

Characterizing the top cycle via strategyproofness

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Gibbard and Satterthwaite have shown that the only single-valued social choice functions (SCFs) that satisfy nonimposition (i.e., the function's range coincides with its codomain) and strategyproofness (i.e., voters are never better off by misrepresenting their preferences) are dictatorships. In this paper, we consider set-valued social choice correspondences (SCCs) that are strategyproof according to Fishburn's preference extension and, in particular, the *top cycle*, an attractive SCC that returns the maximal elements of the transitive closure of the weak majority relation. Our main theorem shows that, under mild conditions, the top cycle is the *only* non-imposing strategyproof SCC whose outcome only depends on the quantified pairwise comparisons between alternatives. This result effectively turns the Gibbard–Satterthwaite impossibility into a complete characterization of the top cycle by moving from SCFs to SCCs. We also leverage key ideas of the proof of this statement to obtain a more general characterization of strategyproof SCCs.

KEYWORDS. Top cycle, strategyproofness, Condorcet, preference extension.

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1. INTRODUCTION

One of the most influential results in microeconomic theory, the Gibbard–Satterthwaite theorem, states that dictatorships are the only single-valued social choice functions (SCFs) that are nonimposing (i.e., every alternative is returned for some preference profile) and strategyproof (i.e., voters are unable to obtain a better outcome by misrepresenting their preferences) when there are at least three alternatives. The convenient but rather restrictive assumption of single-valuedness has been criticized by various scholars. For instance, [Gärdenfors \(1976\)](#) asserts that “[resoluteness] is a rather restrictive and unnatural assumption.” In a similar vein, [Kelly \(1977\)](#) writes that “the Gibbard–Satterthwaite theorem [...] uses an assumption of singlevaluedness, which is unreasonable” and [Taylor \(2005\)](#) that “if there is a weakness to the Gibbard–Satterthwaite theorem, it is the assumption that winners are unique.” The problem with single-valuedness

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is that it is in conflict with the basic fairness notions of anonymity and neutrality, which require that all voters and all alternatives are treated equally. For example, if half of the voters favor a and the other half b , there is no fair way of selecting a single winner because both alternatives are equally acceptable. In the context of social choice, these fairness conditions are imperative because elections should be unbiased. One way to deal with this problem is to identify a set of winning candidates with the understanding that one of these candidates will eventually be selected by some tie-breaking rule independent of the voters' preferences. Ties can, for example, be broken by lottery or by letting a chairperson or a committee pick the winner.¹

As a result, a large body of research investigates so-called social choice correspondences (SCCs), which return sets of alternatives. In particular, several papers have shown statements that mimic the negative consequences of the Gibbard–Satterthwaite theorem (e.g., Duggan and Schwartz (2000), Barberà, Dutta, and Sen (2001), Ching and Zhou (2002), Benoît (2002), Sato (2014)). These results are based on relatively strong assumptions about the manipulators' preferences over sets (which in turn are based on the voters' beliefs about how ties are broken). For example, all of these results rely on the assumption that a voter who prefers a to b to c will engage in a manipulation in which the outcome changes from set $\{a, c\}$ to set $\{b\}$. However, it is quite possible that no voter entertains such preferences over sets. By contrast, the voters' preferences over sets we surmise in this paper are systematically deduced from their preferences over alternatives, which leads to a weaker notion of strategyproofness. In more detail, we consider a preference extension attributed to (Fishburn (1972)), according to which a manipulation is only successful if the manipulator can change the outcome from a set Y to another set X such that he prefers all alternatives in $X \setminus Y$ to all alternatives in Y and all alternatives in X to all alternatives in $Y \setminus X$. Two natural justifications for this extension are the existence of a chairperson who breaks ties or of a priori probabilities of the voters how ties are broken (see, e.g., Gärdenfors (1979), Ching and Zhou (2002), Erdamar and Sanver (2009), Brandt, Saile, and Stricker (2022)). The resulting notion of strategyproofness, often called Fishburn strategyproofness, allows for positive results. For example, the rather indecisive *omnination rule*, which returns all alternatives that are top-ranked by at least one voter, is strategyproof according to this notion.

A particularly promising approach to construct attractive strategyproof SCCs is to focus on the pairwise comparisons between alternatives (see, e.g., Gärdenfors (1976), MacIntyre and Pattanaik (1981), Bandyopadhyay (1983), Campbell and Kelly (2003), Brandt (2015)). For instance, Brandt (2015) shows that several attractive SCCs that only rely on the pairwise majority relation, such as the uncovered set and the bipartisan set, satisfy a strategyproofness notion which is slightly weaker than Fishburn strategyproofness. In this paper, we thus focus on the class of pairwise (aka C2) SCCs, whose outcome only depends on the weighted majority comparisons. This class was introduced by Fishburn (1977) and includes many important SCCs such as Borda's rule, Copeland's

¹These tie-breaking rules are common in real-world elections. Tied elections on the state level within the U.S. are sometimes decided by lottery. The U.S. Vice President acts as the President of the Senate and frequently breaks ties in the Senate. If no candidate in a U.S. presidential election obtains an absolute majority of the voters, then the House of Representatives elects the winner among the best three candidates.

rule, the top cycle, the essential set, the Simpson–Kramer rule, Kemeny’s rule, ranked pairs, and Schulze’s rule (see Chapters 3 and 4 in Brandt, Conitzer, Endriss, Lang, and Procaccia (2016), for an overview of these SCCs). Indeed, it is well known that almost all other SCCs (e.g., positional scoring rules or runoff rules) are manipulable according to Fishburn’s preference extension (see, e.g., Taylor (2005, pp. 44–51)).

A prominent concept that arises from pairwise comparisons between alternatives is that of a Condorcet winner, an alternative that is preferred to every other alternative by a majority of voters (Condorcet (1785)). Condorcet winners need not exist, but many scholars agree that an SCC should uniquely return the Condorcet winner whenever one exists. The nonexistence of Condorcet winners can be addressed by extending the notion of Condorcet winners to so-called dominant sets of alternatives. A set of alternatives X is dominant if every element of X is preferred to every element not in X by a majority of voters. Dominant sets are guaranteed to exist since the set of all alternatives is trivially dominant, and they can be ordered by set inclusion.² These observations have led to the definition of the *top cycle*, an SCC that returns the unique smallest dominant set for any given preference profile. This set consists precisely of the maximal elements of the transitive closure of the weak majority relation. The top cycle has been reinvented several times and is known under various names such as Good set (Good (1971)), Smith set (Smith (1973)), weak closure maximality (Sen (1977)), and GETCHA (Schwartz (1986)).

In this paper, we characterize the class of strategyproof pairwise SCCs under relatively mild and common technical assumptions, namely the conditions of nonimposition, homogeneity, and neutrality. Our first result shows that every strategyproof pairwise SCC that satisfies these conditions always returns a dominant set. An important variant of this characterization is obtained when replacing nonimposition and neutrality with set not-imposition (every set of alternatives is returned for some preference profile): the top cycle is the *only* strategyproof pairwise SCC that satisfies set nonimposition and homogeneity. This result effectively turns the Gibbard–Satterthwaite impossibility theorem into a complete characterization of the top cycle by moving from SCFs to SCCs.

On top of strategyproofness, the top cycle is very robust in terms of changes to the set of feasible alternatives and preferences of the voters: it is invariant under removing losing alternatives as well as modifications of preferences between losing alternatives, and has been characterized repeatedly by choice consistency conditions implied by the weak axiom of revealed preference. Finally, it is one of the most straightforward Condorcet extensions and can be easily computed. The main disadvantage of the top cycle is its possible inclusion of Pareto-dominated alternatives. We believe that this drawback is tolerable because empirical results suggest that the top cycle only rarely contains Pareto-dominated alternatives. This is due to the fact that Pareto dominances are increasingly unlikely for large numbers of voters and the persistent observation that an overwhelming number of real-world elections admit Condorcet winners (see, e.g., Regenwetter, Grofman, Marley, and Tsetlin (2006), Laslier (2010), Gehrlein and Lepelley (2011), Brandt and Seedig (2016)). In these cases, the top cycle consists of a single

²Assume for contradiction that two dominant sets, X , Y , are not contained in each other. Then there exists $x \in X \setminus Y$ and $y \in Y \setminus X$. The definition of dominant sets requires that x is majority-preferred to y and that y is majority-preferred to x , a contradiction.

Pareto-optimal alternative. Moreover, as we point out in Remark 8, the SCC that returns the top cycle of the subset of Pareto-optimal alternatives satisfies all our axioms except pairwise-ness. In particular, this SCC satisfies strategyproofness with respect to Fishburn's extension.

2. RELATED WORK

Gärdenfors (1979) initiated the study of strategyproofness with respect to Fishburn's preference extension. He attributed this extension to Fishburn because it is the weakest extension that satisfies a set of axioms proposed by Fishburn (1972). A small number of SCCs were shown to be Fishburn strategyproof, sometimes by means of stronger strategyproofness notions: the Pareto rule—which returns all Pareto-optimal alternatives (Feldman (1979)), the omninomination rule—which returns all top-ranked alternatives (Gärdenfors (1976)), the Condorcet rule—which returns the Condorcet winner whenever one exists and all alternatives otherwise (Gärdenfors (1976)), the SCC that returns the Condorcet winner whenever one exists and all Pareto-optimal alternatives otherwise (Brandt and Brill (2011)), and the top cycle (Bandyopadhyay (1983), Brandt and Brill (2011), Sanver and Zwicker (2012)). All other commonly studied SCCs fail Fishburn strategyproofness (see, e.g., Taylor (2005), Brandt, Brill, and Harrenstein (2016)). A universal example showing the Fishburn manipulability of many SCCs is given in Figure 2 of Section 3.

More recently, the limitations of Fishburn strategyproofness were explored. Brandt and Geist (2016) studied majoritarian SCCs, that is, SCCs whose outcome only depends on the pairwise majority relation, and showed that no majoritarian SCC satisfies Fishburn strategyproofness and Pareto optimality. The condition of majoritarianism can be replaced with the much weaker condition of anonymity when allowing for ties in the preferences (Brandt, Saile, and Stricker (2022)). Both results were obtained with the help of computer-aided theorem proving techniques. Brandt and Geist (2016, Remark 3) observed that, in the absence of majority ties, the top cycle could be the finest majoritarian Condorcet extension that satisfies Fishburn strategyproofness when there are at least five alternatives. A computer verified this claim for five, six, and seven alternatives using 24 hours of runtime. The claim now follows immediately from our Theorem 1 (see also Remark 3), irrespective of majority ties.

Ching and Zhou (2002) considered a much stronger notion of strategyproofness based on Fishburn's preference extension: they require that the outcome when voting honestly is comparable and preferred to every choice set obtainable by a manipulation according to Fishburn's extension. Their main result shows that only constant and dictatorial SCCs are strategyproof according to this definition. Barberà, Dutta, and Sen (2001) derive a similar conclusion for a weaker notion of strategyproofness based on Fishburn's extension (but still stronger than the one considered in this paper). In their model, voters submit preference relations over sets of alternatives that adhere to certain structural restrictions. When these restrictions are given by Fishburn's extension, they prove that only dictatorships satisfy strategyproofness and unanimity.

Several choice-theoretic characterizations of the top cycle exist. When assuming that choices from two-element sets are made according to majority rule, the influential characterization by Bordes (1976) entails that the top cycle is the finest SCC satisfying β^+ , an expansion consistency condition implied by the weak axiom of revealed preference.³ Ehlers and Sprumont (2008) have shown that, in the absence of majority ties, the refinement condition can be replaced with two contraction consistency conditions. Brandt (2011), Houy (2011), and Brandt, Brill, Seedig, and Suksompong (2018) provide further characterization using choice consistency conditions. We are not aware of a characterization of the top cycle using strategyproofness.

3. PRELIMINARIES

Let $\mathbb{N} = \{1, 2, \dots\}$ denote an infinite set of voters and A a finite set of m alternatives. Moreover, let $\mathcal{F}(\mathbb{N})$ denote the set of all finite and nonempty subsets of \mathbb{N} . Intuitively, \mathbb{N} is the set of all possible voters, whereas an element $N \in \mathcal{F}(\mathbb{N})$ represents a concrete electorate. Given an electorate $N \in \mathcal{F}(\mathbb{N})$, each voter $i \in N$ has a *preference relation* represented by a strict total order \succ_i on A . The set of all preference relations on A is denoted by $\mathcal{R}(A)$. A *preference profile* R is a vector of preference relations, that is, $R \in \mathcal{R}(A)^N$ for some electorate $N \in \mathcal{F}(\mathbb{N})$. The set of all preference profiles on A is denoted by $\mathcal{R}^*(A) = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \mathcal{R}(A)^N$.

For a preference profile $R \in \mathcal{R}^*(A)$, let

$$g_R(x, y) = |\{i \in N : x \succ_i y\}| - |\{i \in N : y \succ_i x\}| \quad (\text{Majority margin})$$

be the *majority margin* of x over y in R . It describes how many more voters prefer x to y than y to x . Whenever $g_R(x, y) \geq 0$ for some pair of alternatives, we say that x weakly (majority) dominates y , denoted by $x \succsim_R y$. Note that the relation \succsim_R , which we call *majority relation*, is complete. Its strict part will be denoted by \succ_R , that is, $x \succ_R y$ if and only if $x \succsim_R y$ and not $y \succsim_R x$, and its indifference part by \sim_R , that is, $x \sim_R y$ if and only if $x \succsim_R y$ and $y \succsim_R x$. Whenever the number of voters is odd, there can be no majority ties and \succsim_R is antisymmetric. We will extend both individual preference relations and the majority relation to sets of alternatives using the shorthand notation $X \succ Y$ whenever $x \succ y$ for all $x \in X$ and $y \in Y$.

The majority relation gives rise to a number of important concepts in social choice theory. A *Condorcet winner* is an alternative x such that $x \succ_R A \setminus \{x\}$. In a similar vein, a *Condorcet loser* is an alternative x with $A \setminus \{x\} \succ_R x$. Neither Condorcet winners nor Condorcet losers need to exist, but whenever they do, each of them is unique. A natural extension of these ideas to sets of alternatives is formalized via the notion of dominant sets. A nonempty set $X \subseteq A$ is *dominant* if $X \succ_R A \setminus X$. Whenever a Condorcet winner exists, it forms a singleton dominant set. In contrast to Condorcet winners, dominant sets are guaranteed to exist since the set of all alternatives A is trivially dominant. For every majority relation \succsim_R , the set of dominant sets is totally ordered by set inclusion, that

³This result was recently rediscovered by Evren, Nishimura, and Ok (2019).

1:	a	\succ_1	e	\succ_1	b	\succ_1	c	\succ_1	d
2:	b	\succ_2	c	\succ_2	a	\succ_2	d	\succ_2	e
3:	c	\succ_3	a	\succ_3	b	\succ_3	e	\succ_3	d
4:	d	\succ_4	b	\succ_4	c	\succ_4	a	\succ_4	e

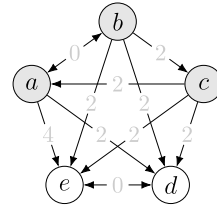


FIGURE 1. A preference profile with $N = \{1, 2, 3, 4\}$ and $A = \{a, b, c, d, e\}$ (left-hand side) and the corresponding weighted majority graph (right-hand side). An edge from x to y with weight w denotes that $g_R(x, y) = w$. Edges with weight 0 are bidirectional since, in this case, both alternatives weakly dominate each other. The smallest dominant set, $\{a, b, c\}$, is highlighted in gray.

is, each majority relation induces a hierarchy of dominant sets that are strictly contained in each other.²

For an illustration of these concepts, consider the example given in Figure 1, which shows a preference profile R and its majority relation. The weights on the edges of the majority relation indicate the majority margins. The profile R neither admits a Condorcet winner nor a Condorcet loser since each alternative has at least one incoming and outgoing edge. There are two dominant sets, $\{a, b, c\}$ and $\{a, b, c, d, e\}$. Note that the notions of Condorcet winners and dominant sets are independent of the exact weights of the edges but only depend on their directions.

Social choice correspondences

This paper is concerned with *social choice correspondences* (SCCs). An SCC maps a preference profile to a nonempty subset of alternatives called the choice set, that is, it is a function of the form $f : \mathcal{R}^*(A) \rightarrow 2^A \setminus \{\emptyset\}$. Note that we employ a so-called variable population framework, so SCCs are defined for all electorates. In this paper, we focus on two important classes of SCCs: *majoritarian* SCCs and *pairwise* SCCs. An SCC f is called *majoritarian* if its outcome merely depends on the majority relation, that is, $f(R) = f(R')$ for all $R, R' \in \mathcal{R}^*(A)$ with $\succ_R = \succ_{R'}$. Furthermore, an SCC f is *pairwise* if its outcome merely depends on the majority margins, that is, $f(R) = f(R')$ for all $R, R' \in \mathcal{R}^*(A)$ with $g_R = g_{R'}$. Majoritarian SCCs can be interpreted as functions that map an unweighted graph (A, \succ_R) to a nonempty subset of its vertices, while pairwise SCCs may additionally use the majority margins $g_R(x, y)$ as weights of the edges. The classes of majoritarian and pairwise SCCs are very rich and contain a variety of well-studied SCCs. For instance, Copeland’s rule, the uncovered set, and the bipartisan set are majoritarian, and Borda’s rule, the Simpson–Kramer rule, the essential set, Kemeny’s rule, ranked pairs, and Schulze’s rule are pairwise (the interested reader may consult Chapters 3 and 4 in Brandt et al. (2016) for definitions of these SCCs). All SCCs listed above, except Borda’s rule, are *Condorcet extensions*, that is, they uniquely return the Condorcet winner whenever one exists.

We say that an SCC f is a *refinement* of an SCC g if $f(R) \subseteq g(R)$ for all preference profiles $R \in \mathcal{R}^*(A)$. In this case, g is also said to be *coarser* than f . For example, Copeland’s

rule, the uncovered set, the essential set, Kemeny’s rule, ranked pairs, and Schulze’s rule are known to be refinements of the top cycle while the Condorcet rule is a coarsening of the top cycle.

The top cycle is a majoritarian SCC that returns the smallest dominant set for a given preference profile. Every preference profile admits a unique smallest dominant set because dominant sets are ordered by set inclusion. Alternatively, the top cycle can be defined based on paths with respect to the majority relation. A path from an alternative x to an alternative y in \succsim_R is a sequence of alternatives (x_1, \dots, x_k) such that $x_1 = x$, $x_k = y$, and $x_i \succsim_R x_{i+1}$ for all $i \in \{1, \dots, k - 1\}$. The transitive closure \succsim_R^* of the majority relation contains all pairs of alternatives (x, y) such that there is a path from x to y in \succsim_R . Then the top cycle can be defined as the set of alternatives that are maximal according to \succsim_R^* :

$$TC(R) = \bigcap \{X \subseteq A : X \succ_R A \setminus X\} = \{x \in A : x \not\succsim_R^* A\}. \tag{Top cycle}$$

In other words, the top cycle consists precisely of those alternatives that reach every other alternative on some path in the majority graph. For instance, the top cycle of the example profile in Figure 1 is $\{a, b, c\}$. Here, it is important that we interpret majority ties as bidirectional edges as otherwise there would be no path from a to b .⁴

On top of majoritarianism and pairwiseism, which restrict the informational basis of SCCs, we now introduce a number of additional properties of SCCs.

- An SCC is *nonimposing* if for every alternative $x \in A$ there is a profile $R \in \mathcal{R}^*(A)$ such that $f(R) = \{x\}$.
- An SCC is *neutral* if $f(R') = \pi(f(R))$ for all electorates $N \in \mathcal{F}(\mathbb{N})$, profiles $R, R' \in \mathcal{R}(A)^N$, and permutations $\pi : A \rightarrow A$ such that $x \succ_i y$ if and only if $\pi(x) \succ'_i \pi(y)$ for all alternatives $x, y \in A$ and voters $i \in N$.
- An SCC is *homogeneous* if for all preference profiles $R \in \mathcal{R}^*(A)$, $f(R) = f(kR)$ where the profile kR consists of k copies of R .

Nonimposition is a mild decisiveness requirement demanding that every alternative will be selected uniquely for some configuration of preferences. It is weaker than Pareto optimality and unanimity (an alternative that is top-ranked by all voters has to be elected uniquely). Neutrality requires that if alternatives are relabeled in a preference profile, the alternatives in the corresponding choice set are relabeled accordingly. Homogeneity states that cloning the entire electorate will not affect the choice set. All three of these properties are very mild and satisfied by all SCCs typically considered in the literature, including the top cycle.

