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# ∀ or ∃ ?

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This paper shows that in some axioms regarding the mixture of random variables, the requirement that the conclusions hold for all values of the mixture parameter can be weakened by requiring the existence of only one nontrivial value of the parameter, which need not be fixed. This is the case for the independence, betweenness, and mixture symmetry axioms. Unlike the standard axioms, these weaker versions cannot be refuted by experimental methods.

Keywords. Betweenness, independence axiom, mixture symmetry.

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## 1. Introduction

Typical axiomatic models of preferences offer a set of axioms $\mathcal{A}$ and a functional $V$, and prove that a preference relation $\succeq$ can be represented by $V$ if and only if it satisfies all the axioms of $\mathcal{A}$. For example, expected utility theory states that a preference (that is, a complete and transitive) relation over a mixture set of lotteries can be represented by $V(F) = \int u(x)\,dF(x)$ with a continuous function $u$ if and only if it is continuous and satisfies the independence axiom: For all $F, G, H$ and $\alpha \in (0, 1]$, $\alpha F + (1-\alpha)H \succeq \alpha G + (1-\alpha)H$ if and only if $F \succeq G$.

Axioms can be used to justify certain types of preferences. Savage (1972) tells how the sure thing principle convinced him not to follow the Allais paradox. The adaptation of his explanation to the independence axiom will argue that as $H$ is common to $\alpha F + (1-\alpha)H$ and $\alpha G + (1-\alpha)H$, and is not affected by the decision maker's choice, attention should concentrate on that part of the procedure that is affected by the choice, but this turns out to be a choice between $F$ and $G$. Another argument in favor of axiomatic approaches is that it is sometimes easier to observe violations of axioms rather than of general theories. Moreover, violations that can be traced down to specific axioms can lead to the construction of new axioms and, hence, of new theories.[1] For example, the Allais paradox (Allais (1953)) violates the independence axiom. Let $A = (5, 0.1; 0, 0.9) \succ B = (1, 0.11; 0, 0.89)$ while $D = 1 \succ C = (5, 0.1; 1, 0.89; 0, 0.01)$. Let $F = (5, \frac{10}{11}; 0, \frac{1}{11})$, $G = H' = 1$, and $H = 0$. Then, by the independence axiom, $A = 0.11F + 0.89H \succ B = 0.11G + 0.89H$ if and only if $F \succ G$ if and only if $C = 0.11F +$

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[1] For a detailed discussion of the behavioral foundations of decision models and further references, see Wakker (2010).


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$0.89H' \succ 0.11G + 0.89H'$. Many alternatives to expected utility theory therefore replace independence with weaker axioms.

Experiments violate the independence axiom, but they cannot nullify its appeal. What they do show is that decision makers may have other considerations, e.g., the fear of deep future remorse if after choosing $C$ over $D$ in the Allais paradox, the event leading to the zero outcome happens. This psychological concern may be stronger than Savage's analysis, but it does not void it. But what happens if we impose the spirit of the independence axiom in some but not all cases? This paper shows that this will not do. Independence and other mixture axioms, and, hence, the theories they imply, can be obtained from much weaker conditions over preferences, provided continuity and monotonicity are assumed. Not only is it not required to impose the independence axiom for *all* values of $\alpha$, it is not necessary to require it for any specific, or even for an unknown, fixed value of $\alpha$. All that is needed is that for each appropriate triplet $F, G, H$ there exists at least *one* value of $\alpha$ (which may change from one triplet to another) satisfying the axiom. To some extent it is similar to the fact that a continuous function $f : \Re \to \Re$ is concave if and only if for all $x < y$ there is at least one $\alpha \in (0, 1)$ for which $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$.[2]

The independence axiom is easy to refute. All one needs is one value of $\alpha$ for which it is violated. But the weaker axiom suggested in this paper can never be refuted, because a finite set of observations cannot prove that there is no value of $\alpha$ for which it is satisfied. Continuity too is not a testable axiom, but assuming monotonicity, the joint imposition of these two non-testable axioms is testable. Experiments like the Allais paradox therefore do not necessarily show a violation of a specific axiom. Rather, they show that the combination of attractive rules leads to unattractive decisions. So instead of trying to find faults with the underlying rules, we should probably try to understand why attractive rules do not coexist.

The paper is organized as follows. Section 2 deals with weakening the independence and the betweenness axioms: If $F \sim G$, then for all $\alpha \in [0, 1]$, $F \sim \alpha F + (1-\alpha)G$ (see Chew (1983) and Dekel (1986)). It shows that assuming continuity and monotonicity, "for all $\alpha$" in these axioms can be replaced with "there exists $\alpha$." Section 3 analyzes parallel results for the mixture symmetry axiom: If $F \sim G$, then for all $\alpha \in [0, 1]$, $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$ (Chew, Epstein, and Segal (1991)). Section 4 shows, by means of an example, the importance of the continuity assumption, as without it the results of the paper do not hold. Section 5 concludes with some remarks on the observability of violations of axioms and their relative strengths.

