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Experimentation in organizations

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We consider a moral hazard problem in which a principal provides incentives to a team of agents to work on a risky project. The project consists of two milestones of unknown feasibility. While working unsuccessfully, the agents' private beliefs regarding the feasibility of the project decline. This learning requires the principal to provide rents to prevent the agents from procrastinating and free-riding on others' discoveries. To reduce these rents, the principal stops the project inefficiently early and gives identical agents asymmetric experimentation assignments. The principal prefers to reward agents with better future contract terms or task assignments rather than monetary bonuses.

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1. Introduction

Innovation within firms often involves the joint efforts of a group of workers who build on each others' achievements and ideas to complete a common goal. Innovative projects are also often uncertain, having many possible points of failure and steps that may fail to come through. As workers participate in uncertain but potentially lucrative projects, they naturally learn privately from their own experience, and from their coworkers', about the projects' quality and feasibility. These sources of dynamic private information complicate the provision of incentives by a principal or a manager who wants to induce the workers to exert effort.

We develop a model of experimentation in organizations in which a manager (principal) contracts with a group of identical workers (agents) to complete a project. The project consists of two tasks of unknown feasibility, each of which has to be completed for the project to yield a final payoff. We model the sequence of uncertain tasks as a sequence of experiments. Agents experiment simultaneously by exerting costly private effort. Due to their private effort, as agents work they also learn privately about the feasibility of each task. When one worker completes a task, all workers may start work on

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the next task that is required to complete the project. The principal chooses how many workers to hire subject to a recruitment cost, and chooses a history-contingent payoff scheme to incentivize agents to exert her desired amount of effort at each time, subject to a limited liability constraint. We ask: What are the features of optimal contracts and effort provision in this setting? How do workers' payments and terms of employment vary over time? And how do optimal contracts depend on how much workers are able to privately learn while doing their job?

The literature on contracts for experimentation focuses mainly on principal-agent relationships with a *single* agent in which all uncertainty is resolved after a *single* success.[1] However, projects typically involve many intermediate milestones and possible points of failure. Workers in the organization interact through multiple stages until a project is abandoned or completed.

As an example, consider a group of software developers creating a computer game. The workers will need to design the storyline and graphics, create a prototype, program the stages of the game, test the game with different types of users and resolve issues and bugs. Each of these steps is uncertain but necessary for the successful completion of the new product. Workers' beliefs in the feasibility of the project will increase after they achieve milestones and may decrease as time passes without progress. Moreover, once a worker develops the right concept or discovers how to overcome a hurdle, other workers learn from it and the team shifts focus to the next challenge.

Our main results indicate that, in our setting, private learning and the sequential resolution of uncertainty lead to asymmetries in the optimal contract in two important ways. First, symmetric agents may be allocated asymmetrically from the start, in the first task. Second, agents who succeed receive second-task allocations that are closer to efficiency than the allocations they would receive in a project that is comprised only by the second task, while agents who do not succeed receive allocations that are distorted down from this benchmark. These asymmetries are inefficient and may translate to persistently higher earnings for some players than for others, even though no learning about agents' abilities occurs.[2]

Let us give an intuition for our results. Due to his private learning, an agent's reward for success must provide two types of rents.[3] The first type is required to prevent agents from delaying effort. As agents can privately shift the timing of their effort, the principal is unable to exactly fine-tune the contract to appropriate all rents. We refer to this first type of rents as "procrastination rents." Procrastination rents increase in the amount of effort that an agent is expected to exert or length of time that he is expected to work, and decrease in the amount of competition generated by other workers' efforts. The second type of rents, which we call "public good rents," arise due to the public good nature of the

[1]See, for example, Bergemann and Hege (2005), Bergemann and Hege (1998), Hörner and Samuelson (2013), and Halac, Kartik, and Liu (2016).

[2]Winter (2004) also finds that optimal contracts are asymmetric when a principal's objective is to implement high effort as a unique Nash equilibrium outcome and there are increasing returns to scale in the number of agents.

[3]We say that an agent receives rents if his reward for success is higher than what exactly compensates him for the cost of effort.

early-task success. An agent can free-ride on his coworkers' efforts and successes in the first task while reaping the rents from experimenting in the next task. This temptation translates into an increased reward that the agent must receive upon success in the early task. In contrast, in a project with no uncertainty, in which both tasks are known to be feasible, the principal does not need to provide rents to agents even though she does not observe their efforts.

The principal must trade off procrastination and public good rents against allocating agents efficiently. The trade-off with procrastination rents produces contracts that are distorted down from the first-best, in which the principal stops experimentation inefficiently early. In the one-task case, where only procrastination rents are present, contracts are symmetric and agents' rents decrease with the number of agents hired. The trade-off with public good rents introduces asymmetrical distortions. The first task allocation may be asymmetric because reducing an agent's experimentation attenuates other agents' temptation to free-ride. The second-task allocations are asymmetric for two reasons. First, the principal distorts the unsuccessful agents' experimentation to reduce their public good rents. Second, an agent who succeeds receives an experimentation assignment that is closer to efficiency because the principal prefers to reward him with a longer experimentation assignment, which yields him more procrastination rents, rather than a bonus. The reason is that an assignment that costs as much as a bonus, in addition to rewarding a successful agent, generates extra surplus due to the agent's extended experimentation.

The incentive to delay effort is reduced as the number of agents involved in the project increases. Competition reduces the temptation to procrastinate, as another agent may succeed while an agent shirks. In contrast, the free-riding incentive increases with the number of agents in the early stages of the project and decreases with the number of agents in the later stages of the project. Thus, it is always optimal to add more agents in the last stage of the project but there may be no additional gain from allowing these agents to participate in early stages. As the cost of recruitment goes to zero, the principal hires an unboundedly increasing number of agents and her payoff approaches the first-best.

Our finding that the principal prefers to reward agents with an increased experimentation assignment is reminiscent of a feature of real-life employment contracts: firms often use job assignments or promotions to reward workers instead of only bonuses.[4] If after a promotion a worker is allocated to tasks that must confer more information rents due to private learning, our results can rationalize why a manager would choose to reward the worker via a promotion instead of (or in addition to) a bonus.

Related literature

This work adds to the literature of experimentation, (see, for instance, Bolton and Harris (1999), Keller, Rady, and Cripps (2005), and Klein and Rady (2011)) and is related to the

[4]Baker, Jensen, and Murphy (1988) pose the question of why promotions are so widely used to provide incentives in firms, given the common wisdom that bonuses should be preferable.

literature on contests and the literature on the provision of incentives for teams of agents under moral hazard.

Bonatti and Hörner (2011) and Bonatti and Hörner (2009) consider a game in which a group of players, who exert private effort, collaborate in a risky project with one stage of uncertainty.[5] The players receive the same (fixed) payoff when they succeed. They find that the players free-ride to learn others' breakthroughs and that effort is inefficiently delayed. In contrast, we consider a principal-agents model in which the agents' payoffs are designed by the principal and there are two stages of uncertainty. Our results show that the principal prevents delayed effort at a cost in the form of procrastination rents.

Bergemann and Hege (1998), Bergemann and Hege (2005), and Hörner and Samuelson (2013) consider principal one-agent experimentation settings, with one stage of uncertainty, in which the principal must provide funds to an agent who may appropriate them. Thus, the contract must satisfy a "no diversion constraint," which is stronger than our limited liability requirement. The limited liability constraint is intended to capture features of a firm that employs workers and does not provide funding for ventures.

Uncertainty about project quality is crucial to our analysis. Green and Taylor (2016), Hu (2014), and Shan (2017) consider principal one-agent models with multistage projects without uncertainty about the quality of each stage. In our setting, under no uncertainty, the principal can implement efficient effort at no informational cost. In subsequent work, Wolf (2017) considers a two-stage experimentation principal one-agent setting in which the timing of the first success is informative about the quality of the second task. He finds that the principal may want to reward an agent with bonuses, rather than assignments for late first-task successes.

Our paper also relates to the literature on contests. In complementary work, Halac, Kartik, and Liu (2017) ask how to design a contest for experimentation for a group of symmetric agents.[6]

The fact that players subject to limited liability must receive information rents is reminiscent of the classical literature on efficiency wages (Shapiro and Stiglitz (1984), Acemoglu and Newman (2002)). However, under efficiency wages, rents arise due to a principal's imperfect effort monitoring technology, while in our model agents receive rents due to their dynamic private learning about the quality of a project.

This paper is related more broadly to other dynamic models with a single agent who is privately informed about effort and can manipulate the principal's beliefs by choice of effort, leading to information rents (Halac, Kartik, and Liu (2016), He, Wei, Yu, and

[5]Bonatti and Hörner (2009) also ask the closely related question of designing a contract for a one-task project when the principal cannot observe individual successes.

[6]In their paper, the principal maximizes the amount of experimentation subject to a bound on the maximum prize. They find that it is sometimes optimal to not disclose breakthroughs to other participants. One may worry that, in our paper, the principal would also like to hide agents' successes. In the prior working-paper version, Moroni (2017), we show that the principal prefers to disclose breakthroughs as soon as they arise. In our paper, in contrast to Halac, Kartik, and Liu (2017), there is no budget constraint, but the principal wishes to maximize profits. The dual of our problem is the maximization of total experimentation subject to a constraint on expected expenditure, rather than on maximum expenditure as in Halac, Kartik, and Liu's (2017) setting.

Gao (2017), Prat and Jovanovic (2014), Bhaskar (2014)). It is also related to models in which agents can privately choose between a safe and risky action (Manso (2011), Ederer (2016)), and to models of incentives for multiple agents without private learning about project quality (Campbell, Ederer, and Spinnewijn (2014), Georgiadis (2015)). Wagner (2016) considers an experimentation setting with multiple agents who are each assigned to different imperfectly correlated one-stage projects. Georgiadis, Lippman, and Tang (2014) consider a principal with limited commitment power managing a team of workers. Our work is distinguished by the presence of multiple tasks about which agents may privately learn.

Finally, this paper provides a rationale for the use of allocations and promotions in a dynamic relationship to provide incentives. Fairburn and Malcomson (1994) show that promotions may be preferable to bonuses when the employees can bribe their supervisor. Prendergast (1993) shows that promotions can facilitate unobservable investments in human capital. Barron and Powell (2019) show that a principal may implement biased promotions to credibly commit to rewarding a successful agent in a relational contract. In Board (2011), a principal delays rents to economize on the provision of incentives in a setting involving a hold-up problem.

2. Model

A principal hires $n \in \mathbb{N}$ symmetric agent(s) that participate in a project. One can think of the agents as workers that are employed by a firm (the principal) seeking to create a new product. To hire an agent, the principal must incur a recruitment cost $c > 0$. The project consists of 2 stages or *tasks* each of which has to be completed in sequential order for the project to succeed. However, it is not known ex ante whether each task can be completed. A task may be "good" or "bad" (or else "feasible" and "impossible"). The probability that task $j \in \{1, 2\}$ is good is $\bar{p}^{(j)} \in (0, 1)$, which is commonly known by all participants.

Time is continuous in $[0, \infty)$. At each time, t agents exert privately observed and costly effort in the current task. If task j is being performed at time t, each agent i exerts effort $a_{i,t}^{(j)} \in [0, \bar{a}]$ at time t at cost $\kappa_j a_{i,t}^{(j)}$, where $\kappa_j > 0$.

Agents can start work on task 2 only after task 1 is completed successfully by some agent. If task j is good and agent i exerts effort $a_{i,t}^{(j)}$ at time t, he completes the task at instantaneous Poisson rate $a_{i,t}^{(j)}$. If task j is bad, it can never be completed, no matter how much effort the players put in.

We refer to the completion of task j as achieving a *breakthrough* on task j. When a breakthrough is achieved in task $j \in \{1, 2\}$, the principal receives an instantaneous payoff π_j, where $\pi_2 > 0$.[7] As long as no breakthrough has occurred, the principal does not reap any benefits from the project. All players observe the timing of each breakthrough and the identity of the player who attained it. The game ends after the second breakthrough.

All players discount the future at a common rate $r \geq 0$.

[7]The payoff after the first breakthrough, π_1, is not necessarily positive. In an application π_1 may be negative if initiating the second task requires the purchase of equipment or capital.

The set of public histories at time t is denoted $\mathcal{H}^t$. A public history specifies which tasks have produced breakthroughs, the times at which breakthroughs were attained, and which agent attained each breakthrough. Formally, a history $h^t \in \mathcal{H}^t$ contains a sequence of time and agent pairs $(\tau^{(j)}, i^{(j)})$ for $j \in \{1, 2\}$, where $\tau^{(j)} \leq t$ is the time at which the j'th breakthrough was attained by agent $i^{(j)}$. Let $\mathcal{H}$ denote the set of realized (terminal) histories of breakthroughs. A history $h \in \mathcal{H}$ with $J \in \{0, 1, 2\}$ breakthroughs contains a sequence of time-agent pairs $\{(\tau^{(j)}, i^{(j)})\}_{j=1}^{J}$ corresponding to the breakthroughs that were attained throughout the game (where $h = \emptyset$ if $J = 0$). Let $\bar{\mathcal{H}}^t$ denote the set of public histories at time t in which some breakthrough is attained at time t.

At the beginning of the game, the principal offers each agent a wage schedule that is contingent on the public history. The contracts offered are publicly observed, and the principal has the ability to fully commit to these contracts. As shown in Appendix A.2, it is without loss to restrict attention to contracts that compensate agents only at time zero and at histories in which a breakthrough is achieved. We refer to these contracts as *bonus contracts*. Formally, a bonus contract specifies for each t and every history $h^t \in \bar{\mathcal{H}}^t$ a transfer $w_{i,t}(h^t) \in \mathbb{R}$ to each agent i at time t and a transfer to each agent i at time zero, $W_{i,0}$. Bonus contracts provide no transfers nor flow payoffs at histories $h^t \notin (\bar{\mathcal{H}}^t \cup \mathcal{H}^0)$.[8] Even though the agents are symmetric, we allow the principal to offer different contracts to different agents.

Let $\mathcal{H}_i^{(j),t}$ be the set of private histories of agent i at time t in task j. A private history of agent i consists of the public history and the effort exerted by i up to time t. Agent i's strategy is a measurable function $a_{i,t}^{(j)} : \mathcal{H}_i^{(j),t} \to [0, \bar{a}]$ from private histories to pure actions. $a_{i,t}^{(j)}(h_i^t)$ is the instantaneous effort that agent i exerts at time t in task j, at private history $h_i^t \in \mathcal{H}_i^{(j),t}$. If the agents are not working in task j, the effort in task j must be zero, that is, at history $h_i^t \in \mathcal{H}_i^{(k),t}$, for $k \neq j$, $a_{i,t}^{(j)}(h_i^t) = 0$.

The payoffs of the players are as follows. Consider a history $h = \{(\tau^{(j)}, i^{(j)})\}_{j=1}^{J} \in \mathcal{H}$. Under history h, J tasks were completed at times $\{\tau^{(j)}\}_{j=1}^{J}$. Define $\tau^{(0)} = \tau^{(-1)} = 0$, $\tau^{(J+1)} = \infty$, and $a_{i,s}^{(3)} = 0$ for all s. Let $w_{i,\tau^{(j)}}(h)$, for $j \in \{0, \dots, J\}$, denote the realized bonuses received by agent i at times $\{\tau^{(j)}\}_{j=0}^{J}$ under h. The payoff to the principal after recruiting n players is

$$\sum_{j=0}^{J} \left(\pi^j - \sum_{i \in N} w_{i,\tau^{(j)}}(h) \right) e^{-r\tau^{(j)}},$$

where $N := \{1, \dots, n\}$, and $\pi^0 = 0$. Agent i's payoff from exerting effort $(a_{i,t}^{(j)})_{t \geq 0}$ at each task j is

$$\sum_{j=0}^{J+1} \left(w_{i,\tau^{(j)}}(h) e^{-r\tau^{(j)}} - \int_{\tau^{(j-1)}}^{\tau^{(j)}} e^{-rs} \kappa_j a_{i,s}^{(j)} \, ds \right).$$

The bonus contracts offered by the principal define a game between the agents. We will look for *Perfect Bayesian equilibria* of this game. Formally, each agent i chooses $a_{i,t}^{(j)}$

[8] The definition of bonus contracts is adapted from Halac, Kartik, and Liu (2016).

to maximize his expected payoff, given his beliefs about the quality of each task, which evolve according to Bayes' rule as follows.[9] Conditional on strategies $(a_{1,t}^{(j)}, \dots, a_{n,t}^{(j)})$ on task j, player i's private belief that j is good at time t, $p_{i,t}^{(j)}$, satisfies *Bayes' rule* if it evolves according to the differential equation

$$\frac{dp_{i,t}^{(j)}}{dt} = \dot{p}_{i,t}^{(j)} = -p_{i,t}^{(j)}\big(1 - p_{i,t}^{(j)}\big)\big(\tilde{a}_{i,t}^{(j)} + a_{-i,t}^{(j)}\big),$$

where $\tilde{a}_{i,t}^{(j)}$ is i's private effort function, $a_{-i,t}^{(j)} = \sum_{k\neq i} a_{k,t}^{(j)}$, and $p_{i,\tau^{(j-1)}}^{(j)} = \bar{p}^{(j)}$. On path, the belief is common to all, as we restrict attention to pure actions, and it is denoted $p_t^{(j)}$. We have $\dot{p}_t^{(j)} < 0$ whenever we have $a_t^{(j)} := \sum_k a_{k,t}^{(j)} > 0$. Thus, the agents become pessimistic about the feasibility of a task as they and their coworkers exert effort in it.

The objective of the principal is to offer contracts to each agent so as to maximize her expected payoff.

We assume throughout that the agents are subject to limited liability, that is, the principal cannot extract a negative transfer after any history. This assumption is reasonable for agents who are credit constrained, or cannot legally commit to the contract, as is typically the case in employment contracts.

Definition 1. A bonus contract satisfies *limited liability (LL)* if the time zero bonus $W_{i,0}$ is weakly positive and at every time t and history $h^t \in \bar{\mathcal{H}}^t$ the bonus $w_{i,t}(h^t)$ received by each agent i at history h^t is weakly positive.[10]

2.1 *The first-best*

We begin with the characterization of the first-best, which implements the efficient level of experimentation for a fixed number of agents. Suppose task 1 was completed at time $\tau^{(1)}$. In the first-best, $\mathbf{a}^{(2)} = (a_{i,t}^{(2)})_{t,i\in N}$ must maximize the sum of the second-task payoffs of all players:[11]

$$\Pi_2 = \max_{\mathbf{a}^{(2)}} \int_{\tau^{(1)}}^{\infty} \big(p_t^{(2)}\pi_2 - \kappa_2\big)a_t^{(2)} e^{-\int_{\tau^{(1)}}^{t}(p_s^{(2)}a_s^{(2)}+r)\,ds}\,dt,$$

with $\dot{p}_t^{(2)} = -p_t^{(2)}(1-p_t^{(2)})a_t^{(2)}$ and $p_{\tau^{(1)}}^{(2)} = \bar{p}^{(2)}$. In the previous expression, $e^{-\int_{\tau^{(1)}}^{t} p_s^{(2)}a_s^{(2)}\,ds}$ is the probability that no breakthrough has occurred yet and $p_t^{(2)}a_{i,t}^{(2)}e^{-\int_{\tau^{(1)}}^{t} p_s^{(2)}a_s^{(2)}\,ds}$ is the probability density that i obtains the first breakthrough in task 2 at time t. The integrand is positive if and only if $p_t^{(2)}\pi_2 > \kappa_2$. Therefore, the payoff is maximized by setting $a_{i,t}^{(2)} = \bar{a}$ for $i \in N$ if $p_t^{(2)}\pi_2 > \kappa_2$, and $a_{i,t}^{(2)} = 0$, otherwise. Thus, each agent exerts effort as long as the expected marginal gain from effort is above its marginal cost.

[9]The payoff of each agent can be written as a function of the instantaneous belief about the feasibility of each task. For completeness, these payoffs are derived and presented in Section A.1 of the Appendix.

[10]It is without loss to restrict attention to bonus contracts also under limited liability as shown in Section A.2 of the Appendix.

[11]A change of variables shows that Π_2 does not depend on $\tau^{(1)}$.

Now, at time zero, the first-best experimentation solves

$$\Pi_1 = \max_{\mathbf{a}^{(1)}} \int_0^\infty \left(p_t^{(1)}(\pi_1 + \Pi_2) - \kappa_1\right) a_t^{(1)} e^{-\int_0^t (p_s^{(1)} a_s^{(1)} + r)\, ds}\, dt,$$

with $\mathbf{a}^{(1)} = (a_{i,t}^{(1)})_{t,i\in N}$, $\dot{p}_t^{(1)} = -p_t^{(1)}(1 - p_t^{(1)})a_t^{(1)}$, and $p_0^{(1)} = \bar{p}^{(1)}$. As before, the solution is a threshold strategy for each agent: $a_{i,t}^{(1)} = \bar{a}$ when $p_t^{(1)}(\pi_1 + \Pi_2) > \kappa_1$ and $a_{i,t}^{(1)} = 0$ when $p_t^{(1)}(\pi_1 + \Pi_2) \leq \kappa_1$.

Thus, if $\bar{p}^{(j)}(\pi_j + \Pi_{j+1}) > \kappa_j$, in the first-best solution the agents work at full speed in task j for length of time

$$\bar{T}^{(j)} = \frac{-\ln\left(\dfrac{1 - \bar{p}^{(j)}}{\bar{p}^{(j)}}\right) + \ln\left(\dfrac{\pi_j + \Pi_{j+1} - \kappa_j}{\kappa_j}\right)}{n\bar{a}}, \tag{1}$$

with $\Pi_3 := 0$. Otherwise, if $\bar{p}^{(j)}(\pi_j + \Pi_{j+1}) \leq \kappa_j$ agents do not exert effort. When positive, the total amount of work exerted conditional on no breakthrough is given by $-\ln((1 - \bar{p}^{(j)})/\bar{p}^{(j)}) + \ln((\pi_j + \Pi_{j+1} - \kappa_j)/\kappa_j)$. This amount does not depend on the number of agents nor on their maximum effort $\bar{a}$. The total amount of work decreases in the cost of effort κ_j and increases in the initial belief $\bar{p}^{(j)}$. We will sometimes write $\bar{T}^{(j)}(n)$ to express $\bar{T}^{(j)}$'s dependence on n.

We will assume throughout that the instantaneous payoffs satisfy $\bar{p}^{(2)} \cdot \pi_2 > \kappa_2$ and $\bar{p}^{(1)}(\pi_1 + \Pi_2) > \kappa_1$ to ensure that the principal implements positive experimentation in both tasks.

2.2 *Procrastination rents*

Before characterizing the optimal contract, let us give an intuition for why under limited liability the agents must receive payoffs in excess of their cost of effort. These excess payoffs are information rents.

If agent i receives no rents while exerting effort in task j, then i must receive bonus $w_{i,t}^{\mathrm{NR}}$ satisfying $p_t^{(j)} w_{i,t}^{\mathrm{NR}} - \kappa_j = 0$, when successful at time t. If agent i receives less than $w_{i,t}^{\mathrm{NR}}$, he would not exert effort at time t. We call $w_{i,t}^{\mathrm{NR}}$ the *no-rent contract*.

No nonzero effort function $a_{i,t}$ is incentive compatible under its corresponding no-rent contract $w_{i,t}^{\mathrm{NR}}$, as i can guarantee a strictly positive payoff via a deviation that involves exerting no effort in a time interval. In fact, during the interval i's payoff is zero, the same payoff he would have obtained during those times under effort $a_{i,t}$. After the deviation, however, the agent is more optimistic than he would have been if he had behaved according to $a_{i,t}$, implying $p_{i,t}^{(j)} w_{i,t}^{\mathrm{NR}} > \kappa_j$. This means that i is able to extract an information rent by delaying effort.

