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Optimal allocations with α-MaxMin utilities, Choquet expected utilities, and prospect theory

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The analysis of optimal risk sharing has been thus far largely restricted to nonexpected utility models with concave utility functions, where concavity is an expression of ambiguity aversion and/or risk aversion. This paper extends the analysis to α-maxmin expected utility, Choquet expected utility, and cumulative prospect theory, which accommodate ambiguity seeking and risk seeking attitudes. We introduce a novel methodology of quasidifferential calculus of Demyanov and Rubinov (1986, 1992) and argue that it is particularly well suited for the analysis of these three classes of utility functions, which are neither concave nor differentiable. We provide characterizations of quasidifferentials of these utility functions, derive first-order conditions for Pareto optimal allocations under uncertainty, and analyze implications of these conditions for risk sharing with and without aggregate risk.

Keywords. Quasidifferential calculus, ambiguity, Pareto optimality, α-MaxMin expected utility, Choquet expected utility, rank-dependent expected utility, cumulative prospect theory.

JEL classification. C02, D61, D81.

1. Introduction

The expected utility hypothesis, with risk aversion and common beliefs, leads to clear-cut results on optimal risk sharing with and without aggregate risk. Motivated by the evidence—empirical and experimental—that expected utility fails to properly describe people's preference in many situations involving risk or uncertainty, the analysis of optimal risk sharing has been extended in the last two decades to nonexpected utility models such as the multiple-prior model of Gilboa and Schmeidler (1989), the variational preferences of Maccheroni, Marinacci, and Rustichini (2006), the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005), the Knightian uncertainty model of Bewley

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(1986), the Choquet (nonadditive) expected utility model of Schmeidler (1989), and others.[1] An important assumption in many of these extensions has been concavity of the utility functions. Concavity implies that preferences exhibit ambiguity aversion and risk aversion.

Ambiguity seeking and risk seeking are two behavioral phenomena frequently observed in empirical and experimental studies.[2] The most popular models in applied and theoretical research that accommodate ambiguity seeking and mixed attitude toward ambiguity are the α-maxmin expected utility (α-MEU), the smooth ambiguity model with nonconcave "second-order" utility, the Choquet expected utility (CEU) with nonconvex capacity, and the Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992). The utility functions of these models are nonconcave, and—with exception of the smooth model—nondifferentiable. This renders the standard methods of differential calculus and convex analysis inapplicable to the analysis of optimal risk sharing.

This paper develops a novel methodology for studying (first-order) optimality conditions for utility functions under uncertainty that are neither concave nor differentiable. The methodology is based on quasidifferential calculus advanced in the 1980's by V. Demyanov and A. Rubinov and others; see Demyanov and Rubinov (1986, 1992). We argue that it is particularly well suited for α-MEU, CEU, and CPT utility functions, and superior to the occasionally used subdifferential of Clarke (1983).[3] We provide characterizations of the quasidifferentials of these three classes of utility functions, derive first-order conditions for optimal risk sharing, and analyze their implications.

Quasidifferential calculus focuses on directional derivatives, and can be seen as an extension of sub and superdifferential calculus of convex analysis (see Rockafellar (1970)) beyond concave and convex functions. It is well known that the directional derivative is a linear function of the directional vector for a (Gateaux) differentiable function. For a concave function, the directional derivative is a sublinear function[4] of the directional vector, while for a convex function, it is superlinear. In quasidifferential calculus, the directional derivative is represented as the sum of a sublinear function and a superlinear function. There is a pair of convex sets—identified in a nonunique way[5]—such that the sublinear part is the support function (maximum) of one set and the superlinear part is the negative support function (minimum) of the second set. The two sets are called superdifferential and subdifferential because they coincide with those of convex analysis for concave and convex functions, respectively. Examples of quasidifferentiable functions include concave and convex functions, their linear combinations, and maxima and minima of arbitrary collections of differentiable functions.

[1] We review the literature on optimal risk sharing at the end of this section.

[2] See, for example, Trautmann and van de Kuilen (2015) and Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010).

[3] See Ghirardato and Sinischalchi (2012).

[4] A function is sublinear (superlinear) if it is subadditive (superadditive, resp.) and positively homogeneous.

[5] Sub- and superdifferentials are identified up to an equivalence class of a relation between pairs of convex sets that we introduce in Section 2. All results for quasidifferentiable functions hold independently of the choice of sub- and superdifferentials unless explicitly stated.

Important results of quasidifferential calculus are statements of first-order conditions for unconstrained and constrained optimization problems. For example, the necessary first-order condition for unconstrained maximum of a quasidifferentiable function on an open set is that the negative of the subdifferential is a subset of the superdifferential. It is a unified statement of the known first-order conditions for differentiable, concave, and convex functions; see Section 2. A strict form of this condition, which requires that the negative of the subdifferential is a subset of the interior of the superdifferential, is a sufficient condition for local maximum. First-order conditions—necessary, and sufficient—for constrained optimization problems have similar statements featuring Lagrange multipliers.

Quasidifferential calculus is an alternative to the method of generalized subdifferential of Clarke (1983).[6] Both methods provide first-order conditions in optimization problems. A drawback of the Clarke subdifferential is its lack of additivity. The subdifferential of a sum of two functions need not be equal to the sum of subdifferentials—it is merely a subset thereof. For example, the α-MEU function is a sum of maximum and minimum functions, but there is no known characterization of the Clarke subdifferential of it. In contrast, the basic rules of differentiation—in particular, additivity—continue to hold for quasidifferentiation; see Appendix B.

In the first part of the paper, we show that utility functions of α-MEU, CEU, and CPT models, all with arbitrary utility-of-wealth functions, are quasidifferentiable, and we derive their quasidifferentials. The α-MEU model is a generalization of the multiple-prior expected utility in a way that the utility function is a weighted sum of minimum and maximum of expected utilities over a set of priors.[7] Relative weight between the minimum and the maximum provides a parametrization of attitudes toward ambiguity. The maximum term, which stands for the ambiguity-seeking attitude, leads typically to nonconcavity of the resulting utility function. The quasidifferential of an α-MEU function consists of the superdifferential equal to the minimizing probabilities scaled by marginal utilities of wealth and the subdifferential equal to the maximizing probabilities scaled by the marginal utilities. Thus, the superdifferential is the same as for the multiple-prior expected utility with a concave utility-of-wealth function.

The CEU model takes the form of the Choquet integral of a utility-of-wealth function with respect to a capacity (or nonadditive probability measure). While a convex capacity reflects ambiguity aversion and a concave capacity reflects ambiguity seeking, a general capacity leads to mixed ambiguity attitude. The CEU function with nonconvex capacity is typically nonconcave. We show that the CEU function with arbitrary capacity is quasidifferentiable, and derive its quasidifferential by making use of a representation of the Choquet integral by the Möbius inverse of a capacity. An important special case of the CEU model is the Rank-Dependent Expected Utility (RDEU) model of Quiggin (1982) and Yaari (1987) where the capacity is a distortion of the reference probability measure. Convexity of the distortion implies convexity of the resulting capacity, and hence ambiguity aversion. Similarly, concavity of the distortion implies ambiguity seeking. In

[6] We discuss the relationship between these two methods in detail in Section 2.1.

[7] See Ghirardato, Maccheroni, and Marinacci (2004).

applications of the RDEU model, the most important is an inverse S-shaped distortion function, which is neither convex nor concave. It reflects overweighting the worst and the best outcomes, which has been documented in empirical work; see Wakker (2010, Chapter 7). We present a novel representation of an RDEU function with inverse S-shaped distortion as a weighted sum of minimum and maximum of expected utility functions with different sets of beliefs for minimum and maximum, and show that its quasidifferential takes a similar form to the quasidifferential of an α-MEU function.

The CPT model postulates a utility function that is a sum of two RDEU functions—one for gains and one for losses—so as to accommodate reference dependence of preferences. In the most popular formulation, distortion functions of the RDEU's are inverse S-shaped and the utility-of-wealth function is convex over losses and concave over gains. Convexity over losses reflects risk-seeking behavior in regard to losses. We show that the CPT utility function with inverse S-shaped distortions and a convex-concave utility-of-wealth function is quasidifferentiable, and we present a method of deriving its quasidifferential.

The second part of the paper is concerned with optimal risk sharing for quasidifferentiable utility functions. An important result is a statement of first-order necessary conditions for an interior Pareto optimal allocation for general quasidifferentiable utility functions. They require that for every profile of vectors in the subdifferentials at an optimal allocation there exists a profile of vectors in the superdifferentials such that, for every agent, the sum of sub and superdifferentials is a scale-multiple of the same vector. These conditions are an extension of the standard first-order conditions on marginal rates of substitution for differentiable functions and the more general conditions for concave or convex functions. We provide a statement of sufficient first-order conditions for local Pareto optima as well.

Several interesting implications emerge when our first-order necessary conditions are applied to the α-MEU model. If the utility-of-wealth functions are concave, then every Pareto optimal allocation with α-MEU functions is an optimal allocation for expected utility functions with the same utility-of-wealth functions and heterogeneous beliefs taken from the agents' sets of priors. For small sets of priors, this is a significant restriction on allocations that can be optimal. Further, we show that there cannot exist a risk-free Pareto optimal allocation in a no-aggregate-risk economy unless the sets of priors have nonempty intersection, that is, unless there is a common prior. Pareto optimal allocations with RDEU functions with inverse S-shaped distortions have similar properties as with the α-MEU. We analyze the first-order conditions of Pareto optimality for the CEU and the CPT models as well.

The paper is organized as follows. Section 2 introduces quasidifferential calculus and provides a discussion of the relationship with the method of Clarke (1983). In Section 3, we analyze quasidifferentiability of α-MEU, CEU, RDEU, and CPT utility functions. We present the first-order conditions for Pareto optimal allocations in Section 4, and derive some implications for the utility functions of Section 3. Section 5 contains concluding remarks. The Appendix consists of three parts: Part A contains proofs omitted from Sections 3 and 4. Part B provides some useful results of quasidifferential calculus, and part C contains a discussion of the class of capacities of Jaffray and Phillippe (1997) introduced in Section 3.

Related literature

There is vast empirical and experimental literature documenting heterogeneous attitudes toward ambiguity. Trautmann and van de Kuilen (2015) survey evidence from Ellsberg-style experiments. Bossaerts et al. (2010) and Ahn, Choi, Gale, and Kariv (2013) find evidence of heterogeneous attitudes toward ambiguity in an asset market experiment hypothesizing that subjects follow the α-MEU model.

