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On rank dominance of tie-breaking rules

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Lotteries are a common way to resolve ties in assignment mechanisms that ration resources. We consider a model with a continuum of agents and a finite set of resources with heterogeneous qualities, where the agents' preferences are generated from a multinomial-logit (MNL) model based on the resource qualities. We show that all agents prefer a common lottery to independent lotteries at each resource if every resource is *popular*, meaning that the mass of agents ranking that resource as their first choice exceeds its capacity. We then prove a stronger result where the assumption that every resource is popular is not required and agents' preferences are drawn from a (more general) nested MNL model. By appropriately adapting the notion of popularity to resource types, we show that a hybrid tie-breaking rule in which the objects in each popular type share a common lottery dominates independent lotteries at each resource.

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1. INTRODUCTION

Lotteries are often used to allocate scarce resources without monetary transfers. How lotteries are conducted naturally affects distributional outcomes. This problem arises, for example, in the assignment of students to schools, when ties must be resolved in overdemanded schools (Erdil and Ergin (2008a)) or in on-campus housing where students list residences or categories of housing to which they wish to be assigned to.[1]

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[1]For example, lottery numbers are used to determine the housing assignments on Stanford and Columbia campuses. (See https://rde.stanford.edu/studenthousing/assignment-rounds#lottery and https://housing.columbia.edu/content/point-values-lottery-numbers-selection-appointments.)

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A common way to allocate resources applies the Deferred Acceptance (DA) algorithm of Gale and Shapley (1962) after resolving ties using lottery numbers. This approach is strategy-proof and creates assignments without justified envy[2] and, in particular, provides a transparent reference for explaining the (potentially) unequal assignments of equals.

We consider allocation problems where a set of *resources* is allocated to unit-demand agents. In these problems, there are two common types of lotteries used along with DA: the *single tie-breaking* (STB) and the *multiple tie-breaking* (MTB) rules. The STB rule assigns each agent a single random number, which is used to break the ties between agents at every resource, whereas the MTB rule assigns each agent a different, independently drawn, random number at each resource. These lotteries naturally result in different assignments. This paper uncovers distributional properties of agents' ranks in stable assignments under these lotteries when resources have heterogeneous qualities and agents have random multinomial-logit-based preferences.

Previous empirical (Abdulkadiroğlu, Pathak, and Roth (2009), De Haan, Gautier, Oosterbeek, and Van der Klaauw (2015)) and theoretical (Arnosti (2015), Ashlagi, Nikzad, and Romm (2019)) studies in the setting of school choice find that a single lottery assigns more students to their top-ranked choices, but also more students to lower-ranked choices. Ashlagi and Nikzad (2016) further identify that in random markets with short supply, this trade-off vanishes and STB is in fact preferable to MTB in the sense of "approximate" first-order stochastic dominance. Their stylized model assumes that students' preference lists are generated independently and uniformly at random and every school has one seat.

This paper considers a general model for rationing resources using lotteries, where resources have heterogeneous qualities and capacities and agents have a rich model of random preferences that takes into account resource qualities. Note that if resources have identical qualities, preferences are generated uniformly at random as in Ashlagi and Nikzad (2016). We are interested in comparing agents' rank distributions at each resource (and not just in the entire market as done in previous studies).

In the setup, we consider there are n resources and a continuum of agents (Abdulkadiroğlu, Che, and Yasuda (2015), Azevedo and Leshno (2016)). Each resource j has a fixed quality $\mu_j > 0$ and capacity $q_j > 0$. Agents have complete and strict preference orders over resources, drawn independently from a multinomial-logit (MNL) model induced by the resource qualities. That is, an agent's top choice is resource j with probability $\frac{\mu_j}{\sum_{k=1}^{n} \mu_k}$; more generally, the probability that resource j is an agent's most preferred resource from a subset of resources $S \ni j$ equals $\frac{\mu_j}{\sum_{k \in S} \mu_k}$.[3] It is assumed (for simplicity and tractability) that resources are indifferent between all agents, so priorities of agents

[2]No justified envy means that no individual prefers another assignment over her assignment and has a higher priority than someone else assigned to the preferred assignment.

[3]Such preferences were also used in Kojima and Pathak (2009) and also referred to as popularity-based preferences, for example, in Gimbert, Mathieu, and Mauras (2019). Moreover MNL preferences are often used to estimate demand in school choice problems (e.g., Shi (2015), Agarwal and Somaini (2018), Pathak and Shi (2013)).

at each resource are solely determined by lotteries. We compare agent-optimal stable matchings under the STB and MTB lotteries.

A key notion in the results is related to the demand for a resource. A resource is *popular* if the mass of agents who rank it as their first choice exceeds its capacity. For example, when all resources have the same capacity and also the same quality, then any resource is popular if and only if the mass of agents is larger than the sum of the capacities of the resources. This is an adaptation of the notion of popularity for uniformly generated preferences from Ashlagi and Nikzad (2016) to the continuum setup here with MNL preferences.[4] We first consider the case where all resources are popular, and find that every agent prefers STB to MTB in the sense of first-order stochastic dominance (Theorem 1, Part 1). Moreover, every resource admits agents that rank it higher under STB than under MTB in a first-order stochastic dominance sense (Theorem 1, Part 2).

When the market also includes nonpopular resources, the first part of the theorem no longer holds, which is consistent with empirical evidence. However, we show that, when the agents' preferences are (more generally) drawn from a nested MNL model,[5] a *hybrid tie-breaking* rule that uses the same lottery number within each popular resource *type* dominates MTB in the sense of first-order stochastic dominance. This leads to a strengthening of the theorem applicable to settings where there is excess supply, as detailed next.

In the more general setup with the nested MNL model, the set of resources is partitioned to different *types*, and the resources of the same type are ranked consecutively by every agent (but not necessarily in the same order). An agent's ordering over the resource types is drawn from an arbitrary distribution, and her ordering over the resources within each type is drawn from a type-specific MNL model. Under an appropriate adaptation of the notion of popularity to resource types, in Proposition 1 we show that a hybrid tie-breaking rule, in which the resources in each popular type share the same lottery number, dominates MTB. That is, every agent prefers a hybrid tie-breaking rule to MTB, and every resource is assigned to agents that rank it higher, in the sense of first-order stochastic dominance (as in the first and second parts of Theorem 1).

The adapted notion defines popular resource types as follows. The *demand* for a resource type is the set of agents who are assigned to that type or prefer a resource of that type to their assignment, under MTB.[6] A type t is popular if, for every resource of type t, the mass of agents who demand t and rank the resource first among the resources of type t is no less than the capacity of the resource. For example, when all resources of

[4]In the setup of Ashlagi and Nikzad (2016), there are a finite number of schools and students, and a student's preference order is drawn independently and uniformly at random from the set of all strict orderings of schools. There, when the number of schools is less than the number of students, the expected number of students that rank a fixed school as their first choice is larger than one, the capacity of the school. In our continuum model, this expectation equals the realized value, and a school is popular when the realized value exceeds the school's capacity.

[5]So, an agent's preferences can be interpreted as if there are two levels of choice, with the first level being the choice of the category (or type) and the second level the choice within the category.

[6]We show that the notion of demand is in fact invariant to the choice of the tie-breaking rules that Proposition 1 concerns with.

a given type have the same capacity and quality, that type is popular if the demand for it is more than the sum of the capacities of its resources.

A rough intuition for Theorem 1 is that when resources are sufficiently popular, a coordinated lottery essentially determines which agents will be assigned, and among these agents the allocation is efficient. A separate lottery for each resource results in inefficiencies among assigned agents who may wish to trade their assignments. For further intuition consider the following simple example. Consider a market with two resources that have qualities $\mu_1 \geq \mu_2$. In the execution of the DA, at each round every unassigned agent applies to her favorite choice to which she has not yet applied. Observe that, under STB, an agent who is rejected from resource 2 cannot be admitted to resource 1, because her lottery number did not suffice to allow her to be admitted to a resource in lower demand. This implies that, under STB, agents assigned to the first resource must rank it as their first choice. Moreover, under both lotteries, the same mass of agents whose first choice is resource 1 are rejected from that resource after all agents apply to their first choice. But more of these agents will be admitted to their second choice under MTB than under STB, because under MTB they receive a new lottery number for resource 2. DA will terminate after 2 rounds for STB, but for MTB the process will continue, with each round assigning more agents to resources that they rank second. Observe that the outcome satisfies the results in Theorem 1.[7]

A direct analysis is already challenging with only three resources, and more subtle arguments are required that build on the cutoff characterization of stable matchings. Under a single lottery and MNL preferences, the outcome has a simple structure and cutoffs have a closed form (Ashlagi and Shi (2014)). Under MTB, however, the cutoffs do not have a closed-form expression. We compare the two tie-breaking rules by developing bounds on the distribution of agents' ranks.

This paper contributes to the analysis of rank distributions in matching markets where agents' preferences are generated from a rich and empirically relevant model (Agarwal and Somaini (2018)). Techniques and intermediate results, which establish properties about the cutoff structures, may be of independent interest.

1.1 *Related literature*

Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu, Pathak, and Roth (2009), Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) apply matching theory to develop strategy-proof mechanisms for school choice. Policy decisions surrounding school choice also involve resolving tie-breaking (Erdil and Ergin (2008b), Abdulkadiroğlu, Pathak, and Roth (2009), Feigenbaum, Kanoria, Lo, and Sethuraman (2020)), design of menus and priorities (Ashlagi and Shi (2016), Dur, Duke Kominers, Pathak, and Sönmez (2013), Shi (2021)) and diversity-related constraints (Ehlers, Hafalir, Yenmez, and Yildirim (2014), Echenique and Bumin Yenmez (2015), Duke Kominers and Sönmez (2016)).

Several papers analyze trade-offs between STB and MTB in addition to those discussed above. Arnosti (2019) compares single and multiple lotteries in a model in which

[7]In fact, in this simple example, the result holds true even if the second resource is not popular.

there is a continuum of schools, each of which has capacity for a finite number of students. The paper analyzes the effect of students having preference lists of varying length, and establishes a single crossing property between the cumulative rank distribution of students under STB and MTB (see also Ashlagi, Nikzad, and Romm (2019), which explains why STB assigns more students to their top choices in a model with random preferences).[8] Additionally, he shows that among students who submit short lists, the rank distribution under a single lottery stochastically dominates the corresponding distribution under independent lotteries. Our paper assumes a rich preference model over qualities but distinguishes between popular and nonpopular schools to explain the source of these trade-offs.

Shi (2021) optimizes over the space of all priority-based mechanisms and finds that a single lottery maximizes the total utility of students when the utilities follow an MNL model. Our paper in contrast looks at the rank distributions under common and independent lotteries and identifies when these distributions exhibit a rank dominance relation.

Arnosti and Shi (2020) compare common and multiple lotteries in a dynamic model where agents have heterogeneous values for distinct items and heterogeneous outside options. They show that using independent lotteries for each item is equivalent to using a waitlist in which agents lose priority when they reject an offer, and that using a common lottery for each item improves the quality of matches.

2. Model

We study a large matching model based on the framework in Azevedo and Leshno (2016). There is a finite set of *resources* $S = \{1, \dots, n\}$ and a continuum of *agents* with mass N. Each agent demands to be allocated a resource, and each resource $j \in S$ has capacity $q_j > 0$, meaning that it can be allocated to a mass of at most q_j agents. Each agent has a strict preference ranking, which is a linear order over all resources. Let Π_n be the set of all permutations of n elements. A *matching market* is given by $C = (m, q, N)$ where m is a probability measure over Π_n and $q = (q_1, \dots, q_n)$ is the vector of the capacities of the resources.

Tie breakers For tractability, we assume that resources do not have any exogenous priorities over the agents; rather, their priorities are solely determined by lotteries. Each agent i is assigned a vector of lottery numbers $L^i \in [0, 1]^n$, where L^i_j is agent i's lottery number at resource j. Each resource j is assumed to prefer agents in decreasing order of their lottery number at j. To generate lottery numbers for agents, the following definition will be helpful.

Definition 1. A *tie-breaking rule* is a probability measure ν defined on $[0, 1]^n$ where each marginal of ν is nonatomic.[9]

[8] Abdulkadiroğlu, Che, and Yasuda (2015) analyze the cutoffs that clear the market in a continuum model and establish that STB is ordinally efficient (see also Che and Kojima (2010), Liu and Pycia (2012), and Ashlagi and Shi (2014)).

[9] where ν is defined on the Lebesgue σ-algebra on $[0, 1]^n$

Requiring that the marginals of a tie-breaking rule are nonatomic ensures that each resource has strict preferences over agents. A *tie-broken market* is given by $E = (C, \nu)$, where C is a matching market and ν is a tie-breaking rule.