Thus, in order to induce positive effort the principal must provide information rents. We call these rents *procrastination rents* because they stem from an agent's ability to delay effort.[12]

[12] When there is no uncertainty about the feasibility of the task, that is, $p_t^{(j)} = 1$ for all t, the agent does not learn, and the principal is able to induce any effort function by setting $w_{i,t}^{\mathrm{NR}} = \kappa_j$, for each agent i.

3. Benchmark: Project with a single task

In this section, we characterize the optimal contract in the benchmark one-task project. As there is only one task, we omit the reference to the task in our notation. Let $w_{i,t}$ denote the transfer agent i receives when he achieves a breakthrough at time t.[13] Paying an agent for another agent's success reduces the incentives to exert effort, and negative transfers are not permitted by the limited liability constraint. Therefore, the only bonus payment allocated at the breakthrough is given to the agent who achieves it. $W_{i,0}$ denotes the transfer at time zero.

Having recruited the agents, the principal seeks to maximize her payoff over bonus contracts and effort functions, solving the following program:

$$\max_{a_{i,\cdot},w_{i,\cdot},W_{i,0}} \sum_i \int_0^\infty p_t a_{i,t}(\pi - w_{i,t}) e^{-\int_0^t (p_s a_s + r)\,ds}\,dt - W_{i,0}, \tag{OB}$$

subject to

$$a_{i,\cdot} \in \operatorname*{argmax}_{\tilde{a}_{i,\cdot}\in[0,\bar{a}]} \int_0^\infty (p_t \tilde{a}_{i,t} w_{i,t} - \kappa \tilde{a}_{i,t}) e^{-\int_0^t (p_s(a_{-i,s}+\tilde{a}_{i,s})+r)\,ds}\,dt, \tag{IC}$$

$$\int_0^\infty (p_t a_{i,t} w_{i,t} - \kappa a_{i,t}) e^{-\int_0^t (p_s a_s + r)\,ds}\,dt + W_{i,0} \geq 0, \tag{IR}$$

$$w_{i,s} \geq 0, \qquad W_{i,0} \geq 0, \tag{LL}$$

for $i \in N$ and time $s \geq 0$. The principal's objective function (OB) is the expected payoff of the principal when she offers bonus contract $(w_{i,t}, W_{i,0})_{i,t}$ and implements effort $(a_{i,t})_{i,t}$. Since the effort of the agents is unobserved, the (IC) constraint requires that the effort function $a_{i,t}$ be optimal for each agent i. The (IR) constraint says that each agent's expected payoff must be greater than his outside option, which is set to zero, and (LL) is the limited liability constraint.[14]

The principal designs the optimal bonus contract so as to pay as little rents as possible while preventing procrastination from her desired effort profile. The following proposition characterizes the optimal contract as the solution of a differential equation and shows that the principal chooses to implement full-speed experimentation up to a deadline.

Proposition 1 (Optimal single-task contract). *Given effort functions* $\mathbf{a} = \{(a_{i,t})_{t\geq 0}\}_{i=1}^n$ *with* $T_i := \sup\{t | a_{i,t} > 0\} < \infty$*, an optimal bonus contract for agent* i*,* $(w_{i,t}, W_{i,0})$*, that implements* $\mathbf{a}$ *is characterized by the following conditions:*

$$\dot{w}_{i,t} = (a_{-i,t} + r)(w_{i,t} - \kappa) - r\kappa e^{x_t}, \qquad w_{i,T_i} = \kappa\big(e^{x_{T_i}} + 1\big) = \kappa/p_{T_i}, \qquad W_{i,0} = 0, \tag{2}$$

[13] In the one task project, there is only one possible history preceding a breakthrough, the history in which no breakthrough has occurred yet. Thus, the contract can condition only on the timing of the breakthrough and the agent who attained it.

[14] In the absence of a limited liability constraint, the principal can extract full surplus and implement the first-best effort by "selling each agent his own arm." That is, each agent pays the expected value of his experimentation and receives π if he is first to attain a breakthrough.

for $t \le T_i$, *where* $x_t = \int_0^t (a_{i,s} + a_{-i,s})\,ds + \log((1-\bar{p})/\bar{p})$, *and satisfies* $w_{i,t} = 0$ *for* $t > T_i$. *This bonus contract is unique, up to almost sure equivalence at every t such that* $a_{i,t} > 0$.[15]

The optimal effort function is given by $a_{i,t} = \bar{a}$ *for* $t \le T^*(n) := \ln[(\pi - \kappa)\bar{p}/(\kappa(1-\bar{p}))]/(1+n)\bar{a} = \frac{n}{n+1}\bar{T}(n)$, *and* $a_{i,t} = 0$, *otherwise. The marginal value of each additional agent is decreasing and converges to zero. Therefore, the optimal number of experimenting agents,* n^* *is finite and generically unique.*

All proofs are in the Appendix.

An important observation is that when rewarded by this optimal contract the agents are indifferent between exerting effort now and at the next instant. Intuitively, if an agent's incentive to exert effort is strict at an instant, the principal can reduce bonuses and continue to induce the same level of effort at that instant. Changing the contract at one instant, however, can affect the incentives at all times. In the proof of Proposition 1 we show, using Pontryagin's principle, that the contract that satisfies these "local" incentive compatibility constraints is, in fact, globally incentive compatible. Therefore, in this continuous time setting the constraints that "bind" are, in a sense, the local, one-instant-to-another constraints.

For an intuition for why the optimal bonus contract prevents each agent from shifting effort from the present instant to the next, notice that equation (2) requires that the agent's bonus increase just fast enough to compensate him for the loss in the prize due to the chance that the opponents achieve a success at the current instant, plus the discount on the prize he is more likely to get at the next instant if the arm is good, minus the cost savings from delaying effort to the future.

For an alternative intuition based on mechanism design, notice that the agent's instantaneous payoff $u_{i,t} := (1-\bar{p})[(w_{i,t} - \kappa)e^{-x_t} - \kappa]e^{-rt}$, with $w_{i,t}$ characterized by (2), satisfies the differential equation $\dot{u}_{i,t} = -a_{i,t}(1-\bar{p})(w_{i,t} - \kappa)e^{-x_t - rt} = -a_{i,t}(u_{i,t} + \kappa e^{-rt}(1-\bar{p}))$ with boundary condition $u_{i,T_i} = 0$. By choosing not to experiment any longer, the agent who has experimented without success according to the equilibrium strategies up to time t—the type t player—can later "imitate" any lower type $s > t$, with $p_s < p_t$. As in the classical mechanism design result, he must be compensated in the amount of the payoff of the lowest type (type T_i) plus the integral of the partial derivative of the payoff with respect to type t, $\dot{u}_{i,t}$. The instantaneous payoff, $u_{i,t}$, decreases with the reduced probability of attaining the prize after exerting effort for an additional instant. Notice that $u_{i,t}$ does not depend on the effort function of the agents in $-i$, implying, together with the boundary condition, that at the optimal contract each player's expected payoff is independent of the opponents' experimentation assignment. This independence will play a crucial role in the characterization of the two-task contract in the next section.

The differential equation (2) can be solved explicitly for any choice of effort functions $\mathbf{a} = \{(a_{i,t})_{t\ge 0}\}_{i=1}^n$. Its solution is given by

$$w_{i,t}(T_i, \kappa, \mathbf{a}, \bar{p}) = \kappa\left(e^{\int_t^{T_i}(a_{i,s} - r)\,ds + x_t} + 1\right) + e^{rt + \int_0^t a_{-i,s}\,ds} \int_t^{T_i} \kappa r e^{-r\tau + \int_0^\tau a_{i,s}\,ds + x_0}\,d\tau. \tag{3}$$

[15] If $a_{i,t} = 0$, then any transfer $\tilde{w}_{i,t} < w_{i,t}$ implements $a_{i,t}$ at time t.

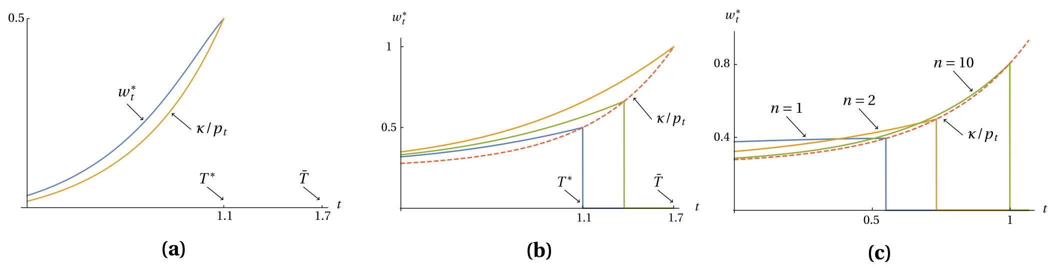

FIGURE 1. (a) w_t^*: Optimal bonus contract. κ/p_t: no-rent contract $(n, r, \bar{a}) = (2, 5, 1)$. (b) Solid curves: bonus contracts for different stopping times. Dashed curve: no-rent contract $(n, r, \bar{a}) = (2, 0.5, 1)$. (c) Bonus payment for different numbers of agents $(r, \bar{a}) = (0.5, 3/n)$. $(\kappa, \bar{p}, \pi) = (1/4, 9/10, 1)$.

Proposition 1 shows that the effort functions do not differ across agents, implying that in the one-task benchmark, optimal contracts offered to symmetric agents are symmetric. In Figure 1(a), w_t^* denotes the symmetric bonus contract. It increases in order to compensate the agents as they become more pessimistic over time but it cannot increase so fast as to make the agents want to delay their effort. w_t^* is the lowest bonus contract that provides incentives to exert maximal effort up to time $T^*(n^*)$, as the principal aims to minimize the information rents paid out to the agents. As discussed in Section 2.2, an agent's bonus must satisfy $w_t^* \geq \kappa/p_t = \kappa(1 + e^{x_t})$ for him to be willing to exert effort at time t. Therefore, from (2), w_t^* is increasing.[16] $T^*(n^*)$ does not depend on r, and hence neither does $w_{T^*(n^*)}^*$, due to the boundary condition in (2). However, $\dot{w}_t^*$ increases in r for fixed w_t^* implying that w_t^* decreases in r for each t. As agents become more impatient, they value future bonuses less, and thus their temptation to procrastinate is reduced. When $r = 0$ and $n^* = 1$, the principal offers a constant bonus.

The gross marginal value of adding an agent—ignoring the recruitment cost c—is positive no matter how many agents the principal hires (she can always induce zero effort). In the proof of Proposition 1, we show that the marginal value of an additional agent decreases in the number of agents hired. The principal's payoff is bounded above by the total surplus of the relationship in the limit as $n \to \infty$. This means that when she has hired sufficiently many agents the value of an additional agent must be close to zero. Therefore, whenever $c > 0$ the principal hires a finite number of agents n^*. The number of agents n^* is the smallest n such that the (decreasing) marginal value of adding agent $n + 1$ is less than the recruitment cost c.

To gain intuition for why the principal induces maximal effort, notice that the principal's payoff from time t can be written as $\tilde{\Pi}_t := \tilde{\Pi}_{i,t} + \tilde{\Pi}_{-i,t}$, where $\tilde{\Pi}_{i,t} = \int_t^\infty (p_t\pi - \kappa)e^{-\int_0^t a_s\, ds - rt} a_{i,s}\, ds - V_{i,t}$ and $\tilde{\Pi}_{-i,t} = \int_t^\infty (p_t\pi - \kappa)e^{-\int_0^t a_s\, ds - rt} a_{-i,s}\, ds - \sum_{j \neq i} V_{j,t}$, and $V_{k,t}$

[16] For comparison, in Bonatti and Hörner (2009) the bonus is decreasing. In their model, the principal does not observe individual successes and must compensate all agents when there is a success. As a result, the public-good rents (as introduced in the next section) are large enough that the optimal contract is decreasing in time. In Klein (2016), the optimal contract is also decreasing for small enough r because an impending deadline reduces the costs of providing incentives as time progresses.

denotes the payoff of agent $k \in N$. From our previous discussion, we know that the effect on $V_{i,t}$ of shifting agent i's effort is zero up to the first order. Similarly, the effect on $V_{j,t}$ for $j \neq i$ is zero as, under the optimal contract, j's payoff does not depend on i's experimentation assignment. The value of marginally shifting i's effort from time interval $[t, t + \Delta t]$ to interval $[t + \Delta t, t + 2\Delta t]$ on $\tilde{\Pi}_{i,t}$ can be inferred from (2). In (2), the value of shifting effort from one instant to the next is equated to the bonus change. Therefore, if $w_{i,t}$ were constant and equal to π the value of delaying the effort of agent i on $\tilde{\Pi}_{i,t}$ is, up to the first order, equal to $\Delta t(-\bar{p}(a_{-i,t} + r)(\pi - \kappa)e^{-\int_0^t a_s\,ds - rt} + r\kappa(1 - \bar{p})e^{-rt})$. The effect on $\tilde{\Pi}_{-i,t}$ is $\bar{p}a_{-i,t}(\pi - \kappa)e^{-\int_0^t a_s\,ds - rt}\Delta t$—as the shift in effort increases the probability that the opponents achieve a success in $[t, t + \Delta t]$ by decreasing the rate of reduction of the belief. Aggregating these effects, we obtain that the value of shifting i's effort to a later instant is $-\bar{p}e^{-\int_0^t a_s\,ds} r(\pi - \kappa/p_t)\Delta t < 0$, which means the principal prefers all effort to be exerted as early as possible.[17]

The stopping time $T^*(n)$ satisfies $T^*(n) = \frac{n}{n+1}\bar{T}(n) < \bar{T}(n)$ and, therefore, experimentation stops inefficiently early. As illustrated in Figure 1(b), procrastination rents increase with the threshold of experimentation. Thus, the principal trades off longer experimentation with increased rents and opts to stop experimentation at an early, inefficient time. To see why the principal prefers to stop experimentation before the efficient threshold, recall that at the first-best experimentation stops when $p_t\pi = \kappa$. Thus, close to the efficient deadline, the principal has to pay a bonus that is close to π to induce effort. If the principal stops experimentation slightly earlier than $\bar{T}$, she incurs a loss in profits from experimentation of second order, since her profits are negligible at times close to $\bar{T}$. At the same time, by reducing the experimentation threshold she reduces w_t^* at every time in which the agents experiment, generating a first-order gain.[18] As n^* increases, the experimentation threshold converges to the efficient one as shown in Figure 1(c).[19]

4. Project with two tasks

In a project with two tasks, the agents have to obtain a success in the first task before they can start experimentation in the second task. If they achieve a breakthrough in the second task, they complete the project. There are many real-world settings in which innovation requires the completion of sequential uncertain tasks. As an example, software is typically developed by groups of engineers who participate in multiple steps of the creation process. These steps may include planning the functionalities and characteristics of the product, creating a prototype, improving performance, and solving issues

[17] In Section A.3.1 of the Appendix, we formalize the intuition that the optimal contract generates indifference for agent i by showing that the gain from shifting effort from one Δt length interval to the next is at most of second order in Δt. The previous discussion does not take into account that the shift in effort changes the bonus of agent i at times before t as well. In Section A.3.1, it is also shown that this last effect is at most of the third order.

[18] The derivative of w_t^* with respect to T^* is given by $\kappa\bar{a}e^{\bar{a}((n-1)t+T^*)+r(t-T^*)+x_0}$, which is bounded away from zero.

[19] In Figure 1(c), $\bar{a} \cdot n$ and, therefore, $\bar{T}$ is kept constant in n.

and bugs. If any of these steps cannot be completed satisfactorily, the project may be abandoned. Medical researchers working on the development of a drug must first find a promising class of compounds and then experiment to find a drug that has the desired effect.

If there are two contracts that give the same discounted payoffs after every history—and, thus, implement the same effort—we assume without loss that the principal chooses the contract that pays each agent at the earliest possible time. For this class of contracts, in analogy to the one-task case, it is not profitable for the principal to reward an agent when another agent succeeds.

4.1 *The second-task contract*

In the following proposition, we show that the second-task contract has the same form as the one-task contract, with agents exerting maximal effort, but with experimentation deadlines that depend on the realized first-task history.

Proposition 2 (Second-task contract). *After each first-task history $h^{(1)}$ and timing of first-task breakthrough $\tau^{(1)}$, and for each $i \in N$, the optimal contract implements effort function $a_{i,t}^{(2)} = \bar{a}$ if $t \in [\tau^{(1)}, \tau^{(1)} + T_i(h^{(1)})]$ and $a_{i,t}^{(2)} = 0$ otherwise, for some history-dependent experimentation threshold $T_i(h^{(1)}) \geq 0$. Agent i receives bonus $w_{i,t-\tau^{(1)}}(T_i(h^{(1)}), \kappa_2, \{(a_{j,t-\tau^{(1)}}^{(2)})_{t\geq\tau^{(1)}}\}_{j=1}^{n}, \bar{p}^{(2)})$, as defined in equation (3), if he succeeds in the second task at time t.*

Proposition 2 is key in allowing us to solve the two-task model. It implies that although the set of feasible second-task continuation contracts is large, the optimal one can be parametrized by a single variable for each agent i: i's experimentation length in the second task, denoted $T_{i,\tau}^{k'}$, following a first-task history in which player k' succeeds at time τ. This result allows us to write the two-task problem as an optimal control problem in which the agents' second-task experimentation thresholds are control variables.

To establish the result in Proposition 2, we show that the optimal way to provide a given promised utility after the first breakthrough is realized involves a nonnegative bonus at the beginning of the second task and a second-task bonus contract that is analogous to the one-task contract. The reason is that, as shown in Proposition 1, this is the bonus schedule that minimizes the incentive costs for any given effort function profile, and the principal prefers to implement effort as early as possible.

4.2 *The first task contract*

The expected payoff of the agents after the realization of the first-task breakthrough has two components: the bonuses that they receive at the time of the first breakthrough and the expected payoff from their second-task experimentation. Unlike in the one-task case, an agent's payoff following an opponent's success is not necessarily zero. Rather, it is strictly positive if he receives an experimentation assignment with positive effort in the second task.

Let $w_{j,t}^{(1)}$ denote the bonus that agent j receives when he achieves a breakthrough in the first task at time t and let $\nu_{i,t}^{j}$ denote the expected payoff that agent i obtains in the second task after that history. $\nu_{i,t}^{j}$ consists of i's procrastination rents in the second task. Let $\mathbf{T}_{\tau}^{j} = (T_{i,\tau}^{j})_{i=1}^{n}$ be a vector of second-task experimentation thresholds, where $T_{i,\tau}^{j}$ denotes the length of experimentation in task 2 of player i after player j achieves a breakthrough at time τ. From the characterization in Proposition 2, $\nu_{i,t}^{j}$ can be calculated explicitly, as a function of $\mathbf{T}_{\tau}^{j}$, and depends *only on i's* second-task threshold and not on opposing players' thresholds.[20] We will denote it $\nu_i(T_{i,\tau}^{j})$ when we want to make the dependence explicit.[21]

Let $x_0^{(1)} = \log((1-\bar{p}^{(1)})/\bar{p}^{(1)})$, $x_t^{(1)} = x_0^{(1)} + \int_0^t a_s^{(1)}\,ds$, and $u_{i,t} = w_{i,t}^{(1)} + \nu_{i,t}^{i}$. $u_{i,t}$ is the total reward that agent i receives when he succeeds in the first task at time t. The following lemma characterizes the rent-minimizing $u_{i,t}$ that implements a first-task effort function, given second-task rents $\{\nu_{i,\tau}^{j}\}_{i,j,\tau\geq 0}$.

Lemma 1 (Lower bound on first-task rents). *The minimum reward, $u_{i,t}^{\min}$, that each agent i must receive to implement effort schedules $\mathbf{a}^{(1)} = \{(a_{j,t}^{(1)})_{t\geq 0}\}_{j=1}^{n}$, with $T_i^{(1)} = \sup\{t|a_{i,t}^{(1)} > 0\} < \infty$, is given by*

$$u_{i,t}^{\min}(\mathbf{a}^{(1)}) = \underbrace{w_{i,t}(T_i^{(1)}, \kappa_1, \mathbf{a}^{(1)}, \bar{p}^{(1)})}_{\textit{single task contract}} + \underbrace{\sum_{j\neq i}\int_t^{\infty} e^{-\int_t^{\tau}(r+a_{-i,s}^{(1)})\,ds} a_{j,\tau}^{(1)}\nu_{i,\tau}^{j}\,d\tau}_{\textit{exp. payoff when slacking in the first task}}. \tag{4}$$

As in the one-task case, $u_{i,t}^{\min}$ makes agent i exactly indifferent between exerting effort in one instant and the next, and it consists of two terms. The first term is the bonus of a one-task contract, which contains all the necessary procrastination rents. The second term is the payoff that i would obtain if he were to stop his first-task effort at time t and act according to his equilibrium strategy after an opponent completes the first task. The second term is positive whenever agent i receives a positive experimentation assignment in the second task after another agent's success. Thus, whenever agent i's expected payoff is strictly positive after any other agent succeeds he must receive—in addition to the procrastination rents—a compensation that prevents him from shirking in the first task. This means that the second-task rents generate an *endogenous* free-riding concern, which raises the principal's costs of providing incentives. We will see that the agents do not free-ride when offered the rent-minimizing contract. The principal raises their compensation to prevent free-riding and, at the same time, distorts their second-task allocations to reduce the extra rents. These *public-good rents* are not present in a one-task project or in the last task of a two-task project.

[20] As discussed after Proposition 1, the independence of $\nu_{i,t}^{j}$ on the opponents' experimentation can be seen directly by replacing $w_{i,t}$, from equation (3) into each agent i's instantaneous utility provision $u_{i,t}$. There is a unique instantaneous utility provision that makes the agent indifferent between shifts of effort across consecutive instants and it is characterized by a differential equation. This payoff, as it is unique, must be independent of the opponents' experimentation.

[21] $\nu_i(T)$ has a closed-form expression, which is given by equation (37) in the Appendix.

The analysis now divides into cases based on whether limited liability or incentive compatibility binds in setting the reward for completing the first task. To understand why different constraints may bind, recall that by Lemma 1, $u_{i,t}^{\min}$ minimizes the expected bonuses for the completion of the first task, subject to incentive compatibility. If the agent's second-task experimentation payoff after a success, $\nu_{i,t}^{i}$ is too high ($u_{i,t}^{\min} - \nu_{i,t}^{i} < 0$), then in order to reward the agent according to $u_{i,t}^{\min}$, the principal would have to give i a negative bonus upon success. However, the limited liability constraint restricts the bonus to be positive and, therefore, the actual reward at time t, $u_{i,t}$, must be weakly greater than $u_{i,t}^{\min}$.

In the case in which the bonus implied by $u_{i,t}^{\min}$, $u_{i,t}^{\min} - \nu_{i,t}^{i}$, is strictly positive for all t, the payoff in the second task is not sufficient to provide incentives in the first task and $u_{i,t}^{\min} = u_{i,t}$. We refer to this case as the *costly incentives case*. In contrast, when $\nu_{i,t}^{i} \geq u_{i,t}^{\min}$ for some t we will say that the parameters fall in the *noncostly incentives case*.

4.2.1 *Costly first-task incentives* In the costly first-task incentives case, the minimum first-task reward binds and we have $u_{i,t} = u_{i,t}^{\min}$ as defined in (4) and $\nu_{i,t}^{i} < u_{i,t}$. Thus, the successful agent receives a bonus upon succeeding in the first task.