Properties of efficient allocations for preferences that exhibit ambiguity aversion have been extensively studied in the literature over the past two decades. Billot, Chateauneuf, Gilboa, and Tallon (2000) show that if agents with concave multiple-prior expected utilities have at least one prior in common and there is no aggregate risk, then all interior Pareto optimal allocations are risk-free. Rigotti, Shannon, and Strzalecki (2008) extend that result to other models of convex preferences under ambiguity. Ghirardato and Sinischalchi (2018) study optimal risk sharing with no aggregate risk assuming supportability of preferred sets at risk-free consumption plans instead of convexity. Even this weaker assumption excludes ambiguity seeking in most models of preferences under ambiguity. General properties of efficient allocations when there is aggregate risk, such as comonotonicity and measurability with respect to aggregate endowment, have been studied in Chateauneuf, Dana, and Tallon (2000) and Dana (2004) for CEU functions with convex capacities, and in Strzalecki and Werner (2011) for general concave utility functions including multiple-prior utilities, variational preferences, and the smooth ambiguity model. Werner (2021) considers participation in risk sharing among agents with multiple-prior expected utilities, and shows that agents with the highest ambiguity (i.e., the largest sets of priors) and low risk aversion are most likely to hold risk-free consumption in any Pareto optimal allocation. De Castro and Chateauneuf (2011) and Strzalecki and Werner (2011) explore efficient risk sharing among ambiguity averse agents when the aggregate risk is unambiguous.

It should be noted that the results of this paper on optimal risk sharing with mixed attitudes toward ambiguity and risk are rather modest in comparison to the aforementioned results for convex preferences. In particular, only the necessity part of the elegant result of Billot et al. (2000) extends to our setting.[8] This appears to be an inevitable consequence of dealing with nonconvex preferences.

First-order necessary conditions for Pareto optimal allocations without differentiability can be found in Rigotti, Shannon, and Strzalecki (2008) for concave utility functions. Those conditions are stated in terms of subjective beliefs but can alternatively be stated in terms of the standard superdifferential of convex analysis, as in Aubin (1998). Ghirardato and Sinischalchi (2018) provide first-order conditions for interior Pareto optimal allocations without concavity using the Clarke subdifferential. The difficulty in applying this result to α-MEU or CEU functions is that there is no known characterization of the Clarke subdifferential of these functions. First-order conditions for Pareto optimal allocations with production in terms of the Clarke normal cone instead of the subdifferential can be found in Khan and Vohra (1987) and Bonnisseau and Cornet (1988).

[8]The sufficiency part extends for locally optimal allocations; see Section 4.

There has been some recent interest in general equilibrium theory and welfare theorems in economies with nonconvex preferences. Araujo, Chateauneuf, Gama, and Novinski (2018) study existence of an equilibrium in complete markets under uncertainty when a subset of agents have convex utility functions while the remaining agents have concave utility functions. Araujo, Bonnisseau, Chateauneuf, and Novinski (2017) study efficient allocations when there is aggregate risk, continuum of states, and a subset of agents have convex utility functions. They show that (strongly) risk averse agents have comonotone consumption plans in efficient allocations. Richter and Rubinstein (2015) introduce methods of abstract geometric convexity to general equilibrium theory and extend the welfare theorems by replacing the assumption of convexity in the standard sense by abstract convexity. They provide examples of economies with nonconvexities in the standard sense—mostly with indivisible goods—where the extended general equilibrium theory applies.

2. Quasidifferential calculus

Quasidifferential calculus is an extension of sub- and superdifferential calculus beyond convex and concave functions. We present basic concepts and results that will be used later.

Let $f : X \to \mathbb{R}$ be a real-valued function on an open subset X of $\mathbb{R}^S$. Function f is said to be directionally differentiable at $x \in X$ in the direction of $\hat{x} \in \mathbb{R}^S$ if the limit

$$f'(x; \hat{x}) = \lim_{t \to 0+} \frac{f(x + t\hat{x}) - f(x)}{t}, \tag{1}$$

exists. If the limit exists for every direction $\hat{x} \in \mathbb{R}^S$, then f is directionally differentiable at x. If f is Gateaux differentiable, then the directional derivative $f'(x; \hat{x})$ is the scalar product $\nabla f(x)\hat{x}$, where $\nabla f(x) \in \mathbb{R}^S$ is the gradient vector.

A function f is said to be *quasidifferentiable* at x if it is directionally differentiable and, furthermore, there exist two compact and convex sets A and B in $\mathbb{R}^S$ such that

$$f'(x; \hat{x}) = \max_{z \in A} \hat{x}z + \min_{z \in B} \hat{x}z \tag{2}$$

for every $\hat{x} \in \mathbb{R}^S$. Relation (2) is a representation of the directional derivative by the sum of a sublinear function and a superlinear function. Sets A and B in this representation are not unique. For example, the pair[9] $[A - S, B + S]$ satisfies (2) for every convex and compact set S as well. More generally, any two pairs of convex and compact sets $[A, B]$ and $[A', B']$ give the same representation as long as[10]

$$A - B' = A' - B. \tag{3}$$

[9]Recall the set addition $A + B = \{a + b : a \in A, b \in B\}$ and subtraction $A - B = \{a - b : a \in A, b \in B\}$.

[10]Note that (2) can be written as $f'(x; \hat{x}) = \mathrm{s}_A(\hat{x}) - \mathrm{s}_{-B}(\hat{x})$, where s_A denotes the support function of the set A. Pairs $[A, B]$ and $[A', B']$ satisfy (3) if and only if $\mathrm{s}_{A-B'}(\hat{x}) = \mathrm{s}_{A'-B}(\hat{x})$ for every $\hat{x} \in \mathbb{R}^S$. This can equivalently be written as $\mathrm{s}_A(\hat{x}) - \mathrm{s}_{-B}(\hat{x}) = \mathrm{s}_{A'}(\hat{x}) - \mathrm{s}_{-B'}(\hat{x})$ for every $\hat{x} \in \mathbb{R}^S$. Thus, (2) holds for $[A, B]$ if and only if it holds for $[A', B']$.

Equation (3) induces an equivalence relation among pairs of convex and compact sets in $\mathbb{R}^S$. Equivalence classes of that relation are in one-to-one correspondence to sums of sublinear and superlinear functions on $\mathbb{R}^S$; see Demyanov and Rubinov (1986). We refer to relation (3) as *DR-equivalence*.[11] Any pair of sets $[A, B]$ from the DR-equivalence class satisfying (2) is denoted by $\underline{\partial} f(x)$ for A and $\bar{\partial} f(x)$ for B, and written as

$$Df(x) = \left[\underline{\partial} f(x), \bar{\partial} f(x)\right].$$

A function f is said to be *subdifferenitiable* at x if it is quasidifferentiable and the superdifferential $\bar{\partial} f(x)$ is a singleton set for some DR-equivalent representation of the quasidifferential $Df(x)$. A subdifferentiable function has a sublinear directional derivative. Every convex function is subdifferentiable at every x with $\underline{\partial} f(x)$ being the subdifferential in the sense of convex analysis (and zero superdifferential). Similarly, f is *superdifferenitiable* at x if it is quasidifferentiable and the subdifferential $\underline{\partial} f(x)$ is a singleton set for some representation of $Df(x)$. The directional derivative of a superdifferenitiable function is superlinear. Every concave function is superdifferentiable, with $\bar{\partial} f(x)$ being the superdifferential of convex analysis. If the quasidifferential $Df(x)$ has a representation with singleton sets as sub- and superdifferentials, then f is Gateaux differentiable at x. Any pair of vectors $(\underline{d}, \bar{d})$ such that $\underline{d} + \bar{d} = \nabla f(x)$ is the quasidifferential of f at x.

For later use, we demonstrate now that a maximum function over a compact set of parameters is subdifferentiable. Let φ be defined by

$$\varphi(x) = \max_{y \in Y} f(x, y), \tag{4}$$

where f is continuous in (x, y) and continuously differentiable in x. The set $Y \subset \mathbb{R}^n$ is compact. Note that function φ may be neither convex nor concave.

Let $\varphi^*(x)$ denote the set of maximizers in (4) at x. It follows from the Danskin's envelope theorem that the directional derivative of φ is

$$\varphi'(x, \hat{x}) = \max_{y^* \in \varphi^*(x)} \nabla_x f\left(x, y^*\right)\hat{x}, \tag{5}$$

for every $\hat{x}$, where $\nabla_x f$ denotes the gradient of f with respect to x. Equation (5) implies that φ is quasidifferentiable with subdifferential given by

$$\underline{\partial}\varphi(x) = \operatorname{co}\left\{\nabla_x f\left(x, y^*\right) : y^* \in \varphi^*(x)\right\}, \tag{6}$$

where co denotes the convex hull, and zero superdifferential. Therefore, φ is subdifferentiable at x.

Summing up, the class of quasidifferentiable functions includes differentiable, concave, convex functions, and maxima and minima of differentiable functions. Sums, scale multiples, and compositions of quasidifferentiable functions are quasidifferentiable. Further, maxima and minima of finite collections of quasidifferentiable functions

[11] See Pallaschke and Urbanski (1994) for an extensive discussion of the relation of DR-equivalence and the problem of minimal representation of an equivalence class.

are quasidifferentiable as well. Most of the rules of differentiation continue to hold for quasidifferentiation; see Appendix B.

Necessary first-order conditions for solutions to optimization problems can be nicely stated for quasidifferentiable function. For example, the necessary condition for the unconstrained maximum x^* of a quasidifferentiable function f on $\mathbb{R}^S$ is

$$-\underline{\partial} f(x^*) \subset \bar{\partial} f(x^*),$$

which can be equivalently expressed as that for every $\underline{z} \in \underline{\partial} f(x^*)$ there exists $\bar{z} \in \bar{\partial} f(x^*)$ such that $\underline{z} + \bar{z} = 0$. The necessary condition for unconstrained minimum is $-\bar{\partial} f(x^*) \subset \underline{\partial} f(x^*)$, with interchanged roles of the sub and superdifferentials. These are unified statements of the standard first-order conditions for differentiable, concave, and convex functions. Strict forms of these conditions —with $\bar{\partial} f(x^*)$ replaced by its interior for a maximum, and $\underline{\partial} f(x^*)$ replaced by its interior for a minimum—are sufficient for local solutions. Note that these first-order conditions do not depend on the choice of DR-equivalent pairs of sets for sub and superdifferentials.

