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Maskin meets Abreu and Matsushima

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The theory of full implementation has been criticized for using integer/modulo games, which admit no equilibrium (Jackson (1992)). To address the critique, we revisit the classical Nash implementation problem due to Maskin (1977, 1999) but allow for the use of lotteries and monetary transfers as in Abreu and Matsushima (1992, 1994). We unify the two well-established but somewhat orthogonal approaches in full implementation theory. We show that Maskin monotonicity is a necessary and sufficient condition for (exact) mixed-strategy Nash implementation by a finite mechanism. In contrast to previous papers, our approach possesses the following features: finite mechanisms (with no integer or modulo game) are used; mixed strategies are handled explicitly; neither undesirable outcomes nor transfers occur in equilibrium; the size of transfers can be made arbitrarily small; and our mechanism is robust to information perturbations.

Keywords. Complete information, full implementation, information perturbations, Maskin monotonicity, mixed-strategy Nash equilibrium, social choice function.

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1. Introduction

Implementation theory can be seen as reverse engineering game theory. Suppose that a society has decided on a social choice rule—a recipe for choosing the socially-optimal alternatives on the basis of individuals' preferences over alternatives. The individuals' preferences vary across states and the realized state is common knowledge among the agents but unknown to a social planner/mechanism designer. To (fully) implement the social choice rule, the designer chooses a mechanism so that at each state, the equilibrium outcomes of the mechanism coincide with the outcomes designated by the social choice rule.

We study Nash implementation by a finite mechanism where agents report only their preferences and preference profiles. We focus on the monotonicity condition (hereafter, Maskin monotonicity), which Maskin shows is necessary and "almost sufficient" for Nash implementation. We aim to implement *social choice functions* (henceforth, SCFs) that are Maskin-monotonic in mixed-strategy Nash equilibria without making use of the *integer game* or the *modulo game*, which prevails in the full implementation literature.

In the integer game, each agent announces some integer and the person who announces the highest integer gets to name his favorite outcome. When the agents' favorite outcomes differ, an integer game has no pure-strategy Nash equilibria. The questionable feature is also shared by modulo games. The modulo game is regarded as a finite version of the integer game in which agents announce integers from a finite set. The agent who matches the modulo of the sum of the integers gets to name an allocation. In order to "knock out" undesirable equilibria in general environments, most constructive proofs in the literature, following Maskin (1999) (circulating as a working paper in 1977), have either taken advantage of the fact that the integer/modulo game has no solution in pure strategies and/or restricted attention to pure-strategy Nash equilibria.

Instead of invoking integer/modulo games, we study Nash implementation in a restricted domain where the designer can invoke both lotteries and (off-the-equilibrium) transfers in designing the implementing mechanism. We study a finite environment in which a finite mechanism is to be anticipated.[1] Finite mechanisms are also bounded in the sense of Jackson (1992) and have no aforementioned questionable features. Indeed, Jackson (1992, Example 4) shows that when no domain restriction on the environment is imposed, some Maskin-monotonic SCF is not implementable in mixed-strategy Nash equilibria by any finite mechanism. It raises the question as to whether *every* Maskin-monotonic SCF is mixed-strategy Nash implementable with domain restrictions imposed by lotteries and transfers.[2]

Our main result (Theorem 1) shows that when the designer can make use of lotteries and transfers off the equilibrium, Maskin monotonicity is indeed a necessary and sufficient condition for mixed-strategy Nash implementation by a finite mechanism. In

[1] In Chen, Kunimoto, Sun, and Xiong (2022), we consider infinite environments in which we construct an infinite yet "well-behaved" implementing mechanism to achieve the same goal.

[2] Another direction one can take is to characterize, without making any domain restriction, the subclass of Maskin-monotonic SCFs, which can be implemented in mixed-strategy Nash equilibria in a finite mechanism. For that goal, our exercise serves as a clarification of whether in certain environments, the class of implementable SCFs is as permissive as it can be.

our finite mechanism, each agent is asked to report only his preference and a preference profile. That is, we replace the integer announcement in Maskin's mechanism with an announcement of each agent's own preference. The preference announcement plays the same role as an integer in knocking out unwanted equilibria, albeit in a different manner. Following the idea of Abreu and Matsushima (1994), we design the mechanism so that whenever an "unwanted equilibrium" occurs, the agents' reports must be "truthful," namely they announce their own preference and preference profile as prescribed under the true state. That in turn implies, through cross-checking the (truthful) preferences and the preference profiles reported by the agents, that the unwanted equilibrium could not have happened. In our finite mechanism, a pure-strategy (truth-telling) equilibrium exists, and all mixed-strategy equilibria achieve the desirable social outcome at each state.

We also provide several extensions of our main results. First, we show that our implementation is robust to information perturbations. Second, we extend Theorem 1 to cover social choice correspondences (i.e., multivalued social choice rules), studied in Maskin (1999) as well as in many subsequent papers. Formally, we show that when there are at least three agents, every Maskin-monotonic social choice correspondence is mixed-strategy Nash implementable (Theorem 2). Moreover, as long as the social choice correspondence is finite-valued, our implementing mechanism remains finite. Third, we show that if there are at least three agents and the SCF satisfies Maskin monotonicity in the restricted domain without any transfer, then it is implementable in mixed-strategy Nash equilibria by a finite mechanism in which the size of transfers remains zero on the equilibrium and can be made arbitrarily small off the equilibrium (Theorem 3).

The rest of the paper is organized as follows. In Section 2, we position our paper in the literature. In Section 3, we present the basic setup and definitions. Section 4 proves our main result. We discuss the extensions of our main result in Section 5. The Appendix contains all proofs, which are omitted from the main text.

2. Related literature

Maskin (1999) proposes the notion of Maskin monotonicity and implements a Maskin-monotonic social choice correspondence by constructing an infinite mechanism with integer games. While integer games are useful in achieving positive results in general settings, the hope has been that for more specific environments, more realistic mechanisms, or mechanisms without the "questionable features," may suffice. The research program has been proposed by Jackson (1992). One such class of specific environments is the one with lotteries and transfers which our paper, as well as the partial implementation literature, focuses on.

In environments with lotteries and transfers, Abreu and Matsushima (1992, 1994) obtain permissive full implementation results using finite mechanisms without the aforementioned questionable features. Like our implementing mechanisms, their mechanisms also make use of only payoff relevant messages, such as preferences or preference profiles. However, Abreu and Matsushima (1992, 1994) do not investigate

Nash implementation but rather appeal to a different notion of implementation: virtual implementation (in Abreu and Matsushima (1992)) or exact implementation under iterated weak dominance (in Abreu and Matsushima (1994)).[3]

Virtual implementation means that the planner contents herself with implementing the SCF with arbitrarily high probability. In contrast, by studying *exact* Nash implementation in the specific setting, we unify the two well-established but somewhat orthogonal approaches to implementation theory, which are due to Maskin (1999) and to Abreu and Matsushima (1992, 1994). Our exercise is directly comparable to Maskin (1999) and highlights the pivotal trade-off between domain restrictions and the feature of implementing mechanisms. We consider it to be one step in advancing the research program proposed by Jackson (1992).

An alternative approach to handling mixed-strategy equilibria is to resort to refinements such as undominated Nash equilibria or subgame-perfect equilibria. With such refinements, essentially every SCF, whether Maskin-monotonic or not, is implementable in a complete-information environment; see, for example, Moore and Repullo (1988) and Abreu and Matsushima (1994). However, according to Chung and Ely (2003) and Aghion, Fudenberg, Holden, Kunimoto, and Tercieux (2012), if we were to achieve exact implementation in these refinements, which are also robust to a *small amount of incomplete information*, then Maskin monotonicity would be restored as a necessary condition. Those permissive implementation results, which are driven by the lack of the closed-graph property of the refinements, cast doubt on the success of taking care of non-Maskin-monotonic SCFs by resorting to equilibrium refinements. In contrast, our Theorem 1 establishes exact and *robust* implementation in mixed-strategy Nash equilibria to the maximal extent that every Maskin-monotonic SCF is implementable (Proposition 3).[4]

Ollár and Penta (2017) study a full implementation problem using transfers both on and off the equilibrium. Specifically, Theorem 2 of Ollár and Penta (2017) provides a sufficient condition restricting agents' beliefs via moment conditions under which their notion of robust full implementation is possible in a direct mechanism. Their notion of robustness is a "global notion" which accommodates arbitrary information structures consistent with a fixed payoff environment.[5] In contrast, our paper follows the classical implementation literature in dealing with the specific belief restriction of complete information, and our notion of robustness (in Section 5.1) accommodates only perturbations around the complete-information benchmark. With the specific belief restriction, we prove that Maskin monotonicity is both necessary and sufficient for implementation in mixed-strategy Nash equilibria in a finite, indirect mechanism with only off-the-equilibrium transfers.

[3] Iterated weak dominance in Abreu and Matsushima (1994) also yields the unique undominated Nash equilibrium outcome. For undominated Nash implementation by "well-behaved" mechanisms, see also Jackson, Palfrey, and Srivastava (1994) and Sjöstrom (1994).

[4] Harsanyi (1973) shows that a mixed-strategy Nash equilibrium outcome may occur as the limit of a sequence of pure-strategy Bayesian Nash equilibria for "nearby games" in which players are uncertain about the exact profile of preferences. Hence, ignoring mixed-strategy equilibria would be particularly problematic if we were to achieve implementation which is robust to information perturbations.

[5] See Ollár and Penta (2021) for a further extension of the approach of Ollár and Penta (2017).

3. Preliminaries

3.1 *Environment*

Consider a finite set of agents $\mathcal{I} = \{1, 2, \ldots, I\}$ with $I \geq 2$, a finite set of possible states Θ, and a set of pure alternatives A. We consider an environment with lotteries and transfers. Specifically, we work with the space of allocations/outcomes $X \equiv \Delta(A) \times \mathbb{R}^I$ where $\Delta(A)$ denotes the set of lotteries on A that have a countable support, and $\mathbb{R}^I$ denotes the set of transfers to the agents. We identify $a \in A$ with a degenerate lottery in $\Delta(A)$.

For each $x = (\ell, (t_i)_{i\in\mathcal{I}}) \in X$, agent i receives the utility $\tilde{u}_i(x, \theta) = v_i(\ell, \theta) + t_i$ for some bounded expected utility function $v_i(\cdot, \theta)$ over $\Delta(A)$. That is, we work with an environment with a transferable utility (TU) on agents' preferences, which Maskin (1999) does not impose. We abuse notation to identify $\Delta(A)$ with a subset of X, that is, each $\ell \in \Delta(A)$ is identified with the allocation $(\ell, 0, \ldots, 0)$ in X.

We focus on a *complete-information* environment in which a true state θ is common knowledge among the agents but unknown to a mechanism designer. The designer's objective is specified by a *social choice function* $f : \Theta \to X$, namely, if the state is θ, the designer would like to implement the social outcome $f(\theta)$.

3.2 *Mechanism and solution*

We denote a (finite) mechanism by $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ where M_i is a nonempty finite *set of messages* available to agent i; $g : M \to X$ (where $M \equiv \times_{i=1}^I M_i$) is the *outcome function*, and $\tau_i : M \to \mathbb{R}$ is the *transfer rule* which specifies the payment to agent i. At each state $\theta \in \Theta$, the environment and the mechanism together constitute a *game with complete information*, which we denote by $\Gamma(\mathcal{M}, \theta)$. Note that the restriction of M_i to a finite set rules out the use of integer games à la Maskin (1999). Throughout the paper, we only make use of finite mechanisms and call them mechanisms for simplicity.

Let $\sigma_i \in \Delta(M_i)$ be a mixed *strategy* of agent i in the game $\Gamma(\mathcal{M}, \theta)$. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I) \in \times_{i\in\mathcal{I}} \Delta(M_i)$ is said to be a mixed-strategy *Nash equilibrium* of the game $\Gamma(\mathcal{M}, \theta)$ if, for all agents $i \in \mathcal{I}$ and all messages $m_i \in \operatorname{supp}(\sigma_i)$ and $m_i' \in M_i$, we have

$$\sum_{m_{-i}\in M_{-i}} \prod_{j\neq i} \sigma_j(m_j)\big[\tilde{u}_i\big(g(m_i, m_{-i}), \theta\big) + \tau_i(m_i, m_{-i})\big]$$
$$\geq \sum_{m_{-i}\in M_{-i}} \prod_{j\neq i} \sigma_j(m_j)\big[\tilde{u}_i\big(g\big(m_i', m_{-i}\big), \theta\big) + \tau_i\big(m_i', m_{-i}\big)\big].$$

A pure-strategy Nash equilibrium is a mixed-strategy Nash equilibrium σ such that each agent i's mixed-strategy σ_i assigns probability one to some $m_i \in M_i$. For any message profile $m \in M$, let $\sigma(m) \equiv \prod_{j\in\mathcal{I}} \sigma_j(m_j)$.

Let $\operatorname{NE}(\Gamma(\mathcal{M}, \theta))$ denote the set of mixed-strategy Nash equilibria of the game $\Gamma(\mathcal{M}, \theta)$. We also denote by $\operatorname{supp}(\operatorname{NE}(\Gamma(\mathcal{M}, \theta)))$ the set of message profiles that can be played with positive probability under some mixed-strategy Nash equilibrium $\sigma \in \operatorname{NE}(\Gamma(\mathcal{M}, \theta)$, that is,

$$\operatorname{supp}\big(\operatorname{NE}\big(\Gamma(\mathcal{M}, \theta)\big)\big) = \big\{m \in M : \text{there exists } \sigma \in \operatorname{NE}\big(\Gamma(\mathcal{M}, \theta)\big) \text{ such that } \sigma(m) > 0\big\}.$$

We now define our notion of Nash implementation.

Definition 1. An SCF f is *implementable in mixed-strategy Nash equilibria by a finite mechanism* if there exists a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ such that for every state $\theta \in \Theta$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta)$ and (ii) $m \in \text{supp}(\text{NE}(\Gamma(\mathcal{M}, \theta))) \Rightarrow g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.

Our definition is adapted from mixed-strategy Nash implementation in Maskin (1999) to (1) require that the implementing mechanism be finite and (2) accommodate our quasilinear environments with transfers. In particular, Condition (ii) requires that transfers be imposed only off the equilibrium. Mezzetti and Renou (2012a) propose another definition of Nash implementation that keeps Condition (ii) but weakens Condition (i) in requiring only the existence of a mixed-strategy Nash equilibrium, which,by Nash's theorem, is guaranteed in a finite mechanism. In their sufficiency result, Mezzetti and Renou (2012a) do not use transfers, however, they construct an infinite mechanism with integer games.

3.3 *Maskin monotonicity*

We now restate the definition of Maskin monotonicity that Maskin (1999) proposes for Nash implementation.

