a pure strategy of choosing a signaling cost that induces, in equilibrium, the dividing belief, μ^*_π. This equilibrium maximizes the high type's payoff among all equilibria in which the low type's payoff is zero.[16] This characterization allows us to frame the equilibrium analysis in the subgame that a test induces in the following way: When the dividing belief is strictly larger than the prior, the good type is choosing a signaling cost to implement the equilibrium that, from its perspective, optimally uses the test.

4. Receiver's preferences over tests

We now study how the sender's ability to signal affects the receiver preferences over tests. In a signaling environment, the receiver's preference relation over tests depends not only

[16] This equilibrium need not necessarily be the optimal equilibrium for the high type if we do not restrict the equilibrium payoff of the low type to be zero. It could be the case that the fully pooling PBE in which both types do not signal provides the high type with a higher payoff than the equilibrium that D1 selects. The reason is that D1 selects the equilibrium that maximizes the difference between the equilibrium payoffs of the high type and the low type. However, while the payoff of the low type is zero whenever the equilibrium that D1 selects involves some separation, it is greater than zero in the fully pooling equilibrium with no signaling.

on the intrinsic informativeness of the tests, but also on how tests incentivize the sender to signal. Our results in the previous section imply that the informativeness level of the sender's signaling strategy in the equilibrium that a test induces depends on its dividing belief: the higher the test's dividing belief is, the more informative the signaling strategy is. In this section, we present the main insight of the paper: the key property that determines the test's dividing belief is the relative informativeness of its low and high grades.[17] Specifically, the more informative the low grades are relative to the high grades, the higher the dividing belief is and the more information arises through the signaling channel. Intuitively, a test whose low grades are more informative than its high grades is better at identifying the low type. Therefore, to use such a test optimally, the high type chooses a signaling cost that induces a belief that places a small probability on the low type to manifest the test's relative advantage.

We establish this idea through a series of formal results. These results also illustrate the tendency of the receiver to prefer tests whose low grades are relatively more informative than their high grades, as such tests incentivize more aggressive signaling by the sender. We start with the following lemma that presents a condition that determines the test's dividing belief.

Lemma 3. *Let $\pi \in \Pi$ and let $\mu \in [0, 1]$ be an initial belief. Then $\mu^*_\pi \lesseqgtr \mu$ if and only if*

$$\sum_{s\in S} \pi(s|0) \cdot \left(\frac{1}{2} - \text{pos}_\pi(s, \mu)\right)^2 \lesseqgtr \sum_{s\in S} \pi(s|1) \cdot \left(\frac{1}{2} - \text{pos}_\pi(s, \mu)\right)^2.$$

To deduce the condition in Lemma 3, we exploit the property established in Lemma 1 and Lemma 2 that $V^1_\pi(\mu) - V^1_\pi(\mu)$ is maximized when $\mu = \mu^*_\pi$. The condition in Lemma 3 is general in the sense that it applies to every possible test. Thus, it is not straightforward to interpret. Nonetheless, one can interpret it in a way that is consistent with our main idea. To see this, note that if the prior is equal to $\frac{1}{2}$, then the sum of the right-hand side and the left-hand side of the inequality is equal to the variance of the posteriors' distribution. It follows from the condition in Lemma 3 that the dividing belief is above (below) $\frac{1}{2}$ if and only if the low (high) type contributes to the variance more than the high (low) type. Roughly speaking, since the low (high) type is more exposed to the low (high) grades, its contribution to the variance is greater when the low (high) grades are more informative. In what follows, we use this rather technical condition to establish our main idea in a starker way. To this end, we first introduce the family of binary tests.

Definition 5. We say that a test π is *binary* if it has only two grades. We denote by h the grade that the high type is more exposed to, i.e., $\pi(h|1) > \pi(h|0)$, and by l the grade that the low type is more exposed to, i.e., $\pi(l|0) > \pi(l|1)$.

We now introduce a way to decompose a test π into three tests:

[17] A high (low) grade is a grade that the high (low) type receives with a higher probability than the low (high) type.

- $\tilde{\pi}_\beta(\pi):\{0,1\}\to\Delta\{h,l\}$, where $\tilde{\pi}_\beta(\pi)(h|1)\geq\ \ \tilde{\pi}_\beta(\pi)(h|0)$
- $\tilde{\pi}_h(\pi):\{0,1\}\to\Delta S^+_\pi$, where $S^+_\pi:=\{s\in\text{supp}(\pi)\mid\pi(s|1)>\pi(s|0)\}$
- $\tilde{\pi}_l(\pi):\{0,1\}\to\Delta S^-_\pi$, where $S^-_\pi:=\{s\in\text{supp}(\pi)\mid\pi(s|1)<\pi(s|0)\}$.

It is useful to think of the decomposition of a test π as a sequential procedure that works as follows: First we apply the binary test $\tilde{\pi}_\beta(\pi)$ and then, if the grade is h, we apply the test $\tilde{\pi}_h(\pi)$ and if the grade is l, we apply the test $\tilde{\pi}_l(\pi)$.[18] In the rest of the analysis, we denote Blackwell's partial order over tests by $\succ_B$ and the receiver's preference relation over tests by $\succ_R$. Note that the receiver's preference relation depends, in principle, on the prior μ_0.

The next lemma uses our decomposition and Lemma 3 to directly establish a connection between a test's dividing belief and the relative informativeness of its high and low grades.

Lemma 4. *Consider a test π and a test π' such that $\tilde{\pi}_\beta(\pi)=\tilde{\pi}_\beta(\pi')$:*

(i) If $\mu^*_\pi>\frac{1}{2}$, $\tilde{\pi}_l(\pi)=\tilde{\pi}_l(\pi')$, and $\tilde{\pi}_h(\pi')\succ_B\tilde{\pi}_h(\pi)$, then $\mu^*_{\pi'}<\mu^*_\pi$.

(ii) If $\mu^*_\pi<\frac{1}{2}$, $\tilde{\pi}_h(\pi)=\tilde{\pi}_h(\pi')$, and $\tilde{\pi}_l(\pi')\succ_B\tilde{\pi}_l(\pi)$, then $\mu^*_{\pi'}>\mu^*_\pi$.

Lemma 4 shows that the value of a test's dividing belief depends not on the informativeness of the test, but rather on the relative informativeness of its high and low grades. To see this, note that the test π' is more informative than the test π. Condition (i) shows that when the test π's dividing belief is high and when the test π' increases the relative informativeness of the high grades, we get that $\mu^*_{\pi'}<\mu^*_\pi$; i.e., the signaling strategy is less informative under the more informative test π'. On the other hand, condition (ii) shows that when the test π's dividing belief is low and when π' increases the relative informativeness of the low grades, we get that $\mu^*_{\pi'}>\mu^*_\pi$; i.e., the signaling strategy is more informative under the more informative test π'. The next proposition formalizes the subtle positive connection that condition (ii) captures by providing a sufficient condition for the receiver to prefer the more informative test for every prior.

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[18] For clarity we give here the formal definition of each test:

$$\tilde{\pi}_\beta(\pi)(h|1)=\sum_{s\in S^+_\pi}\pi(s|1)\qquad\tilde{\pi}_\beta(\pi)(l|1)=\sum_{s\in S^-_\pi}\pi(s|1)$$

$$\tilde{\pi}_\beta(\pi)(h|0)=\sum_{s\in S^+_\pi}\pi(s|0)\qquad\tilde{\pi}_\beta(\pi)(l|0)=\sum_{s\in S^-_\pi}\pi(s|0)$$

$$\tilde{\pi}_h(\pi)\big(\hat{s}\in S^+_\pi|1\big)=\frac{\pi\big(\hat{s}\in S^+_\pi|1\big)}{\sum_{s\in S^+_\pi}\pi(s|1)}\qquad\tilde{\pi}_h(\pi)\big(\hat{s}\in S^+_\pi|0\big)=\frac{\pi\big(\hat{s}\in S^+_\pi|0\big)}{\sum_{s\in S^+_\pi}\pi(s|0)}$$

$$\tilde{\pi}_l(\pi)\big(\hat{s}\in S^-_\pi|1\big)=\frac{\pi\big(\hat{s}\in S^-_\pi|1\big)}{\sum_{s\in S^-_{pi}}\pi(s|1)}\qquad\tilde{\pi}_l(\pi)\big(\hat{s}\in S^-_\pi|0\big)=\frac{\pi\big(\hat{s}\in S^-_\pi|0\big)}{\sum_{s\in S^-_\pi}\pi(s|0)}.$$

Proposition 2. *Assume that π and π' satisfy condition (ii) of Lemma 4; then for any prior $\mu_0 \in (0, 1)$, it holds that $\pi' \succ_R \pi$.*

In the proof, we show that for every prior, the effective signal that π' induces, i.e., the signal that incorporates the information that arises from both the signaling channel and the test, is more informative than the effective signal that π induces. This relation implies that any receiver would prefer π' to π, independently of her loss function. Specifically, it is true for our receiver, whose loss function is quadratic. To prove the above order in the informativeness level of the effective signals, we use Lemma 4, which implies that for every prior, the signal that arises from the signaling channel under π' is (weakly) more informative than the one that arises from the signaling channel under π. We complete the proof by combining this result with the fact that under the conditions of the proposition, the test π' is more informative than the test π.