Definition 2. We say that the parameters fall in the *costly incentives case for thresholds* $\mathbf{T}^{(1)} = (T_k^{(1)})_{k\in\{1,\dots n\}}$ *and* $\mathbf{T}^{(2)} = (T_{k,t}^{j})_{k,j\in\{1,\dots n\}}$ if for every player i and at each time t,

$$\nu_i\big(T_{i,t}^{i}\big) < \kappa_1\big(1 + e^{x_t^{(1)}}\big) + e^{x_t^{(1)}} \int_t^{T_i^{(1)}} e^{\int_t^{\tau}(a_{i,s}^{(1)} - r)\,ds} a_{i,\tau}^{(1)} \kappa_1 \, d\tau$$
$$+ \sum_{j\neq i} \int_t^{\infty} \nu_i\big(T_{i,t}^{j}\big) a_{j,\tau}^{(1)} e^{-\int_t^{\tau}(a_{-i,s}^{(1)} + r)\,ds}\, d\tau, \tag{5}$$

where for each k, $a_{k,\tau}^{(1)} = \bar{a}$ for $\tau \leq T_k^{(1)}$ and $a_{k,\tau}^{(1)} = 0$, otherwise.

The right-hand side in equation (5) is the minimal expected payoff that agent i must receive to experiment at full speed until time $T_i^{(1)}$. The first two terms in the right-hand side are the rents that the agent must receive due to procrastination concerns. These terms increase in κ_1 and decrease in $\bar{p}^{(1)}$ (for fixed $T_i^{(1)}$). The third term is the public good rents. The left-hand side in (5) is agent i's second-task payoff as a function of the deadline $T_{i,t}^{i}$, which decreases in $\bar{p}^{(2)}$ and increases in κ_2.

Let $\mathbf{T}^{(2)} = (T_{i,t}^{j})_{i,j\in\{1,\dots,n^*\},t}$ be the second-task thresholds that are uniquely characterized by the first-order condition (33), in the Appendix, as a function of first-task thresholds $\mathbf{T}^{(1)} = (T_i^{(1)})_{i\in\{1,\dots,n^*\}}$.[22] Let $\mathbf{T}^{(1)}$ and n^* be the first-task thresholds and number of agents that are optimal given $\mathbf{T}^{(2)}$ (as a function of $\mathbf{T}^{(1)}$) and bonuses $(w_{i,t}^{(1)} = u_{i,t}^{\min} - \nu_i(T_{i,t}^{i}))_{i,t}$.[23]

[22] The first-order conditions for $\mathbf{T}^{(2)}$ are derived from applying Pontryagin's principle to these control variables, given the task one-effort function implied by $\mathbf{T}^{(1)}$. These thresholds cannot be calculated explicitly, but for each $\mathbf{T}^{(1)}$ they are uniquely defined by the first-order conditions. The latter follows from the concavity of the principal's second-task payoff as a function of promised utilities that is shown in B.3.4.

[23] $T_i^{(1)}$ must satisfy that $h_{i,T_i^{(1)}}^{(2)} = 0$ ($h_{i,t}^{(2)}$ defined in equation (31)), with $\tilde{\gamma}_{i,t} = 0$. It is straightforward that the optimal number of agents n^* is finite for every $c > 0$ as in the single-task benchmark (see Corollary 1).

Let $T^{*(2)}$ the principal's optimal second-task threshold in the absence of a first task.

Proposition 3. *If parameters fall in the costly incentives case for thresholds* $\mathbf{T}^{(1)}$ *and* $\mathbf{T}^{(2)}$*, then the optimal contract exists and is such that:*[24]

(i) The agents exert maximum effort in the first task until thresholds $(T_j^{(1)})_{j\in\{1,\dots,n^*\}}$.

(ii) If agent i *achieves the first breakthrough at time* t*, each agent* $j\neq i$ *such that* $T_j^{(1)}>t$ *receives the same experimentation threshold,* $T_{-i,t}^i<T^{*(2)}$. $T_{-i,t}^i$ *decreases and converges to zero in* t*. For each player* j *with* $T_j^{(1)}<t$, $T_{j,t}^i>T_{-i,t}^i$*, and* $T_{j,t}^i$ *is strictly decreasing in* $T_j^{(1)}$*. Agent* i*'s experimentation threshold is such that* $p^{(2)}_{T_{i,t}^i}\pi_2=\kappa_2$.

(iii) The expected payoff of player i *if he succeeds at time* t *is* $u_{i,t}^{\min}$ *and his bonus is* $w_{i,t}^{(1)}=u_{i,t}^{\min}-\nu_{i,t}^i>0$.

(iv) There are parameter values for which players' allocations are asymmetric (i.e., $T_i^{(1)}\neq T_j^{(1)}$ *for* $i\neq j$*).*

A simple sufficient condition for the parameters to fall in the costly incentives case is $\bar p^{(1)}\nu_i(\bar T^{(2)}(n)\cdot n)\le\kappa_1$, where $\bar T^{(2)}(n)$ denotes the first-best threshold in equation (1) as a function of n. This follows from condition (5), since $T_{i,t}^i\le\bar T^{(2)}(n)\cdot n$. Intuitively, if the payoff of agent i at the efficient experimentation length—assuming the opponents do not experiment—$\nu_i(\bar T^{(2)}(n)\cdot n)$, is less than the minimum bonus that induces effort in the absence of information rents, $\kappa_1/\bar p^{(1)}$, then the principal must reward a successful agent via a bonus in addition to allocating an efficient experimentation threshold. When $r>\bar a$, then $\nu_i(\bar T^{(2)}n)\le(1-\bar p^{(2)})\kappa_2\bar a^2/(r(r-\bar a))$ and, therefore, $\bar p^{(1)}(1-\bar p^{(2)})\kappa_2\bar a^2\le\kappa_1 r(r-\bar a)$ is sufficient for the parameters to fall in the costly incentives case.[25]

From point (ii) in the proposition, in the costly incentives case, a successful agent is rewarded with a longer second-task experimentation that accords him higher procrastination rents. This means that he faces less competition and receives higher bonuses upon second-task success at any time t than his opponents receive for a success at the same time. An experimenting agent j whose coworker, agent i, succeeds at time t, is assigned experimentation threshold $T_{-i}^{i,t}\in(0,T^{*(2)})$ (symmetric across unsuccessful players), which is decreasing in the total effort exerted by j up to time t, $\bar at$, and converges to zero as t converges to infinity.

Notice that it is not optimal for the principal to bar unsuccessful agents from second-task experimentation. A short enough experimentation period requires public-good rents that are close to zero while producing a gain from experimentation that is bounded

[24]The contract is uniquely optimal among bonus contracts. By risk neutrality, there are equivalent contracts that induce the same effort and give the same expected payoff after every history but provide payments at different times.

[25]See Corollary 4 in the Appendix.

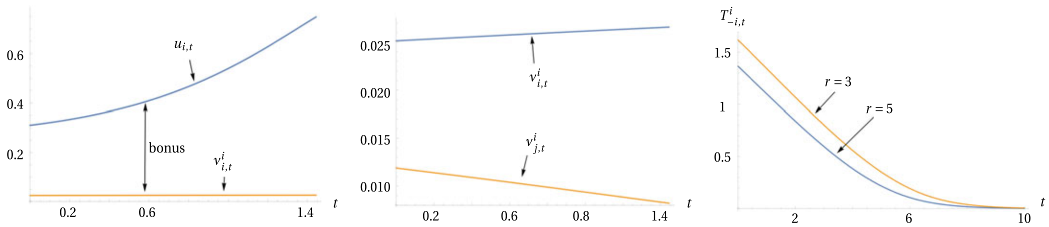

FIGURE 2. (Left): Expected payoff ($u_{i,t}$), bonus and continuation payoff after the first discovery ($\nu^i_{i,t}$) as a function of time t. (Middle): Expected payoff in the second task for agents i ($\nu^i_{i,t}$) and j ($\nu^i_{j,t}$) when agent i succeeds at time t. (Right): Second-task threshold of unsuccessful agent as a function of the timing of the first breakthrough. $(\kappa_1, \kappa_2, \bar{a}, \bar{p}^{(1)}, \bar{p}^{(2)}, \pi_2, \pi_1, n) = (1/4, 1/4, 1, 9/10, 9/10, 5, 0, 2)$.

away from zero.[26] Similarly, it is not optimal to allocate unsuccessful agents to the second-task optimal experimentation amount $T^{*(2)}$ for two reasons. First, given that the successful agent experiments up to the efficient threshold, the threshold that is optimal for an unsuccessful agent in the absence of the first task is strictly less than $T^{*(2)}$. Second, second-task experimentation requires the provision of public good rents in the first task. Due to optimality, the loss from reducing second-task experimentation from the optimum is of second order while its gain is of first order due to the reduction of public-good rents in task one.

Figure 2 (right) illustrates how the experimentation threshold of an unsuccessful agent varies as a function of the timing of the first breakthrough. Intuitively, the optimal distortion increases over time due to the principal's discounting of second-task payoffs and due to the fact that agents who slack expect the first breakthrough to arrive relatively later as they are not exerting effort. Also, a higher continuation payoff when an opponent succeeds at time t increases the rents that must be given to agents for success at every $s < t$. Thus, continuation payoffs have a greater effect on rents when they follow successes that occur later.

An important observation drawn from Proposition 3 is that the principal prefers to reward a successful agent *via an assignment rather than a bonus*. In fact, a successful agent experiments until the efficient experimentation threshold while his opponents experiment until a threshold that is strictly less than the one-task optimal threshold. Alternatively, the principal could have given the successful agent the same reward by offering a larger bonus, while letting him experiment only until the one-task optimal threshold. However, an assignment that gives the agent the same payoff as a bonus also generates additional surplus (from the agent's longer experimentation), and is therefore preferable. More generally, if a second-best solution is such that information asymmetries lead the principal to optimally distort the agents' allocations, she then has room to reward them by undistorting their allocations when they succeed. The intuition that

[26]The latter follows from the assumption that there is strictly positive experimentation in each task in the first-best allocation.

it may be more profitable to reward an agent with tasks that involve more information rents may be present in other second-best settings.[27,28]

It is commonly observed in labor markets that workers are rewarded via promotions rather than just monetary bonuses.[29,30] Our results may provide an explanation for these facts in settings where a promotion allows a worker to garner more private information that is relevant for the performance of his tasks. Because it is optimal for the principal to undistort a successful agent's second-task allocation and distort an unsuccessful agent's, a worker who succeeds on the first task is more likely to succeed on the second task than an identical worker who did not obtain the first-task success. This observation may be related to the empirical finding that workers who are promoted or receive raises are likely to receive another promotion in rapid succession.[31]

An unsuccessful agent's second-task experimentation threshold is decreasing in the time of the first breakthrough. Thus, the successful agent's second-task expected payoff is increasing in the time of the first breakthrough. Figure 2 (middle) shows second-task expected payoffs for successful and unsuccessful agents. The successful agent's overall payoff, $u_{i,t}$, in contrast with the one-task problem, may increase or decrease in the timing of the first breakthrough. The reason is that public-good rents decrease as a function of this timing. If the variation of these rents dominates that of the procrastination rents, the expected reward decreases in the timing of the first breakthrough. Figure 3 (right) shows an example in which $u_{i,t}$ is non-monotonic.

Another surprising feature of the optimal contract, as stated in point (iv) of Proposition 3, is that even though the agents are symmetric, their first task experimentation thresholds may differ. In Lemma 4 and Corollary 4 in the Appendix, we show that for $n = 2$ there are parameter values for which the principal's problem is nonconcave in the choice of $T_1^{(1)}$ and $T_2^{(1)}$, and that the symmetric solution of the first-order condition can be improved upon. A numerical example in which the optimal thresholds are asymmetric is found in Figure 3 (left). In the example, depending on the value of π_1, an agent

[27] In Moroni (2017), it is shown that when agents have an outside option and the value of the outside option increases, agents receive a persistently better contract: their thresholds in the first and second task are longer. The difference in the assignments of successful and unsuccessful agents decreases. Therefore, the gains from a higher outside option persist beyond receiving a higher signing bonus. They translate into higher chances of success, better rewards for success, and better terms of employment in both tasks in every eventuality. These predictions are consistent with the empirical finding that the initial conditions of the labor market for generations of workers have persistent effects. That is to say, workers who face better labor market conditions when they graduate tend to experience persistently better outcomes throughout their careers. See Beaudry and DiNardo (1991) and Baker, Gibbs, and Holmstrom (1994), and Kahn (2010) for evidence on persistent effects of labor market conditions.

[28] In concurrent work, Che, Iossa, and Rey (2021) ask how to incentivize innovation in a procurement setting either via prizes or allocations. They find that allocative differences are preferable and prices are only used when the value of innovation is sufficiently high.

[29] See, for example, Baker, Jensen, and Murphy (1988) and Gibbons and Waldman (1999) for a discussion of the bonuses versus promotions issue.

[30] A prima facie these facts may appear puzzling, since instead of promoting an agent as a reward, the principal can instead give him a bonus and allocate all agents optimally.

[31] Baker, Gibbs, and Holmstrom (1994) find serial correlation in raises and promotions in their study of personnel data of a firm. As an explanation, they propose persistent unobserved heterogeneity, which drives fast advancement of some workers.

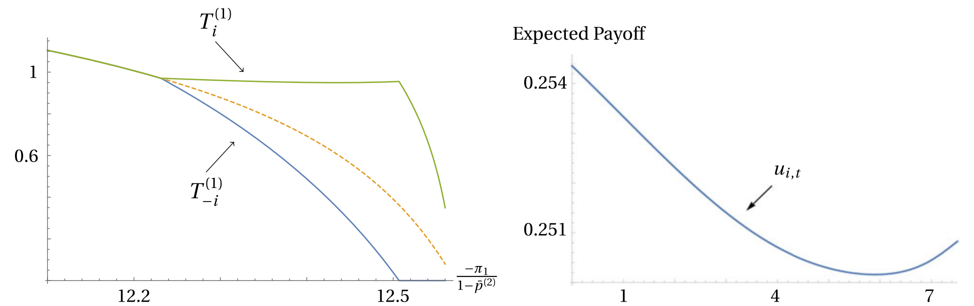

FIGURE 3. (Left): Asymmetric experimentation thresholds in the first task. Agent i's threshold is greater than agent $-i$'s for small values of π_1. Dashed line: Optimal symmetric experimentation threshold. $(\kappa_1, \kappa_2, \bar{a}, \bar{p}^{(1)}, \bar{p}^{(2)}, \pi_2, n, r) = (1/4, 1/9, 1, 0.99, 0.9, 3, 2, 1.5)$. (Right): Expected payoff (including bonus) of agent who succeeds at time t. $(\kappa_1, \kappa_2, \bar{a}, \bar{p}^{(1)}, \bar{p}^{(2)}, \pi_2, \pi_1, n, r) = (1/4, 1/4, 1, 1-10^{-9}, 9/10, 5, 0, 2, 1.5)$.

may receive a shorter threshold or may be barred from participating in the first task altogether. The asymmetries arise because of the public good rents. When one agent is scheduled to work more, he has less incentives to free-ride than the opponent. When the public-good rents are large relative to the rent reductions generated by competition, the principal may prefer to let an agent sit idle to avert the externality he has on a fellow worker who is tempted to slack. This is a rationale for similar workers' (or types of workers') to receive different job assignments.

In Lemma 4, we show that in a two-player game the optimal contract is asymmetric if $\nu^i_{j,T^{S,(1)}} > \kappa_1 e^{2\bar{a}T^{S,(1)}+x_0^{(1)}}$ where $T^{S,(1)}$ denotes the optimal symmetric threshold. This condition is more likely to be satisfied when π_1 is small, the first task is relatively safer ($x_0^{(1)} = \ln[1-\bar{p}^{(1)}/\bar{p}^1]$ is small), $\bar{p}^{(2)}$ is small, and π_2 is large, that is, when the prior probability that the first task is good, the cost of initiating the second task, the uncertainty of the second task, and the value of completing the second task are high.

The following corollary relates to the number of agents that the principal chooses to hire.

COROLLARY 1. *As $c \to 0$, $n^* \to \infty$ and the principal's payoff converges to*

$$\hat{\Pi}^\infty := \bar{p}^{(1)}\Pi^{\infty,(2)} - \kappa_1\left(1 - \left(1-\bar{p}^{(1)}\right)\ln\left(\frac{\Pi^{\infty,(2)}-\kappa_1}{\kappa_1}\cdot\frac{\bar{p}^{(1)}}{1-\bar{p}^{(1)}}\right)\right),$$

where $\Pi^{\infty,(2)} = \bar{p}^{(2)}\pi_2 + \pi_1 - \kappa_2(1-(1-\bar{p}^{(2)})\ln[(\pi_2-\kappa_2)\bar{p}^{(2)}/(\kappa_2(1-\bar{p}^{(2)}))])$. $\hat{\Pi}^\infty$ is also the first-best surplus as $n \to \infty$.

Corollary 1 shows that as the cost of recruitment shrinks to zero, the principal hires more and more agents. In the limit, her payoff approaches the first-best surplus, implying that the agents receive no rents. The result follows from the analysis of the one-task benchmark case. In the second task, the principal's payoff approaches the first-best surplus. In the first task, the agents' payoff can be written as the sum of procrastination rents and public good rents. The former rents approach zero as in the one-task case. The latter approach zero because the second-task payoff after an opponent's success approaches zero as the number of agents increases.

From our analysis, we can straightforwardly draw the following conclusions for two modified versions of the model. First, if the firm has a fixed number of spots and agents can be replaced costlessly after the first round, the principal would fire all agents who do not succeed and keep only the agent who does. The latter would receive a better allocation than the newcomers. The reason is that the unsuccessful agents' allocations are distorted from the principal preferred as a function of the history of play but the new agents' allocation is not. These types of contract are often called *up-or-out contracts*.[32]

Second, if the principal had other potential tasks to which to allocate unsuccessful agents that provide less rents relative to their value, she may choose to allocate agents to these tasks after some histories. These tasks might not be pursued other than as a means to punish agents.[33]

4.2.2 *Noncostly first-task incentives* We now turn to the case in which second-task rents are sufficient to provide incentives in the first task after some histories. This occurs when condition (5) is not satisfied for the costly incentives experimentation thresholds for some time t and agent $i \in N$. In this section, we provide a characterization of symmetric contracts and qualitative results for the potentially asymmetric case. The latter results are stated in the following proposition.

Proposition 4. *If parameters do not fall in the costly incentives case for thresholds $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ in an optimal contract,*

(a) there is an agent i and time thresholds $t^1, t^2 \geq 0$ such that for $t \in [t^1, t^2]$, $u_{i,t} = u^{\min}_{i,t} = \nu^i_{i,t}$.

(b) a player's expected reward, $u_{i,t}(\mathbf{a}^{(1)})$ for optimal effort $\mathbf{a}^{(1)}$, is such that $u_{i,t}(\mathbf{a}^{(1)}) \geq u^{\min}_{i,t}(\mathbf{a}^{(1)})$. If $u_{i,t}(\mathbf{a}^{(1)}) > u^{\min}_{i,t}(\mathbf{a}^{(1)})$, then $u_{i,t}(\mathbf{a}^{(1)}) = \nu^i_{i,t}$, and i has strict incentives to exert effort at time t.[34]

(c) the second-task experimentation thresholds are characterized by equation (33) for each $i, k \in N$, which imply that if agent i achieves the first breakthrough at time t,

- *each agent $j \neq i$ receives experimentation threshold, $T^i_{j,t} \leq T^{*(2)}$. $T^i_{j,t} = T^{*(2)}$ if $u_{j,\tau}(\mathbf{a}^{(1)}) > u^{\min}_{j,\tau}(\mathbf{a}^{(1)})$ almost surely in $\tau \leq t$ for every $j \in N$. Otherwise, $T^i_{j,t} < T^{*(2)}$ for $j \neq i$.*

[32]Tenure track academic contracts and making partner are often cited as examples of up-or-out contracts. See Waldman (1990) and Kahn and Huberman (1988).

[33]More formally, let task 2′ be identical to task 2 except that it gives transfer $\tilde{\pi}_2$ when completed and has prior probability of being good given by $\tilde{p}^{(2)}$. For every $\tilde{\pi}_2 < \pi_2$, there is a time $\bar{t}$ and $\tilde{p}^{(2)} > \bar{p}^{(2)}$ such that if agent i works at time $t > \bar{t}$ and agent $-i$ completes the first task at time t, then the principal assigns agent i to task 2′.

[34]The definition of $u^{\min}_{i,t}(\mathbf{a}^{(1)})$ is given in equation (4).

– *Agent i experiments until a stopping time $T^i_{i,t} \in [T^{*(2)}, \bar{T}^{(2)}n - \sum_{j\neq i} T^i_{j,t}]$. If $u_{i,t}(\mathbf{a}^{(1)}) > u^{\min}_{i,t}(\mathbf{a}^{(1)})$, $T^i_{i,t}$ is the one-task profit maximizing second-task threshold given $T^i_{j,t}$ for $j \neq i$. Therefore, if $T^i_{j,t} = T^{*(2)}$ for every $j \neq i$, then $T^i_{i,t} = T^{*(2)}$. If $u_{i,t} = u^{\min}_{i,t} > \nu^i_{i,t}$, then $T^i_{i,t} = \bar{T}^{(2)}n - \sum_{j\neq i} T^i_{j,t}$.*

Proposition 4 says (a) that the principal must fine-tune the agent's threshold to induce incentives after some histories; (b) that an agent's reward for success takes a similar functional form as in the costly incentives case except that there can be a set of times $\mathcal{T}$ such that the expected reward coincides with $\nu^i_{i,t}$, and not $u^{\min}_{i,t}$, for $t \in \mathcal{T}$ and the agent has strict incentives to exert effort, that is, he is not indifferent between shifting effort from one instant to the next; (c) as in the costly incentives case, the principal distorts the unsuccessful agents' allocations down from the one-task optimal and undistorts the allocation of the successful agent. If all agents have strict incentives to exert effort up to time t, then the time-t second-task assignment corresponds to the one task optimal. At times at which an agent receives a positive transfer upon success, he must be assigned an efficient experimentation deadline.

Intuitively, suppose that at time t the principal can provide incentives relatively cheaply (because the second-task rents are high enough). Then the contract at a time $s > t$ must be distorted less because the effort at time s has a smaller cost (in terms of increased rents) on the cost of effort at time t. When incentives are free because the agent has strict incentives to exert effort at every time $\tau \leq t$—from anticipating high second-task rents—the time-t bonus does not include distortions due to previous times' efforts.

Proposition 4 does not characterize the effort allocation implemented by the principal in the first task. In the special case of symmetric contracts, the following proposition shows that players exert full effort until a deadline.

Proposition 5 (Symmetric contracts in noncostly incentives case). *Suppose parameters do not fall in the costly incentives case for thresholds $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$. Then:*

(i) If the principal is restricted to choose over symmetric contracts, the optimal contract exists and it is such that all agents exert full effort until a deadline.

(ii) If the unrestricted contract is symmetric, then it implements full effort until a deadline.

From the proposition, in the optimal symmetric contract players experiment until a fixed threshold. Figure 4 illustrates possible shapes of optimal contracts. In the left panel of Figure 4, at times in $[0, \bar{t}]$, the second-task rents are sufficient to provide incentives in the first task. From Proposition 4, at these times the second-task thresholds correspond to the one-task optimal. For $t \in [\tilde{t}, \bar{t}]$, the principal fine tunes the contract and does not provide bonuses. For $t \geq \bar{t}$, the agent receives bonuses and is allocated an efficient experimentation deadline in the second task. The right panel of Figure 4 shows an example in which the agent is rewarded via assignments for some intermediate times. Notice that even though the agent has strict incentives to exert effort at $t \in [\tilde{t}, \bar{t}]$, the principal distorts the allocation of the unsuccessful agents, because doing so reduces her costs when an agent achieves a breakthrough between $t = 0$ and $\tilde{t}$.