Necessary first-order conditions for constrained maximization of a quasidifferentiable function can be found in Demyanov and Dixon (1986) for various types of constraints. To illustrate, we present a first-order condition for maximization of a quasidifferentiable utility function f subject to the budget constraint. The budget set is $B(p) = \{x \in \mathbb{R}^S_+ | px \leq pe\}$, where $p \in \mathbb{R}^S_+$ is a vector of prices and e is an endowment. The necessary condition for a strictly positive solution $x^* \in \mathbb{R}^S_{++}$ is

$$-\underline{\partial} f(x^*) \subset \bar{\partial} f(x^*) - \{\lambda p | \lambda \geq 0\}. \tag{7}$$

This can be equivalently stated as that for every $\underline{z} \in \underline{\partial} f(x^*)$ there exists $\bar{z} \in \bar{\partial} f(x^*)$ and a multiplier $\lambda^* \geq 0$ such that $\underline{z} + \bar{z} = \lambda^* p$. Condition (7) with $\bar{\partial} f(x^*)$ replaced by its interior is sufficient for a local constrained maximum. The relative simplicity of condition (7) stems from the fact that the constraint function px is linear and, therefore, differentiable.

2.1 *Clarke subdifferential and quasidifferential*

The quasidifferential is related to, but different from the Clarke (1983) subdifferential. While quasidifferential calculus is concerned with representation of the standard (Dini) directional derivative (1), Clarke's theory introduces extensions of the directional derivative called Clarke lower and upper directional derivatives. The Clarke upper and lower directional derivatives of a Lipschitz continuous function f at x in the direction of $\hat{x}$ are defined, respectively, as

$$f'_+(x; \hat{x}) = \limsup_{y \to x, t \searrow 0} \frac{f(y + t\hat{x}) - f(x)}{t}, \quad \text{and} \quad f'_-(x; \hat{x}) = \liminf_{y \to x, t \searrow 0} \frac{f(y + t\hat{x}) - f(x)}{t}.$$

The Clarke subdifferential of f at x is

$$\partial_{\mathrm{CL}} f(x) = \mathrm{co}\Big\{\lim_{k \to \infty} \nabla f(x_k) : x_k \to x,\, x_k \in T(f)\Big\},$$

where $T(f) \subset \mathbb{R}^S$ is the set of points of differentiability of f. It holds

$$f'_+(x;\hat{x}) = \max_{z\in\partial_{\mathrm{CL}} f(x)} \hat{x}z, \quad \text{and} \quad f'_-(x;\hat{x}) = \min_{z\in\partial_{\mathrm{CL}} f(x)} \hat{x}z.$$

Therefore, the upper directional derivative is a sublinear function while the lower is superlinear. Since $f'_-(x;\hat{x}) \le f'(x;\hat{x}) \le f'_+(x;\hat{x})$, for every x and $\hat{x}$ (see Demyanov and Rubinov (1986, p. 74)), it follows that

$$\min_{z\in\partial_{\mathrm{CL}} f(x)} \hat{x}z \le f'(x;\hat{x}) \le \max_{z\in\partial_{\mathrm{CL}} f(x)} \hat{x}z.$$

Thus, the Clarke subdifferential provides a sublinear majorization and a superlinar minorization of the directional derivative. The quasidifferential provides an exact representation in equation (2). If function f is convex or concave, then the Clarke subdifferential is equal to, respectively, the sub or superdifferential of f.

As mentioned in the Introduction, the Clarke subdifferential lacks additivity. Further, the envelope theorem, such as (6), has merely an approximate statement for the Clarke subdifferential.

3. Quasidifferentiable utility functions

Uncertainty is described by a finite set of states S. The set of all subsets of S is denoted by Σ, and Δ is the probability simplex on (S, Σ). There is a single consumption good. State contingent consumption plans (or acts) are vectors in $\mathbb{R}^S_+$.

3.1 *α-Maxmin expected utility*

The α-MEU function is defined as

$$V(x) = \alpha \min_{P\in\mathcal{P}} E_P\big[v(x)\big] + (1-\alpha) \max_{P\in\mathcal{P}} E_P\big[v(x)\big], \tag{8}$$

for $x \in \mathbb{R}^S_+$, where $\mathcal{P} \subseteq \Delta$ is a (closed and convex) set of probability priors, $v : \mathbb{R}_+ \to \mathbb{R}$ is a utility index, and $\alpha \in [0, 1]$.[12] We assume throughout that v is strictly increasing. The relative weight α is a parameter of ambiguity attitude. If $\alpha = 1$, function V is the ambiguity-averse multiple-prior expected utility of Gilboa and Schmeidler (1989). If $\alpha = 0$, V is the ambiguity-seeking multiple-prior expected utility.

There is an apparent similarity between the α-MEU representation (8) and the representation (2) of a directional derivative in quasidifferential calculus. Indeed, an α-MEU function is the sum of superlinear function $\alpha \min_{P\in\mathcal{P}} E_P[\cdot]$ and sublinear $(1-\alpha)\max_{P\in\mathcal{P}} E_P[\cdot]$ applied to the utility vector $v(x)$. It follows that the α-MEU representation is determined up to DR-equivalence relation (3), and hence the parameter α and

[12] Axiomatizations of the α-MEU representation have been provided for special sets of priors by Gul and Pesendorfer (2015) and Chateauneuf, Eichberger, and Grant (2007), and—for general sets—by Frick, Iijima, and Le Yaouanq (2022) in the setting of two preference relations: subjectively rational and objectively rational. α-MEU functions of Gul and Pesendorfer (2015) and Chateauneuf, Eichberger, and Grant (2007) are also CEU functions, and will be discussed in Section 3.2.

the set of priors $\mathcal{P}$ are typically nonunique.[13] More precisely, two pairs $(\alpha, \mathcal{P})$ and $(\alpha', \mathcal{P}')$ give the same utility function V in (8) if and only if the pair of sets $[\alpha\mathcal{P}, (1-\alpha)\mathcal{P}]$ is DR-equivalent to $[\alpha'\mathcal{P}', (1-\alpha')\mathcal{P}']$. Proposition 1 in Frick, Iijima, and Le Yaouanq (2022) provides necessary and sufficient conditions for DR-equivalence of such pairs of sets.

Let $\mathcal{P}_{\min}(x) \subset \mathcal{P}$ be the closed and convex subset of priors for which the minimum expected utility of x is attained in (8). That is,

$$\mathcal{P}_{\min}(x) = \arg\min_{P\in\mathcal{P}} E_P\big[v(x)\big]. \tag{9}$$

Similarly, let

$$\mathcal{P}_{\max}(x) = \arg\max_{P\in\mathcal{P}} E_P\big[v(x)\big].$$

The following proposition establishes quasidifferentiability of α-MEU functions and derives its quasidifferential.[14]

Proposition 1. *The α-MEU function V is quasidifferentiable on $\mathbb{R}^S_{++}$ for every convex and compact $\mathcal{P} \subset \Delta$, every $\alpha \in [0, 1]$, and every continuously differentiable utility index v. The sub and superdifferentials of V at $x \in \mathbb{R}^S_{++}$ are*[15]

$$\underline{\partial} V(x) = (1-\alpha)v'(x)\mathcal{P}_{\max}(x), \tag{10}$$

$$\text{and} \quad \bar{\partial} V(x) = \alpha v'(x)\mathcal{P}_{\min}(x). \tag{11}$$

Proof. To demonstrate quasidifferentiability of the α-MEU function V, it suffices to show (by Proposition B.1.(i)) that the two summands are quasidifferentiable. The second summand, $(1-\alpha)\max_{P\in\mathcal{P}} E_P[v(x)]$, is the maximum over a compact set of continuously differentiable functions. By the results of Section 2, it is quasidifferentiable. Its quasidifferential is the subdifferential $(1-\alpha)v'(x)\mathcal{P}_{\max}(x)$ and zero superdifferential; see equation (6). The first summand, $\alpha\min_{P\in\mathcal{P}} E_P[v(x)]$, is the minimum over a compact set of continuously differentiable functions. By the same argument, it is quasidifferentiable with the superdifferential $\alpha v'(x)\mathcal{P}_{\min}(x)$ and zero subdifferential. This implies (10) and (11). □

It follows that the quasidifferential of the α-MEU function can be written as

$$\mathrm{DV}(x) = v'(x)\big[(1-\alpha)\mathcal{P}_{\max}(x), \alpha\mathcal{P}_{\min}(x)\big].$$

Clearly, if the sets $\mathcal{P}_{\min}(x)$ and $\mathcal{P}_{\max}(x)$ are singletons, or DR-equivalent to singletons, then V is (Gateaux) differentiable at x.

The ambiguity-averse multiple-prior expected utility with $\alpha = 1$ is superdifferentiable with $\bar{\partial} V(x) = v'(x)\mathcal{P}_{\min}(x)$ and $\underline{\partial} V(x) = \{0\}$. The ambiguity-seeking multiple-prior expected utility with $\alpha = 0$ is subdifferentiable with $\underline{\partial} V(x) = v'(x)\mathcal{P}_{\max}(x)$ and $\bar{\partial} V(x) = \{0\}$.

[13]The fact that α-MEU often has a nonunique parametric specification $(\alpha, \mathcal{P})$ has been pointed out in Siniscalchi (2006).

[14]Proposition 1 can be easily extended to α-MEU-like functions that feature different sets of probabilities in the maximum and the minimum terms.

[15]We use the notation $v'(x)\mathcal{P}_{\max}(x)$ for the set $\{z \in \mathbb{R}^S : z_s = v'(x_s)P(s), P \in \mathcal{P}_{\max}(x)\}$.

3.2 *Choquet expected utility*

Nonadditive probabilities provide another way for preferences under uncertainty to accommodate different attitudes toward ambiguity. The mathematical concept to describe nonadditive probabilities is a *capacity*. A capacity is a set function $\mu : \Sigma \to [0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(S) = 1$, and $\mu(A) \leq \mu(B)$ for every $A \subset B$, $A, B \in \Sigma$. The Choquet expected utility (CEU) with utility index $v : \mathbb{R}_+ \to \mathbb{R}$ is defined as the Choquet integral of v under μ, that is,

$$E_\mu\big[v(x)\big] = \sum_{k=1}^{S} v(x_{(k)})\big[\mu\big(\{s : x_s \geq x_{(k)}\}\big) - \mu\big(\{s : x_s \geq x_{(k-1)}\}\big)\big], \tag{12}$$

where $x_{(k)}$ denotes the kth highest consumption level from among all x_s. An axiomatization of CEU has been provided by Schmeidler (1989).

An important feature of the CEU representation is rank-dependence of weights assigned to utilities of consumption in different states; see Wakker (2010, Chapter 10). A decision weight assigned to $v(x_s)$ in (12) depends on the ranking of x_s among all states. Note that those weights add up to one. Different attitudes toward ambiguity can be described in the CEU model by different properties of the capacity. As shown by Schmeidler (1989) and discussed later in this section, a convex capacity reflects ambiguity aversion while a concave one reflects ambiguity seeking. Capacities that are neither convex nor concave reflect mixed ambiguity attitudes. For an additive capacity,[16] CEU is the standard expected utility, and reflects ambiguity neutrality.