4.1 *Incompliance with Blackwell's order*

Condition (i) of Lemma 4 shows that a test that is more informative in the sense of Blackwell (1951) may induce an equilibrium with a less informative signaling strategy. Thus, a natural question that arises is whether or not the receiver's preference relation complies with Blackwell's partial order. That is, if two tests are ranked according to Blackwell's order, is it true that the receiver prefers the more informative test to the less informative test for every prior? In this subsection, we show that this is not the case. Moreover, we construct an example with two tests that are ranked according to Blackwell's order where, due to signaling considerations, the receiver prefers the less informative test to the more informative test independently of the prior. To construct the example, we first introduce two special subsets of binary tests. The first subset consists of binary tests that do not admit false negatives and the second subset consists of binary tests that do not admit false positives. The following lemma shows that each test in the first subset induces a fully separating equilibrium, while each test in the second subset induces a fully pooling equilibrium. Besides their role in the example we construct, these subsets of tests are yet another manifestation of our main idea as the low (high) grade is more informative than the high (low) grade in tests that do not admit false negatives (positives).

Lemma 5. *Let $\pi \in \Pi$ be a binary test:*

- $\mu^*_\pi = 1$ *if and only if* $\pi(h|1) = 1$
- $\mu^*_\pi = 0$ *if and only if* $\pi(l|0) = 1$.

The main argument in the proof is the following. In a binary test π that does not admit false negatives (positives), the posterior belief of the low (high) grade identifies type 0 (1) with certainty. Hence, the only posterior that depends on the initial belief μ is that of the high (low) grade, which type 1 (0) is more likely to receive. Therefore, we get that $V^1_\pi(\mu)$ ($V^0_\pi(\mu)$) is more sensitive to an increase in the initial belief than $V^0_\pi(\mu)$ ($V^1_\pi(\mu)$) for every $\mu \in [0, 1)$. This implies that $V_\pi^1(\mu) - V_\pi^0(\mu)$ is an increasing (a decreasing) function in $[0, 1]$, which implies that $\mu_\pi^* = 1$ ($\mu_\pi^* = 0$).

Lemma 5 shows that each binary test that does not admit false negatives induces an equilibrium with a fully informative signaling strategy and so it is optimal for the receiver independently of the prior. Note that there exist binary tests that do not admit false negatives that are almost not informative at all.

Equipped with Lemma 5, we return to the relationship between the receiver's preferences and Blackwell's order. The following proposition shows that the receiver's preference relation does not comply with Blackwell's order.

Proposition 3. *There exist two tests π and π' such that $\pi' \succ_B \pi$ and for every prior $\mu_0 \in (0, 1)$, it holds that $\pi \succ_R \pi'$.*

Proof. We construct an example that proves the proposition. Consider a test π' with grades {l, hL, hH} and the probability functions

$$\pi'(\mathrm{l}|1) = 0 \qquad \pi'(\mathrm{l}|0) = 1 - \beta$$
$$\pi'(\mathrm{hL}|1) = \gamma \qquad \pi'(\mathrm{hL}|0) = \beta$$
$$\pi'(\mathrm{hH}|1) = 1 - \gamma \qquad \pi'(\mathrm{hH}|0) = 0,$$

where $\beta > \gamma > 0$. In this test, only the grade hL is partially informative; i.e., it is the only grade that is affected by the value of μ. Now, $\beta > \gamma$, i.e., type 0 is more likely to receive the grade hL than type 1. Therefore, the function $V_{\pi'}^1(\mu) - V_{\pi'}^0(\mu)$ is strictly decreasing in $[0, 1]$, which implies that $\mu_{\pi'}^* = 0$. Now consider a binary test π with the grades {l, h} and the probability functions

$$\pi(\mathrm{l}|1) = 0 \qquad \pi(\mathrm{l}|0) = 1 - \beta$$
$$\pi(\mathrm{h}|1) = 1 \qquad \pi(\mathrm{h}|0) = \beta.$$

The test π is binary and does not admit false negatives. Therefore, by Lemma 5, we get that $\mu_\pi^* = 1$. Since the test π induces a fully separating equilibrium, the effective signal it induces, i.e., the signal that includes the information that arises through both the signaling channel and the test, is fully informative. On the other hand $\mu_{\pi'}^* = 0$ implies that $\mu_{\pi'}^* < \mu_0$ for every prior $\mu_0 \in (0, 1)$. Therefore, the effective signal that the test π' induces is identical to applying the test π' on the prior μ_0. Since the test π' includes a partially informative grade, we get that for every prior $\mu_0 \in (0, 1)$, the effective signal that the test π' induces is not fully informative. We conclude that for every prior $\mu_0 \in (0, 1)$, the test π' provides a strictly lower payoff to the receiver than the test π. Moreover, the effective signal that is induced by the test π Blackwell dominates the effective signal that is induced by the test π'. Now we show that the test π' is more informative than the test π in the sense of Blackwell. To see this, note that the test π' is constructed by activating the test π and then, conditional on the realization of the grade h, activating a binary test π''

with grades {L,H} and probability functions

$$\pi''(\mathrm{L}|1) = \gamma \qquad \pi''(\mathrm{L}|0) = 1$$
$$\pi''(\mathrm{H}|1) = 1-\gamma \qquad \pi''(\mathrm{H}|0) = 0. \qquad \square$$

4.2 *Symmetric tests*

In this subsection, we want to further establish the point that the receiver tends to prefer tests whose low grades are more informative to tests whose high grades are more informative. To do so, we compare the receiver's preferences over pairs of symmetric tests, one whose low grades are more informative than its high grades and one whose high grades are more informative than its low grades. We show that in the absence of signaling, the receiver prefers the latter test to the former when the prior is smaller than $\frac{1}{2}$ and the former test to the latter when the prior is greater than $\frac{1}{2}$. However, in a signaling environment, the receiver prefers the former test to the latter, independently of the prior. In our analysis, we concentrate on a subset of tests that consist of tests that are convex combinations of binary tests. We use the following definitions and notations.

Definition 6. Tests π and π' are *symmetric* if for every $s \in S$, we have that $\pi(s|1) = \pi'(s|0)$ and $\pi(s|0) = \pi'(s|1)$. We denote by $\hat{\pi}$ the symmetric test of π.

Definition 7. We say that a binary test is *L-informative* (*H-informative*) if $\pi(h|1) > \pi(l|0)$ ($\pi(h|1) < \pi(l|0)$) and that it is *N-informative* if $\pi(h|1) = \pi(l|0)$.

We now introduce the notion of a convex combination of binary tests: given a set of binary tests $\{\pi_1, \dots, \pi_k\}$ and a set of real numbers $\{p_1, \dots, p_k\}$ such that $p_i > 0$ for every $i \in \{1, \dots, k\}$ and $\sum_{i=1}^k p_i = 1$, we denote by $\oplus_{i=1}^k p_i \pi_i$ the test such that with probability p_i, the receiver observes a realization of the test π_i.

