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All sequential allotment rules are obviously strategy-proof

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For division problems with single-peaked preferences, we show that all sequential allotment rules, a large subfamily of strategy-proof and efficient rules, are also obviously strategy-proof. Although obvious strategy-proofness is, in general, more restrictive than strategy-proofness, this is not the case in this setting.

Keywords. Obvious strategy-proofness, sequential allotment rules, division problems, single-peaked preferences.

JEL classification. D71.

1. Introduction

We consider the class of division problems where k indivisible units of a good have to be allotted among a set of agents. Each agent has single-peaked preferences over the set of its possible assignments $\{0, \dots, k\}$: There is a preferred or top assignment: up the top, more is preferred to less; beyond the top, the opposite holds. Monetary transfers are not possible.

Different real-world problems can be framed within this model. These include situations where a set of agents must share a good, or a task like the surplus of a joint venture, the cost of a public good, the division of a job, or a rationed good traded at a fixed price. For example, agents could be investors with different risk preferences and wealth, and the units of the good could be shares in a risky project. Agents' risk attitudes and wealth

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induce single-peaked preferences over their assigned shares. Agents could also be workers who have collectively agreed to complete a project requiring a given number of hours paid at a fixed wage. Agents' quasi-concave preferences over money and hours of leisure induce single-peaked preferences over their assigned number of working hours. Finally, the good might be a plot of land that needs to be fully divided among hobby gardeners, each of whom wishes to cultivate some land, but not necessarily all of it.[1]

A solution to division problems is a rule that, for a fixed positive integer k, chooses an allotment for each profile of single-peaked preferences over $\{0, \dots, k\}$. But preferences are agents' private information and they have to be elicited. A rule is strategy-proof if, for each agent, truth-telling is always optimal, regardless of the preferences declared by the other agents. A rule is efficient if the chosen allotment is Pareto optimal at each profile of single-peaked preferences. A rule is replacement monotonic if it satisfies a weak solidarity principle requiring that if an agent obtains a different assignment by changing the reported preference, then all other agents' assignments should change in the opposite direction.

Barberà, Jackson, and Neme (1997) consider the class of continuous division problems where agents might begin with natural claims to minimal or maximal assignments, or might be treated with different priorities, due, for example, to their seniorities, and these initial entitlements should be attended as far as possible. They characterize the class of strategy-proof, efficient and replacement monotonic rules on the domain of single-peaked preferences as the family of sequential allotment rules.

Sequential allotment rules are complex and agents may have difficulties identifying that a strategy is dominant, particularly if they have limited contingent reasoning capabilities. In this paper we ask, "How might efficient allotments be implemented and, at the same time, promote solidarity among agents who may have problems with contingent reasoning?" Li (2017) proposes the stronger incentive notion of obvious strategy-proofness for general settings where agents' types (that coincide with single-peaked preferences in the division problem) are private information. Given a rule, this notion requires that there exist an extensive game form and a type-strategy profile (a behavioral strategy for each agent and for each of its types) that induce the rule. Namely, for every profile of types, when each agent plays the strategy that corresponds to its type, the outcome of the game is the outcome the rule would have chosen at this profile. Moreover, for each agent and for each of its types, the strategy that corresponds to its type is obviously dominant. This means that whenever the agent has to take a decision in the game in extensive form, it evaluates the consequence of the strategy that corresponds to its type in a pessimistic way (thinking that the worst possible outcome will follow) and the consequence of deviating in an optimistic way (thinking that the best possible outcome will follow); moreover, the pessimistic outcome associated with the strategy that corresponds to its type is at least as good as the optimistic outcome associated with deviating. Hence, whenever an agent plays, the decision prescribed by the strategy that corresponds to its type appears as unmistakably optimal, i.e., obviously dominant. The

[1]The continuous version of this model, when the good is perfectly divisible and k is a strictly positive real number, was first studied by Sprumont (1991).

difficulty of establishing whether a rule is obviously strategy-proof lies in the fact that its implementation in obviously dominant strategies must be through an extensive game form. But now the extensive game form is not given by a general revelation principle as it is for strategy-proofness in the form of the direct revelation mechanism. The main difficulty lies then in identifying, for each rule, the extensive game form that implements the rule in obviously dominant strategies.

The result of this paper is the following: Any efficient and replacement monotonic rule that can be implemented in dominant strategies can, moreover, be done so in obviously dominant strategies. That is, in the implementation, we can accommodate agents who have troubles with contingent reasoning because obvious strategy-proofness is no more restrictive than strategy-proofness. Namely, we show that all sequential allotment rules (a quite large class of rules) are obviously strategy-proof. Moreover, our proof is constructive: For each sequential allotment rule, we explicitly show how to construct, by means of the monotonous and individualized algorithm (MIA), the extensive game form that implements the rule in obviously dominant strategies. In addition, our extensive game forms provide full implementation of all sequential allotment rules in obviously dominant strategies (the games in extensive form have the property that all equilibria in obviously dominant strategies induce the same allotment).