## 2. Betweenness and independence

Let $\mathcal{F}$ be the set of distributions over $[0, a]$, $a \in (0, \infty)$. Consider a complete and transitive preference relation $\succeq$, satisfying continuity and monotonicity with respect to

[2] Suppose that for some $\beta \in (0, 1)$, $f(\beta x + (1-\beta)y) < \beta f(x) + (1-\beta)f(y)$. Let $\alpha_* = \sup\{\alpha < \beta : f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)\}$ and $\alpha^* = \inf\{\alpha > \beta : f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)\}$ to get a contradiction.

first-order stochastic dominance,[3] and let $V : \mathcal{F} \to \Re$ represent it. For $F, G \in \mathcal{F}$, let $[F, G] = \{\alpha F + (1-\alpha)G : \alpha \in [0, 1]\}$ and $(F, G) = \{\alpha F + (1-\alpha)G : \alpha \in (0, 1)\}$. For $F \neq G$, the line through $F$ and $G$ is the set $L_{F,G} = \{H : F \in [H, G] \text{ or } G \in [H, F]\}$. The key axiom used in the formalization of expected utility theory is the independence axiom (e.g., Samuelson (1952)).

Independence (I). For all $F, G, H \in \mathcal{F}$, $F \succeq G$ iff for all $\alpha \in (0, 1)$, $\alpha F + (1-\alpha)H \succeq \alpha G + (1-\alpha)H$.

The standard normative justification for this assumption suggests that using a random device that can produce the $\alpha : 1-\alpha$ probabilities, the decision maker should realize that if the $1-\alpha$ event happens, his choice between $\alpha F + (1-\alpha)H$ and $\alpha G + (1-\alpha)H$ is of no consequence: he will get $H$ regardless of his choice. His choice will affect his outcome in case the $\alpha$ event happens: according to his choice he will win $F$ of $G$. His choice between the two compound lotteries should therefore be the same as his choice between $F$ and $G$.

The assumption that the order between $F$ and $G$ is preserved even when receiving either of them becomes uncertain may be too strong. An appealing relaxation was suggested by Chew (1983) and Dekel (1986). A lottery over two outcomes cannot be better or worse than both of them, and, in particular, if the decision maker is indifferent between two options, he should not care which one of them he wins. This concept can be formalized as follows.

Betweenness (B). For all $F, G \in \mathcal{F}$, if $F \succeq G$, then for all $\alpha \in [0, 1]$, $F \succeq \alpha F + (1-\alpha)G \succeq G$.

The betweenness axiom is implied by the independence axiom (set $H = G$ in (I)), but it does not imply it. For example, Chew's (1983) weighted utility theory satisfies betweenness, but not independence.

I offer next much weaker versions of these axioms.

Weak Independence (WI). For all $F, G, H \in \mathcal{F}$, if $F \sim G$, then there exists $\alpha \in (0, 1)$ such that $\alpha F + (1-\alpha)H \sim \alpha G + (1-\alpha)H$.

Weak Betweenness (WB). For all $F, G \in \mathcal{F}$, if $F \sim G$, then there exists $\alpha \in (0, 1)$ such that $\alpha F + (1-\alpha)G \sim F$.

The weak versions discussed here only require the existence of one value of $\alpha$, and this value may depend on the underlying distributions. It turns out that together with continuity and monotonicity, these weaker versions imply the stronger axioms and, therefore, any theory implied by them.

Theorem 1. *Assuming continuity and monotonicity, (WB) implies (B).*

[3] The preference relation $\succeq$ is continuous if for all $F, G, H$, the sets $\{\alpha : \alpha F + (1-\alpha)G \succeq H\}$ and $\{\alpha : H \succeq \alpha F + (1-\alpha)G\}$ are closed (Herstein and Milnor (1953)). It is monotonic with respect to first-order stochastic dominance if $F \succ G$ whenever $F \neq G$ and for all $x$, $F(x) \leq G(x)$.

Proof. We show first that if $F \sim G$, then for all $\alpha \in [0,1]$, $\alpha F + (1-\alpha)G \sim F$. Using a method introduced by Hardy, Littlewood, and Pòlya (1952, Observation 88 in Section 3.7),[4] suppose first that there are $F \sim G$ and $\alpha_0$ such that $\alpha_0 F + (1-\alpha_0)G \nsim F$. Let $\alpha^* = \sup_\alpha\{\alpha < \alpha_0 : \alpha F + (1-\alpha)G \sim F\}$ and $\alpha_* = \inf_\alpha\{\alpha > \alpha_0 : \alpha F + (1-\alpha)G \sim F\}$. By continuity, $F^* := \alpha^* F + (1-\alpha^*)G \sim F_* := \alpha_* F + (1-\alpha_*)G \sim F$. Hence, by (WB) there is $\beta \in (0,1)$ such that $\beta F^* + (1-\beta)F_* \sim F^* \sim F$, a contradiction.

Next we show that if $F \in (H, G)$ and $F \sim G$, then $H \sim G$, and, hence, for all $\alpha \in [0,1]$, $H \sim \alpha H + (1-\alpha)G \sim F \sim G$. Suppose, by way of negation, that for some $F \in (H, G)$, $F \sim G$ yet $H \nsim G$, without loss of generality (wlg) $G \succ H$. Let $F'$ dominate $H$ and $G$ (and, hence, $F$) by first-order stochastic dominance. By continuity there is $H' \in (H, F')$ such that $H' \sim G$. By the first part of the proof, $[H', G] \sim G$ (that is, $\forall \alpha \in [0,1]$, $\alpha H' + (1-\alpha)G \sim G$). Clearly, $H', F \in \Delta(F', H, G) := \{\alpha F' + \beta H + (1-\alpha-\beta)G : \alpha, \beta \geq 0, \alpha + \beta \leq 1\}$. Let $D$ be the intersection point of $[H', G]$ and $[F', F]$. Then $D$ dominates $F$, yet $D \sim G \sim F$, a violation of monotonicity.