Definition 8. We say that a test $\bigoplus_{i=1}^k p_i \pi_i$ is *L-informative* (*H-informative*) if π_i is either L-informative (H-informative) or N-informative for every $i \in \{1, \dots, k\}$, and there exists $j \in \{1, \dots, k\}$ for which the test π_j is L-informative (H-informative). We say that the test π is *N-informative* if π_i is N-informative for every $i \in \{1, \dots, k\}$. We denote the union over all tests that are L-informative, N-informative, and H-informative by Π^d.

We are now ready to formulate a lemma that describes a connection between L-informative/H-informative tests and their dividing beliefs.

Lemma 6. *If $\pi \in \Pi^d$ is an L-informative (H-informative) test, then $\mu^*_\pi > \frac{1}{2}$ ($\mu^*_\pi < \frac{1}{2}$), and if $\pi \in \Pi^d$ is an N-informative test, then $\mu^*_\pi = \frac{1}{2}$.*

We prove the lemma by directly using Lemma 3. Intuitively, since an L-informative (H-informative) test has an advantage at identifying the low (high) type, the high type sets the initial probability of the low (high) type to be lower than $\frac{1}{2}$ to utilize the test optimally.

It is easy to see that if a test is L-informative, then its symmetric test is H-informative and vice versa. In the following proposition, we compare the receiver's preferences over such pairs of L-informative and H-informative symmetric tests in environments with and without signaling.

Proposition 4. *Consider an L-informative test π and its symmetric (H-informative) test $\hat{\pi}$.*

- *In an environment without signaling, if $\mu_0 < \frac{1}{2}$, then $\hat{\pi} \succ_R \pi$, and if $\mu_0 > \frac{1}{2}$, then $\pi \succ_R \hat{\pi}$.*
- *In an environment with signaling, for every $\mu_0 \in (0, 1)$, we have that $\pi \succ_R \hat{\pi}$.*

Proposition 4 shows that in an environment without signaling, the receiver prefers the H-informative (L-informative) test if the prior is lower (greater) than $\frac{1}{2}$; i.e., the receiver's preference over symmetric tests is symmetric. The intuition for this result is the same intuition that accompanies Lemma 6. An H-informative (L-informative) test is better at identifying the high (low) type. Therefore, it provides more information to the receiver when the initial probability of the high (low) type is low. However, in a signaling environment, the symmetry breaks, i.e., the receiver prefers the L-informative test independently of the prior. The intuition for this result is that although the tests are symmetric in terms of the direct information they provide, they are not symmetric in terms of the information they provide through the signaling channel, where the L-informative test has an advantage over the H-informative test. Specifically, consider a low prior in which without the signaling stage, the receiver prefers the H-informative test to the L-informative test. The key point is that when there is signaling, under the L-informative test, the signaling cost effectively moves the prior to a prior that is greater than $\frac{1}{2}$, in which the L-informative test is better, and the action of no signaling, which perfectly reveals that the sender's type is low, is selected with a higher probability than under the H-informative test.

To further explain this last point, we present a sketch of the proof of the second part of the proposition. Let $\mu_0 \in (0, 1)$. Assume first that $\mu_0 \leq \mu^*_{\hat{\pi}}$. Consider the equilibrium under π and $\hat{\pi}$. Since π and $\hat{\pi}$ are symmetric, we get that conditional on observing a positive signaling cost, the equilibrium payoff of the receiver is the same, i.e., $V^R_{\hat{\pi}}(\mu^*_{\hat{\pi}}) = V^R_{\pi}(\mu^*_{\pi})$. When the receiver observes a signaling cost of 0, then she learns that the sender is of type 0 with certainty. Now, since $\mu^*_{\hat{\pi}} < \mu^*_{\pi}$, the ex ante probability that the receiver would observe the signaling cost 0 is greater under π than under $\hat{\pi}$. Therefore, the expected payoff of the receiver is greater under π than under $\hat{\pi}$. The proof in the case when $\mu^*_{\hat{\pi}} < \mu_0 < \mu^*_{\pi}$ relies on a similar but a more subtle argument. If $\frac{1}{2} < \mu^*_{\pi} \leq \mu_0$, then under both $\hat{\pi}$ and π, we get a fully pooling equilibrium, and the receiver's payoff is $V^R_{\hat{\pi}}(\mu_0)$ and $V^R_{\pi}(\mu_0)$, respectively. Since $\frac{1}{2} < \mu_0$, we have that $V^R_{\hat{\pi}}(\mu_0) < V^R_{\pi}(\mu_0)$.

5. Discussion

5.1 *The receiver's commitment power*

A natural question that arises in our model is whether the receiver benefits from her ability to commit to the test. It is straightforward to see that this is indeed the case. A more informative test induces more information for every signaling strategy of the sender. Therefore, if the receiver cannot commit to the test, then, in equilibrium, the receiver chooses a more informative test over a less informative test. Hence, the property that in our setting the receiver's preference relation does not comply with Blackwell's partial order implies that the receiver strictly benefits from her commitment power.

5.2 *Signaling costs*

Another natural question that arises concerns the connection between the receiver's test choice and the expected signaling cost. At first glance, this connection is negative; i.e., a higher informativeness level of a test leads to a lower expected signaling cost. The intuition for this connection is the following. As the test becomes more informative, the high type chooses to differentiate itself more through the test and less through costly signaling. This insight is established in Daley and Green (2014), who analyze the effect of the informativeness level of binary symmetric tests on the expected signaling cost in equilibrium. However, our results show that it could be the case that a less informative test's dividing belief is smaller than the prior, while the more informative test's dividing belief is higher than the prior. In such a case, the less informative test induces a fully pooling equilibrium with a signaling cost of 0, while the more informative test induces a separating equilibrium with a positive signaling cost. That is, our results show that in some cases, the negative connection between the test's informativeness level and the expected signaling cost breaks.

6. Conclusion

We have studied the receiver's learning problem in a signaling model à la Spence with type-independent signaling costs. Our main insight is that tests whose low grades are more informative than their high grades induce more information to be conveyed through the channel of the sender's signaling costs. We established this insight through a series of formal results. We showed that because of this property, the receiver tends to prefer tests whose low grades are more informative.

Appendix

Proof of Lemma 1 Consider the difference in the expected posteriors of the high type and the low type as a function of the initial belief:

$$V_\pi^1(\mu) - V_\pi^0(\mu) = \sum_{s\in S}\big(\pi(s|1) - \pi(s|0)\big) \cdot \frac{\mu \cdot \pi(s|1)}{(1-\mu)\cdot \pi(s|0) + \mu \cdot \pi(s|1)}.$$

Define $R_\pi(s) := \frac{\pi(s|1)}{\pi(s|0)}$ and rewriting we get

$$V_\pi^1(\mu) - V_\pi^0(\mu) = \sum_{s\in S} \pi(s|0)\big(R_\pi(s) - 1\big) \cdot \frac{\mu \cdot R_\pi(s)}{1-\mu+\mu\cdot R_\pi(s)}.$$

Now

$$\text{pos}_\pi(s,\mu)=\frac{\mu\cdot R_\pi(s)}{1-\mu+\mu\cdot R_\pi(s)}$$

and

$$\frac{\partial^2 \text{pos}_\pi(s,\mu)}{\partial\mu^2}=\frac{2\cdot\big(1-R_\pi(s)\big)\cdot R_\pi(s)}{\big(1-\mu+\mu\cdot R_\pi(s)\big)^3}$$

so we get that

$$\text{if } \infty > R_\pi(s) > 1, \text{ then } \frac{\partial^2\big[\text{pos}_\pi(s,\mu)\big]}{\partial\mu^2}<0,$$

i.e., $\text{pos}_\pi(s,\mu)$ is a strictly concave function of μ;

$$\text{if } 0 < R_\pi(s) < 1, \text{ then } \frac{\partial^2\big[\text{pos}_\pi(s,\mu)\big]}{\partial\mu^2}>0,$$

i.e., $\text{pos}_\pi(s,\mu)$ is a strictly convex function of μ.

Therefore, for every grade $s\in S$ that satisfies that $R_\pi(s)\neq 1$, we get that $(R_\pi(s)-1)\cdot\frac{(\mu\cdot R_\pi(s))}{(1-\mu+\mu\cdot R_\pi(s))}$ is a strictly concave function. Since $\pi\in\Pi$ is informative, we get that there exists $s\in S$ for which $R_\pi(s)\neq 1$. Therefore, we get that $V_\pi^1(\mu)-V_\pi^0(\mu)$ is a sum of concave and strictly concave functions, and, therefore, it is a strictly concave function. Since $V_\pi^1(\mu)-V_\pi^0(\mu)$ is strictly concave and defined on a closed interval, we get that it has a unique maximizer.