Two commonly applied tie-breaking rules are studied, Single Tie-Breaking (STB) and Multiple Tie-Breaking (MTB). Under STB each agent receives the same lottery number for all resources, uniformly distributed on $[0, 1]$. So, STB is the uniform measure on the line $\{(x, x, \dots, x) : x \in [0, 1]\}$. Under MTB each agent receives a lottery number independently for each resource, where each number is chosen uniformly on $[0, 1]$. So, MTB is the uniform measure on $[0, 1]^n$.

Matching, stability, and cutoffs Consider a tie-broken market $E = (m, q, N, \nu)$ with tie-breaking ν. Let $\Lambda = \Pi_n \times [0, 1]^n$ be the set of all pairs of agent preferences and lottery numbers. A *matching* is a function $f : S \cup \Lambda \to 2^\Lambda \cup S \cup \{\emptyset\}$ such that:

i. For all $i \in \Lambda$, $f(i) \in S \cup \{\emptyset\}$.

ii. For all $j \in S$, $f(j) \subseteq \Lambda$ is $(m \times \nu)$-measurable and $(m \times \nu)(f(j)) \leq q_j/N$, where $(m \times \nu)$ is the product measure between m and ν.

iii. For all $i \in \Lambda$ and $j \in S$, $j = f(i)$ if and only if $i \in f(j)$.

iv. For any sequence $i^k = (P^k, L^k) \in \Lambda$ and $i = (P, L) \in \Lambda$, with L^k converging to L and weakly decreasing with k in each component, there is some K such that $f(i^k) = f(s)$ for all $k > K$.

The first condition ensures that an agent is assigned either to a resource (and thus matched) or to the empty set (remaining unmatched). The second condition ensures that the mass of agents assigned to each resource does not exceed its capacity. The third condition ensures that if an agent is assigned to a resource, the resource is matched to the agent. The fourth technical condition eliminates multiplicities of matchings that differ by a set of measure 0 (Azevedo and Leshno (2016)).

A matching f is *stable* if there is no agent i and resource j such that i strictly prefers j to $f(i)$, and either there is some $i' \in f(j)$ such that $L^s_j > L^{i'}_j$ or j has excess capacity. Azevedo and Leshno (2016) show that every stable matching corresponds to a set of cutoffs $c = (c_1, \dots, c_n) \in [0, 1]^n$, where every agent i is matched to her most preferred resource j for which her lottery number exceeds the cutoff ($L^i_j \geq c_j$). Furthermore, they show that there exists a unique *agent-optimal* stable match, wherein each agent is matched to their most preferred resource that they can be matched to in any stable match.[10]

Denote by f^ν the agent-optimal stable matching for the tie-broken market. So, f^{STB} and f^{MTB} denote that agent-optimal matching when the tie-breaking rules are STB and MTB, respectively. When it is clear from the context we denote by $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, the cutoffs under the matchings f^{STB} and f^{MTB}, respectively.

[10]Azevedo and Leshno (2016) also show that under some regularity conditions the stable matching is unique.

The STB cutoffs can be calculated in closed form (Ashlagi and Shi (2014)). This is not the case for the MTB cutoffs, but these can be computed through an iterative algorithm, which progressively increases the cutoffs to clear the market and converges to β.

Ranks and dominance For a given matching, the *rank* of an agent is the position on the agent's preference list of the resource to which the agent is assigned. For example, if an agent is matched to her second choice, then her rank is two.

Consider a matching market and a tie-breaking rule $\nu \in \{\text{STB}, \text{MTB}\}$. Denote by R_j^ν the distribution of agent ranks at resource j in the stable matching f^ν. That is, R^ν is the n-dimensional vector in which its kth element is the fraction of agents assigned to resource j with rank k in f^ν.

For a preference order $P \in \Pi_n$, let R_P^ν denote the distribution of ranks of agents with preference P who are matched in f^ν. That is, R_P^ν is the n-dimensional vector in which its kth element is the fraction of agents with preference order P who are assigned their kth-ranked resource in f^ν.

Observe that for any $j \in S$ and $P \in \Pi_n$ the vectors R_j^ν and R_P^ν are *stochastic vectors*, that is, vectors with nonnegative entries that sum to 1.

Definition 2. If $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ are stochastic vectors with equal length n, v *rank-dominates* w, indicated by $v \succeq w$ if for all $k \in \{1, \dots, n\}$,

$$\sum_{j=1}^{k} v_j \geq \sum_{j=1}^{k} w_j.$$

Observe that rank dominance is equivalent to first-order stochastic dominance but with the order of the two vectors reversed. This definition is adopted for clarity of exposition, since in this setting lower agent ranks are preferable to higher ranks in terms of welfare.

Definition 3. *STB dominates MTB for agents with preference order* $P \in \Pi_n$ if

$$R_P^{\text{STB}} \succeq R_P^{\text{MTB}}.$$

Definition 4. *STB dominates MTB at resource* j if

$$R_j^{\text{STB}} \succeq R_j^{\text{MTB}}.$$

3. Main results

We first consider a setting where there is excess demand and the agents' preferences are drawn from a multinomial-logit (MNL) model. We then extend this setting to one with a nested MNL preference model for the agents, where there can be an excess of resources.

3.1 *MNL preferences*

We suppose that each resource j has *quality* $\mu_j > 0$. Informally, an agent's first choice (most preferred resource) is drawn independently from the multinomial distribution with a weight for each resource that equals its quality. The agent's second choice is then drawn similarly from all remaining resources, and so forth.

Definition 5. A matching market $C = (m, q, N)$ has *MNL preferences* with resource qualities $\mu_1, \dots, \mu_n$ if for any preference order $P = (r_1, \dots, r_n) \in \Pi_n$,

$$m(P) = \prod_{k=1}^{n} \frac{\mu_{r_k}}{\sum_{j=k}^{n} \mu_{r_j}}.$$

When clear from the context, we sometimes say that agents have MNL preferences.

We note that this definition is just multiplying the chance that r_1 is the agent's first choice by the chance that r_2 is the agent's second choice, and so on. Throughout the paper, when a matching market has MNL preferences, we assume that resources are indexed such that

$$\frac{\mu_1}{q_1} \geq \cdots \geq \frac{\mu_n}{q_n}.$$

We also assume without loss of generality that the qualities sum to one.

Definition 6. A resource j is *popular* if the mass of agents who rank it as their first choice is at least the capacity of j.

Note that under MNL preferences, resource $j \in S$ is popular if and only if $N\mu_j \geq q_j$.

Theorem 1. *Consider a matching market that has MNL preferences and every resource is popular. Then:*

i. STB dominates MTB for agents with any preferences.

ii. STB dominates MTB at every resource.

The assumption that all resources are popular can be natural in some applications such as housing allocation, but unrealistic in other applications such as school choice. In Section 3.2, we relax this limitation by considering a more general model with multiple resource types and adapt the notion of popularity to resource types.

The intuition for the dominance of STB over MTB when there are only two resources is relatively simple. Under STB, agents who first apply to resource 2 but are rejected remain unassigned, whereas under MTB these agents have a chance to apply to resource 1 and displace agents who otherwise would have been assigned there. So, agents are more likely to receive their second choice resource under MTB, and thus are less likely to receive their first choice. In fact, Theorem 1 applies when there are only two resources even if the resources are not popular.

When there are at least three resources, this intuition breaks down, and even proving the theorem for when there are three resources is nontrivial. We were unable to find a simple or inductive proof of the theorem, as adding an additional resource to a matching market has a complicated effect on the resources and the agents of different types.

In Appendix B, we demonstrate the necessity of the conditions in Theorem 1 through Example 1, which shows that STB may not dominate MTB at resources that are nonpopular.[11] In general, it is possible that neither STB nor MTB would dominate the other at a nonpopular resource. Moreover, when there are both popular and nonpopular resources, the theorem does not imply that STB dominates MTB in every popular resource.

3.2 *Nested MNL preferences*

We now generalize the MNL preference model of Section 3.1 to a nested preference model in which resources are of multiple types. There are $\bar{t}$ resource types, with $T = \{1, \dots, \bar{t}\}$ denoting the set of all types. Each resource $j \in \{1, \dots, n\}$ has a type $t_j \in T$. Denote by n_t the number of resources of type t. Every agent has a complete strict preference order over resource types. The agent prefers a resource of type t to a resource of type t' if she prefers the resource type t to t'. Thus, if $P \in \Pi_n$ is the preference order of an agent over resources, then resources of the same type are ranked consecutively on P.

As in Section 3.1, a resource j has a quality μ_j. An agent's preference order over the resources of the same type are defined by a type-specific MNL model, as formalized next. This model first draws a preference order over the resource types for an agent from an arbitrary distribution $\bar{m}$, and then ranks the resources within each type independently, according to a MNL model based on the corresponding qualities.

Definition 7. A matching market $C = (m, q, N)$ with resource types $1, \dots, \bar{t}$ specified as above and resource qualities $\mu_1, \dots, \mu_n$ has *nested MNL preferences over the resource types* if it satisfies the following conditions. For all preference orders $P \in \Pi_n$, if P does not rank all resources of the same type consecutively, then $m(P) = 0$. Otherwise, let $\bar{m}(P)$ denote the probability that an agent ranks the resource types in the order that they are ranked in P. For each $t \in T$, let $m_t(P)$ denote the probability that an agent with MNL preferences would rank the resources of type t in the order that they are ranked in P, as in Definition 5. Then

$$m(P) = (\bar{m} \times m_1 \times \cdots \times m_{\bar{t}})(P).$$

Without loss of generality, we will assume that the qualities of the resources of the same type sum to one. We will extend Theorem 1 to this multitype setting, proving that when the resources of a given type are at sufficiently high demand, then using a common lottery for resources of that type (HTB) dominates MTB. First, we give some preliminaries formalizing what we mean by sufficiently high demand.

[11] In the example, MTB dominates STB at a nonpopular resource, even though that resource is fully allocated by the end of DA.

Definition 8. For a given matching market and tie-breaking rule, the *demand* for a subset R of the resources is the mass of agents in the agent-optimal stable assignment who are either assigned to a resource in R or who prefer a resource in R to their assignment. When R is the set of resources of the same type t, we use D_t to denote the demand for R.

Definition 9. In a given matching market with nested MNL preferences and tie-breaking rule ν, a resource type t is *popular* if for every resource j of that type,

$$\mu_j D_t \geq q_j,$$

where D_t is the demand for the set of resources of type t.

The left-hand side of the above inequality $\mu_j D_t$ is the mass of agents who demand type t and rank j first among resources of type t. To see why, let S_t denote the set of resources of type t. As the agents' preferences over the resources of type t are drawn from a MNL model, then $\frac{\mu_j}{\sum_{j' \in S_t} \mu_{j'}} D_t$ equals the mass of agents who demand type t and rank j first among resources of type t. Recall that we normalized the qualities of the resources of the same type so that they sum to one. Thus, $\mu_j D_t$ is the mass of agents who demand type t and rank j first among resources of type t. The above inequality asserts that this mass is at least the capacity of j. Notably, if all resources have the same type, then that type is popular if and only if all of the resources are popular in the sense of Definition 6.

We next describe a natural class of hybrid tie-breaking rules that interpolate between STB and MTB, and then show a dominance relation between them.

Definition 10. For a subset of resource types $X \subseteq T$, we denote by $\mathrm{HTB}(X)$ the hybrid tie-breaking rule that (i) for each $t \in X$, assigns to each agent the same independently drawn lottery number at all resources of type t, and (ii) at each resource with type not belonging to X assigns to each agent an independently drawn lottery number.[12]

We note that, according to this definition, an agent is assigned the same lottery number at two resources if they belong to the same popular resource type, but she is assigned independently drawn lottery numbers at resources that belong to different popular types. The next lemma shows that the set of popular resource types under $\mathrm{HTB}(X)$ is invariant to the choice of X, as the demand for a resource type does not depend on X.

Lemma 1. *For a matching market with set T of resource types and nested MNL preferences over T, for every $T_1, T_2 \subseteq T$, a resource type is popular under* $\mathrm{HTB}(T_1)$ *if and only if it is popular under* $\mathrm{HTB}(T_2)$.

[12]As in the previous tie-breaking rules, every lottery number is distributed uniformly over the unit interval.