⁴There is a refinement of the top cycle, sometimes called the Schwartz set or GOCHA, which is defined as the union of undominated sets, or alternatively, as the set of alternatives that reach every other alternative on a path according to \succ_R (rather than \succsim_R) (see, e.g., Schwartz (1972), Deb (1977), Schwartz (1986)). We will not consider it further because it violates rather mild consistency and strategyproofness conditions.

Fishburn's extension and strategyproofness

An important desirable property of SCCs is strategyproofness, which demands that voters should never be better off by lying about their preferences. To make this formally precise for social choice *correspondences*, we need to make assumptions about the voters' preferences over sets. In this paper, we extend the voters' preferences over alternatives to incomplete preference over sets by using Fishburn's preference extension. Given two sets of alternatives $X, Y \subseteq A$, $X \neq Y$, and a preference relation \succ_i , Fishburn's extension is defined by

$$X \succ_i^F Y \quad \text{if and only if} \quad X \setminus Y \succ_i Y \quad \text{and} \quad X \succ_i Y \setminus X. \quad (\text{Fishburn's extension})$$

Fishburn's extension is frequently considered in social choice theory and can be justified in various ways (see, e.g., [Gärdenfors \(1979\)](#), [Ching and Zhou \(2002\)](#), [Erdamar and Sanver \(2009\)](#), [Brandt, Saile, and Stricker \(2022\)](#)). For example, one motivation assumes that a single alternative will eventually be selected from each choice set according to a linear tie-breaking ordering (such as the preference relation of a chairperson) and that voters are unaware of the concrete ordering used to break ties. Then set X is preferred to set Y if and only if for all tie-breaking orderings, the voter weakly prefers the alternative selected from X to that selected from Y and there is at least one ordering for which this comparison is strict. Another motivation is based on a function that assigns an a priori weight to each alternative such that each choice set can be mapped to a lottery over the chosen alternatives such that the probabilities are proportional to the alternatives' weights. Again the voters are unaware of the concrete weight function and prefer set X to set Y if and only if for all utility functions consistent with their ordinal preferences and all a priori weight functions, the expected utility derived from X is higher than that derived from Y .

Strategyproofness based on Fishburn's extension can be defined as follows. An SCC f is (*Fishburn*) *strategyproof* if for all electorates $N \in \mathcal{F}(\mathbb{N})$ and profiles $R \in \mathcal{R}(A)^N$, there is no profile $R' \in \mathcal{R}(A)^N$ such that $\succ_j = \succ'_j$ for all $j \in N \setminus \{i\}$ and $f(R') \succ_i^F f(R)$.

Even though Fishburn strategyproofness seems like a relatively weak strategyproofness notion, it is only satisfied by very few SCCs. In particular, the omninomination rule, the Pareto rule, the top cycle, and the Condorcet rule are Fishburn strategyproof, while virtually all other commonly studied SCCs fail Fishburn strategyproofness. As an example of this claim, we give an example demonstrating the Fishburn manipulability of plurality rule (which chooses the alternatives top-ranked by most voters) and many other SCCs in Figure 2.

4. RESULTS

Our first result is a complete characterization of pairwise strategyproof SCCs in terms of dominant set rules. An SCC is a *dominant set rule* if for every preference profile R , it returns a dominant set with respect to \succsim_R . Examples of dominant set rules are the top cycle, the Condorcet rule (which returns the Condorcet winner if it exists and all alternatives otherwise), and the Condorcet nonloser rule (which returns all alternatives



FIGURE 2. Example showing the Fishburn manipulability of plurality rule. Plurality rule chooses $\{a, c\}$ for the left profile and $\{c\}$ for the right profile, so voter 5 can manipulate by deviating from the profile on the left to the one on the right. The same example shows that many other popular SCCs are Fishburn manipulable, for example, Borda’s rule, Nanson’s rule, Black’s rule, the maximin rule, Bucklin’s rule, Young’s rule, and Kemeny’s rule (we refer to Brandt et al. (2016) for definitions of these SCCs).

but a Condorcet loser). Observe that—even though dominant set rules may seem rather restricted—they allow for a fair degree of freedom in the choice of dominant sets. For instance, dominant set rules can take the majority margins into account. This is demonstrated by the SCC that returns an alternative x as unique winner if $g_R(x, y) > 2$ for all $y \in A \setminus \{x\}$ and otherwise returns all alternatives. It is also possible to define rather unnatural dominant set rules such as the SCC that returns the smallest dominant set whenever the majority graph contains a cycle with identical weights and the second smallest dominant set otherwise.

For our analysis, it suffices to consider the particularly simple subclass of robust dominant set rules: a dominant set rule f is *robust* if $f(R') \subseteq f(R)$ for all preference profiles R, R' such that $f(R)$ is dominant in R' . In other words, if the choice set $f(R)$ for some profile R is also dominant in another profile R' , then no alternative outside of $f(R)$ can be chosen for R' . It is easily seen that the top cycle, the Condorcet rule, and the Condorcet nonloser rule are robust dominant set rules, whereas the two artificial examples given above fail this condition. Moreover, robust dominant set rules are majoritarian and, therefore, also homogeneous.

Our first theorem shows that—under mild additional assumptions—robust dominant set rules are the *only* pairwise SCCs that satisfy strategyproofness.

THEOREM 1. *Let f be a pairwise SCC that satisfies nonimposition, homogeneity, and neutrality. Then f is strategyproof if and only if it is a robust dominant set rule.*

The direction from right to left is relatively straightforward: every robust dominant set rule f is strategyproof because robustness prohibits successful manipulations of dominant set rules. Consider, for example, that voter i manipulates from a profile R to another profile R' such that $f(R') \subsetneq f(R)$. According to Fishburn’s extension, we have that $f(R') \succ_i f(R) \setminus f(R')$. Moreover, $f(R') \succ_{R'} A \setminus f(R')$ because $f(R')$ is dominant in $\succ_{R'}$. Since voter i can only weaken alternatives in $f(R')$ against those in $f(R) \setminus f(R')$ when moving from R to R' , $f(R')$ will only be strengthened against $f(R) \setminus f(R')$ when moving from R' to R . Hence, $f(R') \succ_R f(R) \setminus f(R')$. This implies that $f(R')$ is also dominant in \succ_R because $f(R') \subsetneq f(R)$ and $f(R)$ is dominant in R . Since $f(R')$ is dominant in both R and R' , robustness implies $f(R) \subseteq f(R')$, which is at variance with our initial assumption that $f(R') \subsetneq f(R)$. A similar argument applies to the case that $f(R') \not\subseteq f(R)$ where strategyproofness now implies that $f(R') \setminus f(R) \succ_i f(R)$.

The converse direction—every pairwise SCC that satisfies nonimposition, homogeneity, neutrality, and strategyproofness is a robust dominant set rule—is much more difficult to prove. As a first step, we investigate the consequences of strategyproofness for pairwise SCCs. It turns out that we can abstract away from the concrete preferences of the voters and derive multiple axioms that describe how the choice set is affected when modifying the pairwise comparisons between pairs of alternatives. For instance, we show that rearranging unchosen alternatives in the voters' preferences does not affect the choice set of strategyproof and pairwise SCCs. The proofs of these implications heavily use the fact that two voters with preferences inverse to each other can be added to a preference profile without affecting the outcome of pairwise SCCs. As a second step, we use these axioms to derive some insights on the structure of choice sets returned by pairwise SCCs that satisfy strategyproofness, nonimposition, homogeneity, and neutrality. In more detail, we show for such an SCC f that (i) it chooses a single winner if and only if it is the Condorcet winner and (ii) for every alternative $x \in A$, either $f(R) = \{x\}$ or there is an alternative $y \in f(R) \setminus \{x\}$ such that $y \succ_R x$. The first condition is called strong Condorcet consistency, and we show that every pairwise, homogeneous, strategyproof, and strongly Condorcet-consistent coarsening of the top cycle is a robust dominant set rule. As the last step, we show that every pairwise SCC that satisfies the given axioms is a coarsening of the top cycle, which completes the proof of the theorem.

Theorem 1 has a number of important and perhaps surprising consequences. For instance, it implies that every strategyproof and pairwise SCC that satisfies the given conditions has to be majoritarian. In other words, every such SCC completely ignores the absolute values of majority margins, even though these values allow for the definition of more sophisticated SCCs.⁵ Moreover, Theorem 1 entails that many strategyproofness notions are equivalent under the assumptions of the theorem because robust dominant set rules satisfy much stronger notions of strategyproofness than Fishburn strategyproofness (see Remark 6).

Another interesting consequence of Theorem 1 is that the top cycle is the finest pairwise SCC that satisfies strategyproofness, nonimposition, homogeneity, and neutrality since it returns the smallest dominant set for any given preference profile. As shown in the sequel, we can turn this observation into a characterization of the top cycle by replacing nonimposition and neutrality with set nonimposition. An SCC f satisfies *set nonimposition* if for every nonempty set $X \subseteq A$, there is a profile R such that $f(R) = X$. In other words, every set is chosen for some preference profile R , which is in line with the original motivation of nonimposition for SCFs: the function's image coincides with its codomain. For neutral and pairwise SCCs, set nonimposition can be interpreted as a weak efficiency notion. To see this, assume that there is some set $X \subseteq A$ that is never returned by f . Now, consider a profile R such that $X \succ_i A \setminus X$ for all $i \in N$ and $x \sim_R y$ for all $x, y \in X$ and $x, y \in A \setminus X$. Neutrality and pairwise SCCs imply that f can only return X , A , or $A \setminus X$. Since f never returns X by assumption, we have that $A \setminus X \subseteq f(R)$. However, every voter $i \in N$ prefers X to $A \setminus X$ and the choice set of f is thus very inefficient.

⁵This insight resembles the fact that individual preference intensities can usually not be used by strategyproof voting rules (see, e.g., Nandeiham (2013), Ehlers, Majumdar, Mishra, and Sen (2020)). Note, however, that majority margins represent collective preference intensities.

The following lemma shows how set nonimposition can be used to single out the top cycle among all robust dominant set rules.

LEMMA 1. *The top cycle is the only robust dominant set rule that satisfies set nonimposition.*

PROOF. It has already been stated that TC is a robust dominant set rule. Moreover, TC satisfies set nonimposition because every set X is the smallest dominant set for every profile R such that $x \sim_R y$ for all $x, y \in X$, and $X \succ_R A \setminus X$; the existence of such a profile follows from McGarvey's construction (McGarvey (1953)).

For the other direction, consider a robust dominant set rule $f \neq TC$. Since f is not the top cycle, there is a profile R with dominant sets D_1, \dots, D_k such that $D_i \subseteq D_j$ if and only if $i \leq j$ and $f(R) = D_i$ with $i \geq 2$. This means that there is no profile R' such that $f(R') = D_1$ since otherwise, robustness from R' to R implies that $f(R) \subseteq f(R') = D_1$ because $f(R')$ is dominant in \succsim_R . In other words, f violates set nonimposition and the top cycle is consequently the only robust dominant set rule that satisfies this axiom. \square

The combination of Theorem 1 and Lemma 1 already characterizes the top cycle as the only pairwise SCC that satisfies strategyproofness, set nonimposition, homogeneity, and neutrality. It turns out that neutrality is not required for this characterization as the insights of the proof of Theorem 1 can be leveraged to establish that only robust dominant set rules satisfy pairwise SCC, strategyproofness, set nonimposition, and homogeneity. In particular, the axioms of Theorem 2 also imply strong Condorcet consistency, which together with our previous insights and Lemma 1 yields the following characterization.

THEOREM 2. *The top cycle is the only pairwise SCC that satisfies strategyproofness, set nonimposition, and homogeneity.*

We conclude the paper with a number of remarks.

REMARK 1 (Independence of the axioms). We can show that all of the axioms, except nonimposition, are required for the direction from left to right of Theorem 1. If we only omit pairwise SCC, the omninomination rule satisfies all required axioms but is no dominant set rule. If we dismiss neutrality, the following SCC based on two special alternatives a and b satisfies all requirements, but is no dominant set rule: f^{ab} returns $\{a\}$ if $a \succ_R A \setminus \{a, b\}$ and $a \succsim_R b$; otherwise it returns the outcome of the Condorcet rule. All axioms except homogeneity are satisfied by the SCC TC^* , which returns the top cycle with respect to the relation $x \succsim_R^* y$ if and only if $g_R(x, y) \geq -1$. However, TC^* is no robust dominant set rule because it depends on the majority margins. It is open whether nonimposition is required for Theorem 1. We discuss a variant of Theorem 1, which uses strong Condorcet consistency instead of neutrality and nonimposition in the Appendix. For this variant, it is easy to prove that all axioms are indeed required.

For the converse direction of Theorem 1, none of the auxiliary axioms is required. In particular, every robust dominant set rule is homogeneous and pairwise because robustness entails majoritarianism. Moreover, these SCCs satisfy strategyproofness regardless of whether they are neutral or nonimposing. For instance, the SCC that chooses the set $\{a, b, c\}$ if it is a dominant set and all alternatives otherwise is neither neutral nor nonimposing, but it is a robust dominant set rule and strategyproof.

For Theorem 2, we can show the independence of all axioms. Borda's rule only violates strategyproofness, the Condorcet rule only violates set nonimposition, the omninomination rule only violates pairwise, and TC^* only violates homogeneity.

REMARK 2 (Tournaments). A significant part of the literature focuses on the special case when there are no majority ties and the majority graph is a tournament (see, e.g., Laslier (1997), Brandt, Brill, and Harrenstein (2016)). This, for example, happens when the number of voters is odd. In the absence of majority ties and when $m \leq 4$, there is a strategyproof SCC known as the uncovered set, which satisfies all requirements of Theorem 1 but is no dominant set rule. When $m \geq 5$, the uncovered set violates strategyproofness and Theorem 1 holds even in the absence of majority ties.

REMARK 3 (Dropping homogeneity). The example given in Remark 1 for the independence of homogeneity only shows that robustness might be violated if we dismiss homogeneity, but the considered SCC is still a dominant set rule. It turns out that this observation is true in general if we mildly strengthen nonimposition to *unanimity* (a unanimously top-ranked alternative will be selected uniquely): every pairwise SCC that satisfies strategyproofness, unanimity, and neutrality is a dominant set rule if $m \neq 4$. The last condition is required because of the uncovered set discussed in Remark 2. By weakening robustness, one can thus obtain an alternative characterization of strategyproof SCCs based on *weak robustness*: an SCC f is weakly robust if $f(R') \subseteq f(R)$ for all preference profiles R, R' such that $g_R(x, y) \leq g_{R'}(x, y)$ for all $x \in f(R), y \in A \setminus f(R)$. Then, if $m \neq 4$, every pairwise SCC that satisfies unanimity and neutrality is strategyproof if and only if it is a weakly robust dominant set rule.

REMARK 4 (Weakening neutrality). Another variant of Theorem 1 can be obtained by weakening neutrality to the following condition: $x \in f(R)$ if and only if $y \in f(R)$ for every preference profile R and all pairs of alternatives $x, y \in A$ such that $g_R(x, y) = 0$ and $g_R(x, z) = g_R(y, z)$ for all $z \in A \setminus \{x, y\}$. This condition is void if there is an odd number of voters. As a result, homogeneity becomes more important for the proof as some steps only work for an even number of voters.

REMARK 5 (Weakening nonimposition). In the presence of neutrality, nonimposition can be weakened to a condition that merely requires that the SCC returns a singleton set for at least one profile. If we weaken neutrality, this is no longer possible and our proof suggests that, among the three auxiliary axioms, nonimposition plays the most important role as it is crucial for deriving strong Condorcet consistency.

REMARK 6 (Strengthening strategyproofness). Fishburn strategyproofness is a rather weak strategyproofness notion, which makes the direction from left to right in our characterizations strong. However, robust dominant set rules—especially the top cycle—are actually much more resistant against manipulation. To formalize this, we introduce a new preference extension, denoted by \succsim^{F+} , based on the relation \succsim_i^{\exists} over the subsets of A . This relation is defined as $X \succsim_i^{\exists} Y$ if and only if $X = \emptyset, Y = \emptyset$, or there are alternatives $x \in X, y \in Y$ such that $x \succ_i y$. Then

$$X \succsim_i^{F+} Y \quad \text{if and only if}$$

$$X \setminus Y \succ_i Y \setminus X \quad \text{and} \quad X \setminus Y \succsim_i^{\exists} X \cap Y \quad \text{and} \quad X \cap Y \succsim_i^{\exists} Y \setminus X.$$

Clearly, $X \succsim_i^F Y$ implies $X \succsim_i^{F+} Y$, and consequently, \succsim^{F+} -strategyproofness is stronger than Fishburn strategyproofness. We define an even stronger notion of strategyproofness based on the \succsim^{F+} extension as follows: an SCC f is *strongly* \succsim^{F+} -strategyproof if $f(R) \succsim_i^{F+} f(R')$ for all voters $i \in N$ and preference profiles R, R' with $\succ_j = \succ'_j$ for all $j \in N \setminus \{i\}$. Strong \succsim^{F+} -strategyproofness requires that all choice sets for manipulated preference profiles are comparable to the original choice set, making it much stronger than both \succsim^{F+} -strategyproofness and Fishburn strategyproofness. Strong Fishburn strategyproofness can be defined analogously. The top cycle is strongly \succsim^{F+} -strategyproof. Interestingly, Ching and Zhou (2002) have shown that only dictatorial and constant SCCs satisfy the slightly stronger notion of strong Fishburn strategyproofness, which obviously rules out the top cycle.

REMARK 7 (Group strategyproofness). An SCC f is group strategyproof if for all preference profiles R, R' and sets of voters $G \subseteq N$ such that $\succ_j = \succ'_j$ for $j \in N \setminus G$, it holds that $f(R') \not\succeq_i^F f(R)$ for some voter $i \in G$. Since every robust dominant set rule is group strategyproof, it follows from Theorem 1 that strategyproofness is equivalent to group strategyproofness for pairwise SCCs that satisfy homogeneity, nonimposition, and neutrality.

REMARK 8 (Pareto optimality). The main disadvantage of the top cycle is that it may return Pareto-dominated alternatives. In fact, every strategyproof pairwise SCC that satisfies our assumptions violates Pareto optimality. However, it is possible to circumvent this impossibility by first removing all Pareto-dominated alternatives and then computing the top cycle of the remaining alternatives. This SCC, $TC(PO)$ where PO stands for the Pareto rule, was already considered by Bordes (1979) and can be shown to be strategyproof. In fact, it satisfies all conditions of Theorem 1 except pairwiseness since it is not possible to compute the set of Pareto-dominated alternatives based on the majority margins only. Interestingly, the “converse” SCC, $PO(TC)$, which first computes the top cycle and then removes all Pareto-dominated alternatives, is nested in between $TC(PO)$ and TC but violates strategyproofness.

REMARK 9 (Fishburn efficiency). As discussed in the previous remark, the top cycle fails Pareto optimality. However, the top cycle satisfies the weaker notion of *Fishburn efficiency*, which requires that for every profile R , there is no set of alternatives X such that

$X \succ_i^F f(R)$ for all $i \in N$. Fishburn efficiency can be seen as a weak form of ex ante efficiency, where outcomes are compared before ties are broken. It is easy to see that the top cycle is the only robust dominant set rule satisfying this axiom since every other such rule already violates set nonimposition. It can moreover be shown that the top cycle is the coarsest majoritarian SCC that satisfies Fishburn efficiency, that is, every majoritarian SCC f that is Fishburn efficient satisfies that $f(R) \subseteq TC(R)$ for all preference profiles R . Since the top cycle is also the finest majoritarian SCC that satisfies Fishburn strategyproofness, neutrality, and nonimposition, it can be completely characterized using strategyproofness and efficiency.