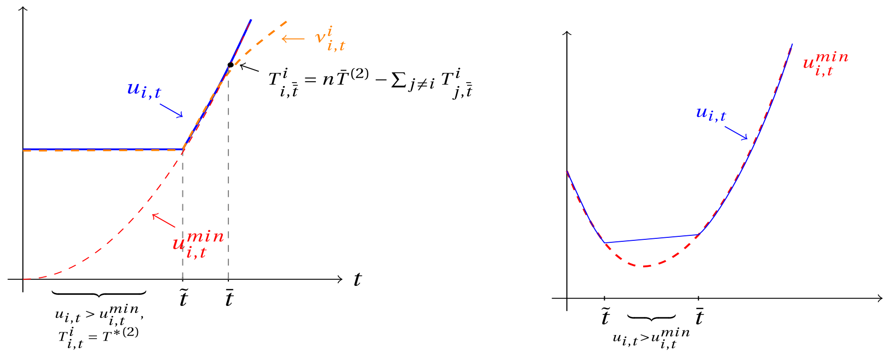

Figure 4. Examples of shapes of optimal contracts in the noncostly incentives case.

In contrast with the costly incentives case, we are not able to show that it is in the principal's best interest to implement full effort until a deadline when one allows for general (potentially asymmetric) contracts in the noncostly incentives case. That is to say, if the optimal contract is asymmetric we have not ruled out that a player's effort may jump back and forth from $\bar{a}$ to 0 before a final stopping time. When this happens, the optimal contract can have the shapes illustrated in Figure 4, except that $u_{i,t}$ may jump down at intervals in which zero effort is exerted.[35]

5. Conclusions

This paper asks how to optimally design contracts that give incentives to innovate to groups of agents. The principal chooses how many symmetric agents to hire subject to a recruitment cost. We show that incentives can be provided by simple history contingent bonus contracts. Agents must receive information rents, called *procrastination rents*, to prevent them from delaying effort. These rents are increasing in the amount of experimentation that agents are expected to perform in equilibrium. In order to reduce these rents, the principal stops experimentation early compared to the first-best and allocates the agents asymmetrically.

In our setting, projects require multiple successful experiments to succeed and contracts have two novel characteristics. First, the agents receive *public-good rents* to prevent them from free-riding on other agents' discoveries in early periods. Second, rewards and punishments are implemented by experimentation assignments. To reduce the public-good rents, the principal may exclude some agents from working, even in the absence of another profitable project. Further, agents' contracts are sensitive to early successes. Agents who succeed early receive bigger bonuses when they succeed later on. They also have a higher chance of success due to a reduced competition from coworkers and an extended, more rent plentiful, experimentation assignment.

[35] Proposition 4 does not establish existence in the noncostly incentives case under potentially asymmetric contracts. In order to prove existence in Section B.3, we consider a relaxed problem in which we omit a necessary condition for the agents' effort. This condition is satisfied when the principal wishes to set the effort at the maximum. Pontryagin's conditions continue to be necessary in the unrelaxed problem and imply the results in Proposition 4.

Our model predicts that in settings subject to private learning optimal contracts may have the following features. First, from Proposition 3, within a firm workers who obtain successes or promotions are likely to be credited with future successes in the same project as long as the project remains risky even if all workers are equally productive. This serial correlation in raises and promotions has been observed in real-world firms (Baker, Gibbs, and Holmstrom (1994)).

Second, as shown in Lemma 4, the asymmetry of contracts is related to the size of the public good rents the players receive in the later stage relative to the procrastination rents. In our model, the principal is able to correctly attribute a success to the agent who attained it. If the principal were less able to observe individual performance, we speculate that incentives to free-ride would be stronger, and hence we should expect even fewer agents in early stages of projects. Third, successful workers should be rewarded with promotions earlier in the project and bonuses in a project's final stage. Workers who do not succeed are assigned to tasks that give them less information rents—compared to the successful ones. Tasks that give less information rents can be tasks that carry less risk or are easier to perform (less costly).

Appendix A: Model and benchmark

A.1 *Agents' expected payoff as a function of beliefs*

Let $\mathbf{a}^{(j)} = \{(a^{(j)}_{k,t})_{t\geq 0}\}_{k\in N}$, let $\tilde{p}^{(j)}_{i,s}$ denote player i's belief about the feasibility of task j at a time $s \geq 0$ and let $w^{(j)}_{i,k,t}$ denote the bonus player i receives when player k achieves a breakthrough at time t in task j. The expected payoff of agent i on task 2, if task 1 was completed at time τ by player k, and the second-task effort profile is $\mathbf{a}^{(2)}$, can be written as

$$V^{(2)}_{i,k,\tau}\big(\mathbf{a}^{(2)}\big) = \int_{\tau}^{\infty} \sum_{k} \big(\tilde{p}^{(2)}_{i,t} w^{(2)}_{i,k,t} - \kappa_2\big) a^{(2)}_{k,t} e^{-\int_{\tau}^{t} (\tilde{p}^{(2)}_{i,s} a^{(2)}_{s} + r)\, ds}\, dt,$$

where, in the previous expression, the term $e^{-\int_{\tau}^{t} \tilde{p}^{(2)}_{i,s} a^{(2)}_{k,s}\, ds}$ is the probability that no breakthrough has occurred yet in task 2 and $\tilde{p}^{(2)}_{i,t} a^{(2)}_{i,t} e^{-\int_{\tau}^{t} \tilde{p}^{(2)}_{i,s} a^{(2)}_{s}\, ds}$ is the probability density that i obtains the first breakthrough at time t.

Similarly, the expected payoff of agent i under effort profile $(\mathbf{a}^{(1)}, \mathbf{a}^{(2)})$ can be written recursively as

$$V^{(1)}_{i}\big(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}\big) = \int_{0}^{\infty} \sum_{k} \big(\tilde{p}^{(1)}_{i,t} \big(w^{(1)}_{i,k,t} + V^{(2)}_{i,k,t}\big(\mathbf{a}^{(2)}\big)\big) - \kappa_1\big) a^{(1)}_{k,t} e^{-\int_{0}^{t} (\tilde{p}^{(1)}_{i,s} a^{(1)}_{s} + r)\, ds}\, dt.$$

A.2 *Bonus contracts are without loss*

To see that it is without loss to restrict attention to bonus contracts, suppose the principal could offer each agent i a general wage schedule $\tilde{w}_i : \mathcal{H}^t \to \mathbb{R}$ contingent on each public history. This wage schedule can be represented by a a flow payoff $\tilde{w}^{f}_{i,t} \in \mathbb{R}$ and lump-sum transfer $\tilde{w}^{l}_{i,t} \in \mathbb{R}$ at each time t. That is, heuristically the revenue accruing to agent i over the time interval $[t, t+dt]$ is $\tilde{w}^f_{i,t}\,dt + \tilde{w}^l_{i,t}$. The wage schedule $(\tilde{w}^f_{i,t}, \tilde{w}^l_{i,t})_t$ is adapted to the σ-algebra induced by the public histories in set $\mathcal{H}^t$ and maps public histories to $\mathbb{R}$.

Let us see that for each contract $\tilde{w}_i$ there is a bonus contract of the form $w_i = (w_{i,t}, W_{i,0})_t$, where $w_{i,t}(h^t)$ denotes the transfer that agent i receives at each history $h^t \in \bar{\mathcal{H}}^t$ and $W_{i,0}$ denotes the transfer at time zero, that gives the same payoff to principal and agent after each history.

In what follows, we assume the project has $\hat{J} \geq 2$ uncertain tasks. Let $h \in \mathcal{H}$ be a terminal history with breakthroughs that realize at times $\tau_1, \dots, \tau_J$ for $0 \leq J \leq \hat{J}$. If $J = 0$, $h = \emptyset$. Let h^{τ_j} denote the history h truncated to time τ_j, including time τ_j. Let $\bar{w}_i(h^{\tau_{j-1}})$ denote the discounted payoff that contract $\tilde{w}_i$ gives to agent i at the history in which the game ends with no breakthroughs at task j after history $h^{\tau_{j-1}}$. Now, defining

$$W_{i,0} = \bar{w}_i(\emptyset), \qquad w_{i,\tau_j}\big(h^{\tau_j}\big) = \big(\bar{w}_i\big(h^{\tau_j}\big) - \bar{w}_i\big(h^{\tau_{j-1}}\big)\big)e^{r\tau_j},$$

where $h^{\tau_0} = \emptyset$, we obtain a bonus contract that gives agent i and the principal the same discounted payoff after each history as the original contract $\tilde{w}_i$.

To see that bonus contracts are without loss even under limited liability, note that if the original contract satisfies limited liability, that is, $\tilde{w}^f_{i,t}, \tilde{w}^l_{i,t} \geq 0$ at every history, then $\bar{w}_i(h^{\tau_{j-1}}) \geq 0$ for every history $h^{\tau_{j-1}}$. Thus, the only additional contracts that (LL) rules out, by requiring $w_{i,\tau_j}, W_{i,0} \geq 0$, are those in which the payoff of achieving an additional breakthrough is less than the payoff of not achieving it. These contracts are not optimal when the principal wishes to incentivize positive effort in every task.

A.3 *Proof of proposition 1*

Optimal contract for a given effort function The characterization of the optimal bonus $w_{i,t}$ offered to each agent i is done in the following steps. First, we derive necessary conditions that $(w_{i,s})_{s\geq 0}$ must satisfy to implement agent i effort schedule $(a_{i,s})_{s\geq 0}$, given $(a_{-i,s})_{s\geq 0}$. Second, we find the bonus contract that minimizes the principal's cost among the class of bonus contracts that satisfy the necessary conditions. Finally, we show that the only effort function that satisfies agent i's necessary conditions under the contract that we identified in the second step is $(a_{i,s})_{s\geq 0}$. Hence, since each agent's program has a solution, this must be the optimal contract.

The agent's problem We now write the agent's problem, given a bonus contract $w_{i,t}$, and derive necessary conditions for the agent's choice of effort using Pontryagin's maximum principle.

Suppose the principal wants to implement effort function $(a_{i,s})_{s\geq 0}$ for each agent i. Let $T_i := \sup\{\tau | a_{i,\tau} > 0\}$ denote the time at which the principal stops agent i's effort. We will see that for the optimal effort functions, $\{(a_{i,s})_{s\geq 0}\}_i$, T_i is finite for each i. Agent i's problem can be written as[36]

$$\max_{a_{i,\cdot}} \int_0^{T_i} \big(w_{i,t}\bar{p}e^{-\int_0^t a_s\,ds} - \kappa\big(\bar{p}e^{-\int_0^t a_s\,ds} + (1-\bar{p})\big)\big)a_{i,t}e^{-rt}\,dt.$$

Define $y_t = \int_0^t a_s\,ds$ so that

$$\dot{y}_t = a_{i,t} + a_{-i,t}. \tag{6}$$

We now write the agent's optimal control problem with state variable y_t and control variable $a_{i,t}$. The Hamiltonian is

$$H(a_{i,t}, y_t, \eta_{i,t}) = \big(\bar{p}(w_{i,t}-\kappa)e^{-y_t} - \kappa(1-\bar{p})\big)a_{i,t}e^{-rt} + \eta_{i,t}(a_{i,t} + a_{-i,t}),$$

where $\eta_{i,t}$ is the costate variable associated to y_t. From Theorem 22.26 on page 465 of Clarke (2013), the following conditions (7)–(10) below are necessary for the agent's choice of effort. For any measurable $w_{i,t}$, $\eta_{i,t}$ is an absolutely continuous function that evolves according to

$$\dot{\eta}_{i,t} = -\frac{\partial H(a_{i,t}, y_t, \eta_{i,t})}{\partial y_t} = \bar{p}(w_{i,t}-\kappa)e^{-y_t}a_{i,t}e^{-rt}, \tag{7}$$

and, moreover, $a_{i,t}$ maximizes $H(a_{i,t}, y_t, \eta_{i,t})$. Define

$$\gamma_{i,t} := \big[\big(\bar{p}w_{i,t}e^{-y_t} - \kappa\bar{p}e^{-y_t} - \kappa(1-\bar{p})\big)e^{-rt} + \eta_{i,t}\big]\frac{1}{1-\bar{p}}. \tag{8}$$

Since $a_{i,t}$ maximizes $H(a_{i,t}, y_t, \gamma_{i,t})$, we have

$$\gamma_{i,t} > 0 \quad \Longrightarrow \quad a_{i,t} = \bar{a}_i \quad \text{and} \quad \gamma_{i,t} < 0 \quad \Longrightarrow \quad a_{i,t} = 0. \tag{9}$$

The transversality condition, as y is unrestricted at T_i, is $\eta_{i,T_i} = 0$, which defining, $x_t = y_t + \log((1-\bar{p})/\bar{p})$, implies

$$\gamma_{i,T_i} = e^{-rT_i}\big(e^{-x_{T_i}}w_{i,T_i} - \kappa\big(1 + e^{-x_{T_i}}\big)\big). \tag{10}$$

Now, suppose $\gamma_{i,t} > 0$ in the interval $[T_i - dt, T_i]$ for small dt. From equation (8), the principal can reduce $\gamma_{i,t}$ slightly in that instant by reducing $w_{i,t}$ without affecting the effort $a_{i,t}$ that maximizes $H(a_{i,t}, y_t, \eta_{i,t})$. From equation (7), after the change, $\eta_{i,t}$ must increase in the previous instant, $[T_i - 2dt, T_i - dt]$, and so must $\gamma_{i,t}$ from (8). This allows the principal to reduce $w_{i,t}$ at the instant $[T_i - 2dt, T_i - dt]$ as well. Arguing recursively, we conclude that of all wage schedules that satisfy agent i's necessary conditions for effort function $a_{i,s}$, the principal's preferred one—the one that yields the least rents to the agent—is such that $\gamma_{i,t} = 0$ or, equivalently, such that $\eta_{i,t}$ satisfies

$$\eta_{i,t} = -\big(\bar{p}w_{i,t}e^{-y_t} - \kappa\bar{p}e^{-y_t} - \kappa(1-\bar{p})\big)e^{-rt}. \tag{11}$$

[36]To understand how to derive this equation from the agent's payoff in the (IC) condition notice that by the law of motion of p_t, $\dot{p}_t/(1-p_t) = -\frac{d}{dt}\log(1-p_t) = -p_t a_t$ and, therefore, $e^{-\int_0^t p_s a_s\,ds} = (1-\bar{p})/1-p_t$. Replacing $p_t = \bar{p}e^{-\int_0^t a_s\,ds}/(\bar{p}e^{-\int_0^t a_s\,ds} + 1 - \bar{p})$, which is obtained via Bayes' rule, yields the desired expression.

Setting $\gamma_{i,t} = 0$ and calculating $\dot{\eta}_{i,t}$ from (11) and replacing into (7), we obtain

$$0 = r\kappa + e^{-x_t}\big(r(\kappa - w_{i,t}) + (\kappa - w_{i,t})a_{-i,t} + \dot{w}_{i,t}\big), \qquad w_{i,T_i} = \kappa\big(1 + e^{x_{T_i}}\big). \tag{12}$$

There is a unique bonus contract that satisfies (12) and it is given by equation (3).

We next show that a solution to the agent's problem exists and that these necessary conditions are also sufficient at the contract $w_{i,t}$ defined by (12). This establishes that the agent's choice of effort under $w_{i,t}$ is indeed $a_{i,\cdot}$.

The solution to the agent's problem exists By Theorem 23.11 in Clarke (2013), agent i's problem has a solution. In fact, the bonus wage $w_{i,t}$, defined by (12), is Lebesgue measurable in t and $\Lambda(t, y, a) := -(\bar{p}w_{i,t}e^{-y_t} - \kappa\bar{p}e^{-y_t} - \kappa(1-\bar{p}))a_{i,t}e^{-rt}$ is Lebesgue measurable, convex in a_i, and continuous in (y, a_i). Also, the set of controls is bounded; the process $\dot{y} = a_{-i}$, for fixed function a_{-i} and $a_i = 0$ is admissible and makes the agent's objective finite.

The necessary conditions are also sufficient Let $w_{i,t}$ be a bonus contract associated to effort functions $\{(a_{k,s})_{s\geq 0}\}_k$ that satisfies equation (12) for $t \leq T_i$ and is equal to zero for $t > T_i$. We now show that $(a_{i,s})_{s\geq 0}$ satisfies the agent i's (IC) constraint. Since the agent's problem's solution exists, if the effort function $a_{i,s}$ does not satisfy (IC) there must be another function $\tilde{a}_{i,s}$—that differs from $a_{i,s}$ in a positive measure set—and costate variable $\tilde{\eta}_{i,s}$ that satisfies the necessary conditions (7)–(10), such that $\tilde{a}_{i,s}$ improves the agent's payoff. Let us see that such effort function and associated costate variable do not exist.

Replacing $\gamma_{i,t}$ from (8) into equation (7), we obtain that a necessary condition for effort $a_{i,t}$ is

$$\dot{\gamma}_{i,t}e^{rt} = r\kappa + e^{-x_t}\big(r(\kappa - w_{i,t}) + (\kappa - w_{i,t})a_{-i,t} + \dot{w}_{i,t}\big). \tag{13}$$

An analogous necessary condition must be satisfied for effort $\tilde{a}_{i,t}$, for an associated multiplier $\tilde{\gamma}_{i,t}$ (defined analogously from $\tilde{\eta}_{i,t}$).

Let us see that only $\gamma_{i,t} = 0$ for all t and effort function $a_{i,t}$ (up to Lebesgue measure zero sets in t) can satisfy (13). Define $\tilde{x}_t = x_0 + \sum_i \int_0^t \tilde{a}_{i,s}\,ds$ and $x_t = x_0 + \sum_i \int_0^t a_{i,s}\,ds$. x_t and $\tilde{x}_t$ are continuous.

By continuity, there must be an interval (t_1, t_2) such that either (a) $x_t < \tilde{x}_t$ for $t \in (t_1, t_2)$ and $a_{i,t} < \tilde{a}_{i,t}$ for t in $\hat{\mathcal{T}}_1$, where $\hat{\mathcal{T}}_1$ is a positive measure subset of (t_1, t_2) or (b) $x_t > \tilde{x}_t$ for $t \in (t_1, t_2)$ and $a_{i,t} > \tilde{a}_{i,t}$ for $t \in \hat{\mathcal{T}}_2$, where $\hat{\mathcal{T}}_2$ is a positive measure subset of (t_1, t_2). If (a) occurs then $\tilde{\gamma}_{i,t} \geq 0$ for every $t \in \hat{\mathcal{T}}_1$, by the agent's maximization over effort. Fix $\hat{t} \in \hat{\mathcal{T}}_1$. Equation (13) implies that $\dot{\tilde{\gamma}}_{i,\tau} > 0$ at $\tau \in [\hat{t}, t_2)$ and, therefore, $\tilde{\gamma}_{i,\tau} > 0$ and $\tilde{a}_{i,\tau} = \bar{a}$ for $\tau \in (\hat{t}, t_2)$. Now, by continuity we have also $x_{t_2} < \tilde{x}_{t_2}$ and $\dot{\tilde{\gamma}}_{i,t_2}, \tilde{\gamma}_{i,t_2} > 0$, and applying the same argument recursively we obtain that $\tilde{\gamma}_{i,\tau} > 0$ and $x_\tau < \tilde{x}_\tau$ for $\tau \in (t, T_i]$. However,

$$\tilde{\gamma}_{i,T_i} = e^{-rT_i}\big(-\kappa - e^{-\tilde{x}_{T_i}}\kappa + e^{-\tilde{x}_{T_i}}w_{i,T_i}\big) \leq e^{-rT_i}\big(-\kappa - e^{-x_{T_i}}\kappa + e^{-x_{T_i}}w_{i,T_i}\big) = \gamma_{i,T_i} = 0,$$

which contradicts $\tilde{\gamma}_{i,T_i} > 0$. An analogous argument shows that (b) yields a contradiction.

The principal's effort choice Let $v_{i,t} := u_{i,t}/(1-\bar{p}) = e^{-rt-x_t}(w_{i,t}-\kappa) - \kappa e^{-rt}$, where $w_{i,t}$ is defined by (12), and $z := e^{-rt-x_t}$. We have

$$\dot{v}_{i,t} = -(r+a_t)e^{-rt-x_t}(w_{i,t}-\kappa) + re^{-rt}\kappa + e^{-rt-x_t}\dot{w}_{i,t} = -a_{i,t}\big(v_{i,t}+\kappa e^{-rt}\big),$$

where the second equality is obtained by replacing $\dot{w}_{i,t}$ from equation (12).

The Hamiltonian of the principal's problem, with state variables $v_{i,t}$ and z_t, and control variables $\{a_{i,t}\}_{i\in N}$, is given by

$$\begin{aligned} &H_t^P\big((v_{i,t})_i, z_t, (\eta_{i,t})_i, \gamma_t, (a_{i,t})_i\big) \\ &\quad = \sum_{i=1}^n \big[\big((\pi-\kappa)z_t - \kappa e^{-rt} - v_{i,t}\big)a_{i,t} - \eta_{i,t}a_{i,t}\big(v_{i,t}+\kappa e^{-rt}\big)\big] - \gamma_t z_t(a_t+r), \end{aligned}$$

where $\eta_{i,t}$ and γ_t are the costate variables associated with $v_{i,t}$ and z_t, respectively. The law of motion of the costates is

$$\dot{\eta}_{i,t} = a_{i,t} + \eta_{i,t}a_{i,t}, \qquad \dot{\gamma}_t = -(\pi-\kappa)a_t + \gamma_t(a_t+r).$$

The control $a_{i,t}$ enters linearly in the Hamiltonian and its factor is given by

$$h_{i,t} := (\pi-\kappa)z_t - \kappa e^{-rt} - v_{i,t} - \eta_{i,t}\big(v_{i,t}+\kappa e^{-rt}\big) - \gamma_t z_t.$$

By the maximization of Hamiltonian with respect to controls, if $h_{i,t} > 0$ then $a_{i,t} = \bar{a}$ and if $h_{i,t} < 0$, then $a_{i,t} = 0$.

Let us see that the principal sets all the players' efforts at the maximum up to a symmetric deadline. We have

$$\begin{aligned} \frac{dh_{i,t}}{dt} &= -(\pi-\kappa)(a_t+r)z_t + r\kappa e^{-rt} - \dot{v}_{i,t} - \dot{\eta}_{i,t}\big(v_{i,t}+\kappa e^{-rt}\big) \\ &\quad - \eta_{i,t}\big(\dot{v}_{i,t} - r\kappa e^{-rt}\big) - \dot{\gamma}_t z_t + (a_t+r)z_t\gamma_t, \end{aligned}$$

which after replacing the laws of motion, simplifies to

$$\begin{aligned} \frac{dh_{i,t}}{dt} &= -(\pi-\kappa)rz_t + \kappa re^{-rt+\int_0^t a_{i,s}\,ds} \\ &= -re^{-rt+\int_0^t a_{i,s}\,ds}\big[(\pi-\kappa)e^{-x_t-\int_0^t a_{i,s}\,ds} - \kappa\big]. \end{aligned} \tag{14}$$

Define $x_{i,t} = x_t + \int_0^t a_{i,s}\,ds$. Then, if $x_{i,t} < \log[(\pi-\kappa)/\kappa] := \bar{x}_i$, $\frac{dh_{i,t}}{dt} < 0$, and if $x_{i,t} > \bar{x}_i$, $\frac{dh_{i,t}}{dt} > 0$.

Claim 1. *The principal stops each player i's experimentation at a finite time. That is,* $\sup\{t \mid \int_0^t a_{i,s}\,ds$ *strictly increasing at* $t\} := T_i < \infty$*, for each player i.*

Proof. Suppose $T_i = \infty$. There are two cases: **(1)** $x_{i,T_i} := \lim_{t\to T_i} x_{i,t} > \bar{x}_i$ and **(2)** $x_{i,T_i} \le \bar{x}_i$.