Proof. Fix a subset of types $X \subseteq T$ and consider the assignment of agents under HTB(X). For resource type t, let Q_t be the sum of the capacities of resources of type t. Conditional on an agent demanding resource type t, regardless of her preferences she has the same probability of being assigned to a resource of type t. This holds because the tie-breakers for resources of type t are independent of the tie-breakers for the other resources. Call this probability p_t. See that then each agent has an independent probability p_t of having at least one resource of type t available to her, and each agent will be assigned to a resource of her most preferred type among the resource types for which at least one resource is available to her. Now, consider the assignment of the agents to resource *types*, where the capacity of each type is Q_t and types have independent random preferences over the agents. In this assignment, the cutoffs of the types as defined in Azevedo and Leshno (2016) must be unique. The values p_t must satisfy the same set of "market clearing" conditions as these cutoffs, namely that the expected mass of agents assigned to any type t does not exceed Q_t, and for any type t if the expected mass of agents assigned to t is strictly less than Q_t then $p_t = 1$. So, the values p_t must equal the cutoff values, which are unique and do not depend on X. □

Proposition 1. Consider a matching market with a set T of resource types and nested MNL preferences over T. Let $T_1 \subseteq T_2 \subseteq T$, where every type in T_2 is popular. Then:

i. HTB(T_2) dominates HTB(T_1) for agents with any preferences.

ii. HTB(T_2) dominates HTB(T_1) at every resource.

We remark that the above proposition is a generalization of Theorem 1, as HTB$(\emptyset)$ coincides with MTB, and HTB(T) coincides with STB when T contains a single type $\bar{t} = 1$. Nevertheless, the two latter properties alone do not imply Proposition 1. In Example 4, we discuss other hybrid rules that satisfy these two properties, but not the properties described in Proposition 1.

We next discuss two examples and then present the proof of the proposition.

Example: Student housing Consider a student housing problem where there are h housing *options* available (such as low-rise and mid-rise apartments, and houses), each with limited capacity. Each housing option is available either on campus or off campus. A student first determines whether she prefers an on-campus type of housing or an off-campus type. The options in each type are ranked for her according to a type-specific MNL preference model. When on-campus housing is a popular type of housing, Proposition 1 implies that the same lottery numbers should be used in all housing options of that type to break ties between students.

Example: School choice Consider a school choice problem, where each school is a resource and is associated with a specific type (which could represent the school's academic focus or language immersion program, for example). Each student ranks the types according to a complete strict preference order. If two schools are of distinct types, then the student prefers one over the other if and only if she prefers its type to the other one's type. The student ranks the schools of the same type according to a type-specific MNL preference model.

Proof of Proposition 1. Fix $T_1 \subseteq T_2 \subseteq T$ where every type in T_2 is popular. It suffices to prove the result for the case that $|T_2 \backslash T_1| = 1$, since the result for this case can be used iteratively to prove the general case. So, assume that $|T_2 \backslash T_1| = \{t\}$ for some type t. Then each agent has the same probability of being assigned to a resource of type t under HTB(T_1) and under HTB(T_2), because of the uniqueness of the p_t values described in the proof of Lemma 1. So, we can create a coupling between the assignments under HTB(T_1) and under HTB(T_2) as follows. Each agent is assigned to a resource of the same type under the two tie-breaking rules, and for each type $t' \neq t$, agents assigned to a resource of type t' are assigned to the same resource under the two tie-breaking rules. Finally, agents assigned to a resource of type t are assigned independently according to the respective tie-breaking rules.

Considering this coupling between the two assignments, in both assignments the same set of agents are assigned to a resource of type t; denote this set by S_t. Moreover, the distribution of ranks under the two tie-breaking rules only differs for agents in S_t and for resources of type t. So, HTB(T_2) would dominate HTB(T_1) for agents with any preferences if and only if HTB(T_2) dominates HTB(T_1) for agents in S_t with any preferences. Similarly, HTB(T_2) would dominate HTB(T_1) at every resource if and only if HTB(T_2) dominates HTB(T_1) at every resource of type t. We will complete the proof by showing that these equivalent conditions hold.

Consider the assignment of agents under HTB(T_1). Let the random variable X_P^1 denote the number of resources that an agent with preference order P over resources of type t prefers to *all* resources in t, conditional on the agent being in S_t. (The random elements are the agent's tie-breaking lottery numbers in the coupled process and the order that the agent prefers the resource types.) Also, let the random variable Y_P^1 denote the number of resources of type t that the agent weakly prefers to her assigned resource. Then the rank of the agent can be written as

$$Z_P^1 = X_P^1 + Y_P^1.$$

Similarly, we can write $Z_P^2 = X_P^2 + Y_P^2$ under HTB(T_2), where each variable with a superscript 2 corresponds to its counterpart variable with superscript 1 but under HTB(T_2).

From the definition of the coupling, it follows that X_P^1 and X_P^2 have the same distribution. We thus have reduced the problem to the single-type setting: consider the matching market C with MNL preferences containing only the resources of type t, and a total mass of D_t agents. Then Y_P^1 and Y_P^2, respectively, have the same distribution as the rank distributions for an agent in C with preferences P, under MTB and STB. Since type t is popular, every resource is popular in C, and so Theorem 1 implies that the distribution of Y_P^2 rank dominates the distribution of Y_P^1 for every P. Thus, the distribution of Z_P^2 rank dominates the distribution of Z_P^1 for every P, which concludes the proof of point 1 of the proposition. Similarly, point 2 follows from the fact that the ranks of agents assigned to the resources of type t are equal to the ranks of the agents assigned to the resources in C plus values drawn from the distribution of X_P^1. □

4. Analysis

4.1 *Preliminary results*

This section presents preliminary results that will be useful in our proofs. Consider a matching market $C = (m, q, N)$ that has MNL preferences with resource qualities $\mu_1, \ldots, \mu_n$. The next result shows that agents' (MNL) preferences can be generated equivalently using a stochastic process involving exponential clocks.

Claim 1. *Consider drawing an agent's preferences by the following process. For each resource j, let X_j be an independent exponential random variable with rate μ_j. For each $k \in [n]$, let $X^{(k)}$ be the kth-smallest value in $X_1, \ldots, X_n$. For each resource j, if $X_j = X^{(k)}$ then set resource j as the agent's kth-ranked resource. The distribution of preferences generated by this process is equivalent to the distribution of preferences generated by the MNL preference model.*

The above process can be interpreted as n exponential clocks, where X_j is the time that clock j rings, and the agent ranks the resources in the order of the time the clocks ring. We call this method of drawing agent preferences the *clock process*.

Proof. For an agent i and resource j, the probability that i ranks resource j as her first choice in the clock process is

$$P\{X_j = X^{(1)}\} = P\{X_j = \min(X_1, \ldots, X_n)\} = \frac{\mu_j}{\sum_{p=1}^{n} \mu_p}.$$

Thus, the distribution of i's first choice is the same in both the clock process and the MNL preference model. Now suppose that in the clock process, clock k is the first clock to ring and $X_k = t$. Conditional on this event, by the memoryless property of exponential random variables, for each resource $k' \neq k$ we see that $X_{k'} - t$ is exponentially distributed with rate $\mu_{k'}$. So, the probability that resource $k' \neq k$ is the next clock to ring is

$$\frac{\mu_{k'}}{\left(\sum_{p=1}^{n} \mu_p\right) - \mu_k},$$

which is the same as in the MNL preference model. Continuing this reasoning inductively proves the claim. □

Next, we provide properties of the STB and MTB cutoffs.

Proposition 2 (Ashlagi and Shi (2014)). The STB cutoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$ are the following:

$$\alpha_k = 1 - \min\left\{\sum_{j=1}^{k-1} \frac{q_j}{N} + \frac{r_k q_k}{N\mu_k}, 1\right\}, \quad k = 1, \ldots, n, \tag{1}$$

where $r_k = \sum_{j=k}^{n} \mu_j$.

Note that the α_k values are decreasing in k.

Proposition 3. Suppose all resources are popular. Then for all $k \in [n]$ the MTB cutoffs $\beta = (\beta_1, \dots, \beta_n)$ satisfy

$$\prod_{j=1}^{k} \beta_j \geq 1 - \frac{\sum_{j=1}^{k} q_j}{N} \cdot \frac{\sum_{j=1}^{n} \mu_j}{\sum_{j=1}^{k} \mu_j}.$$

Proof. Fix a randomly chosen agent i. For a subset of resources $G \subseteq S$, let Z_G be the event that i is not assigned to a resource in G. Let $L = [k]$.

Since $Z_L \cap Z_{S \setminus L}$ is the event that i is unassigned, $P(Z_L \cap Z_{S\setminus L}) = \prod_{j=1}^{n} \beta_j$. Moreover, i will not be assigned to a resource in $S \setminus L$ if her lottery number at each of these resources does not exceed the cutoff. Therefore, $P(Z_{S\setminus L}) \geq \prod_{j=k+1}^{n} \beta_j$. By Bayes' rule,

$$P(Z_L | Z_{S\setminus L}) \leq \prod_{j=1}^{k} \beta_j. \tag{2}$$

Let M be the mass of agents that are not assigned to resources in $S \backslash L$. Since the total capacity of the resource in L is $\sum_{j=1}^{k} q_j$,

$$M = N - \sum_{j=k+1}^{n} q_j \quad \text{and} \quad M\big(1 - P(Z_L | Z_{S\setminus L})\big) = \sum_{j=1}^{k} q_j. \tag{3}$$

Since all resources are popular, $N\mu_j \geq q_j$ for all $j \in [n]$, implying that

$$\sum_{j=k+1}^{n} q_j \leq N \sum_{j=k+1}^{n} \mu_j. \tag{4}$$

By (2), (3), and (4), we obtain that

$$\prod_{j=1}^{k} \beta_j \geq 1 - \frac{\sum_{j=1}^{k} q_j}{M} = 1 - \frac{\sum_{j=1}^{k} q_j}{N - \sum_{j=k+1}^{n} q_j} \geq 1 - \frac{\sum_{j=1}^{k} q_j}{N\left(1 - \sum_{j=k+1}^{n} \mu_j\right)} = 1 - \frac{\sum_{j=1}^{k} q_j}{N \sum_{k=1}^{k} \mu_j}.$$ □

The next property is a simple observation about stochastic vectors. For convenience, define the notation $[n] = \{1, \dots, n\}$.

Definition 11. Let v and w be stochastic vectors of length n. Vector v *crosses under* w if there is some $k \in [n]$ such that $v(p) \leq w(p)$ when $1 \leq p \leq k$, and $v(p) \geq w(p)$ when $k < p \leq n$.

Claim 2. *Let v and w be stochastic vectors and suppose v crosses under w. Then $w \succeq v$.*

Proof. Suppose that v crosses under w, and $k \in [n]$ satisfies $v(p) \le w(p)$ for all $1 \le p \le k$ and $v(p) \ge w(p)$ for all $k < p \le n$. If $t \le k$, then

$$\sum_{p=1}^{t} v(p) \ge \sum_{p=1}^{t} w(p).$$

If $t \ge k$, then

$$\sum_{p=1}^{t} v(p) = 1 - \sum_{p=t+1}^{n} v(p) \le 1 - \sum_{p=t+1}^{n} w(p) = \sum_{p=1}^{t} w(p). \qquad \square$$

4.2 *Proof of Theorem 1, Part 1*

Consider a matching market $C = (m, q, N)$ with n resources, satisfying MNL preferences, with resource qualities μ. Without loss of generality, assume $\sum_{j=1}^{n} \mu_j = 1$. Fix an agent i with preferences $P = (r_1, \dots, r_n)$. Let $(R_P^{\text{STB}})_{\le k}$ denote the probability that i will be assigned to one of her top k choices under STB, and let $(R_P^{\text{MTB}})_{\le k}$ denote the same under MTB. Then $R_P^{\text{STB}} \succeq R_P^{\text{MTB}}$ if and only if for all $k \in [n]$,

$$\left(R_P^{\text{STB}}\right)_{\le k} \ge \left(R_P^{\text{MTB}}\right)_{\le k}.$$

Fix an arbitrary integer k, where $1 \le k \le n$, and let $m_k = \max\{r_1, \dots, r_k\}$. Since the STB cutoffs are weakly decreasing in the index (of resources), i will be assigned to a resource in $\{r_1, \dots, r_k\}$ if and only if she has lottery number at least α_{m_k}, so

$$\left(R_P^{\text{STB}}\right)_{\le k} = 1 - \alpha_{m_k}.$$

Under MTB, i will not be assigned to a resource in $\{r_1, \dots, r_k\}$ if and only if for each resource $j \in \{r_1, \dots, r_k\}$ her lottery number for j is below β_j. So,

$$\left(R_P^{\text{MTB}}\right)_{\le k} = 1 - \prod_{j \in \{r_1, \dots, r_k\}} \beta_j.$$

Since k is chosen arbitrarily, it is sufficient to show that

$$\prod_{j \in \{r_1, \dots, r_k\}} \beta_j \ge \alpha_{m_k}. \tag{5}$$

Since $m_k \ge k$, then $\alpha_{m_k} \le \alpha_k$. Therefore, it is sufficient to show that

$$\prod_{j=1}^{k} \beta_j \ge \alpha_k. \tag{6}$$

This will be done by comparing the cutoffs for the matching market C to the cutoffs for another matching market C', which is similar to C but contains additional resources and a larger mass of agents.