REMARK 10 (Beyond the majority relation). Dominant set rules can be defined with respect to any complete binary relation derived from the preference profile. To formalize this idea, let the *information base* $I(R)$ denote a function that maps $R \in \mathcal{R}^*(A)$ to a complete binary relation $\succsim_{I(R)}$ on A . Applying a dominant set rule to $\succsim_{I(R)}$ clearly results in an SCC. Moreover, if $I(R)$ is *local* (i.e., $a \succsim_{I(R)} b$ if and only if $a \succsim_{I(R')} b$ for all $a, b \in A$ and $R, R' \in \mathcal{R}(A)^N$ such that $a \succ_i b$ if and only if $a \succ_i' b$ for all $i \in N$) and *monotone* (i.e., $a \succsim_{I(R)} b$ implies $a \succsim_{I(R')} b$ for all $a, b \in A$, and $R, R' \in \mathcal{R}^*(A)$ such that R' is derived from R by reinforcing a against b in the preference relation of a voter i), then every robust dominant set rule on $\succsim_{I(R)}$ is strategyproof. This proves, for instance, that dominant set rules based on *supermajority relations* (i.e., $a \succsim_{I(R)} b$ if and only if $g_R(a, b) \geq -k$ for some $k \in \mathbb{N}$) or on *shifted majority relations* (i.e., $a \succ_{I(R)} b$ if $g_R(a, b) > k$, $a \sim_{I(R)} b$ if $g_R(a, b) = k$, and $b \succ_{I(R)} a$ otherwise) are strategyproof.

For some information bases $I(R)$, it is even possible to prove statements analogous to Theorem 1 when demanding exclusive dependence on $I(R)$. To this end, we say an SCC is *$I(R)$ -based* for some information basis $I(R)$ if $f(R) = f(R')$ for all preference profiles $R, R' \in \mathcal{R}^*(A)$ such that $\succsim_{I(R)} = \succsim_{I(R')}$. This extends the definition of majoritarianism. For instance, it is easy to derive from our proof that robust dominant set rules on a supermajority relation $I(R)$ are the only neutral, nonimposing, strategyproof, and $I(R)$ -based SCCs. An equivalent statement holds for shifted majority relations $I(R)$ when defining neutrality based on $\succsim_{I(R)}$.

REMARK 11 (Fixed electorates). A nonstandard assumption in our model is that of a variable electorate. This assumption is necessary, because when fixing the number of voters, the Pareto rule satisfies all axioms of Theorem 1 and Theorem 2 but fails to be a dominant set rule. We now sketch two approaches to adapt our results to a fixed electorate framework. First, we may replace pairwise and homogeneity with majoritarianism. Since the construction of McGarvey (1953) allows us to build every majority relation with at most m^2 voters, we need at most $m^2 + 2$ voters for our results to hold under majoritarianism. The second approach is to restrict attention to profiles whose maximal majority margin is bounded by a constant $c \geq 2$. This is possible because we never need to increase the maximal majority margin to a value larger than c in our proofs.⁶ Using again

⁶This is not in conflict with the fact that we sometimes use homogeneity to duplicate preference profiles in the proof, because it is either possible to entirely avoid these homogeneity applications, or to ensure that all majority margins are 1 before duplicating the profile.

McGarvey's construction, every profile with a majority margin of at most c can be built with cm^2 voters and we thus need at most $cm^2 + 2$ voters for our proof. Hence, we can show that every nonimposing, neutral, strategyproof, homogeneous, and pairwise SCC is a robust dominant set rule for profiles with maximal majority margin of at most c if there are $cm^2 + 2$ voters (here, homogeneity is defined for majority margins).

APPENDIX: OMITTED PROOFS

This Appendix contains the proofs of Theorems 1 and 2. Proof sketches for these results were given in Section 4 and we here focus on the details. Since the proofs are rather involved, we divide them into multiple lemmas, which are organized in subsections to highlight related ideas. In particular, we discuss additional notation in Appendix A.1, some general results on the structure of the top cycle in Appendix A.2, implications of Fishburn strategyproofness for pairwise SCCs in Appendix A.3, a variant of Theorem 1 that relies on strong Condorcet consistency in Appendix A.4, and finally the proofs of our main results in Appendix A.5.

A.1 Notation

Before discussing our proofs, we need to introduce some additional notation. First, we specify how we denote preference relations. We usually write preference relations as comma-separated lists. In these lists, we use $\mathbf{lex}(X)$ and $\mathbf{lex}(X)^{-1}$ to indicate that the alternatives in a set X are ordered lexicographically or inversely lexicographically. For instance, $a, \mathbf{lex}(\{b, c\}), d$ is equivalent to a, b, c, d and means that a is preferred to b , b to c , and c to d . Similarly, $a, \mathbf{lex}(\{b, c\})^{-1}, d$ is equivalent to a, c, b, d . Furthermore, we occasionally interpret a voter's preference relation as a set of tuples and use set operations such as set intersections and set differences to form new preference relations. In particular, we write $\succ_i|_X$ to denote the restriction of \succ_i to X , that is, $\succ_i|_X = \succ_i \cap X^2$. We use the same notation for the majority relation, that is, $\succsim_R|_X$ denotes the restriction of \succsim_R to X . For instance, $\succsim_R|_X = \succsim'_R|_X$ means that the majority relations of R and R' agree on the alternatives in X .

The second important concept is that of cycles in the majority relation. A *cycle* in a majority relation \succsim_R is a sequence of $q \geq 2$ alternatives (a_1, \dots, a_q) such that $a_i \succsim_R a_{i+1}$ for all $i \in \{1, \dots, q-1\}$, $a_q \succsim_R a_1$, and $a_i \neq a_j$ for all distinct $i, j \in \{1, \dots, q\}$. Informally, a cycle is a path in \succsim_R that starts and ends at the same alternative and visits every alternative on the cycle (except the first one) only once. For instance, in the majority graph in Figure 1, $C = (a, b, c)$ is a cycle. While slightly overloading notation, we denote with C both the ordered sequence of alternatives that defines a cycle and the set of alternatives contained in the cycle.

Finally, we introduce the notions of connectors and connected sets. The *connected set* A_x of an alternative $x \in A$ in a profile R contains all alternatives (except x) that drop out of the top cycle if we remove x from the preference profile, that is, $A_x = TC(R) \setminus (TC(R|_{A \setminus \{x\}}) \cup \{x\})$. The notion of connected sets helps us to distinguish

the alternatives in the top cycle further: we say an alternative $x \in A$ is a *connector* in R if $A_x \neq \emptyset$. This means intuitively that, if we remove x from the preference profile, x and additional alternatives drop out of the top cycle. In other words, x connects the alternatives in A_x to the rest of the top cycle. Note that an alternative $x \notin TC(R)$ cannot be a connector since $A_x = \emptyset$ for these alternatives and that connectors only exist if $|TC(R)| \geq 3$.

A.2 Structure of the top cycle

For the proofs of our results, it will be helpful to have a deeper understanding of the structure of the top cycle. In more detail, we first show that the top cycle is closely connected to cycles in the majority relation. [Moulin \(1986\)](#) has shown such a statement under the assumption that there are no majority ties: there is a cycle in the majority relation \succsim_R that connects all the alternatives in $TC(R)$. Since we need to allow for majority ties, we generalize this result by interpreting majority ties as bidirectional edges.

LEMMA 2. *Let R be a preference profile. It holds for a set $X \subseteq A$ with $|X| \geq 2$ that $TC(R) = X$ if and only if there is a cycle $C = (a_1, \dots, a_{|X|})$ in \succsim_R such that $C = X$ and $X \succ_R A \setminus X$. Furthermore, $TC(R) = \{x\}$ if and only if x is the Condorcet winner in R .*

PROOF. We first prove that $TC(R) = \{x\}$ if and only if x is the Condorcet winner in R . Thus, note that $\{x\} = TC(R)$ implies that $x \succ_R A \setminus \{x\}$ because the top cycle returns a dominant set. Hence, x is the Condorcet winner if it is the unique winner of the top cycle. Next, let x denote the Condorcet winner in a preference profile R . It follows that $x \succ_R A \setminus \{x\}$ and, therefore, $\{x\}$ is a dominant set. Even more, it is obviously the smallest dominant set, and thus $TC(R) = \{x\}$, which proves the first claim.

Next, we focus on sets of alternatives $X \subseteq A$ with $|X| \geq 2$ and show first that if $X = TC(R)$, there is a cycle C in \succsim_R such that $C = X$ and $X \succ_R A \setminus X$. Since the latter condition directly follows from the definition of the top cycle, we only have to show that there is a cycle in \succsim_R containing all alternatives in X . Note for this that if there is an alternative $x \in X$ with $x \succ_R X \setminus \{x\}$, this alternative is the Condorcet winner and $TC(R) = \{x\} \neq X$. Consequently, for every alternative $x \in X$, there is another alternative $y \in X \setminus \{x\}$ such that $y \succsim_R x$. This means that there is a cycle in $\succsim_R|_X$. Let $C = (a_1, \dots, a_q)$ denote an inclusion maximal cycle in $\succsim_R|_X$ and assume for contradiction that there is an alternative $y \in X \setminus C$.

As a first step, consider the case that there are two distinct alternatives $a_i, a_j \in C$ such that $a_i \succsim_R y$ and $y \succsim_R a_j$. In this case, we can extend the cycle C by adding y , which contradicts the inclusion maximality of C . Note for this that we can find two alternatives $a_k, a_{k+1} \in C$ such that a_{k+1} is the successor of a_k in C , $a_k \succsim_R y$, and $y \succsim_R a_{k+1}$. Otherwise, it holds for all $a_l \in C$ that $a_l \succsim_R y$ implies for its successor a_{l+1} in C that $a_{l+1} \succ_R y$. If we start at a_i and subsequently apply this argument along the cycle C , we derive eventually that $a_l \succ_R y$ for all $a_l \in C$, which contradicts that $y \succsim_R a_j$. Hence, there must be such alternatives a_k and a_{k+1} and we can extend the cycle C to $C' = (a_1, \dots, a_k, y, a_{k+1}, \dots, a_q)$.

As a consequence of the last case, it holds for all alternatives $x \in X \setminus C$ that either $x \succ_R C$ or $C \succ_R x$. We partition the alternatives in $X \setminus C$ with respect to these two options into the sets $X_1 = \{x \in X \setminus C : x \succ_R C\}$ and $X_2 = \{x \in X \setminus C : C \succ_R x\}$. If $X_1 = \emptyset$, then $C \succ_R A \setminus C$, which contradicts that $X = TC(R)$ because C is a smaller dominant set than X . If $X_2 = \emptyset$ or $X_1 \succ_R X_2$, then $X_1 \succ_R A \setminus X_1$, which again contradicts that $X = TC(R)$ because X_1 is now a smaller dominant set than X . Thus, both X_1 and X_2 are nonempty and there is a pair of alternatives $x_1 \in X_1, x_2 \in X_2$ such that $x_2 \succsim_R x_1$. However, this means that we can extend the cycle C by adding x_1 and x_2 as $a_1 \succ_R x_2, x_2 \succsim_R x_1$, and $x_1 \succ_R a_2$. This contradicts the inclusion maximality of C and, therefore, the initial assumption that $C \neq X$ was incorrect.

Finally, we prove that $TC(R) = X$ for a set $X \subseteq A$ with $|X| \geq 2$ if there is a cycle $C = (a_1, \dots, a_{|X|})$ in \succsim_R with $C = X$ and $X \succ_R A \setminus X$. Note for this that X is a dominant set in \succsim_R if it satisfies these conditions. Since dominant sets are totally ordered by set inclusion and the top cycle is the smallest dominant set, it follows that $TC(R) \subseteq X$. Next, assume that $X \setminus TC(R) \neq \emptyset$, which means that $TC(R) \succ_R X \setminus TC(R)$ because of the definition of the top cycle. However, then there cannot be a cycle in \succsim_R that connects all alternatives in X because there is no path from an alternative in $X \setminus TC(R)$ to an alternative in $TC(R)$. This contradicts our assumptions, and thus the assumption $X \setminus TC(R) \neq \emptyset$ was incorrect. Hence, it follows that $X = TC(R)$. □

Lemma 2 is one of the most important insights for our subsequent proofs as the existence of the cycle provides paths between all alternatives $x, y \in TC(R)$. This insight will also be used in the next lemma, where we investigate connected sets.

LEMMA 3. *Let R be a preference profile and suppose that x is a connector in R . Moreover, let $y \in A_x$ denote an alternative in the connected set of x . It holds that $A_y \subseteq A_x$ unless $x \succ_R A \setminus \{x, y\}$.*

PROOF. Consider an arbitrary preference profile R and a connector x in R . Note that the existence of a connector implies that $k = |TC(R)| \geq 3$. Thus, let $C = (a_1, \dots, a_k)$ denote a cycle connecting the alternatives in $TC(R)$; such a cycle exists because of Lemma 2. Since connectors need to be in the top cycle, it follows that there is an index i such that $x = a_i$. In the sequel, we assume without loss of generality that $x = a_1$ since we can decide on the starting point of the cycle.

As a first step, we show that there is an index $l \in \{2, \dots, k - 1\}$ such that $A_x = \{a_{l+1}, \dots, a_k\}$. Consider for this the profile $R^{-x} = R|_{A \setminus \{x\}}$ derived from R by removing x from the preference profile. We next determine the top cycle in R^{-x} because $A_x = TC(R) \setminus (TC(R^{-x}) \cup \{x\})$. First, note that $TC(R^{-x}) \subseteq TC(R) \setminus \{x\}$ because all alternatives in $TC(R) \setminus \{x\} \neq \emptyset$ still strictly dominate all alternatives outside of this set. This implies that a_2 , the successor of $x = a_1$ on C , is in $TC(R^{-x})$ because it can reach every other alternative $a_i \in TC(R) \setminus \{a_1, a_2\}$ via $\succsim_{R^{-x}}$: we can simply traverse the cycle C to go from a_2 to a_i . Now, if $a_2 \succ_R TC(R) \setminus \{a_1, a_2\}$, then a_2 is the Condorcet winner in R^{-x} , and thus $A_x = \{a_3, \dots, a_k\}$, so $l = 3$ satisfies our condition. Otherwise, let $h_1 \in \{3, \dots, k\}$ denote the largest index such that $a_{h_1} \succsim_R a_2$. It follows from the definition of the top cycle

that $a_{h_1} \in TC(R^{-x})$ because $a_2 \in TC(R^{-x})$, and thus $\{a_2, \dots, a_{h_1}\} \subseteq TC(R^{-x})$ because all these alternatives can reach a_{h_1} by traversing the cycle C . It is easy to see that we can repeat this argument: if $\{a_2, \dots, a_{h_1}\} \succ_R \{a_{h_1+1}, \dots, a_k\}$, then $TC(R^{-x}) = \{a_2, \dots, a_{h_1}\}$ and $A_x = \{a_{h_1+1}, \dots, a_k\}$. Otherwise, we can find the largest index $h_2 \in \{h_1 + 1, \dots, k\}$ such that a_{h_2} dominates an alternative in $\{a_2, \dots, a_{h_1}\}$. Then it follows that $a_{h_2} \in TC(R^{-x})$, and consequently, $\{a_2, \dots, a_{h_2}\} \subseteq TC(R^{-x})$ because we can traverse the cycle C to find a path for every such alternative a_i to a_{h_2} . By repeating this argument, we eventually arrive at an index l such that $A_x = \{a_{l+1}, \dots, a_k\}$ since $A_x \neq \emptyset$.

Next, consider an arbitrary alternative $y \in A_x$. Our goal is to prove that $A_y \subseteq A_x$, and thus we consider the profile $R^{-y} = R|_{A \setminus \{y\}}$. We will show that $TC(R^{-x}) \cup \{x\} \subseteq TC(R^{-y})$ because then $A_y = TC(R) \setminus (TC(R^{-y}) \cup \{y\}) \subseteq TC(R) \setminus (TC(R^{-x}) \cup \{x\}) = A_x$. For this, we employ a case distinction with respect to y and first suppose that y is not the direct predecessor of x on C , that is, $y = a_i$ for some $i < k$. Hence, let $y' = a_{i+1}$ denote the successor of y on C and note that our previous insights show that $y' \in A_x$, too. It holds that $y' \in TC(R^{-y})$ because y' can reach every alternative $a_j \in TC(R) \setminus \{y, y'\}$ in $\succsim_{R^{-y}}$ by traversing the cycle C . Next, note that $TC(R^{-x}) \succ_{R^{-x}} y'$ because $y' \notin TC(R^{-x})$ and, therefore, also $TC(R^{-x}) \succ_{R^{-y}} y'$. This proves that $TC(R^{-x}) \subseteq TC(R^{-y})$ because $y' \in TC(R^{-y})$. In particular, $x' = a_2$, the successor of $x = a_1$ on the cycle C , is in $TC(R^{-y})$ because $x' \in TC(R^{-x})$. Since $x \succsim_{R^{-y}} x'$, it follows also that $x \in TC(R^{-y})$, which proves that $TC(R^{-x}) \cup \{x\} \subseteq TC(R^{-y})$, and thus $A_y \subseteq A_x$.

As second case, suppose that $y = a_k$, that is, y is the direct predecessor of $x = a_1$ on C . In this case, we immediately derive that $x \in TC(R^{-y})$ because we can again traverse the cycle C to find a path from x to every other alternative $a_i \in TC(R) \setminus \{x, y\}$ in $\succsim_{R^{-y}}$. Next, it is important that there is an alternative $z \in A \setminus \{x, y\}$ such that $z \succsim_R x$. If there is no such alternative, then $x \succ_R A \setminus \{x, y\}$ and we have nothing to show as this is the exception stated in the lemma. Since $z \neq y$ and $z \succsim_R x$, it follows also that $z \in TC(R)$ and $z \in TC(R^{-y})$. Now, if $z \in A_x$, then $TC(R^{-x}) \subseteq TC(R^{-y})$ because $TC(R^{-x}) \succ_R z$. Conversely, if $z \in TC(R^{-x}) = TC(R) \setminus (A_x \cup \{x\})$, we use the fact that there is a cycle C' connecting the alternatives $TC(R^{-x})$ in $\succsim_{R^{-x}}$. This cycle exists also in \succsim_R , and since $y \notin TC(R^{-x})$, also in $\succsim_{R^{-y}}$. Hence, there is a path from every alternative $a_i \in TC(R^{-x})$ to z , which proves that $TC(R^{-x}) \cup \{x\} \subseteq TC(R^{-y})$. Thus, it follows also in this case that $A_y \subseteq A_x$, which proves the lemma. \square

A.3 Implications of strategyproofness

In the context of pairwise SCCs, it is inconvenient to work with the preference relations of individual voters since the main idea of these SCCs is to abstract away from profiles. However, strategyproofness requires information about a voter's preference relation to deduce which choice sets are possible before and after a manipulation. To mitigate this tradeoff, we analyze the implications of strategyproofness for pairwise SCCs in this section. This leads to the definition of four axioms, all of which are satisfied by every pairwise and strategyproof SCC. Also, the first three of these axioms are weakened versions of a property known as set-monotonicity (see Brandt (2015), Brandt, Brill, and Harrenstein (2016)).

In more detail, we investigate how the choice set of a strategyproof and pairwise SCC is allowed to change if a voter reinforces or weakens an alternative against some other alternatives. Formally, reinforcing an alternative a against some other alternative b in the preference relation of voter i means that voter i switches from $b \succ_i a$ to $a \succ'_i b$ and nothing else changes in voter i 's preference relation or in the preference relations of other voters. Conversely, weakening an alternative a against some other alternative b in the preference relation of voter i means that voter i reinforces b against a . Note that weakening or reinforcing an alternative a against another alternative b requires that a and b are adjacent in \succ_i , that is, there is no alternative $z \in A \setminus \{a, b\}$ such that $a \succ_i z \succ_i b$ or $b \succ_i z \succ_i a$.

Depending on whether the alternatives a and b are chosen, strategyproofness has different consequences when reinforcing a against another alternative b . The first case that we consider is to reinforce a chosen alternative a against another alternative b . A natural requirement in this situation is monotonicity, which demands that a chosen alternative is still chosen after reinforcing it (see, e.g., [Moulin \(1988\)](#)). Unfortunately, we cannot show that strategyproofness implies monotonicity for pairwise SCCs. For instance, assume that a voter submits b, a, c , and $\{a, c\}$ is chosen. Next, voter i reinforces a against b and as result $\{b, c\}$ is chosen. In this example, Fishburn's set extension does not allow to compare $\{a, c\}$ to $\{b, c\}$, and hence, this is no violation of strategyproofness. As a consequence, we consider a weakened variant of monotonicity, which we refer to as weak monotonicity (WMON). This axiom requires that, if a voter reinforces a chosen alternative a against another alternative b , then a is still in the choice set unless b is chosen after the manipulation but not before.