Case (1). There is t such that $x_{i,t} > \bar{x}_i$, and $a_{i,t} > 0$. Therefore, $h_{i,t} \geq 0$. Since $x_{i,t} > \bar{x}_i$, $\frac{dh_{i,t}}{dt} > 0$ and $h_{i,\tau} > 0$, $\forall \tau > t$. This implies that $a_{i,\tau} = \bar{a}$, $\forall \tau > t$, implying $x_{T_i} = \infty$. However, since it is unprofitable for the principal to implement effort beyond the efficient one, this is a contradiction.

Case (2). Since x_{i,T_i} is finite, we must have $\limsup_{t\to T_i} a_{i,t} = 0$. Let t be such that $x_{i,t} < \bar{x}_i$ and $a_{i,t} \in (0, \bar{a}]$. By the principal's optimization, $h_{i,t} \geq 0$, and since $x_{i,t} < \bar{x}_i$, $\frac{dh_{i,t}}{dt} < 0$. This implies $h_{i,\tau} > 0$ for every $\tau < t$ and $a_{i,\tau} = \bar{a}$. This contradicts $\limsup_{t\to T_i} a_{i,t} = 0$. □

The boundary conditions are $\eta_{i,0} = 0$, $\gamma_T = 0$, $v_{i,T_i} = 0$, $z_0 = e^{-x_0}$, where $T = \max_{i\in N} T_i < \infty$, by the previous claim. The laws of motion and boundary conditions yield

$$
\begin{aligned}
\eta_{i,t} &= \int_0^t a_{i,t} e^{\int_\tau^t a_{i,s}\, ds}\, d\tau = -\big(1 - e^{\int_0^t a_{i,s}\, ds}\big), \\
\gamma_t &= \int_t^T (\pi - \kappa) a_\tau e^{-\int_t^\tau a_s\, ds - r(\tau - t)}\, d\tau, \\
v_{i,t} &= \int_t^{T_i} \kappa a_{i,\tau} e^{-r\tau + \int_\tau^{T_i} a_{i,s}\, ds}\, d\tau.
\end{aligned}
\tag{15}
$$

Let us see that $T_i = T$ for each i. Suppose not, and let i be the player who stops last. By the optimality of $a_{i,\cdot}$ and continuity of $h_{i,t}$, T_i is such that $h_{i,T_i} = 0$. Replacing (15) and the boundary conditions into the expression for h_{i,T_i} and from (14), we obtain that if T_i satisfies $h_{i,T_i} = 0$ then $\frac{dh_{i,T_i}}{dt} = 0$ and $\frac{dh_{j,t}}{dt} < 0$ for $t < T_i$ for every $j \in N$. This implies that all players experiment at maximum effort up to a deadline. Now, suppose player $k \neq i$ stops at time $T_k < T_i$. Then, by (15), $v_{k,t} = 0 \leq v_{i,t}$ and $0 \leq \eta_{k,t} \leq \eta_{i,t}$ for $t \in (T_k, T_i)$. Therefore, $h_{k,t} \geq h_{i,t} > 0$ for $t \in (T_k, T_i)$. This contradicts $a_{k,t} = 0$ for $t \in (T_k, T_i)$.

We conclude that players exert full effort until the deadline, T, characterized by equation $\frac{dh_{i,T}}{dt} = 0$, which is satisfied if and only if $(\pi - \kappa)e^{-(n+1)\bar{a}T} - \kappa = 0$.

Choice of number of agents Let us see that the marginal value of adding an agent is decreasing in n. In fact, ignoring the recruitment cost, the principal's expected payoff when she hires n agents is given by

$$
\bar{p}\bar{a}n\bigg(\frac{\kappa e^{T^*(n)(\bar{a}-r)}}{r - \bar{a}} + \frac{(\kappa - \pi)e^{-T^*(n)(\bar{a}n+r)-x_0}}{\bar{a}n + r} + \frac{e^{-x_0}(\pi - \kappa)}{\bar{a}n + r} - \frac{\kappa}{r - \bar{a}}\bigg).
$$

The second derivative of the previous expression with respect to n is given by

$$
\begin{aligned}
&-\bigg(\bar{a}e^{-T^*(n)(\bar{a}n+r)-x_0}\bigg(\bar{a}(\pi - \kappa)\bigg(-\frac{nT^*(n)^2(r - \bar{a})(\bar{a}n + r)^2}{n+1} - 2rT^*(n)(\bar{a}n + r) \\
&\quad + 2r\big(e^{T^*(n)(\bar{a}n+r)} - 1\big)\bigg)\bigg)\bigg)\Big/(\bar{a}n + r)^3.
\end{aligned}
\tag{16}
$$

To see that (16) is negative consider the term in parenthesis in its numerator, which we denote $B(T^*(n))$. B satisfies $B(0) = 0$, $B'(0) = 0$ and $B''(T^*(n)) > 0$ for $T^*(n) > 0$. Since

$B(\cdot)$'s first derivative increases strictly in $T^*(n)$ and $B'(0)=0$, then $B'(T^*(n))>0$ for $T^*(n)>0$. Similarly, since $B(0)=0$ then $B(T^*(n))>0$ for $T(n^*)>0$. This shows that (16) is negative.

A.3.1 *Justification for footnote 17* Consider a shift in effort by ε from time $[t-\Delta t, t]$ to time $[t, t+\Delta t]$. Let us see that up to the second order in Δt, under the optimal contract that satisfies equation (2), each agent's payoff is unaffected by the change. The change in agent i's expected payoff is approximately

$$\begin{aligned}\Delta t e^{-rt}\Big[&(a_{i,t}-\varepsilon)(w_{i,t}-\kappa)\bar{p}e^{-\int_0^t a_s\,ds+\varepsilon\Delta t}-(1-\bar{p})\kappa(a_{i,t}-\varepsilon)\\&+(a_{i,t+\Delta t}+\varepsilon)(w_{i,t+\Delta t}-\kappa)\bar{p}e^{-\int_0^t a_s\,ds-\Delta t(r+a_{i,t+\Delta t}+a_{-i,t+\Delta t})}\\&-(1-\bar{p})e^{-r\Delta t}\kappa(a_{i,t+\Delta t}+\varepsilon)\Big].\end{aligned}\tag{17}$$

The payoff after $t+\Delta t$ does not change since the belief from $t+\Delta t$ onwards is unaffected after the shift in effort. If the agent is indifferent between effort in two consecutive instants, the derivative of the previous expression with respect to ε is zero at $\varepsilon=0$. Calculating the derivative and evaluating at $\varepsilon=0$, we obtain

$$\begin{aligned}-(w_{i,t}-\kappa)\bar{p}e^{-\int_0^t a_s\,ds}&+\Delta t a_{i,t}(w_{i,t}-\kappa)\bar{p}e^{-\int_0^t a_s\,ds}\\&+(w_{i,t+\Delta t}-\kappa)\bar{p}e^{-\int_0^t a_s\,ds-\Delta t(r+a_{i,t+\Delta t}+a_{-i,t+\Delta t})}\\&+(1-\bar{p})\kappa-(1-\bar{p})\kappa e^{-r\Delta t}=0.\end{aligned}$$

Replacing the exponentials by their first-order Taylor expansion with respect to Δt, and multiplying by $e^{\int_0^t a_s\,ds}/\bar{p}$ yields

$$\begin{aligned}(w_{i,t+\Delta t}-w_{i,t})=\Delta t(w_{i,t+\Delta t}-\kappa)(r+a_{i,t+\Delta t}+a_{-i,t+\Delta t})-\Delta t a_{i,t}(w_{i,t}-\kappa)\\-\Delta t(1-\bar{p})\kappa r\cdot e^{\int_0^t a_s\,ds}/\bar{p}.\end{aligned}$$

Now, dividing by Δt and taking the limit as $\Delta t\to 0$ yields the differential equation (2).

Let us see that the effect of the shift of effort on the agents' expected payoff from time 0 to t is at most of third order. Consider now a shift of effort from $[t, t+\Delta t]$ to $[t+\Delta t, t+2\Delta t]$. The shift does not affect the payoff of agents in $-i$. It increases agent i's payoff, as it affects the term $\int_0^\tau a_{i,s}\,ds$ inside the integral in equation (3). Let $a_{i,t}$ denote agent i's effort function and let $\tilde{a}_{i,t}$ denote i's effort after the shift. Replacing the expression for the optimal bonus contract from equation (3), we obtain that i's discounted payoff from period 0 to t under effort functions $(\tilde{a}_{i,s})_{i,s}$ is given by

$$\begin{aligned}V_i^t&:=(1-\bar{p})\int_0^t\left[\left(\int_{\tilde{\tau}}^{T_i}\kappa re^{-r\tau+\int_{\tilde{\tau}}^{\tau}\tilde{a}_{i,s}\,ds}\,d\tau\right)+\kappa e^{-rT_i+\int_{\tilde{\tau}}^{T_i}\tilde{a}_{i,s}\,ds}-\kappa e^{-r\tilde{\tau}}\right]\tilde{a}_{i,\tilde{\tau}}\,d\tilde{\tau}\\&=(1-\bar{p})\int_0^{T_i}\left(\int_0^{\min\{t,\tau\}}\kappa re^{-r\tau+\int_{\tilde{\tau}}^{\tau}\tilde{a}_{i,s}\,ds}d\tilde{\tau}\right)\tilde{a}_{i,\tau}\,d\tau\\&\quad+\int_0^t\left(\kappa e^{-rT_i+\int_\tau^{T_i}\tilde{a}_{i,s}\,ds}-\kappa e^{-r\tau}\tilde{a}_{i,\tau}\right)d\tau.\end{aligned}\tag{18}$$

Since $a_{i,\tau} = \tilde{a}_{i,\tau}$ and $\int_\tau^{T_i} \tilde{a}_{i,s}\,ds = \int_\tau^{T_i} a_{i,s}\,ds$ for $\tau \le t$, only the first term changes after the shift in effort, therefore,

$$\begin{aligned}\frac{\partial V_i^t}{\partial \varepsilon} &= \frac{\partial}{\partial \varepsilon}\biggl(\bar{p}\kappa r \int_0^{T_i} \int_0^{\min\{t,\tau\}} e^{-r\tau + \int_s^\tau \tilde{a}_{i,s} + x_0} \tilde{a}_{i,s}\,ds\,d\tau\biggr) \\ &= \frac{\partial}{\partial \varepsilon}\biggl(\bar{p}\kappa r \int_t^{t+2\Delta t} \int_0^t e^{-r\tau + \int_s^\tau \tilde{a}_{i,s} + x_0} a_{i,s}\,ds\,d\tau,\biggr) \\ &\approx \tilde{c}\cdot \frac{\partial}{\partial \varepsilon}\biggl(\Delta t \int_0^t e^{-rt + \int_s^{t+\Delta t} a_{i,l}\,dl + x_0}\bigl(a_{i,s}e^{-\Delta t \varepsilon} + a_{i,s}e^{-\Delta t(r - a_{t+\Delta t} - \varepsilon)}\bigr)\biggr)\,ds,\end{aligned}$$

where $\tilde{c}$ is a constant and where the second equality is due to $\int_s^\tau a_{i,s}\,ds = \int_s^\tau \tilde{a}_{i,s}\,ds$ and $a_{i,s} = \tilde{a}_{i,s}$ for $\tau, s \notin [t, t+2\Delta t]$.

Taking the derivative with respect to ε, evaluating at $\varepsilon = 0$, and approximating the exponentials by their first-order Taylor expansions yields

$$\begin{aligned}\left.\frac{\partial V_i^t}{\partial \varepsilon}\right|_{\varepsilon=0} &\approx \tilde{c}\cdot \Delta t \int_0^t e^{-rt + \int_s^{t+\Delta t} a_{i,l}\,dl + x_0}\bigl(-a_{i,s}\Delta t + a_{i,s}\Delta t\bigl(1 - (r + a_{t+\Delta t})\Delta t\bigr)\bigr)\,ds \cdot \tilde{c} \\ &= O\bigl(\Delta t^3\bigr).\end{aligned}$$

Appendix B: Appendix: Two-task project

B.1 *Second task: Proof of Proposition 2*

Suppose that under the optimal contract each agent i gets expected utility $V_i(h^1)$ from experimenting in the second task after history h^1 in the first task. Let us see that it is optimal for the principal to offer a contract of the form of equation (3), and to implement maximum effort until a deadline.

Suppose that the principal implements effort functions $\{(a_{i,t}^{(2)})_{t\ge 0}\}_{i\in N}$ in task 2. The principal's payoff can be written as $\Pi^2 - \sum_{i=1}^n V_i(h^1)$. Thus, any contract that implements the same effort and gives the same expected payoff to the agents gives the principal the same expected payoff. From Proposition 1, the contract that satisfies (3) given effort functions $\{(a_{i,t}^{(2)})_{t\ge 0}\}_{i\in N}$ gives the least expected payoff to the agents while implementing these efforts. Therefore, such a contract—in addition to a bonus at the start of the second task, so as to yield the same payoff to each agent i—would be a weak improvement over any contract. From the discussion after Claim 1—or else, the discussion after Proposition 1 and footnote 17—it is optimal for the principal to implement the maximum effort until a deadline.

B.2 *First task. The agents' problem: Proof of Lemma 1*

As before, we first derive necessary conditions for each agent's problem using optimal control. Let $z_t = e^{-rt - x_t^{(1)}}$ with $x_t^{(1)} = \int_0^t a_s^{(1)}\,ds + \log((1-\bar{p}^{(1)})/\bar{p}^{(1)})$. The Hamiltonian of agent i's problem with state variable z_t and control variable $a_{i,t}^{(1)}$ is given by

$$H_t^A = (u_{i,t} - \kappa_1) z_t a_{i,t}^{(1)} - \kappa_1 a_{i,t}^{(1)} e^{-rt} + \sum_{j\ne i} \nu_{i,t}^j a_{j,t}^{(1)} z_t - \eta_{i,t}\bigl(a_t^{(1)} + r\bigr) z_t,$$

where $\nu_{i,t}^{j}$ denotes the expected payoff of agent i in the second task when agent j achieves a success at time t, $u_{i,t} = \nu_{i,t}^{i} + w_{i,t}^{(1)}$, and η_i is the costate variable associated to z. By Pontryagin's principle, η_i is absolutely continuous in t and evolves according to the differential equation

$$\dot{\eta}_{i,t} = -(u_{i,t} - \kappa_1)a_{i,t}^{(1)} - \sum_{j \neq i} \nu_{i,t}^{j} a_{j,t}^{(1)} + \eta_{i,t}\big(a_t^{(1)} + r\big). \tag{19}$$

Define $\tilde{\gamma}_{i,t} := (u_{i,t} - \kappa_1)z_t - \kappa_1 e^{-rt} - \eta_{i,t} z_t$. Then $a_{i,t}^{(1)}$ maximizes the Hamiltonian if $a_{i,t}^{(1)} = \bar{a}$ when $\tilde{\gamma}_{i,t} > 0$ and $a_{i,t}^{(1)} = 0$ when $\tilde{\gamma}_{i,t} < 0$. Calculating the derivative of $\tilde{\gamma}_{i,t}$ and replacing into equation (19) yields

$$\dot{\tilde{\gamma}}_{i,t} = \dot{u}_{i,t} z_t - (u_{i,t} - \kappa_1)\big(a_{-i,t}^{(1)} + r\big)z_t + \sum_{j \neq i} \nu_{i,t}^{j} a_{j,t}^{(1)} z_t + \kappa_1 r e^{-rt}. \tag{20}$$

Let $T_i^{(1)} = \sup\{t \in \mathbb{R}_+ | a_{i,t}^{(1)} > 0\}$. If i's opponents continue to experiment after time $T_i^{(1)}$, i has a salvage value at time $T_i^{(1)}$:

$$G\big(z_{T_i^{(1)}}, T_i^{(1)}\big) := z_{T_i^{(1)}} \int_{T_i^1}^{\infty} \sum_{j \neq i} \nu_{i,t}^{j} a_{j,t}^{(1)} e^{-\int_{T_i^{(1)}}^{t} a_s^{(1)}\, ds - r(t - T_i^{(1)})}\, dt.$$

The transversality condition for η_i is

$$\eta_{i,T_i^{(1)}} = \frac{\partial G\big(z_{T_i^{(1)}}, T_i^{(1)}\big)}{\partial z_{T_i^{(1)}}} = \int_{T_i^{(1)}}^{\infty} \sum_{j \neq i} \nu_{i,t}^{j} a_{j,t}^{(1)} e^{-\int_{T_i^{(1)}}^{t} a_s^{(1)}\, ds - r(t - T_i^{(1)})}\, dt,$$

which implies

$$\tilde{\gamma}_{i,T_i^{(1)}} = (u_{i,T_i^{(1)}} - \kappa_1)z_{T_i^{(1)}} - \kappa_1 e^{-rT_i^{(1)}} - G\big(z_{T_i^{(1)}}, T_i^{(1)}\big). \tag{21}$$

Setting $\tilde{\gamma}_{i,\cdot} = 0$ yields a differential equation for $u_{i,t}^{\min}$

$$\begin{aligned} \dot{u}_{i,t}^{\min} &= \big(u_{i,t}^{\min} - \kappa_1\big)\big(a_{-i,t}^{(1)} + r\big) - \sum_{j \neq i} \nu_{i,t}^{j} a_{j,t}^{(1)} - \kappa_1 r e^{x_t^{(1)}}, \\ u_{i,T_i^{(1)}}^{\min} &= \frac{\kappa_1}{p_{T_i^{(1)}}^{(1)}} + \int_{T_i^{(1)}}^{\infty} \sum_{j \neq i} \nu_{i,t}^{j} a_{j,t}^{(1)} e^{-\int_{T_i^{(1)}}^{t} (a_s^{(1)} + r)\, ds}\, dt. \end{aligned} \tag{22}$$

It's solution is equation (4).

For fixed effort functions $(a_{i,t}^{(1)}, a_{-i,t}^{(1)})_t$, let us define the time-t instantaneous rent as $\hat{v}_{i,t} := z_t(u_{i,t} - \kappa_1) - \kappa_1 e^{-rt}$. From equation (20), we can write

$$\dot{\hat{v}}_{i,t} = -a_{i,t}^{(1)}\big(\hat{v}_{i,t} + \kappa_1 e^{-rt}\big) - z_t \sum_{j \neq i} a_{j,t}^{(1)} \nu_{i,t}^{j} + \dot{\tilde{\gamma}}_{i,t}. \tag{23}$$

Let $\hat{v}^{\min}_{i,t} := z_t(u^{\min}_{i,t} - \kappa_1) - \kappa_1 e^{-rt}$. From equation (23) and the boundary condition (21), we obtain

$$\hat{v}_{i,t} - \hat{v}^{\min}_{i,t} = \tilde{\gamma}_{i,t}. \tag{24}$$

Since $\hat{v}_{i,t}$ increases in $u_{i,t}$, $u^{\min}_{i,t}$ is the rent minimizing $u_{i,t}$ that satisfies the necessary condition for the agent's effort choice with $\tilde{\gamma}_{i,t} \geq 0$, for every t. Notice that $\hat{v}_{i,t}$ and $\hat{v}^{\min}_{i,t}$ depend on the effort that each agent has exerted up to time t through z.

B.3 *First task: The principal's problem*

B.3.1 *The principal's optimal control problem* By equation (15) and Proposition 2, the second-task payoff of an agent i as a function of his experimentation deadline $T^{(2)}_i$ is given by

$$\nu_i\big(T^{(2)}_i\big) = \big(1 - \bar{p}^{(2)}\big) \int_0^{T^{(2)}_i} \bar{a} \int_t^{T^{(2)}_i} \kappa \bar{a} e^{-r\tau + \int_\tau^{T^{(2)}_i} \bar{a}\, ds}\, d\tau\, dt.$$

It does not depend on the experimentation of i's opponents and is strictly increasing in $T^{(2)}_i$. Therefore, for each value of agent i's payoff after j's time-t first-task success, $\nu^j_{i,t}$, there is a unique second-task experimentation threshold, denoted $T^{(2)}(\nu^j_{i,t})$, that yields i payoff $\nu^j_{i,t}$. Let $\boldsymbol{\nu}^j_t = (\nu^j_{i,t})_{i\in N}$. The second-task surplus after a history in which agent j succeeds in the first task at time t are given by

$$W\big(\boldsymbol{\nu}^j_t\big) := \sum_i \int_0^{T^{(2)}(\nu^j_{i,t})} \big(p^{(2)}_\tau \pi_2 - \kappa_2\big) \bar{a} e^{-\int_0^\tau (p^{(2)}_s a^{(2)}_s + r)\, ds}\, d\tau + \pi_1,$$

where $a^{(2)}_{i,s} = \bar{a}$ if and only if $s \leq T^{(2)}(\nu^j_{i,t})$. With this notation, the principal's instantaneous time-t payoff from agent i's first-task experimentation can be written as $[(W(\boldsymbol{\nu}^i_t) - \kappa_1 - \sum_{j\neq i} \nu^i_{j,t}) z_t a^{(1)}_{i,t} - \kappa_1 e^{-rt} a^{(1)}_{i,t} - \hat{v}_{i,t} a^{(1)}_{i,t}](1 - \bar{p}^{(1)})$.

The principal faces the following constraints when designing the first-task contract for each agent i. First, in the proof of Lemma 1 we saw that i's choice of effort satisfies equation (20). Second, the limited liability constraint requires that $u_{i,t} - \nu^i_{i,t} \geq 0$. Otherwise, his implied bonus would be negative.

When stating the principal's problem, we will let the multiplier $\tilde{\gamma}_{i,t}$ in the agent's necessary condition (24) be controlled by the principal so as to maximize her payoff. It is without loss to assume $\tilde{\gamma}_{i,t} \geq 0$ since for any contract with $\tilde{\gamma}_{i,t} < 0$ for t in some set Θ there is a payoff equivalent contract that gives the same incentives to the agents with $\tilde{\gamma}_{i,t} = 0$ for $t \in \Theta$. Because a large $\tilde{\gamma}_{i,t}$ implies a large cost to the principal, it is without loss to assume that $\tilde{\gamma}_{i,t}$ is bounded above by a large constant $M > 0$.

We next write the principal's Hamiltonian using state variables z_t and $\hat{v}^{\min}_{i,t}$, and controls $(a_i, \boldsymbol{\nu}^i, \tilde{\gamma}_i)_{i\in N}$ in the set $\hat{U} = \{(a_i, \boldsymbol{\nu}^i, \tilde{\gamma}_i)_{i\in N} | a_{i,t} \in [0, \bar{a}], \tilde{\gamma}_{i,t} \in [0, M], \boldsymbol{\nu}^i_t \geq 0, \forall t\}$. As we do not impose that $a_{i,t} = \bar{a}$ if $\tilde{\gamma}_{i,t} > 0$, what we state is a relaxation of the principal's problem.[37] From equation (24) each agent i's total reward upon success at time t is given by

[37] In Section B.3.5, we show that in the optimal contracts we characterize this condition holds.

$\hat{v}_{i,t}^{\min} + \tilde{\gamma}_{i,t}$. The principal's Hamiltonian can then be written as

$$\begin{aligned}\tilde{H}_t^P = &-\alpha_t\big(a_t^{(1)} + r\big)z_t \\ &+ \sum_i \Bigg[\bigg(W\big(\boldsymbol{\nu}_t^i\big) - \kappa_1 - \sum_{j\neq i} \nu_{j,t}^i\bigg) z_t a_{i,t}^{(1)} - \kappa_1 e^{-rt} a_{i,t}^{(1)} - \big(\hat{v}_{i,t}^{\min} + \tilde{\gamma}_{i,t}\big)a_{i,t}^{(1)} \\ &+ \lambda_{i,t}\bigg(-a_{i,t}^{(1)}\big(\hat{v}_{i,t}^{\min} + \kappa_1 e^{-rt}\big) - z_t \sum_{j\neq i} a_{j,t}^{(1)} \nu_{i,t}^j\bigg) \\ &+ \xi_{i,t}\big(\hat{v}_{i,t}^{\min} + \tilde{\gamma}_{i,t} - \big(\nu_{i,t}^i - \kappa_1\big)z_t + \kappa_1 e^{-rt}\big)\Bigg],\end{aligned} \tag{25}$$

where $\lambda_{i,t}$ is the costate associated to $\hat{v}_{i,t}^{\min}$, α_t is the costate for z_t, and $\xi_{i,t}$ is a multiplier associated to the limited liability constraint at time t, constraint which we denote (LL_t).[38] For ease of analysis, we assume that the principal must end experimentation at some large time $\bar{T} > 0$. The solution is invariant when we take $\bar{T} \to \infty$. Let $T_i^{(1)} = \sup\{t \in [0, \bar{T}] | a_{i,t}^{(1)} > 0\}$ and let $\bar{T}^{(1)} = \max_{i\in N} T_i^{(1)}$.