Proof of Proposition 1 Given a test $\pi\in\Pi$, our model is a special case of Daley and Green's (2014) model in which the signaling cost is type-independent. Therefore, our characterization of equilibria in the second-period subgame stems from their characterization (see Proposition 3.8 in Daley and Green (2014)).

Proof of Lemma 2 Assume by contradiction that the lemma is not true; i.e., there exists a test $\pi\in\Pi$ such that $\mu_\pi\neq\mu_\pi^*$. Assume first that there exists $\pi\in\Pi$ such that $\mu_\pi<\mu_\pi^*$. Consider a prior $\hat\mu$ with $\mu_\pi<\hat\mu<\mu_\pi^*$. From Proposition 1, we have that for such a prior the unique equilibrium is fully pooling without any signaling. The payoff of the low type in this equilibrium is $V_\pi^0(\hat\mu)$. Define $\hat c := V_\pi^0(\mu_\pi^*)-V_\pi^0(\hat\mu)>0$, and consider a deviation to the signaling cost $\hat c$. The receiver observes the deviation and forms a belief regarding the type of sender that deviated. Assume that the belief of the receiver given this deviation is μ_π^*. Given this belief, we can see that the expected payoff of the low type from such a deviation is

$$V_\pi^0(\mu_\pi^*)-\hat c = V_\pi^0(\hat\mu).$$

It follows that given this belief, the low type is indifferent between its equilibrium payoff and its payoff from this deviation. Let us compute now the expected payoff of the high type from this deviation given the same belief μ_π^*:

$$V_\pi^1(\mu_\pi^*)-\hat c = V_\pi^1(\mu_\pi^*)-\big(V_\pi^0(\mu_\pi^*)-V_\pi^0(\hat\mu)\big).$$

It follows that the difference between the payoff of the high type given this deviation (when the belief is μ_π^*) and its equilibrium payoff is

$$V_\pi^1(\mu_\pi^*) - (V_\pi^0(\mu_\pi^*) - V_\pi^0(\hat{\mu})) - V_\pi^1(\hat{\mu}) = (V_\pi^1(\mu_\pi^*) - V_\pi^0(\mu_\pi^*)) - (V_\pi^1(\hat{\mu}) - V_\pi^0(\hat{\mu})).$$

From the definition of μ_π^* as the unique belief that maximizes the difference $V_\pi^1(\mu) - V_\pi^0(\mu)$ and the last equation, we get that the expected payoff of the high type given this deviation when the belief is μ_π^* is higher than its equilibrium payoff. Now, from the fact that the expected payoff from this deviation is strictly increasing in the belief of the receiver given the deviation, we get that the set of beliefs that satisfy that the payoff of the high type is strictly larger than its equilibrium payoff is a superset of the set of beliefs that satisfy that the payoff of the low type is strictly larger than its equilibrium payoff. From the definition of the D1 criterion, we get that the belief after such a deviation must be that the type that deviated is the high type. Clearly, under this belief, this deviation is a profitable one at least for the high type. It follows that the equilibrium we considered is not an equilibrium in contradiction to Proposition 1.

Assume now that there exists a test $\pi \in \Pi$ such that $\mu_\pi > \mu_\pi^*$. Consider a prior $\hat{\mu}$ with $\mu_\pi > \hat{\mu} > \mu_\pi^*$. From Proposition 1, we have that under this prior the unique equilibrium is the following: The high type chooses the signaling cost $c := V_\pi^0(\mu_\pi)$ and the low type mixes between the same signaling cost c and zero, such that the equilibrium interim belief after the receiver observes the signaling cost c is μ_π. Consider now a deviation to the signaling cost $\hat{c} := V_\pi^0(\mu_\pi^*)$. Notice that because $\mu_\pi > \mu_\pi^*$, we have that $V_\pi^0(\mu_\pi) > V_\pi^0(\mu_\pi^*)$, it follows that $c > \hat{c}$. Assume that the belief of the receiver after observing a deviation to the signaling cost $\hat{c}$ is μ_π^*. From the definition of $\hat{c}$ it is clear that the payoff of the low type after this deviation and given the interim belief μ_π^* is zero, which is equal to its equilibrium payoff. The payoff of the high type from this deviation given the belief μ_π^* is

$$V_\pi^1(\mu_\pi^*) - \hat{c} = V_\pi^1(\mu_\pi^*) - V_\pi^0(\mu_\pi^*).$$

Notice that the equilibrium payoff of the high type is

$$V_\pi^1(\mu_\pi) - c = V_\pi^1(\mu_\pi) - V_\pi^0(\mu_\pi).$$

From the definition of μ_π^* as the unique belief that maximizes the difference $V_\pi^1(\mu) - V_\pi^0(\mu)$, we get that the deviation given the interim belief μ_π^* strictly improves the high type's payoff relative to its equilibrium payoff. Now, from the fact that the expected payoff from this deviation is strictly increasing in the belief of the receiver given the deviation, we get that the set of beliefs such that the payoff of the high type is strictly larger than its equilibrium payoff is a superset of the set of beliefs such that the payoff of the low type is strictly larger than its equilibrium payoff. From the definition of the D1 criterion, we get that the belief after such a deviation must be that the type that deviated is the high type. Clearly, under this belief, this deviation is a profitable one for both types. It follows that the equilibrium we considered is not an equilibrium in contradiction to Proposition 1.

Proof of Lemma 3 To analyze whether some belief μ is greater than, less than, or equal to μ^*_π, we develop the expression $\frac{\partial[V^1_\pi(\mu)-V^0_\pi(\mu)]}{\partial\mu}$. First, we develop the expression

$$\frac{\partial\,\mathrm{pos}_\pi(s,\mu)}{\partial\mu}=\frac{\pi(s|1)\big((1-\mu)\pi(s|0)+\mu\pi(s|1)\big)-\big(\pi(s|1)-\pi(s|0)\big)\pi(s|1)}{\big((1-\mu)\pi(s|0)+\mu\pi(s|1)\big)^2}.$$

Rearranging the expression, we get that

$$\frac{\partial\,\mathrm{pos}_\pi(s,\mu)}{\partial\mu}=\frac{\pi(s|1)\pi(s|0)}{\big((1-\mu)\pi(s|0)+\mu\pi(s|1)\big)^2}.$$

After further development, we get that

$$\frac{\partial\,\mathrm{pos}_\pi(s,\mu)}{\partial\mu}=\frac{1}{\mu(1-\mu)}\big(\mathrm{pos}_\pi(s,\mu)-\mathrm{pos}_\pi(s,\mu)^2\big)$$

and so

$$\frac{\partial\big[V^1_\pi(\mu)-V^0_\pi(\mu)\big]}{\partial\mu}=\sum_{s\in S}\big(\pi(s|1)-\pi(s|0)\big)\cdot\frac{1}{\mu(1-\mu)}\cdot\big(\mathrm{pos}_\pi(s,\mu)-\mathrm{pos}_\pi(s,\mu)^2\big).$$

Given μ, we want to see whether

$$\sum_{s\in S}\big(\pi(s|1)-\pi(s|0)\big)\cdot\frac{1}{\mu(1-\mu)}\cdot\big(\mathrm{pos}_\pi(s,\mu)-\mathrm{pos}_\pi(s,\mu)^2\big)\lesseqqgtr 0,$$

which is equivalent to

$$\sum_{s\in S}\pi(s|0)\cdot\left(\frac{1}{2}-\mathrm{pos}_\pi(s,\mu)\right)^2\lesseqqgtr\sum_{s\in S}\pi(s|1)\cdot\left(\frac{1}{2}-\mathrm{pos}_\pi(s,\mu)\right)^2.$$

Proof of Lemma 4 We prove part (i) of the lemma; part (ii) can be proved analogously. Consider $\mu=\mu^*_\pi>\frac{1}{2}$. From Lemma 3, we have that

$$\sum_{s\in\mathrm{supp}(\pi)}\pi(s|0)\cdot\left(\frac{1}{2}-\mathrm{pos}_\pi(s,\mu)\right)^2=\sum_{s\in\mathrm{supp}(\pi)}\pi(s|1)\cdot\left(\frac{1}{2}-\mathrm{pos}_\pi(s,\mu)\right)^2. \tag{1}$$

We can write this equation in an equivalent way:

$$\begin{aligned}&\sum_{s\in S^-_\pi}\big(\pi(s|0)-\pi(s|1)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_\pi(s,\mu)\right)^2\\&\quad=\sum_{s\in S^+_\pi}\big(\pi(s|1)-\pi(s|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_\pi(s,\mu)\right)^2.\end{aligned} \tag{2}$$

We now use the conditions in the lemma to show that

$$\sum_{s\in S^-_{\pi'}}\big(\pi'(s|0)-\pi'(s|1)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_{\pi'}(s,\mu)\right)^2 < \sum_{s\in S^+_{\pi'}}\big(\pi'(s|1)-\pi'(s|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_{\pi'}(s,\mu)\right)^2. \tag{3}$$

This will end the proof, because according to Lemma 3, if the right-hand side is bigger than the left-hand side, then $\mu^*_{\pi'}<\mu=\mu^*_{\pi}$.