Suppose now that $F \succ G$ and that for some $\alpha_0 \in (0,1)$, $F_0 := \alpha_0 F + (1-\alpha_0)G \succeq F$. Then by continuity there is $F^* \in [F_0, G)$ such that $F^* \sim F$. But then, by the last paragraph, $F \sim G$, a contradiction. The proof of the case $G \succeq F_0$ is similar. □

Remark 1. Assuming monotonicity, (WB) implies that if $F \succ G$, then for all $1 \geq \alpha > \beta \geq 0$, $\alpha F + (1-\alpha)G \succ \beta F + (1-\beta)G$. Otherwise, there is $\gamma \in [\alpha, 1]$ such that $\gamma F + (1-\gamma)G \sim \beta F + (1-\beta)G$; hence, by the second paragraph in the proof of Theorem 1, $F \sim \beta F + (1-\beta)G$. This implies $F \sim G$, a contradiction.

Theorem 2. *Assuming continuity and monotonicity, (WI) implies (I).*

Proof. Let $F \sim G$ and $H = G$ in the definition of (WI) to obtain that it implies (WB) and, hence, (B).

Consider first the case $F \sim G$ and their mixtures with an arbitrary $H$. If $F \sim H \sim G$, then by (B), for all $\alpha \in (0,1)$, $\alpha F + (1-\alpha)H \sim H \sim \alpha G + (1-\alpha)H$. Suppose wlg that $F \sim G \succ H$. By (WI), there is a decreasing sequence $\alpha_n$ such that $F_n^\alpha := \alpha_n F + (1-\alpha_n)H \sim G_n^\alpha := \alpha_n G + (1-\alpha_n)H$. Let $\bar{\alpha} = \lim_{n\to\infty} \alpha_n$ (it exists as $\{\alpha_n\}$ is a decreasing and bounded sequence). By (WI), if $\bar{\alpha} > 0$, then there is $\alpha < \bar{\alpha}$ such that $\alpha F + (1-\alpha)H \sim \alpha G + (1-\alpha)H$. Choose, therefore, a sequence $\alpha_n$ such that $\bar{\alpha} = 0$.

By (B), for all $\alpha \in (0,1)$ and $D = F, G$, $D \succ \alpha F + (1-\alpha)H \succ H$. It follows, therefore, by continuity that for all $\alpha \in (0,1)$ there is $\beta \in (0,1)$ such that $\alpha F + (1-\alpha)H \sim \beta G + (1-\beta)H$. By Remark 1, this $\beta$ is unique. Suppose now that for a certain $\tilde{\alpha} \in (0,1)$ there is $\tilde{\beta} \in (0,1)$, $\tilde{\beta} \neq \tilde{\alpha}$, such that $\tilde{F} := \tilde{\alpha} F + (1-\tilde{\alpha})H \sim \tilde{G} := \tilde{\beta} G + (1-\tilde{\beta})H$. As before, there is a sequence $\beta_n \downarrow 0$ such that for all $n$, $F_n^\beta := \beta_n \tilde{F} + (1-\beta_n)H \sim G_n^\beta := \beta_n \tilde{G} + (1-\beta_n)H$. Since $\tilde{\alpha} \neq \tilde{\beta}$, the line $L_n$ through $F_n^\alpha$ and $G_n^\alpha$, and the line $\tilde{L}_n$ through $F_n^\beta$ and $G_n^\beta$ are not parallel. Without loss of generality, $H$ is in the interior of a probability triangle (see Machina (1982)) that also contains $F$ and $G$. Otherwise, let $H_n \to H$, where for every $n$, $H_n$ is in the interior of the triangle formed by $F, G, H$. The limit of the intersection points of $L_n$ and $\tilde{L}_n$ is $H$. Therefore, for a sufficiently large $n$, there is such an intersection point

[4] I am grateful to Peter Wakker for this reference.

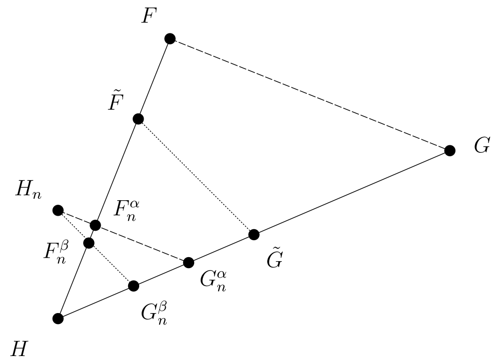


FIGURE 1. Wide dash denotes $\alpha$-lines; dense dash denotes $\beta$-lines.

in the triangle. By the second paragraph in the proof of Theorem 1, $F_n^\alpha \sim G_n^\alpha$ implies $H \sim G_n^\alpha$ and $F_n^\beta \sim G_n^\beta$ implies $H \sim G_n^\beta$. By transitivity, $G_n^\alpha \sim G_n^\beta$, a violation of Remark 1 (see Figure 1).