Let $C' = (m', q', N')$ be a matching market with $n' > n$ resources, where agents have MNL preferences. Let $\mu'_1, \dots, \mu'_n$ be the resource qualities in C'. For $j \in [n]$, let $q'_j = q_j$ and $\mu'_j = \mu_j$, and assume that

$$\frac{\mu_n}{q_n} \geq \frac{\mu'_{n+1}}{q'_{n+1}} \geq \cdots \geq \frac{\mu'_{n'}}{q'_{n'}}.$$

Let $N' = N \sum_{j=1}^{n'} \mu'_j$. Note that for each resource $j \in [n]$, the mass of agents who rank j as their top choice in C is equal to the mass of agents who rank j as their top choice in C'. Let $\alpha' = (\alpha'_1, \dots, \alpha'_{n'})$ and $\beta' = (\beta'_1, \dots, \beta'_{n'})$ be the STB and MTB cutoffs for C', respectively. For each $j \in [n]$, let

$$\gamma_j = \prod_{p=1}^{j} \beta_p$$

and for each $j \in [n']$, let

$$\gamma'_j = \prod_{p=1}^{j} \beta'_p.$$

Then we must show that $\gamma_k \geq \alpha_k$. Note that since α_n and γ_n are the probabilities that an agent will be assigned to any resource under STB and MTB, respectively, we have

$$\alpha_n = \gamma_n = 1 - \frac{\sum_{j=1}^{n} q_j}{N}. \tag{7}$$

So, assume $k < n$. In the remainder of the proof, we present two lemmas, show how the lemmas imply the theorem, and finally prove the lemmas.

Lemma 2. *For all* $j \in [n]$, $\alpha'_j \leq \alpha_j$.

Lemma 3. $\gamma'_n \geq \gamma_n$.

We apply the two lemmas to complete the proof. The lemmas essentially say that when resources are added to a matching market and the mass of agents is increased accordingly, the STB cutoffs decrease while the MTB cutoffs increase. We will use this fact in reverse: when resources are removed and the mass of agents decreased accordingly, the STB cutoffs increase while the MTB cutoffs decrease.

Consider the matching market $C'' = (m'', q'', N'')$ satisfying MNL preferences, containing $k < n$ resources with qualities $\mu_1, \dots, \mu_k$. For all $j \in [k]$, let $q''_j = q_j$, and let

$$N'' = N \sum_{j=1}^{k} \mu_j.$$

We know $\alpha''_k = \gamma''_k$ since C'' has k resources; thus, Lemmas 2 and 3 imply that $\gamma_k \geq \alpha_k$, which completes the proof.

In the remainder of this section, we prove the two lemmas.

Proof of Lemma 2. For a given matching market with n resources satisfying MNL preferences, consider computing the α values by the following "water-filling algorithm." Let each resource j be able to hold a mass q_j of water, and a total mass N of water needs to be poured. The algorithm starts at time 0 and in stage 1. At time 0, all resources are empty. During stage 1, water is poured into each resource j at a rate of $\frac{N\mu_j}{\mu_1+\cdots+\mu_n}$. Stage 1 concludes when resource 1 is filled to capacity. Then the next stage begins. During stage k, resources $k-1$ have already been filled to capacity, and the water poured into them "spills over" into resources $k, \dots, n$: during stage k each resource $j \in \{k, \dots, n\}$ fills at a rate of $\frac{N\mu_j}{\mu_k+\cdots+\mu_n}$. It can be shown that for each resource j, $1-\alpha_j$ is the time that j becomes full (i.e., the time that stage $j+1$ begins).

Now, consider using the water-filling algorithm on C and C' to compute α and α'. We prove the lemma inductively over k, showing that $\alpha'_k \le \alpha_k$ for all $1 \le k \le n$. As a base case, in both problems resource 1 fills at a rate of $N\mu_1$, so $\alpha'_1 = \alpha_1$. Now for the inductive step, assume for some $k < n$, $\alpha'_k \le \alpha_k$. Note that for both problems, before time $1-\alpha_k$ the ratio of the rate that resource $k+1$ fills to the rate that resource k fills is

$$\frac{\mu_{k+1}}{\mu_k + \mu_{k+1}}.$$

Furthermore, at the end of stage k, resource j has been filled with mass q_k. Thus, in both problems, at the end of stage k resource $k+1$ has been filled with mass

$$\frac{q_k\mu_{k+1}}{\mu_k + \mu_{k+1}}.$$

For C, stage k concludes at time $1-\alpha_k$, and in stage $k+1$ resource $k+1$ fills at rate

$$r = N\sum_{j=1}^{k}\mu_j\frac{\mu_{k+1}}{\mu_{k+1}+\cdots+\mu_n} + N\mu_{k+1}.$$

In C', stage k concludes at time $1-\alpha'_k$, and then in stage $k+1$ resource $k+1$ fills at rate

$$r' = N(\mu_1+\cdots+\mu_{n'})\sum_{j=1}^{k}\mu_j\frac{\mu_{k+1}}{\mu_{k+1}+\cdots+\mu_{n'}} + N\mu_{k+1}.$$

Observe that $r' < r$ since

$$\frac{\mu_1+\cdots+\mu_{n'}}{\mu_{k+1}+\cdots+\mu_{n'}} \le \frac{1}{\mu_{k+1}+\cdots+\mu_n},$$

which follows from the following inequality: if $a, b, c > 0$, and $b \le a$, then

$$\frac{a+c}{b+c} \le \frac{a}{b}.$$

Now, by assumption $\alpha'_j \le \alpha_j$, so for C' stage $k+1$ begins at a later time than stage $k+1$ begins for C. Furthermore, since $r \ge r'$, during stage $k+1$ resource $k+1$ fills at a slower rate for C' than for C. Thus, $1-\alpha'_{k+1} \geq 1-\alpha_{k+1}$, so $\alpha'_{k+1} \leq \alpha_{k+1}$. This concludes the induction and the proof of the lemma. □

Proof of Lemma 3. Observe that α_n is the probability that an agent is not assigned to any resource under STB, and γ_n is the same under MTB. Thus,

$$\gamma_n = \alpha_n = 1 - \frac{\sum_{j=1}^{n} q_j}{N}.$$

Proposition 3 gives

$$\gamma'_n \geq 1 - \frac{\sum_{j=1}^{n} q'_j}{N'} \cdot \frac{\sum_{j=1}^{n'} \mu'_j}{\sum_{j=1}^{n} \mu'_j} = 1 - \frac{\sum_{j=1}^{n} q_j}{N} = \gamma_n,$$

where the last equality follows from equation (7). □

4.3 *Proof of Theorem 1, Part 2*

The proof uses an auxiliary assignment process, referred to as virtual MTB (VMTB), which assigns (or leaves unassigned) *each* agent independently as follows.

VMTB independent assignment process

Input: vector of cutoffs $\beta' = (\beta'_1, \beta'_2, \ldots, \beta'_n)$. Initialize: $k = 1$.

Step k: Let j be the resource that is the agent's kth rank. The agent applies to resource j. With probability $1-\beta'_j$, the resource admits the agent and the process ends. Otherwise, the agent is rejected from j. If $k = n$, the agent remains unassigned and the process terminates. Otherwise, increase k by one, and go to the next step.

We refer to the VMTB assignment process with inputs β' simply as VMTB(β'). Note that the VMTB process may violate resources' capacities. However, due to a result by Azevedo and Leshno (2016), the process generates the MTB assignment with the "correct" input:[13] Observe that by construction, if β are the MTB cutoffs, then the assignment under VMTB(β) is equivalent to the assignment under MTB.

We fix the notation for the distribution of agent ranks under VMTB. Let β' be a vector of cutoffs, and $C = (m, q, N)$ be a matching market. Let $q^{\beta'} = (q_1^{\beta'}, \ldots, q_n^{\beta'})$, where $q_j^{\beta'}$ is the mass of agents assigned to resource j under VMTB(β'). For each resource $j \in S$, let $R_j^{\beta'}$ denote the value of R_j^{MTB} for the matching market $C^{\beta'} = (m, q^{\beta'}, N)$. That is, $R_j^{\beta'}$ is

[13] Azevedo and Leshno (2016) show that a stable matching corresponds to a set of cutoffs where each agent is assigned to her most preferred resource, in which her lottery number exceeds the cutoff.

the value of R_j^{MTB} when the agents are assigned according to VMTB(β'). So, $R_j^{\beta} = R_j^{\text{MTB}}$ for every resource j.

Consider a matching market $C = (m, q, N)$ with n resources, satisfying MNL preferences, with resource qualities μ. Without loss of generality, assume $\sum_{k=1}^{n} \mu_k = 1$. Fix a resource j; we will show that $R_j^{\text{STB}} \succeq R_j^{\text{MTB}}$. Let β be the market-clearing cutoffs for C under MTB, let $\beta^0 = (\beta_1, \dots, \beta_j, 0, \dots, 0)$, and let $V_j = R_j^{\beta^0}$. The proof of the theorem proceeds by first showing that $R_j^{\text{STB}} \succeq V_j$ in Lemma 4, and then showing that $V_j \succeq R_j^{\text{MTB}}$ in Lemma 5. By the transitivity of the rank dominance relation, it will follow that $R_j^{\text{STB}} \succeq R_j^{\text{MTB}}$. Here, we briefly sketch the proofs of these two lemmas, and leave the full proofs for the Appendix.

Lemma 4. $R_j^{\text{STB}} \succeq V_j$.

Proof sketch for Lemma 4. The proof of the lemma proceeds by constructing a stochastic vector W of length n, and then showing that both $R_j^{\text{STB}} \succeq W$ and $W \succeq V_j$. These relations are shown by proving that W crosses under R_j^{STB} and that V_j crosses under W as in Definition 11, and then applying Claim 2.

Observe that under both VMTB(β^0) and STB, an agent can only be assigned to resource j if she prefers j to all resources $j' > j$. Let A_j denote the set of agents who prefer j to all resources $j' > j$. We construct W by setting $W(1) = R_j^{\text{STB}}(1)$, and for $k \geq 2$ we set $W(k)$ to be proportionate to the mass of agents who rank resource j their kth choice and are in A_j. Note that for $k > j$, this results in $W(k) = 0$.

We show that W crosses under R_j^{STB} by proving that for $k \geq 2$, conditional on an agent being in A_j and ranking j her kth choice, the probability that the agent is assigned to j is weakly decreasing in k. But $W(1) = R_j^{\text{STB}}(1)$, and for $k \geq 2$ the $W(k)$ values were chosen as if conditional on an agent being in A_j and ranking j her kth choice, the probability she is assigned to j is equal over k. Since W and R_j^{STB} are normalized to have the same sum, it is straightforward to show that then W crosses under R_j^{STB}.

To complete the proof of the lemma, we show that V_j crosses under W by proving that $W(1) \geq V_j(1)$, proving a lower bound on $V_j(k)$ for $k \geq 2$, and finally showing that $W(k)$ does not exceed this lower bound for $k \geq 2$. □

To complete the proof of the theorem, it remains to show that $V_j \succeq R_j^{\text{MTB}}$. Recall that β are the MTB cutoffs, and $\beta^0 = (\beta_1, \dots, \beta_j, 0, \dots, 0)$. Recall that we have $R_j^{\text{MTB}} = R_j^{\beta}$, and we defined $V_j = R_j^{\beta^0}$. Since $\beta_k \geq \beta_k^0$ for all $k \in [n]$, the following lemma implies that $V_j \succeq R_j^{\text{MTB}}$.

Lemma 5. *Let β and β' be vectors of cutoffs such that $\beta_k \leq \beta'_k$ for all $k \in [n]$. Then $R_j^{\beta} \succeq R_j^{\beta'}$.*

Proof sketch for Lemma 5. To prove the lemma, we fix a resource $j' \neq j$ and show that $R_j^{\beta} \succeq R_j^{\beta'}$ when $\beta'_k = \beta_k$ for all $k \neq j'$. This would prove the claim because this special case can be applied successively. Let β and β' be sets of cutoffs such that $\beta_k = \beta'_k$

for all $k \neq j'$, and $\beta_{j'} \leq \beta'_{j'}$. The major step in the proof is to show that $R_j^{\beta'} \succeq R_j^{\beta}$ when $\beta_{j'} = 0$ and $\beta'_{j'} = 1$. Intuitively, if the cutoff for j' is reduced from 1 to 0, then the mass of agents who would be assigned to j who prefer j' will now be assigned to j' instead. We show that this mass of agents is rank-dominated by the rest of the agents who are assigned to j, so reducing the cutoff for j' improves the rank distribution of agents at j. Finally, we show how this fact can be used to prove $R_j^{\beta'} \succeq R_j^{\beta}$ for general values of $\beta'_{j'}$, by interpolating between $R_j^{\beta'}$ when $\beta_{j'} = 0$, and $R_j^{\beta'}$ when $\beta_{j'} = 1$. □

5. Experiments

We ran simulations to verify whether our results hold in small discrete markets involving finite numbers of agents and resources. We focused on the setting of Theorem 1, and considered discrete markets with 3 resources, each of the same capacity q and $N = 3q + k$ agents. The markets had excess demand, that is, $k > 0$. Setting $q = k = 50$, we performed 100 simulations as follows. (In the end, we repeated these simulations for other parameterizations as well.)