In light of the extreme behavioral criterion on which the notion of obviously dominance is founded, it is not surprising that the literature has already identified settings for which just a few, if any, and very special strategy-proof rules satisfy the stronger requirement. Li (2017) already shows that the rule associated to the top-trading cycles algorithm in the house allocation problem of Shapley and Scarf (1974) is not obviously strategy-proof, and Troyan (2019) identifies a domain of acyclic preferences that is necessary and sufficient for that rule to be obviously strategy-proof. Ashlagi and Gonczarowski (2018) show that the rule associated to the deferred acceptance algorithm is not obviously strategy-proof for the agents belonging to the offering side, but it is on the domain of acyclic preferences defined by Ergin (2002). Li (2017) also contains a positive result in which monotone price mechanisms (generalizations of ascending auctions) are characterized as those implementing all obviously strategy-proof rules on the domain of quasi-linear preferences for a general binary allocation problem that encompasses, for example, private-valued auctions with unit demand or binary public goods. Specifically, he shows that the rule induced by the mechanism that selects the efficient allocation and the Vickrey–Clarke–Groves payment is obviously strategy-proof.

Our paper contributes to the possibility strand of this literature by showing that, despite the fact that in many settings, obvious strategy-proofness becomes significantly more restrictive than just strategy-proofness, for division problems with single-peaked preferences, each sequential allotment rule (i.e., each strategy-proof, efficient, and replacement monotonic rule) is indeed obviously strategy-proof, and, as we have already said, we show it by exhibiting an extensive game form that implements each sequential allotment rule in obviously dominant strategies.

The paper is organized as follows. Section 2 contains the preliminaries. Section 3 presents the notion of obvious strategy-proofness adapted to our setting. Section 4 contains Theorem 1, stating that all sequential allotment rules are obviously strategy-proof, and the description of the algorithm that, for each sequential allotment rule, defines an extensive game form that implements the rule in obviously dominant strategies. Section 5 contains the proof of Theorem 1. Section 6 contains four final remarks. The Appendix collects omitted proofs.

2. Preliminaries

Agents are the elements of a finite set $N = \{1, \dots, n\}$, where $n \geq 2$. They have to share k indivisible units of a good, where $k \geq 1$ is a positive integer.[2] An *allotment* is a vector $x = (x_1, \dots, x_n) \in \{0, \dots, k\}^N$ such that $\sum_{i=1}^{n} x_i = k$. We refer to $x_i \in \{0, \dots, k\}$ as agent i's *assignment*. Let X be the set of allotments. Each agent $i \in N$ has a (weak) *preference* R_i over $\{0, \dots, k\}$, the set of i's possible assignments. Let P_i be the strict preference associated with R_i. The preference R_i is *single-peaked* if (i) it has a unique most preferred assignment $\tau(R_i)$, the *top* of R_i, such that for all $x_i \in \{0, \dots, k\} \setminus \{\tau(R_i)\}$, $\tau(R_i)\ P_i\ x_i$, and (ii) for any pair $x_i, y_i \in \{0, \dots, k\}$, $y_i < x_i < \tau(R_i)$ or $\tau(R_i) < x_i < y_i$ implies $x_i\ R_i\ y_i$. We assume that agents have single-peaked preferences. Often, only $\tau(R_i)$ about R_i will be relevant and if R_i is understood, we will refer to its top as τ_i. We denote by $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{k}$ the vectors $(0, \dots, 0)$, $(1, \dots, 1)$, $(k, \dots, k) \in \{0, \dots, k\}^N$ and, given $S \subset N$, denote by $\mathbf{0}_S$, $\mathbf{1}_S$, and $\mathbf{k}_S$ the corresponding subprofiles of assignments where all agents in S receive 0, 1, or k, respectively. Given $x = (x_1, \dots, x_n)$, we denote $(x_i)_{i \in S}$ and $(x_i - 1)_{i \in S}$ by x_S and $(x - \mathbf{1})_S$, respectively.

Let $\mathcal{R}$ be the set of all single-peaked preferences. *Profiles*, denoted by $R = (R_1, \dots, R_n) \in \mathcal{R}^N$, are n-tuples of single-peaked preferences. To stress the role of agent i or agents in S, we will represent a profile R by (R_i, R_{-i}) or by (R_S, R_{-S}), respectively.

A (discrete) *division problem* is a pair (k, N), where k is the number of units of the good that have to be allotted among the agents in N with single-peaked preferences over $\{0, \dots, k\}$.[3]

A solution of the division problem (k, N) is a *rule* $\Phi : \mathcal{R}^N \to X$ that selects, for each profile $R \in \mathcal{R}^N$, an allotment $\Phi(R) \in X$. We now present several desirable properties of rules.

A rule $\Phi : \mathcal{R}^N \to X$ is *efficient* if, for each $R \in \mathcal{R}^N$, there is no $y \in X$ such that $y_i\ R_i\ \Phi_i(R)$ for all $i \in N$ and $y_j\ P_j\ \Phi_j(R)$ for at least one $j \in N$.