A useful concept for establishing quasidifferentiability of a CEU function is the *Möbius inverse* of a capacity μ. This is a set function $\mathtt{m}_\mu : \Sigma \to \mathbb{R}$ such that

$$\mu(A) = \sum_{\{B \in \Sigma : B \subset A\}} \mathtt{m}_\mu(B). \tag{13}$$

The set function $\mathtt{m}_\mu$ satisfies (i) $\mathtt{m}_\mu(\emptyset) = 0$, (ii) $\sum_{B\in\Sigma} \mathtt{m}_\mu(B) = 1$, and (iii) $\mathtt{m}_\mu(\{s\}) \geq 0$ for all $s \in S$. It can be obtained as

$$\mathtt{m}_\mu(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \mu(B) \tag{14}$$

for every $A \in \Sigma$, where $|A|$ denotes the number of states in A. Equations (13) and (14) define a one-to-one mapping between capacities and set functions satisfying conditions (i)–(iii). A capacity with positive Möbius inverse is called *belief function*. Belief functions have been extensively studied in Dempster (1967) and in the theory of evidence of Shafer (1976). A capacity is a belief function if and only if it is totally monotone.[17]

The Choquet integral (12) can be expressed using the Möbius inverse as

$$E_\mu\big[v(x)\big] = \sum_{A \in \Sigma} \mathtt{m}_\mu(A) \min_{s \in A} v(x_s); \tag{15}$$

[16] A capacity μ is additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \in \Sigma$ such that $A \cap B = \emptyset$.

[17] A capacity is totally monotone if $\mu(\bigcup_{i=1}^n A_i) \geq \sum_{I \subset \{1,\dots,n\}} (-1)^{|I|+1} \mu(\bigcap_{i \in I} A_i)$ for every $A_1, \dots, A_n \in \Sigma$ and every n.

see Gilboa and Schmeidler (1994, Section 4). Let Δ^A denote the set of all probability measures on (S, Σ) with the support on A. Equation (15) can be rewritten as

$$E_\mu\big[v(x)\big] = \sum_{A\in\Sigma} \mathtt{m}_\mu(A) \min_{P\in\Delta^A} E_P\big[v(x)\big].$$

As in (9), let $\Delta^A_{\min}(x)$ denote the subset of Δ^A for which the minimum of expected utility of x is attained. That is,

$$\Delta^A_{\min}(x) = \arg\min_{P\in\Delta^A} E_P\big[v(x)\big].$$

Further, let Σ^+_μ (Σ^-_μ) denote the subset of the set of events Σ on which the Möbius inverse of μ is positive (negative, resp.). We have the following.

Proposition 2. *The CEU function $E_\mu[v(x)]$ is quasidifferentiable on $\mathbb{R}^S_{++}$ for every capacity μ and every differentiable*[18] *utility index v. The quasidifferential $[\underline{\partial} E_\mu[v(x)], \bar{\partial} E_\mu[v(x)]]$ at $x \in \mathbb{R}^S_{++}$ is given by*

$$\underline{\partial} E_\mu\big[v(x)\big] = v'(x) \sum_{A\in\Sigma^-_\mu} \mathtt{m}_\mu(A)\Delta^A_{\min}(x), \tag{16}$$

$$\text{and} \quad \bar{\partial} E_\mu\big[v(x)\big] = v'(x) \sum_{A\in\Sigma^+_\mu} \mathtt{m}_\mu(A)\Delta^A_{\min}(x). \tag{17}$$

Proof. The function $E_\mu[v(x)]$ is the finite sum of minimum functions; see equation (15). It follows from Corollary B.1 in Appendix B that the summand $\min_{s\in A} v(x_s)$ in equation (15) is quasidifferentiable with the quasidifferential equal to $v'(x)[0, \Delta^A_{\min}(x)]$. Using the rules of quasidifferential calculus for sums of functions (see Appendix B), we obtain Proposition 2. □

Proposition 2 implies that a CEU function is differentiable at every injective $x \in \mathbb{R}^S_+$, that is, $x_s \neq x_{s'}$ for every $s \neq s'$. Indeed, if x is injective, then $\Delta^A_{\min}(x)$ is a singleton for every $A \in \Sigma$. Further, it implies that if the Möbius inverse of μ is positive, that is, μ is a belief function, then CEU is superdifferentiable. If the Möbius inverse is negative except for singletons (i.e., $\mathtt{m}_\mu(A) \leq 0$ for every A with $|A| \geq 2$), then it is subdifferentiable. Indeed, if Σ^+_μ consists of singletons, then the superdifferential of (17) is a single vector.

There are some capacities for which CEU functions have an α-MEU representation. For these capacities, the quasidifferential of the CEU function can be obtained from Proposition 1. This avoids using the combinatorial Möbius inverse and maximization or minimization over probabilities on every event in Σ featured in Proposition 2.

First, we consider convex and concave capacities. A capacity is *convex* (or supermodular) if

$$\mu(A\cup B) + \mu(A\cap B) \geq \mu(A) + \mu(B) \tag{18}$$

[18] Since the proof relies on the envelope theorem of Appendix B, Corollary B.1, for a finite set of functions instead of the result of Section 2, the assumption of continuous differentiability from Proposition 1 can be weakened to differentiability.

for every $A, B \in \Sigma$. It is *concave* (or submodular) if the reverse inequality holds in (18). Every belief function is convex. Every capacity whose Möbius inverse is negative except for singletons is concave. The Choquet integral with respect to a convex capacity is

$$\begin{aligned} E_\mu\big[v(x)\big] &= \min_{P \in \mathrm{core}(\mu)} E_P\big[v(x)\big], \\ \text{where} \quad \mathrm{core}(\mu) &= \big\{P \in \Delta : P(A) \geq \mu(A), \forall A \in \Sigma\big\} \end{aligned} \tag{19}$$

is the core of μ; see Gilboa and Schmeidler (1994). By Proposition 1, it is superdifferentiable with the superdifferential equal to $v'(x)\mathcal{P}_{\min}(x)$ for $\mathcal{P} = \mathrm{core}(\mu)$.

Similarly, the Choquet integral with respect to a concave capacity is

$$E_\mu\big[v(x)\big] = \max_{P \in \mathrm{core}(\bar{\mu})} E_P\big[v(x)\big], \tag{20}$$

where $\bar{\mu}$ is the *conjugate* capacity defined by $\bar{\mu}(A) = 1 - \mu(A^c)$, where $A^c = S \setminus A$. It is subdifferentiable with subdifferential $v'(x)\bar{\mathcal{P}}_{\max}(x)$ for $\bar{\mathcal{P}} = \mathrm{core}(\bar{\mu})$.

Capacities as convex combinations of convex capacities and their concave conjugates have been introduced by Jaffray and Phillippe (1997). For a capacity μ_α defined by

$$\mu_\alpha = \alpha\mu + (1-\alpha)\bar{\mu} \tag{21}$$

where μ is a convex capacity and $\alpha \in [0, 1]$, the CEU function is

$$E_{\mu_\alpha}\big[v(x)\big] = \alpha \min_{P \in \mathrm{core}(\mu)} E_P\big[v(x)\big] + (1-\alpha) \max_{P \in \mathrm{core}(\mu)} E_P\big[v(x)\big], \tag{22}$$

that is, the α-MEU function with $\mathrm{core}(\mu)$ as the set of priors. Examples of Jaffray and Philippe capacities are the *Hurwicz capacity* of Gul and Pesendorfer (2015) and the *neo-additive capacity* of Chateauneuf, Eichberger, and Grant (2007); see Appendix C.

We conclude this section with an example of a parametric set of capacities for which the quasidifferential of the CEU can be obtained either from Proposition 1 or Proposition 2.

Example 1. Consider a capacity μ on three states given by $\mu(\{s\}) = \eta$, $\mu(\{s, s'\}) = 3\eta$, for $s \neq s'$, where $0 \leq \eta \leq \frac{1}{3}$. One can verify that the capacity μ is convex for every $\eta \leq \frac{1}{5}$. The core of μ is

$$\mathrm{core}(\mu) = \big\{P \in \Delta : P(s) \geq \eta, P(s) + P\big(s'\big) \geq 3\eta, \forall s, s', s \neq s'\big\}.$$

The core is nonempty for every $\eta \leq \frac{2}{9}$. It is a hexagon for $\eta < \frac{1}{5}$, and a triangle for $\frac{1}{5} \leq \eta \leq \frac{2}{9}$. The Möbius inverse of μ is

$$\mathrm{m}_\mu\big(\{s\}\big) = \eta, \qquad \mathrm{m}_\mu\big(\{s, s'\}\big) = \eta, \qquad \mathrm{m}_\mu(S) = 1 - 6\eta, \quad \forall s, s', s \neq s'.$$

It is positive for every $\eta \leq \frac{1}{6}$.

Let us consider the risk-free consumption plan $\bar{x} = (1, 1, 1)$ and the linear utility $v(z) = z$. If $\eta \leq \frac{1}{5}$ and μ is convex, then $E_\mu[x]$ is superdifferentiable and the superdifferential at $\bar{x}$ equals the core of μ, by Proposition 1.

The quasidifferential of $E_\mu[x]$ at $\bar{x}$ can be derived using Proposition 2 for every $\eta \leq \frac{1}{3}$. The set of minimizing probabilities $\Delta^A_{\min}(\bar{x})$ at $\bar{x}$ is equal to Δ^A for every $A \subset S$. If $\eta \leq \frac{1}{6}$, then the set Σ_- of events with negative Möbius inverse is empty and the subdifferential (16) is zero. Let $\Delta_{s,s'}$ be the set of probabilities with support on two states s and s'. The superdifferential (17) equals $\eta\bar{x} + \eta\{\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}\} + (1 - 6\eta)\Delta$. It can be shown that this set is equal to the core of μ. If $\frac{1}{6} < \eta \leq \frac{1}{5}$, then Σ_- consist of the event S and the subdifferential is nonzero and equal to $(1 - 6\eta)\Delta$. The superdifferential is $\eta\bar{x} + \eta\{\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}\}$. It can be shown that this pair of sets is DR-equivalent to the pair $[0, \operatorname{core}(\mu)]$ resulting from Proposition 1.