Definition 2. An SCF f satisfies *Maskin monotonicity* if, for every pair of states $\tilde{\theta}$ and θ with $f(\tilde{\theta}) \neq f(\theta)$, some agent $i \in \mathcal{I}$ and some allocation $x \in X$ exist such that

$$\tilde{u}_i(x, \tilde{\theta}) \leq \tilde{u}_i\big(f(\tilde{\theta}), \tilde{\theta}\big) \quad \text{and} \quad \tilde{u}_i(x, \theta) > \tilde{u}_i\big(f(\tilde{\theta}), \theta\big). \tag{1}$$

To illustrate how the idea of Maskin monotonicity is applied, suppose that the SCF f is implemented in Nash equilibria by a mechanism. When $\tilde{\theta}$ is the true state, there exists a pure-strategy Nash equilibrium $m \in M$ in $\Gamma(\mathcal{M}, \tilde{\theta})$, which induces $f(\tilde{\theta})$. If $f(\tilde{\theta}) \neq f(\theta)$ and θ is the true state, then m cannot be a Nash equilibrium, that is, there exists some agent i who has a profitable deviation. Suppose that the deviation induces outcome x, that is, agent i strictly prefers x to $f(\tilde{\theta})$ at state θ. Since m is a Nash equilibrium at state $\tilde{\theta}$, such a deviation cannot be profitable at state $\tilde{\theta}$; that is, agent i weakly prefers $f(\tilde{\theta})$ to x at state $\tilde{\theta}$. In other words, x belongs to agent i's lower contour set at $f(\tilde{\theta})$ of state $\tilde{\theta}$, whereas it belongs to the strict upper-contour set at $f(\tilde{\theta})$ of state θ. Therefore, Maskin monotonicity is a necessary condition for Nash implementation; in fact, it is a necessary condition even for Nash implementation that restricts attention to pure-strategy equilibria (i.e., to require that condition (ii) of Definition 1 hold only for pure-strategy Nash equilibria).

4. Main result

In this section, we present our main result, which shows that Maskin monotonicity is necessary and sufficient for mixed-strategy Nash implementation. We formally state the result as follows.

THEOREM 1. *An SCF f is implementable in mixed-strategy Nash equilibria by a finite mechanism if and only if it satisfies Maskin monotonicity.*

In the rest of this section, we will establish Theorem 1 and discuss the issues regarding the theorem. Section 4.1 details how our implementing mechanism is constructed. In Section 4.2, we prove Theorem 1 by making use of the implementing mechanism constructed in Section 4.1. Section 4.3 illustrates two special cases in which our implementing mechanism can be made into a direct mechanism where each agent reports a state. In Section 4.4, we discuss the necessity of domain restrictions in establishing Theorem 1.

4.1 *The mechanism*

We construct a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$, which will be used to prove Theorem 1. The mechanism shares a number of features of the implementing mechanisms in Maskin (1999) and in Abreu and Matsushima (1992, 1994), which we summarize at the end of the subsection. The construction involves two major building blocks that we call the *best challenge scheme* and *dictator lotteries*, respectively. After introducing these building blocks, we will define the message space, allocation rule, and transfer rule of our implementing mechanism.

For each agent i, as a preliminary step, we define

$$\Theta_i \equiv \{v_i(\cdot, \theta) : \theta \in \Theta\}.$$

That is, Θ_i is the set of expected utility functions of agent i induced by some state θ. Denote by $\theta_i \in \Theta_i$ the expected utility function of agent i obtained at state $\theta \in \Theta$, namely that $\theta_i = v_i(\cdot, \theta)$. We call θ_i the type of player i at state θ. We denote by $u_i(\cdot, \theta_i)$ the quasilinear utility function, which corresponds to type θ_i, namely that for each $x = (\ell, (t_i)_{i\in\mathcal{I}}) \in X$, we have $u_i(x, \theta_i) \equiv v_i(\ell, \theta) + t_i$. For a Maskin-monotonic SCF f, we have $f(\theta) = f(\tilde{\theta})$ if states θ and $\tilde{\theta}$ induce the same type profile (i.e., $\theta_i = \tilde{\theta}_i$ for every i). Hence, if a type profile $(\tilde{\theta}_i)_{i\in\mathcal{I}}$ is induced by some state $\theta \in \Theta$, we may abuse the notation to write $(\tilde{\theta}_i)_{i\in\mathcal{I}} \in \Theta$ and set $f((\tilde{\theta}_i)_{i\in\mathcal{I}}) \equiv f(\theta)$.

REMARK 1. It is possible no state in Θ induces a given type profile. For example, suppose we that have two states α and β and two agents A and B who have an identical expected utility function that varies across the states, namely $\alpha_A = \alpha_B \neq \beta_A = \beta_B$. In this example, there are four type profiles: (α_A, α_B), (α_A, β_B), (β_A, α_B), and (β_A, β_B), and yet neither the type profile (α_A, β_B) nor (β_A, α_B) corresponds to a state.

4.1.1 *Best challenge scheme* For $(x, \theta_i) \in X \times \Theta_i$, we use $\mathcal{L}_i(x, \theta_i)$ to denote the lower-contour set at allocation x in X for type θ_i, that is,

$$\mathcal{L}_i(x, \theta_i) = \{x' \in X : u_i(x, \theta_i) \geq u_i(x', \theta_i)\}.$$

We use $\mathcal{SU}_i(x, \theta_i)$ to denote the strict upper-contour set of $x \in X$ for type θ_i, that is,

$$\mathcal{SU}_i(x, \theta_i) = \{x' \in X : u_i(x', \theta_i) > u_i(x, \theta_i)\}.$$

Hence, according to Definition 2, an SCF f satisfies Maskin monotonicity if and only if for every pair of states $\tilde{\theta}$ and θ in Θ,

$$f(\tilde{\theta}) \neq f(\theta) \quad \Rightarrow \quad \exists i \in \mathcal{I} \quad \text{such that} \quad \mathcal{L}_i\big(f(\tilde{\theta}), \tilde{\theta}_i\big) \cap \mathcal{SU}_i\big(f(\tilde{\theta}), \theta_i\big) \neq \emptyset. \tag{2}$$

Agent i in (2) is called a "whistle-blower" or a "test agent," and an allocation in $\mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$ is called a "test allocation" for agent i and the ordered pair of states $(\tilde{\theta}, \theta)$. We now define a notion called *the best challenge scheme*, which plays a crucial role in proving Theorem 1. We say that a mapping $x : \Theta \times \Theta_i \to X$ is a *challenge scheme* for an SCF f if and only if, for each pair of state $\tilde{\theta} \in \Theta$ and type $\theta_i \in \Theta_i$,

$$\begin{cases} x(\tilde{\theta}, \theta_i) \in \mathcal{L}_i\big(f(\tilde{\theta}), \tilde{\theta}_i\big) \cap \mathcal{SU}_i\big(f(\tilde{\theta}), \theta_i\big), & \text{if } \mathcal{L}_i\big(f(\tilde{\theta}), \tilde{\theta}_i\big) \cap \mathcal{SU}_i\big(f(\tilde{\theta}), \theta_i\big) \neq \emptyset; \\ x(\tilde{\theta}, \theta_i) = f(\tilde{\theta}), & \text{if } \mathcal{L}_i\big(f(\tilde{\theta}), \tilde{\theta}_i\big) \cap \mathcal{SU}_i\big(f(\tilde{\theta}), \theta_i\big) = \emptyset. \end{cases}$$

We may think of state $\tilde{\theta}$ as an announcement made by one or more other agents that agent i of type θ_i could "challenge" (as a whistle-blower). The following lemma shows that there is a challenge scheme in which each whistle-blower i facing state announcement $\tilde{\theta}$ finds it weakly best to challenge $\tilde{\theta}$ by simply reporting his true type θ_i.

Lemma 1. *There is a challenge scheme $x(\cdot, \cdot)$ for an SCF f such that for every state $\tilde{\theta}$ and type θ_i,*

$$u_i\big(x(\tilde{\theta}, \theta_i), \theta_i\big) \geq u_i\big(x\big(\tilde{\theta}, \theta_i'\big), \theta_i\big), \quad \forall \theta_i' \in \Theta_i. \tag{3}$$

We relegate its formal proof to Appendix A.1.[6] In defining the implementing mechanism, we shall invoke a challenge scheme, which satisfies (3). We call such a challenge scheme *the best challenge scheme*. In words, under the best challenge scheme, for any state $\tilde{\theta}$, agent i of type θ_i weakly prefers the allocation $x(\tilde{\theta}, \theta_i)$ to any other $x(\tilde{\theta}, \theta_i')$ induced by announcing $\theta_i' \neq \theta_i$.

4.1.2 *Dictator lotteries* Let $\tilde{X} \equiv A \cup \bigcup_{i \in \mathcal{I}, \theta_i \in \Theta_i, \tilde{\theta} \in \Theta} x(\tilde{\theta}, \theta_i)$. We can then conclude that $\tilde{X}$ is a set over which all agents' utilities are bounded, because $v_i(\cdot, \theta)$ is bounded, Θ is finite, and we prespecify $x(\tilde{\theta}, \theta_i)$ for each $i \in \mathcal{I}$, type $\theta_i \in \Theta_i$, and state $\tilde{\theta} \in \Theta$. Hence, we can choose $\eta' > 0$ as an upper bound on the monetary value of a change of allocation in $\tilde{X}$, that is,

$$\eta' > \sup_{i \in \mathcal{I}, \theta_i \in \Theta_i, x, x' \in \tilde{X}} \big| u_i(x, \theta_i) - u_i\big(x', \theta_i\big) \big|. \tag{4}$$

We now state a result, which ensures the existence of what we call dictator lotteries for agent i. A collection of lotteries are called *dictator lotteries* of agent i if they satisfy Conditions (5) and (6) stated in Lemma 2. Condition (5) says that in selecting the dictator lotteries, each agent has a strict incentive to reveal his true type. Condition (6) says that these dictator lotteries are strictly less preferred to any allocations in $\tilde{X}$.

[6] We owe special thanks to Phil Reny for suggesting the lemma, which simplifies the implementing mechanism adopted in an earlier version of our paper.

Lemma 2. *For each agent* $i \in \mathcal{I}$*, there exists a collection of lotteries* $\{y_i(\theta_i)\}_{\theta_i \in \Theta_i}$ *such that for all types* $\theta_i, \theta_i' \in \Theta_i$ *with* $\theta_i \neq \theta_i'$*, we have*

$$u_i\big(y_i(\theta_i), \theta_i\big) > u_i\big(y_i(\theta_i'), \theta_i\big); \tag{5}$$

moreover, for each $j \in \mathcal{I}$ *and type* $\theta_j' \in \Theta_j$*, we also have that, for every* $x \in \tilde{X}$*,*

$$u_i\big(y_j(\theta_j'), \theta_i\big) < u_i(x, \theta_i). \tag{6}$$

Since two distinct types in Θ_i induce different expected utility functions over $\Delta(A)$, it follows from Abreu and Matsushima (1992, Lemma, p. 999) that we can prove the existence of lotteries $\{y_i'(\cdot)\} \subset \Delta(A)$ that satisfy Condition (5). To satisfy Condition (6), we simply add a penalty of η' to each outcome of the lotteries $\{y_i'(\theta_i)\}_{\theta_i \in \Theta_i}$. More precisely, for each $\theta_i \in \Theta_i$, we set

$$y_i(\theta_i) = \big(y_i'(\theta_i), -\eta', \dots, -\eta'\big) \in X.$$

4.1.3 *Message space* A generic message of agent i is described as follows:

$$m_i = \big(m_i^1, m_i^2\big) \in M_i = M_i^1 \times M_i^2 = \Theta_i \times \left[\bigtimes_{j=1}^{I} \Theta_j\right].$$

That is, agent i is asked to make (1) a report of his own type (which we denote by m_i^1); and (2) a report of a type profile (which we denote by m_i^2). To simplify the notation, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent j is of type $\tilde{\theta}_j$. Recall that agents have complete information about the true state. If the true state is θ, we say that agent i sends a truthful first report if $m_i^1 = \theta_i$ and a truthful second report if $m_i^2 = (\theta_j)_{j \in \mathcal{I}}$. Note that each agent is asked to report a type profile in M_i^2 instead of a state. Hence, the mechanism must take care of the difficulties in identifying the state from a type profile, which we explain at the beginning of Section 4.1.

It is useful to compare the message space of our mechanism with that of the implementing mechanism in Maskin (1999). In Maskin's mechanism (see Maskin (1999, p. 31)), each agent is asked to report a preference profile and an integer, as well as an allocation. The allocation need not be specified in the case of SCFs, since there is no ambiguity about the socially desirable outcome assigned to each state. In contrast, we ask each agent to report a preference/type profile and a *type*. The type component of the message space plays the role of an integer in Maskin's mechanism in knocking out unwanted equilibria, albeit in a different manner. As the integer game admits no equilibrium when there is disagreement over most preferred outcomes, it is used to assure that undesirable message profiles do not form an equilibrium. However, the logic of that argument no longer works when the goal is to achieve implementation in mixed-strategy Nash equilibria by a finite mechanism. Indeed, there is no a priori way to rule out *any* message profile because any of them might be played with positive probability in a mixed-strategy Nash equilibrium.

By making use of the type component m_i^1 in the message space, we appeal to the approach of Abreu and Matsushima (1992, 1994) to resolve the issue. More precisely, we design the mechanism so that when an unwanted message profile is triggered in equilibrium, the type report m_i^1 must coincide with agent i's preference under the true state. Through the cross-checking of the preferences and preference profiles reported by the agents (in a similar manner to Abreu and Matsushima (1992, 1994)), it further implies that the unwanted message profile could not have happened. Unlike Abreu and Matsushima (1992, 1994), however, to ensure that m_i^1 is truthful, we must guarantee that the designer's twin goals of allowing for whistle-blowing/challenges (as in Maskin (1999)) and eliciting the truth (from the dictator lotteries, as in Abreu and Matsushima (1992, 1994)) can be aligned perfectly. It is achieved through Lemmas 1 and 2: since truth-telling is weakly optimal for the former and strictly optimal for the latter, we can make the truth-telling in m_i^1 strictly optimal by taking a convex combination of the best challenge scheme and dictator lotteries. Hence, Maskin meets Abreu and Matsushima. We formalize the idea in Section 4.1.4.

4.1.4 *Outcome function* For each message profile $m \in M$, the allocation is determined as follows:

$$g(m) = \frac{1}{I(I-1)} \sum_{i\in\mathcal{I}} \sum_{j\neq i} \left[e_{i,j}(m_i, m_j) \left(\frac{1}{2} \sum_{k=i,j} y_k\left(m_k^1\right) \right) \oplus \left(1 - e_{i,j}(m_i, m_j)\right) x\left(m_i^2, m_j^1\right) \right],$$

where $\{y_k(\cdot)\}$ are the dictator lotteries for agent k obtained from Lemma 2, and $\alpha x \oplus (1-\alpha)x'$ denotes the outcome, which corresponds to the compound lottery that outcome x occurs with probability α, and outcome x' occurs with probability $1-\alpha$;[7] moreover, we define

$$e_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m_i^2 \in \Theta,\ m_i^2 = m_j^2, \text{ and } x\left(m_i^2, m_j^1\right) = f\left(m_i^2\right); \\ \varepsilon, & \text{if } m_i^2 \in \Theta, \text{ and } \left[m_i^2 \neq m_j^2 \text{ or } x\left(m_i^2, m_j^1\right) \neq f\left(m_i^2\right)\right]; \\ 1, & \text{if } m_i^2 \notin \Theta. \end{cases}$$

To explain the outcome function, hereafter we say that the second reports of agent i and agent j are *consistent* if $m_i^2 = m_j^2$ *and* the common type profile identifies a state in Θ; moreover, we say that agent j *does not challenge* agent i if $x(m_i^2, m_j^1) = f(m_i^2)$.