Consider now the case $F \succ G$. If $F \succeq H \succ G$, then by (B), for all $\alpha \in (0, 1)$, $\alpha F + (1 - \alpha)H \succeq H \succ \alpha G + (1 - \alpha)H$. The proof of the case $F \succ H \succeq G$ is similar. If $F \succ G \succ H$, then there is $\alpha^* \in (0, 1)$ such that $F^* := \alpha^* F + (1 - \alpha^*)H \sim G$. By Remark 1 and the first part of the proof, for all $\alpha \in (0, 1)$, $\alpha F + (1 - \alpha)H \succ \alpha F^* + (1 - \alpha)H \sim \alpha G + (1 - \alpha)H$. The proof of the case $H \succ F \succ G$ is similar. □

## 3. Mixture symmetry

Quadratic utility, one of the early alternatives to expected utility theory, was suggested by Machina (1982, footnote 45). A preference relation $\succeq$ is quadratic if can be represented by

$$V(F) = \int \int \varphi(x, y)\, dF(x)\, dF(y).$$

For some continuous, monotonic, and symmetric function, $\varphi : \Re_+^2 \to \Re$. For finite lotteries $(x_1, p_1; \dots; x_n, p_n)$ this functional becomes

$$V(x_1, p_1; \dots; x_n, p_n) = \sum_i \sum_j p_i p_j \varphi(x_i, x_j).$$

This model was axiomatized by Chew, Epstein, and Segal (1991) (henceforth CES) and was extended to social choice theory by Epstein and Segal (1992). The key axiom in CES is as follows.

Strong Mixture Symmetry (SMS). For all $F, G \in \mathcal{F}$, if $F \sim G$, then for all $\alpha \in [0, 1]$, $\alpha F + (1 - \alpha)G \sim (1 - \alpha)F + \alpha G$.

If the decision maker is indifferent between $F$ and $G$, and decides which of them to play by flipping a biased coin, then indifference follows between the option of playing $F$ if heads, $G$ if tails, and the option of playing $G$ if heads, $F$ if tails. Moreover, this holds for any biased coin. Theorem 4 in CES states that if, in addition, preferences are either quasi-concave or quasi-convex, then they can be represented by a quadratic functional. This section replaces both requirements with weaker axioms. Instead of quasi-concavity (or quasi-convexity), I require only that preferences along chords have a single extreme, and the "for all $\alpha$" in the strong mixture symmetry axiom is replaced with "there exists $\alpha$." Here too, $\alpha$ may vary from one pair of distributions to another. It turns out that these weaker assumptions still imply the quadratic representation.

Weak Mixture Symmetry (WMS). For every $F, G \in \mathcal{F}$, if $F \sim G$, then there exists $\alpha \in (0, \frac{1}{2})$ such that $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$.

As mentioned above, quasi-concavity and quasi-convexity of preferences play a crucial role in the analysis of quadratic functions.

Single Peak/Single Trough on $[F, G]$ (SP)/(ST). Let $F \sim G$, $F \neq G$. There is $\beta \in (0,1)$ such that the preferences $\succeq$ over $\alpha F + (1-\alpha)G$ are strictly increasing [decreasing] in $\alpha$ on $(0, \beta)$ and strictly decreasing [increasing] in $\alpha$ on $(\beta, 1)$.

Single Extreme (SE). Let $F \sim G$ be such that there is no $\alpha \in (0,1)$ for which $F \sim \alpha F + (1-\alpha)G$. Then $\succeq$ is either (SP) or (ST) on $[F, G]$.

Strict Quasi-Concavity/Quasi-Convexity (SQC)/(SQX). For all $F \neq G$ and $\alpha \in (0,1)$, $F \succeq G$ implies $\alpha F + (1-\alpha)G \succ G$ / $F \succ \alpha F + (1-\alpha)G$.

Nonlinearity (NL). For all $F \sim G$, $F \neq G$, there is $H \in [F, G]$ such that $F \nsim H$.

Clearly (SQC) implies (SP) and (SQX) implies (ST) on $[F, G]$ for all $F$ and $G$, and both (SP) and (ST), and, hence (SE), imply (NL).[5] However, neither (SQC) nor (SQX) is implied by (SE). For example, let $\succeq$ on $\Re^2_+$ be represented by

$$V(p,q) = \begin{cases} \dfrac{2p+q+\sqrt{4pq-3q^2}}{4}, & q \leq p, \\ \dfrac{p^2+q^2}{2q}, & q > p, \end{cases}$$

(see Figure 2).

Theorem 3. *Let the continuous and monotonic preference relation $\succeq$ satisfy (SE). Then the following three conditions are equivalent.*

*(i) It satisfies (WMS)*

*(ii) It satisfies (SMS)*

*(iii) It can be represented by a quadratic function.*

*Additionally, in all three cases, it either satisfies (SQC) or it satisfies (SQX).*

[5] Although I never assume (NL) without assuming (SE), it is sometimes illustrative to use it directly.

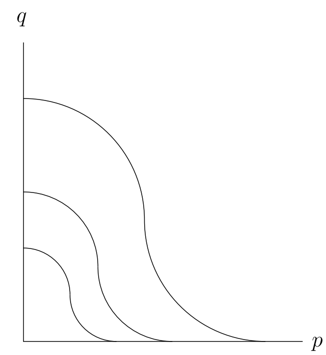


Figure 2. (SE) and (NL) together do not imply (SQC) or (SQX).

The arguments of the proof are presented in the interiors of probability triangles $\{p\bar{F}+q\bar{G}+(1-p-q)\delta_0 : p, q \geq 0,\ p+q \leq 1\}$, where $\bar{F}$, $\bar{G}$, and $\delta_0$ are not on the same line $L$. By continuity, the claims apply to the boundaries of the triangles as well, since $[F_n \to F, G_n \to G\ \forall n \alpha F_n + (1-\alpha)G_n \sim (1-\alpha)F_n + \alpha G_n]$ implies $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$. Observe that since the set of outcomes is (a subset of) $\Re_+$, monotonicity with respect to first-order stochastic dominance implies that preferences are increasing in $p$ and $q$.