Evolution of costate variables By Pontryagin's principle,

$$\begin{aligned}\dot{\alpha}_t &= -\frac{\partial \tilde{H}_t^P}{\partial z_t} \\ &= \sum_i \Bigg[\bigg(-W\big(\boldsymbol{\nu}_t^i\big) + \kappa_1 + \sum_{j\neq i} \nu_{j,t}^i (1 + \lambda_{j,t})\bigg) a_{i,t}^{(1)} + \xi_{i,t}\big(\nu_{i,t}^i - \kappa_1\big)\Bigg] \\ &\quad + \alpha_t\big(a_t^{(1)} + r\big),\end{aligned} \tag{26}$$

$$\dot{\lambda}_{i,t} = -\frac{\partial H_t^P}{\partial \hat{v}_{i,t}^{\min}} = a_{i,t}^{(1)} + \lambda_{i,t} a_{i,t}^{(1)} - \xi_{i,t}.$$

From (23) and (21), $\hat{v}_{i,t}^{\min}$ must satisfy

$$\dot{\hat{v}}_{i,t}^{\min} = -a_{i,t}^{(1)}\big(\hat{v}_{i,t}^{\min} + \kappa_1 e^{-rt}\big) - z_t \sum_{j\neq i} a_{j,t}^{(1)} \nu_{i,t}^j, \qquad \hat{v}_{i,T_i^{(1)}}^{\min} = G\big(z_{T_i^{(1)}}, T_i^{(1)}\big). \tag{27}$$

Using the boundary conditions, $\lambda_{i,0} = 0$, $\alpha_{\bar{T}^{(1)}} = 0$, we obtain:

$$\lambda_{i,t} = \int_0^t \big(a_{i,\tau}^{(1)} - \xi_{i,\tau}\big) e^{\int_\tau^t a_{i,s}^{(1)}\, ds}\, d\tau,$$

$$\alpha_t = \sum_i \int_t^{\bar{T}^{(1)}} \Bigg(\bigg(W\big(\boldsymbol{\nu}_\tau^i\big) - \kappa_1 - \sum_{j\neq i} \nu_{j,\tau}^i (1 + \lambda_{j,\tau})\bigg) a_{i,\tau}^{(1)} - \xi_{i,\tau}\big(\nu_{i,\tau}^i - \kappa_1\big)\Bigg)$$

[38]The limited liability constraint case holds if $\nu_{i,t}^i \le u_{i,t}$ for every $t \le T_i^{(1)}$. This is equivalent to requiring

$$\big(\nu_{i,t}^i - \kappa_1\big)z_t - \kappa_1 e^{-rt} \le \hat{v}_{i,t}^{\min} + \tilde{\gamma}_{i,t}. \tag{LL_t}$$

$$\times e^{-\int_t^\tau (a_s^{(1)}+r)\,ds}\,d\tau, \tag{28}$$

$$\begin{aligned}\hat{v}_{i,t}^{\min} &= G\big(z_{T_i^{(1)}}, T_i^{(1)}\big)e^{\int_t^{T_i^{(1)}} a_{i,s}^{(1)}\,ds} + \int_t^{T_i^{(1)}} \Big(a_{i,\tau}^{(1)}\kappa_1 e^{-r\tau} + \sum_{j\neq i} \nu_{i,\tau}^j a_{j,\tau}^{(1)} z_\tau\Big)e^{\int_t^\tau a_{i,s}^{(1)}\,ds}\,d\tau \\ &= z_t \int_t^\infty \sum_{j\neq i} \nu_{i,\tau}^j a_{j,\tau}^{(1)} e^{-\int_t^\tau a_{-i,s}^{(1)}\,ds - r(\tau-t)}\,d\tau + \int_t^{T_i^{(1)}} a_{i,\tau}^{(1)}\kappa_1 e^{-r\tau} e^{\int_t^\tau a_{i,s}^{(1)}\,ds}\,d\tau.\end{aligned}$$

Maximization with respect to $\tilde{\gamma}_{i,t}$ $\tilde{\gamma}_{i,t}$ enters linearly into the Hamiltonian and its factor is $-a_{i,t}^{(1)} + \xi_{i,t}$. We have the following cases.

Case 1: (LL_t) does not bind which implies $\xi_{i,t} = 0$ and by the principal's maximization $\tilde{\gamma}_{i,t} = 0$, whenever $a_{i,t}^{(1)} > 0$.

Case 2: (LL_t) binds and $\tilde{\gamma}_{i,t} > 0$, which implies $\xi_{i,t} = a_{i,t}^{(1)}$.

Case 3: (LL_t) binds and $\tilde{\gamma}_{i,t} = 0$, which implies $\xi_{i,t} \in [0, a_{i,t}^{(1)}]$ by the principal's maximization.

In all cases, by (28), $\lambda_{i,t} \geq 0$, and if $\tilde{\gamma}_{i,\tau} > 0$, for all $\tau \leq t$ then $\lambda_{i,t} = 0$ from Case 2 and (28).

Maximization with respect to $\nu_{i,t}^j$ By the maximization with respect to $\nu_{i,t}^i$, and the definition of $W(\boldsymbol{\nu}_t^i)$, $\nu_{i,t}^i$ solves

$$\max_{\nu_{i,t}^i} \bigg(\int_0^{T^{(2)}(\nu_{i,t}^i)} \bar{a}\big(p_\tau^{(2)}\pi_2 - \kappa_2\big)e^{-\int_0^\tau (p_s^{(2)}a_s^{(2)}+r)\,ds}\,d\tau\bigg) \cdot a_{i,t}^{(1)} z_t - \xi_{i,t}\nu_{i,t}^i z_t. \tag{29}$$

Notice that if $\xi_{i,t} = 0$ (which occurs whenever $u_{i,t} > \nu_{i,t}^i$), then $T^{(2)}(\nu_{i,t}^i)$ is the efficient experimentation threshold. If $\xi_{i,t} = a_{i,t}$, then $T^{(2)}(\nu_{i,t}^i)$ is the one-task optimal given the opponents' thresholds.

The maximization with respect to $\nu_{i,t}^k$ for $k \neq i$ yields

$$\max_{\nu_{i,t}^k} \bigg(\int_0^{T^{(2)}(\nu_{i,t}^k)} \bar{a}\big(p_\tau^{(2)}\pi_2 - \kappa_2\big)e^{-\int_0^\tau (p_s^{(2)}a_s^{(2)}+r)\,ds}\,d\tau\bigg) \cdot a_{k,t}^{(1)} z_t - (1+\lambda_{i,t})\nu_{i,t}^k a_{k,t}^{(1)} z_t. \tag{30}$$

If $\lambda_{k,t} = 0$ and $a_{k,t}^{(1)} > 0$ for all $k \in N$, then the maximization coincides with that of the one-task case, where the optimal experimentation thresholds are symmetric and equal to $T^{(2)*} = \bar{T}^{(2)}\frac{n}{n+1}$.[39]

Since $\lambda_{k,t} \geq 0$ and $\xi_{i,t} \leq a_{i,t}^{(1)}$, $T^{(2)}(\nu_{i,t}^i) \geq T^{(2)}(\nu_{k,t}^i)$ for $k \neq i$.

[39] To see this, note that replacing the expression for $T_i'(\nu_i)$ from equation (38) into the FOC's for the maximizations with respect to $\nu_{i,t}^i$ and $\nu_{i,t}^j$ yields the same stopping time as in the one-task case obtained by setting $\frac{dh_{i,t}}{dt} = 0$ in equation (14).

Maximization with respect to $a^{(1)}_{i,t}$ Define

$$h^{(2)}_{i,t} = \left(W(\boldsymbol{\nu}^i_t) - \kappa_1 - \sum_{j\neq i}(1+\lambda_{j,t})\nu^i_{j,t}\right)z_t - \kappa_1 e^{-rt} - \hat{v}^{\min}_{i,t} - \tilde{\gamma}_{i,t}$$
$$- \lambda_{i,t}\left(\hat{v}^{\min}_{i,t} + \kappa_1 e^{-rt}\right) - \alpha_t z_t. \tag{31}$$

Since the Hamiltonian $\tilde{H}^P_t$ is linear in $a^{(1)}_{i,t}$ with factor $h^{(2)}_{i,t}$, then $h^{(2)}_{i,t} > 0$ implies $a^{(1)}_{i,t} = \bar{a}$ and $h^{(2)}_{i,t} < 0$ implies $a^{(1)}_{i,t} = 0$.

B.3.2 *Costly first-task incentives: Proof of Proposition 3* The parameters fall in the costly incentives case if and only if $\tilde{\gamma}_{i,t} = 0$ and $u_{i,t} > \nu^i_{i,t}$ for every t and, therefore, $\xi_{i,t} = 0$ for each t.

From equation (28), we have

$$\lambda_{i,t} = e^{\int_0^t a^{(1)}_{i,s}\,ds} - 1. \tag{32}$$

From (30), the following first-order necessary condition for $\hat{T}^{(2)} = T^{(2)}(\nu^k_{i,t})$ holds:[40]

$$\left(-(1+\lambda_{i,t})\left(e^{\hat{T}^{(2)}\bar{a}} - 1\right)\kappa_2 + (\pi_2 - \kappa_2)e^{-x^{(2)}_{\hat{T}^{(2)}}} - \kappa_2\right)$$
$$- \int_{\hat{T}^{(2)}}^{T^{(2)}(\nu^k_{k,t})} a^{(2)}_s(\pi_2 - \kappa_2)e^{-\bar{a}\hat{T}^{(2)} - \int_0^s a^{(2)}_{-i,\tau}\,d\tau - r(s-\hat{T}^{(2)}) - x^{(2)}_0}\,ds = 0, \tag{33}$$

where $T^{(2)}(\nu^k_{k,t}) = V_{\text{eff}}/\bar{a} - \sum_{j\neq k} T^{(2)}(\nu^k_{j,t})$, with $V_{\text{eff}} = -x^{(2)}_0 + \text{Log}((\pi_2 - \kappa_2)/\kappa_2)$, since the successful player experiments up to the efficient threshold by (29). By Claim 3 in Section B.3.4, the program in which the successful agent experiments until the efficient threshold is strictly concave.[41] Therefore, equation (33) for each i is also a sufficient condition. And by the implicit function theorem, the solution to first-order condition (33) is continuously differentiable in t, and hence, $h^{(2)}_{i,t}$ in (31) is continuously differentiable in t.

Let $\tilde{I}_t := \{i | \int_0^t a^{(1)}_{i,s}\,ds$ is strictly increasing in $t\}$.

Lemma 2. *If* $\int_0^t a^{(1)}_{i,s}\,ds = \int_0^t a^{(1)}_{j,s}\,ds$ *for every* $i, j \in \tilde{I}_t$ *(and, therefore,* $\lambda_{i,t} = \lambda_{j,t}$*) then* $\nu^k_{i,t} = \nu^{\tilde{k}}_{\tilde{i},t}$ *for* $i, k, \tilde{i}, \tilde{k} \in \tilde{I}_t$ *with* $i \neq k$ *and* $\tilde{i} \neq \tilde{k}$*, and if* $\lambda_{i,t} > \lambda_{j,t}$ *then* $\nu^k_{i,t} < \nu^{\tilde{k}}_{j,t}$ *for every* $k, \tilde{k} \in \tilde{I}_t$*,* $i \neq k$*,* $j \neq \tilde{k}$*.*

Proof. Suppose there are $k \in \tilde{I}_t$, $\hat{i} \in \tilde{I}_t \setminus \{k\}$, and $\hat{j} \in \tilde{I}_t \setminus \{k, \hat{i}\}$ such that $\lambda_{\hat{i},t} = \lambda_{\hat{j},t}$ and $T^{(2)}(\nu^k_{\hat{i},t}) < T^{(2)}(\nu^k_{\hat{j},t})$. $\hat{T}^{(2)} = T^{(2)}(\nu^k_{\hat{i},t})$ satisfies equation (33) with $i = \hat{i}$. Now, let us see that this last observation implies that the left-hand side of (33) for $i = \hat{j}$ evaluated at $\hat{T}^{(2)} = T^{(2)}(\nu^k_{\hat{i},t})$ is negative. In fact, the first two terms in (33) are equal for $i = \hat{j}$ and $i = \hat{i}$,

[40] In this first-order condition, we have replaced the expression for $T'_i(\nu_i)$ from equation (38).

[41] This is due to $\hat{J}(\boldsymbol{\nu}^i)_{(-i,-i)}$ positive definite when g^i is not the identity.

but the integral term is strictly greater when $i=\hat{j}$. The latter statement follows from $\int_0^s a_{-\hat{j},\tau}\,d\tau < \int_0^s a_{-\hat{i},\tau}\,d\tau$ for $s > T^{(2)}(\nu^k_{\hat{i},t})$, since $T^{(2)}(\nu^k_{\hat{i},t}) < T^{(2)}(\nu^k_{\hat{j},t})$. This contradicts the concavity of $W(\cdot)$, established in Lemma 6 below, which requires the left-hand side of (33) to be positive for $i=\hat{j}$ at $\hat{T}^{(2)} = T^{(2)}(\nu^k_{\hat{i},t}) < T^{(2)}(\nu^k_{\hat{j},t})$ and, therefore, we must have $\nu^k_{\hat{i},t} = \nu^k_{\hat{j},t}$ for $\hat{i}, \hat{j} \in \tilde{I}_t \setminus \{k\}$.

Now, the optimization in equations (29) and (30) does not depend on the identity of player k but only on the values of the multipliers $\lambda_{i,t}$ for each $i \in N$ and $k \in \tilde{I}_t$. Since the solution to the program is unique by the strict concavity established in Claim 3, from our previous argument, we must have $\nu^k_{i,t} = \nu^{\tilde{k}}_{\tilde{i},t}$ for $i, k, \tilde{i}, \tilde{k} \in \tilde{I}_t$ with $i \neq k$ and $\tilde{i} \neq \tilde{k}$.

By a similar argument, $\lambda_{\hat{i},t} > \lambda_{\hat{j},t}$ implies $T^{(2)}(\nu^i_{\hat{i},t}) < T^{(2)}(\nu^k_{\hat{j},t})$. □

Lemma 3. *For every $i, j \in \tilde{I}_t$, $\int_0^t a^{(1)}_{i,s}\,ds = \int_0^t a^{(1)}_{j,s}\,ds$.*

Proof. Let $\bar{t} = \inf\{t \mid \int_0^t a^{(1)}_{i,s}\,ds \neq \int_0^t a^{(1)}_{j,s}\,ds$, for some $i, j \in \tilde{I}_t\}$ and suppose $\bar{t} < \infty$.

Define

$$Dh^S_{i,t} = r\left(e^{-rt}\kappa_1 e^{\int_0^t a^{(1)}_{i,s}\,ds} + \sum_{j\neq i} \nu^i_{j,t} e^{\int_0^t a^{(1)}_{j,s}\,ds} z_t - \left(W\left(\boldsymbol{\nu}^i\right) - \kappa_1\right) z_t\right), \tag{34}$$

Replacing the laws of motion and applying Lemma 2, we obtain

(a) $Dh^S_{i,t} = \frac{dh^{(2)}_{i,t}}{dt}$ for every $t \leq \bar{t}$ and $i \in N$, and

(b) $Dh^S_{i,t}$ is continuous, $Dh^S_{i,t} \cdot e^{rt}$ is strictly increasing in t, and it is strictly negative at $t=0$, for $i \in \tilde{I}_0$.[42]

(c) For each i, let $\tilde{T}_i$ be such that $Dh^S_{i,t}$ is zero (if no such $\tilde{T}_i$ exists, set $\tilde{T}_i = -1$ if $Dh^S_{i,0} > 0$ and $\tilde{T}_i = \infty$, otherwise), then $Dh^S_{i,t} < 0$ for $t < \tilde{T}_i$ and $Dh^S_{i,t} > 0$ for $t > \tilde{T}_i$.

Property (c) follows directly from (b).

By the hypothesis, there is a sequence $\{t^n\}_{n\in\mathbb{N}}$, with $t^n \downarrow \bar{t}$ and players $i, j \in \tilde{I}_{t^n}$, for each n, such that $\int_0^{t^n} a^{(1)}_{i,s}\,ds \neq \int_0^{t^n} a^{(1)}_{j,s}\,ds$. Notice that $i, j \in \tilde{I}_{t_n}$ implies $T^{(1)}_i, T^{(1)}_j > \bar{t}$, and $h^{(2)}_{i,t^n}, h^{(2)}_{j,t^n} \geq 0$ due to the maximization with respect to instantaneous effort. Since $h^{(2)}_{k,t}$ is continuous for each $k \in N$, $h^{(2)}_{i,\bar{t}}, h^{(2)}_{j,\bar{t}} \geq 0$.

Suppose first that $\tilde{T}_k \neq \bar{t}$ for each $k \in \{i, j\}$. From (a) and (c), there is $\tilde{\varepsilon}$ such that for every $t \in [\bar{t}, \bar{t}+\tilde{\varepsilon})$, **(1)** $h^{(2)}_{k,t} < h^{(2)}_{k,\bar{t}}$ for every $k \in N$ such that $\bar{t} < \tilde{T}_k$, and **(2)** $h^{(2)}_{k,t} > h^{(2)}_{k,\bar{t}}$ for every $k \in N$ such that $\bar{t} > \tilde{T}_k$.

If $\bar{t} < \tilde{T}_k$ holds for $k \in \{i, j\}$, $h^{(2)}_{k,\bar{t}} > 0$ since $h^{(2)}_{k,t^n} \geq 0$ for $k \in \{i, j\}$ and there is n_0 such that $t^{n_0} \in [\bar{t}, \bar{t}+\tilde{\varepsilon})$. By continuity, there is $\tau > 0$, such that $h^{(2)}_{k,t} > 0$ for $t \in [\bar{t}, \bar{t}+\tau)$, and hence, $a_{k,t} = \bar{a}$ for $t \in [\bar{t}, \bar{t}+\tau)$. If $\bar{t} > \tilde{T}_k$ holds for $k \in \{i, j\}$, $a_{k,t} = \bar{a}$ for $t \in [\bar{t}, \bar{t}+\tilde{\varepsilon})$. In both cases, we obtain $\int_0^t a^{(1)}_{i,s}\,ds = \int_0^t a^{(1)}_{j,s}\,ds$ for $t \in [\bar{t}, \bar{t}+\min\{\tilde{\varepsilon}, \tau\})$, which is a contradiction.

[42]Notice that the derivative of $Dh^S_{i,t}$ with respect to $\nu^i_{j,t}$ and $\nu^i_{i,t}$ are zero by the first-order conditions. If $Dh^S_{i,0} > 0$, then it is suboptimal for the principal to allow i to experiment at times close to 0.

Now, suppose, without loss, that $\tilde{T}_i = \bar{t}$, and hence $Dh^S_{i,\bar{t}} = 0$. Since $T^{(1)}_j > \tilde{T}_i$, then $\alpha_{\tilde{T}_i} > 0$ and $\hat{v}^{\min}_{i,t} > 0$ from (28) and, therefore, from (31) and $Dh^S_{i,\bar{t}} = 0$, $h^{(2)}_{i,\bar{t}} < 0$, which is a contradiction. □

Corollary 2. *If $i \in \tilde{I}_t$, then $a_{i,\tau} = \bar{a}$ for every $\tau < t$.*

Proof. Since $\bar{t} = \infty$, (a) in the proof of Lemma 3, implies that $Dh^S_{i,t} = \frac{dh^{(2)}_{i,t}}{dt}$ for every t. If there is a player i and $\tau > \tilde{T}_i$ such that $i \in \tilde{I}_\tau$, then by (c) in the proof of Lemma 3, $h^{(2)}_{i,t} > 0$ for $t > \tau$ and, therefore, $a^{(1)}_{i,t} = \bar{a}$ for $t \in (\tau, \infty)$. This is a contradiction as it is suboptimal for the principal to implement experimentation beyond the efficient amount.

Let $i \in \tilde{I}_t$ for $t \le \tilde{T}_i$. Since $\frac{dh^{(2)}_{i,t}}{dt} < 0$ for every $t < \tilde{T}_i$, $h^{(2)}_{i,t} \ge 0$ implies $h^{(2)}_{i,\tau} > 0$ for every $\tau < t$. □

Corollary 2 shows that players must exert maximum effort up to a deadline.

To see that $T^{(2)}(\nu^i_{j,t})$ decreases in t and converges to zero as $\int_0^t a_{i,s}\,ds \to \infty$ notice that the first term in (33) decreases in t. The derivative of the left-hand side of (33) with respect to $\hat{T}^{(2)}$, evaluated at $T^{(2)}(\nu^k_{j,t})$ is given by[43]

$$
\begin{aligned}
&-\bar{a}(1+\lambda_{j,t})\kappa_2 e^{\bar{a}T^{(2)}(\nu^k_{j,t})} - \bar{a}(\pi_2-\kappa_2)e^{-x_{T^{(2)}(\nu^k_{j,t})}} \\
&\quad + (\bar{a}-r)\int_{T^{(2)}(\nu^k_{j,t})}^{T^{(2)}(\nu^k_{k,t})} a^{(2)}_s(\pi_2-\kappa_2)e^{-\bar{a}T^{(2)}(\nu^k_{j,t})-\int_0^s a^{(2)}_{-j,\tau}\,d\tau - r(s-T^{(2)}(\nu^k_{j,t}))-x^{(2)}_0}\,ds \\
&< 0.
\end{aligned}
\tag{35}
$$

Thus, to keep the equality in (33) after an increase in $\lambda_{j,t}$, $T^{(2)}(\nu^k_{j,t})$ must strictly decrease. Furthermore, as $\lambda_{i,t} \to \infty$ as $\int_0^t a_{i,s}\,ds \to \infty$, $T^{(2)}(\nu^k_{j,t})$ converges to zero as $\int_0^t a_{i,s}\,ds \to \infty$. Finally, at $t = 0$, $\lambda_{j,t} = 0$ and since the integral term in (33) is strictly positive $T^{(2)}(\nu^k_{j,0}) < T^{*(2)}$ and, therefore, $T^{(2)}(\nu^k_{j,t}) < T^{(2)}(\nu^k_{j,0}) < T^{*(2)}$ for all t.

The following corollary summarizes our findings from Lemmas 2, 3, and Corollary 2.

Corollary 3. *The experimentation deadlines of each player i after a player j in $-i$ succeeds at time t, $(T^j_{i,t})_{i,j}$ uniquely solve (33) with $T^j_{i,t} = \hat{T}^{(2)}$ and $\lambda_{i,t} = e^{-\bar{a}\min\{t, T^{(1)}_i\}} - 1$, for each $i, j \in N$, $i \neq j$. The deadline $T^j_{i,t}$ does not depend on the identity of player $j \in -i$ and for players $i, \tilde{i} \in \tilde{I}_t \setminus \{j\}$, $T^j_{i,t} = T^j_{\tilde{i},t}$.*

$T^{(1)}_i$ is chosen to maximize the principal's payoff given the continuation deadlines $T^j_{i,t}$, which in turn also depend on $\{T^{(1)}_j\}_{j\in N}$ via $\{\lambda_{j,t}\}_{j\in N}$.