In words, the designer first chooses an ordered pair of distinct agents (i, j) with equal probability. The outcome function distinguishes three cases: (1) if the second reports of agent i and agent j are consistent *and* agent j does not challenge agent i, then we implement $f(m_i^2)$; (2) if agent i reports a type profile which does not identify a state in Θ, then we implement the dictator lottery $\frac{1}{2}\sum_{k=i,j} y_k(m_k^1)$;[8] (3) otherwise, we implement the compound lottery:

$$C_{i,j}^{\varepsilon}(m_i, m_j) \equiv \varepsilon \left(\frac{1}{2} \sum_{k=i,j} y_k\left(m_k^1\right) \right) \oplus (1-\varepsilon) x\left(m_i^2, m_j^1\right).$$

[7] More precisely, if $x = (\ell, (t_i)_{i\in\mathcal{I}})$ and $x' = (\ell', (t_i')_{i\in\mathcal{I}})$ are two outcomes in X, we identify $\alpha x \oplus (1-\alpha)x'$ with the outcome $(\alpha\ell \oplus (1-\alpha)\ell', (\alpha t_i + (1-\alpha)t_i')_{i\in\mathcal{I}})$. For simplicity, we also write the compound lottery $\frac{1}{2} y_i(m_i^1) \oplus \frac{1}{2} y_j(m_j^1)$ as $\frac{1}{2}\sum_{k=i,j} y_k(m_k^1)$.

[8] Observe that we make the first report of both agents i and j effective (through affecting the compound lottery $\frac{1}{2}\sum_{k=i,j} y_k(m_k^1)$), regardless of whether pair (i, j) or pair (j, i) is picked. The construction will be used in proving Claim 1, which in turn, is used to prove Claim 4.

Note that $C^{\varepsilon}_{i,j}(m_i, m_j)$ is an $(\varepsilon, 1-\varepsilon)$-combination of (i) the two dictator lotteries—$y_i(m^1_i)$ and $y_j(m^1_j)$—which occur with equal probability and (ii) the allocation specified by the best challenge scheme $x(m^2_i, m^1_j)$.

By (4), we can choose $\varepsilon > 0$ sufficiently small, and $\eta > 0$ sufficiently large[9] such that first, we have

$$\eta > \sup_{i\in\mathcal{I},\, \theta_i\in\Theta_i, m, m'\in M} \left| u_i\big(g(m), \theta_i\big) - u_i\big(g(m'), \theta_i\big)\right|; \tag{7}$$

second, it does not disturb the "effectiveness" of agent j's challenge: due to (6), we can have

$$\begin{aligned} & x\big(m^2_i, m^1_j\big) \neq f\big(m^2_i\big) \\ \Rightarrow \quad & u_j\big(C^{\varepsilon}_{i,j}(m_i, m_j), m^2_{i,j}\big) < u_j\big(f\big(m^2_i\big), m^2_{i,j}\big) \quad \text{and} \\ & u_j\big(C^{\varepsilon}_{i,j}(m_i, m_j), m^1_j\big) > u_j\big(f\big(m^2_i\big), m^1_j\big). \end{aligned} \tag{8}$$

It means that whenever agent j challenges agent i, the lottery $C^{\varepsilon}_{i,j}(m_i, m_j)$ is strictly worse than $f(m^2_i)$ for agent j when agent i tells the truth about agent j's preference in m^2_i; moreover, the lottery $C^{\varepsilon}_{i,j}(m_i, m_j)$ is strictly better than $f(m^2_i)$ for agent j when agent j tells the truth in m^1_j, which implies that agent i tells a lie about agent j's preference.

4.1.5 *Transfer rule* We now define the transfer rule. For every message profile $m \in M$ and every agent $i \in \mathcal{I}$, we specify the transfer received by agent i as follows:

$$\tau_i(m) = \sum_{j\neq i} \big[\tau^1_{i,j}(m_i, m_j) + \tau^2_{i,j}(m_i, m_j)\big],$$

where for each agent $j \neq i$, we define

$$\tau^1_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m^2_{i,j} = m^2_{j,j}; \\ -\eta, & \text{if } m^2_{i,j} \neq m^2_{j,j} \text{ and } m^2_{i,j} \neq m^1_j; \\ \eta, & \text{if } m^2_{i,j} \neq m^2_{j,j} \text{ and } m^2_{i,j} = m^1_j, \end{cases} \tag{9}$$

$$\tau^2_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m^2_{i,i} = m^2_{j,i}; \\ -\eta, & \text{if } m^2_{i,i} \neq m^2_{j,i}. \end{cases} \tag{10}$$

Recall that $\eta > 0$ is chosen to be larger than the maximal utility difference from the outcome function $g(\cdot)$; see (7).

In words, for each pair of agents (i, j), if their second reports on agent j's type coincide ($m^2_{ij} = m^2_{jj}$), then no transfer will be made; if their second reports on agent j's type differ ($m^2_{ij} \neq m^2_{jj}$), then we consider the following two subcases: (i) if agent i's second report about agent j's type matches agent j's first report ($m^2_{i,j} = m^1_j$), then agent j pays η

[9] Instead of using η' defined in (4), we choose η because the mechanism may produce a strictly larger finite set of alternatives than those contained in $\tilde{X}$. For instance, allocations from the dictator lotteries may occur from the mechanism but are not contained in $\tilde{X}$.

to agent i; (ii) if agent i's second report about agent j's type does not match agent j's first report ($m^2_{i,j} \neq m^1_j$), then both agents pay η to the designer. Note that the first report m^1_i does not affect the transfer to agent i.

4.2 *Proof of Theorem 1*

As we argue in Section 3.3, Maskin monotonicity is a necessary condition for Nash implementation. We therefore focus on the "if" part of the proof. Fix an arbitrary true state θ throughout the proof. Recall that θ_i stands for agent i's type at state θ and $(\theta_i)_{i\in\mathcal{I}}$ denotes the true type profile.

We argue that the truth-telling message profile m (i.e., $m_i = (\theta_i, \theta)$ for each agent i) constitutes a pure-strategy Nash equilibrium. Since m is truthful, for all agents i and j, we have $e_{i,j}(m_i, m_j) = 0$ and $\tau_i(m) = 0$ (consistency and no challenge). Consider a possible deviation $\tilde{m}_i$ of agent i from m. First, if $\tilde{m}^2_{i,j} = \theta'_j \neq \theta_j$ for some $j \in \mathcal{I}$, then the message profile $(\tilde{m}_i, m_{-i})$ induces the penalty of η from rule $\tau^1_{i,j}(\cdot)$ if $j \neq i$, and rule $\tau^2_{i,j}(\cdot)$ if $j = i$. As a result, $\tilde{m}_i$ is strictly worse against m_{-i} than m_{-i}.

Second, if $\tilde{m}^1_i \neq \theta_i$ and $\tilde{m}^2_i = \theta$, $(\tilde{m}_i, m_{-i})$ leads either to $x(\theta, \tilde{m}^1_i) = f(\theta)$ and thereby the same payoff, or to $x(\theta, \tilde{m}^1_i) \neq f(\theta)$. In the latter case, the message profile $(\tilde{m}_i, m_{-i})$ results in the outcome $C^{\varepsilon}_{i,j}(\tilde{m}_i, m_j)$, which by (8), is strictly worse than $f(\theta)$ induced by m. Furthermore, deviating from m_i to $\tilde{m}_i$ does not affect the transfer of agent i. Therefore, the truth-telling message profile m constitutes a pure-strategy Nash equilibrium.

We next show that for every Nash equilibrium σ of the game $\Gamma(\mathcal{M}, \theta)$ and every message profile m reported with positive probability under σ, we must achieve the socially desirable outcome, that is, $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every agent i. The proof is divided into three steps.

Step 1 (Contagion of truth). If agent j announces his type truthfully in his first report with probability one, then everyone must also report agent j's type truthfully in their second report.

Step 2 (Consistency). Every agent reports the same state $\tilde{\theta}$ in the second report.

Step 3 (No challenge). No agent challenges the common reported state $\tilde{\theta}$, that is, $x(\tilde{\theta}, m^1_j) = f(\tilde{\theta})$ for every agent $j \in \mathcal{I}$.

Consistency implies that $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$, whereas no challenge together with Maskin monotonicity of the SCF f implies that $g(m) = f(\tilde{\theta}) = f(\theta)$. It completes the proof of Theorem 1. We now proceed to establish these three steps. In the rest of the proof, we fix σ as an arbitrary mixed-strategy Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$.

As a consequence of Lemmas 1 and 2, the mechanism has the following crucial property, which we will make use of in establishing the implementation.

Claim 1. *Let σ be a Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$. If $m^1_i \neq \theta_i$ for some $m_i \in \operatorname{supp}(\sigma_i)$, then for every agent $j \neq i$, we have $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$ with σ_j-probability one.*

The claim essentially follows from Lemmas 1 and 2. Indeed, the two lemmas together imply that agents must have a strict incentive to tell the truth in their first report, as long as switching from a lie to truth affects the allocation with positive probability. A detailed verification of the claim, however, is tedious, as it involves checking different cases of the value of functions $e_{i,j}(\cdot)$ and $e_{j,i}(\cdot)$. We relegate its formal proof to Appendix A.4.

Step 1: Contagion of truth

Claim 2. *The following two statements hold:*

(a) If agent j sends a truthful first report with σ_j-probability one, then every agent $i \neq j$ must report agent j's type truthfully in his second report with σ_i-probability one.

(b) If every agent $i \neq j$ reports the same type $\tilde{\theta}_j$ of agent j in his second report with σ_i-probability one, then agent j must also report the type $\tilde{\theta}_j$ in his second report with σ_j-probability one.

Proof. We first prove (a). Suppose instead that there exist some agent $i \in \mathcal{I}$ and some message m_i played with σ_i-positive probability such that m_i misreports agent j's type in the second report, that is, $m_{i,j}^2 \neq \theta_j$. Let $\tilde{m}_i$ be a message that differs from m_i only in reporting j's type truthfully $\tilde{m}_{i,j}^2 = \theta_j$. Such a change affects only $\tau_{i,j}^1(\cdot)$. For every m_{-i} played with σ_{-i}-positive probability, we consider the following two cases.

Case 1. $m_{j,j}^2 = \theta_j$.

Since agent j sends a truthful first report with σ_j-probability one, due to the construction of $\tau_{i,j}^1(\cdot)$, we have $\tau_{i,j}^1(m_i, m_{-i}) = -\eta$ whereas $\tau_{i,j}^1(\tilde{m}_i, m_{-i}) = 0$.

Case 2. $m_{j,j}^2 \neq \theta_j$.

Since agent j sends a truthful type in the first report with σ_j-probability one, according to the construction of $\tau_{i,j}^1(\cdot)$, we have $\tau_{i,j}^1(m_i, m_{-i})$ is either 0 or $-\eta$ whereas $\tau_{i,j}^1(\tilde{m}_i, m_{-i}) = \eta$.

Thus, in terms of transfers, the gain from reporting $\tilde{m}_i$ rather than m_i is at least η, which is larger than the maximal utility loss from the outcome function $g(\cdot)$ by (7). Hence, $\tilde{m}_i$ is a profitable deviation from m_i against σ_{-i}. As it contradicts the hypothesis that $m_i \in \text{supp}(\sigma_i)$, we have established (a).

We now prove (b). Suppose, on the contrary, that there exists some message m_j played with σ_j-positive probability such that $m_{j,j}^2 \neq \tilde{\theta}_j$. Let $\tilde{m}_j$ be a message that is identical to m_j except that $\tilde{m}_{j,j}^2 = \tilde{\theta}_j$. Such a change affects only $\tau_{j,i}^2(\cdot)$. According to the construction of $\tau_{j,i}^2(\cdot)$ and since every agent $i \neq j$ reports $\tilde{\theta}_j$ in the second report with σ_i-probability one, agent j saves the penalty of $(I-1)\eta$ from reporting $\tilde{m}_j$ instead of m_j. Again, since η is greater than the maximal utility difference by (7), we conclude that $\tilde{m}_j$ is a profitable deviation from m_j against σ_{-i}. It contradicts the hypothesis that $m_j \in \text{supp}(\sigma_j)$. Hence, we prove (b). □

Step 2: Consistency Claim 3 shows that in equilibrium, all agents must announce the same state $\tilde{\theta}$ with probability one.

Claim 3. *There exists a state $\tilde{\theta} \in \Theta$ such that every agent announces $\tilde{\theta}$ in their second report with probability one.*

Proof. We consider the following two cases.

Case 1. *Everyone tells the truth in the first report with probability one, that is, $m_i^1 = \theta_i$ with σ_i-probability one for every agent $i \in \mathcal{I}$.*

It follows directly from Claim 2 that $m_i^2 = \theta$ with σ_i-probability one for every agent $i \in \mathcal{I}$.

Case 2. *There exists agent i who tells a lie in the first report with σ_i-positive probability.*

That is, there exists $m_i \in \text{supp}(\sigma_i)$ such that $m_i^1 \neq \theta_i$. By Claim 1, (m_i, m_{-i}) is consistent with σ_{-i}-probability one. In particular, there exists $\tilde{\theta} \in \Theta$ such that every agent $j \neq i$ must report

$$m_j^2 = m_i^2 = \tilde{\theta} \quad \text{with } \sigma_j\text{-probability one.} \tag{11}$$

Hence, by Claim 2(b), for every $\tilde{m}_i \in \text{supp}(\sigma_i)$, we have

$$\tilde{m}_{i,i}^2 = m_{i,i}^2 = \tilde{\theta}_i. \tag{12}$$

We now prove that for every $\tilde{m}_i \in \text{supp}(\sigma_i)$, we have $\tilde{m}_i^2 = m_i^2 = \tilde{\theta}$, which would complete the proof. We prove it by contradiction, that is, suppose there exists $\tilde{m}_i \in \text{supp}(\sigma_i)$ such that

$$\tilde{m}_i^2 \neq m_i^2. \tag{13}$$

Furthermore, (11) and (13) imply that for every agent $j \neq i$, $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$ with σ_j-probability one. Hence, by Claim 1, every agent $j \neq i$ must tell the truth in the first report, that is, $m_j^1 = \theta_j$ with σ_j-probability one. As a result, Claim 2(a) implies for every agent $j \neq i$,

$$\tilde{m}_{i,j}^2 = m_{i,j}^2 = \theta_j \quad \text{with } \sigma_i\text{-probability one.} \tag{14}$$

Finally, (12) and (14) imply $\tilde{m}_i^2 = m_i^2$, contradicting (13). □

Step 3: No challenge By Claim 3, there exists a common state $\tilde{\theta} \in \Theta$ with σ_i-probability one for every agent $i \in \mathcal{I}$. We now show in Claim 4 that no one challenges the common state $\tilde{\theta}$.