In each simulation, first the resource qualities μ_1, μ_2, μ_3 were drawn independently from the uniform distribution over the unit interval. If there was a resource i that is not popular (i.e., $\frac{\mu_i}{\mu_1+\mu_2+\mu_3} N < q$), then we redrew all resource qualities. This was repeated until all of the resources were popular according to the drawn qualities. Then the next step of the simulation proceeded as follows. For each tie-breaking rule $\tau \in \{\text{MTB}, \text{STB}\}$, we constructed 100,000 tie-broken matching markets (i.e., *samples*) by drawing the agents' preferences and the tie-breakers according to the setup of Section 3.1. Then we computed the outcome of DA in each sample. For a tie-breaking rule $\tau \in \{\text{MTB}, \text{STB}\}$ and each rank r, we computed the average number of agents assigned to a rank at least as good as r in the outcome of DA, where the average is taken over the 100,000 samples associated with the tie-breaking rule τ. Denote this average by A_r^{τ}. We observed that $A_r^{\text{MTB}} \leq A_r^{\text{STB}} + 0.001$ holds for every rank r, in all of the 100 simulations that we performed. (The second summand on the right-hand side is a slack variable, which can be reduced to a smaller constant with a larger number of samples.)

This observation confirms the predictions of Theorem 1 in reasonably small markets. We also repeated the same set of simulations for two other parameterizations $q = 50$ and $k = 10$, and $q = 20$ and $k = 1$, and observed the same result.

6. Extensions

The result of the first part of Theorem 1 can be extended to apply to a much broader set of distributions of agent preferences, when the mass of agents relative to the capacity of the resources is sufficiently large. The distribution of agent preferences needs only to satisfy a minor technical condition we call *nonordered*.

Definition 12. A matching market has *nonordered* agent preferences if there is no resource $j < n$ such that the full mass of agents prefer j to all resources $j' > j$.

Theorem 2. *For any matching market $C = (m, q, N)$ with nonordered agent preferences, there exists $N' \in \mathbb{R}$ such that if $N \geq N'$, STB dominates MTB for agents with any preference order.*

The following corollary relaxes the nonordered condition of Theorem 2 at the expense of a slightly weaker result. First, a few definitions are needed. For fixed measure m and capacities q, define $R_P^{\text{MTB}}(N)$ to be R_P^{MTB} for matching market $C = (m, q, N)$. Define

$$D_P = \lim_{N' \to \infty} R_P^{\text{MTB}}(N'),$$

which is the distribution of ranks of agents with preferences P under MTB in the limit as the mass of agents approaches infinity.

Corollary 2.1. *For any matching market with n resources, for all $P \in \Pi_n$,*

$$R_P^{\text{STB}} \succeq D_P.$$

The proofs of Theorem 2 and Corollary 2.1 appear in Appendices A.3 and A.4.

7. Conclusion

This paper considered the problem of resolving ties when assigning agents to resources with heterogeneous qualities using the deferred acceptance mechanism. It is shown that when resources are "popular," a single lottery used by all resources is preferable to having each resource use a separate lottery, in a first stochastic order sense, for all agents and resources.

The above result also extends to scenarios where there is excess supply. In particular, when the set of resources is partitioned into different types and the agents have nested MNL preferences over resources, we adapt the notion of popularity to resource types and show that a hybrid rule in which resources in each popular type use the same lottery number dominates the multiple tie-breaking rule.

The notion of popularity defined for resources types exploits the nested MNL structure of the preferences. It remains an interesting direction to develop well-grounded measures for popularity that relax this assumption on preferences. For more general preference structures, our theory is silent and it is unknown, for example, whether there are tie-breaking rules that dominate MTB. Moreover, in some markets agents are given priorities at different resources and lotteries are used to resolve ties between agents with equal priorities. Defining the notion of popularity becomes more involved in such settings, as the notion would depend also on the priority structure.[14]

[14]Consider, for example, a school choice problem where a subset of the students are given priority at every school. If no school is ranked first by more students with high priority than its capacity, then the results here are applicable after assigning prioritized students to their first choices. Otherwise, while the problem can still be approached sequentially, the challenge is that the choice of lotteries for the prioritized group affects which schools are popular in the residual problem.

Appendix A: Proofs

A.1 *Proof of Lemma 4*

Let A_j be the set of agents who prefer j to all resources $k > j$, and see that all agents assigned to j under both VMTB(β^0) and STB are in A_j. Recall that $m(A_j)$ denotes the probability that a randomly chosen agent is in A_j. For $k \in [n]$, let Q_j^k denote the set of agents for whom j is their kth choice. We construct a stochastic vector W of length j as follows. Let $W(1) = R_j^{\text{STB}}(1)$. For each $k \in [2, \dots, j]$, let

$$W(k) = d \cdot m\big(A_j \cap Q_j^k\big),$$

for some constant d, and for $k > j$, let $W(k) = 0$.

First, we show that $R_j^{\text{STB}} \succeq W$. For a stochastic vector D of length n and constant $k \in [n]$, we define the simplifying notation

$$P(D \geq k) = \sum_{p=k}^{n} D(p).$$

We also define

$$\tilde{Q}_j^k = \bigcup_{p=k}^{n} Q_j^k.$$

Let M_j be the set of agents assigned to j under STB. Since $M_j \subseteq A_j$, for any $k \in [j]$ we have

$$P\big(R_j^{\text{STB}} \geq k\big) = \frac{m\big(M_j \cap \tilde{Q}_j^k\big)}{m(M_j)} = \frac{m\big(M_j | A_j \cap \tilde{Q}_j^k\big) m\big(A_j \cap \tilde{Q}_j^k\big)}{m(M_j)}, \tag{8}$$

and

$$P(W \geq k) = d \sum_{p=k}^{j} m\big(A_j \cap Q_j^p\big) = d \cdot m\big(A_j \cap \tilde{Q}_j^k\big). \tag{9}$$

Next, since $W(1) = R_j^{\text{STB}}(1)$ we have $P(W \geq 2) = P(R_j^{\text{STB}} \geq 2)$, so

$$d \cdot m\big(A_j \cap \tilde{Q}_j^2\big) = \frac{m\big(A_j \cap \tilde{Q}_j^2\big) m\big(M_j | A_j \cap \tilde{Q}_j^2\big)}{m(M_j)},$$

and thus

$$d = \frac{m\big(M_j | A_j \cap \tilde{Q}_j^2\big)}{m(M_j)}. \tag{10}$$

Then from equations (8), (9), and (10), we get

$$\frac{P(W \geq k)}{P\big(R_j^{\text{STB}} \geq k\big)} = \frac{d \cdot m(M_j)}{m\big(M_j | A_j \cap \tilde{Q}_j^k\big)} = \frac{m\big(M_j | A_j \cap \tilde{Q}_j^2\big)}{m\big(M_j | A_j \cap \tilde{Q}_j^k\big)}.$$

The following claim then implies that $P(W \geq k) \geq P(R_j^{\text{STB}} \geq k)$ for all $k \in [n]$, and hence $R_j^{\text{STB}} \succeq W$.

Claim 3. *For any* $k \in [2, \dots, j]$,

$$m\big(M_j | A_j \cap \tilde{Q}_j^k\big) \leq m\big(M_j | A_j \cap \tilde{Q}_j^2\big).$$

Proof. Fix a value of $k \in [2, \dots, j]$. Fix a lottery number $L \geq \alpha_j$, and let i be a randomly chosen agent with lottery number L. Let $P = (r_1, \dots, r_n)$ denote the preferences of i. Let i' be a randomly chosen agent in A_j with lottery number L, who does not rank j her top choice. Let B_L be the set of resources in $[j-1]$ with STB cutoffs above L. Then for each resource $s \in [j-1]$,

$$\begin{aligned} P\big(P_1' = s\big) = P\big(r_1 = s | i \in A_j \cap \tilde{Q}_j^2\big) &= \frac{P\big(i \in A_j \cap \tilde{Q}_j^2 | r_1 = s\big) P(r_1 = s)}{P\big(i \in A_j \cap tilde Q_j^2\big)} \\ &= \frac{P(i \in A_j | r_1 = s) P(r_1 = s)}{P\big(i \in A_j \cap \tilde{Q}_j^2\big)} \\ &= \frac{P(i \in A_j | r_1 = s) P(r_1 = s)}{\sum_{p=1}^{j-1} P(i \in A_j | r_1 = p) P(r_1 = p)}. \end{aligned} \tag{11}$$

The value of $P(i \in A_j)$ and $P(i \in A_j | P_1' = p)$ can be determined as follows. When an agent's preferences are being drawn from the MNL model, her first choice is drawn first, then her second choice, and so on. When a resource in $\{j, \dots, n\}$ is first drawn, the probability that resource j is drawn is

$$\frac{\mu_j}{\sum_{p=j}^{n} \mu_p}.$$

Thus,

$$P(i \in A_j) = \frac{\mu_j}{\sum_{p=j}^{n} \mu_p}, \tag{12}$$

and by the same argument, for any resource $k' \in [j-1]$,

$$P\big(i \in A_j | r_1 = k'\big) = \frac{\mu_j}{\sum_{p=j}^{n} \mu_p}.$$

So, from (11) we get

$$P(r_1' = s) = \frac{P(r_1 = s)}{\sum_{p=1}^{j-1} P(r_1 = p)} = \frac{\mu_s}{\sum_{p=1}^{j-1} \mu_p}. \tag{13}$$

Observe that for any $p \in B_L$,

$$P(i' \in M_j | P_1' = p) = P(i \in M_j | i \in A_j). \tag{14}$$

This follows from the independence of irrelevant alternatives of the MNL preference model. So, from (13) and (14) we get

$$\begin{aligned} P(i' \in M_j) &= \sum_{p \in B_L} P(i' \in M_j | P_1' = p) P(P_1' = p) \\ &= \sum_{p \in B_L} P(i \in M_j | i \in A_j) P(P_1' = p) \\ &= P(i \in M_j | i \in A_j) \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}. \end{aligned} \tag{15}$$

Now, let i^* be a randomly chosen agent in $A_j \cap \tilde{Q}_j^k$ with lottery number L and preferences $P^* = (r_1^*, \dots, r_n^*)$. We define the notation

$$r_{[p]} = \{r_1, \dots, r_p\}$$

for $p \in [n]$. See that if $i^* \in M_j$, then $r_{[k-1]} \subseteq B_L$. So,

$$\begin{aligned} P(i^* \in M_j) &= P(i \in M_j | i \in A_j \cap \tilde{Q}_j^k) \\ &= P(r_{[k-2]} \subseteq B_L | i \in A_j \cap \tilde{Q}_j^k) \\ &\quad \times P(r_{[k-1]} \in B_L | i \in A_j \cap \tilde{Q}_j^k, r_{[k-2]} \subseteq B_L) \\ &\quad \times P(i \in M_j | i \in A_j). \end{aligned} \tag{16}$$

For every set $G \subseteq B_L$ such that $|G| = k - 2$, let

$$p(G) = P(P_{[k-2]}^* = G | P_{[k-2]}^* \subseteq B_L).$$

Then

$$P\big(r_{[k-1]} \in B_L | i \in A_j \cap \tilde{Q}_j^k, r_{[k-2]} \subseteq B_L\big) = \sum_{G \subseteq B_L, |G|=k-2} p(G) \frac{\sum_{p \in B_L} \mu_p - \sum_{p \in G} \mu_p}{\sum_{p=1}^{j-1} \mu_p - \sum_{p \in G} \mu_p}$$

$$\leq \sum_{G \subseteq B_L} p(G) \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}$$

$$= \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}, \tag{17}$$

and thus from (16),

$$P\big(i^* \in M_j\big) \leq P\big(r_{[k-2]} \subseteq B_L | i \in A_j \cap \tilde{Q}_j^k\big) \frac{\sum_{p \in B_L} \mu_p}{P(i \in M_j | i \in A_j)} \sum_{p=1}^{j-1} \mu_p,$$

$$\leq P(i \in M_j | i \in A_j) \frac{\sum_{p \in B_L} \mu_p}{\sum_{p=1}^{j-1} \mu_p}. \tag{18}$$