Because we have that $\tilde{\pi}_l(\pi')=\tilde{\pi}_l(\pi)$ and that $\tilde{\pi}_\beta(\pi)=\tilde{\pi}_\beta(\pi')$, we can deduce that the left-hand side of (2) is equal to the left-hand side of (3), so it is sufficient to show that

$$\sum_{s\in S^+_{\pi}}\big(\pi(s|1)-\pi(s|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_{\pi}(s,\mu)\right)^2 < \sum_{s\in S^+_{\pi'}}\big(\pi'(s|1)-\pi'(s|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_{\pi'}(s,\mu)\right)^2. \tag{4}$$

From the fact that $\tilde{\pi}_h(\pi')\succ_B\tilde{\pi}_h(\pi)$, we can deduce that there exists a set of tests $\{\hat{\pi}_s\}_{s\in S^+_\pi}$ such that the test $\tilde{\pi}_h(\pi')$ is equivalent to a test that is defined according to the next sequential procedure: first the test $\tilde{\pi}_h(\pi)$ is activated and then after every possible result $s\in S^+_\pi$, the test $\hat{\pi}_s$ is activated. This equivalence allows as to write (4) in the following equivalent way:

$$\sum_{s\in S^+_{\pi}}\big(\pi(s|1)-\pi(s|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}_{\pi}(s,\mu)\right)^2 < \sum_{s\in S^+_{\pi}}\sum_{\hat{s}\in S_{\hat{\pi}_s}}\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1)-\pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}(s,\hat{s},\mu)\right)^2. \tag{5}$$

We prove that (5) is true case by case. That is, for every $s\in S^+_\pi$, we prove that

$$\big(\pi(s|1)-\pi(s|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}(s,\mu)\right)^2 < \sum_{\hat{s}\in S_{\hat{\pi}_s}}\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1)-\pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)\cdot\left(\frac{1}{2}-\mathrm{pos}(s,\hat{s},\mu)\right)^2. \tag{6}$$

To see why this is true, first notice that clearly we have that

$$\big(\pi(s|1)-\pi(s|0)\big)=\sum_{\hat{s}\in S_{\hat{\pi}_s}}\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1)-\pi(s|0)\hat{\pi}_s(\hat{s}|0)\big). \tag{7}$$

It follows that

$$1 = \sum_{\hat{s}\in S_{\hat{\pi}_s}} \frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) - \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1) - \pi(s|0)\big)}. \tag{8}$$

From this it follows that we can write (6) in the following equivalent way:

$$\left(\frac{1}{2} - \text{pos}(s, \mu)\right)^2 < \sum_{\hat{s}\in S_{\hat{\pi}_s}} \frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) - \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1) - \pi(s|0)\big)} \cdot \left(\frac{1}{2} - \text{pos}(s, \hat{s}, \mu)\right)^2. \tag{9}$$

Now notice that because we have that $\tilde{\pi}_\beta(\pi) = \tilde{\pi}_\beta(\pi')$ and because $s \in S^+_\pi$, it must be that $\text{pos}(s, \mu) > \mu > \frac{1}{2}$ and also for every $\hat{s} \in S_{\hat{\pi}_s}$ we have that $\text{pos}(s, \hat{s}, \mu) > \mu > \frac{1}{2}$. Additionally, notice that the function $(\frac{1}{2} - x)^2$ is concave and increasing when $x > \frac{1}{2}$. Last, notice that by construction the distribution of posteriors that corresponds to $\{\text{pos}(s, \hat{s}, \mu)\}_{\hat{s}\in S_{\hat{\pi}_s}}$ is a mean preserving spread of $\text{pos}(s, \mu)$; that is, we have that

$$\text{pos}(s, \mu) = \sum_{\hat{s}\in S_{\hat{\pi}_s}} \frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) + \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1) + \pi(s|0)\big)} \cdot \text{pos}(s, \hat{s}, \mu). \tag{10}$$

From the concavity of the function $(\frac{1}{2} - x)^2$, we get that

$$\left(\frac{1}{2} - \text{pos}(s, \mu)\right)^2 < \sum_{\hat{s}\in S_{\hat{\pi}_s}} \frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) + \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1) + \pi(s|0)\big)} \cdot \left(\frac{1}{2} - \text{pos}(s, \hat{s}, \mu)\right)^2. \tag{11}$$

In the last part of the proof we will establish that

$$\begin{aligned}&\sum_{\hat{s}\in S_{\hat{\pi}_s}} \frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) + \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1) + \pi(s|0)\big)} \cdot \left(\frac{1}{2} - \text{pos}(s, \hat{s}, \mu)\right)^2 \\ &\quad < \sum_{\hat{s}\in S_{\hat{\pi}_s}} \frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) - \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1) - \pi(s|0)\big)} \cdot \left(\frac{1}{2} - \text{pos}(s, \hat{s}, \mu)\right)^2. \end{aligned} \tag{12}$$

To do this we will prove the next lemma.

Lemma 7. *For every* $\hat{s} \in S_{\hat{\pi}_s}$,

- *if* $\text{pos}(s, \hat{s}, \mu) > \text{pos}(s, \mu)$, *then* $\frac{(\pi(s|1)\hat{\pi}_s(\hat{s}|1) - \pi(s|0)\hat{\pi}_s(\hat{s}|0))}{(\pi(s|1) - \pi(s|0))} > \frac{(\pi(s|1)\hat{\pi}_s(\hat{s}|1) + \pi(s|0)\hat{\pi}_s(\hat{s}|0))}{(\pi(s|1) + \pi(s|0))}$;
- *if* $\text{pos}(s, \hat{s}, \mu) < \text{pos}(s, \mu)$, *then* $\frac{(\pi(s|1)\hat{\pi}_s(\hat{s}|1) - \pi(s|0)\hat{\pi}_s(\hat{s}|0))}{(\pi(s|1) - \pi(s|0))} < \frac{(\pi(s|1)\hat{\pi}_s(\hat{s}|1) + \pi(s|0)\hat{\pi}_s(\hat{s}|0))}{(\pi(s|1) + \pi(s|0))}$.