Asymmetric contracts Throughout the following analysis of asymmetric contracts, we assume $n = 2$. The stopping time of player $i \in \{1, 2\}$, $T^{(1)}_i$, must satisfy $h^{(2)}_{i,T^{(1)}_i} = 0$, which

[43]To understand the inequality, notice that $(\pi_2-\kappa_2)z_T \ge \int_T^\infty a^{(2)}_s(\pi_2-\kappa_2)z_s\,ds$.

replacing (28) and the boundary conditions yields, for $j \neq i$,

$$\begin{aligned}
&\big(W\big(\boldsymbol{\nu}^i_{T_i^{(1)}}\big) - \kappa_1 - e^{\int_0^{T_i^{(1)}} a_{j,s}\,ds} \nu^i_{j,T_i^{(1)}}\big) z_{T_i^{(1)}} - \kappa_1 e^{\int_0^{T_i^{(1)}} (a_{i,s}-r)\,ds} \\
&\quad - z_{T_i^{(1)}} \int_{T_i^{(1)}}^{\infty} \big(W\big(\boldsymbol{\nu}^j_\tau\big) - \kappa_1\big) a^{(1)}_{j,\tau} z_\tau \, d\tau = 0.
\end{aligned} \tag{36}$$

It can be verified that these are also the first-order conditions of the effort choice problem of the principal chooses the deadlines $T_i^{(1)}$ for each $i \in \{1, 2\}$. If the principal finds it optimal to offer a symmetric first-period experimentation deadline, $T^{S,(1)}$, then $T^{S,(1)}$ solves (36) for $T_j^{(1)} = T^{S,(1)}$ for each $i \in \{1, 2\}$.

Lemma 4. *The optimal contract is asymmetric if* $\nu^i_{j,T^{S,(1)}} > \kappa_1 e^{2\bar{a}T^{S,(1)} + x_0^{(1)}}$.

Proof. At $T_1^{(1)} = T_2^{(1)} = T^{S,(1)}$, (36) becomes

$$\big(W\big(\boldsymbol{\nu}^i_{T_i^{(1)}}\big) - \kappa_1 - e^{\bar{a}T_j^{(1)}} \nu_{j,T_i^{(1)}}\big) e^{-\bar{a}(T_j^{(1)} + T_i^{(1)}) - rT_i^{(1)}} - \kappa_1 e^{-rT_i^{(1)}} e^{\bar{a}T_i^{(1)}}.$$

Let $D = -\bar{a}((W(\boldsymbol{\nu}^i_{T^{S,(1)}}) - \kappa_1 - e^{\bar{a}T^{S,(1)}} \nu^i_{j,T^{S,(1)}}) e^{-T^{S,(1)}(2\bar{a}+r) - x_0^{(1)}} + \kappa_1 e^{-T^{S,(1)}(r-\bar{a})})$ and $P = -\bar{a}(W(\boldsymbol{\nu}^i_{T^{S,(1)}}) - \kappa_1) e^{-T^{S,(1)}(2\bar{a}+r) - x_0^{(1)}}$. From the previous expression, the Hessian matrix with respect to $T_1^{(1)}, T_2^{(1)}$ at $T^{S,(1)}$ is given by

$$\begin{pmatrix} D & P \\ P & D \end{pmatrix}.$$

If $D^2 - P^2 < 0$, the Hessian matrix is not negative semidefinite, implying that the principal can improve on $T_1^{(1)} = T_2^{(1)} = T^{S,(1)}$. To conclude note that $D^2 - P^2 < 0$ if and only if $-D = |D| < -P = |P|$, which occurs if and only if $\nu^i_{j,T^{S,(1)}} > \kappa_1 e^{2\bar{a}T^{S,(1)} + x_0^{(1)}}$. □

Sufficient conditions for costly first-task incentives and asymmetric contracts

Lemma 5. *If condition (5) is satisfied for the optimal deadlines as characterized in Corollary 3, the parameters fall in the costly first-task incentives case.*

Proof. The costly first-task incentives case holds if $\nu^i_{i,t} \leq u^{\min}_{i,t}$ for every $t \leq T_i^{(1)}$. This is equivalent to requiring $(\nu^i_{i,t} - \kappa_1) z_t - \kappa_1 e^{-rt} \leq \hat{v}^{\min}_{i,t}$. The conclusion follows from the expression $\hat{v}^{\min}_{i,t}$ in (28). □

Corollary 4. *If* $r > \bar{a}$ *and*

$$\kappa_1 > \kappa_2 \bar{p}^{(1)}\big(1 - \bar{p}^{(2)}\big) \frac{\bar{a}^2}{r(r - \bar{a})} > \kappa_1\big(1 - \bar{p}^{(1)}\big),$$

there are choices of π_1 *and* π_2 *such that the parameters fall in the costly incentives case and the optimal contract is asymmetric.*

Proof. When $r > \bar{a}$, from equation (37), ν_i is bounded by and converges to $\bar{a}^2\kappa_2(1-\bar{p}^{(2)})/(r(r-\bar{a}))$ as the experimentation deadline converges to ∞. Therefore, there are choices of π_1 and π_2, such that $T^{S,(1)}$ is small enough that $\kappa_2(1-\bar{p}^{(2)})\bar{a}^2/(r(r-\bar{a})) > \kappa_1 e^{2\bar{a}T^{S,(1)}+x_0^{(1)}} = \kappa_1 e^{2\bar{a}T^{S,(1)}}(1-\bar{p}^{(1)})/\bar{p}^{(1)}$ and $\nu^i_{j,T^{S,(1)}}$ and $\nu^i_{i,T^{S,(1)}}$ are close enough to $\kappa_2(1-\bar{p}^{(2)})\bar{a}^2/(r(r-\bar{a}))$, that (5) and the condition in Lemma 4 hold. □

B.3.3 *Noncostly incentives case* We now show that in every optimal contract in the noncostly incentives case there must be an interval of times t such that $\xi_{i,t} > 0$ and $\tilde{\gamma}_{i,t} = 0$, implying that the principal provides incentives by fine-tuning the second-task threshold.

Fine-tuning of experimentation threshold Let us see that, generically, in the optimal contract $\tilde{\gamma}_{i,t} = 0$ for some t and player i. Suppose not, then we are in Case 2 (of maximization with respect to $\tilde{\gamma}_{i,t}$), and equation (28) implies that $\lambda_{i,t} = 0$ for every t and $i \in N$. Then, from (31), $h^{(2)}_{i,t}$ becomes

$$h^{(2)}_{i,t} = z_t\Bigg(\big(W(\boldsymbol{\nu}^*) - n\nu^*\big) - \sum_j \int_t^\infty \big(W(\boldsymbol{\nu}^*) - n\nu^*\big) a^{(1)}_{j,\tau} e^{-\int_t^\tau (a_s^{(1)}+r)\,ds}\,d\tau\Bigg),$$

where ν^* denotes the agents' second-task expected payoff under threshold $T^{*(2)}$ and $\boldsymbol{\nu}^* = (\nu^*)_{i=1}^n$. For any fixed $T_j^{(1)}$ for $j \neq i$, the term in parenthesis is strictly positive. Therefore, $h^{(2)}_{i,t} > 0$ and $a^{(1)}_{i,t} = \bar{a}$ for every t. This is a contradiction.

Optimal symmetric contract We now show that when the optimal contract is symmetric, or else, when we restrict attention to symmetric contracts, agents experiment at full speed up to a deadline. In fact, suppose that $0 < a_{i,t} < \bar{a}$ for $t \in (t_1, t_2)$. We will show that the principal increases her profits by shifting effort in (t_1, t_2). As in the justification of footnote 17, we calculate the effect of shifting ε effort from one instant to the previous one. Suppose $\tilde{\gamma}_{i,t} = 0$ for $t \in (t_1, t_2)$. The effect on $\hat{\Pi}_i = \int_0^\infty a^{(1)}_{i,t}[(W(\boldsymbol{\nu}^i_t) - \sum_{j\neq i}\nu^i_{j,t} - \kappa_1)z_t - \kappa_1 e^{-rt} - \hat{v}_{i,t}]\,dt$, is up to the second order equal to[44]

$$\bar{p}e^{-\int_0^t a_s^{(1)}\,ds - rt}\left(-\frac{d\Big(W(\boldsymbol{\nu}^i_t) - \sum\limits_{j\neq i}\nu^i_{j,t}\Big)}{dt} + \big(a^{(1)}_{-i,t} + r\big)\Big(W(\boldsymbol{\nu}^i_t) - \kappa_1 - \sum_{j\neq i}\nu^i_{j,t}\Big) - r\kappa_1 e^{x_t^{(1)}}\right).$$

The effect on $\sum_{j\neq i}\hat{\Pi}_j$ is up to the second order, equal to

$$-\bar{p}e^{-\int_0^t a_s^{(1)}\,ds - rt}\sum_{j\neq i} a^{(1)}_{j,t}\Big(W(\boldsymbol{\nu}^j_t) - \sum_{k\neq j}\nu^j_{k,t} - u_{j,t}\Big).$$

[44] Recall that the effect on $\hat{v}_{i,t}$ is zero up to the second order.

Now,

$$\frac{d\left(W(\boldsymbol{\nu}_t^i)-\sum_{j\neq i}\nu_{j,t}^i\right)}{dt}=\sum_{j\neq i}\frac{\partial\left(W(\boldsymbol{\nu}_t^i)-\sum_{j\neq i}\nu_{j,t}^i\right)}{\partial\nu_{j,t}^i}\frac{d\nu_{j,t}^i}{dt}+\frac{\partial W(\boldsymbol{\nu}_t^i)}{\partial\nu_{i,t}^i}\frac{d\nu_{i,t}^i}{dt}.$$

The first term is negative since $T^{(2)}(\nu_{j,t}^i)$ is strictly decreasing in t and $T^{(2)}(\nu_{j,t}^i)\leq T^{*(2)}$, which implies $\frac{\partial W(\boldsymbol{\nu}_t^i)}{\partial\nu_{j,t}^i}\geq 0$ by the concavity of W shown in Lemma 6, below. By the principal's optimization over $\nu_{i,t}^i$ and since the (LL_t) constraint binds at t, the second term is equal to

$$\bar{\xi}_{i,t}\left(\left(a_{-i,t}^{(1)}+r\right)(u_{i,t}-\kappa_1)-r\kappa_1 e^{x_t^{(1)}}-\sum_{j\neq i}\nu_{j,t}^i a_{j,t}^{(1)}\right),$$

where $\bar{\xi}_{i,t}=\xi_{i,t}/a_{i,t}^{(1)}$. Notice that, because $\xi_{i,t}\in[0,a_{i,t}^{(1)}]$, $\bar{\xi}_{i,t}\in[0,1]$. Combining all the terms, for a symmetric contract, the effect of shifting effort is given by

$$\begin{aligned}\bar{p}e^{-\int_0^t a_s^{(1)}\,ds-rt}\Bigg[&-\sum_{j\neq i}\frac{\partial\left(W(\boldsymbol{\nu}_t^i)-\sum_{j\neq i}\nu_{j,t}^i\right)}{\partial\nu_{j,t}^i}\frac{d\nu_{j,t}^i}{dt}\\&+\sum_{k\neq i}a_{k,t}^{(1)}\left(u_{k,t}-\kappa_1-\bar{\xi}_{i,t}\left(u_{i,t}-\kappa_1-\nu_{k,t}^i\right)\right)\\&+r\left(\left(W(\boldsymbol{\nu}_t^i)-\kappa_1-\sum_{j\neq i}\nu_{j,t}^i-\kappa_1 e^{x_t^{(1)}}\right)-\left(\bar{\xi}_{i,t}\left(u_{i,t}-\kappa_1-\kappa_1 e^{x_t^{(1)}}\right)\right)\right)\Bigg]>0,\end{aligned}$$

where the last term is positive due to $W(\boldsymbol{\nu}_t^i)-\kappa_1-\sum_{j\neq i}\nu_{j,t}^i-e^{x_t^{(1)}}\kappa_1-u_{i,t}\geq 0$ (otherwise, $a_{i,t}=0$ at time t). This shows that the principal would profit from shifting effort from one instant to the previous one.

Now, if $\tilde{\gamma}_{i,t}>0$ in an interval contained in (t_1,t_2), then we can write $\hat{\Pi}_i=\int_0^\infty a_{i,t}^{(1)}\times(W(\boldsymbol{\nu}_t^i)-\sum_j\nu_{j,t}^i)z_t\,dt$. The effect on $\hat{\Pi}_i$ of the shift in effort is up to the second order equal to

$$\bar{p}e^{-\int_0^t a_s^{(1)}\,ds-rt}\left(-\frac{d\left(W(\boldsymbol{\nu}_t^i)-\sum_j\nu_{j,t}^i\right)}{dt}+\left(a_{-i,t}^{(1)}+r\right)\left(W(\boldsymbol{\nu}_t^i)-\sum_j\nu_{j,t}^i\right)\right).$$

The effect on $\sum_{j\neq i}\hat{\Pi}_j$ is

$$-\bar{p}e^{-\int_0^t a_s^{(1)}\,ds-rt}\sum_{j\neq i}a_{j,t}^{(1)}\Big(W\big(\boldsymbol{\nu}_t^j\big)-\sum_k \nu_{k,t}^j\Big).$$

From the principal's optimization problem over $\boldsymbol{\nu}^i$, since $\tilde{\gamma}_{i,t}>0$, $\xi_{i,t}=a_{i,t}^{(1)}$ and $\lambda_{j,t}\geq 0$ for $j\in N$, then $\frac{d(W(\boldsymbol{\nu}_t^i)-\sum_j \nu_{j,t}^i)}{dt}\leq 0$. The total effect of shifting effort is strictly positive.

B.3.4 *Existence of a solution to the principal's problem* Let ν_i be a second-task expected utility promised to player i, then $T^{(2)}(\nu_i)$ solves

$$\frac{e^{-rT^{(2)}(\nu_i)}\kappa_2\bar{a}\big(r-e^{T^{(2)}(\nu_i)\bar{a}}r+\big(-1+e^{rT^{(2)}(\nu_i)}\big)\bar{a}\big)}{r(r-\bar{a})}\big(1-\bar{p}^{(2)}\big)=\nu_i, \tag{37}$$

as the left-hand side of the previous equation is the expected payoff of i from experimenting up to $T^{(2)}(\nu_i)$. From equation (37), we have

$$\frac{dT^{(2)}(\nu_i)}{d\nu_i}=\frac{e^{rT^{(2)}(\nu_i)}}{\kappa_2\bar{a}\big(e^{\bar{a}T^{(2)}(\nu_i)}-1\big)\big(1-\bar{p}^{(2)}\big)}, \tag{38}$$

and

$$\begin{aligned}\frac{d^2T^{(2)}(\nu_i)}{d^2\nu_i}&=-\frac{e^{2rT^{(2)}(\nu_i)}\big((\bar{a}-r)e^{\bar{a}T^{(2)}(\nu_i)}+r\big)}{(\kappa_2\bar{a})^2\big(e^{\bar{a}T^{(2)}(\nu_i)}-1\big)^3\big(1-\bar{p}^{(2)}\big)^2}\\ &=-\Big(\frac{dT^{(2)}(\nu_i)}{d\nu_i}\Big)^2\Big(\frac{\bar{a}e^{\bar{a}T^{(2)}(\nu_i)}}{\big(e^{\bar{a}T^{(2)}(\nu_i)}-1\big)}-r\Big).\end{aligned} \tag{39}$$

Define $\boldsymbol{\mathcal{V}}_{\text{eff}}=\{\boldsymbol{\nu}^i\in\mathbb{R}^n|\boldsymbol{\nu}^i\geq 0,\sum_k T^{(2)}(\boldsymbol{\nu}_k^i)\leq n\bar{T}^{(2)}\}$ and $\boldsymbol{\mathcal{V}}_b^i=\{\boldsymbol{\nu}^i\in\mathbb{R}^n|\boldsymbol{\nu}^i\geq 0,\boldsymbol{\nu}_k^i\leq (T^{(2)})^{-1}(T^{(2)*}),k\neq i\}$ and define, for each $i\in N$, $g^i:\boldsymbol{\mathcal{V}}_b^i\to\boldsymbol{\mathcal{V}}_b^i\cap\boldsymbol{\mathcal{V}}_{\text{eff}}$ as $g^i(\boldsymbol{\nu}^i)=\boldsymbol{\nu}^i$ if $\boldsymbol{\nu}^i\in\operatorname{int}(\boldsymbol{\mathcal{V}}_{\text{eff}})$, and, otherwise, $g^i(\boldsymbol{\nu}^i)_k=\boldsymbol{\nu}_k^i$ for $k\neq i$ and $g^i(\boldsymbol{\nu}^i)_i=(T^{(2)})^{-1}(n\bar{T}^{(2)}-\sum_{k\neq i}T^{(2)}(\boldsymbol{\nu}_k^i))$. The domain of the function g^i is the set of second period procrastination rents that allocate experimentation threshold of at most $T^{(2)*}$ to an unsuccessful agent. g^i is the identity if the total experimentation implied by these rents is less than or equal than the efficient experimentation and it shortens the successful agent i's experimentation, otherwise. Because it is suboptimal, we can restrict the principal's problem to optimize over second-task rents that do not implement total experimentation exceeding the efficient one, and such that an unsuccessful agent's threshold is no longer than $T^{(2)*}$.

Let $f_0(\mathbf{x},\mathbf{a},t)=\sum_{i\in N}((W(g^i(\boldsymbol{\nu}_t^i))-\kappa_1-\sum_{j\neq i}\nu_{j,t}^i)z_t-\kappa_1e^{-rt}-\hat{v}_{i,t}^{\min}-\tilde{\gamma}_{i,t})a_{i,t}$, where $\mathbf{x}=(z,(\hat{v}_i^{\min})_{i\in N})$ is a vector containing the state variables and $\mathbf{a}\in\hat{U}=\{(a_i,\boldsymbol{\nu}^i,\tilde{\gamma}_i)_{i\in N}|a_{i,t}\in[0,\bar{a}],\tilde{\gamma}_{i,t}\in[0,M],\boldsymbol{\nu}_t^i\in\boldsymbol{\mathcal{V}}_b^i\cap\operatorname{co}(\boldsymbol{\mathcal{V}}_{\text{eff}}),\forall t\}$ is a vector of control variables that lives in the closed and convex set $\hat{U}$.[45] Let $f_1(\mathbf{x},\mathbf{a},t)=-a_{i,t}(\hat{v}_{i,t}^{\min}+\kappa_1e^{-rt})-$

[45]$\operatorname{co}(\boldsymbol{\mathcal{V}}_{\text{eff}})$ is closed as $\boldsymbol{\mathcal{V}}_{\text{eff}}$ is compact.

$z_t \sum_{j\neq i} \nu^j_{i,t} a_{j,t}$, $f_2(\mathbf{x}, \mathbf{a}, t) = -z_t(a_t + r)$, and $f(\mathbf{x}, \mathbf{a}, t) = (f_i(\mathbf{x}, \mathbf{a}, t))^2_{i=1}$. Let $h(\mathbf{x}, \mathbf{a}, t) = (h_i(\mathbf{x}, \mathbf{a}, t))^n_{i=1}$ be the vector of constraints of the principal's optimal control problem, with $h_i(\mathbf{x}, \mathbf{a}, t) = \hat{v}^{\min}_{i,t} + \tilde{\gamma}_{i,t} - (g^i(\boldsymbol{\nu}^i_t)_i - \kappa_1)z_t - \kappa_2 e^{-rt}$, for each $i \in N$.

In order to establish existence, we refer to Theorem 6.18 in Seierstad and Sydsæter (1987). Define the set $N(\mathbf{x}, t) := \{(f_0(\mathbf{x}, \mathbf{a}, t) + \nu, f(\mathbf{x}, \mathbf{a}, t)) : \nu \leq 0, h(\mathbf{x}, \mathbf{a}, t) \geq 0, \mathbf{a} \in \hat{U}\}$. From the theorem, a solution of the principal's problem exists holds if $N(\mathbf{x}, t)$ is convex. In what follows, we show that f_0 is concave in $(\boldsymbol{\nu}^i)_{i\in N}$ and that this condition is sufficient for the convexity of $N(\mathbf{x}, t)$.

Lemma 6. *$W(g^i(\boldsymbol{\nu}^i))$ is concave in $\boldsymbol{\nu}^i \in \boldsymbol{\mathcal{V}}^i_b \cap \operatorname{co}(\boldsymbol{\mathcal{V}}_{\text{eff}})$.*

Proof. To show that $W \circ g^i$ is concave in $\boldsymbol{\nu}^i = (\nu^i_j)_{j\in N}$, we calculate the Hessian matrix of $-W \circ g^i$ and show that it is positive definite. The derivative of $W \circ g^i(\boldsymbol{\nu}^i)$ with respect to ν^i_j is

$$\frac{dW \circ g^i(\boldsymbol{\nu}^i)}{d\nu^i_j} = \bar{a}\frac{dT^{(2)}(\nu^i_j)}{d\nu^i_j}\Bigg[(\pi_2 - \kappa_2)z^{(2)}_{T^{(2)}(\nu^i_j)} - \kappa_2 e^{-rT^{(2)}(\nu^i_j)} - \int_{T^{(2)}(\nu^i_j)}^{\infty} a^{(2)}_{-j,s}(\pi_2 - \kappa_2)z^{(2)}_{-j,s}e^{-T^{(2)}(\nu^i_j)\bar{a}}\, ds\Bigg],$$

where $z^{(2)}_{-j,s} = e^{-\int_0^s a_{-j,\tau}\, d\tau - rs}$ and which, from (39), yields

$$\frac{d^2W \circ g^i(\boldsymbol{\nu}^i)}{d^2\nu^i_j} = \bar{a}\left(\frac{dT^{(2)}(\nu^i_j)}{d\nu^i_j}\right)^2\Bigg[-\bar{a}\tilde{c}_j(\boldsymbol{\nu}^i) - \frac{\bar{a}e^{\bar{a}T^{(2)}(\nu^i_j)}}{(e^{\bar{a}T^{(2)}(\nu^i_i)} - 1)}\big(\hat{c}\big(T^{(2)}(\nu^i_j)\big) - \kappa_2 e^{-rT^{(2)}(\nu^i_j)}\big) - r\int_{T^{(2)}(\nu^i_j)}^{\infty} a^{(2)}_{-j,s}(\pi_2 - \kappa_2)z^{(2)}_{-j,s}e^{-T^{(2)}(\nu^i_i)\bar{a}}\, ds\Bigg],$$

where $\hat{c}(T) := (\pi_2 - \kappa_2)z^{(2)}_T - \int_T^\infty a^{(2)}_{-i,s}(\pi_2 - \kappa_2)z^{(2)}_{-i,s}e^{-T\bar{a}}\, ds$ and $\tilde{c}_j(\boldsymbol{\nu}^i) = \hat{c}(T^{(2)}(\nu^i_j))$ if g^i is the identity and $\tilde{c}_j(\boldsymbol{\nu}^i) = \hat{c}(T^{(2)}(\nu^i_j)) - \kappa_2 e^{-rT^{(2)}(g^i(\boldsymbol{\nu}^i)_i)}\mathbf{1}\{i \neq j\}$, otherwise.[46] Analogously, we obtain for $j \neq k$,

$$\frac{d^2W \circ g^i(\boldsymbol{\nu}^i)}{d\nu^i_j\, d\nu^i_k} = -\frac{dT^{(2)}(\nu^i_j)}{d\nu^i_j}\frac{dT^{(2)}(\nu^i_k)}{d\nu^i_k}\bar{a}^2 \cdot \tilde{c}_{\hat{m}}(\boldsymbol{\nu}^i),$$

where $\hat{m} = \operatorname{argmax}_{j,k}\{\nu^i_j, \nu^i_k\}$.