If $\frac{1}{5} < \eta$, then capacity μ is not convex. The quasidifferential of $E_\mu[x]$ at $\bar{x}$ is, by Proposition 2, the pair of sets $[(1-6\eta)\Delta,\ \eta\bar{x}+\eta\{\Delta_{1,2}+\Delta_{1,3}+\Delta_{2,3}\}]$, a symmetric triangle, and a symmetric hexagon. ◊

3.3 *Rank-dependent expected utility*

The Rank-Dependent Expected Utility (RDEU) model is a special case of the CEU model with the capacity being a distorted probability measure. For a reference (subjective) probability measure π on Σ and a *probability distortion (or weighting)* function $w : [0, 1] \to [0, 1]$, assumed increasing and satisfying $w(0) = 0$ and $w(1) = 1$, the distorted probability μ_w is a capacity defined by

$$\mu_w(A) = w\big(\pi(A)\big), \quad \text{for all } A \in \Sigma.$$

An RDEU is the Choquet integral of a utility index v with respect to μ_w, that is,

$$V_{\mathrm{RD}}(x) = \sum_{i=1}^{S} v(x_{(i)})\big[w\big(\pi\big(\{(1), \dots, (i)\}\big)\big) - w\big(\pi\big(\{(1), \dots, (i-1)\}\big)\big)\big], \tag{23}$$

where $x_{(k)}$ is the kth highest consumption level from among all x_s. Proposition 2 implies that the RDEU is quasidifferentiable for every differentiable utility index v.

The feature of the RDEU model, which distinguishes it from the general CEU model, is probabilistic sophistication, that is, distribution invariance under the reference probability measure π. Properties of the distortion function in RDEU correspond to certain behavioral phenomena just like properties of the capacity in CEU. For example, convexity of a distortion function w, which amounts to w being relatively flat for low values of probability and steep for high values, implies underweighting the best outcomes and overweighting the worst outcomes. It reflects pessimism. The resulting capacity μ_w is convex, and the RDEU function can be expressed as an ambiguity-averse multiple-prior utility (19) with $\mu = \mu_w$. Similarly, concavity of w reflects optimism. The resulting capacity μ_w is concave, and RDEU is an ambiguity-seeking multiple-prior utility (20) with $\bar{\mu} = \bar{\mu}_w$.

Empirical investigations of the RDEU model point to *inverse S-shaped* distortion functions; see Wakker (2010, Chapter 7). An inverse S-shaped function is concave on an interval $[0, B]$ and convex on $[B, 1]$ for some inflection point $B \in [0, 1]$. It reflects

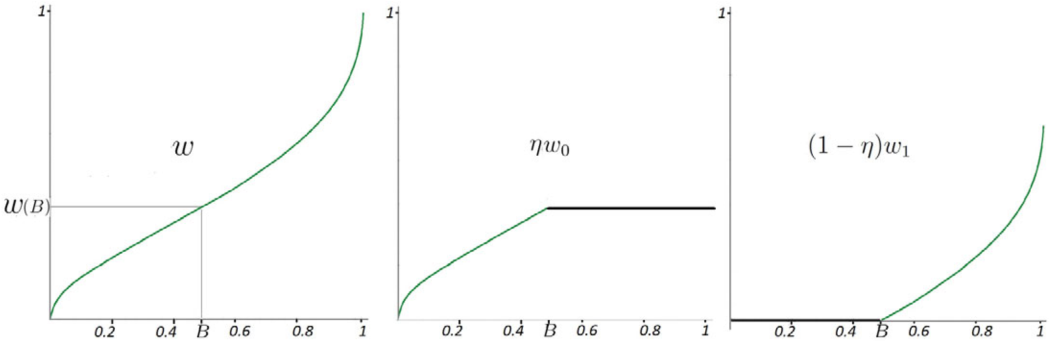

FIGURE 1. Decomposition of the inverse S-shaped distortion function w of (24) with parameter $r = 0.61$ and inflection point $B \approx 0.48$.

overweighting the worst and the best outcomes. It plays an important role in the cumulative prospect theory. An example of an inverse S-shaped distortion is the normalized power function of Tversky and Kahneman (1992). It is given by

$$w(p) = \frac{p^r}{\left(p^r + (1-p)^r\right)^{\frac{1}{r}}}, \tag{24}$$

with parameter $r \in [0, 1]$, and is shown in the left panel of Figure 1.

We show that an arbitrary inverse S-shaped distortion can be written as a convex combination of convex and concave distortions. This leads to a representation of the RDEU function similar to an α-MEU function, and an expression for the quasidifferential of RDEU similar to the one in Proposition 1.

Let w be any inverse S-shaped distortion function with inflection point B where $0 \leq B \leq 1$. Define distortion functions w_0 and w_1 as

$$w_0(p) = \frac{1}{w(B)} \min\{w(p), w(B)\}$$

and

$$w_1(p) = \frac{1}{1-w(B)} \max\{w(p) - w(B), 0\},$$

for every $p \in [0, 1]$. Distortion w_0 is concave while w_1 is convex. It holds

$$w(p) = \eta w_0(p) + (1-\eta) w_1(p), \tag{25}$$

for every $p \in [0, 1]$, where $\eta = w(B)$. Figure 1 illustrates this decomposition.

Equation (25) implies that $\mu_w = \eta \mu_{w_0} + (1-\eta)\mu_{w_1}$. Applying (19) and (20) of Section 3.2, we obtain the following.

Proposition 3. *For every inverse S-shaped distortion w, every $\pi \in \Delta$, and every differentiable*[19] *utility index v, the RDEU function V_{RD} has a representation*

$$V_{\mathrm{RD}}(x) = (1-\eta) \min_{P \in \mathcal{P}^1} E_P\big[v(x)\big] + \eta \max_{P \in \mathcal{P}^0} E_P\big[v(x)\big], \tag{26}$$

where $\mathcal{P}^0 = \mathrm{core}(\bar{\mu}_{w_0})$, $\mathcal{P}^1 = \mathrm{core}(\mu_{w_1})$, and $\eta = w(B)$. Further, V_{RD} is quasidifferentiable at every $x \in \mathbb{R}^S_{++}$, and the quasidifferential $[\underline{\partial} V_{\mathrm{RD}}(x), \bar{\partial} V_{\mathrm{RD}}(x)]$ is given by

$$\underline{\partial} V_{\mathrm{RD}}(x) = \eta v'(x) \mathcal{P}^0_{\max}(x),$$
$$\text{and} \quad \bar{\partial} V_{\mathrm{RD}}(x) = (1-\eta) v'(x) \mathcal{P}^1_{\min}(x).$$

It can be shown that if distortion w in Proposition 3 is symmetric, that is, $w(p) = 1 - w(1-p)$ every $p \in [0, 1]$, then $\mathcal{P}^0 = \mathcal{P}^1$, and representation (26) is a genuine α-MEU with $\eta = 1/2$.

3.4 *Cumulative prospect theory*

The Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992) is a refinement of the RDEU model to accommodate reference dependence of preferences. The CPT differentiates between gains and losses, and permits different risk attitudes over gains and losses. The utility index $v : \mathbb{R} \to \mathbb{R}$ is concave over gains and convex over losses. Further, there are two probability distortion functions—one for gains and one for losses—both inverse S-shaped, which reflects overweighting extreme outcomes.

Gains and losses in the CPT model are defined relative to a *reference point* $\bar{x} \in \mathbb{R}^S_+$. For an arbitrary consumption plan $x \in \mathbb{R}^S_+$, gains are $(x-\bar{x})^+ = (x-\bar{x}) \vee 0$, and losses are $(x-\bar{x})^- = (x-\bar{x}) \wedge 0$. The CPT utility function V_{CP}, with a reference probability measure $\pi \in \Delta$, two distortion functions w^+ for gains and w^- for losses, and a utility index v, is the sum of two RDEUs. That is,

$$V_{\mathrm{CP}}(x) = V^+_{\mathrm{RD}}(x) + V^-_{\mathrm{RD}}(x) = E_{\mu_{w+}}\big[v\big((x-\bar{x})^+\big)\big] + E_{\bar{\mu}_{w-}}\big[v\big((x-\bar{x})^-\big)\big], \tag{27}$$

where the Choquet integrals are from (23). Note that the RDEU function over losses in (27) is taken with respect to the conjugate capacity $\bar{\mu}_{w-}$ of distortion function w^-. Thus, the decision weight assigned in $E_{\bar{\mu}_{w-}}[v((x-\bar{x})^-)]$ to a loss-outcome $x_{(i)}$, with $x_{(i)} - \bar{x} < 0$, is $w^-(\pi(\{(i), \dots, (S)\})) - w^-(\pi(\{(i+1), \dots, (S)\}))$.

Tversky and Kahneman (1992) specification of distortion functions w^+ and w^- takes the form of inverse S-shaped normalized-power functions of (24) with parameters $r^+ = 0.61$ for w^+ and $r^- = 0.69$ for w^-. Note that the conjugate $\bar{w}^-$ is inverse S-shaped, as well. The utility index is the power function

$$v(z) = \begin{cases} z^b & \text{if } z \geq 0 \\ -\theta(-z)^b, & \text{if } z < 0. \end{cases} \quad \text{where } b \in (0, 1] \text{ and } \theta > 1.$$

[19]As in Proposition 2, the proof relies on the envelope theorem of Appendix B, Corollary B.1 because the sets $\mathcal{P}^0$ and $\mathcal{P}^1$ are convex polytopes, and the weaker assumption of differentiability is sufficient.

It is concave over gains and convex over losses. In absence of probability distortions, this would induce risk aversion for gains and risk seeking for losses. The parameter θ reflects *loss aversion*; see Wakker (2010, Chapter 8). Kahneman and Tversky (1979) found that parameters $b = 0.88$ and $\theta = 2.25$ fit the experimental data best.

The power function is "problematic" (see Wakker (2010, p. 267)) because of infinite derivative at zero. This leads to problems in studying loss aversion, but also makes it not suitable for the study of optimal choices. We shall assume that the utility index v is differentiable for every $z \neq 0$ and has a well-defined right- and left-hand derivatives at zero. An example of such a function occasionally used in the CPT is the shifted power function

$$v(z) = \begin{cases} \dfrac{(1+z)^b}{b} - \dfrac{1}{b} & \text{if } z \geq 0 \\ -\theta \dfrac{(1-z)^b}{b} + \dfrac{\theta}{b}, & \text{if } z < 0. \end{cases} \tag{28}$$

with $b \in (0, 1]$ and $\theta > 1$; see Wakker (2010, p. 271). It is concave on gains, convex on losses, and superdifferentiable at zero.

The following proposition, proved in Appendix A, establishes quasidifferentiability of CPT utility function with inverse S-shaped distortion functions.

Proposition 4. *For every inverse S-shaped and differentiable distortions w^+ and w^-, every probability measure π on Σ, and every utility index v that is differentiable on $\mathbb{R} \setminus \{0\}$ and has well-defined right- and left-hand derivatives at 0, the CPT utility function V_{CP} is quasidifferentiable.*[20]

The quasidifferential of the CPT utility function in Proposition 4 can be derived using representation (26) of the RDEU functions $V^+_{\mathrm{RD}}(x)$ and $V^-_{\mathrm{RD}}(x)$ and the rules of quasidifferential calculus of Appendix B. Because of nondifferentiability of gains and losses at zero, expressions for quasidifferentials of $V^+_{\mathrm{RD}}(x)$ and $V^-_{\mathrm{RD}}(x)$ are more complex than those in Proposition 3. We omit exact derivations. For consumption plans that involve strictly positive gains in every state or strictly positive losses in every state, the quasidifferential of the CPT utility function is the quasidifferential of the respective RDEU summand; see Section 3.3.