Claim 4. *No agent challenges with positive probability the common state $\tilde{\theta}$ announced in the second report.*

Proof. Suppose by way of contradiction that $x(\tilde{\theta}, m_i^1) \neq f(\tilde{\theta})$ for some message $m_i \in \text{supp}(\sigma_i)$. By Claim 3, we have $x(m_j^2, m_i^1) \neq f(m_j^2)$ for every message $m_j \in \text{supp}(\sigma_j)$ and every agent $j \neq i$. It implies that $e_{j,i}(m_j, m_i) = \varepsilon$ with σ_j-probability one for every $j \neq i$ and $m_j \in \text{supp}(\sigma_j)$. By Claim 1, we have $m_j^1 = \theta_j$ with σ_j-probability one and $m_i^1 = \theta_i$. Thus, we obtain $x(\tilde{\theta}, \theta_i) \neq f(\tilde{\theta})$. By the construction of the best challenge scheme, we also have $x(\tilde{\theta}, \theta_i) \in \mathcal{L}_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$. Then, by (8), every message $\bar{m}_i$ with $x(\tilde{\theta}, \bar{m}_i^1) = f(\tilde{\theta})$ cannot be a best response against σ_{-i}. Indeed, since $x(\tilde{\theta}, \theta_i) \in \mathcal{SU}_i(f(\tilde{\theta}), \theta_i)$, it is a profitable deviation to replace $\bar{m}_i^1$ by θ_i. Hence, $x(\tilde{\theta}, \tilde{m}_i) \neq f(\tilde{\theta})$ and $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$ for every $\tilde{m}_i \in \text{supp}(\sigma_i)$. Once again, by Claim 1, we have $\tilde{m}_i^1 = \theta_i$ with σ_i-probability one. Therefore, every agent's first report is truthful with probability one. By Claim 2, we conclude that $\tilde{\theta} = \theta$. Since $x(\tilde{\theta}, \theta_i) \neq f(\tilde{\theta})$, it follows that $x(\tilde{\theta}, \theta_i)$ belongs to the empty intersection $\mathcal{L}_i(f(\theta), \theta_i) \cap \mathcal{SU}_i(f(\theta), \theta_i)$, which is impossible. □

4.3 *Implementation in a direct mechanism*

In this section, we present two special cases in which our implementing mechanism can be made into a direct mechanism. Both cases require three or more agents. A direct (revelation) mechanism is a mechanism $((M_i), g, (\tau_i))_{i \in \mathcal{I}}$ in which (i) agents are asked to report the state (i.e., $M_i = \Theta$ for every agent i), and (ii) a unanimous report leads to the socially desirable outcome with no transfers (i.e., $g(\theta, \dots, \theta) = f(\theta)$, and $\tau_i(\theta) = 0$, for every $i \in \mathcal{I}$ and $\theta \in \Theta$). Our notion of direct mechanism is adopted in, for example, Dutta and Sen (1991) and Osborne and Rubinstein (1994, Definition 179.2) both of which ask each agent to report a state.

Although direct mechanisms invoke a simpler message space than the augmented mechanisms used in the full implementation literature, the literature on partial implementation has attempted to construct mechanisms that are simpler or easier to implement than direct mechanisms, allowing lotteries and transfers. See, for example, Dasgupta and Maskin (2000) and Perry and Reny (2002). While our result complements these papers, our main focus is to study full implementation in mixed-strategy Nash equilibrium without making use of integer or modulo games.

The first case shows that every Maskin-monotonic SCF is (fully) implementable in pure-strategy Nash equilibria in a direct mechanism. Pure-strategy Nash implementation means that we only require that each pure-strategy Nash equilibrium achieve desirable outcomes, that is, condition (ii) of Definition 1 holds only for pure-strategy Nash equilibria. Indeed, one might expect that by penalizing disagreement with transfers, the designer can easily obtain a unanimous state announcement without using integer/modulo games. Once there is a unanimous state announcement in equilibrium, Maskin monotonicity will ensure implementation, as it does in Maskin (1999). The following proposition formalizes the idea; see Appendix A.2 for a proof.

Proposition 1. *Suppose that there are at least three agents and the SCF f satisfies Maskin monotonicity. Then f is implementable in pure-strategy Nash equilibria by a direct mechanism.*

The idea of "penalizing disagreement" becomes problematic once we consider mixed-strategy equilibria. Indeed, the direct mechanism, which we construct in proving Proposition 1 is reminiscent of modulo games, which as is well known, admit unwanted mixed-strategy equilibria. Thus, it should come at no surprise that the direct mechanism also admits unwanted mixed-strategy equilibria.

The second case establishes (full) mixed-strategy Nash implementation in direct mechanisms by considering a state space of a "product form," that is, $\Theta = \times_{i=1}^{I} \Theta_i$. We state the following result and relegate its proof to Appendix A.3.

Proposition 2. *Suppose that there are at least three agents, $\Theta = \times_{i=1}^{I} \Theta_i$, and the SCF f satisfies Maskin monotonicity. Then f is implementable in mixed-strategy Nash equilibria by a direct mechanism.*

Proposition 2 represents an extreme case in which mixed-strategy Nash implementation can be achieved in a direct mechanism. Product state space naturally arises in a Bayesian setup with a full-support common prior. While such a full-support prior is precluded by the complete-information assumption, it is consistent with "almost complete information," which we will introduce in Section 5.1.

4.4 *Implementation without off-the-equilibrium transfers*

The following example illustrates the fact that without any domain restriction such as quasilinear preferences with transfers, some Maskin-monotonic SCF cannot be implemented by mixed-strategy Nash equilibria in finite mechanisms.

Example 1 (Example 4 of Jackson (1992)). Consider the environment with two agents 1 and 2. Suppose that there are four alternatives a, b, c, and d and two states θ and θ'. Suppose that agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \sim_1 d$, and agent 2 has the preference $a \succ_2^{\theta} b \succ_2^{\theta} d \succ_2^{\theta} c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \sim_2^{\theta'} d$ at state θ'. Consider the SCF f such that $f(\theta) = a$ and $f(\theta') = c$. $\diamondsuit$

With no restrictions on agents' preferences, Jackson (1992) shows that for every finite mechanism, which implements f in pure-strategy Nash equilibria, there must also exist a "bad" mixed-strategy Nash equilibrium such that at state θ' the equilibrium outcome differs from c with positive probability.[10] Since f satisfies Maskin monotonicity, the example shows that without imposing any domain restrictions on the environment,

[10]We briefly recap the argument here. Let $\mathcal{M}$ be a finite mechanism, which implements the SCF f in pure-strategy Nash equilibria. Consider a mechanism, which restricts the message space of $\mathcal{M}$ such that, against any message of agent i, the opponent agent j can choose a message that induces either outcome a or b. The restricted set of messages is nonempty since the equilibrium message profile at state θ leads to outcome a. It follows that at state θ', the game induced by the restricted mechanism must have a mixed-strategy Nash equilibrium. Moreover, the equilibrium outcome must be a or b with positive probability; otherwise, agent 2 can deviate to induce outcome a or b with positive probability. Since c and d are ranked lowest by both agents at state θ', the mixed-strategy equilibrium must remain an equilibrium at state θ' in the game induced by $\mathcal{M}$; moreover, the equilibrium fails to achieve $f(\theta') = c$.

it is impossible to implement any Maskin-monotonic SCF in mixed-strategy equilibria by a finite mechanism. However, regardless of the cardinal representation of the preferences in Jackson's example, the SCF f can actually be implemented in mixed-strategy equilibria with arbitrarily small transfers off the equilibrium; for more discussion, see Section 5.3 and in particular, footnote 16.

5. Extensions

We now establish several extensions of our main result (Theorem 1). In Section 5.1, we show that the implementation result is robust to information perturbations. That is, we establish that our implementation result remains valid in any incomplete-information environment that is close to our complete-information benchmark. In Section 5.2, we extend our result to the case of social choice correspondences (henceforth, SCCs). Section 5.3 clarifies how the designer can modify the implementing mechanism to make the size of transfers arbitrarily small. For the sake of clarity, we will not discuss any combination of multiple extensions. For instance, we will study the case of SCCs only in Section 5.2 but focus entirely on SCFs in the rest of the paper.

The extensions involve more technical details. Thus, we assume, in this section, that the set A (of pure alternatives) is finite and relegate all the proofs to the Appendix.

5.1 *Robustness to information perturbations*

Chung and Ely (2003) and Aghion et al. (2012) consider a designer who not only wants all equilibria of her mechanism to yield a desirable outcome under complete information, but is also concerned about the possibility that agents may entertain small doubts about the true state. They argue that such a designer should insist on implementing the SCF in the closure of a solution concept as the amount of incomplete information about the state vanishes. Chung and Ely (2003) adopt undominated Nash equilibrium and Aghion et al. (2012) adopt subgame-perfect equilibrium as a solution concept in studying the robustness issue.

To allow for information perturbations, suppose that the agents do not observe the state directly but are informed of the state via signals. The set of agent i's signals is denoted as S_i, which is identified with Θ, that is, $S_i \equiv \Theta$.[11] A signal profile is an element $s = (s_1, \dots, s_I) \in S \equiv \bigtimes_{i \in \mathcal{I}} S_i$. When the realized signal profile is s, agent i observes only his own signal s_i. Let s_i^θ denote the signal which corresponds to state θ, and we write $s^\theta = (s_i^\theta)_{i \in I}$. State and signals are drawn from some prior distribution over $\Theta \times S$. In particular, complete information can be modeled as a prior μ such that $\mu(\theta, s) = 0$ whenever $s \neq s^\theta$. Such a μ will be called a *complete-information prior*. We assume that for each agent i, the marginal distribution on i's signals places a strictly positive weight on each of i's signals, that is, $\text{marg}_{S_i} \mu(s_i) > 0$ for every $s_i \in S_i$, so that the posterior belief given every signal is well-defined. For every prior ν, we also write $\nu(\cdot | s_i)$ for the conditional distribution of ν on signal s_i.

[11] We adopt the formulation from Chung and Ely (2003) and Aghion et al. (2012). Our result holds for any alternative formulation under which the (Bayesian) Nash equilibrium correspondence has a closed graph.

The distance between two priors is measured by the uniform metric. That is, for every two priors μ and ν, we have $d(\mu, \nu) \equiv \max_{\theta,s} |\mu(\theta, s) - \nu(\theta, s)|$. Write $\nu^\varepsilon \to \mu$ if $d(\nu^\varepsilon, \mu) \to 0$ as $\varepsilon \to 0$. A prior ν together with a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ induces an incomplete-information game, which we denote by $\Gamma(\mathcal{M}, \nu)$. A (mixed-)strategy of agent i is now a mapping $\sigma_i : S_i \to \Delta(M_i)$.

The designer may resort to a solution concept $\mathcal{E}$ for the game $\Gamma(\mathcal{M}, \nu)$ (such as Bayesian Nash equilibrium), which induces a set of mappings from $\Theta \times S$ to X, which we call *acts*, following Chung and Ely (2003). For instance, each Bayesian Nash equilibrium σ induces the act α_σ with $\alpha_\sigma(\theta, s) \equiv \sigma(s) \circ (g, (\tau_i)_{i\in\mathcal{I}})^{-1}$, where we abuse the notation to identify the finite-support distribution $\sigma(s) \circ (g, (\tau_i)_{i\in\mathcal{I}})^{-1}$ on X with an allocation in X. We denote the set of acts induced by the solution concept $\mathcal{E}$ as $\mathcal{E}(\mathcal{M}, \nu)$. We endow X with a topology with respect to which the utility function u_i is continuous on X.[12] We now define $\overline{\mathcal{E}}$-implementation.

Definition 3. An SCF f is $\overline{\mathcal{E}}$-implementable under the complete-information prior μ if there exists a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$ such that for every $(\theta, s) \in \text{supp}(\mu)$ and every sequence of priors $\{\nu^n\}$ converging to μ, the following two requirements hold: (i) there is a sequence of acts $\{\alpha_n\}$ with $\alpha_n \in \mathcal{E}(\mathcal{M}, \nu_n)$ such that $\alpha_n(\theta, s) \to f(\theta)$ and (ii) for every sequence of acts $\{\alpha_n\}$ with $\alpha_n \in \mathcal{E}(\mathcal{M}, \nu_n)$, we have $\alpha_n(\theta, s) \to f(\theta)$.

Chung and Ely (2003) and Aghion et al. (2012) show that Maskin monotonicity is a necessary condition for $\overline{\text{UNE}}$-implementation and $\overline{\text{SPE}}$-implementation, respectively.[13] The result of Chung and Ely (2003) implies that implementation of a non-Maskin-monotonic SCF in undominated Nash equilibria such as the result in Abreu and Matsushima (1994) is necessarily vulnerable to information perturbations. Moreover, both Chung and Ely (2003, Theorem 2) and Aghion et al. (2012) establish the sufficiency result by using an infinite mechanism with an integer game and restricting attention to pure-strategy equilibria. It raises the question as to whether their robustness test may be too demanding when it is applied to finite mechanisms where mixed-strategy equilibria have to be taken seriously. In particular the implementing mechanism of Jackson, Palfrey, and Srivastava (1994), that of Abreu and Matsushima (1994), or the simple mechanism in Section 5 of Moore and Repullo (1988) are considered examples of such finite mechanisms.

The canonical mechanism, which we propose in the proof of Theorem 1 is indeed finite, and we show that our finite mechanism implements every Maskin-monotonic SCF in mixed-strategy Nash equilibria. Since the solution concept of Bayesian Nash equilibrium, viewed as a correspondence on priors, has a closed graph, that finite mechanism also achieves $\overline{\text{NE}}$-implementation. We now obtain the following result as a corollary of Theorem 1 in our setup with lotteries and transfers.

[12]For instance, it is the case if A is a (Hausdorff) topological space, $v_i(a, \theta)$ is bounded and continuous in a, and $\Delta(A)$ is endowed with the weak*-topology. Then $X \equiv \Delta(A) \times \mathbb{R}^I$, endowed with the product topology, is also a Hausdorff topological space.

[13]Aghion et al. (2012) adopt sequential equilibrium as the solution concept for the incomplete-information game $\Gamma(\mathcal{M}, \nu)$.

PROPOSITION 3. *Let $\mathcal{E}$ be a solution concept such that $\emptyset \neq \mathcal{E}(\mathcal{M}, \mu) \subseteq \mathrm{NE}(\mathcal{M}, \mu)$ for each finite mechanism $\mathcal{M}$ and a complete-information prior μ. Then every Maskin-monotonic SCF f is $\overline{\mathcal{E}}$-implementable.*

The condition $\emptyset \neq \mathcal{E}(\mathcal{M}, \mu) \subseteq \mathrm{NE}(\mathcal{M}, \mu)$ is satisfied for virtually every refinement of Nash equilibrium, because we allow for mixed-strategy equilibria and $\Gamma(\mathcal{M}, \mu)$ is a finite game.

5.2 *Social choice correspondences*

A large portion of the implementation literature strives to deal with social choice correspondences (hereafter, SCCs), that is, multivalued social choice rules. In this section, we extend our Nash implementation result to cover the case of SCCs. We suppose that the designer's objective is specified by an SCC $F : \Theta \rightrightarrows X$; and for simplicity, we assume that $F(\theta)$ is a finite set for each state $\theta \in \Theta$. It includes the special case where the codomain of F is A. Following Maskin (1999), we first define the notion of Nash implementation for an SCC.