The first step in the proof (Claim 1) shows that on a chord $[F, G]$, (WMS) together with (SE) implies (SMS). Claims 2 and 3 show that such chords exist and that lines cannot intersect indifference curves at more than two points. Claim 4 shows that (WMS) and (SE) imply either strict quasi-concavity or strict quasi-convexity globally. The theorem then follows by Theorem 4 of Chew, Epstein, and Segal (1991).

Claim 1. *Let $F \sim G$. If $\succeq$ satisfies (WMS) and (SE) on $[F, G]$, then for all $\alpha \in [0, 1]$, $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$.*

Proof. Assume (SP) on $[F, G]$ (the proof for (ST) is similar) and that $F \neq G$ (otherwise the claim is trivial). Let

$$\bar{\alpha} = \sup\left\{\alpha \in \left(0, \frac{1}{2}\right) : \alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G\right\}. \tag{1}$$

By continuity, $F_0 := \bar{\alpha}F + (1-\bar{\alpha})G \sim G_0 := (1-\bar{\alpha})F + \bar{\alpha}G$. If $\bar{\alpha} \neq \frac{1}{2}$, then by (WMS) there exists $0 < \tilde{\alpha} < \frac{1}{2}$ such that $\tilde{\alpha}F_0 + (1-\tilde{\alpha})G_0 \sim (1-\tilde{\alpha})F_0 + \tilde{\alpha}G_0$; hence, $\bar{\alpha} = \frac{1}{2}$.

Next we show that for all $\alpha \in (0, \frac{1}{2})$, $\frac{1}{2}F + \frac{1}{2}G \succ \alpha F + (1-\alpha)G$. Suppose not. Without loss of generality, there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)G \succeq \frac{1}{2}F + \frac{1}{2}G$, and since $\succeq$ is (SP) on $[F, G]$, there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)G \succ \frac{1}{2}F + \frac{1}{2}G$. It follows that $\alpha F + (1-\alpha)G$ is decreasing in $\alpha$ on $[\beta, 1]$ for some $\beta < \frac{1}{2}$, in contradiction to the above conclusion that $\bar{\alpha} = \frac{1}{2}$. It thus follows that $\succeq$ is increasing in $\alpha$ on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$.

Let $F_1 = \alpha_1 F + (1-\alpha_1)G$ for some $\alpha_1 \in (0, \frac{1}{2})$. By continuity and the last conclusion there is $\alpha' \in (0, \frac{1}{2})$ such that $F_1 \sim G_1 := (1-\alpha')F + \alpha' G$. Since $\succeq$ is (SP) on $[F_1, G_1]$, it follows as above that $\frac{1}{2}F_1 + \frac{1}{2}G_1 \succ \alpha F_1 + (1-\alpha)G_1$ for all $\alpha \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. But since $\frac{1}{2}F + \frac{1}{2}G \in [F_1, G_1]$, it must be the midpoint of this segment; hence, $\alpha' = \alpha_1$. □

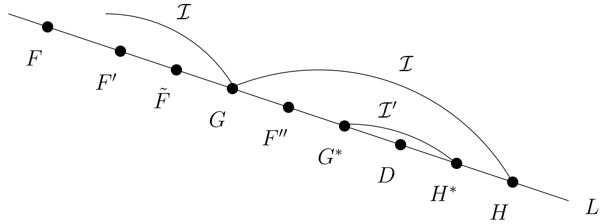


FIGURE 3. Distributions along $L$.

CLAIM 2. *Assume (SE). Let $\mathcal{I}$ be an indifference curve of $\succeq$ and let $L$ be a line. If $|\mathcal{I} \cap L| \geq 3$, then there exists an indifference curve $\mathcal{I}'$ and $F^*, G^*, H^* \in \mathcal{I}' \cap L$ such that $(F^*, G^*) \cap \mathcal{I}'$ and $(G^*, H^*) \cap \mathcal{I}'$ are empty.*

PROOF. By (NL) there exists $D \in L \backslash \mathcal{I}$ (see Figure 3). By the continuity of $\succeq$, there are $G, H \in L \cap \mathcal{I}$ such that $D \in (G, H)$ and $(G, H) \cap \mathcal{I} = \emptyset$ (see the first part of the proof of Theorem 1 above). Without loss of generality, $G \succ D$ and there is $F \in L \cap \mathcal{I}$ such that $G \in (F, H)$. If $(F, G) \cap \mathcal{I} = \emptyset$, we are through and the three desired points are $F, G, H$. Otherwise, there is in $(F, G) \cap \mathcal{I}$ a sequence $G_n \to G$. Without loss of generality, for all $n$, $G_{n+1} \in (G_n, G)$. Assume that there exists $F' \in (F, G)$ such that $G \succ F'$. If not, that is, if for all $F' \in (F, G)$, $F' \succeq G$, then start over by choosing $\bar{D}, \bar{G}, \bar{H} \in (F, G)$ such that $\bar{D} \in (\bar{G}, \bar{H})$, $\bar{G}, \bar{H} \in \mathcal{I}$, $\bar{D} \succ \bar{G}$, and there exists $F' \in (F, \bar{G})$ such that $F' \succ \bar{G}$. The proof then continues with the opposite preference signs.