By (15) and (18), we obtain

$$P\big(i^* \in M_j\big) \leq P\big(i' \in M_j\big). \tag{19}$$

Finally, if $L < \alpha_j$, then

$$P\big(i' \in M_j\big) = P\big(i^* \in M_j\big) = 0.$$

So, $P(i' \in M_j) \leq P(i^* \in M_j)$ for all $L \in [0, 1]$, and we have proven Claim 3. □

Next, it needs to be shown that $W \succeq V_j$. Fix an agent i with preferences $P = (r_1, \dots, r_n)$. If $i \in A_j$, $r_j = j$ and $j \in M_j$, then $r_{[j-1]} = [j-1]$ so i must have been rejected by every resource in $[j-1]$. Let

$$K = \frac{V_j(j)}{P\big(A_j \cap Q_j^j\big) \prod_{p=1}^{j-1} \beta_p},$$

so

$$V_j(j) = K \cdot P\left(A_j \cap Q_j^j\right) \prod_{p=1}^{j-1} \beta_p.$$

If $i \in A_j$ and $r_k = j$ for some $k \le j$, then i only needs to be rejected by a subset of the resources in $[j-1]$ for her to apply to resource j. Therefore, for $k \in [j-1]$,

$$V_j(k) \ge K \cdot P\left(A_j \cap Q_j^k\right) \prod_{p=1}^{j-1} \beta_p.$$

Recall that for all $p \in \{2, \dots, j\}$,

$$\frac{W(p)}{P\left(A_j \cap Q_j^p\right)} = d.$$

Thus, if it is shown that

$$d \le K \cdot \prod_{p=1}^{j-1} \beta_p, \tag{20}$$

then $W(k) \le V_j(k)$ for all $2 \le k \le j$. It will then follow that $W \succeq V_j$ by Claim 2. So, it remains to show that inequality (20) holds. For all $k \in [j-1]$, we have

$$V_j(k) \le K \cdot m\left(A_j, Q_j^k\right)$$

and

$$\sum_{p=1}^{j} V_j(p) = 1.$$

So,

$$1 = \sum_{p=1}^{j} V_j(p) \le K \sum_{p=1}^{j} m\left(A_j \cap Q_j^p\right) = K \cdot m(A_j),$$

and thus

$$K \ge \frac{1}{m(A_j)}.$$

Proposition 3 gives that

$$\prod_{p=1}^{j-1} \beta_p \ge 1 - \frac{\sum_{i=1}^{j-1} q_p}{N \sum_{p=1}^{j-1} \mu_p},$$

and so

$$K \cdot \prod_{p=1}^{j-1} \beta_p \geq \frac{1}{m(A_j)} \left(1 - \frac{\sum_{i=1}^{j-1} q_p}{N \sum_{p=1}^{j-1} \mu_p} \right). \tag{21}$$

Next, we need an upper bound for d. Observe that

$$\begin{aligned} 1 &= \sum_{p=1}^{j} W(p) \\ &= W(1) + \sum_{p=2}^{j} W(p) \\ &= R_j^{\text{STB}}(1) + d \sum_{p=2}^{j} m\big(A_j \cap Q_j^p\big). \end{aligned} \tag{22}$$

The mass of agents who rank resource j as their first choice is $N\mu_j$, and these agents are accepted to resource j with probability $1 - \alpha_j$. Since the total mass of agents assigned to resource j is q_j, we have

$$R_j^{\text{STB}}(1) = \frac{N\mu_j(1 - \alpha_j)}{q_j}. \tag{23}$$

Moreover,

$$\begin{aligned} \sum_{p=2}^{j} m\big(A_j \cap Q_j^p\big) &= m\big(A_j, \tilde{Q}_j^2\big) \\ &= m(A_j) - m\big(A_j \cap Q_j^1\big) \\ &= m(A_j) - \mu_j. \end{aligned} \tag{24}$$

Then (22), (23), and (24) give

$$1 = N\frac{\mu_j}{q_j}(1 - \alpha_j) + \big(m(A_j) - \mu_j\big)d. \tag{25}$$

Let $r_j = \sum_{p=j}^{n} \mu_p$. Then by (12),

$$m(A_j) = \frac{\mu_j}{r_j}.$$

From Proposition 1,

$$1 - \alpha_j = \frac{1}{N}\left(\sum_{p=1}^{j-1} q_p + \frac{r_j q_j}{\mu_j}\right).$$

Thus, equation (25) becomes

$$1 = \frac{\mu_j}{q_j}\sum_{p=1}^{j-1} q_p + r_j + \left(\frac{\mu_j}{r_j} - \mu_j\right)d$$
$$= \frac{\mu_j}{q_j}\sum_{p=1}^{j-1} q_p + r_j + \frac{\mu_j}{r_j}(1-r_j)d,$$

which implies

$$d = \frac{r_j}{\mu_j}\cdot\frac{1}{1-r_j}\left(1 - r_j - \frac{\mu_j}{q_j}\sum_{p=1}^{j-1} q_p\right)$$
$$= \frac{r_j}{\mu_j}\left(1 - \frac{\mu_j}{q_j(1-r_j)}\sum_{p=1}^{j-1} q_p\right).$$

Since resource j is popular, $N\mu_j \geq q_j$, so

$$d \leq \frac{r_j}{\mu_j}\left(1 - \frac{\sum_{p=1}^{j-1} q_p}{N(1-r_j)}\right)$$
$$= \frac{1}{m(A_j)}\left(1 - \frac{\sum_{p=1}^{j-1} q_p}{N(1-r_j)}\right). \tag{26}$$

By (21) and (26), we obtain (20), which gives $V_j \succeq W$. This concludes the proof of Lemma 4.

A.2 *Proof of Lemma 5*

The proof makes use of the following definitions. Recall that we fixed a randomly chosen agent i with preferences $(r_1, \ldots, r_n)$.

Definition 13. For a set of cutoffs β^* and $k \in [n]$, let $H_k^{\beta^*}$ be the event that agent i is assigned to resource k under VMTB(β^*).

Definition 14. For $k \in [n]$, let q_k^i be the rank of resource k in agent i's preference order.

Now, suppose that $\beta_{j'} = 0$ and $\beta'_{j'} = 1$. The following claim shows a convenient reformulation of the problem.

CLAIM 4. *Suppose that for all $k \in [n]$,*

$$P\big(q^i_{j'} < q^i_j | H^{\beta'}_j, q^i_j \geq k\big) \geq P\big(q^i_{j'} < q^i_j | H^{\beta'}_j\big). \tag{27}$$

Then $R^{\beta}_j \succeq R^{\beta'}_j$.

PROOF. Suppose inequality (27) holds for all $k \in [n]$. It needs to be shown that for all $k \in [n]$,

$$\sum_{p=k}^{n} R^{\beta'}_j(p) \geq \sum_{p=k}^{n} R^{\beta}_j(p).$$

Consider initially assigning agents according to VMTB(β'). This initial assignment can be transformed to an assignment according to VMTB(β), by lowering the cutoff for resource j' to 0, and any agent who prefers j' to her initial assignment becomes reassigned to j'. Then if i was initially assigned to resource j with rank p, she will be reassigned to resource j' with probability

$$P\big(q^i_{j'} < q^i_j | H^{\beta'}_j, i_j = p\big).$$

Thus, for all $p \in [n]$,

$$R^{\beta}_j(p) = C \cdot \big(1 - P\big(q^i_{j'} < q^i_j | H^{\beta'}_j, i_p = j\big)\big) \cdot R^{\beta'}_j(p),$$

where C is a normalizing constant so that R^{β}_j has a total mass of one. For $k \in [n]$, conditioned on i being assigned to resource j and having rank no better than k, we have

$$\sum_{p=k}^{n} R^{\beta}_j(p) = C \cdot \big(1 - P\big(q^i_{j'} < q^i_j | H^{\beta'}_j, q^i_j \geq k\big)\big) \cdot \sum_{p=k}^{n} R^{\beta'}_j(p).$$

Setting $k = 1$ in the above equation, we get

$$\sum_{p=1}^{n} R^{\beta}_j(p) = C \cdot \big(1 - P\big(q^i_{j'} < q^i_j | H^{\beta'}_j\big)\big) \cdot \sum_{p=1}^{n} R^{\beta'}_j(p).$$

Since

$$\sum_{p=1}^{n} R^{\beta}_j(p) = \sum_{p=1}^{n} R^{\beta'}_j(p) = 1,$$

this gives

$$C = \frac{1}{1 - P\big(q^i_{j'} < q^i_j | H^{\beta'}_j\big)}.$$

So, for $k \in [n]$, by inequality (27),

$$\sum_{pkm}^{n} R^{\beta}_j(p) = \frac{1 - P\big(q^i_{j'} < q^i_j | H^{\beta'}_j, q^i_j \geq k\big)}{1 - P\big(q^i_{j'} < q^i_j | H^{\beta'}_j\big)} \cdot \sum_{p=k}^{n} R^{\beta'}_j(p)$$

$$\geq \sum_{p=k}^{n} R_j^{\beta'}(p).$$

Thus, $R_j^{\beta} \succeq R_j^{\beta'}$. □

The next step is to prove that inequality (27) holds for all $k \in [n]$. Consider the exponential clock process of drawing agent preferences described in Claim 1, where for each $p \in [n]$, X_p is the time that clock p rings and $X^{(p)}$ is the pth earliest clock to ring. Then inequality (27) is equivalent to

$$P\big(X_{j'} < X_j | H_j^{\beta'}, X_j \geq X^{(k)}\big) \geq P\big(X_{j'} < X_j | H_j^{\beta'}\big). \tag{28}$$

We will now show that inequality (28) holds for all $k \in [n]$. For $k \in [n-1]$ and $S' \subseteq S$ where $|S'| = k$ and $j, j' \notin S'$, let

$$p(S') = P\big(i_{[k]} = S' | H_j^{\beta'}, q_j^i \geq k+1\big).$$

That is, $p(S')$ is the probability that i's top k ranked resources are S', conditional on i being assigned to j with rank $k+1$. For a set of resources $B \subseteq S$ where $j', j \notin B$, we define a set of cutoffs β^B as follows: for all resources $p \in B$, $(\beta^B)_p = 1$, and for all resources $p \notin B$, $(\beta^B)_p = \beta'_p$. Note that VMTB(β^B) is equivalent to VMTB(β') where resources in B have been removed from the matching market. Let

$$P_B = P\big(X_{j'} < X_j | H_j^{\beta^B}\big).$$

Observe that if i is assigned to j with rank no better than k, then she must have been rejected by each resource in $i_{[k-1]}$. If $j' \in i_{[k-1]}$, then the probability that i prefers j' to j is one. Otherwise, the probability that i prefers j' to j will be $P_{i_{[k-1]}}$. Let

$$b_k = P\big(j' \in i_{[k-1]} | H_j^{\beta'}, X_j \geq X^{(k)}\big).$$

Then

$$P\big(X_i < X_j | H_j^{\beta'}, X_j \geq X^{(k)}\big) = b_k + (1 - b_k)\Bigg(\sum_{S' \subseteq S : |S'| = k-1, j, j' \notin S'} p(S') P_{S'}\Bigg). \tag{29}$$

The following proposition, along with equation (29) gives inequality (28).