Proof. We will prove the first claim of the lemma; the proof of the second claim is analogous. We start with rewriting the inequality in the lemma:

$$\frac{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) - \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1) + \pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)} > \frac{\big(\pi(s|1) - \pi(s|0)\big)}{\big(\pi(s|1) + \pi(s|0)\big)}. \tag{13}$$

Rewriting again we get

$$\frac{\pi(s|1)\hat{\pi}_s(\hat{s}|1)}{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1)+\pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}-\frac{\pi(s|0)\hat{\pi}_s(\hat{s}|0)}{\big(\pi(s|1)\hat{\pi}_s(\hat{s}|1)+\pi(s|0)\hat{\pi}_s(\hat{s}|0)\big)}$$
$$>\frac{\pi(s|1)}{\big(\pi(s|1)+\pi(s|0)\big)}-\frac{\pi(s|0)}{\big(\pi(s|1)+\pi(s|0)\big)}, \tag{14}$$

and again,

$$\operatorname{pos}\left(s,\hat{s},\frac{1}{2}\right)-\left(1-\operatorname{pos}\left(s,\hat{s},\frac{1}{2}\right)\right)>\operatorname{pos}\left(s,\frac{1}{2}\right)-\left(1-\operatorname{pos}\left(s,\frac{1}{2}\right)\right). \tag{15}$$

This is true if and only if

$$\operatorname{pos}\left(s,\hat{s},\frac{1}{2}\right)>\operatorname{pos}\left(s,\frac{1}{2}\right). \tag{16}$$

This is clearly true because we have that $\operatorname{pos}(s,\hat{s},\mu)>\operatorname{pos}(s,\mu)$. □

Combining this lemma and the fact that the function $(\frac{1}{2}-x)^2$ is increasing when $x>\frac{1}{2}$, we get that (12) is true. To see this, notice that both the left-hand side and the right-hand side of (12) are convex combinations over the same numbers. From the lemma and the fact that the function $(\frac{1}{2}-x)^2$ is increasing when $x>\frac{1}{2}$, we can deduce that the right-hand side puts larger weights on large numbers, i.e., $(\frac{1}{2}-\operatorname{pos}(s,\hat{s},\mu))^2$ when $\operatorname{pos}(s,\hat{s},\mu)>\operatorname{pos}(s,\mu)>\frac{1}{2}$, and lower weights on small numbers, i.e., $(\frac{1}{2}-\operatorname{pos}(s,\hat{s},\mu))^2$ when $\frac{1}{2}<\operatorname{pos}(s,\hat{s},\mu)<\operatorname{pos}(s,\mu)$. This ends the proof.

Proof of Proposition 2 We first partition the set of possible priors $(0,1)$ into three regions: $(0,\mu^*_\pi]$, $(\mu^*_\pi,\mu^*_{\pi'})$, $[\mu^*_{\pi'},1)$.

If $\mu_0\in(\mu^*_{\pi'},1]$, then we have, according to Proposition 1 and Lemma 2, that under both tests the unique equilibrium is the fully pooling equilibrium. It follows that under such priors, the fact that the sender can signal plays no role and, therefore, the receiver would prefer the more informative test π'.[19]

If $\mu_0\in[\mu^*_\pi,\mu^*_{\pi'})$, then we have that under the test π the unique equilibrium is a fully pooling equilibrium, while under the test π' the unique equilibrium is a semi-pooling equilibrium. That is, under the test π' the receiver first gets an informative signal through the signaling channel and then she gets another signal directly from the test. Clearly, the effective signal that the receiver observes through this procedure is more informative than π' and we know that π' is more informative than π. The proof follows from the transitivity of the Blackwell relation.

If $\mu_0\in(0,\mu^*_\pi)$, then we have that under both tests the unique equilibrium is a semi-pooling equilibrium. We first want to show that the informative signal that the receiver observes through the signaling strategy is more informative under the test π' than under the test π. Denote by sig_π and $\mathrm{sig}_{\pi'}$ the test that corresponds to the informative signal

[19]Recall that when we say that a test/signal is more informative than another test/signal, we mean that in the sense of Blackwell (1951).

that the receiver gets through the signaling strategy under π and π', respectively. After the signaling stage under π' the prior is spread into two posteriors 0 and $\mu^*_{\pi'}$ while under the test π the prior is spread into the posteriors 0 and μ^*_{π}. Because we have that $\mu^*_{\pi'} > \mu^*_{\pi}$, it follows that the posteriors distribution under π' is a mean preserving spread of the posteriors distribution under π. Now, because we are in a binary state environment, it follows that $\text{sig}_{\pi'} \succ_B \text{sig}_{\pi}$. The effective signal that the receiver observes in equilibrium under the test π' (π) is the result of a procedure in which first the receiver observes $\text{sig}_{\pi'}$ (sig_{π}) and then for each posterior (interim belief) the test π' (π) is activated. Note that we have that $\pi' \succ_B \pi$ and also that $\text{sig}_{\pi'} \succ_B \text{sig}_{\pi}$. It follows that both the first stage and the second stage of the procedure are more informative under π' than under π and, therefore, the receiver prefers π' also under priors in the set $(0, \mu^*_{\pi})$.

Proof of Lemma 5 Applying the condition of Lemma 3 to binary tests, we get the following result. Let π be a binary test, and let $\mu \in [0, 1]$ be an initial distribution, $\mu^*_{\pi} \lesseqqgtr \mu$ if and only if

$$\left|\frac{1}{2} - \text{pos}_{\pi}(l, \mu)\right| \lesseqqgtr \left|\text{pos}_{\pi}(h, \mu) - \frac{1}{2}\right|.$$

Now if $\pi(h|1) = 1$, then $\text{pos}_{\pi}(l, \mu) = 0$, which implies that the left-hand side is equal to $\frac{1}{2}$. Since π is partially informative (as we assume in the model), we get that $\text{pos}_{\pi}(h, \mu) < 1$ for every $\mu < 1$ and $\text{pos}_{\pi}(h, \mu) = 1$ only if $\mu = 1$. Therefore, we get that the left-hand side is equal to the right-hand side if and only if $\mu = 1$, and so $\mu^*_{\pi} = 1$.

If $\pi(l|0) = 1$, then $\text{pos}_{\pi}(h, \mu) = 1$, which implies that the right-hand side is equal to $\frac{1}{2}$. Since π is partially informative (as we assume in the model), we get that $\text{pos}_{\pi}(l, \mu) > 0$ for every $\mu > 0$ and $\text{pos}_{\pi}(h, \mu) = 0$ if and only if $\mu = 0$. Therefore, we get that the left-hand side is equal to the right-hand side if and only if $\mu = 0$, and so $\mu^*_{\pi} = 0$.

To complete the proof we show that if $\pi \in \Pi$ does not include a fully informative grade, i.e., a grade that some type receives with probability 0, then $\mu^*_{\pi} \in (0, 1)$. The argument is the following. If a grade $s \in S$ is partially informative, i.e., both type 0 and type 1 receive it with positive probabilities, then $\text{pos}_{\pi}(s, 0) = 0$ and $\text{pos}_{\pi}(s, 1) = 1$. Since both grades in the support of π are partially informative, we get that $V^1_{\pi}(0) = V^0_{\pi}(0) = 0$ and $V^1_{\pi}(1) = V^0_{\pi}(1) = 0$. Hence, we get that the function $V^1_{\pi}(\mu) - V^0_{\pi}(\mu)$ is strictly concave with $V^1_{\pi}(0) - V^0_{\pi}(0) = V^1_{\pi}(1) - V^0_{\pi}(1) = 0$. Therefore, $\mu^*_{\pi} = \arg\max_{\mu \in [0,1]} V^1_{\pi}(\mu) - V^0_{\pi}(\mu) \in (0, 1)$.

Proof of Lemma 6 Again, from Lemma 3 we have that for binary test π we have that $\mu^*_{\pi} \lesseqqgtr \mu$ if and only if

$$\left|\frac{1}{2} - \text{pos}_{\pi}(l, \mu)\right| \lesseqqgtr \left|\text{pos}_{\pi}(h, \mu) - \frac{1}{2}\right|.$$

A test π has a $\mu^*_{\pi} \lesseqqgtr \frac{1}{2}$ if and only if

$$\frac{1}{2} - \frac{1 - \pi(h|1)}{1 - \pi(h|1) + \pi(l|0)} \lesseqqgtr \frac{\pi(h|1)}{\pi(h|1) + 1 - \pi(l|0)} - \frac{1}{2},$$

which is equivalent to

$$\pi(h|1) \lesseqqgtr \pi(l|0),$$

so we derived the result for binary tests. Consider an L-informative test π. This test corresponds to a probability distribution of L-informative and N-informative binary tests with at least one L-informative binary test in its support. We denote the binary tests in the support of the L-informative test π by $\{\pi_1, \dots, \pi_k\}$ with a probability distribution $\{p_i\}_{i=1}^k$. Now $V^1_\pi(\mu) - V^0_\pi(\mu) = \sum_{i=1}^k p_i[V^1_{\pi_i}(\mu) - V^0_{\pi_i}(\mu)]$ and $\frac{\partial [V^1_{\pi_i}(\frac{1}{2}) - V^0_{\pi_i}(\frac{1}{2})]}{\partial \mu} \geq 0$ for all $i \in \{1, \dots, k\}$ with at least one $j \in \{1, \dots, k\}$ for which $\frac{\partial [V^1_{\pi_j}(\frac{1}{2}) - V^0_{\pi_j}(\frac{1}{2})]}{\partial \mu} > 0$. This implies that $\frac{\partial [V^1_{\pi}(\frac{1}{2}) - V^0_{\pi}(\frac{1}{2})]}{\partial \mu} > 0$, which implies that $\mu^*_\pi > \frac{1}{2}$. An analogous argument proves the results for H-informative and N-informative tests.