Since all points in $(G, H)$ are inferior to $G$, as is $F'$, it follows by continuity that there is an indifference curve $\mathcal{I}'$, sufficiently close to $\mathcal{I}$, and three points $\tilde{F}, G^*, H^* \in \mathcal{I}'$ such that $[G^*, H^*] \subset (G, H)$ and $\tilde{F} \in (F, G)$. By (SE) on $[G, H]$, $(G^*, H^*) \cap \mathcal{I}' = \emptyset$ and for all $F'' \in [G, G^*)$, $G^* \succ F''$. Moreover, there is $F^* \in [\tilde{F}, G]$ such that $F^* \in \mathcal{I}'$ and $(F^*, G] \cap \mathcal{I}' = \emptyset$. Otherwise, there is a sequence $\tilde{F}_n \to G$ such that for all $n$, $\tilde{F}_{n+1} \in (\tilde{F}_n, G) \cap \mathcal{I}'$, a violation of continuity, as $G \notin \mathcal{I}'$. It follows that $F^* \sim G^* \sim H^*$, for all $D' \in (F^*, G^*)$, $D' \succ F^*$, and for all $D' \in (G^*, H^*)$, $G^* \succ D'$; hence, $F^*, G^*, H^*$ satisfy the requirements of the claim. □

CLAIM 3. *Assume (WMS) and let $G \in (F, H)$ be such that $F \sim G \sim H$. If $\succeq$ satisfies (SE) on $[F, G]$, then it does not satisfy (SE) on $[G, H]$.*

PROOF. Let $G = \alpha_0 F + (1 - \alpha_0)H$, and suppose wlg that $\alpha_0 \leq \frac{1}{2}$ and that $\succeq$ satisfies (ST) on $[G, H]$ (see Figure 4, where the indifference curve between $H$ and $G$ is depicted by the solid curve, and its two possible continuations to $F$ are depicted by the dashed lines).

Case 1: $\succeq$ satisfies (SP) on $[F, G]$. $F \sim H$; hence, by (WMS) there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1 - \alpha)H \sim (1 - \alpha)F + \alpha H$. For $\alpha \leq \alpha_0$, $(1 - \alpha)F + \alpha H \succ F \sim H \succ \alpha F + (1 - \alpha)H$; therefore, $\alpha_0 < \alpha < \frac{1}{2}$. In that case, both $\alpha F + (1 - \alpha)H$ and $(1 - \alpha)F + \alpha H$ are between $F$ and $G$. By Claim 1, $\beta F + (1 - \beta)G \sim (1 - \gamma)F + \gamma G$ if and only if $\beta = \gamma$. Let $\beta \in (0, 1)$ be such that $\alpha F + (1 - \alpha)H = \beta F + (1 - \beta)G$; hence,

$$\alpha F + (1 - \alpha)H = \beta F + (1 - \beta)\big[\alpha_0 F + (1 - \alpha_0)H\big]$$

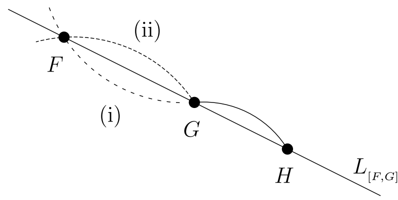


FIGURE 4. $G = \alpha_0 F + (1-\alpha_0)H$, $\alpha_0 \leq \frac{1}{2}$.

$$= \big[\beta + (1-\beta)\alpha_0\big]F + (1-\beta)(1-\alpha_0)H \quad \Longrightarrow$$

$$\beta = \frac{\alpha - \alpha_0}{1-\alpha_0}. \tag{2}$$

On the other hand, by (SP) on $[F, G]$, the only points in $[F, G]$ to be indifferent to $\alpha F + (1-\alpha)H$ and $\beta F + (1-\beta)G$ are $(1-\alpha)F + \alpha H$ and $(1-\beta)F + \beta G$, which must be the same. We get

$$(1-\alpha)F + \alpha H = (1-\beta)F + \beta\big[\alpha_0 F + (1-\alpha_0)H\big]$$

$$= \big[(1-\beta + \beta\alpha_0\big]F + \beta(1-\alpha_0)H \quad \Longrightarrow$$

$$\beta = \frac{\alpha}{1-\alpha_0}. \tag{3}$$

Equations (2) and (3) imply $\alpha_0 = 0$, a contradiction, as $G \neq H$.

Case 2: $\succeq$ satisfies (ST) on $[F, G]$. Here too, there is $\alpha < \frac{1}{2}$ such that $\alpha F + (1-\alpha)H \sim (1-\alpha)F + \alpha H$. If $\alpha > \alpha_0$, a contradiction is created as above. Otherwise, creating a sequence of points as in the proof of Claim 1, we eventually get to points in $[F, G]$ that are indifferent to each other but are not in symmetrical position on this segment, a contradiction to the assumption that $\succeq$ on $[F, G]$ is (ST). □

Claim 2 show that under (SE), if an indifference curve $\mathcal{I}$ intersects line $L$ in more than two points, then there is an indifference curve $\mathcal{I}'$ that intersects $L$ at three points but not between them. Claim 3 shows that under (WMS), such $\mathcal{I}'$ does not exist.

CONCLUSION 1. *Let $\succeq$ satisfy (SE) and (WMS), and let $\mathcal{I}$ be an indifference curve of $\succeq$. Then for any line $L$, $|\mathcal{I} \cap L| \leq 2$.*

CLAIM 4. *If $\succeq$ satisfies (WMS) and (SE), then it satisfies either (SQC) or (SQX).*