4. Pareto optimal allocations

We consider the setting with I agents whose preferences over state-contingent consumption plans in $\mathbb{R}^S_+$ are described by strictly increasing utility functions V_i. The aggregate endowment of the economy is $e \in \mathbb{R}^S_+$. Recall that an allocation $\{x_i\}$, where $x_i \in \mathbb{R}^S_+$ for every i, is feasible if $\sum_{i=1}^I x_i \leq e$. A feasible allocation is Pareto optimal if there is no other feasible allocation $\{\tilde{x}_i\}$ such that $V_i(\tilde{x}_i) \geq V_i(x_i)$ with at least one strict inequality.

[20] Proposition 4 can be extended to CPT utility functions with arbitrary distortions. The argument relies on the representation (15) of the Choquet integrals in the definition of RDEU functions $V^+_{\mathrm{RD}}(x)$ and $V^-_{\mathrm{RD}}(x)$. These functions can be represented as weighted sums of minimum functions and, therefore, are quasidifferentiable.

Since the utility functions need not be concave, Pareto optimal allocations cannot be characterized as solutions to the problem of maximizing a weighted sum of individual utilities subject to the feasibility constraint. Instead, we consider the problem of maximizing one agent's utility subject to constraints on other agents' utilities and feasibility. Choosing agent 1 without loss of generality, we have

$$\max_{\{x_i\}\in\mathbb{R}_+^{\mathrm{SI}}} V_1(x_1)$$
$$\text{subject to}\quad V_i(x_i)\geq \bar{v}_i,\quad i=2,\dots,I,\qquad \sum_{i=1}^{I} x_i \leq e, \tag{29}$$

for some bounds $\bar{v}_i\in\mathbb{R}$. Every allocation solving (29) is Pareto optimal. Conversely, every Pareto optimal allocation is a solution to (29) for some bounds $\bar{v}_i$. The following necessary first-order conditions for a Pareto optimal allocation are derived from (29).

Proposition 5. *Suppose that utility functions V_i are quasidifferentiable. If $\{x_i\}$ is an interior Pareto optimal allocation, then for every profile $\{\underline{z}_i\}$ with $\underline{z}_i\in\underline{\partial}V_i(x_i)$ there exist a corresponding profile $\{\bar{z}_i\}$ with $\bar{z}_i\in\bar{\partial}V_i(x_i)$, positive multipliers $\lambda_i\in\mathbb{R}_+$, and a positive vector $q\in\mathbb{R}_+^S$, not all zero, such that*

$$\lambda_i[\bar{z}_i+\underline{z}_i]=q, \tag{30}$$

for every i. Further, the complementary slackness conditions hold.[21]

Proof. The result follows from Proposition 1.1 in Gao (2000a). A statement of it, and details of the derivation can be found in Appendix A. □

A strict form of the first-order condition (30) is sufficient for local Pareto optimality. An allocation is locally Pareto optimal if it cannot be improved upon by a feasible allocation that lies in a small neighborhood of that allocation for every agent. The strict form of (30) requires that for every profile $\{\underline{z}_i\}$ with $\underline{z}_i\in\underline{\partial}V_i(x_i)$ there exist a profile $\{\bar{z}_i\}$ with $\bar{z}_i\in\operatorname{int}\bar{\partial}V_i(x_i)$, and positive multipliers λ_i and a positive vector $q\in\mathbb{R}_+^S$, not all zero, such that (30) holds. The result follows from Proposition 3.1 in Gao (2000b).

If every utility function V_i is differentiable at x_i, then the first-order condition (30) states that $\lambda_i\nabla V_i(x_i)=q$ for the gradient vector $\nabla V_i(x_i)$, for every i, which is the standard condition of common marginal rates of substitution. If every function V_i is concave, so that the quasidifferential has the representation $[0,\bar{\partial}V_i(x)]$ with zero subdifferential, then condition (30) states that there exist a profile $\{\bar{z}_i\}$ with $\bar{z}_i\in\bar{\partial}V_i(x)$ such that $\lambda_i\bar{z}_i=q$. This is the standard necessary and sufficient condition for Pareto optimality of an interior allocation for concave utility functions; see Aubin (1998).

In the reminder of this section, we present statements of Proposition 5 specialized to all agents with α-MEU or all with RDEU, and discussions of applications to CEU and CPT. Settings with mixed utility functions can be easily analyzed using those results.

[21]Those are $\lambda_i(V_i(x_i)-\bar{v}_i)=0$ for $i=2,\dots I$ and $q_s(\sum_{i=1}^{I}x_{i,s}-e_s)=0$ for $s=1,\dots,S$. We omit the slackness conditions from all subsequent refinements of Proposition 5.

4.1 *Optimal allocations with α-MEU utilities*

Suppose that agents have α-MEU functions with agent-specific weights $\alpha_i \in [0, 1]$ and sets of priors $\mathcal{P}_i \subset \Delta$, assumed closed and convex. Utility indexes $v_i : \mathbb{R}_+ \to \mathbb{R}$ are strictly increasing and continuously differentiable. Using Propositions 1 and 5, we obtain the following.

Proposition 6. *If $\{x_i\}$ is an interior Pareto optimal allocation with α-MEU functions, then for every profile of beliefs $\{\underline{P}_i\}$ with $\underline{P}_i \in \mathcal{P}^i_{\max}(x_i)$ there exist a corresponding profile of beliefs $\{\bar{P}_i\}$ with $\bar{P}_i \in \mathcal{P}^i_{\min}(x_i)$, strictly positive multipliers $\lambda_i \in \mathbb{R}_+$, and a positive vector $q \in \mathbb{R}^S_+$, $q \neq 0$, such that for every i*

$$\lambda_i v_i'(x_i)\big[\alpha_i \bar{P}_i + (1-\alpha_i)\underline{P}_i\big] = q. \tag{31}$$

If $\alpha_i = 1$ for every i, so that agents have ambiguity-averse multiple-prior expected utilities, then condition (31) says that

$$\lambda_i v_i'(x_i)\bar{P}_i = q, \tag{32}$$

for every i, for some profile of beliefs $\{\bar{P}_i\}$ with $\bar{P}_i \in \mathcal{P}^i_{\min}(x_i)$. Rigotti, Shannon, and Strzalecki (2008) show that condition (32) is necessary and sufficient for Pareto optimality of an interior allocation with ambiguity-averse multiple-prior expected utilities with concave utility indexes. Proposition 6 shows that it remains necessary without concavity. A strict version of (32) with $\bar{P}_i \in \operatorname{int}\mathcal{P}^i_{\min}(x_i)$ for every i is sufficient for local Pareto optimality with arbitrary ambiguity-averse multiple-prior expected utilities.

Proposition 6 implies that every interior Pareto optimal allocation with α-MEU functions with concave utility indexes is Pareto optimal for expected utility functions with heterogeneous beliefs taken from the agents' sets of priors.

Corollary 1. *Suppose that v_i is concave for every i. If $\{x_i\}$ is an interior Pareto optimal allocation with α-MEU functions, then there exists a profile of beliefs $\{P_i\}$ with $P_i \in \mathcal{P}_i$ such that the allocation $\{x_i\}$ is Pareto optimal with expected utilities $E_{P_i}[v_i(x)]$.*

Proof. For arbitrary beliefs $\bar{P}_i$ and $\underline{P}_i$ satisfying (31), define probability measures $P_i = \alpha_i \bar{P}_i + (1-\alpha_i)\underline{P}_i$. Note that $P_i \in \mathcal{P}_i$. The allocation $\{x_i\}$ satisfies the first-order conditions of Pareto optimality for expected utilities with beliefs P_i. Because of concavity of v_i, those conditions are sufficient, and hence the allocation is Pareto optimal. □

For small sets of priors with a nonempty intersection, the set of Pareto optimal allocations with heterogeneous beliefs taken from those sets is a limited set of allocations. Another corollary to Proposition 6 establishes the necessity of a common prior for the existence of a risk-free Pareto optimal allocation. Of course, there can be a risk-free allocation only if the aggregate endowment is risk-free, that is, there is no aggregate risk.

Corollary 2. *If there exists an interior risk-free Pareto optimal allocation with α-MEU functions, then*

$$\bigcap_{i=1}^{I} \mathcal{P}_i \neq \emptyset. \tag{33}$$

Proof. Let $\{x_i\}$ be an interior risk-free Pareto optimal allocation and let $P_i \in \mathcal{P}_i$ be as defined in the proof of Corollary 1. Since x_i is state-independent, the first-order condition

$$\lambda_i v_i'(x_i) P_i = q$$

implies that $P_i = P$, and hence $P \in \mathcal{P}_i$ for every i. Therefore, (33) holds. □

In the absence of concavity of α-MEU functions, the common prior condition (33) is clearly not sufficient for Pareto optimality of risk-free allocations. However, the strict version of (33)—that is, $\operatorname{int} \bigcap_{i=1}^{I} \mathcal{P}_i \neq \emptyset$—is sufficient for local Pareto optimality of risk-free allocations. Billot et al. (2000) show that condition (33) is necessary and sufficient for all Pareto optimal allocations with concave ambiguity-averse multiple-prior expected utilities to be risk-free if there is no aggregate risk. Rigotti, Shannon, and Strzalecki (2008) extend that result to general convex, ambiguity-averse preferences.[22]

The next corollary shows limitations to the possibility of subdifferentiability of α-MEU functions at a Pareto optimal allocation. Recall from Section 2 that a function is subdifferentiable at x if its quasidifferential has a representation with zero superdifferential. An α-MEU function with $\alpha = 0$ is subdifferentiable everywhere and has a set-valued subdifferential equal to the set of priors at any risk-free consumption plan.

Corollary 3. *Let $\{x_i\}$ be an interior Pareto optimal allocation with α-MEU functions. If there are two or more agents whose utility functions are subdifferentiable at their respective consumption plans x_i, then these functions are differentiable at x_i.*

Corollary 3 implies that, for any interior Pareto optimal allocation $\{x_i\}$, there can be at most one agent i whose α-MEU function is subdifferentiable but not differentiable at x_i. Further, if there are at least two agents with ambiguity-seeking α-MEU functions with $\alpha_i = 0$ and there is no aggregate risk, then no Pareto optimal allocation can be risk-free.