DEFINITION 1 (Weak monotonicity (WMON)). An SCC f satisfies weak monotonicity (WMON) if $a \in f(R)$ implies $a \in f(R')$ or $b \in f(R') \setminus f(R)$ for all alternatives $a, b \in A$, and preference profiles R, R' for which there is a voter i such that $\succ'_j = \succ_j$ for all $j \in N \setminus \{i\}$ and $\succ'_i = \succ_i \setminus \{(b, a)\} \cup \{(a, b)\}$.

WMON has multiple important consequences. First, if we reinforce a chosen alternative a against another chosen alternative b , it guarantees that a remains chosen because $b \notin f(R') \setminus f(R)$. Second, if we reinforce a chosen alternative a against an unchosen alternative b , either $a \in f(R')$ and $b \notin f(R')$, or $a \notin f(R')$ and $b \in f(R')$. If both alternatives were chosen after this step, we could reinforce b against a in voter i 's preference relation to revert back to the original preference profile R , and WMON implies that b remains chosen. However, this is in conflict with the assumption that b is not chosen for R . Conversely, it follows directly from the definition of WMON that it is not possible that $a, b \notin f(R')$ if $a \in f(R)$. Finally, it should be mentioned that monotonicity implies weak monotonicity because it requires that a chosen alternative a remains chosen after reinforcing it. Or, put differently, monotonicity excludes additionally the case that a becomes unchosen and b becomes chosen after reinforcing a against b .

Unfortunately, WMON does not guarantee that weakening an unchosen alternative means that the unchosen alternative remains unchosen. We thus introduce weak set-monotonicity (WSMON) as our second axiom, which is concerned with what happens if

we weaken an unchosen alternative not against a single alternative but against all other alternatives.

DEFINITION 2 (Weak set-monotonicity (WSMON)). An SCC f satisfies weak set-monotonicity (WSMON) if $f(R) = f(R')$ for all preference profiles R, R' for which a voter $i \in N$ and an alternative $a \notin f(R)$ exist such that $\succ_j = \succ'_j$ for all $j \in N \setminus \{i\}$, $\succ_i|_{A \setminus \{a\}} = \succ'_i|_{A \setminus \{a\}}$, $a \succ_i A \setminus \{a\}$, and $A \setminus \{a\} \succ'_i a$.

The idea of WSMON is that moving an unchosen alternative from the first place to the last place in a voter's preference relation should not affect the outcome. This is a weaker variant of set-monotonicity, which requires that weakening an unchosen alternative against a single alternative does not affect the choice set. Unfortunately, we cannot prove this stronger variant because we cannot even prove monotonicity at this point. However, pushing the top-ranked alternative to the bottom of the preference ranking is a rather common operation in the analysis of strategyproof SCCs, which is often referred to as push-down lemma (see, e.g., Zwicker (2016)).

The third situation that we are concerned with is that a voter only reorders unchosen alternatives. Intuitively, such an operation should not change the choice set as no relevant comparisons change. This idea is formalized as independence of unchosen alternatives (IUA).

DEFINITION 3 (Independence of unchosen alternatives (IUA)). An SCC f satisfies independence of unchosen alternatives (IUA) if $f(R) = f(R')$ for all preference profiles R, R' for which a voter $i \in N$ and alternatives $B \subseteq A \setminus f(R)$ exist such that $\succ_j = \succ'_j$ for all voters $j \in N \setminus \{i\}$ and $\succ_i \setminus \succ_i|_B = \succ'_i \setminus \succ'_i|_B$.

Independence of unchosen alternatives, also called independence of losers, is a well-known axiom (see, e.g., Laslier (1997), Brandt (2011, 2015)), which requires that the choice set is invariant with respect to modifications of preferences between unchosen alternatives. In particular, if a voter reinforces an unchosen alternative against another unchosen alternative, the choice set is not allowed to change. Just like WMON and WSMON, IUA is implied by set-monotonicity.

Finally, we introduce an axiom with a different spirit than the previous ones: instead of asking whether an alternative a is chosen after weakening or reinforcing it, we ask whether alternatives that are not involved in the swap are chosen or not. Intuitively, it seems plausible that if an alternative is not affected by a manipulation, its membership in the choice set should not change. However, this condition, whose spirit is similar to the localizedness property used in the characterization of strategyproof randomized social choice functions by Gibbard (1977), is extremely restrictive. Here, we consider a weaker variant: if a voter changes his preference relation between some alternatives B and the inclusion of the alternatives in B in the choice set is unaffected by this modification, then the choice set should not change at all. This idea leads to weak localizedness (WLOC), which is formalized below.

DEFINITION 4 (Weak localizedness (WLOC)). An SCC f satisfies weak localizedness (WLOC) if $f(R) = f(R')$ for all preference profiles R, R' for which a voter $i \in N$ and alternatives $B \subseteq A$ exist such that $\succ_j = \succ'_j$ for all voters $j \in N \setminus \{i\}$, $\succ_i \setminus \succ_i|_B = \succ'_i \setminus \succ'_i|_B$, and $B \cap f(R) = B \cap f(R')$.

To the best of our knowledge, neither WLOC nor similar axioms have been studied before for social choice correspondences. Furthermore, it should be mentioned that WLOC—even though it might seem weak when considered in isolation—is quite powerful when combined with other axioms. For instance, the combination of WMON and WLOC implies that swapping two chosen alternatives can only affect the choice set if the weakened alternative becomes unchosen.

We now prove that strategyproof and pairwise SCCs satisfy all axioms discussed in this section.

LEMMA 4. *Every strategyproof and pairwise SCC satisfies WMON, WSMON, IUA, and WLOC.*

PROOF. Let f denote a strategyproof and pairwise SCC. We consider each axiom listed in the lemma separately, but each proof relies on the same idea: we assume for contradiction that f fails the considered axiom, which means that there are two profiles R and R' that differ in the preference relation of a single voter i and $f(R)$ and $f(R')$ violate the conditions of the axiom. Next, we add two new voters i^* and j^* with inverse preferences such that voter i^* can make the same modification as voter i . This leads to new preference profiles R^1 and R^2 such that $f(R^1) = f(R)$ and $f(R^2) = f(R')$ due to pairwise-ness. Finally, we can choose the preference relation of voter i^* such that deviating from R^1 to R^2 is a manipulation, and thus obtain a contradiction to the strategyproofness of f .

WMON: Following the idea explained above, we assume for contradiction that f violates WMON. This means that there are preference profiles R, R' , alternatives $a \in f(R)$, $b \in A \setminus \{a\}$, and a voter $i \in N$ such that $\succ'_j = \succ_j$ for all $j \in N \setminus \{i\}$ and $\succ'_i = \succ_i \setminus \{(b, a)\} \cup \{(a, b)\}$, but $a \notin f(R')$ and $b \notin f(R') \setminus f(R)$. Next, we let R^1 denote the profile derived from R by adding the voters i^* and j^* . The preference relations of these voters are shown below, where $\bar{A} = A \setminus \{a, b\}$ and $\bar{f}(R) = f(R) \setminus \{a, b\}$. Moreover, the profile R^2 evolves out of R^1 by letting voter i^* swap a and b :

$$\begin{aligned} \succ_{i^*}^1 &= \mathbf{lex}(\bar{A} \setminus f(R), \mathbf{lex}(\bar{f}(R) \cap f(R')), b, a, \mathbf{lex}(\bar{f}(R) \setminus f(R'))) \\ \succ_{i^*}^2 &= \mathbf{lex}(\bar{f}(R) \setminus f(R'))^{-1}, a, b, \mathbf{lex}(\bar{f}(R) \cap f(R'))^{-1}, \mathbf{lex}(\bar{A} \setminus f(R))^{-1} \end{aligned}$$

Since the preference relations of voter i^* and j^* are inverse in R^1 , pairwise-ness implies that $f(R^1) = f(R)$. Moreover, this axiom also requires that $f(R^2) = f(R')$. It then follows that voter i^* can manipulate by deviating from R^1 to R^2 as he prefers all alternatives in $f(R') \setminus f(R)$ to those in $f(R)$ and all alternatives in $f(R')$ to those in $f(R) \setminus f(R')$. This can be seen by making a case distinction on whether $b \in f(R)$: if $b \notin f(R)$, then our contradiction assumption implies that $b \notin f(R')$, too. Hence, no alternative in $(f(R) \setminus f(R')) \cup \{b\}$ is chosen, which ensures that this is a manipulation for voter i^*

since $a \in f(R) \setminus f(R')$. Conversely, if $b \in f(R)$, then b is either in $f(R) \cap f(R')$ or in $f(R) \setminus f(R')$. Both cases constitute again a manipulation as $b \succ_{i^*} f(R) \setminus (f(R') \cup \{b\})$ and $(f(R) \cap f(R')) \setminus \{b\} \succ_{i^*} b$. Hence, switching from R^1 to R^2 is in all cases a manipulation for voter i^* , which contradicts the strategyproofness of f . Consequently, the initial assumption that f violates WMON was incorrect.

WSMON: As second case, assume that f fails WSMON. Thus, there are preference profiles R, R' , a voter $i \in N$, and an alternative $a \notin f(R)$ such that R and R' only differ in the fact that $a \succ_i A \setminus \{a\}$ and $A \setminus \{a\} \succ'_i a$, but $f(R) \neq f(R')$. Consider the profile R^1 , which is derived from R by adding the voters i^* and j^* with the preferences shown below. Moreover, R^2 evolves out of R^1 by letting voter i^* make a into his least preferred alternative:

$$\begin{aligned} \succ_{i^*}^1 &= a, \mathbf{lex}(A \setminus (\{a\} \cup f(R))), \mathbf{lex}(f(R) \cap f(R')), \mathbf{lex}(f(R) \setminus f(R')) \\ \succ_{j^*}^1 &= \mathbf{lex}(f(R) \setminus f(R'))^{-1}, \mathbf{lex}(f(R) \cap f(R'))^{-1}, \mathbf{lex}(A \setminus (\{a\} \cup f(R)))^{-1}, a \end{aligned}$$

It is again easy to verify that $f(R^1) = f(R)$ and $f(R^2) = f(R')$ because of pairwise-ness. Thus, voter i^* can manipulate f by switching from R^1 to R^2 because he prefers all alternatives in $A \setminus f(R)$ to all alternatives in $f(R)$ and all alternatives in $f(R) \cap f(R')$ to all alternatives in $f(R) \setminus f(R')$. This contradicts the strategyproofness of f and, therefore, the initial assumption that f violates WSMON was incorrect.

IUA: Third, assume that f violates IUA, which means that there are preference profiles R, R' , a voter $i \in N$, and a set of alternatives $B \subseteq A \setminus f(R)$ such that $\succ_j = \succ'_j$ for all voters $j \in N \setminus \{i\}$, $\succ_i \setminus \succ_i|_B = \succ'_i \setminus \succ'_i|_B$, and $f(R) \neq f(R')$. Now, consider the profile R^1 derived from R by adding two voters i^* and j^* . The preference relations of these two voters are shown below, where $\bar{A} = A \setminus B$, $\succ_i|_B$ indicates that the alternatives in B are ordered as in \succ_i , and $\succ_i^{-1}|_B$ that the alternatives in B are ordered exactly inverse to \succ_i . Moreover, let R^2 denote the profile derived from R^1 by letting voter i^* order the alternatives in B as voter i does in R' :

$$\begin{aligned} \succ_{i^*}^1 &= \succ_i|_B, \mathbf{lex}(\bar{A} \setminus f(R)), \mathbf{lex}(f(R) \cap f(R')), \mathbf{lex}(f(R) \setminus f(R')) \\ \succ_{j^*}^1 &= \mathbf{lex}(f(R) \setminus f(R'))^{-1}, \mathbf{lex}(f(R) \cap f(R'))^{-1}, \mathbf{lex}(\bar{A} \setminus f(R))^{-1}, \succ_i|_B^{-1} \end{aligned}$$

Just as before, we infer from pairwise-ness that $f(R^1) = f(R)$ and $f(R^2) = f(R')$. However, this means that voter i^* can manipulate by deviating from R^1 to R^2 : by construction, he prefers all alternatives in $A \setminus f(R)$ to all alternatives in $f(R)$, and all alternatives in $f(R) \cap f(R')$ to all alternatives in $f(R) \setminus f(R')$. Since $f(R) \neq f(R')$, this is in conflict with strategyproofness.

WLOC: Finally, suppose for contradiction that f violates WLOC. Hence, there are two preference profiles R, R' , a nonempty set of alternatives $B \subseteq A$, and a voter $i \in N$ such that $\succ_j = \succ'_j$ for all $j \in N \setminus \{i\}$, $\succ_i \setminus \succ_i|_B = \succ'_i \setminus \succ'_i|_B$, $f(R) \cap B = f(R') \cap B$, but $f(R) \neq f(R')$. Once again, we derive a new profile R^1 from R by adding two voters i^* and j^* . The preferences of these voters are shown below, where $\bar{A} = A \setminus B$ and $\bar{f}(R) = f(R) \setminus$

B. Moreover, let R^2 denote the profile derived from R^1 by letting voter i^* arrange the alternatives in B according to \succ_i^1 :

$$\begin{aligned} \succ_{i^*}^1 &= \mathbf{lex}(\bar{A} \setminus f(R)), \succ_i|_B, \mathbf{lex}(\bar{f}(R) \cap f(R')), \mathbf{lex}(\bar{f}(R) \setminus f(R')) \\ \succ_{j^*}^1 &= \mathbf{lex}(\bar{f}(R) \setminus f(R'))^{-1}, \mathbf{lex}(\bar{f}(R) \cap f(R'))^{-1}, \succ_i^{-1}|_B, \mathbf{lex}(\bar{A} \setminus f(R))^{-1} \end{aligned}$$

Also in this case, pairwise-ness shows that $f(R^1) = f(R)$ and $f(R^2) = f(R')$. Thus, voter i^* can manipulate by switching from R^1 to R^2 because $f(R') \succ_{i^*}^1 f(R) \setminus f(R')$ and $f(R') \setminus f(R) \succ_{i^*}^1 f(R)$. For both claims, it is important that B is disjoint to both $f(R) \setminus f(R')$ and $f(R') \setminus f(R)$ since an alternative $x \in B$ is in $f(R)$ if and only if it is in $f(R')$. Hence, the first claim follows directly as the alternatives in $f(R) \setminus f(R')$ are the least preferred ones of voter i^* , and the second claim follows since $f(R') \setminus f(R) \subseteq A \setminus (B \cup f(R))$. Thus, deviating from R^1 to R^2 is a manipulation for voter i^* , which contradicts the strategyproofness of f . \square

A.4 Consequences of strong Condorcet consistency

In this section, we prove a variant of Theorem 1 which relies on *strong Condorcet consistency* instead of neutrality and nonimposition. This axiom requires of an SCC f that $f(R) = \{x\}$ if and only if x is the Condorcet winner in R . Less formally, strongly Condorcet-consistent SCCs have to elect the Condorcet winner whenever there is one, and cannot elect a single alternative in the absence of a Condorcet winner. The main result of this section states that robust dominant set rules are the only SCCs that satisfy pairwise-ness, homogeneity, strong Condorcet consistency, and strategyproofness. As we will see in Appendix A.5, the combination of pairwise-ness, homogeneity, strategyproofness, neutrality, and nonimposition implies strong Condorcet consistency, which means that this auxiliary claim is actually more general than Theorem 1. Moreover, the axioms of Theorem 2 imply strong Condorcet consistency and we can therefore use the results of this section also to characterize the top cycle.

For proving the results of this section, we rely on the lemmas of the previous subsections. In particular, we often say that we reinforce an alternative x against another alternative y without specifying which voter reinforces x against y . This is possible because we can always add two voters with inverse preferences such that one of them can perform the required manipulation. Adding these two voters does not affect the choice set because of pairwise-ness and the consequences of the deviation will be specified by the axioms of Appendix A.3. Hence, we can abstract away from the exact preference profiles and focus on the majority margins. For the readers' convenience, we repeat the four axioms of the last section because they form the basis of the following proofs. Let f be a pairwise SCC, $a, b \in A$, and $R, R' \in \mathcal{R}^*(A)$.

- **WMON:** If R' is derived from R by reinforcing a against b and $a \in f(R)$, then $a \in f(R)$ or $b \in f(R') \setminus f(R)$.
- **WSMON:** If R' is derived from R by weakening a against all other alternatives $x \in A \setminus \{a\}$ and $a \notin f(R)$, then $f(R) = f(R')$.

- IUA: If R' is derived from R by reordering some alternatives in $A \setminus f(R)$, then $f(R) = f(R')$.
- WLOC: If R' is derived from R by reordering the alternatives in $B \subseteq A$ such that $f(R) \cap B = f(R') \cap B$, then $f(R) = f(R')$.

As shown in Appendix A.3, every strategyproof and pairwise SCC satisfies these axioms. We now use these properties to show our first key insight, namely that all such SCCs that satisfy strong Condorcet consistency also satisfy a new property called Condorcet stability (COS). This axiom requires that there should be no alternative—within or outside of the choice set—that strictly dominates all other alternatives in the choice set. Note that COS implies that an alternative can only be a single winner if it weakly dominates every other alternative.⁷

DEFINITION 5 (Condorcet stability (COS)). An SCC f satisfies Condorcet stability (COS) if for every preference profile R , there is no alternative $x \in A$ such that $x \succ_R f(R) \setminus \{x\}$ whenever $f(R) \setminus \{x\}$ is nonempty.

This condition is equivalent to requiring that every alternative is weakly dominated by another chosen alternative unless it is the unique winner, that is, for every alternative $x \in A$ with $f(R) \neq \{x\}$ there is an alternative $y \in f(R) \setminus \{x\}$ such that $y \succeq_R x$. It is thus closely connected to the notion of external stability, which requires that for every alternative $x \in A \setminus f(R)$, there is an alternative $y \in f(R)$ such that $y \succeq_R x$ (see, e.g., Miller, Grofman, and Feld (1990), Duggan (2013)). Indeed, Condorcet stability is a stronger requirement than external stability as it also includes a notion of internal stability.

As we show next, the conjunction of our axioms implies COS.

LEMMA 5. *Every pairwise SCC that is strategyproof and strongly Condorcet consistent satisfies COS.*

PROOF. Let f denote a pairwise SCC that satisfies strategyproofness and strong Condorcet consistency. First, recall that strong Condorcet consistency requires that $f(R) = \{x\}$ if and only if x is the Condorcet winner in R . Hence, strong Condorcet consistency implies COS for all profiles R with a Condorcet winner x because $f(R) = \{x\}$ entails $x \succ_R A \setminus \{x\}$. Next, we focus on profiles without a Condorcet winner and assume for contradiction that f fails COS for such a profile. More formally, this assumption means that there is a profile R without a Condorcet winner and an alternative $a \in A$ such that $a \succ_R f(R) \setminus \{a\}$. It follows from the absence of a Condorcet winner that there is at least one alternative x with $x \succeq_R a$, but no such alternative is chosen. Hence, we can repeatedly use WSMON to weaken the alternatives x with $x \succeq_R a$ against all other alternatives until we arrive at a profile R' such that $a \succ_{R'} A \setminus \{a\}$. Now, WSMON implies that the choice set does not change during these steps, and thus it holds that $f(R') = f(R)$. This

⁷This axiom is quite useful for characterizing SCCs that are strategyproof in profiles that admit a Condorcet winner: a majoritarian and nonimposing SCC is strategyproof in such profiles if and only if it satisfies COS.

means in particular that $|f(R')| \geq 2$ because strong Condorcet consistency requires that $|f(R)| \geq 2$. However, a is the Condorcet winner in R' , and consequently, strong Condorcet consistency also shows that $f(R') = \{a\}$. These two observations contradict each other, and consequently, the assumption that f violates COS was incorrect. \square

COS plays an important role in our proofs because we can use it to force an alternative into the choice set. In particular, the combination of strong Condorcet consistency and COS have rather strong consequences: the first axiom states that we choose a single winner if and only if it is the Condorcet winner and the second one requires therefore that every alternative is weakly dominated by a chosen alternative if there is no Condorcet winner. We now use this interaction to prove our first lemma for pairwise SCCs that satisfy homogeneity, strategyproofness, and strong Condorcet consistency.