DEFINITION 4. An SCC F is *implementable in mixed-strategy Nash equilibria by a finite mechanism* if there exists a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i\in\mathcal{I}}, g)$ such that for every state $\theta \in \Theta$, the following two conditions are satisfied: (i) for every $x \in F(\theta)$, there exists a pure-strategy Nash equilibrium m in the game $\Gamma(\mathcal{M}, \theta)$ with $g(m) = x$ and $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$ and (ii) for every $m \in \mathrm{supp}(\mathrm{NE}(\Gamma(\mathcal{M}, \theta)))$, we have $\mathrm{supp}(g(m)) \subseteq F(\theta)$ and $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$.

Second, we state the definition of Maskin monotonicity for an SCC.

DEFINITION 5. An SCC F satisfies *Maskin monotonicity* if for each pair of states $\tilde{\theta}$ and θ and $z \in F(\tilde{\theta})\backslash F(\theta)$, some agent $i \in \mathcal{I}$ and some allocation $z' \in X$ exist such that

$$\tilde{u}_i\big(z', \tilde{\theta}\big) \le \tilde{u}_i(z, \tilde{\theta}) \quad \text{and} \quad \tilde{u}_i\big(z', \theta\big) > \tilde{u}_i(z, \theta).$$

We now state our Nash implementation result for SCCs and relegate the proof to Appendix A.5.[14]

THEOREM 2. *Suppose there are at least three agents. An SCC F is implementable in mixed-strategy Nash equilibria by a finite mechanism if and only if it satisfies Maskin monotonicity.*

Compared with Theorem 1 for SCFs, Theorem 2 needs to overcome additional difficulties. In the case of SCFs, when the agents' second reports are consistent at a common state $\tilde{\theta}$, they will be associated with a single outcome $f(\tilde{\theta})$. Hence, if agent i's second

[14] When there are only two agents, we can still show that every Maskin-monotonic SCC F is *weakly* implementable in Nash equilibria, that is, there exists a mechanism, which has a pure-strategy Nash equilibrium and satisfies requirement (ii) in Definition 4.

report is challenged, then *every* second report, which is played with positive probability by any agent, must also be challenged in equilibrium. Together with Claim 1, it implies that every agent must tell the truth in their first and second report, which leads to a contradiction in the proof of Claim 4.

In the case of SCCs, each allocation $x \in F(\theta)$ has to be implemented as the outcome of some pure-strategy equilibrium. Hence, each agent must also report an allocation to be implemented. It also follows that a challenge scheme for an SCC must be defined for a type θ_i to challenge a pair $(\tilde{\theta}, x)$ with $x \in F(\tilde{\theta})$. As a result, even when the agents' second reports are consistent at state $\tilde{\theta}$ (which still holds by Claim 3), they might still be randomizing between two allocations x and x' in $F(\tilde{\theta})$ such that $(\tilde{\theta}, x)$ is challenged and yet $(\tilde{\theta}, x')$ is not. Hence, we cannot follow a similar argument as in Claim 1 to derive a contradiction. Instead, we build on the implementing mechanism in Section 4.1 and show that agent i will not report $(\tilde{\theta}, x)$, which can be challenged either by (i) agent $j \neq i$ or by (ii) agent i himself. We deal with Case (i) by imposing a large penalty on agent i conditional on agent j's challenging $(\tilde{\theta}, x)$, whereas we deal with Case (ii) by allowing agent i to challenge himself without having to pay the penalty.

Remark. Mezzetti and Renou (2012b) also consider deterministic SCCs in a separable environment studied in Jackson, Palfrey, and Srivastava (1994). Mezzetti and Renou (2012b) identify a condition (which they call top-D inclusiveness) under which an SCC is implementable in mixed-strategy Nash equilibria in finite mechanisms if and only if it satisfies *set-monotonicity* (proposed by Mezzetti and Renou (2012a)). There are several differences between our Theorem 2 and their result. First, Mezzetti and Renou (2012b) require only the existence of mixed-strategy equilibria but we follow Maskin (1999) in requiring the existence of pure-strategy equilibria in part (i) of Definition 4. Second, Mezzetti and Renou (2012b) consider an ordinal setting, while we consider a cardinal setting. These two features of Mezzetti and Renou (2012b) are the reason why they use set-monotonicity as a necessary condition for characterizing their ordinal Nash implementation.[15] Third, our quasilinear environments with transfers are more restrictive than the separable environments considered by Mezzetti and Renou (2012b). Finally, Mezzetti and Renou (2012b) need “top D-inclusiveness” as an additional condition, which requires that there exist at least one agent for whom the SCC contains the agent's best outcome within the range of the SCC for every state of the world, whereas we impose no conditions beyond Maskin monotonicity for the SCC.

5.3 *Small transfers*

One potential drawback of the mechanism we propose for Theorem 1 is that the size of transfers may be large. To tackle the problem, we use the technique introduced by

[15] In Chen et al. (2022), we study the concept of ordinal Nash implementation proposed by Mezzetti and Renou (2012a). The notion requires that the implementing mechanism achieve mixed-strategy Nash implementation for every cardinal representation of preferences over lotteries. We show that ordinal almost monotonicity, as defined in Sanver (2006), is a necessary and sufficient condition for ordinal Nash implementation.

Abreu and Matsushima (1994) to show that if the SCF satisfies Maskin monotonicity in the restricted domain without any transfer, then it is Nash-implementable with arbitrarily small transfers.

We first propose a notion of Nash implementation with bounded transfers off the equilibrium and still no transfers on the equilibrium.

Definition 6. An SCF $f : \Theta \to \Delta(A)$ is implementable in mixed-strategy Nash equilibria by a finite mechanism *with transfers bounded by* $\bar{\tau}$ if there exists a mechanism $\mathcal{M} = ((M_i, \tau_i)_{i \in \mathcal{I}}, g)$ such that for every state $\theta \in \Theta$, (i) there exists a pure-strategy Nash equilibrium in the game $\Gamma(\mathcal{M}, \theta)$ and (ii) for each m in $\text{supp}(\text{NE}(\Gamma(\mathcal{M}, \theta)))$, we have $g(m) = f(\theta)$ and $\tau_i(m) = 0$ for every agent $i \in \mathcal{I}$ and (iii) $|\tau_i(m)| \leq \bar{\tau}$ for every $m \in M$ and every agent $i \in \mathcal{I}$.

Next, we propose a notion of Nash implementation in which there are no transfers on the equilibrium and only arbitrarily small transfers off the equilibrium.

Definition 7. An SCF f is implementable in mixed-strategy Nash equilibria by a finite mechanism *with arbitrarily small transfers* if, for every $\bar{\tau} > 0$, the SCF f is implementable in Nash equilibria by a finite mechanism with transfers bounded by $\bar{\tau}$.

We say that an SCF f satisfies Maskin monotonicity in the restricted domain $\Delta(A)$ if $f(\tilde{\theta}) \neq f(\theta)$ implies that there are an agent i and some lottery $x(\tilde{\theta}, \theta_i)$ in $\Delta(A)$ such that $x(\tilde{\theta}, \theta_i)$ belongs to $L_i(f(\tilde{\theta}), \tilde{\theta}_i) \cap SU_i(f(\tilde{\theta}), \theta_i)$. Here, for $(\ell, \theta_i) \in \Delta(A) \times \Theta_i$, we use $L_i(\ell, \theta_i)$ to denote the lower-contour set at allocation ℓ in $\Delta(A)$ for type θ_i, that is,

$$L_i(\ell, \theta_i) = \{\ell' \in \Delta(A) : v_i(\ell, \theta) \geq v_i(\ell', \theta)\}.$$

In a similar fashion, SU_i is defined. Clearly, Maskin monotonicity in the restricted domain $\Delta(A)$ is stronger than Maskin monotonicity in the domain X, as the former requires that the test allocation be a lottery over alternatives without any transfer. In Appendix A.6, we assume there are at least three agents, and prove the following result.[16]

Theorem 3. *Suppose there are at least three agents. An SCF $f_A : \Theta \to \Delta(A)$ is implementable in mixed-strategy Nash equilibria by a finite mechanism with arbitrarily small transfers if f_A satisfies Maskin monotonicity in the restricted domain.*

Appendix

In this Appendix, we provide the proofs omitted from the main body of the paper.

[16] In the case with only two agents, Theorem 3 still holds if there exists an alternative $w \in A$, which is the worst alternative for any agent at any state. In that case, we can simply modify the "voting rule" ϕ in the proof of Theorem 3 to be $\phi(m^h) = f(\tilde{\theta})$ if both agents announce a common type profile, which identifies a state $\tilde{\theta}$ in m^h, and $\phi(m^h) = w$ otherwise. In particular, $w = c$ in Example 4 of Jackson (1992), and thus the SCF can be implemented with arbitrarily small transfers. Moreover, the conclusion holds regardless of the utility representation of the agents' preferences. However, note that we assume that agents have quasilinear utilities while Jackson's example does not make such an assumption.

A.1 *Proof of Lemma 1*

First, we elaborate the proof of Lemma 1 here.

Proof. Consider a challenge scheme $\bar{x}(\cdot,\cdot)$. First, we show that we can modify $\bar{x}(\cdot,\cdot)$ into a new challenge scheme $x(\cdot,\cdot)$ such that

$$x(\tilde{\theta},\theta_i)\neq f(\tilde{\theta}) \quad \text{and} \quad x(\tilde{\theta},\theta_i')\neq f(\tilde{\theta}) \quad \Rightarrow \quad u_i\big(x(\tilde{\theta},\theta_i),\theta_i\big)\geq u_i\big(x(\tilde{\theta},\theta_i'),\theta_i\big). \tag{15}$$

To construct $x(\cdot,\cdot)$, for each player i, we distinguish two cases: (a) if $\bar{x}(\tilde{\theta},\theta_i)=f(\tilde{\theta})$ for all $\theta_i\in\Theta_i$, then set $x(\tilde{\theta},\theta_i)=\bar{x}(\tilde{\theta},\theta_i)=f(\tilde{\theta})$; (b) if $\bar{x}(\tilde{\theta},\theta_i)\neq f(\tilde{\theta})$ for some $\theta_i\in\Theta_i$, then define $x(\tilde{\theta},\theta_i)$ as the most preferred allocation of type θ_i in the finite set

$$X(\tilde{\theta})=\big\{\bar{x}\big(\tilde{\theta},\theta_i'\big):\theta_i'\in\Theta_i \text{ and } \bar{x}\big(\tilde{\theta},\theta_i'\big)\neq f(\tilde{\theta})\big\}.$$

Since $\bar{x}(\tilde{\theta},\theta_i')\in\mathcal{L}_i(f((\tilde{\theta}),\tilde{\theta}_i)$, we have $u_i(x(\tilde{\theta},\theta_i),\tilde{\theta}_i)\leq u_i(f(\tilde{\theta}),\tilde{\theta}_i)$; moreover, since $x(\tilde{\theta},\theta_i)$ as the most preferred allocation of type θ_i in $X(\tilde{\theta})$ and $\bar{x}(\tilde{\theta},\theta_i)\in\mathcal{SU}_i(f(\tilde{\theta}),\theta_i)$, it follows that $u_i(x(\tilde{\theta},\theta_i),\theta_i)>u_i(f(\tilde{\theta}),\theta_i)$. In other words, $x(\cdot,\cdot)$ remains a challenge scheme. Moreover, $x(\cdot,\cdot)$ satisfies (15) by construction.

Next, for each state $\tilde{\theta}$ and type θ_i, we show that $x(\cdot,\cdot)$ satisfies (3). We proceed by considering the following two cases. First, suppose that $x(\tilde{\theta},\theta_i)\neq f(\tilde{\theta})$. Then, by (15), it suffices to consider type θ_i' with $x(\tilde{\theta},\theta_i')=f(\tilde{\theta})$. Since $x(\tilde{\theta},\theta_i')=f(\tilde{\theta})$ and $x(\tilde{\theta},\theta_i)\neq f(\tilde{\theta})$, then it follows from $x(\tilde{\theta},\theta_i)\in\mathcal{SU}_i(f(\tilde{\theta}),\theta_i)$ that $u_i(x(\tilde{\theta},\theta_i),\theta_i)>u_i(x(\tilde{\theta},\theta_i'),\theta_i)$. Hence, (3) holds. Second, suppose that $x(\tilde{\theta},\theta_i)=f(\tilde{\theta})$. Then it suffices to consider type θ_i' with $x(\tilde{\theta},\theta_i')\neq f(\tilde{\theta})$. Since $x(\tilde{\theta},\theta_i)=f(\tilde{\theta})$, we have $\mathcal{L}_i(f(\tilde{\theta}),\tilde{\theta}_i)\cap\mathcal{SU}_i(f(\tilde{\theta}),\theta_i)=\emptyset$. Moreover, $x(\tilde{\theta},\theta_i')\neq f(\tilde{\theta})$ implies that $x(\tilde{\theta},\theta_i')\in\mathcal{L}_i(f(\tilde{\theta}),\tilde{\theta}_i)$. Hence, we must have $x(\tilde{\theta},\theta_i')\notin\mathcal{SU}_i(f(\tilde{\theta}),\theta_i)$. That is, $u_i(x(\tilde{\theta},\theta_i),\theta_i)\geq u_i(x(\tilde{\theta},\theta_i'),\theta_i)$, that is, (3) holds. □

A.2 *Proof of Proposition 1*

To facilitate the comparison with Maskin (1999), we assume that there are three or more agents and define the following direct mechanism, denoted by $\mathcal{M}^{\mathrm{D}}$, according to three rules.

Rule 1. If there exists state $\tilde{\theta}$ such that every agent announces $\tilde{\theta}$, then implement the outcome $f(\tilde{\theta})$.

Rule 2. If there exists state $\tilde{\theta}$ such that everyone except agent i announces $\tilde{\theta}$ and agent i announces $\tilde{\theta}'$, then implement a test allocation $x(\tilde{\theta},\tilde{\theta}_i')$ for agent i and the ordered pair of states $(\tilde{\theta},\tilde{\theta}')$; and if there is no such test allocation, implement $f(\tilde{\theta})$. Moreover, charge agent $i+1$ (mod I) a large penalty 2η, where the scale η dominates any difference in utility from allocation.

Rule 3. Otherwise, implement $f(m_1)$. Moreover, charge each agent i a penalty of η if i reports a state which is not reported by the unique majority (i.e., $\{m_i\}\neq\arg\max_{\tilde{\theta}}|\{j\in\mathcal{I}: m_j=\tilde{\theta}\}|$).[17]

[17] Note that Rule 3 penalizes every agent by η, if each of them reports a different state.

Now let the true state be θ.

It follows from Rule 2 that since θ is the true state, $x(\theta, \tilde{\theta}'_i) \neq f(\theta)$ implies that $x(\theta, \tilde{\theta}'_i) \in \mathcal{L}_i(f(\theta), \theta_i)$. Hence, everyone reporting the true state constitutes a pure-strategy Nash equilibrium.