From our calculations, the i'th block of the Hessian of $-W \circ g^i$, corresponding to $\boldsymbol{\nu}^i$, is of the form $J = D(\boldsymbol{\nu}^i)\hat{J}(\boldsymbol{\nu}^i)D(\boldsymbol{\nu}^i)$, where $D(\boldsymbol{\nu}^i)$ is the diagonal matrix with $D(\boldsymbol{\nu}^i)_{k,k} = \bar{a} \cdot \frac{dT^{(2)}(\boldsymbol{\nu}^i_k)}{d\boldsymbol{\nu}^i_k}$. Let $\tilde{J}(\boldsymbol{\nu}^i)$ denote the matrix obtained from $\hat{J}$ by interchanging rows and columns so that the cell corresponding to the j'th row and k'th column of $\tilde{J}$ is associated to the derivative with respect to the j'th and k'th highest elements of $\boldsymbol{\nu}^i$. Let $\hat{\boldsymbol{\nu}}^i$ be

[46] For this last equality, we used $\kappa_2 e^{-rT^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} = (\pi_2 - \kappa_2)z_{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)}$ when g^i is not the identity.

$$
\tilde{J}=\begin{pmatrix}
c_1+b_1 & c_1 & c_1 & \cdots & \cdots & c_1 \\
c_1 & c_2+b_2 & c_2 & \cdots & \cdots & c_2 \\
\vdots & c_2 & c_3+b_3 & c_3 & \cdots & c_3 \\
\vdots & \vdots & c_3 & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & \cdots & \cdots & \cdots & c_n+b_n
\end{pmatrix}
\qquad
\hat{\hat{J}}^{(k-1)}=\begin{pmatrix}
d_1^{(k-1)} & c_1^{(k-1)} & \cdots & c_1^{(k-1)} & \cdots & \cdots & c_1^{(k-1)} \\
0 & d_2^{(k-1)} & \cdots & c_2^{(k-1)} & \cdots & \cdots & c_2^{(k-1)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & \cdots & d_k^{(k-1)} & c_k^{(k-1)} & \cdots & c_k^{(k-1)} \\
\vdots & \vdots & & c_k^{(k-1)} & d_{k+1}^{(k-1)} & \cdots & c_{k+1}^{(k-1)} \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & & c_k^{(k-1)} & c_{k+1}^{(k-1)} & \cdots & d_{\hat{n}}^{(k-1)}
\end{pmatrix}
$$

FIGURE 5. Matrices $\hat{J}(\hat{\boldsymbol{\nu}}^i)$ and $\hat{\hat{J}}^{(k-1)}$.

the vector that corresponds to the scrambling of $\boldsymbol{\nu}^i$ that has all its elements in descending order. The matrix $\tilde{J}$ is shown in Figure 5, where we define $c_k = \tilde{c}_k(\hat{\boldsymbol{\nu}}^i)$. From the following claim, it follows that b_k—which can be inferred from the second derivative of $W \circ g^i$—satisfies $b_k > 0$ for $k \neq i$ and for $k = i$ if $\boldsymbol{\nu}^i \in \operatorname{int} \boldsymbol{\mathcal{V}}_{\text{eff}}$.

CLAIM 2. *$\hat{c}(T^{(2)}(\nu^i_j)) - \kappa_2 e^{-rT^{(2)}(\nu^i_j)} \geq 0$ for each $j \in N$, $\tilde{c}_j(\boldsymbol{\nu}^i) > 0$ for each $j \in N \setminus \{i\}$, and for $j = i$ if $\boldsymbol{\nu}^i \in \operatorname{int} \boldsymbol{\mathcal{V}}_{\text{eff}}$, and c_k is nondecreasing in k.*

PROOF. By the definition of $\hat{c}$,

$$
\begin{aligned}
\hat{c}\big(T^{(2)}\big(\nu^i_j\big)\big) - \kappa_2 e^{-rT^{(2)}(\nu^i_j)} &= (\pi_2 - \kappa_2) z^{(2)}_{T^{(2)}(\nu^i_j)} - \kappa_2 e^{-rT^{(2)}(\nu^i_j)} \\
&\quad - \int_{T^{(2)}(\nu^i_j)}^{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} a^{(2)}_{-j,s}(\pi_2 - \kappa_2) z^{(2)}_{-j,s} e^{-T^{(2)}(\nu^i_j)\bar{a}}\, ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&- \int_{T^{(2)}(\nu^i_j)}^{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} a^{(2)}_{-j,s}(\pi_2 - \kappa_2) z^{(2)}_{-j,s} e^{-T^{(2)}(\nu^i_j)\bar{a}}\, ds \\
&= - \int_{T^{(2)}(\nu^i_j)}^{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} \big(a^{(2)}_{-j,s} + r\big)(\pi_2 - \kappa_2) z^{(2)}_{-j,s} e^{-T^{(2)}(\nu^i_j)\bar{a}}\, ds \\
&\quad + \int_{T^{(2)}(\nu^i_j)}^{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} r(\pi_2 - \kappa_2) z^{(2)}_{-j,s} e^{-T^{(2)}(\nu^i_j)\bar{a}}\, ds \\
&\geq (\pi_2 - \kappa_2) z^{(2)}_{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} - (\pi_2 - \kappa_2) z^{(2)}_{T^{(2)}(\nu^i_j)} + \kappa_2 e^{-rT^{(2)}(\nu^i_j)} - \kappa_2 e^{-rT^{(2)}(g^i(\boldsymbol{\nu}^i)_i)},
\end{aligned}
$$

where the inequality follows from $(\pi_2-\kappa_2)z_s^{(2)} \geq \kappa_2 e^{-rs}$ for $s \in [T^{(2)}(\nu_j^i), T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)]$.[47] This shows that $\hat{c}(T^{(2)}(\nu_j^i)) - \kappa_2 e^{-rT^{(2)}(\nu_j^i)} \geq 0$, with strict inequality for $j \neq i$. Since $\kappa_2 e^{-rT^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} \leq \kappa_2 e^{-rT^{(2)}(\nu_j^i)}$ for $\boldsymbol{\nu}^i \in \boldsymbol{\mathcal{V}}_b^i$ (with strict inequality for $j \neq i$ and $\boldsymbol{\nu}^i \in \operatorname{int} \boldsymbol{\mathcal{V}}_{\text{eff}}$), $\tilde{c}_j(\boldsymbol{\nu}^i) > 0$ for each $j \in N \setminus \{i\}$ and $j = i$ if $\boldsymbol{\nu}^i \in \operatorname{int} \boldsymbol{\mathcal{V}}_{\text{eff}}$.

To see that c_k is nondecreasing in k, notice that for $T_1 < T_2$,

$$\begin{aligned}\hat{c}(T_1) - \hat{c}(T_2) &= (\pi_2-\kappa_2)\big(z_{T_1}^{(2)} - z_{T_2}^{(2)}\big) - \int_{T_1}^{T_2} a_s^{(2)}(\pi_2-\kappa_2)z_s^{(2)}\,ds \\ &\geq (\pi_2-\kappa_2)\big(z_{T_1}^{(2)} - z_{T_2}^{(2)}\big) - \int_{T_1}^{T_2} \big(a_s^{(2)}+r\big)(\pi_2-\kappa_2)z_s^{(2)}\,ds = 0. \qquad \square\end{aligned}$$

Claim 3. *If $\boldsymbol{\nu}^i \in \operatorname{int} \boldsymbol{\mathcal{V}}_{eff}$, then $\hat{J}(\boldsymbol{\nu}^i)$ is positive definite and if $\boldsymbol{\nu}^i \in \boldsymbol{\mathcal{V}}_b^i \cap \operatorname{co}(\boldsymbol{\mathcal{V}}_{\text{eff}}) \setminus \operatorname{int} \boldsymbol{\mathcal{V}}_{\text{eff}}$ the i'th row and column of $\hat{J}(\boldsymbol{\nu}^i)$ are zero and $\hat{J}(\boldsymbol{\nu}^i)_{(-i,-i)}$ (the submatrix formed by deleting i'th row and column from $\hat{J}(\boldsymbol{\nu}^i)$) is positive definite.*

Proof. Fix $\boldsymbol{\nu}^i$ and let $\hat{\hat{J}} = \tilde{J}(\boldsymbol{\nu}^i)$ if $\boldsymbol{\nu}^i \in \operatorname{int} \boldsymbol{\mathcal{V}}_{\text{eff}}$ and $\hat{\hat{J}} = \tilde{J}(\boldsymbol{\nu}^i)_{(-1,-1)}$, otherwise. Let $\hat{n}$ be the dimension $\hat{\hat{J}}$.

Let us first show that the LU decomposition of $\hat{\hat{J}}$, obtained using Gaussian elimination, yields a lower triangular matrix L with ones in its diagonal (a unit lower triangular matrix) and an upper diagonal matrix U with strictly positive elements in its diagonal. This shows that $\det(\hat{\hat{J}}) = \det(L)\det(U) > 0$.

We show by induction on the step of the Gaussian elimination algorithm that in step $k-1$ the resulting matrix, $\hat{\hat{J}}^{(k-1)}$, as seen in Figure 5, has the following properties: $(\mathbf{1})_{k-1}$ $\hat{\hat{J}}^{(k-1)}_{m,\tilde{m}} = 0$ if $m > \tilde{m}$ and $\tilde{m} < k$ (columns 0 through $k-1$ have zeros below the diagonal), $(\mathbf{2})_{k-1}$ $\hat{\hat{J}}^{(k-1)}_{j,j} := d_j^{(k-1)} > \hat{\hat{J}}^{(k)}_{\tilde{k},j} = \hat{\hat{J}}^{(k)}_{j,\tilde{k}} = \hat{\hat{J}}^{(k)}_{\tilde{\tilde{k}},j} := c_j^{(k-1)} \geq 0$ for , $\tilde{\tilde{k}}$, $\tilde{k} > j$, and $j \geq k$ (as in Figure 5 the elements in column $j \geq k$ below the diagonal coincide with the elements in the row j to the right of the diagonal), $(\mathbf{3})_{k-1}$ $c_j^{(k-1)} \leq c_{\hat{j}}^{(k-1)}$ for $\hat{j} > j \geq k$. Condition $(\mathbf{2})_{\hat{n}}$ guarantees that the diagonal elements of $U = \hat{\hat{J}}^{(\hat{n})}$ are strictly positive.

Properties $(\mathbf{1})_0$ through $(\mathbf{3})_0$ hold for $\hat{\hat{J}}^{(0)} = \hat{\hat{J}}$ by Claim 2. Properties $(\mathbf{1})_k$ through $(\mathbf{3})_k$ hold if $(\mathbf{1})_{k-1}$ through $(\mathbf{3})_{k-1}$ hold. In fact, to eliminate the constant $c_k^{(k-1)} = \hat{\hat{J}}^{(k-1)}_{\hat{k},k}$ below the diagonal in rows $k+1$ through $\hat{n}$, the algorithm subtracts $\lambda_k = c_k^{(k-1)}/d_k^{(k-1)}$ times the k'th row of $\hat{J}^{(k-1)}$ from rows $k+1$ through $\hat{n}$. $\lambda_k \in (0,1)$ and $\hat{\hat{J}}^{(k-1)}_{\hat{k},k}$ is independent of $\hat{k}$ by $(\mathbf{2})_{k-1}$. Now, $(\mathbf{1})_k$ always holds in the Gaussian elimination algorithm. $(\mathbf{2})_k$ and $(\mathbf{3})_k$ hold because of $(\mathbf{2})_{k-1}$ and $(\mathbf{3})_{k-1}$, as $\lambda_k \in (0,1)$ and for each element with column index $j > k$ and row index $\hat{j} > k$ the same constant is subtracted from it in the k'th step of the algorithm.

[47]Notice that the first integral after the equality can be computed explicitly and equals $(\pi_2 - \kappa_2)z^{(2)}_{T^{(2)}(g^i(\boldsymbol{\nu}^i)_i)} - (\pi_2-\kappa_2)z^{(2)}_{T^{(2)}(\nu_j^i)}$ while applying $(\pi_2-\kappa_2)z_s^{(2)}(\nu_j^i) \geq \kappa_2 e^{-rs}(\nu_j^i)$ to the second integral yields $\kappa_2 e^{-rT^{(2)}(\nu_j^i)} - \kappa_2 e^{-rT^{(2)}(g^i(\boldsymbol{\nu}^i)_i)}$.

Now, by a similar argument, every principal submatrix of $\hat{\hat{J}}$ has a strictly positive determinant since it satisfies properties $(\mathbf{1})_0$ through $(\mathbf{3})_0$. Therefore, every principal minor of $\hat{\hat{J}}$ is strictly positive. This establishes that $\hat{\hat{J}}$ is positive definite. Finally, the matrix $\tilde{J}$ was obtained by interchanging rows and columns of $\hat{J}$ in a manner that does not change the determinant, which establishes our desired results.[48] $\square$

To conclude that $W \circ g$ is concave, let $\boldsymbol{\nu}^{i,1}, \boldsymbol{\nu}^{i,2} \in \boldsymbol{\mathcal{V}}^i_b \cap \mathrm{co}(\boldsymbol{\mathcal{V}}_{\mathrm{eff}})$, $\boldsymbol{\nu}^{i,1} \neq \boldsymbol{\nu}^{i,2}$. From what we have just shown, $W \circ g$ is continuously differentiable, and thus, $\frac{dW\circ g^i(\alpha\boldsymbol{\nu}^{i,1}+(1-\alpha)\boldsymbol{\nu}^{i,2})}{d\alpha} = \nabla W \circ g^i(\boldsymbol{\nu}^{i,2} + \alpha(\boldsymbol{\nu}^{i,1} - \boldsymbol{\nu}^{i,2})) \cdot (\boldsymbol{\nu}^{i,1} - \boldsymbol{\nu}^{i,2})$. Claim 3 shows that $\hat{J}$ and, therefore, J is positive semidefinite, which implies $\frac{dW\circ g^i(\alpha\boldsymbol{\nu}^{i,1}+(1-\alpha)\boldsymbol{\nu}^{i,2})}{d\alpha}$ is nonincreasing in α. Therefore, we conclude by noting that for $\lambda \in [0, 1]$,[49]

$$\begin{aligned} &W \circ g^i\big(\boldsymbol{\nu}^{i,2} + \lambda\big(\boldsymbol{\nu}^{i,1} - \boldsymbol{\nu}^{i,2}\big)\big) - W \circ g^i\big(\boldsymbol{\nu}^{i,2}\big) - \lambda\big(W \circ g^i\big(\boldsymbol{\nu}^{i,1}\big) - W \circ g^i\big(\boldsymbol{\nu}^{i,2}\big)\big) \\ &\quad = \lambda \int_0^1 \big(\nabla W \circ g^i\big(\lambda r \boldsymbol{\nu}^{i,1} + (1-\lambda r)\boldsymbol{\nu}^{i,2}\big) - \nabla W \circ g^i\big(r\boldsymbol{\nu}^{i,1} + (1-r)\boldsymbol{\nu}^{i,2}\big)\big) \\ &\qquad \cdot \big(\boldsymbol{\nu}^{i,1} - \boldsymbol{\nu}^{i,2}\big)\, dr \geq 0. \quad \square \end{aligned}$$

Claim 4. *$N(\mathbf{x}, t)$ is convex.*

Proof. To see that $N(\mathbf{x}, t) = \{(f_0(\mathbf{x}, \mathbf{a}, t) + \nu, f(\mathbf{x}, \mathbf{a}, t)) : \nu \leq 0, h(\mathbf{x}, \mathbf{a}, t) \geq 0, \mathbf{a} \in \hat{U}\}$ is convex, consider two controls $\mathbf{a} = (a_i, \boldsymbol{\nu}^i, \tilde{\gamma}_i)_i$ and $\tilde{\mathbf{a}} = (\tilde{a}_i, \tilde{\boldsymbol{\nu}}^i, \tilde{\tilde{\gamma}}_i)_i$ in $\hat{U}$, satisfying $h(\mathbf{x}, \mathbf{a}, t), h(\mathbf{x}, \tilde{\mathbf{a}}, t) \geq 0$, and real numbers $\nu, \nu' \leq 0$ such that $(f_0(\mathbf{x}, \mathbf{a}, t) + \nu, f(\mathbf{x}, \mathbf{a}, t))$, $(f_0(\mathbf{x}, \tilde{\mathbf{a}}, t) + \nu', f(\mathbf{x}, \tilde{\mathbf{a}}, t)) \in N(\mathbf{x}, t)$. We need to show that $\lambda(f_0(\mathbf{x}, \mathbf{a}, t) + \nu, f(\mathbf{x}, \mathbf{a}, t)) + (1-\lambda)(f_0(\mathbf{x}, \tilde{\mathbf{a}}, t) + \nu', f(\mathbf{x}, \tilde{\mathbf{a}}, t))$ is in $N(\mathbf{x}, t)$ for every $\lambda \in [0, 1]$.

For each i, let $\tilde{\beta}_i = \lambda a_i/(\lambda a_i + (1-\lambda)\tilde{a}_i)$, $\hat{\boldsymbol{\nu}}^i = \tilde{\beta}_i \boldsymbol{\nu}^i + (1-\tilde{\beta}_i)\tilde{\boldsymbol{\nu}}^i$, $\hat{\gamma}_i = \tilde{\beta}_i\tilde{\gamma}_i + (1-\tilde{\beta}_i)\tilde{\tilde{\gamma}}_i$ and $\hat{a}_i = \lambda a_i + (1-\lambda)\tilde{a}_i$, and define $\hat{\mathbf{a}} = (\hat{a}_i, \hat{\boldsymbol{\nu}}^i, \hat{\gamma}_i)_i$. The control $\hat{\mathbf{a}}$ also belongs to $\hat{U}$.

Since f is linear in $(a_j)_{j\in N}$, $\lambda f(\mathbf{x}, \mathbf{a}, t) + (1-\lambda) f(\mathbf{x}, \tilde{\mathbf{a}}, t) = f(\mathbf{x}, \hat{\mathbf{a}}, t)$. Also, $h(\mathbf{x}, \hat{\mathbf{a}}, t) \geq 0$ by h's linearity in $\tilde{\gamma}_i$ and $g^i(\boldsymbol{\nu}^i_t)_i$, and $g^i(\boldsymbol{\nu}^i_t)_i \leq \boldsymbol{\nu}^i_i$.

Let $f_{0,i}(\boldsymbol{\nu}^i, \tilde{\gamma}_i, t, \mathbf{x}) = (W(g^i(\boldsymbol{\nu}^i_t)) - \kappa_1 - \sum_{j\neq i} \nu^i_{j,t})z_t - \kappa_1 e^{-rt} - \hat{v}^{\min}_{i,t} - \tilde{\gamma}_{i,t}$. We can write $f_0(\mathbf{x}, \mathbf{a}, t) = \sum_{i\in N} f_{0,i}(\boldsymbol{\nu}^i, \tilde{\gamma}_i, t, \mathbf{x})a_{i,t}$ and $\lambda f_0(\mathbf{x}, \mathbf{a}, t) + (1-\lambda) f_0(\mathbf{x}, \tilde{\mathbf{a}}, t) = \sum_{i\in N}(\lambda f_{0,i}(\boldsymbol{\nu}^i, \tilde{\gamma}_i, t, \mathbf{x})a_{i,t} + (1-\lambda) f_{0,i}(\tilde{\boldsymbol{\nu}}^i, \tilde{\tilde{\gamma}}_i, t, \mathbf{x})\tilde{a}_{i,t})$. Now, we have

$$\begin{aligned} &\lambda a_{i,t} f_{0,i}\big(\boldsymbol{\nu}^i, \tilde{\gamma}_i, t, \mathbf{x}\big) + (1-\lambda)\tilde{a}_{i,t} f_{0,i}\big(\tilde{\boldsymbol{\nu}}^i, \tilde{\tilde{\gamma}}_i, t, \mathbf{x}\big) \\ &\quad = \hat{a}_{i,t}\big(\tilde{\beta}_i f_{0,i}\big(\boldsymbol{\nu}^i, \tilde{\gamma}_i, t, \mathbf{x}\big) + (1-\tilde{\beta}_i) f_{0,i}\big(\tilde{\boldsymbol{\nu}}^i, \tilde{\tilde{\gamma}}_i, t, \mathbf{x}\big)\big) \\ &\quad \leq \hat{a}_{i,t} f_{0,i}\big(\hat{\boldsymbol{\nu}}^i, \hat{\gamma}_i, t, \mathbf{x}\big), \end{aligned}$$

where the inequality follows by the concavity of $W \circ g$ and linearity of $f_{0,i}$ on $\tilde{\gamma}_i$.

[48] A principal submatrix of a matrix A is a submatrix obtained by deleting k rows and the same k columns of A. A determinant of a principal submatrix is called a principal minor of A. If a matrix has strictly positive principal minors, it is positive definite.

[49] A classic result states that if a function is twice differentiable and has a positive semidefinite Hessian matrix then it is convex. We make the present argument for completion since $W \circ g^i$ might not have a continuous second derivative as it is defined by parts.

Therefore, there is $\hat{\nu} \le \lambda\nu + (1-\lambda)\nu'$ such that $\lambda(f_0(\mathbf{x}, \mathbf{a}, t) + \nu, f(\mathbf{x}, \mathbf{a}, t)) + (1-\lambda)(f_0(\mathbf{x}, \tilde{\mathbf{a}}, t) + \nu', f(\mathbf{x}, \tilde{\mathbf{a}}, t)) = (f_0(x, \hat{\mathbf{a}}, t) + \hat{\nu}, f(x, \hat{\mathbf{a}}, t)) \in N(\mathbf{x}, t)$. □

B.3.5 *Necessary conditions for agent's effort are sufficient* At the contracts we characterized, the principal sets each agent's effort to $\bar{a}$ and $u_{i,t}$ is chosen so that $\tilde{\gamma}_{i,t} \ge 0$. Therefore, the necessary condition of the agent holds trivially. Now, for sufficiency, suppose agent i has a deviation to an effort function $\tilde{a}_{i,t}$, that differs from $a_{i,t}$ in a positive measure set, with associated multiplier $\hat{\tilde{\gamma}}_{i,t}$. Because under $a_{i,t}$ player i experiments at full effort it must be that $\tilde{a}_{i,t} \le a_{i,t}$ for all $t \le T_i^{(1)}$ and $\tilde{a}_{i,t} < a_{i,t}$ in a positive measure of times $t \in [0, T_i^{(1)}]$. Therefore, there is a time $\bar{t}$ and a time $\tau \in (\bar{t}, T_i^{(1)}]$ such that $\int_0^t \tilde{a}_{i,s}\, ds < \int_0^t a_{i,s}\, ds$ for every $t \ge \bar{t}$ and such that $\tilde{a}_{i,\tau} < a_{i,\tau}$. From equations (3) and (4), $u_{i,t}^{\min}(\tilde{a}_{i,\cdot}) < u_{i,t}^{\min}(a_{i,\cdot})$ for every $t \ge \bar{t}$. Therefore, by (24), $\hat{\tilde{\gamma}}_{i,\tau} e^{r\tau + \int_0^\tau \tilde{a}_{i,s}\, ds + \int_0^\tau a_{-i,s}\, ds} = u_{i,\tau} - u_{i,\tau}^{\min}(\tilde{a}_{i,\cdot}) > u_{i,\tau} - u_{i,\tau}^{\min}(a_{i,\cdot}) = \tilde{\gamma}_{i,\tau} z_\tau^{-1} \ge 0$. However, $\hat{\tilde{\gamma}}_{i,t} > 0$ contradicts $a_{i,\tau} > \tilde{a}_{i,\tau}$.

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