LEMMA 6. *Let f denote a pairwise SCC that satisfies strong Condorcet consistency, homogeneity, and strategyproofness. If $TC(R) \subseteq f(R)$ for all profiles R , then f is a robust dominant set rule.*

PROOF. Let f denote a pairwise, homogeneous, strategyproof, and strongly Condorcet-consistent SCC. We prove this lemma in three steps: first, we show that if $TC(R) \subseteq f(R)$ for all profiles R , then f is a dominant set rule. Next, we prove that, if f is a dominant set rule, our assumptions require it to be majoritarian. As last point, we show that f is even robust if it is a majoritarian dominant set rule. Combining all three steps thus shows the lemma.

Step 1: If $TC(R) \subseteq f(R)$ for all profiles R , f is a dominant set rule.

We prove this claim by contradiction, and thus assume that f always chooses a superset of TC but is no dominant set rule. This means that there is a profile R such that $f(R)$ is no dominant set in \succsim_R , which implies that $f(R) \neq TC(R)$. Moreover, there is no Condorcet winner in R , because otherwise strong Condorcet consistency would require that f chooses this alternative as unique winner, which would contradict that $f(R)$ is no dominant set. We infer from this observation that $|TC(R)| > 1$ because TC is strongly Condorcet consistent. Next, note that there are alternatives $a \in f(R)$ and $b \in A \setminus f(R)$ such that $b \succsim_R a$ because $f(R)$ is no dominant set. Even more, $a \notin TC(R)$; otherwise, b would also be in $TC(R)$ because $b \succsim_R a$, which would imply that $TC(R) \not\subseteq f(R)$ since $b \notin f(R)$. However, this contradicts our assumptions.

Next, let x denote an alternative in $TC(R) \subseteq f(R)$, which implies that $x \succ_R a$. We repeatedly reinforce a against x until we arrive at a profile R' such that $a \succ_{R'} x$. Moreover, let \bar{R}' denote the last profile constructed before R' , that is, $x \succ_{\bar{R}'} a$ and a single voter needs to reinforce a against x to derive R' . First, we show that $f(\bar{R}') = f(R)$ and for this consider two consecutive profiles \hat{R} and \hat{R}' in the sequence that leads from R to \bar{R}' . This means that \hat{R}' is derived from \hat{R} by reinforcing a against x . Moreover, it holds that $\succsim_R = \succsim_{\hat{R}} = \succsim_{\hat{R}'}$ because the majority relation between a and x has not changed yet. This implies that $TC(R) = TC(\hat{R}) = TC(\hat{R}')$ and, therefore, $x \in TC(\hat{R}) \subseteq f(\hat{R})$ and $x \in TC(\hat{R}') \subseteq f(\hat{R}')$. Now, if $a \in f(\hat{R})$, then WMON implies that $a \in f(\hat{R}')$ since a is reinforced against x to derive \hat{R}' . Finally, WLOC implies then that $f(\hat{R}) = f(\hat{R}')$ because a and x

are both chosen before and after the manipulation. Since we start this process at the profile R with $\{a, x\} \subseteq f(R)$, it follows from a repeated application of this argument that $f(\bar{R}') = f(R)$.

Finally, we show that $\{b, x\} \not\subseteq f(R')$ but $\{b, x\} \subseteq TC(R')$. This contradicts the assumption that TC is always contained in f , and thus proves that our initial assumption that f is no dominant set rule was incorrect. First, we show that $\{b, x\} \not\subseteq f(R')$. Observe for this that $b \notin f(\bar{R}') = f(R)$ and that $\{a, x\} \subseteq f(\bar{R}') = f(R)$. Since R' is derived from \bar{R}' by reinforcing a against x , WMON implies that $a \in f(R')$. Now, if $x \in f(R')$, it follows from WLOC that the choice set is not allowed to change, which implies that $b \notin f(R')$. This means that $b \in f(R')$ is only possible if $x \notin f(R')$, so $\{b, x\} \not\subseteq f(R')$. Next, we prove the second claim that $\{b, x\} \subseteq TC(R')$. For proving this, it is important that $|TC(R)| > 1$ and that $\succ_{R'}$ differs from \succ_R only in the fact that $a \succ_{R'} x$ and $x \succ_R a$. The first point means that there is a cycle C in \succ_R connecting all alternatives in $TC(R)$ because of Lemma 2 and the second one that this cycle also exists in $\succ_{R'}$. Hence, there is a path from x to every alternative $y \in TC(R) \setminus \{x\}$. Moreover, there is a path from x to every alternative $z \in A \setminus TC(R)$ because we can go from x to another alternative $y \in TC(R) \setminus \{x\}$ using the cycle C and from y to z because $y \succ_{R'} z$. This means that $x \in TC(R')$ as it reaches every other alternative on some path. Furthermore, $a \in TC(R')$ because $a \succ_{R'} x$, and $b \in TC(R')$ because $b \succ_{R'} a$. This shows that $\{b, x\} \subseteq TC(R')$, even though $\{b, x\} \not\subseteq f(R')$. Hence, $TC(R') \not\subseteq f(R')$, which contradicts the assumption that $TC(R) \subseteq f(R)$ for all profiles R . This proves that the assumption that f is no dominant set rule was incorrect.

Step 2: If f is a dominant set rule, it is majoritarian.

Our goal in this step is to show that if f is a dominant set rule, it is majoritarian. Thus, assume for contradiction that f is a dominant set rule but violates majoritarianism. The latter point means that there are two preference profiles R and R' such that $\succ_R = \succ_{R'}$ but $f(R) \neq f(R')$. We assume that both R and R' are defined by an even number of voters. This is without loss of generality as we can just duplicate the profiles if required. The majority relations do not change by this step since the majority margins are only doubled, and the choice sets do not change because of homogeneity. Thus, we can also work with these larger profiles instead. Next, observe that $f(R) \neq f(R')$ implies that $g_R \neq g_{R'}$ because f is pairwise. Moreover, $f(R)$ and $f(R')$ are both dominant sets in \succ_R because f is a dominant set rule. Since dominant sets are ordered by set inclusion, it follows that $f(R) \subsetneq f(R')$ or $f(R') \subsetneq f(R)$. We assume without loss of generality that $f(R)$ is a subset of $f(R')$; otherwise, we can just exchange the role of R and R' in the subsequent arguments. Our goal is to transform R into a profile R^* such that $g_{R^*} = g_{R'}$ and $f(R^*) \subseteq f(R) \subsetneq f(R')$. This is in conflict with the pairwise nature of f and shows therefore that the assumption $f(R) \neq f(R')$ was incorrect.

We use the largest majority margin $c = \max_{x, y \in A} g_R(x, y)$ in R for the derivation of R^* . In more detail, we first construct a profile R^1 such that $g_{R^1}(x, y) = c$ for all alternatives $x, y \in A$ with $x \succ_R y$. For this, we repeatedly use the following steps: first, identify a pair of alternatives $x, y \in A$ such that $x \succ_R y$ but the majority margin between x and y is not c yet. Then reinforce x against y . By repeating these steps, we eventually arrive at a profile R^1 , which satisfies $g_{R^1}(x, y) = c$ for all $x, y \in A$ with $x \succ_R y$. We show next that $f(R^1) \subseteq f(R)$ by a case distinction with respect to x and y . For this, consider a single

step of our process, and let \bar{R} denote the profile before reinforcing x against y and \bar{R}' denote the profile after reinforcing x against y . If $x \notin f(\bar{R})$ and $y \notin f(\bar{R})$, it follows from IUA that $f(\bar{R}) = f(\bar{R}')$. If $x \in f(\bar{R})$ and $y \notin f(\bar{R})$, it follows from WMON that $x \in f(\bar{R})$ and $y \notin f(\bar{R}')$ because we have $x \succ_R y$ and, therefore, also $x \succ_{\bar{R}'} y$. Hence, if $y \in f(\bar{R}')$, then $x \in f(\bar{R}')$ as f chooses a dominant set. However, this is in conflict with WMON: if we revert the swap, this axiom implies that $y \in f(\bar{R})$, which contradicts our assumptions. Thus, $x \in f(\bar{R}')$, $y \notin f(\bar{R}')$, and WLOC implies that $f(\bar{R}) = f(\bar{R}')$. As a third point, note that $x \notin f(\bar{R})$ and $y \in f(\bar{R})$ are impossible because we assume that $x \succ_R y$. Hence, this case contradicts that $f(\bar{R})$ is a dominant set. The last case is that $x \in f(\bar{R})$ and $y \in f(\bar{R})$. In this case, it follows from WMON that $x \in f(\bar{R}')$. If now also $y \in f(\bar{R}')$, WLOC implies that $f(\bar{R}) = f(\bar{R}')$. Conversely, if $y \notin f(\bar{R}')$, then $f(\bar{R}') \subsetneq f(\bar{R})$. Otherwise, an alternative $z \in A \setminus f(\bar{R})$ is in $f(\bar{R}')$, which is in conflict with the fact that $f(\bar{R}')$ is a dominant set since $y \succ_{\bar{R}'} z$ because $y \in f(\bar{R})$ and $z \notin f(\bar{R})$. Hence, we derive in all possible cases that $f(\bar{R}') \subseteq f(\bar{R})$. By repeatedly applying this argument, it follows that $f(R^1) \subseteq f(R)$.

Next, note that $\succ_{R^1} = \succ_R$ because we only increase the majority margins between alternatives $x, y \in A$ with $x \succ_R y$. Furthermore, there are only two possible majority margins in R^1 : if $x \sim_{R^1} y$, then $g_{R^1}(x, y) = 0$ and if $x \succ_{R^1} y$, then $g_{R^1}(x, y) = c$. This means that we can use homogeneity to derive a profile R^2 with smaller majority margins: we set $g_{R^2}(x, y) = 2$ for all alternatives $x, y \in A$ with $x \succ_{R^1} y$ and $g_{R^2}(x, y) = 0$ for all alternatives $x, y \in A$, with $x \sim_{R^1} y$. Such a preference profile R^2 exists because we can use McGarvey's construction to build a preference profile for all majority margins that have the same parity (McGarvey (1953)). It follows from homogeneity and pairwise-ness that $f(R^2) = f(R^1)$ because we can just multiply R^2 such that all majority margins are equal to those in R^1 . Note here that the assumption that R is defined by an even number of voters is important because it ensures that c is a multiple of 2. As last point, observe that $\succ_{R^2} = \succ_R = \succ_{R'}$ because we did not change the sign of a majority margin. Moreover, $g_{R^2}(x, y) \leq g_{R'}(x, y)$ for all $x, y \in A$ because R' is defined by an even number of voters. Hence, if $x \succ_{R'} y$, then $g_{R'}(x, y) \geq 2 = g_{R^2}(x, y)$, and if $x \sim_{R'} y$, then $g_{R^2}(x, y) = g_{R'}(x, y) = 0$.

As a last step, we derive a preference profile R^3 with $g_{R^3} = g_{R'}$ from R^2 by applying the same process as in the construction of R^1 : we repeatedly identify a pair of alternatives $x, y \in A$ such that $x \succ_R y$ and the current majority margin between x and y is less than the one in R' , and reinforce x against y . Clearly, this process results in a profile R^3 with $g_{R^3} = g_{R'}$ and the same arguments as for R^1 show that $f(R^3) \subseteq f(R^2)$. We derive therefore from pairwise-ness that $f(R') = f(R^3) \subseteq f(R^2) \subseteq f(R^1) \subseteq f(R) \subsetneq f(R')$, which is a contradiction because the last subset relation is by assumption strict. Hence, our initial assumption was incorrect and f is indeed majoritarian.

Step 3: If f is a majoritarian dominant set rule, it is robust.

As a last step, we show that f is robust if it is a majoritarian dominant set rule. Thus, assume for contradiction that f is a majoritarian dominant set rule that fails robustness. This means that there are two preference profiles R and R' such that $f(R)$ is dominant in $\succ_{R'}$, but $f(R') \not\subseteq f(R)$. As a consequence, there is an alternative $y \in f(R') \setminus f(R)$. Moreover, since $f(R)$ is dominant in $\succ_{R'}$, it follows that $f(R) \succ_{R'} y$, and hence $f(R) \subsetneq f(R')$ as f is a dominant set rule. We derive a contradiction to this

assumption by constructing two preference profiles R^2 and R^3 such that $f(R^2) = f(R)$, $f(R^3) = f(R')$, and $\succsim_{R^2} = \succsim_{R^3}$. These observations are conflicting since $\succsim_{R^2} = \succsim_{R^3}$ requires that $f(R^2) = f(R^3)$ because of majoritarianism, but $f(R) \neq f(R')$. Note that we assume in the sequel that both R and R' are defined by an even number of voters as we want to introduce majority ties. This is without loss of generality as f is homogeneous.

First, we explain how to derive R^2 from R . As a first step, we reorder the alternatives in $A \setminus f(R)$ to derive a profile R^1 with $\succsim_{R^1} \upharpoonright_{A \setminus f(R)} = \succsim_R \upharpoonright_{A \setminus f(R)}$. As a consequence of IUA, it follows that $f(R^1) = f(R)$ since this step does not affect chosen alternatives. Next, let D_{i^*} denote the dominant set in R^1 that is currently chosen, that is, $f(R^1) = D_{i^*}$. We derive the profile R^2 by repeating the following procedure with R^1 as starting profile: in the current preference profile \bar{R} , we choose a pair of alternatives $x, y \in D_{i^*}$ such that $y \succ_{\bar{R}} x$ and reinforce x against y until we arrive at a profile \bar{R}' with $x \sim_{\bar{R}'} y$. It follows from WMON and majoritarianism that $x \in f(\bar{R}')$ if $x, y \in f(\bar{R})$. Moreover, as f is a dominant set rule, $x \in f(\bar{R}')$ implies $y \in f(\bar{R}')$ because $y \succ_{\bar{R}'} x$. Hence, we infer from WLOC that $f(\bar{R}) = f(\bar{R}')$ if $x, y \in f(\bar{R})$. Since $f(R^1) = D_{i^*}$, we can thus repeat this process until we arrive at a profile R^2 with $x \sim_{R^2} y$ for all $x, y \in D_{i^*}$, and it follows from the previous argument that $f(R^1) = f(R^2)$. Moreover, the majority relation of R^2 is completely specified: we have $f(R) \succ_{R^2} A \setminus f(R)$, $x \sim_{R^2} y$ for all $x, y \in f(R)$, and $\succsim_{R^2} \upharpoonright_{A \setminus f(R)} = \succsim_R \upharpoonright_{A \setminus f(R)}$.

Finally, we apply the same construction as for R^2 to derive the profile R^3 from R' . In more detail, observe that, by assumption, $D_{i^*} = f(R)$ is dominant in $\succsim_{R'}$ and $D_{i^*} \subsetneq f(R')$. Hence, we can use the same construction as for R^2 to introduce majority ties between all alternatives in D_{i^*} in $\succsim_{R'}$. The same reasoning as in the previous paragraph shows that this step does not change the choice set, and it hence holds for the resulting profile R^3 that $f(R') = f(R^3)$. In particular, R^3 has now the same majority relation as R^2 , which is in conflict with majoritarianism since $\succsim_{R^2} = \succsim_{R^3}$ but $f(R^2) = f(R) \neq f(R') = f(R^3)$. This is a contradiction to our assumptions and f is therefore robust if it is a majoritarian dominant set rule. □

Lemma 6 presents a simple criterion for deciding when a strategyproof, homogeneous, pairwise, and strongly Condorcet-consistent SCC f is a robust dominant set rule, namely when $TC(R) \subseteq f(R)$ for all profiles R . Our next goal is to prove that every such SCC meets this condition without further assumptions. Hence, suppose for contradiction that this is not the case, that is, there are an SCC f that satisfies all our axioms and a profile R such that $TC(R) \not\subseteq f(R)$. If such a profile R exists, we may as well focus on the profile R^f that minimizes the size of the top cycle among all profiles R with $TC(R) \not\subseteq f(R)$. Furthermore, for every SCC f , we define $k_f \in \{1, \dots, m + 1\}$ as the maximal value such that $TC(R) \subseteq f(R)$ for all preference profiles R with $|TC(R)| < k_f$. Note that $k_f = |TC(R^f)|$ if $TC(R)$ is not always a subset of $f(R)$, and $k_f = m + 1$ otherwise.

As the next step, we show that $k_f \geq 4$ for all pairwise SCCs f that satisfy strategyproofness, homogeneity, and strong Condorcet consistency. In general, this means that such SCCs can only fail to choose a superset of the top cycle if $TC(R)$ is sufficiently large. For the special case where $m \leq 3$, Lemmas 6 and 7 already imply that f needs to be a robust dominant set rule because the size of the top cycle is bounded by the number of alternatives.

LEMMA 7. *Let f denote a pairwise SCC that satisfies strong Condorcet consistency, strategyproofness, and homogeneity. Then $k_f \geq 4$.*

PROOF. Let f denote a pairwise SCC that satisfies strong Condorcet consistency, homogeneity, and strategyproofness. Furthermore, suppose for contradiction that there is a profile R^* such that $k = |TC(R^*)| \leq 3$ but $TC(R^*) \not\subseteq f(R^*)$. We proceed with a case distinction with respect to $|TC(R^*)|$ to derive a contradiction for the three possible cases.

Case 1: $|TC(R^*)| = 1$

If $|TC(R^*)| = 1$, there has to be a Condorcet winner in R^* since TC is strongly Condorcet consistent. Consequently, the strong Condorcet consistency of f requires that $f(R^*) = TC(R^*)$, which contradicts the assumption that $TC(R^*) \not\subseteq f(R^*)$.

Case 2: $|TC(R^*)| = 2$

The top cycle only elects two alternatives $x, y \in A$ if $x \sim_{R^*} y$ and $\{x, y\} \succ_{R^*} A \setminus \{x, y\}$. Hence, strong Condorcet consistency requires that f chooses at least two alternatives. In turn, COS implies then that both x and y are chosen because x is the only alternative that dominates y and y is the only alternative that dominates x . This shows that $TC(R^*) \subseteq f(R^*)$ and we again have a contradiction.

Case 3: $|TC(R^*)| = 3$

Next, assume there are three alternatives a, b , and c such that $TC(R^*) = \{a, b, c\}$, but $\{a, b, c\} \not\subseteq f(R^*)$. First, note that strong Condorcet consistency requires that $|f(R^*)| \geq 2$ because there is no Condorcet winner in R^* ; otherwise, it would hold that $|TC(R^*)| = 1$ as TC uniquely chooses the Condorcet winner whenever it exists. Next, observe that, according to Lemma 2, there has to be a cycle C that connects a, b, c since $TC(R^*) = \{a, b, c\}$. Without loss of generality, suppose that $C = (a, b, c)$, that is, $a \succ_{R^*} b, b \succ_{R^*} c, c \succ_{R^*} a$. Moreover, we also suppose without loss of generality that $a \notin f(R^*)$ because $TC(R^*) \not\subseteq f(R^*)$. COS then implies that $b \sim_{R^*} c$ and $\{b, c\} \subseteq f(R^*)$, because otherwise no chosen alternative dominates b or c . Next, consider the profile R^1 derived from R^* by reinforcing b against c , so we have $b \succ_{R^1} c$ instead of $b \sim_{R^*} c$. First, note that there is no Condorcet winner in R^1 , and thus strong Condorcet consistency requires us to choose at least two alternatives. Hence, COS implies that $a \in f(R^1)$ because it is the only alternative that dominates b in R^1 . Moreover, WMON shows that $b \in f(R^1)$ since we swap two chosen alternatives to derive R^1 from R^* . Finally, the contraposition of WLOC implies that $c \notin f(R^1)$ since $a \in f(R^1) \setminus f(R^*)$. These observations entail that $a \sim_{R^1} b$, because otherwise no chosen alternative dominates a , which violates COS. Thus, we can repeat the previous steps by reinforcing a against b , which results in a profile R^2 such that $a \succ_{R^2} b, b \succ_{R^2} c, \{a, c\} \subseteq f(R^2)$, and $b \notin f(R^2)$. Hence, COS implies once again that $c \sim_{R^2} a$ and we can again break this majority tie to derive the profile R^3 . In more detail, all edges of C are now strict, and $a \notin f(R^3)$. This contradicts COS because $b \succ_{R^3} A \setminus \{a, b\}$. Hence, no chosen alternative dominates b and the assumption that $TC(R^*) \not\subseteq f(R^*)$ was incorrect. □

Due to Lemma 7, it follows that every pairwise SCC that satisfies strategyproofness, homogeneity, and strong Condorcet consistency can only fail to choose a superset of the top cycle if $|TC(R)| \geq 4$. Hence, we subsequently investigate profiles with a top cycle that

contains at least 4 alternatives. For this, we first need to discuss some auxiliary lemmas and start by showing that a pairwise SCC f that satisfies homogeneity, strategyproofness, and strong Condorcet consistency must choose almost all alternatives of the top cycle for profiles R with $|TC(R)| = k_f \geq 4$.