Proof of Proposition 4 To prove the proposition, we first present three lemmas.

Lemma 8. *Consider an initial belief $\mu \in (0, 1)$. Let π and π' be two tests such that $\pi, \pi' \in \Pi$:*

$$V^1_\pi(\mu) - V^0_\pi(\mu) \lesseqqgtr V^1_{\pi'}(\mu) - V^0_{\pi'}(\mu) \quad \textit{if and only if} \quad V^R_\pi(\mu) \lesseqqgtr V^R_{\pi'}(\mu).$$

Proof. Consider a belief μ and a test π. Recall that the receiver's payoff is equal to:

$$V^R_\pi(\mu) = -(\sum_{s\in S} \mu \cdot \pi(s|1) \cdot \left(1 - \mathrm{pos}_\pi(s, \mu)\right)^2 + (1-\mu) \cdot \pi(s|0) \cdot \left(-\mathrm{pos}_\pi(s, \mu)\right)^2)$$

we denote by $\pi(s)$ the probability of a signal $s \in S$, i.e.,

$$\pi(s) := \mu \cdot \pi(s|1) + (1-\mu) \cdot \pi(s|0)$$

so we get that

$$\begin{aligned}
-V^R_\pi(\mu) &= \sum_{s\in S} \pi(s) \cdot \left[\mathrm{pos}_\pi(s, \mu) \cdot \left(1 - \mathrm{pos}_\pi(s, \mu)\right)^2 + \left(1 - \mathrm{pos}_\pi(s, \mu)\right) \cdot \mathrm{pos}_\pi(s, \mu)^2\right] \\
&= \sum_{s\in S} \pi(s) \cdot \mathrm{pos}_\pi(s, \mu) \cdot \left(1 - \mathrm{pos}_\pi(s, \mu)\right) \\
&= \sum_{s\in S} \pi(s) \cdot \mathrm{pos}_\pi(s, \mu) - \sum_{s\in S} \pi(s) \cdot \mathrm{pos}_\pi(s, \mu)^2 \\
&= \mu - \sum_{s\in S} \pi(s) \cdot \mathrm{pos}_\pi(s, \mu)^2 \\
&= \mu - \sum_{s\in S} \mu \cdot \pi(s|1) \cdot \mathrm{pos}_\pi(s, \mu) \\
&= \mu \cdot \left(1 - \sum_{s\in S} \pi(s|1) \cdot \mathrm{pos}_\pi(s, \mu)\right) \\
&= \mu \cdot \left(1 - V^1_\pi(\mu)\right).
\end{aligned}$$

Therefore, we get that

$$V_{\pi}^{R}(\mu) = \mu \cdot \big(V_{\pi}^{1}(\mu) - 1\big),$$

that is, the receiver's payoff is greater whenever $V_{\pi}^{1}(\mu)$ is greater. Now, Bayes' plausibility implies that

$$\mu \cdot V_{\pi}^{1}(\mu) + (1-\mu) \cdot V_{\pi}^{0}(\mu) = \mu,$$

which implies that $V_{\pi}^{1}(\mu) \lesseqqgtr V_{\pi'}^{1}(\mu)$ if and only if $V_{\pi}^{1}(\mu) - V_{\pi}^{0}(\mu) \lesseqqgtr V_{\pi'}^{1}(\mu) - V_{\pi'}^{0}(\mu)$. Therefore, $V_{\pi}^{1}(\mu) - V_{\pi}^{0}(\mu) \lesseqqgtr V_{\pi'}^{1}(\mu) - V_{\pi'}^{0}(\mu)$ if and only if $V_{\pi}^{R}(\mu) \lesseqqgtr V_{\pi'}^{R}(\mu)$. □

Lemma 9. *Let π and π' be symmetric tests. For every $\mu \in [0, 1]$, we have*

$$V_{\pi}^{1}(\mu) - V_{\pi}^{0}(\mu) = V_{\pi'}^{1}(1-\mu) - V_{\pi'}^{0}(1-\mu)$$

and

$$V_{\pi}^{R}(\mu) = V_{\pi'}^{R}(1-\mu).$$

Proof. We first prove that

$$V_{\pi}^{1}(\mu) - V_{\pi}^{0}(\mu) = V_{\pi'}^{1}(1-\mu) - V_{\pi'}^{0}(1-\mu).$$

Note that

$$\begin{aligned} 1 - \mathrm{pos}_{\pi}(s, \mu) &= \frac{(1-\mu) \cdot \pi(s|0)}{(1-\mu) \cdot \pi(s|0) + \mu \cdot \pi(s|1)} \\ &= \frac{(1-\mu) \cdot \pi'(s|1)}{(1-\mu) \cdot \pi'(s|1) + \mu \cdot \pi'(s|0)} = \mathrm{pos}_{\pi'}(s, 1-\mu). \end{aligned}$$

Now

$$\begin{aligned} V_{\pi}^{1}(\mu) - V_{\pi}^{0}(\mu) &= \sum_{s \in S} \big(\pi(s|1) - \pi(s|0)\big) \cdot \mathrm{pos}_{\pi}(s, \mu) \\ &= \sum_{s \in S} \big(\pi'(s|0) - \pi'(s|1)\big) \cdot \big(1 - \mathrm{pos}_{\pi'}(s, 1-\mu)\big) \\ &= \sum_{s \in S} \big(\pi'(s|0) - \pi'(s|1)\big) + \sum_{s \in S} \big(\pi'(s|1) - \pi'(s|0)\big) \mathrm{pos}_{\pi'}(s, 1-\mu) \\ &= V_{\pi'}^{1}(1-\mu) - V_{\pi'}^{0}(1-\mu). \end{aligned}$$

We now prove that

$$\begin{aligned} &V_{\pi}^{R}(\mu) = V_{\pi'}^{R}(1-\mu) \\ &V_{\pi}^{R}(\mu) = -\Bigg[\sum_{s \in S} \pi(s|1) \cdot \mu \cdot \big(1 - \mathrm{pos}_{\pi}(s, \mu)\big)^{2} + \sum_{s \in S} \pi(s|0) \cdot (1-\mu) \cdot \big(-\mathrm{pos}_{\pi}(s, \mu)\big)^{2}\Bigg] \\ &\qquad = -\Bigg[\sum_{s \in S} \pi'(s|0) \cdot \mu \cdot \big(\mathrm{pos}_{\pi'}(s, 1-\mu)\big)^{2} \end{aligned}$$

$$+ \sum_{s \in S} \pi'(s|1) \cdot (1-\mu) \cdot \big(1 - \text{pos}_{\pi'}(s, 1-\mu)\big)^2 \Big]$$
$$= V^R_{\pi'}(1-\mu). \qquad \square$$

Lemma 10. *For every binary test π, if there exists an initial belief $0 < \tilde{\mu} \neq \frac{1}{2}$ such that $V^1_\pi(\tilde{\mu}) - V^0_\pi(\tilde{\mu}) = V^1_\pi(1-\tilde{\mu}) - V^0_\pi(1-\tilde{\mu})$, then $V^1_\pi(\mu) - V^0_\pi(\mu) = V^1_\pi(1-\mu) - V^0_\pi(1-\mu)$ for every $\mu \in [0, \frac{1}{2}]$.*