4.2 *Optimal allocations with CEU, RDEU, and CPT utilities*

First-order conditions for Pareto optimal allocations with CEU, RDEU, and CPT utilities can be obtained from Proposition 5 using the formulas for sub- and superdifferentials of Section 3. For CEU functions, these are equations (16) and (17) of Proposition 2. If agents' capacities are Jaffray and Philippe capacities (21), then CEU functions are α-MEU, and the results of Section 4.1 can be applied.

For the important class of RDEU functions with inverse S-shaped distortions, Proposition 3 established their representation as weighted sums of minimum and maximum

[22] See Ghirardato and Sinischalchi (2018) for further extensions.

of expected utilities over two different sets of beliefs. The first-order conditions for an interior Pareto optimal allocation are similar to conditions (31) for α-MEU utilities. They are

$$\lambda_i v_i'(x_i)\big[(1-\eta_i)\bar{P}_i + \eta_i \underline{P}_i\big] = q, \tag{34}$$

for every i, where $\bar{P}_i \in \mathcal{P}^{1,i}_{\min}(x_i)$ and $\underline{P}_i \in \mathcal{P}^{0,i}_{\max}(x_i)$. The sets of beliefs $\mathcal{P}^{1,i}_{\min}(x_i)$ and $\mathcal{P}^{0,i}_{\max}(x_i)$, as well as scalars η_i are from Proposition 3. Corollary 1 of Section 4.1 can be extended to RDEU functions with inverse S-shaped distortions.[23] We have the following.

Corollary 4. *Suppose that v_i is concave for every i. If $\{x_i\}$ is an interior Pareto optimal allocation with RDEU functions with inverse S-shaped distortion functions, then there exists a profile of beliefs $\{P_i\}$ with $P_i \in (1-\eta_i)\mathcal{P}^{1,i} + \eta_i \mathcal{P}^{0,i}$ such that the allocation $\{x_i\}$ is Pareto optimal with expected utilities $E_{P_i}[v_i(x)]$.*

CPT utility functions are sums of rank-dependent expected utilities of gains and losses. Quasidifferentiability of CPT utilities has been established in Proposition 4. For Pareto optimal allocations that involve strictly positive gains in every state for every agent, the first-order conditions are those for gain RDEU functions; see equation (34). The same holds for optimal allocations with losses in every state, if such allocations exist.

We conclude this section with an example of Pareto optimal allocations in an Edgeworth box with CPT utilities.

Example 2. There are two agents and two states. The aggregate endowment is $e = (6, 6)$. Agents have the same CPT utility function with reference belief $\pi = (\frac{1}{2}, \frac{1}{2})$, state-dependent reference point for gains and losses $\bar{x} = (2, 1)$, and shifted power utility index v of the form (28) with parameters $b = \frac{1}{8}$ and $\theta = 2$, where the latter reflects loss aversion. To simplify, we abstract from distortion of probabilities,[24] that is, we take $w^+ = w^- = \mathrm{id}$. The resulting CPT utility function is nonconcave and nondifferentiable.

Figure 2 shows the Edgeworth box under consideration. Indifference curves are plotted for six different utility levels for each agent. They have kinks on the borderlines between gains and losses, and are (locally) concave in the regions of losses in both states. Pareto optimal allocations in the shaded rectangle between $(2, 1)$ and $(4, 5)$, where both agents experience gains in both states, look like typical optimal allocations for differentiable and concave utility functions. One such allocation is the equal-sharing allocation in the center of the box. Further, there are Pareto optimal allocations such as $\{(1.3, 1), (4.7, 5)\}$ near the bottom-left corner and the corresponding one near the top-right corner where one of the agents experiences losses. The utility function of that agent is not differentiable and her indifference curve has a kink with nonconvex upper-contour set. Allocations where one agent's consumption is the reference point are Pareto optimal and points of nondifferentiability as well. The first-order conditions of Pareto

[23] Recall that concave and convex distortions belong to the class of inverse S-shaped distortions.

[24] Nonlinear distortion functions would lead to kinks in indifference curves on the 45-degree line in Figure 2.

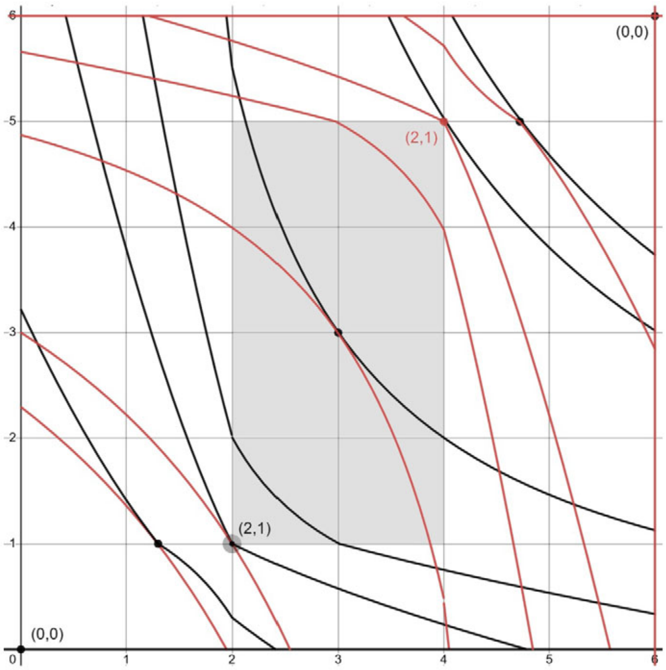

FIGURE 2. Edgeworth box with CPT utility functions.

optimality for these allocations are the conditions of Proposition 5 for quasidifferentiable utility functions.

Note that even though utility functions in this example are nondifferentiable on a small set (of measure zero) of points, those points are of critical importance for optimal allocations. ◊

5. Concluding remarks

We introduced the methodology of quasidifferential calculus to the analysis of optimality conditions for nondifferentiable and nonconcave utility functions arising in contemporary decision theory. Quasidifferential calculus offers transparent statements of first-order optimality conditions in a way that unifies and extends the well-known conditions for differentiable, concave, and convex functions. We argued that it is better suited for α-MEU, CEU, RDEU, and CPT utility functions than the alternative method of the Clarke subdifferential.

We presented first-order conditions for Pareto optimal allocations under uncertainty for these utility functions. The results lead to interesting implications concerning optimal risk sharing with quasidifferentiable utilities. For example, a necessary condition for the existence of risk-free Pareto optimal allocation in an economy with no aggregate risk and arbitrary α-MEU functions is that the sets of priors have a nonempty intersection.

The α-MEU and CEU models are often considered in settings of infinitely many states. Since quasidifferential calculus has been developed in general Banach spaces of functions (see Pallaschke and Rolewicz (1997)), the results of this paper can be extended to infinite state spaces. We leave technical details of such extensions for future research.

Appendix A: Proofs

Proof of Proposition 4. Let us consider first the RDEU function $E_{\mu_{w+}}[v((x-\bar{x})^+)]$ for gains. If w^+ is inverse S-shaped, then the gain-RDEU function has the max-plus-min representation (26) of Proposition 3 with sets $\mathcal{P}^0_+ = \mathrm{core}(\bar{\mu}_{w^+_0})$ and $\mathcal{P}^1_+ = \mathrm{core}(\mu_{w^+_1})$. If function v is strictly increasing, differentiable for $z \neq 0$, and has well-defined right- and left-hand derivatives at 0, then the gain function $v((z-\bar{z})^+)$ for $z \in \mathbb{R}$ is quasidifferentiable because it is the maximum of two quasidifferentiable functions $v(z-\bar{z})$ and 0. It is in fact subdifferentiable; see Section 2.[25] The function $E_P[v((x-\bar{x})^+)]$ is the sum of quasidifferentiable functions, hence it is quasidifferentiable for every $P \in \Delta$. Further, the minimum function $\min_{P \in \mathcal{P}^1_+} E_P[v((x-\bar{x})^+)]$ and the maximum function $\max_{P \in \mathcal{P}^0_+} E_P[v((x-\bar{x})^+)]$ are quasidifferentiable as well. This follows from Proposition B.3 in Appendix B because the sets $\mathcal{P}^0_+$ and $\mathcal{P}^1_+$ are convex polytopes.

The same arguments apply to the loss RDEU function $E_{\bar{\mu}_{w-}}[v((x-\bar{x})^-)]$ with the only difference that the loss $v((z-\bar{z})^-)$ is superdifferentiable as the minimum two quasidifferentiable functions.[26] □

Proof of Proposition 5. Let $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 0, \ldots, m$ be quasidifferentiable. Consider the following constrained maximization problem:

$$\begin{aligned} &\max_x f_0(x) \\ &\text{subject to} \quad f_i(x) \geq 0, \quad i = 1, \ldots, m. \end{aligned} \tag{35}$$

Proposition A.1 (Gao (2000a)). *If x^* is a solution to (35), then for every profile $\{\underline{z}_i\}$ with $\underline{z}_i \in \underline{\partial} f_i(x^*)$, there exist a corresponding profile $\{\bar{z}_i\}$ with $\bar{z}_i \in \bar{\partial} f_i(x^*)$ and positive multipliers $\lambda_i \in \mathbb{R}_+$, $i = 0, \ldots, n$, not all zero, such that*

$$\sum_{i=0}^{m} \lambda_i [\bar{z}_i + \underline{z}_i] = 0,$$

and $\lambda_i f_i(x^*) = 0$ for every $i \geq 1$.

We note that the multiplier λ_i may depend on the selected profile $\{\underline{z}_i\}$ with $\underline{z}_i \in \underline{\partial} f_i(\bar{x})$. Proposition A.1 is an extension of the Fritz John's first-order conditions for differentiable functions in nonlinear programming; see Takayama (1985). Neither Proposition A.1 nor John's result require a constrained qualification condition known from

[25] The subdifferential $\underline{\partial} v((z-\bar{z})^+)$ of the gain function is $v'(z-\bar{z})$ for $z > \bar{z}$, 0 for $z < \bar{z}$, and the interval $[0, v'_+(0)]$ for $z = \bar{z}$, where $v'_+(0)$ is the right-hand derivative at 0.

[26] The superdifferential $\bar{\partial} v((z-\bar{z})^-)$ of the loss function is $v'(z-\bar{z})$ for $z < \bar{z}$, 0 for $z > \bar{z}$, and the interval $[0, v'_-(0)]$ for $z = \bar{z}$, where $v'_-(0)$ is the left-hand derivative at 0.

the Kuhn–Tucker theorems, but they feature a multiplier on the objective function that could be zero. We apply Proposition A.1 to the Pareto problem (29). The function $e_s - \sum_{i=1}^{I} x_{i,s}$ of the feasibility constraint in state s is differentiable. Using $q \in \mathbb{R}^S_+$ for the vector of multipliers of feasibility constraints, we obtain the first-order conditions (30). □

Proof of Corollary 3. If α-MEU function V_i is subdifferentiable, then the pair of sets $[(1-\alpha)\mathcal{P}_{\max}(x_i), \alpha\mathcal{P}_{\min}(x_i)]$ is DR-equivalent to $[A_i, 0]$ for some compact and convex set A_i. It can be easily seen that $A_i \subset \mathcal{P}$.