LEMMA 8. *Let f denote a pairwise SCC that satisfies strong Condorcet consistency, homogeneity, strategyproofness, and $4 \leq k_f \leq m$. It holds that $|TC(R) \cap f(R)| \geq k_f - 1$ for all profiles R with $|TC(R)| = k_f$.*

PROOF. Let f denote a pairwise SCC that satisfies all axioms of the lemma and assume that $k_f \in \{4, \dots, m\}$. Furthermore, suppose for contradiction that there is a profile R such that $|TC(R)| = k_f$ and $|f(R) \cap TC(R)| \leq k_f - 2$. This means that at least two alternatives of the top cycle are not chosen, that is, the set $X = TC(R) \setminus f(R)$ contains at least two alternatives. Next, let $x \in X$ denote one of these alternatives. We proceed with a case distinction with respect to the connected set A_x and first suppose that $X \not\subseteq A_x \cup \{x\}$. Because of the definition of connected sets, this assumption means that $X \cap TC(R^{-x}) \neq \emptyset$, where $R^{-x} = R|_{A \setminus \{x\}}$ denotes the profile derived from R by removing x . We use this fact to derive a contradiction as follows: starting at R , we repeatedly weaken x against all alternatives until we derive a profile R' in which x is the Condorcet loser. WSMON entails for every step that the choice set does not change, which means that $f(R) = f(R')$. Moreover, $TC(R') = TC(R) \setminus (A_x \cup \{x\})$ because for the top cycle it is irrelevant whether an alternative is a Condorcet loser or not present at all. However, this means that $TC(R') \not\subseteq f(R')$ because $X \cap TC(R') \neq \emptyset$, but $X \cap f(R') = X \cap f(R) = \emptyset$. Since $|TC(R')| \leq |TC(R) \setminus \{x\}| < k_f$, this contradicts the definition of k_f , which requires that $TC(\bar{R}) \subseteq f(\bar{R})$ for all profiles \bar{R} with $|TC(\bar{R})| < k_f$.

As second case, suppose that $X \subseteq A_x \cup \{x\}$. In this case, consider a second alternative $y \in X \setminus \{x\}$, which means that $y \in A_x$. We want to use Lemma 3. Note that there is an alternative $z \in f(R)$ with $z \succ_R x$ because of COS and strong Condorcet consistency. Hence, x does not dominate all other alternatives but y and Lemma 3 consequently shows that $A_y \subseteq A_x$. In particular, this means that $x \notin A_y \cup \{y\}$ and, therefore, $X \not\subseteq A_y \cup \{y\}$. Hence, we can use the same argument as in the last case to derive a contradiction by focusing on y . Since both cases result in a contradiction, it follows that the assumption $|TC(R) \cap f(R)| \leq k_f - 2$ was incorrect, that is, $|TC(R) \cap f(R)| \geq k_f - 1$ holds for all profiles R with $|TC(R)| = k_f$. \square

Lemma 8 is important because it implies for all profiles R with $|TC(R) \cap f(R)| < |TC(R)| = k_f$ that there is a single alternative of the top cycle which is unchosen. As we demonstrate next, this insight can be used to strengthen the axioms in Appendix A.3 when we restrict attention to profiles R, R' with $|TC(R)| = |TC(R')| = k_f$. In particular, the next lemma is concerned with what happens when we weaken an alternative $y \in f(R) \cap TC(R)$ against multiple alternatives $X \subseteq f(R) \cap TC(R)$ when $|TC(R)| = k_f$.

LEMMA 9. *Let f denote a pairwise SCC that satisfies strong Condorcet consistency, homogeneity, strategyproofness, and $4 \leq k_f \leq m$, and consider two preference profiles R, R' such*

that $TC(R) = TC(R')$ and $|TC(R)| = k_f$. If there are a set of alternatives $X \subseteq f(R) \cap TC(R)$ and an alternative $y \in (f(R) \cap TC(R)) \setminus X$ such that $g_{R'}(x, y) = 2 + g_R(x, y)$ for all $x \in X$, and $g_{R'}(x', y') = g_R(x', y')$ for all other pairs of alternatives, it holds that $f(R) = f(R')$ or $f(R') \cap TC(R') = TC(R') \setminus \{y\}$.

PROOF. Consider a pairwise SCC f that satisfies homogeneity, strategyproofness, and strong Condorcet consistency and let R, R', X , and y be defined as in the lemma. In particular, it holds that $g_{R'}(x, y) = g_R(x, y) + 2$ for all $x \in X$, and $g_{R'}(x', y') = g_R(x', y')$ for all other pairs of alternatives. This means that we can transform R into a profile R^* with the same majority margins as R' by reinforcing all alternatives in X against y . Consequently, the lemma follows if we show that $X \subseteq f(R')$: if also $y \in f(R')$, then WLOC entails that $f(R) = f(R')$, and if $y \notin f(R')$, then Lemma 8 implies that $f(R') \cap TC(R') = TC(R') \setminus \{y\}$ because $|TC(R')| = k_f \geq 4$ and $y \in TC(R') \setminus f(R')$.

Thus, suppose for contradiction that $X \not\subseteq f(R')$. Then Lemma 8 shows that there is an alternative $z \in X$ such that $f(R') \cap TC(R') = TC(R') \setminus \{z\}$ because $X \subseteq TC(R')$. Moreover, we assume that $X \cup \{y\} \subseteq f(R)$. We can turn these observations into a manipulation of f by adding two voters i^* and j^* with inverse preferences to R . In more detail, the profile R^1 consists of R and the voters i^* and j^* whose preference relations are specified subsequently. In the definitions of these preference relations, we use $\bar{f}(R) = f(R) \setminus (X \cup \{y\})$ and $\bar{X} = X \setminus \{z\}$:

$$\succ_{i^*}^1 = \mathbf{lex}(A \setminus f(R), \mathbf{lex}(\bar{f}(R) \cap f(R')), y, \mathbf{lex}(\bar{X}), z, \mathbf{lex}(\bar{f}(R) \setminus f(R')))$$

$$\succ_{j^*}^1 = \mathbf{lex}(\bar{f}(R) \setminus f(R'))^{-1}, z, \mathbf{lex}(\bar{X})^{-1}, y, \mathbf{lex}(\bar{f}(R) \cap f(R'))^{-1}, \mathbf{lex}(A \setminus f(R))^{-1}$$

Since the preferences of these voters are inverse, it follows from pairwise nature that $f(R^1) = f(R)$. Next, we derive R^2 from R^1 by letting voter i reinforce all alternatives in X against y . Since we derive R' from R by the same modification, pairwise nature shows that $f(R^2) = f(R')$. However, this means that voter i^* can manipulate by deviating from R^1 to R^2 . Note for this that $f(R') \setminus f(R) \succ_{i^*} f(R)$ because $A \setminus f(R) \succ_{i^*} f(R)$. Furthermore, it holds that $f(R') \succ_{i^*} f(R) \setminus f(R')$ because the alternatives in $f(R) \setminus f(R') = (\bar{f}(R) \setminus f(R')) \cup \{z\}$ are bottom-ranked by voter i^* . Finally, since $z \in f(R) \setminus f(R')$, this is indeed a manipulation for voter i^* . Hence, the assumption that $X \not\subseteq f(R')$ was incorrect, which proves the lemma. □

Lemma 9 significantly strengthens WMON for profiles R, R' with $TC(R) = TC(R')$ and $|TC(R)| = k_f \geq 4$ and alternatives in $TC(R)$. In particular, we can now reinforce sets of alternatives against single alternatives and there are only two possible outcomes under the given assumptions. Therefore, we ensure in the following that the premises of Lemma 9 are always true: in all subsequent profiles R , it holds that $|TC(R)| = k_f$, we only modify the preferences between alternatives in the top cycle, and the top cycle will never change. As the next step, we derive a profile R for which all majority margins are known, $|TC(R)| = k_f$, and $TC(R) \not\subseteq f(R)$.

LEMMA 10. Let f denote a pairwise SCC that satisfies strong Condorcet consistency, homogeneity, and strategyproofness. If $4 \leq k_f \leq m$, there is a profile R such that $TC(R) \not\subseteq f(R)$, $|TC(R)| = k_f$, and there is a cycle $C = (a_1, \dots, a_{k_f})$ in $\succsim_{R|TC(R)}$ with $g_R(a_{k_f}, a_1) = 2$ and $g_R(a_i, a_j) = 2$ for all other indices $i, j \in \{1, \dots, k_f\}$ with $i < j$.

PROOF. Let f denote a pairwise SCC that satisfies strong Condorcet consistency, homogeneity, and strategyproofness, and suppose that $k_f \in \{4, \dots, m\}$. Moreover, consider a profile R^* such that $TC(R^*) \not\subseteq f(R^*)$ and $k_f = |TC(R^*)|$; such a profile exists by the definition of k_f . Additionally, we assume in the sequel that R^* is defined by an even number of voters. This is possible as we can simply duplicate the profile R^* if it is defined by an odd number of voters. This step does neither affect the top cycle nor f since both SCCs are homogeneous, and we can thus work with this larger profile if R^* was defined by an odd number.

We prove this lemma in two steps: first, we construct a profile \hat{R}^1 such that $TC(R^*) = TC(\hat{R}^1) \not\subseteq f(\hat{R}^1)$ and there is a pair of alternatives $a, b \in TC(\hat{R}^1)$ with $b \succ_{\hat{R}^1} A \setminus \{a, b\}$. This profile is essential since COS shows now that a must be chosen, even after various manipulations. Based on this insight, we construct as the second step a profile \hat{R}^2 that satisfies all requirements of our lemma.

Step 1: Constructing the profile \hat{R}^1 .

Our first goal is to construct a profile \hat{R}^1 such that $TC(R^*) = TC(\hat{R}^1) \not\subseteq f(\hat{R}^1)$ and there is a pair of alternatives $a, b \in TC(\hat{R}^1)$ with $b \succ_{\hat{R}^1} A \setminus \{a, b\}$. For this, consider a cycle $C = (a_1, \dots, a_{k_f})$ in \succsim_{R^*} that contains all alternatives in $TC(R^*)$; such a cycle exists because of Lemma 2. Furthermore, let $b = a_{i+1}$ denote an arbitrary alternative in $TC(R^*) \cap f(R^*)$ and let $a = a_i$ denote its predecessor on the cycle C . Our goal is to reinforce b against all alternatives $A \setminus \{a, b\}$ such that $b \succ_R A \setminus \{a, b\}$.

The first key insight for this is that strong Condorcet consistency and COS entail that there is always a chosen alternative c that dominates b if there is no Condorcet winner. Based on this observation, we repeat the following steps starting at profile R^* : in the current profile R' , we identify an alternative $c \in f(R') \setminus \{a, b\}$ with $c \succ_{R'} b$ and reinforce b against c . First, note that during all of these steps, b remains chosen because of WMON and the fact that we only swap chosen alternatives. Next, observe that these steps do not affect the cycle C because c is not the predecessor of b . Hence, Lemma 2 implies that the top cycle does not change and that $c \in TC(R^*) = TC(R')$ because $c \succ_{R'} b$. The latter observation and Lemma 9 also entail that not all alternatives in the top cycle are chosen after reinforcing b against c because either c is now unchosen or the choice set is not allowed to change at all. Thus, this process terminates at a profile R^1 such that $b \succ_{R^1} f(R^1) \setminus \{a, b\}$, $TC(R^1) = TC(R^*) \not\subseteq f(R^1)$, and $\{a, b\} \subseteq f(R^1)$. The last point is true since WMON shows that $b \in f(R^1)$ and COS requires that $a \in f(R^1)$ because $a \succ_{R^1} b$ and $b \succ_{R^1} f(R^1) \setminus \{a, b\}$. We are done after this step if $b \succ_{R^1} A \setminus \{a, b\}$ but this is not guaranteed.

Hence, assume that there are alternatives $x \in A \setminus f(R^1)$ with $x \succ_{R^1} b$. Note that this assumption implies that $x \in TC(R^1)$ because $b \in TC(R^*) = TC(R^1)$. We want to repeatedly identify such an alternative $x \in TC(R^1) \setminus f(R^1)$ with $x \succ_{R^1} b$ and reinforce b against x . WMON and WLOC imply for each of these steps that either the choice set does not change, or b becomes unchosen and x chosen. In particular, this means that

after such a step, not all alternatives of the top cycle are chosen because the top cycle is not affected by these changes. However, we cannot guarantee that b remains chosen during these steps and, therefore, we need to treat the case that we arrive at a profile R^2 with $b \notin f(R^2)$ separately. Given such a profile R^2 , we show how we can find another profile R^3 such that $b \in f(R^3)$, $TC(R^*) = TC(R^3) \not\subseteq f(R^3)$, and $g_{R^3}(b, x) = g_{R^2}(b, x)$ for all $x \in A$. For this, note that the cycle $C = (a_1, \dots, a_{k_f})$ exists also in R^2 , and thus $TC(R^2) = TC(R^*)$. Since $|TC(R^*)| = k_f$, Lemma 8 shows that $TC(R^2) \setminus \{b\} \subseteq f(R^2)$ if $b \notin f(R^2)$. In particular, this means that $c = a_{i+2}$ (i.e., the successor of b on C) is in $f(R^2)$. We apply next a similar idea as in the construction of R^1 : at every preference profile R' , we reinforce c against a chosen alternative $x \in f(R') \setminus \{b, c\}$ with $x \succ_{R'} c$. Just as in the first step, Lemma 9 implies that c remains chosen during these steps and that we never choose all alternatives of $TC(R')$. Also, we do not flip any edge in the cycle C during this process because we never reinforce c against its predecessor b . Hence, neither the top cycle nor a majority margin involving b change. Finally, this process terminates at a profile R^3 such that $c \succ_{R^3} f(R^3) \setminus \{b, c\}$ and $c \in f(R^3)$. Moreover, COS now requires that $b \in f(R^3)$ because $c \succ_{R^3} A \setminus \{b, c\}$, that is, if $b \notin f(R^3)$, no chosen alternative dominates c . Hence, profile R^3 indeed satisfies all our requirements.

Thus, if b drops out of the choice set after reinforcing it against an unchosen alternative, we can apply this construction to derive a profile R^3 with $b \in f(R^3)$ and $TC(R^*) = TC(R^3) \not\subseteq f(R^3)$. At this point, we can simply repeat the same constructions used in the derivation of R^1 and R^2 , and eventually, we will arrive at a profile \hat{R}^1 such that $b \succ_{\hat{R}^1} A \setminus \{a, b\}$ because the majority margins of b are non-decreasing during all steps and strictly decreasing during the constructions of R^1 and R^2 . Also, none of the constructions requires us to invert edges of the cycle C , and thus $TC(\hat{R}^1) = TC(R^*)$, whereas Lemma 9 shows that $TC(\hat{R}^1) \not\subseteq f(\hat{R}^1)$.

Step 2: Constructing the profile \hat{R}^2 .

As a second step, we construct the profile \hat{R}^2 that satisfies all requirements of the lemma. In more detail, \hat{R}^2 has to satisfy that $TC(R^*) = TC(\hat{R}^2) \not\subseteq f(\hat{R}^2)$ and that there is a cycle $C = (a_1, \dots, a_{k_f})$ in $\succ_{\hat{R}^2}$ that connects all alternatives in $TC(\hat{R}^2)$ such that $g_{\hat{R}^2}(a_{k_f}, a_i) = 2$ and $g_{\hat{R}^2}(a_i, a_j) = 2$ for all other indices $i, j \in \{1, \dots, k_f\}$ with $i < j$. For the construction of this profile, let R^1 denote the profile constructed in the last step, and let $C = (a_1, \dots, a_{k_f})$ denote a cycle that connects all alternatives $x \in TC(R^*) = TC(R^1)$ in \succ_{R^1} . By construction, there are alternatives $a, b \in TC(R^1)$ such that $b \succ_{R^1} A \setminus \{a, b\}$. In the sequel, we assume without loss of generality that $b = a_1$ since we can pick the starting point of the cycle. This means that $a = a_{k_f}$, that is, a is the predecessor of b on the cycle C , because a is the only alternative that dominates b . Finally, recall that R^* and, therefore, also R^1 are defined by an even number of voters, which implies that the majority margins are even.

The central observation for the construction of \hat{R}^2 is that COS and strong Condorcet consistency guarantee that $a \in f(R^1)$ because a is the only alternative that dominates b . Even more, this is true as long as $b \succ_R A \setminus \{a, b\}$ and $a \succ_R b$. We use this observation to reinforce $a = a_{k_f}$ against the alternatives $x \in TC(R^1) \setminus \{a_1, a_{k_f-1}, a_{k_f}\}$: in each step, we identify an alternative $x \in TC(R^1) \setminus \{a_1, a_{k_f-1}, a_{k_f}\}$ with $x \succ_{R'} a$ in the current profile R' and reinforce a against x . As mentioned before, COS implies that a has to be chosen

during all steps. Moreover, if $x \notin f(R')$, it follows from WMON that x remains unchosen after this step; otherwise, we could revert the modification and WMON entails that $x \in f(R')$, contradicting our previous assumption. Hence, WLOC implies that the choice set cannot change in this case. Conversely, if $x \in f(R')$, it follows either that x is no longer chosen after this step, or that the choice set is not allowed to change because of WLOC. In particular, this shows that not all alternatives in $TC(R^*) = \{a_1, \dots, a_{k_f}\}$ are chosen after this step. Hence, we can repeat these steps until we arrive at a profile R^2 such that $a \succ_{R^2} A \setminus \{a, a_1, a_{k_f-1}\}$. Also note that $g_{R^2}(a, x) = 2$ for all $x \in TC(R^1) \setminus \{a, a_1, a_{k_f-1}\}$ with $x \succ_{R^1} a$ because we only reinforce a against such alternatives x until a strictly dominates them. Finally, none of these steps involves an edge of the cycle C , which implies that $TC(R^2) = TC(R^1) = TC(R^*)$. Hence, $TC(R^2) = TC(R^*) \not\subseteq f(R^2)$.

As the next step, we reinforce a against b if $a \sim_{R^2} b$, and against its predecessor a_{k_f-1} on the cycle C until $g_{R^3}(a, a_{k_f-1}) = -2$ if $g_{R^2}(a, a_{k_f-1}) \leq -4$. This results in a new profile R^3 and, by the same arguments as before, it follows that not all alternatives in $TC(R^*)$ are chosen. Also, it is easy to see that the top cycle did not change since $a \succ_{R^3} b$ and $a_{k_f-1} \succ_{R^3} a$. As a last point, observe that all new outgoing edges $a \succ_{R^2} x$ have weight 2 and that the incoming edge from a_{k_f-1} has a weight of at most 2.

Finally, note that a dominates each alternative $x \in A \setminus \{a, a_{k_f-1}\}$ in R^3 , and thus COS and strong Condorcet consistency imply now that a_{k_f-1} must be chosen. Hence, we can repeat the previous steps for a_{k_f-1} , or more generally, we can traverse along the cycle C using these steps. Thus, we repeat this process until we applied our constructions to a_1 . It follows from the construction that each edge in the final profile \hat{R}^2 has weight 2. Furthermore, the cycle C also exists in the final profile \hat{R}^2 and thus, $TC(\hat{R}^2) = TC(R^*)$. Moreover, it is a consequence of COS, WMON, and WLOC that $TC(\hat{R}^2) \not\subseteq f(\hat{R}^2)$. Finally, it follows for the profile \hat{R}^2 that a_1 dominates each alternative but a_{k_f} , and each alternative a_i with $1 < i < k_f$ dominates all alternatives a_j with $j > i$. This claim follows by inspecting our construction in more detail: if $j = i + 1$, that is, if a_j is the successor of a_i in C , this follows immediately as we do not break the cycle. If $j > i + 1$, we first apply our construction to a_j ensuring that a_j dominates a_i . Later, we apply our construction to a_i , which reverts this edge and ensures that it has a weight of 2. Since this majority margin will not be modified anymore, this proves that the profile \hat{R}^2 indeed satisfies all criteria of the lemma. \square

If we consider a pairwise SCC f that satisfies all required axioms but is no robust dominant set rule, Lemma 10 states the exact majority margins of a profile R^* such that $TC(R^*) \not\subseteq f(R^*)$ and $|TC(R^*)| = k_f \leq m$. As a last step, we derive a contradiction to this by showing that $TC(R^*) \subseteq f(R^*)$ is required. By considering the contraposition of Lemma 10, we infer from this that $k_f \notin \{4, \dots, m\}$. Together with Lemma 7, this means that $k_f = m + 1$, which shows that every pairwise SCC that satisfies strong Condorcet consistency, homogeneity, and strategyproofness is a robust dominant set rule.