PROOF. Suppose that there are two indifference curves $\mathcal{I}$ and $\mathcal{I}'$ with $F, G \in \mathcal{I}$ and $F', G' \in \mathcal{I}'$ such that $\succeq$ is (SP) on $[F, G]$ and (ST) on $[F', G']$. By continuity, $\mathcal{I}$ and $\mathcal{I}'$ can be assumed to be different indifference curves. Also by continuity, if such points exist, then we can find such pairs that are not all on the same line. Therefore, we can assume wlg that $[F, F'] \cap [G, G'] = \emptyset$; otherwise $[F, G'] \cap [G, F'] = \emptyset$ and the roles of $G$ and $G'$ are reversed. By assumption, $\frac{1}{2}F + \frac{1}{2}G \succ F \sim G$ while $F' \sim G' \succ \frac{1}{2}F' + \frac{1}{2}G'$. By continuity, for every $\alpha \in (0,1)$ there exist $\beta_\alpha \in (0,1)$ such that $\alpha F + (1-\alpha)F' \sim \beta_\alpha G + (1-\beta_\alpha)G'$. By continuity, there is $\alpha$ such that

$$\begin{aligned}\alpha F + (1-\alpha)F' &\sim \beta_\alpha G + (1-\beta_\alpha)G' \\ &\sim \frac{1}{2}\big(\alpha F + (1-\alpha)F'\big) + \frac{1}{2}\big(\beta_\alpha G + (1-\beta_\alpha)G'\big),\end{aligned}$$

contradicting Conclusion 1 that a line can intersect an indifference curve at no more than two points. □

Proof of Theorem 3. Obviously, (SMS) implies (WMS) and since we assume (SE), by Claim 1, (WMS) implies (SMS). By Claim 4, $\succeq$ is either (SQC) or (SQX). By Chew, Epstein, and Segal (1991, Theorem 4), if $\succeq$ is either quasi-concave or quasi-convex, then it can be represented by a quadratic function if and only if it satisfies (SMS). □

## 4. Continuity and monotonicity

The proofs of Theorems 1–3 used continuity. These theorems do not hold without this assumption. Consider the following preferences over $\mathcal{F}$.

Example 1. $F \succ G$ if either $E[F] > E[G]$ or if $E[F] = E[G]$ and $F(0) \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ while $G(0) \in [\frac{1}{3}, \frac{2}{3}]$. If $E[F] = E[G]$ and either $F(0), G(0) \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ or $F(0), G(0) \in [\frac{1}{3}, \frac{2}{3}]$, then $F \sim G$. ◊

These preferences are monotonic with respect to first-order stochastic dominance. They satisfy (WB), (WI), and (WMS), but not (B), (I), or (SMS). Let $F$ and $G$ be such that $E[F] = E[G]$, $F(0) = \frac{1}{4}$, and $G(0) = 1$. Then $F \sim G \sim \frac{1}{3}F + \frac{2}{3}G \succ \frac{2}{3}F + \frac{1}{3}G$ implies violations of (B) and (SMS), and for $H = F$, we get $\frac{1}{2}F + \frac{1}{2}H \succ \frac{1}{2}G + \frac{1}{2}H$, a violation of (I).

For (WB) and (WMS), the only nontrivial case is where $F(0) \in [0, \frac{1}{3})$ and $G(0) \in (\frac{2}{3}, 1]$, and a sufficiently small $\alpha > 0$ will obtain the desired properties of the mixtures. Regarding (WI), let $F \sim G$. Then $E[F] = E[G]$, and for all $H$ and $\alpha$, $E[\alpha F + (1-\alpha)H] = E[\alpha G + (1-\alpha)H]$. If $F(0), G(0) \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$, then so are $\alpha F(0) + (1-\alpha)H(0)$ and $\alpha G(0) + (1-\alpha)H(0)$ for sufficiently high $\alpha < 1$. Suppose, on the other hand, that $F(0), G(0) \in [\frac{1}{3}, \frac{2}{3}]$. If $H(0) \in [\frac{1}{3}, \frac{2}{3}]$, then so are $\alpha F(0) + (1-\alpha)H(0)$ and $\alpha G(0) + (1-\alpha)H(0)$ for all $\alpha$, and if $H(0) \in [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$, then so are $\alpha F(0) + (1-\alpha)H(0)$ and $\alpha G(0) + (1-\alpha)H(0)$ for sufficiently small $\alpha > 0$. □

Strict monotonicity is essential in the proof of Theorem 2. Consider the preferences $\succeq$ that are represented by

$$V(F) = \begin{cases} E[F], & E[F] \leq 1, \\ 1, & E[F] > 1. \end{cases}$$

These preferences satisfy (WB), but not (B). For example, $\delta_2 \sim (2, \frac{1}{2}; 0, \frac{1}{2}) \succ \delta_0$.

The analysis of this paper assumed that lotteries are defined over the real line, but it can be easily extended to more general domains. Let $\mathcal{X}$ be a set of outcomes with a preference relation $\succsim$ over it. The preferences $\succeq$ over lotteries are monotonic with respect to $\succsim$ if for every two lotteries $X$ and $Y$, [for all $x \in \mathcal{X}$, $\Pr(x \succsim X) \leq \Pr(x \succsim Y)$] implies $X \succsim Y$.