Proposition 4. For all $S' \subseteq S$ where $j, j' \notin S'$,

$$P_{S'} \geq P\big(X_{j'} < X_j | H_j^{\beta'}\big).$$

Proof. Let $\tilde{X}_j$ be a random variable with distribution equal to the distribution of X_j conditional on $H_j^{\beta'}$. Since $X_{j'}$ is an exponential random variable with rate $\mu_{j'}$, the CDF of $X_{j'}$ is

$$F(x) = 1 - e^{-\mu_{j'} x}.$$

Since $\beta'_{j'} = 1$, the value of $X_{j'}$ does not affect the assignment of i, so $X_{j'}$ is independent of $H_j^{\beta'}$. Furthermore, since $X_{j'}$ is independent of X_j, we have

$$P(X_i < X_j | H_j^{\beta'}) = P(X_i < \tilde{X}_j) = E_{\tilde{X}_j}[F(x)].$$

For a set of resources $B \subseteq S$ where $j, j' \notin B$, let $\tilde{X}_j^B$ be a random variable with distribution equal to the distribution of X_j conditional on $H_j^{\beta^B}$. Then

$$P_{S'} = P(X_i < \tilde{X}_j^{S'}) = E_{\tilde{X}_j^{S'}}[F(x)].$$

Since $F(x)$ is an increasing function, if $\tilde{X}_j \succeq \tilde{X}_j^{S'}$ then this will imply

$$E_{\tilde{X}_j^{S'}}[F(x)] \geq E_{\tilde{X}_j}[F(x)],$$

which gives the proposition. So, it remains to show $\tilde{X}_j \succeq \tilde{X}_j^{S'}$, which will follow from the next claim. The claim implies that for any $B_1 \subseteq B_2 \subseteq S$ where $j, j' \in B_1 \cap B_2$, we have $\tilde{X}_j^{B_1} \succeq \tilde{X}_j^{B_2}$. First, fix an indexing of the resources excluding j and j',

$$S \setminus \{i, j\} = \{a_1, a_2, \dots, a_{n-2}\}.$$

For $p \in \{0, 1, \dots, n-2\}$, let

$$\tilde{X}_j^p = \tilde{X}_j^{\{a_1,\dots,a_p\}}.$$

Claim 5.

$$\tilde{X}_j^0 \succeq \tilde{X}_j^1 \succeq \cdots \succeq \tilde{X}_j^{n-2}.$$

Proof. For $p \in \{0, 1, \dots, n-2\}$, let $f_p(x)$ be the PDF of $\tilde{X}_j^p$. Consider assigning i by VMTB($\beta^{\{a_1,\dots,a_{n-2}\}}$). Since $\beta_p^{\{a_1,\dots,a_{n-2}\}} = 1$ for all resources $p \neq j$, conditional on any value of X_j we have that i will be assigned to resource j with probability $1 - \beta'_j$. Thus, the distribution of $\tilde{X}_j^{n-2}$ is equal to the distribution of X_j, so

$$f_{n-2}(x) = \mu_j e^{-\mu_j x}.$$

Hence, $f(x)$ is decreasing over $x \geq 0$. Now, as an inductive hypothesis, assume that for some $p \in \{1, 2, \dots, n-2\}$, $f_p(x)$ is decreasing over $x \geq 0$. We will show that this implies that both $f_{p-1}(x)$ is decreasing over $x \geq 0$ and that $\tilde{X}_j^{p-1} \succeq \tilde{X}_j^p$. Now, suppose that i is assigned to j by VMTB($\beta^{\{a_1,\dots,a_p\}}$, and $X_j = x$. Since $\beta_{a_p}^{\{a_1,\dots,a_p\}} = 1$, the probability that i prefers resource a_p to j is

$$F_p(x) = P(X_{a_p} < x) = 1 - e^{-\mu_{a_p} x}.$$

Consider now lowering the cutoff for resource a_p to β'_{a_p}, and reassigning agents with high enough lottery numbers to resource a_p if they prefer it to their initial assignment.

Observe that this process will result in the assignment under VMTB($\beta^{\{a_1,\dots,a_{p-1}\}}$). Then for some positive normalizing constant $K' > 0$,

$$f_{p-1}(x) = K' f_p(x)\big[F_{p-1}(x)\big(1-\beta'_{a_{p-1}}\big) + (1 - F_{p-1}(x)\big].$$

Since $1-\beta'_{a_{p-1}} \le 1$ and $F_{p-1}(x)$ is increasing in x, we have that

$$K'\big[F_{p-1}(x)\big(1-\beta'_{a_{p-1}}\big) + \big(1 - F_{p-1}(x)\big)\big]$$

is decreasing in x for $x \ge 0$. Since by hypothesis $f_p(x)$ is decreasing for $x \ge 0$, we have $f_{p-1}(x)$ is decreasing for $x \ge 0$. To show $\tilde{X}_j^{p-1} \succeq \tilde{X}_j^p$, we need the following claim.

Claim 6. *Let Y be a nonnegative random variable with decreasing PDF $f(x)$, and Z be a nonnegative random variable with PDF $h(x) = f(x)g(x)$, where $g(x)$ is a nonnegative decreasing function. Then $Z \succeq Y$.*

Proof. By definition, $Z \succeq Y$ is equivalent to

$$\int_0^t f(x)\,dx \le \int_0^t h(x)\,dx, \quad \forall t \ge 0. \tag{30}$$

Since

$$\int_0^\infty f(x)\,dx = \int_0^\infty h(x)\,dx = 1,$$

we get that (30) is equivalent to

$$\int_t^\infty f(x)\,dx \ge \int_t^\infty h(x)\,dx, \quad \forall t \ge 0. \tag{31}$$

By (30) and (31), we obtain that $Y \succeq Z$ if

$$\frac{\int_0^t f(x)\,dx}{\int_t^\infty f(x)\,dx} \le \frac{\int_0^t h(x)\,dx}{\int_t^\infty h(x)\,dx}.$$

Since $g(x)$ is decreasing,

$$\frac{\int_0^t h(x)\,dx}{\int_t^\infty h(x)\,dx} = \frac{\int_0^t f(x)g(x)\,dx}{\int_t^\infty f(x)g(x)\,dx} \ge \frac{\int_0^t f(x)g(t)\,dx}{\int_t^\infty f(x)g(t)\,dx} = \frac{\int_0^t f(x)\,dx}{\int_t^\infty f(x)\,dx},$$

which gives Claim 6. □

Using Claim 6 with $Y = \tilde{X}_j^p$, $Z = \tilde{X}_j^{p-1}$, $f(x) = f_p(x)$, and

$$g(x) = K\big[F_{p-1}(x)(1-\beta_{a_{k+1}}) + (1 - F_{p-1}(x)\big],$$

we get that if $\tilde{X}_j^p$ has a decreasing PDF, then $\tilde{X}_j^{p-1} \succeq \tilde{X}_j^p$. This completes the proof of Claim 5, and the proof of Proposition 4. □

□

Lemma 5 has now been proven in the special case that $\beta_{j'} = 0$ and $\beta'_{j'} = 1$. It remains to show that the lemma holds in the general case. Now, suppose β and β' satisfy $\beta_{j'} \le \beta'_{j'}$ and $\beta_p = \beta'_p$ for all $p \neq j'$. Let e_i be the vector with one in the ith entry and zero in the other entries. See that $\beta - \beta_i e_i$ is equal to β, but with the ith entry set to zero. Similarly, $\beta + (1-\beta_i)e_i$ is equal to β but with the ith cutoff set to one. Let

$$E = R_j^{\beta - \beta_i e_i}$$

and

$$F = R_j^{\beta + (1-\beta_i)e_i}.$$

By the special case of the lemma, we have that $E \succeq F$. Now, consider assigning agents by VMTB(β) according to the following equivalent process. Each agent is independently put into case 1 with probability $\beta_{j'}$, and put into case 2 with probability $1 - \beta_{j'}$. Then agents in case 1 are assigned according to VMTB($\beta - \beta_i e_i$), and agents in case 2 are assigned according to VMTB($\beta + (1-\beta_i)e_i$). The case 1 agents correspond to the agents who have a lottery number for resource j' below $\beta_{j'}$, and the case 2 agents to the agents who have a lottery number above $\beta_{j'}$. Thus, this process is indeed equivalent to VMTB(β). Let c_1 be the probability that a randomly chosen case 1 agent is assigned to resource j, and let c_2 be the probability that a randomly chosen case 2 agent is assigned to resource j. Then, for some normalizing constant $C' > 0$,

$$R_j^{\beta} = C'\big[(1-\beta_{j'})c_1 E + \beta_{j'} c_2 F\big]. \tag{32}$$

See that for some $\lambda_1 \in [0, 1]$,

$$R_j^{\beta} = \lambda_1 E + (1-\lambda_1)F. \tag{33}$$

To solve for λ_1, from (32) we obtain

$$\lambda_1 = C'(1-\beta_{j'})c_1$$

and

$$1 - \lambda_1 = C'\beta_{j'} c_2.$$

Adding the above two equations together and solving for C' gives

$$C' = \frac{1}{(1-\beta_{j'})c_1 + \beta_{j'} c_2}.$$

Thus,

$$\lambda_1 = \frac{(1-\beta_{j'})c_1}{(1-\beta_{j'})c_1 + \beta_{j'} c_2}.$$

Similarly, for some normalizing constant C'',

$$R_j^{\beta'} = C''\big[(1-\beta'_{j'})c_1 E + \beta'_{j'} c_2 F\big],$$

and so

$$R_j^{\beta'} = \lambda_2 E + (1-\lambda_2)F, \tag{34}$$

where

$$\lambda_2 = \frac{(1-\beta'_{j'})c_1}{(1-\beta'_{j'})c_1 + \beta'_{j'} c_2}.$$

Since $\beta'_{j'} \geq \beta_{j'}$, we then have that $\lambda_1 \geq \lambda_2$. Finally, from (33) and (34) we get that for any $k \in [n]$,

$$\begin{aligned}\sum_{p=1}^{k} R_j^{\beta}(p) - \sum_{p=1}^{k} R_j^{\beta'}(p) &= (\lambda_1 - \lambda_2)\sum_{p=1}^{m} E(p) + (\lambda_2 - \lambda_1)\sum_{p=1}^{m} F(p) \\ &= (\lambda_1 - \lambda_2)\left[\sum_{p=1}^{m} E(p) - \sum_{p=1}^{m} F(p)\right] \\ &\leq 0,\end{aligned}$$

where the inequality follows from $\lambda_1 \geq \lambda_2$ and $E \succeq F$. Thus, $R_j^{\beta} \succeq R_j^{\beta'}$ as desired, which concludes the proof of Lemma 5.

A.3 *Proof of Theorem 2*

Consider a matching market $C = (m, q, N)$. Fix a preference order $P = (r_1, \dots, r_n) \in \Pi_n$ such that $m(P) > 0$, and fix a rank $k \in [n]$. If $j = r_p$, we write $P^{-1}(j) = p$. Index the resources such that for all resources j and j', if $\alpha_j = \alpha_{j'}$ and $P^{-1}(j) < P^{-1}(j')$, then $j > j'$. Let

$$R_{P,k}^{\mathrm{STB}} = \sum_{p=1}^{k} R_P^{\mathrm{STB}}(p)$$

and

$$R_{P,k}^{\mathrm{MTB}} = \sum_{p=1}^{k} R_P^{\mathrm{MTB}}(p).$$

We need to show that

$$R_{P,k}^{\mathrm{MTB}} \leq R_{P,k}^{\mathrm{STB}}$$

for any sufficiently large N. First, we show an upper bound for $R_{P,k}^{\mathrm{MTB}}(P, N)$. Let $Q = \sum_{j=1}^{n} q_j$. Observe that

$$R_{P,n}^{\mathrm{MTB}} = R_{P,n}^{\mathrm{STB}} = \frac{Q}{N},$$

so we assume that $k < n$. Let $\beta^N = (\beta_1^N, \ldots, \beta_n^N)$ be the MTB cutoffs for C. Since $\prod_{j=1}^n \beta_j^N = 1 - \frac{Q}{N}$, it must be that for all $j \in [n]$,

$$\lim_{N\to\infty} \beta_j^N = 1.$$

Now, for a fixed agent i, we say that resource j is *available* to i if i has priority for j at least as large as the cutoff for j. Then i will be assigned to a resource if and only if at least one resource is available for her. For each resource j, let B_j^N be the event that j is the only resource available to i, as a function of N. Let A^N be the event that no resource is available to i, as a function of N. Then

$$P(A^N) = 1 - \frac{Q}{N}$$

and for all $j \in [n]$,

$$P(B_j^N) \geq (1 - \beta_j^N) P(A^N). \tag{35}$$

During the DA algorithm at most a mass of N agents will apply to any given resource. For N sufficiently large, every resource will be filled to capacity. This implies that for all $j \in [n]$,

$$1 - \beta_j^N \geq \frac{q_j}{N}. \tag{36}$$

Then by (35) and (36), we get

$$P(B_j^N) \geq \frac{q_j}{N} P(A^N). \tag{37}$$

Note that if the event B_j^N occurs, then i will be assigned to j regardless of her preferences. Let

$$E = \bar{A}^N \setminus \left(\bigcup_{j=1}^{k} B_j^N \right),$$

that is, E is the event that at least two resources are available to i. Because the events $B_1^N, \ldots, B_n^N$ are all disjoint and contained in A^N,

$$\begin{aligned} P(E) &= P(\bar{A}^N) - \sum_{j=1}^{n} P(B_j^N) \\ &\leq \frac{Q}{N} - \sum_{j=1}^{n} \frac{q_i}{N} P(A^N) = \frac{Q}{N}\left(1 - P(A^N)\right) = \frac{Q^2}{N^2}, \end{aligned}$$

where the inequality follows from (37). Since a mass of $N - Q$ agents is unassigned, at least $N - Q$ agents apply to every resource. So, for all $j \in [n]$,

$$(N - Q)(1 - \beta_j^N) \leq q_j,$$

and thus

$$1 - \beta_j^N \leq \frac{q_i}{N - Q}. \tag{38}$$

For all $j \in [n]$,

$$P(B_j^N) \leq 1 - \beta_j^N. \tag{39}$$

Then (38) and (39) give that

$$P(B_j^N) \leq \frac{q_j}{N - Q}.$$

Letting $Q_k = \sum_{j=1}^k q_{r_j}$, we have

$$\begin{aligned} R_{P,k}^{\text{MTB}} &\leq \sum_{j=1}^k P(B_{r_j}^N) + P(E) \\ &\leq \frac{Q_k}{N - Q} + \frac{Q^2}{N^2}. \end{aligned}$$