LEMMA 11. *Every pairwise SCC that satisfies strong Condorcet consistency, homogeneity, and strategyproofness is a robust dominant set rule.*

PROOF. Assume for contradiction that there is a pairwise SCC f that satisfies strategyproofness, homogeneity, and strong Condorcet consistency, but is no robust dominant set rule. The contraposition of Lemma 6 shows that there is a profile R such that $TC(R) \not\subseteq f(R)$. On the other hand, Lemma 7 shows that $TC(R) \subseteq f(R)$ for all profiles R with $|TC(R)| \leq 3$. These claims contradict each other if $m \leq 3$, and thus we focus on the case that $m \geq 4$. Hence, let $k_f \in \{4, \dots, m\}$ denote the maximal value such that $TC(R) \subseteq f(R)$ for all profiles R with $|TC(R)| < k_f$. Moreover, let \bar{R} denote a profile such that $TC(\bar{R}) \not\subseteq f(\bar{R})$ and $|TC(\bar{R})| = k_f$; such a profile exists because of the definition of k_f . Next, we apply Lemma 10 to derive a profile R^* such that $TC(R^*) = TC(\bar{R})$, $TC(R^*) \not\subseteq f(R^*)$, and the alternatives $TC(R^*) = \{a_1, \dots, a_{k_f}\}$ can be ordered such that $g_{R^*}(a_{k_f}, a_1) = 2$ and $g_{R^*}(a_i, a_j) = 2$ for all other indices $i, j \in \{1, \dots, k_f\}$ with $i < j$. Furthermore, $TC(R^*) \setminus f(R^*)$ contains a single alternative a_j because of Lemma 8.

For deriving a contradiction to this assumption, we will consider a number of profiles related to R^* . In particular, our subsequent construction will mimic neutrality since f needs not be neutral. Thus, we define profile R^π given some permutation $\pi : TC(R^*) \rightarrow TC(R^*)$ as follows: $g_{R^\pi}(x, y) = g_{R^*}(x, y)$ if $x \in A \setminus TC(R^*)$ or $y \in A \setminus TC(R^*)$ and $g_{R^\pi}(\pi(a_i), \pi(a_j)) = g_{R^*}(a_i, a_j)$ for all $a_i, a_j \in TC(R^*)$. Less formally, R^π is constructed as follows: we derive the majority margins of R^π by reordering the edges between alternatives $x, y \in TC(R^*)$ according to π but we do not reorder the edges to alternatives outside of the top cycle. For a better readability, we refer to $\pi(a_i)$ as a_i^π from now on. In particular, the construction of R^π implies that $g_{R^\pi}(a_{k_f}^\pi, a_1^\pi) = 2$ and $g_{R^\pi}(a_i^\pi, a_j^\pi) = 2$ for all $i, j \in \{1, \dots, k_f\}$ with $i < j$. Furthermore, we define Π_π^l as the set of permutations π' with $\pi(a_i) = \pi'(a_i)$ for $i \in \{l, \dots, k_f\}$, that is, all permutations $\pi' \in \Pi_\pi^l$ agree with π on the alternatives $\{a_l, \dots, a_{k_f}\}$.

Based on the profiles R^π , we next prove the lemma. For this, let j^* denote the smallest index such that $a_{j^*}^\pi \notin f(R^\pi)$ for some permutation π on $TC(R^*)$. Moreover, let π^* denote the corresponding permutation, that is, $a_{j^*}^{\pi^*} \notin f(R^{\pi^*})$. Given the value j^* and the profile R^{π^*} , we prove the lemma in three steps. First, we show that $\{a_1^\pi, a_2^\pi, a_{k_f}^\pi\} \subseteq f(R^\pi)$ for all permutations π . This means in particular that $j^* \in \{3, \dots, k_f - 1\}$. Next, we show that $f(R^\pi) = f(R^{\pi^*})$ for all permutations $\pi \in \Pi_{\pi^*}^{j^*}$. This observation implies that $a_{j^*}^\pi = a_{j^*}^{\pi^*} \notin f(R^\pi)$ for all these permutations. Finally, we use this insight to derive a contradiction. All profiles used for Steps 2 and 3 are depicted exemplarily in Figure 3 for the case that $k_f = 6$.

Step 1: $\{a_1^\pi, a_2^\pi, a_{k_f}^\pi\} \subseteq f(R^\pi)$ for all permutations $\pi : TC(R^*) \rightarrow TC(R^*)$

Consider an arbitrary preference profile R^π . First, note that there is no Condorcet winner in this profile since there is no Condorcet winner in R^* , and thus strong Condorcet consistency requires that $|f(R^\pi)| \geq 2$. As a consequence, COS requires that $a_1^\pi \in f(R^\pi)$ and $a_{k_f}^\pi \in f(R^\pi)$ because a_1^π is the only alternative that dominates a_2^π and $a_{k_f}^\pi$ is the only alternative that dominates a_1^π . As a last point, suppose for contradiction that $a_2^\pi \notin f(R^\pi)$. Since $|TC(R^\pi)| = k_f$, it follows from Lemma 8 that $TC(R^\pi) \setminus \{a_2^\pi\} \subseteq f(R^\pi)$,

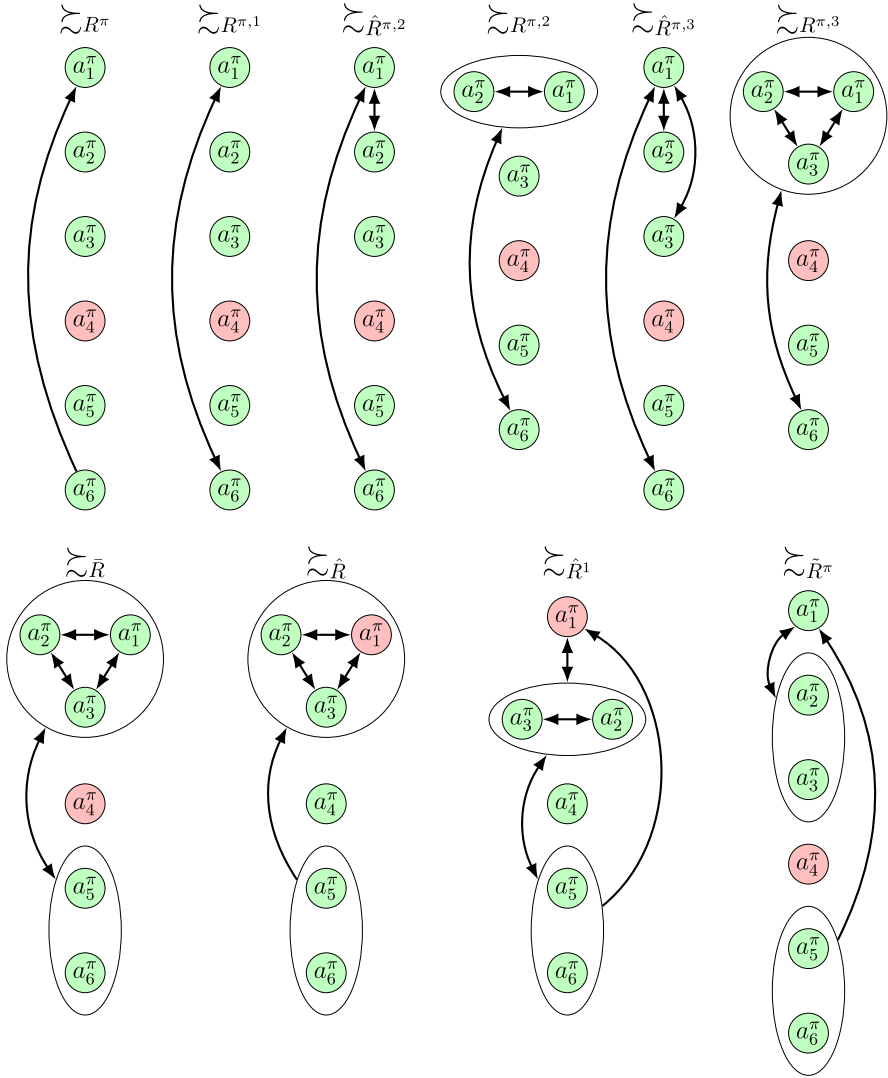


FIGURE 3. The (weighted) majority relations used in the proof of Lemma 11 for $k_f = 6$. Alternatives outside of the top cycle are not depicted, and we assume that $j^* = 4$ and $a_1^\pi \notin f(\hat{R})$, where π denotes a permutation in $\Pi_{\pi^*}^j$. Alternatives placed in an ellipse have identical relationships to all alternatives outside of the ellipse, and all missing edges point downwards. All directed edges indicate a majority margin of 2 and all bidirectional edges indicate a majority margin of 0. Green (light gray) alternatives are chosen and red (dark gray) ones are unchosen by f .

in particular that $a_3^\pi \in f(R^\pi)$. As the next step, we reinforce a_3^π twice against a_1^π to derive a profile R' with $g_{R'}(a_3^\pi, a_1^\pi) = 2$. Note that a_1^π needs to stay chosen during these steps because it is still the only alternative dominating a_2^π and WMON implies that a_3^π remains also chosen. Hence, it follows from WLOC that $f(R') = f(R^\pi)$, which means that $a_2^\pi \notin f(R')$. However, a_2^π is the only alternative that dominates a_3^π in R' , and thus

COS is violated. This is a contradiction, and thus the assumption that $a_2^\pi \notin f(R^\pi)$ was incorrect.

Step 2: $f(R^\pi) = f(R^{\pi^*})$ for all permutations $\pi \in \Pi_{\pi^*}^{j^*}$

For proving this step, we consider the profiles $R^{\pi,l}$ for every $l \in \{1, \dots, j^* - 1\}$, which differs from R^π in the fact that $g_{R^{\pi,l}}(a_i^\pi, a_j^\pi) = 0$ for all $i, j \in \{1, \dots, l, k_f\}$. Intuitively, $R^{\pi,l}$ is derived R^π by introducing a large set of tied alternatives $\{a_1^\pi, \dots, a_l^\pi, a_{k_f}^\pi\}$ in the majority relation. Our goal is to show that $f(R^\pi) = f(R^{\pi, j^* - 1})$ for all permutations $\pi \in \Pi_{\pi^*}^{j^*}$. This implies that $f(R^\pi) = f(R^{\pi'})$ for all such permutation π, π' because $g_{R^{\pi,l}} = g_{R^{\pi',l}}$ for all $\pi, \pi' \in \Pi_{\pi^*}^{l+1}$ and $l \in \{1, \dots, j^* - 1\}$. For deriving this statement, we show inductively that $f(R^\pi) = f(R^{\pi,l})$ for all $\pi \in \Pi_{\pi^*}^{j^*}$ and all $l \in \{1, \dots, j^* - 1\}$.

First, we focus on the induction basis $l = 1$ and consider therefore an arbitrary permutation $\pi \in \Pi_{\pi^*}^{j^*}$. Note that $R^{\pi,1}$ only differs from R^π by the fact that $g_{R^{\pi,1}}(a_{k_f}^\pi, a_1^\pi) = 0$ instead of 2. Hence, we can derive $R^{\pi,1}$ from R^π by reinforcing a_1^π against $a_{k_f}^\pi$. Furthermore, we have shown in the last step that $\{a_1^\pi, a_{k_f}^\pi\} \subseteq f(R^\pi)$, and COS requires that both alternatives are chosen in $f(R^{\pi,1})$ because $a_{k_f}^\pi$ is still the only alternative that dominates a_1^π and a_1^π is the only alternative that dominates a_2^π . Consequently, WLOC shows that $f(R^\pi) = f(R^{\pi,1})$.

For the induction step, assume that there is a value $l \in \{1, \dots, j^* - 2\}$ such that $f(R^\pi) = f(R^{\pi,l})$ for all $\pi \in \Pi_{\pi^*}^{j^*}$. We need to prove that this claim is also true for $l + 1$. Hence, note that for every permutation $\pi \in \Pi_{\pi^*}^{j^*}$, it holds that $\{a_1^\pi, \dots, a_{j^*-1}^\pi, a_{k_f}^\pi\} \subseteq f(R^\pi)$ because of the definition of j^* and Step 1. Next, we explain how to derive $R^{\pi,l+1}$ from R^π for an arbitrary permutation $\pi \in \Pi_{\pi^*}^{j^*}$: first, we reinforce a_1^π against $a_{k_f}^\pi$ to derive the profile \bar{R}^π . It follows from the same argument as in the induction basis that $f(\bar{R}^\pi) = f(R^\pi)$. Next, we reinforce all alternatives a_i^π with $i \in \{2, \dots, l + 1\}$ against a_1^π . COS requires for the resulting profile \hat{R}^π that a_1^π is chosen because it is still the only alternative dominating a_2^π and Lemma 9 shows therefore that $f(\hat{R}^\pi) = f(R^\pi)$. Furthermore, observe that $g_{\hat{R}^\pi}(a_1^\pi, a_i^\pi) = 0$ for all $i \in \{2, \dots, l + 1, k_f\}$. Finally, we can derive $R^{\pi,l+1}$ from \hat{R}^π by letting a voter i with preference relation $\succ_{i=2, \dots, a_{l+1}^\pi, a_{k_f}^\pi}$, $\mathbf{lex}(A \setminus \{a_2^\pi, \dots, a_{l+1}^\pi, a_{k_f}^\pi\})$ change his preference relation to $\succ'_i = a_{k_f}^\pi, a_{l+1}^\pi, \dots, a_2^\pi$, $\mathbf{lex}(A \setminus \{a_2^\pi, \dots, a_{l+1}^\pi, a_{k_f}^\pi\})$. We can assume that such a voter exists since pairwise dominance allows us to add voters with inverse preferences without affecting the choice set. This step ensures that $g_{R^{\pi,l+1}}(a_i^\pi, a_j^\pi) = 0$ for all $i, j \in \{2, \dots, l + 1, k_f\}$ and it therefore transforms \hat{R}^π into $R^{\pi,l+1}$.

Now, since $\{a_2^\pi, \dots, a_{l+1}^\pi, a_{k_f}^\pi\} \subseteq f(R^\pi) = f(\hat{R}^\pi)$, WLOC implies that $f(R^\pi) = f(R^{\pi,l+1})$ if $\{a_2^\pi, \dots, a_{l+1}^\pi, a_{k_f}^\pi\} \subseteq f(R^{\pi,l+1})$. Hence, our next goal is to prove this set inclusion and we assume for contradiction that there is an alternative a_j^π with $j \in \{2, \dots, l + 1, k_f\}$ such that $a_j^\pi \notin f(R^{\pi,l+1})$. First, suppose that $a_j^\pi \in \{a_2^\pi, \dots, a_{l+1}^\pi\}$, which means that $TC(R^\pi) \setminus \{a_j^\pi\} \subseteq f(R^\pi)$ because of Lemma 8. In this case, we derive a contradiction by considering the permutation π' with $a_1^{\pi'} = a_j^\pi, a_j^{\pi'} = a_1^\pi,$

and $a_i^{\pi'} = a_i^{\pi}$ for all other $i \in \{1, \dots, k_f\}$. In more detail, we can use the same construction as for π to transform $R^{\pi'}$ into $R^{\pi', l+1} = R^{\pi, l+1}$. In particular, the analysis of the previous paragraph shows that $\{a_1^{\pi'}, \dots, a_{l+1}^{\pi'}, a_{k_f}^{\pi'}\} \subseteq f(R^{\pi'}) = f(\hat{R}^{\pi'})$ and we derive $R^{\pi', l+1} = R^{\pi, l+1}$ from the profile $\hat{R}^{\pi'}$ by a manipulation that only involves the alternatives $\{a_2^{\pi'}, \dots, a_{l+1}^{\pi'}, a_{k_f}^{\pi'}\}$. Hence, the assumption that $TC(R^{\pi}) \setminus \{a_j^{\pi}\} = TC(R^{\pi'}) \setminus \{a_1^{\pi'}\} \subseteq f(R^{\pi, l+1})$ implies that $f(\hat{R}^{\pi'}) = f(R^{\pi', l+1})$ because of WLOC. However, this contradicts that $a_1^{\pi'} = a_j^{\pi} \notin f(R^{\pi, l+1})$, which proves that $\{a_2^{\pi}, \dots, a_{l+1}^{\pi}\} \subseteq f(R^{\pi, l+1})$.

As a second case, suppose that $a_{k_f}^{\pi} \notin f(R^{\pi, l+1})$. In this case, we derive a contradiction to the induction hypothesis by deriving $R^{\pi, l+1}$ from $R^{\pi, l}$. Thus, note that these two preference profiles only differ in majority margins involving a_{l+1}^{π} : we have $g_{R^{\pi, l}}(a_{l+1}^{\pi}, a_{k_f}^{\pi}) = 2$ but $g_{R^{\pi, l+1}}(a_{l+1}^{\pi}, a_{k_f}^{\pi}) = 0$, and for all $i \in \{1, \dots, l\}$, $g_{R^{\pi, l}}(a_i^{\pi}, a_{l+1}^{\pi}) = 2$ but $g_{R^{\pi, l+1}}(a_i^{\pi}, a_{l+1}^{\pi}) = 0$. Also, observe that the induction hypothesis implies that $f(R^{\pi}) = f(R^{\pi, l})$, which means that $\{a_1^{\pi}, \dots, a_{l+1}^{\pi}, a_{k_f}^{\pi}\} \subseteq f(R^{\pi, l})$. Hence, we can transform $R^{\pi, l}$ into $R^{\pi, l+1}$ as follows: first, we reinforce a_{l+1}^{π} one after another against all alternatives a_i^{π} with $i \in \{1, \dots, l\}$. For each swap, Lemma 9 implies that the choice set does either not change at all, or all alternatives in $TC(R^{\pi}) \setminus \{a_i^{\pi}\}$ are chosen (where a_i^{π} denotes the weakened alternative). In particular, this shows that a_{l+1}^{π} and $a_{k_f}^{\pi}$ stay chosen during this process. Finally, we reinforce $a_{k_f}^{\pi}$ against a_{l+1}^{π} to derive $R^{\pi, l+1}$. Then WMON implies that $a_{k_f}^{\pi} \in f(R^{\pi, l+1})$, contradicting our assumption. Hence, it follows that $\{a_2^{\pi}, \dots, a_{l+1}^{\pi}, a_{k_f}^{\pi}\} \subseteq f(R^{\pi, l+1})$, which proves the induction step. As a consequence, we infer that $f(R^{\pi}) = f(R^{\pi, j^*-1}) = f(R^{\pi', j^*-1}) = f(R^{\pi'})$ for all permutations $\pi, \pi' \in \Pi_{\pi^*}^*$.