To prove the first part, let i and j be the two agents whose utility functions are subdifferentiable with respective subdifferentials $v_i'(x_i)A_i$ and $v_j'(x_j)A_j$, and zero superdifferentials. To simplify the exposition, we disregard agents other than i and j in our arguments. In particular, a Pareto optimal allocation for I agents is Pareto optimal for any pair of agents.

Proposition 6 says that for any selection of $a_i \in A_i$ and $a_j \in A_j$, there exist multipliers λ_i and λ_j, and vector q such that

$$v_i'(x_i)a_i = \lambda_i q \quad \text{and} \quad v_j'(x_j)a_j = \lambda_j q. \tag{36}$$

Let us consider arbitrary $a_i' \in A_i$. We shall prove that $a_i' = a_i$. Applying Proposition 6 to the pair $a_i' \in A_i$ and $a_j \in A_j$, there exist λ_i', λ_j', and q' such that

$$v_i'(x_i)a_i' = \lambda_i' q' \quad \text{and} \quad v_j'(x_j)a_j = \lambda_j' q'. \tag{37}$$

Using the equations for agent j in (36) and (37), it follows that vectors q and q' are scale-multiples of each other, that is, $q' = (\lambda_j'/\lambda_j)q$. This implies that a_i and a_i' are scale-multiples of each other. Since they both lie in the probability simplex Δ, they must be equal. Therefore, the subdifferential A_i is a singleton and V_i is differentiable at x_i. The same argument with reversed roles for i and j shows that the set A_j must be singleton. This concludes the proof. □

Appendix B: Rules of quasidifferential calculus

The quasidifferential of a function is a pair of compact and convex sets. We define first some algebraic operations on pairs of sets. Let A, B, C, D be convex and compact sets in $\mathbb{R}^S$. The operations of addition and multiplication by a scalar are defined as follows:

$$[A+C, B+D] = [A, B] + [C, D] \quad \text{and} \quad c[A, B] = \begin{cases} [cA, cB] & \text{if } c \geq 0 \\ [cB, cA], & \text{if } c < 0. \end{cases}$$

The rules of quasidifferentiation are extensions of the well-known rules of the classical differential calculus. A more detailed and systematic account can be found in Demyanov and Rubinov (1992, Chapters 10–12).

Proposition B.1. *Suppose that functions $f_k : \mathbb{R}^S_+ \to \mathbb{R}$ are quasidifferentiable at $x \in \mathbb{R}^S_+$ for every $k = 1, \ldots, m$. Let $Df_k(x) = [\underline{\partial} f_k(x), \bar{\partial} f_k(x)]$ be the quasidifferential of f_k and $a_k \in \mathbb{R}$ for $k = 1, \ldots, m$. The following rules hold:*

*(i) (*Sum*) Let $f = \sum_{k=1}^{m} a_k f_k$. Then f is quasidifferentiable at x and*

$$Df(x) = \sum_{k=1}^{m} a_k Df_k(x).$$

*(ii) (*Product*) Let $f = f_1 \cdot f_2$. Then f is quasidifferentiable at x and*

$$Df(x) = f_1(x)Df_2(x) + f_2(x)Df_1(x).$$

Proof of Proposition B.1. Part (i) follows from Theorem 10.2(i) and (ii) in Demyanov and Rubinov (1986). For part (ii), see Theorem 10.2(iii). □

Theorem 12.2 of Demyanov and Rubinov (1986) provides an exact formula for the quasidifferential of a composition of two quasidifferentiable functions. We reproduce it in Proposition B.2. Note that the chain rule for the Clarke subdifferential calculus yields only upper bounds on the Clarke subdifferential of the composition; see Section 2.1 in Clarke (1983).

Proposition B.2. *Suppose that functions $f_k : \mathbb{R}_+^S \to \mathbb{R}$ are quasidifferentiable at $x \in \mathbb{R}_+^S$ for each $k = 1, \dots, m$. If function $g : \mathbb{R}^m \to \mathbb{R}$ is uniformly quasidifferentiable*[27] *at $y = (f_1(x), \dots, f_m(x))$, then the composition $V : \mathbb{R}_+^S \to \mathbb{R}$ defined by*

$$V(x) = g\big(f_1(x), \dots f_m(x)\big)$$

is quasidifferentiable at x, and

$$\underline{\partial} V(x) = \left\{ \sum_{k=1}^{m} (\underline{z}_k + \overline{z}_k)\underline{w}_k - \underline{z}_k \underline{\gamma}_k - \overline{z}_k \overline{\gamma}_k : \underline{w} \in \underline{\partial} g(y), \underline{z}_k \in \underline{\partial} f_k(x), \overline{z}_k \in \bar{\partial} f_k(x) \right\},$$

$$\bar{\partial} V(x) = \left\{ \sum_{k=1}^{m} (\underline{z}_k + \overline{z}_k)\overline{w}_k + \underline{z}_k \underline{\gamma}_k + \overline{z}_k \overline{\gamma}_k : \overline{w} \in \bar{\partial} g(y), \underline{z}_k \in \underline{\partial} f_k(x), \overline{z}_k \in \bar{\partial} f_k(x) \right\}.$$

where $\underline{\gamma}, \overline{\gamma} \in \mathbb{R}^m$ are arbitrary vectors such that $\underline{\gamma} \le \underline{\partial} g(y) \cup (-\bar{\partial} g(y)) \le \overline{\gamma}$.

The next result is taken from Demyanov and Rubinov (1992), Theorem 2.2.

Proposition B.3. *Suppose that functions $f_k : \mathbb{R}_+^S \to \mathbb{R}$ are quasidifferentiable at $x \in \mathbb{R}_+^S$ for every $k = 1, \dots, m$. Let*

$$\varphi(x) = \max_{k=1,\dots,m} f_k(x), \quad \text{and} \quad \psi(x) = \min_{k=1,\dots,m} f_k(x)$$

$$\text{and} \quad \varphi^*(x) = \arg\max_k f_k(x), \quad \text{and} \quad \psi^*(x) = \arg\min_k f_k(x).$$

[27] That is, g is uniformly directionally differentiable and quasidifferentiable at y. This holds, for instance, if g is Lipschitz continuous around y; see Proposition 3.4, page 29, in Demyanov and Rubinov (1986).

Then φ and ψ are quasidifferentiable at x and

$$(i) \qquad D\varphi(x) = \left[\mathrm{co}\left\{\bigcup_{k\in\varphi^*(x)}\left(\underline{\partial} f_k(x) - \sum_{i\in\varphi^*(x)\setminus k}\bar{\partial} f_i(x)\right)\right\}, \sum_{k\in\varphi^*(x)}\bar{\partial} f_k(x)\right] \tag{38}$$

$$(ii) \qquad D\psi(x) = \left[\sum_{k\in\psi^*(x)}\underline{\partial} f_k(x), \mathrm{co}\left\{\bigcup_{k\in\psi^*(x)}\left(\bar{\partial} f_k(x) - \sum_{i\in\psi^*(x)\setminus k}\underline{\partial} f_i(x)\right)\right\}\right]. \tag{39}$$

Corollary B.1. *If every function f_k is differentiable, then*

$$D\varphi(x) = \left[\mathrm{co}\left\{\nabla f_k(x) : k \in \varphi^*(x)\right\}, \{0\}\right] \tag{40}$$

$$D\psi(x) = \left[\{0\}, \mathrm{co}\left\{\nabla f_k(x) : k \in \psi^*(x)\right\}\right]. \tag{41}$$

Proof. If f_k is differentiable for every k, then we can set $\underline{\partial} f_k(x) = \nabla f_k(x)$ and $\bar{\partial} f_k(x) = 0$ in equation (38) of Proposition B.3, and this results in (40). If we set $\bar{\partial} f_k(x) = \nabla f_k(x)$ and $\underline{\partial} f_k(x) = 0$ in equation (39), we obtain (41). □

We proved in Section 2 that results similar to (40) and (41) hold for an arbitrary family of continuously differentiable functions; see (6).

Appendix C: Jaffray and Philippe capacities

A special case of a Jaffray and Philippe capacity (21) is the *Hurwicz capacity*; see Gul and Pesendorfer (2015). It obtains when the convex capacity μ is taken as *inner* capacity μ_π associated with a probability measure π on an algebra $\mathcal{F} \subset \Sigma$ of subsets of states (generated by a partition of S). This capacity is defined by

$$\mu_\pi(A) = \max_{B\subset A, B\in\mathcal{F}} \pi(B), \qquad A \in \Sigma.$$

By Proposition 2.4 in Denneberg (1994), μ_π is a convex capacity. The core of μ_π is the set of all probability measures on Σ that coincide with π on $\mathcal{F}$.[28] That is,

$$\mathrm{core}(\mu_\pi) = \left\{P \in \Delta : P(A) = \pi(A), \forall A \in \mathcal{F}\right\}. \tag{42}$$

The resulting CEU function is an α-MEU with the set of priors (42), and is the *Hurwicz expected utility*.

Another special case of a Jaffray and Philippe capacity is the *neo-additive capacity*. It obtains when μ in (21) is taken as $\delta\pi + (1-\delta)\mu^N$, where $\pi \in \Delta$ is an arbitrary probability measure, μ^N is the null capacity, and $\delta \in [0, 1]$; see Chateauneuf, Eichberger, and Grant (2007) and Eichberger, Grant, Kelsey, and Koshevoy (2011). The *null capacity* is defined by $\mu^N(A) = 0$ for every $A \in \Sigma$, $A \neq S$, and $\mu^N(S) = 1$. The core of capacity $\delta\pi + (1-\delta)\mu^N$

[28] The proof of (42) is as follows: A probability measure P is in the core of the capacity μ_π if and only if $P(A) \geq \pi(B)$ for every $A \in \Sigma$ and every $B \subset A$, $B \in \mathcal{F}$. Since algebra $\mathcal{F}$ is generated by a partition of S, the latter holds if and only if P is an extension of π to Σ. Thus, (42) holds.

is the set $\delta\pi + (1-\delta)\Delta$. The CEU in (22) for a neo-additive capacity μ_α^{neo} can be written as

$$E_{\mu_\alpha^{\text{neo}}}\big[v(x)\big] = \delta E_\pi\big[v(x)\big] + (1-\delta)\Big[\alpha \min_{P\in\Delta} E_P\big[v(x)\big] + (1-\alpha)\max_{P\in\Delta} E_P\big[v(x)\big]\Big].$$

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