Now fix an arbitrary pure-strategy Nash equilibrium m. First, we claim that m cannot trigger Rule 2. Suppose that Rule 2 is triggered, and let agent i be the odd man out. Then agent $i+1$ finds it strictly profitable to deviate to announce m_i. After such a deviation, since $I \geq 3$, either Rule 3 is triggered or it remains in Rule 2, but agent i is no longer the odd man out. Thus, agent $i+1$ saves at least η (from paying 2η to paying η or 0). Such a deviation may also change the allocation selected by the outcome function $g(\cdot)$, which induces utility change less than η. Hence, agent $i+1$ strictly prefers deviating to announce m_i, which contradicts the hypothesis that m is a Nash equilibrium.

Second, we claim that m cannot trigger Rule 3 either. Suppose that Rule 3 is triggered. Pick an arbitrary state reported by some (not necessarily unique) majority of agents, that is, $\hat{\theta} \in \arg\max_{\tilde{\theta}} |\{j \in I : m_j = \tilde{\theta}\}|$. Let $\mathcal{I}_{\hat{\theta}}$ be the set of agents who report $\hat{\theta}$. Clearly, $\mathcal{I}_{\hat{\theta}} \subsetneq \mathcal{I}$, because Rule 3 (rather than Rule 1) is triggered. Then we can find an agent $i^* \in \mathcal{I}_{\hat{\theta}}$ such that agent i^*+1 (mod I) is not in $I_{\hat{\theta}}$. Since agent i^*+1 does not belong to the unique majority, he must pay η under m. Then agent i^*+1 will strictly prefer deviating to announce $m_{i^*} = \hat{\theta}$. After such a deviation, either Rule 3 is triggered, and agent i^*+1 falls in the unique majority who reports $\hat{\theta}$; or Rule 2 is triggered, but agent i^* cannot be the odd man out. Thus, agent i^*+1 saves η (from paying η to paying 0) and η' is larger than the maximal utility change induced by different allocations in $g(\cdot)$. The existence of profitable deviation of agent i^*+1 contradicts the hypothesis that m is a Nash equilibrium.

Hence, we conclude that m must trigger Rule 1. It follows that $f(\tilde{\theta}) = f(\theta)$. Otherwise, by Maskin monotonicity, a whistle blower can deviate to trigger Rule 2.

A.3 *Proof of Proposition 2*

The proof is based on modifying the implementing mechanism and the proof of Theorem 1. We only provide a sketch here. Set $M_i = M_i^1 \times M_i^2$ where $M_i^1 = \Theta_i$ and $M_i^2 = \bigtimes_{j \neq i} \Theta_j$. Since $I \geq 3$, the type of each agent is reported by at least two agents in their second report. For each message profile $m = (m_i)_{i=1}^I$, denote by $\tilde{\Theta}(m)$ the set of state induced from the agents' second report, namely that $\tilde{\theta} \in \tilde{\Theta}(m)$ iff for every $i \in \mathcal{I}$, we have $\tilde{\theta}_i = m_{j,i}^2$ for some agent $j \neq i$. Then we modify the outcome function:

$$g(m) = \frac{1}{I\left|\tilde{\Theta}(m)\right|} \sum_{i \in \mathcal{I}} \sum_{\tilde{\theta} \in \tilde{\Theta}(m)} \left[e(m) \frac{1}{I} \sum_{j \in \mathcal{I}} y_j\left(m_j^1\right) \oplus \left(1 - e(m)\right) x\left(\tilde{\theta}, m_i^1\right) \right]$$

where $e(m) = 0$ if (i) $\tilde{\Theta}(m)$ contains a unique state (consistency) and (ii) $x(\tilde{\theta}, m_i^1) = f(\tilde{\theta})$ for every agent i and every $\tilde{\theta} \in \tilde{\Theta}(m)$ (no challenge); otherwise, $e(m) = \varepsilon$.[18] For the transfer rule, we define

$$\hat{\tau}^1_{i,j}(m_i, m_{-i}) = \begin{cases} 0 & \text{if } m^2_{i,j} = m^2_{k,j} \text{ for all } k \in \mathcal{I}\backslash\{i, j\}; \\ -\eta & \text{if } m^2_{i,j} \neq m^2_{k,j} \text{ for some } k \in \mathcal{I}\backslash\{i, j\} \text{ and } m^2_{i,j} \neq m^1_j; \\ \eta & \text{if } m^2_{i,j} \neq m^2_{k,j} \text{ for some } k \in \mathcal{I}\backslash\{i, j\} \text{ and } m^2_{i,j} = m^1_j. \end{cases}$$

Set $\tau_i(m) = \sum_{j \neq i} \hat{\tau}^1_{i,j}(m)$. As the agents no longer report their own type in the second report, we do not need to define $\tau^2_{i,j}(\cdot)$.

The proof of implementation follows the same steps as the proof of Theorem 1 and we only highlight the difference. First, for the contagion of truth argument, we can only establish Claim 2(a) because in the modified mechanism, the agents no longer report their own type in the second report so that we do not have rule $\tau^2_{i,j}(\cdot)$. For the consistency argument, it turns out that Claim 2(a) suffices. Specifically, consider an arbitrary message $m_i \in \text{supp}(\sigma_i)$ such that $m^1_i \neq \theta_i$. The same argument as in the proof of Claim 3 implies that (m_i, m_{-i}) is consistent for every $m_{-i} \in \text{supp}(\sigma_{-i})$. To show that $(\tilde{m}_i, m_{-i})$ is consistent for any other $\tilde{m}_i \in \text{supp}(\sigma_i)$, we make use of the assumption that we have three or more agents. In particular, since (m_i, m_{-i}) is consistent for every $m_{-i} \in \text{supp}(\sigma_{-i})$, if $(\tilde{m}_i, m_{-i})$ is inconsistent, it must be $\tilde{m}^2_{i,k} \neq m^2_{j,k}$ for some $j \neq i$, $k \neq i$, and $k \neq j$. By Claim 1, agent k must report his true type with probability one. Then it follows from Claim 2(a) that $\tilde{m}^2_{i,k} = m^2_{j,k}$ with probability one and we have reached a contradiction. The argument for no challenge remains the same.

A.4 *Proof of Claim 1*

Suppose that $m^1_i \neq \theta_i$ for some $m_i \in \text{supp}(\sigma_i)$. Consider a message $\tilde{m}_i$ which differs from m_i only in sending a truthful first report, that is, $\tilde{m}^1_i = \theta_i$ and $\tilde{m}^2_i = m^2_i$. We prove the claim by showing that $\tilde{m}_i$ is always a weakly better response than m_i against m_j, and is strictly better whenever the following condition does not hold: $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$. Recall that the first report of agent i has no effect on his own transfer.

We consider first the case that the designer uses agent j's report to check agent i's report. In that situation, the first report of agent i has no effect on the function $e_{i,j}(\cdot, m_j)$ for every m_j. Hence, we have $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j)$. Moreover, if $m^2_i \notin \Theta$, then $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j) = 1$; thus, by Lemma 2, $\tilde{m}_i$ is a strictly better response than m_i against m_j. Hence, we may assume $m^2_i \in \Theta$ and consider the following two cases:

Case 1.1. $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j) = \varepsilon$.

It follows from Lemmas 2 and 1 that

$$u_i\big(C^{\varepsilon}_{i,j}(\tilde{m}_i, m_j), \theta_i\big) - u_i\big(C^{\varepsilon}_{i,j}(m_i, m_j), \theta_i\big) > 0.$$

Hence, $\tilde{m}_i$ is a strictly better response than m_i against m_j.

[18]Here, we do not have the case with $e(m) = 1$ since $\Theta = \bigtimes_{i=1}^{I} \Theta_i$ implies that $\tilde{\Theta}(m) \subseteq \Theta$.

Case 1.2. $e_{i,j}(\tilde{m}_i, m_j) = e_{i,j}(m_i, m_j) = 0$.

Since $m_i^2 = \tilde{m}_i^2$, both (m_i, m_j) and $(\tilde{m}_i, m_j)$ lead to the same outcome $x(m_i^2, m_j^1) = x(\tilde{m}_i^2, m_j^1) = f(m_i^2)$.

Next, suppose that the designer uses agent i's report to check agent j's report. Again, if $m_j^2 \notin \Theta$, then $e_{j,i}(m_j, \tilde{m}_i) = e_{j,i}(m_j, m_i) = 1$; thus, by Lemma 2, $\tilde{m}_i$ is a strictly better response than m_i against m_j. Hence, we may assume $m_j^2 \in \Theta$ and consider the following four cases.

Case 2.1. $e_{j,i}(m_j, m_i) = \varepsilon$ *and* $e_{j,i}(m_j, \tilde{m}_i) = 0$.

It follows from (6) and Lemma 1 that

$$u_i\big(f\big(m_j^2\big), \theta_i\big) - u_i\big(C_{j,i}^{\varepsilon}(m_j, m_i), \theta_i\big) > 0,$$

where $f(m_j^2)$ is the outcome induced by $(m_j, \tilde{m}_i)$.

Case 2.2. $e_{j,i}(m_j, m_i) = 0$ *and* $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$.

Since $e_{j,i}(m_j, m_i) = 0$, we have $m_i^2 = \tilde{m}_i^2 = m_j^2$. Hence, $e_{j,i}(m_j, \tilde{m}_i) = \varepsilon$ implies that $x(m_j^2, \tilde{m}_i^1) = x(m_j^2, \theta_i) \neq f(m_j^2)$. Thus, it follows from (8) that

$$u_i\big(C_{j,i}^{\varepsilon}(m_j, \tilde{m}_i), \theta_i\big) - u_i\big(f\big(m_j^2\big), \theta_i\big) > 0,$$

where $f(m_j^2)$ is the outcome induced by (m_j, m_i).

Case 2.3. $e_{j,i}(m_j, m_i) = e_{j,i}(\tilde{m}_j, m_i) = \varepsilon$.

It follows from Lemmas 1 and 2 that

$$u_i\big(C_{j,i}^{\varepsilon}(m_j, \tilde{m}_i), \theta_i\big) - u_i\big(C_{j,i}^{\varepsilon}(m_j, m_i), \theta_i\big) > 0.$$

Case 2.4. $e_{j,i}(m_j, m_i) = e_{j,i}(\tilde{m}_j, m_i) = 0$.

Both (m_j, m_i) and $(m_j, \tilde{m}_i)$ lead to the same outcome $x(m_j^2, \tilde{m}_i^1) = x(m_j^2, m_i^1) = f(m_j^2)$.

In sum, as long as $e_{i,j}(m_i, m_j) = \varepsilon$ or $e_{j,i}(m_j, m_i) = \varepsilon$ (Case 1.1 and Cases 2.1–2.3), $\tilde{m}_i$ is a strictly better response than m_i against m_j. Hence, in order for $\tilde{m}_i$ not to be a profitable deviation, we must have $e_{i,j}(m_i, m_j) = e_{j,i}(m_j, m_i) = 0$.

A.5 *Proof of Theorem 2*

We first extend the notion of a *challenge scheme* for an SCC. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$ and $z \in F(\tilde{\theta})$, if $\mathcal{L}_i(z, \tilde{\theta}_i) \cap \mathcal{SU}_i(z, \theta_i) \neq \emptyset$, we select some $x(\tilde{\theta}, z, \theta_i) \in \mathcal{L}_i(z, \tilde{\theta}_i) \cap \mathcal{SU}_i(z, \theta_i)$; otherwise, we set $x(\tilde{\theta}, z, \theta_i) = z$. In the sequel, we define $F(\Theta) \equiv \bigcup_{\theta \in \Theta} F(\theta)$. Observe that $F(\Theta)$ is a finite set, since each $F(\theta)$ is assumed to be finite.

As in the case of SCFs, the following lemma shows that there is a challenge scheme under which truth-telling induces the best allocation. In addition, we choose the challenge scheme in such a way that for every agent i, type θ_i, and state $\tilde{\theta}$ under which the challenge is effective (i.e., $x(\tilde{\theta}, z, \theta_i) \neq z$), no type $\theta_i'' \in \Theta_i$ is indifferent between $x(\tilde{\theta}, z, \theta_i)$ and any allocation z' in $F(\Theta)$.

Lemma 3. *For any SCC F, there is a challenge scheme $\{x(\tilde{\theta}, z, \theta_i)\}_{i\in\mathcal{I},\tilde{\theta}\in\Theta,z\in F(\tilde{\theta}),\theta_i\in\Theta_i}$ such that for every $i \in \mathcal{I}$, $\tilde{\theta} \in \Theta$, $z \in F(\tilde{\theta})$, and $\theta_i \in \Theta_i$,*

$$u_i\big(x(\tilde{\theta}, z, \theta_i), \theta_i\big) \geq u_i\big(x\big(\tilde{\theta}, z, \theta_i'\big), \theta_i\big), \quad \forall \theta_i' \in \Theta_i; \tag{16}$$

moreover, whenever, $x(\tilde{\theta}, z, \theta_i) \neq z$, we have

$$u_i\big(x(\tilde{\theta}, z, \theta_i), \theta_i''\big) \neq u_i\big(z', \theta_i''\big), \quad \forall \theta_i'' \in \Theta_i, \forall z' \in F(\Theta). \tag{17}$$

Proof. We first prove (17) by constructing a challenge scheme $\{x(\tilde{\theta}, z, \theta_i)\}$. Fix agent i of type θ_i. For each state $\tilde{\theta} \in \Theta$ and $z \in F(\tilde{\theta})$, if $\mathcal{L}_i(z, \tilde{\theta}_i) \cap \mathcal{SU}_i(z, \theta_i) = \emptyset$, we let $x(\tilde{\theta}, z, \theta_i) = z$; otherwise, we define

$$\mathcal{S}(i, z, \tilde{\theta}, \theta) = \big\{z'' \in X : u_i\big(z'', \tilde{\theta}_i\big) < u_i(z, \tilde{\theta}_i) \text{ and } u_i\big(z'', \theta_i\big) > u_i(z, \theta_i)\big\}.$$

Observe that $\mathcal{S}(i, z, \tilde{\theta}, \theta)$ is a nonempty open set, since we can add a small penalty to agent i with an allocation in $\mathcal{L}_i(z, \tilde{\theta}_i) \cap \mathcal{SU}_i(z, \theta_i)$. Now consider

$$\begin{aligned} &\mathcal{S}^*(i, z, \tilde{\theta}, \theta) \\ &\quad \equiv \mathcal{S}(i, z, \tilde{\theta}, \theta) \setminus \bigcup_{\theta_i''\in\Theta_i} \bigcup_{z'\in F(\Theta)} \big\{z'' \in X : u_i\big(z'', \theta_i''\big) = u_i\big(z', \theta_i''\big)\big\}. \end{aligned}$$

Thanks to the finiteness of $F(\Theta)$ and Θ_i, $\mathcal{S}^*(i, z, \tilde{\theta}, \theta)$ remains a nonempty open set after we delete finitely many closed sets $\{z'' \in X : u_i(z'', \theta_i'') = u_i(z', \theta_i'')\}$, one for each $\theta_i'' \in \Theta_i$ and $z' \in F(\Theta)$. Now we choose an element $x(\tilde{\theta}, z, \theta_i) \in \mathcal{S}^*(i, z, \tilde{\theta}, \theta)$. Hence, we obtain (17). The proof of (16) is completed once we apply the proof of Lemma 1 to the challenge scheme $\{x(\tilde{\theta}, z, \theta_i)\}_{i\in\mathcal{I},\tilde{\theta}\in\Theta,z\in F(\tilde{\theta}),\theta_i\in\Theta_i}$. □

Next, we propose a mechanism $\mathcal{M} = ((M_i), g, (\tau_i))_{i\in\mathcal{I}}$, which will be used to prove the if-part of Theorem 2. First, a generic message of agent i is described as follows:

$$m_i = \big(m_i^1, m_i^2, m_i^3\big) \in M_i = M_i^1 \times M_i^2 \times M_i^3 = \Theta_i \times \Big[\bigtimes_{j=1}^{I} \Theta_j\Big] \times F(\Theta) \quad \text{such that}$$

$$m_i^2 \in \Theta \quad \Rightarrow \quad m_i^3 \in F\big(m_i^2\big).$$

That is, agent i is asked to announce (1) agent i's own type (which we denote by m_i^1); (2) a type profile (which we denote by m_i^2); (3) an allocation m_i^3 such that $m_i^3 \in F(m_i^2)$ if m_i^2 is a state. As we do in the case of SCFs, we write $m_{i,j}^2 = \tilde{\theta}_j$ if agent i reports in m_i^2 that agent

j's type is $\tilde{\theta}_j$. Likewise, since F is Maskin-monotonic, we have $F(\theta) = F(\tilde{\theta})$ if $\tilde{\theta}_i = \theta_i$ for every i; hence, for $m_i^2 \in \Theta$, $F(m_i^2)$ is uniquely defined as $F(\tilde{\theta})$ such that $\tilde{\theta}_j = m_{i,j}^2$ for all j.