Step 3: Deriving the contradiction

As a last step, we derive a contradiction by showing that $a_{j^*}^{\pi^*} \in f(R^{\pi^*})$. This claim is in conflict with the definitions of j^* and R^{π^*} , which require that $a_{j^*}^{\pi^*} \notin f(R^{\pi^*})$. For proving this claim, we consider first the profile \bar{R} , which differs in the following majority margins from R^{π^*} : $g_{\bar{R}}(a_i^{\pi^*}, a_j^{\pi^*}) = 0$ for all $i, j \in \{1, \dots, j^* - 1\}$ and $g_{\bar{R}}(a_i^{\pi^*}, a_j^{\pi^*}) = 0$ for all $i \in \{1, \dots, j^* - 1\}$, $j \in \{j^* + 1, \dots, k_f\}$. A similar analysis as in Step 2 shows that $f(R^{\pi^*}) = f(\bar{R})$. In more detail, we can use an induction on the profiles $\bar{R}^{\pi, l}$ for all $l \in \{1, \dots, j^* - 1\}$ and $\pi \in \Pi_{\pi^*}^{j^*}$, which are defined by the following majority margins: $g_{\bar{R}^{\pi, l}}(a_i^{\pi}, a_j^{\pi}) = 0$ for all $i, j \in \{1, \dots, l\}$, $g_{\bar{R}^{\pi, l}}(a_i^{\pi}, a_j^{\pi}) = 0$ for all $i \in \{1, \dots, l\}$, $j \in \{j^* + 1, \dots, k_f\}$, and $g_{\bar{R}^{\pi, l}}(x, y) = g_{R^{\pi}}(x, y)$ for all remaining majority margins. More intuitively, the profiles $\bar{R}^{\pi, l}$ differ from the profiles $R^{\pi, l}$ only in the fact that all alternatives $\{a_1^{\pi}, \dots, a_l^{\pi}\}$ are in a majority tie with all alternatives in $\{a_{j^*+1}^{\pi}, \dots, a_{k_f}^{\pi}\}$ instead of just $a_{k_f}^{\pi}$. Since the last step shows that $TC(R^{\pi}) \setminus \{a_{j^*}^{\pi}\} \subseteq f(R^{\pi})$ for all permutations $\pi \in \Pi_{\pi^*}^{j^*}$, an almost identical induction as in Step 2 shows that $f(R^{\pi}) = f(\bar{R}^{\pi, l})$ for all permutations $\pi \in \Pi_{\pi^*}^{j^*}$ and $l \in \{1, \dots, j^* - 1\}$. This means in particular that $f(R^{\pi^*}) = f(\bar{R}^{\pi^*, j^*-1}) = f(\bar{R})$.

Departing from this observation, we now consider profile \hat{R} which is derived from \bar{R} by reinforcing all alternatives in $X_2 = \{a_{j^*+1}^{\pi^*}, \dots, a_{k_f}^{\pi^*}\}$ against all alternatives in $X_1 =$

$\{a_1^{\pi^*}, \dots, a_{j^*-1}^{\pi^*}\}$. This means for the majority margins that $g_{\hat{R}}(x, y) = 2$ for all $x \in X_2$, $y \in X_1$. As a consequence of this observation, $a_{j^*}^{\pi^*}$ is now the only alternative that dominates $a_{j^*+1}^{\pi^*}$, and thus COS requires that $a_{j^*}^{\pi^*} \in f(\hat{R})$. Next, note that a repeated application of Lemma 9 shows that $X_2 \subseteq f(\hat{R})$ because we can transform \bar{R} into \hat{R} by reinforcing the alternatives X_2 against each alternative $x \in X_1$ individually. For each of these steps, Lemma 9 shows that either the choice set does not change or all alternatives in $TC(R^{\pi^*}) \setminus \{x\}$ are chosen. In particular, this means that $X_2 \subseteq f(\hat{R})$ and that $TC(\hat{R}) \not\subseteq f(\hat{R})$. Hence, there is an alternative $a_j^{\pi^*} \in X_1$ such that $a_j^{\pi^*} \notin f(\hat{R})$.

As the next step, consider the profile \hat{R}^j derived from \bar{R} by reinforcing the alternatives in X_2 only against $a_j^{\pi^*}$. We show that $a_j^{\pi^*} \notin f(\hat{R}^j)$ and assume for the sake of contradiction that this is not the case. Hence, Lemma 9 implies that $f(\hat{R}^j) = f(\bar{R})$. Moreover, we can now transform \hat{R}^j into \hat{R} by reinforcing the alternatives in X_2 once against each alternative $x \in X_1 \setminus \{a_j^{\pi^*}\}$. For every step, Lemma 9 shows that $a_j^{\pi^*}$ needs to stay chosen, and thus we have a contradiction to the assumption that $a_j^{\pi^*} \notin f(\hat{R})$. Hence, it must hold that $a_j^{\pi^*} \notin f(\hat{R}^j)$, which implies that $TC(\hat{R}^j) \setminus \{a_j^{\pi^*}\} \subseteq f(\hat{R}^j)$ because of Lemma 8.

Finally, we derive a contradiction to this observation. Consider for this a permutation $\pi \in \Pi_{\pi^*}^*$ such that $a_1^\pi = a_j^{\pi^*}$. We show that $a_j^{\pi^*} \in f(\hat{R}^j)$ by transforming R^π into \hat{R}^j and observe for this $X_1 \cup X_2 \subseteq f(R^\pi)$ because of Step 2. As a first step, we reinforce all alternatives in $X_2 \setminus \{a_{k_f}^\pi\}$ against a_1^π twice, and the alternatives in $X_1 \setminus \{a_1^\pi\}$ once against a_1^π . During all these steps, COS requires that a_1^π remains chosen because it is the only alternative that dominates a_2^π . In turn, Lemma 9 implies therefore that the choice set cannot change, that is, this process results in a profile \tilde{R}^π with $f(\tilde{R}^\pi) = f(R^\pi)$. Moreover, note that $g_{\tilde{R}^\pi}(a_1^\pi, x) = g_{\hat{R}^j}(a_1^\pi, x)$ for all $x \in A \setminus \{a_1^\pi\}$. For the next step, consider a voter i with the preference relation $\succ_i = a_2^\pi, \dots, a_{j^*-1}^{\pi^*}, a_{j^*+1}^{\pi^*}, \dots, a_{k_f}^\pi, \mathbf{lex}(X)$, where X contains all missing alternatives. We can assume that such a voter exists as pairwise-ness allows us to add pairs of voters with inverse preferences without affecting the choice set. Next, we let voter i deviate to the preference relation $a_{j^*+1}^{\pi^*}, \dots, a_{k_f}^\pi, a_{j^*-1}^{\pi^*}, \dots, a_2^\pi, \mathbf{lex}(X)$, which transforms the profile \tilde{R}^π into \hat{R}^j . In particular, we know that all alternatives in $\{a_2^\pi, \dots, a_{j^*-1}^{\pi^*}, a_{j^*+1}^{\pi^*}, \dots, a_{k_f}^\pi\}$ are chosen both in $f(\tilde{R}^\pi)$ (because $f(R^\pi) = f(\tilde{R}^\pi)$) and in $f(\hat{R}^j)$ (because $TC(R^{\pi^*}) \setminus \{a_j^{\pi^*}\} = TC(R^\pi) \setminus \{a_1^\pi\} \subseteq f(\hat{R}^j)$). Thus, WLOC implies that $f(\tilde{R}^\pi) = f(\hat{R}^j)$, which conflicts with $a_j^{\pi^*} = a_1^\pi \notin f(\hat{R}^j)$. This contradiction proves the lemma. \square

A.5 Proofs of the main results

Finally, we are ready to prove our main results. First, we discuss the proof of Theorem 1: a pairwise, nonimposing, neutral, and homogeneous SCC is strategyproof if and only if it is a robust dominant set rule. To be able to use the results of the previous section, we

show that every SCC which satisfies these requirements is strongly Condorcet consistent.

LEMMA 12. *Every pairwise SCC that satisfies strategyproofness, nonimposition, homogeneity, and neutrality is strongly Condorcet consistent.*

PROOF. Consider a pairwise SCC f that satisfies nonimposition, homogeneity, neutrality, and strategyproofness. We need to show two claims: if there is a Condorcet winner, it is chosen uniquely by f , and if an alternative is the unique winner of f , it is the Condorcet winner. We prove these claims separately.

Claim 1: If x is the Condorcet winner in R , then $f(R) = \{x\}$.

Consider an arbitrary profile R with Condorcet winner x . The claim follows by showing that $f(R) = \{x\}$, and thus let R' denote a profile such that $f(R') = \{x\}$. Such a profile exists because f is nonimposing. Next, we repeatedly use WSMON to push down the best alternative of every voter until we arrive at a profile R^1 such that every voter top-ranks x . Since R^1 is constructed by repeated application of WSMON, it follows that $f(R^1) = f(R') = \{x\}$. As the next step, we let all voters order the alternatives in $A \setminus \{x\}$ lexicographically. This leads to the profile R^2 and IUA implies that the choice set does not change, so $f(R^2) = \{x\}$. Moreover, all voters have the same preference relation in R^2 . Thus, it follows from homogeneity that $f(R^3) = \{x\}$, where R^3 consists of a single voter who has the same preference relation as the voters in R^2 . Next, let $c = \min_{y \in A \setminus \{x\}} g_R(x, y)$ denote the smallest majority margin of x in R and note that $c \geq 1$ because x is the Condorcet winner in R . We use again homogeneity to construct a profile R^4 that consists of c copies of R^3 , which means that $f(R^4) = \{x\}$. Furthermore, observe that the parity of the number of voters used in R^4 is equal to the parity of the number of voters used in R . The reason for this is that c is odd if and only if R is defined by an odd number of voters.

As the last step, we need to set the majority margins to their values in R . For this, we repeat the following procedure on each pair of alternatives y, z with $g_{R^4}(y, z) < g_R(y, z)$ until we arrive at a profile R^5 with $g_{R^5} = g_R$. First, we add two voters i and j to the preference profile such that voter i prefers x the least and ranks z directly over y , and voter j 's preference relation is inverse to voter i 's. Observe that we can assign such a preference relation to voter i because $g_{R^4}(x, z') = c \leq g_R(x, z')$ for all $z' \in A \setminus \{x\}$ implies that $x \neq z$. Since the preference relations of these two voters are inverse, the majority margins do not change and pairwise consistency requires thus that x is still the unique winner. Next, we let voter i swap y and z , which increases the majority margin between y and z by 2. Moreover, the choice set cannot change during this step because x is voter i 's least preferred alternative. Hence, if another set would be chosen, this step is a manipulation for voter i , which contradicts strategyproofness. Therefore, we can repeat this process for every pair of alternatives until we derive a profile R^5 with $g_{R^5} = g_R$ and our arguments show that $f(R^5) = \{x\}$. This proves that $f(R) = \{x\}$ because of pairwise consistency, which shows that f chooses the Condorcet winner uniquely whenever it exists.

Claim 2: If $f(R) = \{x\}$, then x is the Condorcet winner in R .

Next, we focus on the opposite direction and show that if an alternative is chosen as unique winner by f , then it is the Condorcet winner. Assume for contradiction that this is not the case, which means that there is a preference profile R and an alternative x such $f(R) = \{x\}$ even though x is not the Condorcet winner in R . Then there is an alternative $y \in A \setminus \{x\}$ such that $g_R(y, x) \geq 0$. We continue with a case distinction with respect to whether $g_R(y, x) = 0$ or $g_R(y, x) > 0$. First, assume that $g_R(y, x) > 0$. In this case, we can repeatedly reinforce y against all other alternatives $z \in A \setminus \{x\}$. This process eventually results in a profile R' in which y is the Condorcet winner. Since Claim 1 proves that f elects the Condorcet winner whenever it exists, we derive that $f(R') = \{y\}$. On the other side, it follows from IUA that these steps do not change the choice set because we only swap unchosen alternatives, so $f(R') = \{x\}$. These two observations contradict each other, and thus the assumption that $g_R(y, x) > 0$ was incorrect.

Next, assume that $g_R(y, x) = 0$. In this case, we partition the voters N according to their preferences between x and y : we denote with $N_{x>y} = \{i \in N : x \succ_i y\}$ the set of voters who prefer x to y , and with $N_{y>x} = \{i \in N : y \succ_i x\}$ the set of voters who prefer y to x . We let all voters in $N_{x>y}$ change their preferences such that y is directly below x , and all voters in $N_{y>x}$ change their preferences such that y is directly above x . For these steps, IUA implies that x remains the unique winner as we only reorder unchosen alternatives. Hence, it follows for the resulting profile R' that $f(R') = \{x\}$. However, it holds that $g_{R'}(x, z) = g_R(y, z)$ for all $z \in A \setminus \{x, y\}$ and $g_{R'}(x, y) = 0$. Neutrality and pairwise consistency thus require that either $\{x, y\} \subseteq f(R')$ or $\{x, y\} \cap f(R') \neq \emptyset$ because renaming x and y does not change the majority margins. This is in conflict with the previous claim, and hence the assumption that $f(R) = \{x\}$ and $g_R(x, y) = 0$ was incorrect. We have derived a contradiction in both cases, which proves that $f(R) = \{x\}$ can only be true if x is the Condorcet winner in R . \square

Since we established that every SCC that satisfies the requirements of Theorem 1 is strongly Condorcet consistent, this result follows now easily from Lemma 11.

THEOREM 1. *Let f be a pairwise SCC that satisfies nonimposition, homogeneity, and neutrality. Then f is strategyproof if and only if it is a robust dominant set rule.*

PROOF. Let f denote a pairwise SCC that satisfies homogeneity, neutrality, and nonimposition. The direction from left to right follows from Lemma 11 and Lemma 12: if f is additionally strategyproof, Lemma 12 shows that f is strongly Condorcet consistent and, in turn, Lemma 11 implies that f is a robust dominant set rule.

Next, we discuss the direction from right to left and assume thus that f is a robust dominant set rule. Furthermore, suppose for contradiction that f is not strategyproof. Hence, there are two preference profiles R and R' and a voter i such that $\succ_j = \succ'_j$ for all $j \in N \setminus \{i\}$ and $f(R') \succ_i^F f(R)$. First, assume that $f(R') \setminus f(R) \neq \emptyset$ and observe that $f(R) \succ_R f(R') \setminus f(R)$ since $f(R)$ is a dominant set. Deviating from R to

R' is only a manipulation for voter i if $f(R') \setminus f(R) \succ_i f(R)$. However, this means that $f(R) \succ_{R'} f(R') \setminus f(R)$ as voter i can only weaken the alternatives in $f(R') \setminus f(R)$ against those in $f(R)$. Since f is a dominant set rule and $f(R') \setminus f(R) \neq \emptyset$, this implies that $f(R) \subseteq f(R')$. Hence, $f(R) \succ_{R'} A \setminus f(R')$ because $f(R')$ is a dominant set, which proves that $f(R)$ is also a dominant set in $\succ_{R'}$. As a consequence, robustness requires that $f(R') \subseteq f(R)$, which contradicts the assumption that $f(R') \setminus f(R) \neq \emptyset$. Hence, no manipulation is possible in this case.

As a second case, suppose that $f(R') \subsetneq f(R)$. First, note that $f(R') \succ_{R'} A \setminus f(R')$ because $f(R')$ is a dominant set. Moreover, since deviating from R to R' is a manipulation for voter i , it holds that $f(R') \succ_i f(R) \setminus f(R')$. As a consequence of these two observations, it follows that $f(R') \succ_R f(R) \setminus f(R')$ because voter i can only weaken the alternatives in $f(R')$ against those in $f(R) \setminus f(R')$. Finally, since $f(R') \subseteq f(R)$, it follows that $f(R') \succ_R A \setminus f(R)$, and thus $f(R')$ is a dominant set in \succ_R . Hence, robustness from R' to R implies that $f(R) \subseteq f(R')$, which contradicts our assumption that $f(R') \subsetneq f(R)$. Thus, f is also in this case not manipulable, which shows that it is strategyproof. \square

Next, we focus on Theorem 2 and prove this result using Lemma 11. As a first step, we show that pairwise consistency, strategyproofness, homogeneity, and set nonimposition imply strong Condorcet consistency.

LEMMA 13. *Every pairwise SCC that satisfies set nonimposition, homogeneity, and strategyproofness is strongly Condorcet consistent.*

PROOF. Let f denote an SCC as specified by the lemma. First, note the proof of Claim 1 in Lemma 12 does not require neutrality and it thus shows that f is Condorcet consistent. Hence, we focus on the converse direction and show that $f(R) = \{x\}$ can only be true if x is the Condorcet winner in R . For this, suppose for contradiction that there is a profile R and an alternative x such that $f(R) = \{x\}$, but x is not the Condorcet winner in R . This means that there is another alternative $y \in A \setminus \{x\}$ such that $g_R(y, x) \geq 0$. If $g_R(y, x) > 0$, we can use the same construction as in the proof of Lemma 12 to derive that f violates Condorcet consistency since this construction works again without neutrality. Hence, suppose that $g_R(x, y) = 0$. In this case, we first weaken y in the preference relation of every voter $i \in N$ with $y \succ_i x$ such that it is directly over x and reinforce y in the preference relation of every voter $i \in N$ with $x \succ_i y$ such that it is placed directly below x . We infer from IUA that x is still the unique winner. Next, we iterate over the voters $i \in N$ and use WSMON to repeatedly push down voter i 's best alternative until he top-ranks x or y . It follows for the resulting profile R^1 that $f(R^1) = \{x\}$ because of WSMON and that all voters report x and y as their best two alternatives. Thereafter, we let the voters reorder the alternatives in $A \setminus \{x, y\}$ lexicographically. IUA implies that this step does not affect the choice set, and thus it holds for the new profile R^2 that $f(R^2) = f(R^1) = \{x\}$.

Next, we show that $f(R^2) = \{x\}$ is in conflict with set nonimposition. For this, consider a preference profile R^3 with $f(R^3) = \{x, y\}$; such a profile exists since f is

set nonimposing. Our goal is to transform R^3 into R^2 while showing that both x and y must be chosen. As a first step, we repeatedly add voters with inverse preferences and use WSMON to weaken the alternatives $z \in A \setminus \{x, y\}$ until we derive a profile R^4 with $\{x, y\} \succ_{R^4} A \setminus \{x, y\}$. It follows from pairwise and WSMON that $f(R^4) = f(R^3) = \{x, y\}$. This implies that $x \sim_{R^4} y$ because otherwise there is a Condorcet winner, which must be chosen uniquely. Even more, note that, as long as $x \sim_R y$ and $\{x, y\} \succ_R A \setminus \{x, y\}$, it holds that either $\{x, y\} \subseteq f(R)$ or $f(R) \subseteq \{x, y\}$. Otherwise, there is a profile R' and alternatives z_1, z_2 such that $z_1 \in f(R') \setminus \{x, y\}$ and $z_2 \in \{x, y\} \setminus f(R')$. Hence, if a voter reinforces $z_3 \in \{x, y\} \setminus \{z_2\}$ against z_2 , z_3 is the Condorcet winner and it must thus be chosen uniquely. However, this is in conflict with strategyproofness because WLOC (if $z_3 \in f(R')$) or IUA (if $z_3 \notin f(R')$) is violated.

We use the last observation to repeatedly reinforce the alternatives $\{x, y\}$ against the alternatives $A \setminus \{x, y\}$ in R^4 until all voters report x and y as their best two alternatives. For each swap, WMON implies that either $z_1 \in \{x, y\}$ remains chosen and $z_2 \in A \setminus \{x, y\}$ remains unchosen, or z_1 becomes unchosen and z_2 chosen. However, the latter is impossible because of our previous observation, and thus we derive from WMON and WLOC that the choice set is not allowed to change. Hence, it holds for the resulting profile R^5 that $f(R^5) = \{x, y\}$ and that all voters report x and y as their best two alternatives. Thereafter, we derive the profile R^6 from R^5 by arranging the alternatives in $A \setminus \{x, y\}$ in lexicographic order, which does not affect the choice set because of IUA. Finally, note that in R^6 , half of the voters report $\succ_1 = x, y, \mathbf{lex}(A \setminus \{x, y\})$ and the other half report $\succ_2 = y, x, \mathbf{lex}(A \setminus \{x, y\})$. Using homogeneity, it follows therefore that $f(R^7) = f(R^6)$, where R^7 consists of two voters who report \succ_1 and \succ_2 , respectively. Finally, the profile R^2 consists of multiple copies of R^7 , and hence it again follows from homogeneity that $f(R^2) = f(R^7) = \{x, y\}$. However, this contradicts the previous observation that $f(R^2) = \{x\}$, and hence f can only choose a single winner if it is the Condorcet winner. \square

Finally, we prove Theorem 2 based on Lemmas 1, 11, and 13.

THEOREM 2. *The top cycle is the only pairwise SCC that satisfies strategyproofness, set nonimposition, and homogeneity.*

PROOF. We have already shown in Lemma 1 that the top cycle satisfies set nonimposition. Moreover, by definition, TC is majoritarian and, therefore, also pairwise and homogeneous. Finally, the top cycle is a robust dominant set rule and hence strategyproof by Theorem 1. For the other direction, consider an arbitrary pairwise SCC f that satisfies strategyproofness, set nonimposition, and homogeneity. Since all criteria of Lemma 13 are satisfied, it follows that f is strongly Condorcet consistent. Next, we use Lemma 11 to derive that f is a robust dominant set rule. As the last step, Lemma 1 shows that f is the top cycle since this is the only robust dominant set rule that satisfies set nonimposition. \square

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