Let $r_k^N = \frac{Q_k}{N}$. Then we obtain the upper bound

$$R_{P,k}^{\text{MTB}} \leq r_k^N \cdot \frac{N}{N - Q} + \frac{Q^2}{N^2}. \tag{40}$$

Therefore,

$$R_{P,k}^{\text{MTB}} \leq r_k^N + O\left(\frac{1}{N^2}\right).$$

Next, we need to show a lower bound for $R_{P,k}^{\text{STB}}$. If an agent is picked at random, she will be assigned to a resource in $\{r_1, \dots, r_k\}$ with probability r_k^N. For any $P' \in \Pi_n$, let $g_k^N(P, P')$ be the probability that an agent with preferences P' is assigned to a resource in $\{r_1, \dots, r_k\}$. Because in the DA algorithm it is a dominant strategy for the agents to submit their true preferences,

$$g_k^N(P, P) \geq g_k^N(P, P'), \quad \forall P' \in \Pi_n. \tag{41}$$

See that

$$r_k^N = \sum_{P' \in \Pi_n} m(P', [0, 1]^n) g_k^N(P, P') \leq g_k(P, P) = R_{P,k}^{\text{STB}}. \tag{42}$$

Let $m_k = \max\{r_1, \dots, r_k\}$. We know

$$R_{P,n}^{\text{STB}} = 1 - \alpha_n = \frac{Q}{N},$$

which for sufficiently large N is smaller than the upper bound for $R_{P,k}^{\text{MTB}}$ given by (40), since $Q_k < Q$. Now assume $m_k < n$. Because agents' preferences are nonordered, there is some $P^* \in \Pi_n$ and resource some $p > m_k$ such that $m(P^*, [0, 1]^n) > 0$ and $(P^*)^{-1}(p) <$

$(P^*)^{-1}(m_k)$. Note that by the definition of m_k, $P^{-1}(m_k) < P^{-1}(p)$ and $s \notin \{r_1, \dots, r_k\}$. Furthermore, by our indexing of the resources, for all resources j such that $\alpha_j = \alpha_{m_k}$, $P^{-1}(m_k) < P^{-1}(j)$. Thus, an agent with preferences P has a nonzero probability of being assigned to m_k under STB. That is,

$$R_P^{\text{STB}}\big(P^{-1}(m_k)\big) > 0.$$

Assume that $N \geq Q$. Then under STB, only agents with a lottery number of at least $\frac{Q}{N}$ are assigned to a resource. For a given agent i, conditional on i having a lottery number of at least $\frac{Q}{N}$, the probability that i is assigned to any given resource is independent of N. Thus, for some positive constant c,

$$R_P^{\text{STB}}\big(P^{-1}(m_k)\big) = \frac{cQ}{N}.$$

Now, consider two agents i and i', where i has preferences P and i' has preferences P^*, and both agents receive the same lottery number. Suppose their lottery number is such that i will be assigned to m_k, which happens with probability $\frac{cQ}{N}$. Then i' will not be assigned to m_k or any other resource in $\{r_1, \dots, r_k\}$, since i' prefers p to m_k and $\alpha_p \leq \alpha_{m_k}$. If the agents receive a lottery number such that i is not assigned to m_k, i is still at least as likely as i' to be assigned to a resource in $\{r_1, \dots, r_k\}$. Thus,

$$g_k^N\big(P, P^*\big) \leq g_k^N(P, P) - \frac{cQ}{N}. \tag{43}$$

Combining (41), (42), and (43) we get the desired lower bound

$$R_{P,k}^{\text{STB}} \geq r_k^N + \frac{cQ}{N}\big(P^{-1}(m_k)\big) \cdot m\big(P^*, [0, 1]^n\big). \tag{44}$$

The two bounds (40) and (44) give that $R_{P,k}^{\text{MTB}} \leq R_{P,k}^{\text{STB}}$ for N sufficiently large, which concludes the proof.

A.4 *Proof of Corollary 2.1*

Consider a matching market $C = (m, q, N)$. Fix a preference order $P = (r_1, \dots, r_n) \in \Pi_n$ such that $m(P) > 0$, and fix a rank $k \in [n]$. Let

$$R_{P,k}^{\text{STB}} = \sum_{j=1}^{k} R_P^{\text{STB}}(k)$$

and

$$D_{P,k} = \sum_{j=1}^{k} D_P(k).$$

Then to prove the theorem it needs to be shown that

$$D_{P,k} \leq R_{P,k}^{\text{STB}} \tag{45}$$

for all N. Let $Q = \sum_{j=1}^{n} q_j$, and $Q_k = \sum_{j=1}^{k} q_{r_j}$. From the proof of Theorem 2, inequality (44) gives that

$$R_{P,k}^{\text{STB}} \geq \frac{Q_k}{N}.$$

On the other hand, (40) gives that

$$D_{P,k} \leq \frac{1}{N} \lim_{N' \to \infty} N' \left(\frac{Q_k}{N'} \cdot \frac{N'}{N' - Q} + \frac{Q^2}{(N')^2} \right) = \frac{Q_k}{N}.$$

So, $D_{P,k} \leq R_{P,k}^{\text{STB}}$, which completes the proof.

Appendix B: Tightness of results

The examples presented here demonstrate the necessity of the conditions in our theorems. Example 1 shows the necessity of each resource being popular for Theorem 1. In this example, resources 1 and 2 are popular, while resource 3 is competitive but not popular.

Example 1 (Necessity of popularity condition). Consider the following matching market with $n = 3$ and satisfying MNL preferences. Let $N = 4$, $\mu = (3, 2, 1)$, and $q = (1/3, 1/3, 1/3)$. ◊

Claim 7. *In Example 1, STB does not dominate MTB at resource 3.*

Proof. Observe that resources 1 and 2 are popular, but resource 3 is nonpopular. Because $N > \sum_{j=1}^{n} q_j$, resource 3 is competitive. We compute the rank distributions at the resources to be (rounded to the nearest tenth)

$$R_1^{\text{STB}} = (1, 0, 0), \qquad R_2^{\text{STB}} = (0.8, 0.2, 0), \qquad R_1^{\text{STB}} = (0.5, 0.2, 0.3),$$
$$R_1^{\text{MTB}} = (0.9, 0.1, 0), \qquad R_2^{\text{MTB}} = (0.7, 0.3, 0), \qquad R_1^{\text{MTB}} = (0.5, 0.3, 0.2).$$

So, STB dominates MTB at resources 1 and 2, but not at resource 3. □

The next example shows why the nonordered condition is necessary for Theorem 2.

Example 2 (Necessity of nonordered condition). Consider the following matching market $C = (m, q, N)$ with $n = 3$. Let $N \geq \sum_{j=1}^{n} q_j$, and let m be given by

$$m\big((1, 2, 3)\big) = p, \qquad m\big((2, 3, 1)\big) = 1 - p,$$

where p and q are any values such that $\alpha_1 > \alpha_2 > \alpha_3$. ◊

Claim 8. *In Example 2, STB does not dominate MTB for agents with preferences* $(1, 2, 3)$.

PROOF. Let $(R_P^{\text{STB}})_{\leq k}$ be the probability that an agent with preferences P is assigned to a resource in her top k choices under STB. Let $(R_P^{\text{MTB}})_{\leq k}$ be the same under MTB, and let $P^* = (1, 2, 3)$. We will show that

$$\left(R_{P^*}^{\text{STB}}\right)_{\leq 2} < \left(R_{P^*}^{\text{MTB}}\right)_{\leq 2}, \tag{46}$$

which gives that $R_{P^*}^{\text{STB}} \not\succeq R_{P^*}^{\text{MTB}}$. Under both STB and MTB, a randomly chosen agent will be assigned to resources 1 or 2 with probability

$$r = \frac{q_1 + q_2}{N}.$$

Under STB, an agent will be assigned to resources 1 or 2 if and only if she has a lottery number of at least α_2, regardless of her preferences. So, $(R_{P^*}^{\text{STB}})_{\leq 2} = r$. Now, fix a randomly chosen agent i. Under MTB, there is a nonzero probability that i has high enough lottery numbers to be accepted to both resources 1 and 3, but not 2, in which case, i will be assigned to resources 1 or 2 if and only if she has preferences P^*. Thus, an agent with preferences P^* is strictly more likely to be assigned to resources 1 or 2 than an agent with preferences $(2, 3, 1)$. Therefore, agents with preferences P^* are assigned to one of their top two resources with probability strictly greater than r, which shows (46) as desired. □

The next example shows that dominance of STB over MTB at every resource does not hold for arbitrary distributions of agent preferences, even in the limit as the mass of agents grows.

EXAMPLE 3 (No dominance at resource in the limit). Consider the following matching market $C = (m, q, N)$ with $n = 4$. Let $N > 1$ and $q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Let m be given by

$$m\big((1, 2, 3, 4)\big) = p, \qquad m\big((4, 3, 2, 1)\big) = 1 - p,$$

where $p < 1$ is a constant sufficiently close to one so that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. ◊

CLAIM 9. *In Example 3,*

$$R_3^{\text{STB}}(N) \not\succeq R_3^{\text{MTB}}(N).$$

PROOF. Because $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$, an agent can only be assigned to resource 3 under STB if the agent has preferences $(1, 2, 3, 4)$. So,

$$R_3^{\text{STB}}(N) = (0, 0, 1, 0)$$

Since all agents rank resource 3 as their second or third choice, only agents of rank two or three are assigned to resource 3 under MTB. Because $N > 1$, under MTB a nonzero mass of agents will be rejected from all resources, so every MTB cutoff is strictly greater than zero. Thus, a nonzero mass of agents with a rank of 2 will be assigned to resource 3 under MTB, and so

$$R_3^{\text{MTB}} = (0, c, 1 - c, 0)$$

for some constant $c > 0$. Thus, $R_3^{\text{STB}} \not\succeq R_3^{\text{MTB}}$. □

The final example shows that a different version of the hybrid tie-breaking rule that uses the same lottery number for an agent at all popular resource types does not always dominate MTB. We will denote such a hybrid tie-breaking rule by HBT′. To be more precise, HTB′ assigns the same (independently drawn) lottery number to an agent at every resource that belongs to a popular resource type, and assigns her independently drawn lottery numbers at every other resource.

Example 4 (Nondominance of common lottery for popular resources). Consider the following matching market with $n = 3$ and satisfying MNL preferences. Let $N = 1$, $\mu = (1, 1, 1)$, and $q = (1/4, 1/4, 1/2)$. ♢

Claim 10. *In Example 4,*

$$R^{\mathrm{HTB}'}_{(1,2,3)} \not\succeq R^{\mathrm{MTB}}_{(1,2,3)}.$$

Proof. We will show

$$\big(R^{\mathrm{HTB}'}_{(1,2,3)}\big)_{\leq 2} < \big(R^{\mathrm{MTB}}_{(1,2,3)}\big)_{\leq 2}, \tag{47}$$

which implies the claim. Under both MTB and HTB′, resource 3 will have a cutoff of zero. Since resources 1 and 2 are symmetric, let α be the cutoff of these resources under HTB′ and β the cutoff under MTB. Let type 1 agents be the agents with preferences $(1, 2, 3)$ or $(2, 1, 3)$, and type 2 agents be the agents with preferences $(1, 3, 2)$ or $(2, 3, 1)$. See that only type 1 and type 2 agents will be assigned to resources 1 or 2, and that under HTB′, both type 1 and type 2 agents will be assigned to resources 1 or 2 with probability $(1-\alpha)$. On the other hand, under MTB, type 1 agents will be assigned to resources 1 or 2 with probability

$$(1-\beta) + \beta(1-\beta) = (1+\beta)(1-\beta)$$

and type 2 agents will be assigned to resources 1 or 2 with probability $(1 - \beta)$. Let m_1 be the mass of type 1 agents, and m_2 the mass of type 2 agents. Then the capacity constraints of the resources give that

$$(m_1 + m_2)(1-\alpha) = \frac{1}{2}$$

and

$$m_1(1+\beta)(1-\beta) + m_2(1-\beta) = \frac{1}{2}.$$

Since $(1+\beta)(1-\beta) > (1-\beta)$, we get that $(1+\beta)(1-\beta) > (1-\alpha)$, which proves (47). □

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