We define $\phi(m)$ as follows: for each $m \in M$,

$$\phi(m) = \begin{cases} x, & \text{if } \left|\{i \in \mathcal{I} : m_i^3 = x\}\right| \geq I - 1; \\ m_1^3, & \text{otherwise.} \end{cases}$$

We say that $\phi(m)$ is an *effective allocation* under m. In words, the effective allocation is x, if there are $I - 1$ players who agree on allocation x; otherwise, the effective allocation is the allocation announced by agent 1.

The allocation rule g is defined as follows: for each $m \in M$,

$$g(m) = \frac{1}{I^2} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \left[e_{i,j}(m) \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) \oplus \left(1 - e_{i,j}(m)\right) x\left(\tilde{\theta}, \phi(m), m_j^1\right) \right],$$

where $\{y_k(\theta_k)\}_{\theta_k \in \Theta_k}$ are the dictator lotteries for agent k as defined in Lemma 2. Given a message profile m, and a pair of agents i and j, we say that agent j *challenges agent* i if and only if $m_i^3 = \phi(m)$ and $x(m_i^2, \phi(m), m_j^1) \neq \phi(m)$, that is, agent i's reported allocation is an effective one and agent j challenges this effective allocation. We define the $e_{i,j}$-function as follows: for each $m \in M$,

$$e_{i,j}(m) = \begin{cases} 0, & \text{if } m_i^2 \in \Theta,\ m_i^2 = m_j^2, \text{ and agent } j \text{ does not challenge agent } i; \\ \varepsilon, & \text{if } m_i^2 \in \Theta, \text{ and } \left[m_i^2 \neq m_j^2 \text{ or agent } j \text{ challenges agent } i\right]; \\ 1, & \text{if } m_i^2 \notin \Theta. \end{cases}$$

Recall that the $e_{i,j}$-function in Section 4.1.4 for the case of SCFs only depends on m_i and m_j. In contrast, the $e_{i,j}$-function here depends on the entire message profile, as the nature of the challenge also depends on whether the allocation reported by agent i is an effective allocation or not.

Fix $i, j \in \mathcal{I}$, $\varepsilon \in (0, 1)$, and $m \in M$. Then we define

$$C_{i,j}^{\varepsilon}(m) \equiv \varepsilon \times \frac{1}{I} \sum_{k \in \mathcal{I}} y_k\left(m_k^1\right) \oplus (1 - \varepsilon) \times x\left(m_i^2, \phi(m), m_j^1\right).$$

For every message profile m and agent j, we can choose $\varepsilon > 0$ sufficiently small such that (i) $C_{i,j}^{\varepsilon}(m)$ does not disturb the "effectiveness" of agent j's challenge, that is,

$$\begin{aligned} & x\left(m_i^2, \phi(m), m_j^1\right) \neq \phi(m) \\ & \quad \Rightarrow \quad u_j\left(C_{i,j}^{\varepsilon}(m), m_{i,j}^2\right) < u_j\left(\phi(m), m_{i,j}^2\right) \quad \text{and} \quad u_j\left(C_{i,j}^{\varepsilon}(m), m_j^1\right) > u_j\left(\phi(m), m_j^1\right); \end{aligned} \tag{18}$$

moreover, (ii) an "effective self-challenge" of agent j induces a generic outcome such that at each state, no agent is indifferent between the resulting outcome and any outcome in $F(\Theta)$, that is,

$$\begin{aligned} & x\big(m_j^2, \phi(m), m_j^1\big) \neq \phi(m) \\ & \quad \Rightarrow \quad u_j\big(C_{j,j}^{\varepsilon}(m), \theta_j\big) \neq u_j(x, \theta_j) \\ & \qquad \text{for any } \theta \text{ and any } x \in F(\Theta). \end{aligned} \tag{19}$$

Observe that property (ii) can be made satisfied because inequality (17) holds in Lemma 3; moreover, by (6) in Lemma 2, $u_j(C_{j,j}^{\varepsilon}(m), \theta_j)$ is a strictly decreasing function in ε.

The transfer to agent i is specified as follows: for each $m \in M$,

$$\tau_i(m) = \sum_{j \neq i} \big[\bar{\tau}_{i,j}^1(m) + \bar{\tau}_{i,j}^2(m) + \tau_{i,j}^3(m)\big]$$

where we set $\bar{\tau}_{i,j}^1(m) = 2\tau_{i,j}^1(m)$ and $\bar{\tau}_{i,j}^2(m) = 2\tau_{i,j}^2(m)$, while $\tau_{i,j}^1(m)$ and $\tau_{i,j}^2(m)$ are defined as in Section 4.1.5; moreover, we specify $\tau_{i,j}^3(m)$ as follows: for each $m \in M$,

$$\tau_{i,j}^3(m) = \begin{cases} -\eta, & \text{if agent } j \text{ challenges agent } i, \\ 0, & \text{otherwise.} \end{cases}$$

That is, agent i is asked to pay η if his reported outcome m_i^3 is challenged by agent $j \neq i$. Note that we still require that η be greater than the maximal payoff difference, which is guaranteed by (7) in Section 4.1.5.

In the rest of the proof of Theorem 2, we fix θ as the true state and σ as a (possibly mixed strategy) Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$ throughout.

To prove Theorem 2, we use a stronger statement than Claim 1 since each agent i's dictator lotteries are triggered whenever there is an agent j and an agent k (whether $k = j$ or $k \neq j$) such that $e_{j,k}(m_j, m_k) = \varepsilon$. The proof of this stronger claim is identical to the proof of Case 1.1 in Claim 1.

Claim 5. *If $m_i^1 \neq \theta_i$ for some $m_i \in \operatorname{supp}(\sigma_i)$, then we have $e_{j,k}(m_i, m_{-i}) = 0$ for every $m_{-i} \in \operatorname{supp}(\sigma_{-i})$ and every pair of agent(s) $j, k \in \mathcal{I}$.*

We now observe that Claims 2 and 3 used in the proof of Theorem 1 hold with exactly the same proof. As we did in the proof of Theorem 1, by Claim 3, we denote the common state announced in the agents' second report by $\tilde{\theta}$. In the following, we establish Claim 7 as the counterpart of Claim 4 used in the proof of Theorem 1 in the modified mechanism introduced above.

For each allocation $x \in F(\Theta)$, we define the following set of agents:

$$\mathcal{J}(x) \equiv \big\{j \in \mathcal{I} : \mathcal{L}_j(x, \tilde{\theta}_j) \cap \mathcal{SU}_j(x, \theta_j) = \emptyset\big\}.$$

The following preliminary claim will be used in proving Claims 7 and 8.

Claim 6. *For any pair of agent(s) i and j (whether $i = j$ or $i \neq j$) and message profile $m \in \operatorname{supp}(\sigma)$ such that $m_i^3 = \phi(m)$, we have $x(\tilde{\theta}, m_i^3, m_j^1) \neq m_i^3$ if and only if $j \notin \mathcal{J}(m_i^3)$.*

Proof. Fix agent $i \in \mathcal{I}$ and a message profile $m \in \text{supp}(\sigma)$. We first prove the if-part. Suppose, on the contrary, that there exists some agent $j \notin \mathcal{J}(m_i^3)$ such that $x(\tilde{\theta}, m_i^3, m_j^1) = m_i^3$. Then, by (18), the deviation from m_j to $\tilde{m}_j = (\theta_j, m_j^2, m_j^3)$ delivers a strictly better payoff for agent j against m_{-j}, while, by Lemmas 2 and 3, the deviation from m_j to $\tilde{m}_j$ generates no payoff loss for agent j against any $m'_{-j} \neq m_{-j}$. Hence, the deviation $\tilde{m}_j$ is profitable, which contradicts the hypothesis that σ is a Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$.

Next, we prove the only-if-part. Suppose, on the contrary, that there exists some agent $j \in \mathcal{J}(m_i^3)$ such that $x(\tilde{\theta}, m_i^3, m_j^1) \neq m_i^3$. Since $j \in \mathcal{J}(m_i^3)$, we must have $m_j^1 \neq \theta_j$. Define $\tilde{m}_j$ as a deviation which is identical to m_j except that $\tilde{m}_j^3 = \theta_j \neq m_j^3$. Then we have $x(\tilde{\theta}, m_i^3, \theta_j) = m_i^3$ since $j \in \mathcal{J}(m_i^3)$. By (18), $\tilde{m}_j$ generates a strictly better payoff for agent j than m_j against m_{-j}. By Lemmas 2 and 3, we also know that agent j's payoff generated by $\tilde{m}_j$ is at least as good as that generated by m_j against any $m'_{-j} \neq m_{-j}$. Hence, $\tilde{m}_j$ constitutes a profitable deviation, which contradicts the hypothesis that σ is a Nash equilibrium of the game $\Gamma(\mathcal{M}, \theta)$. □

Claim 7. *No one challenges an allocation announced in the third report of any other agent, that is, for any pair of agents $i, j \in \mathcal{I}$ with $i \neq j$ and any $m \in \text{supp}(\sigma)$, if $m_i^3 = \phi(m)$, then $x(\tilde{\theta}, m_i^3, m_j^1) = m_i^3$.*

Proof. Suppose to the contrary that there exist $i, j \in \mathcal{I}$ with $i \neq j$, $m \in \text{supp}(\sigma)$ such that $m_i^3 = \phi(m)$ and $x(\tilde{\theta}, m_i^3, m_j^1) \neq m_i^3$. By Claim 6, $j \notin \mathcal{J}(m_i^3)$. We now derive a contradiction in each of the following two cases.

Case (i). *There is some $\tilde{m} \in \text{supp}(\sigma)$ such that $\mathcal{J}(\phi(\tilde{m})) = \mathcal{I}$.*

Define $\tilde{m}_i$ as the same as m_i except that $\tilde{m}_i^3 = \phi(\tilde{m})$. Fix $\hat{m}_{-i} \in \text{supp}(\sigma_{-i})$. We distinguish two subcases.

Case (i.1). $\phi(m_i, \hat{m}_{-i}) = m_i^3$.

By Claim 6 and the fact that $j \notin \mathcal{J}(m_i^3)$, we have $x(\tilde{\theta}, \phi(m_i, \hat{m}_{-i}), \hat{m}_j^1) \neq \phi(m_i, \hat{m}_{-i})$, that is, $\phi(m_i, \hat{m}_{-i})$ must be challenged by $\hat{m}_j^1$ and agent i is penalized by η according to $\tau_{i,j}^3$. In comparison, if $\phi(\tilde{m}_i, \hat{m}_{-i}) = \phi(\tilde{m})$, since $\mathcal{J}(\phi(\tilde{m})) = \mathcal{I}$, it follows from Claim 6 that no agent challenges $\phi(\tilde{m}_i, \hat{m}_{-i})$; if $\phi(\tilde{m}_i, \hat{m}_{-i}) \neq \phi(\tilde{m})$, then $\tilde{m}_i$ is not effective; hence, agent i avoids paying the penalty η for being challenged.

Case (i.2). $\phi(m_i, \hat{m}_{-i}) \neq m_i^3$.*Then the deviation does not change allocation or transfers from agent i's perspective.*

We know that Case (i.1) happens with positive probability from our hypothesis. Hence, by (7), it is a profitable deviation.

Case (ii). *for every* $\tilde{m} \in \operatorname{supp}(\sigma)$, $\mathcal{J}(\phi(\tilde{m})) \neq \mathcal{I}$.

Fix $\tilde{m} \in \operatorname{supp}(\sigma)$. If $\tilde{m}_k^3 = \phi(\tilde{m})$ for some agent $k \in \mathcal{I}$, then Claim 6 implies that $x(\tilde{\theta}, \tilde{m}_k^3, \tilde{m}_{k'}^1) \neq \tilde{m}_k^3$ for some agent k', namely that agent k' challenges agent k at $\tilde{m}$. Thus, we know that $e_{k,k'}(\tilde{m}) = \varepsilon$. Hence, by Claim 5, every agent reports the true type in their first reports under any $\tilde{m} \in \operatorname{supp}(\sigma)$. Hence, by Claim 2, we conclude that $\tilde{\theta} = \theta$. It implies that $\phi(\tilde{m}) \in F(\theta)$ and $\mathcal{J}(\phi(\tilde{m})) = \mathcal{I}$. Thus, it contradicts that $\mathcal{J}(\phi(\tilde{m})) \neq \mathcal{I}$. □

Claim 8. *No one challenges an allocation announced in his own third report, that is, for every agent i,* $m \in \operatorname{supp}(\sigma)$ *and* $m_i^3 = \phi(m)$ *we have* $x(\tilde{\theta}, m_i^3, m_i^1) = m_i^3$.

Proof. Suppose to the contrary that there exist agent i and some message $m \in \operatorname{supp}(\sigma)$ such that $x(\tilde{\theta}, m_i^3, m_i^1) \neq m_i^3$. By Claim 7 and the construction of ϕ, the agent i must be agent 1. Moreover, $\phi(m) = m_1^3$ and $\phi(m) \neq m_j^3$ for every $j \neq 1$. Hence,

$$\phi(\tilde{m}_1, m_{-1}) = \tilde{m}_1^3 \quad \text{for every } \tilde{m}_1. \tag{20}$$

By Claim 6, we know that $1 \notin \mathcal{J}(m_1^3)$, that is,

$$\mathcal{L}_1\big(m_1^3, \tilde{\theta}_1\big) \cap \mathcal{SU}_1\big(m_1^3, \theta_1\big) \neq \emptyset; \tag{21}$$

moreover, for every $\tilde{m} \in \operatorname{supp}(\sigma)$ with $\phi(\tilde{m}) = m_1^3$, we have $x(\tilde{\theta}, m_1^3, \tilde{m}_1^1) \neq m_1^3$. Next, we shall show that

$$\tilde{m} \in \operatorname{supp}(\sigma) \quad \text{and} \quad \phi(\tilde{m}) = \tilde{m}_1^3 \quad \Rightarrow \quad x\big(\tilde{\theta}, \tilde{m}_1^3, \tilde{m}_1^1\big) \neq \tilde{m}_1^3. \tag{22}$$

To establish (22), suppose on the contrary that $x(\tilde{\theta}, \tilde{m}_1^3, \tilde{m}_1^1) = \tilde{m}_1^3$ for some $\tilde{m} \in \operatorname{supp}(\sigma)$ with $\tilde{m}_1^3 = \phi(\tilde{m})$. We now compare the payoff difference between m_i and $\tilde{m}_i$ against $\hat{m}_{-1}$ by considering the following two different situations.