## 5. Concluding remarks

### 5.1 *Refutable axioms*

Axioms and theories can be divided into those that can be refuted by experiments and those that cannot. Transitivity states that if $F \succ G$ and $G \succ H$, then $F \succ H$. One set of observations where $F \succ G$, $G \succ H$, and $H \succ F$ will prove a violation; so is expected utility theory or axiom (I) (see discussion in the Introduction). Chambers, Echenique, and Shmaya (2014) formalize this distinction. Refutable axioms are written as

$$\forall \nu_1 \dots \forall \nu_n \neg(\phi_1 \wedge \cdots \wedge \phi_m), \tag{4}$$

where $\phi_1, \dots, \phi_m$ are atomic formulas with variables from $\nu_1, \dots, \nu_n$, and each of them represents a direct observation (see Chambers, Echenique, and Shmaya (2014, Definition 4) for details). For example, the independence axiom (I) can be written as

$$\forall F\, \forall G\, \forall H\, \forall \alpha \neg\big([F \succ G] \wedge \big[\alpha G + (1-\alpha)H \succeq \alpha F + (1-\alpha)H\big]\big).$$

Axiom (WI), on the other hand, cannot be represented as a finite set of atomic formulas. Let $F \sim G$. Any finite set of values of $\alpha$ for which $\alpha F + (1-\alpha)H \nsim \alpha G + (1-\alpha)H$ is consistent with (WI). Similarly, (B) and (SMS) can be refuted, yet (WB) and (WMS) cannot. Continuity too cannot be refuted. A statement of the form "if $F \succ G$ and $F'$ is sufficiently close to $F$, then $F' \succ G$" does not specify what "close enough" is. Therefore, no observation $G \succeq F'$ can violate this axiom.

Chew, Epstein, and Segal (1991) included quasi-concavity and quasi-convexity in their axiomatizations. These assumptions can be written as in (4). For example, quasi-concavity states

$$\forall F\, \forall G\, \forall \alpha \neg\big([F \succeq G] \wedge \big[G \succ \alpha F + (1-\alpha)G\big]\big).$$

In contrast, axiom (SE) cannot be refuted by empirical observations, as the restriction to $F \sim G$ such that there is no $\alpha \in (0, 1)$ for which $F \sim \alpha F + (1-\alpha)G$ can tolerate any finite number of values of $\alpha_1, \dots, \alpha_m$ for which $F \sim G \sim \alpha_i F + (1-\alpha_i)G$, without imposing restrictions on the preferences along the intermediate segments.

This paper proves that given monotonicity, the combination of continuity with the weaker versions of the mixture axioms used in the expected utility, betweenness, and quadratic utility theories are sufficient to imply these theories, but none of these axioms can be violated by experiments. It indicates that the combined power of basic rules may be much stronger than each of the rules, because the theories themselves can easily be refuted.

The above axioms are claimed to have a normative appeal. For example, if $F \sim G$, then the decision maker should not care whether he receives $F$ or $G$; hence, $F \sim \frac{1}{2}F +$

$\frac{1}{2}G \sim G$. But this motivation relies on the assumption that decision makers identify $(F, \frac{1}{2}; G, \frac{1}{2})$ with $\frac{1}{2}F + \frac{1}{2}G$, which they probably do not (see Segal (1990) and Starmer (2000)). Such a distinction requires of course a different domain of preferences.

### 5.2 $\exists\,\forall$ *versus* $\forall\,\exists$

Following a suggestion by Debreu, the following weaker version of (I) was offered by Herstein and Milnor (1953).

(HM-I). For all $F, G, H \in \mathcal{F}$, if $F \sim G$, then $\frac{1}{2}F + \frac{1}{2}H \sim \frac{1}{2}G + \frac{1}{2}H$.

Obviously, $\alpha = \frac{1}{2}$ in this axiom can be replaced with any given value of $\alpha$, and, moreover, the given value of $\alpha$ does not need to be known. That is, instead of (HM-I), we can have the following version.

(WHM-I). There exists $\alpha^* \in (0, 1)$ such that for all $F, G, H \in \mathcal{F}$, if $F \sim G$, then $\alpha^* F + (1 - \alpha^*)H \sim \alpha^* G + (1 - \alpha^*)H$.

Both (I) and (HM-I) are refutable, but (WHM-I), like (WI), cannot be refuted by any set of finite observations.[6] Of course, since the preferences of Example 1 do not satisfy (WHM-I), this axiom is mathematically stronger than (WI). Moreover, having $F = H$ or $G = H$ in (WHM-I) implies that for $F \sim G$ there is a dense set of points in $[F, G]$ that are all indifferent to $F$ and $G$. Violations of betweenness under such circumstances are mathematically possible, but seem to be behaviorally implausible. But there is a deeper conceptual difference between these two axioms. Consider the two parallel versions of betweenness for strict preferences: There exists $\alpha^*$ such that for all $F \succ G$, $F \succ \alpha^* + (1 - \alpha^*)G \succ G$ versus for all $F \succ G$ there exists $\alpha$ such that $F \succ \alpha F + (1 - \alpha)G \succ G$. Together with continuity, the former implies betweenness. The latter axiom follows by continuity and, therefore, imposes no further restrictions on $\succeq$.

The $\exists\,\forall$ version of the axioms demands some regularity of the preferences. If $\alpha^* \neq \frac{1}{2}$, then since the requirement $\exists \alpha^* \in (0, 1)$ such that if $F \sim G$, then $F \sim \alpha^* F + (1 - \alpha^*)G \sim G$ is symmetric in $F$ and $G$, it follows that $F \sim (1 - \alpha^*)F + \alpha^* G \sim G$ as well. On the other hand, (WB) does not imply any symmetry. It may well happen that $F \sim \frac{1}{3}F + \frac{2}{3}G \sim G$ yet $F \nsim \frac{2}{3}F + \frac{1}{3}G \nsim G$.

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[6] There are cases where the order of the quantifiers determines whether a theory is falsifiable or not. Mongin (1986) in his discussion of Simon (1985) uses the following example: $(\exists k)(\forall i)(y_i = kx_i)$ says that $y$ is a linear function of $x$ and is falsifiable. But $(\forall i)(\exists k)(y_i = kx_i)$ is of course true and, therefore, not falsifiable.

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