Proof. Assume a binary test π such that there exists a belief $0 < \mu \neq \frac{1}{2}$ such that $V^1_\pi(\mu) - V^0_\pi(\mu) = V^1_\pi(1-\mu) - V^0_\pi(1-\mu)$. We denote $\pi(h|1) \equiv p$ and $\pi(h|0) \equiv q$;

$$V^1_\pi(\mu) - V^0_\pi(\mu) = V^1_\pi(1-\mu) - V^0_\pi(1-\mu)$$

is equivalent to

$$\begin{aligned} &(p-q)\frac{\mu p}{\mu p + (1-\mu)q} - (p-q)\frac{\mu(1-p)}{\mu(1-p) + (1-\mu)(1-q)} \\ &\quad = (p-q)\frac{(1-\mu)p}{(1-\mu)p + \mu q} - (p-q)\frac{(1-\mu)(1-p)}{(1-\mu)(1-p) + \mu(1-q)}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{\mu p}{\mu p + (1-\mu)q} - \frac{\mu(1-p)}{\mu(1-p) + (1-\mu)(1-q)} \\ &\quad = \frac{(1-\mu)p}{(1-\mu)p + \mu q} - \frac{(1-\mu)(1-p)}{(1-\mu)(1-p) + \mu(1-q)}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{\cancel{\mu^2 p(1-p)} + \mu(1-\mu)p(1-q) \cancel{-\mu^2 p(1-p)} - \mu(1-\mu)q(1-p)}{\big(\mu p + (1-\mu)q\big)\big(\mu(1-p) + (1-\mu)(1-q)\big)} \\ &\quad = \frac{\cancel{(1-\mu)^2 p(1-p)} + \mu(1-\mu)p(1-q) \cancel{-(1-\mu)^2 p(1-p)} - \mu(1-\mu)q(1-p)}{\big((1-\mu)p + \mu q\big)\big((1-\mu)(1-p) + \mu(1-q)\big)}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\big(\mu p + (1-\mu)q\big)\big(\mu(1-p) + (1-\mu)(1-q)\big) \\ &\quad = \big((1-\mu)p + \mu q\big)\big((1-\mu)(1-p) + \mu(1-q)\big), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\mu^2 p(1-p) + \cancel{\mu(1-\mu)p(1-q)} + \cancel{(1-\mu)\mu q(1-p)} + (1-\mu)^2 q(1-q) \\ &\quad = (1-\mu)^2 p(1-p) + \cancel{\mu(1-\mu)p(1-q)} + \cancel{(1-\mu)\mu q(1-p)} + \mu^2 q(1-q), \end{aligned}$$

which is equivalent to

$$p(1-p)\big(\mu^2 - (1-\mu)^2\big) = q(1-q)\big(\mu^2 - (1-\mu)^2\big),$$

which is equivalent to

$$p(1-p)=q(1-q),$$

which is independent of μ. $\square$

We now prove that in an environment without signaling, if $\mu_0<\frac{1}{2}$, then $\hat{\pi}\succ_R\pi$, and if $\mu_0>\frac{1}{2}$, then $\pi\succ_R\hat{\pi}$. Lemma 9 and Lemma 10 show that for a binary L-informative test π and its symmetric H-informative test $\hat{\pi}$, the functions $V^1_{\pi}(\mu)-V^0_{\pi}(\mu)$ and $V^1_{\hat{\pi}}(\mu)-V^0_{\hat{\pi}}(\mu)$ are symmetric around the belief μ and that they intersects in the segment (0,1) only at $\mu=\frac{1}{2}$. Therefore, Lemma 6 implies that for an L-informative binary test π and its symmetric H-informative test $\hat{\pi}$, it holds that if $\mu<\frac{1}{2}$, then $V^1_{\hat{\pi}}(\mu)-V^0_{\hat{\pi}}(\mu)>V^1_{\pi}(\mu)-V^0_{\pi}(\mu)$, and if $\mu>\frac{1}{2}$, then $V^1_{\hat{\pi}}(\mu)-V^0_{\hat{\pi}}(\mu)<V^1_{\pi}(\mu)-V^0_{\pi}(\mu)$. Lemma 8 then implies that for an L-informative binary test π and its symmetric H-informative test $\hat{\pi}$, it holds that if $\mu_0<\frac{1}{2}$, then $V^R_{\hat{\pi}}(\mu_0)>V^R_{\pi}(\mu_0)$, and if $\mu_0>\frac{1}{2}$ then $V^R_{\hat{\pi}}(\mu_0)<V^R_{\pi}(\mu_0)$. From the definition of a binary N-informative test π^n we get that $V^R_{\pi^n}(\mu)=V^R_{\pi^n}(1-\mu)$ for every $\mu\in[0,\frac{1}{2}]$. Therefore, we get that for every L-informative test π and its symmetric H-informative test $\hat{\pi}$, it holds that if $\mu_0<\frac{1}{2}$, then $V^R_{\hat{\pi}}(\mu_0)>V^R_{\pi}(\mu_0)$, and if $\mu_0>\frac{1}{2}$, then $V^R_{\hat{\pi}}(\mu_0)<V^R_{\pi}(\mu_0)$.

We now prove that in an environment with signaling, for every $\mu_0\in(0,1)$, we have that $\pi\succ_R\hat{\pi}$. Assume that $\mu_0\le\mu^*_{\hat{\pi}}<\mu^*_{\pi}$. Since the tests are symmetric, we get by Lemma 9 that $\mu^*_{\hat{\pi}}$ and μ^*_{π} are symmetric around $\frac{1}{2}$. Lemma 9 also implies that $V^R_{\hat{\pi}}(\mu^*_{\hat{\pi}})=V^R_{\pi}(\mu^*_{\pi})$, i.e., the payoff of the receiver conditional on observing the positive signaling cost is the same under both tests. Since $\mu^*_{\hat{\pi}}<\mu^*_{\pi}$, the probability that type 0 selects the signaling cost 0 that fully reveals its type is greater under π than under $\hat{\pi}$. Hence, we get that the receiver's payoff is strictly greater under π than under $\hat{\pi}$. Assume that $\mu^*_{\hat{\pi}}<\mu_0<\mu^*_{\pi}$. The equilibrium in the subgame that the test $\hat{\pi}$ induces is fully pooling and the receiver's payoff is $V^R_{\hat{\pi}}(\mu_0)$. We divide into cases. Assuming that $\frac{1}{2}<\mu_0<\mu^*_{\pi}$, by the first part of Proposition 4, we get that $V^R_{\hat{\pi}}(\mu_0)<V^R_{\pi}(\mu_0)$. Now, since $\mu_0<\mu^*_{\pi}$, we get that the effective signal that the receiver obtains is composed of observing both an informative signal through the signaling channel and an informative signal of the test. Hence, the equilibrium payoff of the receiver in the subgame that test π induces is greater than her payoff when the test π is activated on the prior $V^R_{\pi}(\mu_0)$. Hence, the receiver's equilibrium payoff under π is strictly greater than $V^R_{\hat{\pi}}(\mu_0)$. Assume that $\mu^*_{\hat{\pi}}<\mu_0<\frac{1}{2}<1-\mu_0<\mu^*_{\pi}$. The information that the receiver obtains in the equilibrium of the subgame that test π induces can be presented as follows. The prior is μ_0. In the first stage, the receiver observes a binary test with two grades that produce the beliefs 0 and $1-\mu_0$. In the second stage, conditional on the test's grade that produces the interim belief $1-\mu_0$, the receiver observes another binary test with two grades that produce the beliefs 0 and μ^*_{π}. Finally the receiver observes the grade of the test π. Lemma 9 implies that $V^R_{\hat{\pi}}(\mu_0)=V^R_{\pi}(1-\mu_0)$. Therefore, the information that the receiver obtains in the case where the receiver first observes the binary test of the first stage and then the grade of the test π provides a strictly greater payoff than $V^R_{\hat{\pi}}(\mu_0)$. This implies that the information that the receiver obtains in the equilibrium in the subgame that test π induces, where the receiver observes the grade of the test π after both stages, provides a strictly greater payoff than $V^R_{\hat{\pi}}(\mu_0)$. Assume that $\mu^*_{\hat{\pi}} < \frac{1}{2} < \mu^*_{\pi} < \mu_0$. Then the receiver's payoffs from $\hat{\pi}$ and π are $V^R_{\hat{\pi}}(\mu_0)$ and $V^R_{\pi}(\mu_0)$, respectively, and we have proved that in that case $V^R_{\hat{\pi}}(\mu_0) < V^R_{\pi}(\mu_0)$.

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