Case (i). $\phi(m_1, \hat{m}_{-1}) = m_1^3$ *and* $\phi(\tilde{m}_1, \hat{m}_{-1}) = \tilde{m}_1^3$.

In this case, we know that $x(\tilde{\theta}, m_1^3, m_1^1) \neq m_1^3$ and $x(\tilde{\theta}, \tilde{m}_1^3, \tilde{m}_1^1) = \tilde{m}_1^3$. By Claim 7, no agent challenges any of the other agents. Thus, the allocation difference occurs only when agent 1 is chosen to challenge himself.

Case (ii). $\phi(m_1, \hat{m}_{-1}) \neq m_1^3$ *or* $\phi(\tilde{m}_1, \hat{m}_{-1}) \neq \tilde{m}_1^3$.

By the construction of ϕ, there exists $z \in F(\tilde{\theta})$ such that every agent $j \neq 1$ reports $\hat{m}_j^3 = z$. Once again, by the construction of ϕ, we also have $\phi(m_1, \hat{m}_{-1}) = \phi(\tilde{m}_1, \hat{m}_{-1}) = z$. Moreover, by Claim 7, no agent challenges z, that is, $x(\tilde{\theta}, z, \hat{m}_k^1) = z$ for every $k \in \mathcal{I}$. Hence, $(m_1, \hat{m}_{-1})$ and $(\tilde{m}_1, \hat{m}_{-1})$ deliver the same allocation and transfer to agent 1.

In summary, against σ_{-i}, the payoff difference between m_i and $\tilde{m}_i$ lies only in case (i) and is equal to

$$\frac{1}{I^2} u_i\big(C_{i,i}^{\varepsilon}(m_i, \hat{m}_{-1}), \theta_i\big) - \frac{1}{I^2} u_i\big(\tilde{m}_i^3, \theta_i\big).$$

The payoff difference must be zero because both m_i and $\tilde{m}_i$ are played with positive probability in equilibrium. However, it contradicts (17). Hence, (22) holds.

Finally, it follows from (20) and (22) that agent 1 must challenge himself with probability one. By Claim 5, every agent reports the true type in their first reports under any $\tilde{m} \in \text{supp}(\sigma)$. Moreover, by Claim 2, we have $\tilde{\theta} = \theta$, which is a contradiction to (21). $\square$

It only remains to prove the existence of pure-strategy Nash equilibrium.

Claim 9. *For every $x \in F(\theta)$, there exists a pure-strategy Nash equilibrium $m \in M$ of the game $\Gamma(\mathcal{M}, \theta)$ such that $g(m) = x$ and $\tau_i(m) = 0$ for every $i \in \mathcal{I}$.*

Proof. Fix an arbitrary allocation $x \in F(\theta)$. We argue that truth-telling (i.e., $m_i = (\theta_i, \theta, x)$ for each i) constitutes a pure-strategy equilibrium of the game $\Gamma(\mathcal{M}, \theta)$. Note that reporting $\tilde{m}_i$ with $\tilde{m}_i^1 = \theta_i$, $\tilde{m}_i^2 = \theta$, and $\tilde{m}_i^3 \neq x$ instead of m_i affects neither the allocation nor the transfer. The argument for proving that either $\tilde{m}_i^1 \neq \theta_i$ or $\tilde{m}_i^2 \neq \theta$ cannot be a profitable unilateral deviation for every agent i is identical to the relevant portion of the proof of Theorem 1. $\square$

A.6 *Proof of Theorem 3*

Recall that in the mechanism, which we use to prove Theorem 1, agent i's generic message is $m_i = (m_i^1, m_i^2) \in \Theta_i \times [\bigtimes_{j=1}^I \Theta_j]$. We expand m_i^2 into H copies of $[\bigtimes_{j=1}^I \Theta_j]$ and define

$$m_i = \left(m_i^1, m_i^2, \ldots, m_i^{H+1}\right) \in \Theta_i \times \underbrace{\Big[\bigtimes_{j=1}^{I} \Theta_j\Big] \times \cdots \times \Big[\bigtimes_{j=1}^{I} \Theta_j\Big]}_{H \text{ terms}}$$

where H is a positive integer to be chosen later. For each message profile $m \in M$, the allocation is defined as follows:

$$\begin{aligned} g(m) = \frac{1}{I(I-1)} \sum_{i \in \mathcal{I}} \sum_{j \neq i} \Bigg[& e_{i,j}(m_i, m_j) \frac{1}{2} \sum_{k=i,j} y_k\left(m_k^1\right) \\ & \oplus \frac{1 - e_{i,j}(m_i, m_j)}{H} \Bigg[x\left(m_i^2, m_j^{H+2}\right) \oplus \sum_{h=3}^{H+1} \phi\left(m^h\right) \Bigg] \Bigg] \end{aligned}$$

where $\{y_k(\cdot)\}$ are the dictator lotteries[19] for agent k defined in Lemma 2, $\phi(\cdot)$ is an outcome function such that

$$\phi\left(m^h\right) = \begin{cases} f(\tilde{\theta}), & \text{if } m_i^h = \tilde{\theta} \in \Theta \text{ for at least } I - 1 \text{ agents;} \\ b, & \text{otherwise, where } b \text{ is an arbitrary outcome in } A, \end{cases}$$

[19]Although the dictator lotteries may contain transfers, we do not take into account the scale of transfers in it. To dispense with the transfers in the dictator lotteries, we can use an arbitrarily small amount of money to make the best challenge schemes and social choice function generic, that is, any two resulting outcomes are distinct from each other.

and

$$
e_{i,j}(m_i, m_j) = \begin{cases} 0, & \text{if } m_i^2 \in \Theta,\ m_i^2 = m_j^2 = m_i^h = m_j^h \text{ and } x\big(m_i^2, m_j^{H+2}\big) = f\big(m_i^2\big), \\ & \forall h \in \{3, \dots, H+1\}; \\ 1, & \text{if } m_i^2 \notin \Theta; \\ \varepsilon, & \text{otherwise.} \end{cases}
$$

We now define the transfer rule. For every message profile $m \in M$ and agent $i \in \mathcal{I}$, we specify the transfer to agent i as follows:

$$
\tau_i(m) = \sum_{j \neq i} \big[\tau_{i,j}^{1,2}(m) + \tau_{i,j}^{2,2}(m)\big] + \sum_{h=3}^{H+1} \tau_i^h(m) + d_i\big(m^2, \dots, m^{H+1}\big)
$$

where γ, κ, $\xi > 0$ (their size are determined later)

$$
\tau_{i,j}^{1,2}(m) = \begin{cases} 0, & \text{if } m_{i,j}^2 = m_{j,j}^2; \\ -\gamma & \text{if } m_{i,j}^2 \neq m_{j,j}^2 \text{ and } m_{i,j}^2 \neq m_j^1; \\ \gamma & \text{if } m_{i,j}^2 \neq m_{j,j}^2 \text{ and } m_{i,j}^2 = m_j^1, \end{cases}
$$

$$
\tau_{i,j}^{2,2}(m) = \begin{cases} 0, & \text{if } m_{i,i}^2 = m_{j,i}^2; \\ -\gamma, & \text{if } m_{i,i}^2 \neq m_{j,i}^2; \end{cases}
$$

moreover, for every $h \in \{3, \dots, H+1\}$,

$$
\tau_i^h(m) = \begin{cases} -\kappa, & \text{if there exists } \tilde{\theta} \text{ such that } m_i^h \neq \tilde{\theta} \text{ but } m_j^h = \tilde{\theta} \text{ for all } j \neq i; \\ 0, & \text{otherwise,} \end{cases}
$$

and

$$
\begin{aligned} & d_i\big(m^2, \dots, m^{H+1}\big) \\ & \quad = \begin{cases} -\xi, & \text{if there exists } h \in \{3, \dots, H+1\} \text{ such that } m_i^h \neq m_i^2 \text{ and } m_j^{h'} = m_j^2, \\ & \text{for all } h' \in \{2, \dots, h-1\} \text{ and all } j \neq i; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}
$$

Finally, we choose positive numbers γ, ξ, H, κ, and ε such that

$$
\begin{aligned} & \bar{\tau} > \gamma + (H-1)\kappa + \xi \\ & \gamma > \xi + \varepsilon\eta \\ & \kappa > \varepsilon\eta \\ & \xi > \frac{1}{H}\eta + \kappa. \end{aligned}
$$

More precisely, we first fix $\bar{\tau}$ and choose $\gamma < \frac{1}{3}\bar{\tau}$ and $\xi < \min\{\frac{1}{3}\bar{\tau}, \gamma\}$. Second, we choose H large enough so that $\xi > \frac{1}{H}\eta$. Third, we choose κ small enough such that $(H-1)\kappa < \frac{1}{3}\bar{\tau}$

and $\xi > \frac{1}{H}\eta + \kappa$. Fourth, we choose ε small enough such that $\gamma > \xi + \varepsilon\eta$ and $\kappa > \varepsilon\eta$. We can now prove Theorem 3 following the three steps as in the proof of Theorem 1.

A.6.1 *Contagion of truth* First, note that Claims 1 and 2 hold. The proof of Claim 2 applies with only one minor difference: Here, m_i^2 may affect agent i's payoff through $d_i(\cdot)$. However, a similar argument follows, since we have $\gamma > \xi + \varepsilon\eta$.[20] Let θ denote the true state.

Claim 10. *If every agent j reports the truth in his first report σ_j-probability one, then every agent j reports the truth in his 2nd,...,$(H+1)$th report. That is, $m_j^h = \theta$ for $h = 2, \dots, H+1$.*

By Claims 1 and 2, every agent j reports the state truthfully in his 2nd report. Then we can follow verbatim the argument on page 12 of Abreu and Matsushima (1994), which shows that every agent j reports the state truthfully in his hth report for every $h = 2, \dots, H+1$.

A.6.2 *Consistency*

Claim 11. *There exists a state $\tilde{\theta}$ such that every agent announces $\tilde{\theta}$ in the second report all the way to the last/$(H+1)$th report with probability one.*

Proof. We prove consistency by considering the two cases as in the proof of Claim 3. The proof for the first case remains the same. For the second case, suppose that one agent, say i, tells a lie in the first report. As agent i believes that all the other agents report the same state $\tilde{\theta}$ in their second all the way to the last report. By the same argument in the second case in the proof of Claim 3, we can show that agent i announces $\tilde{\theta}$ in the second report with probability one. In addition, for every $h = 2, \dots, H+2$, as agent i believes that all the other agents report the same state $\tilde{\theta}$, by the rule $\phi(m^h)$ and $\tau_i^h(m^h)$, we know $m_i^h = \tilde{\theta}$. □

A.6.3 *No challenge*

Claim 12. *No agent challenges with positive probability the common state $\tilde{\theta}$ announced in the second report.*

Proof. The argument is the same as the proof of Claim 4. □

References

Abreu, Dilip and Hitoshi Matsushima (1992), "Virtual implementation in iteratively undominated strategies: Complete information." *Econometrica*, 60, 993–1008. [1683, 1685, 1686, 1689, 1691, 1692]

[20] It corresponds to property (b) in Abreu and Matsushima (1994).

Abreu, Dilip and Hitoshi Matsushima (1994), "Exact implementation." *Journal of Economic Theory*, 64, 1–19. [1683, 1685, 1686, 1689, 1691, 1692, 1700, 1703, 1715]

Aghion, Philippe, Drew Fudenberg, Richard Holden, Takashi Kunimoto, and Olivier Tercieux (2012), "Subgame-perfect implementation under information perturbations." *The Quarterly Journal of Economics*, 127, 1843–1881. [1686, 1699, 1700]

Chen, Yi-Chun, Takashi Kunimoto, Yifei Sun, and Siyang Xiong (2022), "Maskin meets Abreu and Matsushima." Working paper, https://arxiv.org/abs/2110.06551. [1684, 1702]

Chung, Kim-Sau and Jeffrey C. Ely (2003), "Implementation with near-complete information." *Econometrica*, 71, 857–871. [1686, 1699, 1700]

Dasgupta, Partha and Eric Maskin (2000), "Efficient auctions." *The Quarterly Journal of Economics*, 115, 341–388. [1697]

Dutta, Bhaskar and Arunava Sen (1991), "A necessary and sufficient condition for two-person Nash implementation." *The Review of Economic Studies*, 58, 121–128. [1697]

Harsanyi, John C. (1973), "Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points." *International Journal of Game Theory*, 2, 1–23. [1686]

Jackson, Matthew O. (1992), "Implementation in undominated strategies: A look at bounded mechanisms." *The Review of Economic Studies*, 59, 757–775. [1683, 1684, 1685, 1686, 1698, 1703]

Jackson, Matthew O., Thomas R. Palfrey, and Sanjay Srivastava (1994), "Undominated Nash implementation in bounded mechanisms." *Games and Economic Behavior*, 6, 474–501. [1686, 1700, 1702]

Maskin, Eric (1999), "Nash equilibrium and welfare optimality." *Review of Econconmic Studies*, 66, 23–38. [1683, 1684, 1685, 1686, 1687, 1688, 1689, 1691, 1692, 1697, 1701, 1702, 1704]

Mezzetti, Claudio and Ludovic Renou (2012a), "Implementation in mixed Nash equilibrium." *Journal of Economic Theory*, 147, 2357–2375. [1688, 1702]

Mezzetti, Claudio and Ludovic Renou (2012b), "Mixed Nash implementation with finite mechanisms." Report. [1702]

Moore, John and Rafael Repullo (1988), "Subgame perfect implementation." *Econometrica*, 56, 1191–1220. [1686, 1700]

Ollár, Mariann and Antonio Penta (2017), "Full implementation and belief restrictions." *American Economic Review*, 107, 2243–2277. [1686]

Ollár, Mariann and Antonio Penta (2021), "A network solution to robust implementation: The case of identical but unknown distributions." [1686]

Osborne, Martin J. and Ariel Rubinstein (1994), *A Course in Game Theory*. MIT Press. [1697]

Perry, Motty and Philip J. Reny (2002), "An efficient auction." *Econometrica*, 70, 1199–1212. [1697]

Sanver, M. Remzi (2006), "Nash implementing non-monotonic social choice rules by awards." *Economic Theory*, 28, 453–460. [1702]

Sjöstrom, Tomas (1994), "Implementation in undominated Nash equilibria without integer games." *Games and Economic Behavior*, 6, 502–511. [1686]

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