A rule $\Phi : \mathcal{R}^N \to X$ satisfies *same-sidedness* if, for all $R \in \mathcal{R}^N$,

$$\sum_{j \in N} \tau(R_j) \geq k \quad \text{implies} \quad \Phi_i(R) \leq \tau(R_i) \quad \text{for all } i \in N \tag{1}$$

$$\sum_{j \in N} \tau(R_j) \leq k \quad \text{implies} \quad \Phi_i(R) \geq \tau(R_i) \quad \text{for all } i \in N. \tag{2}$$

[2] For simplicity, and to circumvent the technical difficulties that arise in games in extensive form where agents play in a continuous way (see, for instance, Alós-Ferrer and Ritzberger (2013)), we consider the discrete division problem, first studied by Herrero and Martínez (2011).

[3] Division problems have been studied intensively; see, for instance, Thomson (1994a, 1994b, 1997), Barberà's (2011) survey on strategy-proofness, and, more recently, Moulin (2017), Wakayama (2017), Juarez and You (2019), Bochet and Tumennassan (2020), Bochet, Sakai, and Thomson (2021), or Thomson's (2021) survey.

Namely, all agents are rationed in the same side of the top, i.e., below the tops when there is scarcity and above the tops when there is excess. It is easy to check that under single-peakedness, efficiency is equivalent to same-sidedness.

A rule $\Phi : \mathcal{R}^N \to X$ is *strategy-proof* if for all $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i}) \; R_i \; \Phi_i\big(R'_i, R_{-i}\big).$$

Rules require agents to report single-peaked preferences. A rule is strategy-proof if it is always in the best interest of agents to truthfully report their preferences (i.e., truth-telling is a weakly dominant strategy in the game in normal form obtained from the rule at each profile). If $\Phi_i(R'_i, R_{-i}) \; P_i \; \Phi_i(R_i, R_{-i})$, we say that i manipulates $\Phi : \mathcal{R}^N \to X$ at $R \in \mathcal{R}^N$ via $R'_i \in \mathcal{R}$. Clearly, $\Phi : \mathcal{R}^N \to X$ is strategy-proof if no agent can manipulate it.

A rule $\Phi : \mathcal{R}^N \to X$ is *replacement monotonic* if for all $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$,

$$\Phi_i(R_i, R_{-i}) \leq \Phi_i\big(R'_i, R_{-i}\big) \quad \text{implies} \quad \Phi_j(R_i, R_{-i}) \geq \Phi_j\big(R'_i, R_{-i}\big) \quad \text{for all } j \neq i.$$

Replacement monotonicity is a weak solidarity property (see Thomson (1997)). It requires that if an agent obtains a different assignment by changing the reported preference, then all other agents' assignments should change in the opposite direction.[4]

A rule $\Phi : \mathcal{R}^N \to X$ is *individually rational* with respect to an allotment $q \in X$ if for all $R \in \mathcal{R}^N$ and $i \in N$,

$$\Phi_i(R) \; R_i \; q_i.$$

Individual rationality with respect to an allotment $q \in X$ guarantees that each agent i receives an assignment that is weakly preferred to q_i.

A rule $\Phi : \mathcal{R}^N \to X$ is *tops-only* if for all $R, R' \in \mathcal{R}^N$ such that $\tau(R_i) = \tau(R'_i)$ for all $i \in N$,

$$\Phi(R) = \Phi\big(R'\big).$$

Tops-onlyness is a basic requirement of simplicity. In the division problem, it follows from efficiency and strategy-proofness. Abusing notation, a tops-only rule $\Phi : \mathcal{R}^N \to X$ can be written as $\Phi : \{0, \ldots, k\}^N \to X$, and so we will often interchange $\Phi(\tau_1, \ldots, \tau_n)$ and $\Phi(R_1, \ldots, R_n)$.

Barberà, Jackson, and Neme (1997) consider the class of continuous division problems where agents might begin with natural claims to minimal or maximal assignments, or might be treated with different priorities, due, for example, to their seniorities, and these initial entitlements should be attended as far as possible. They characterize the class of strategy-proof, efficient, and replacement monotonic rules on the domain of single-peaked preferences as the family of sequential allotment rules.

[4]As Barberà, Jackson, and Neme (1997) argue, the normative justification for this property relies on efficiency and single-peakedness. The condition has a clear solidarity-based normative content and it is equivalent to a weakening of the welfare version called one-sided welfare domination under preference replacement Thomson (1997). It is a form of non-bossiness: An agent, without affecting its assignment, cannot transfer units among the other agents. A rule $\Phi : \mathcal{R}^N \to X$ is *non-bossy* if for all $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, $\Phi_i(R_i, R_{-i}) = \Phi_i(R'_i, R_{-i})$ implies $\Phi(R_i, R_{-i}) = \Phi(R'_i, R_{-i})$. Replacement monotonicity implies non-bossiness.

A sequential allotment rule may be understood as a process (or calculation procedure) that starts from two reference allotments: The scarcity-guaranteed allotment $\overline{q}$, to be used whenever the sum of agents' tops is larger than k, and the excess-guaranteed allotment $\underline{q}$, to be used whenever the sum of agents' tops is smaller than k. Fix a preference profile. If the corresponding guaranteed allotment is not efficient, the rule corrects it to select an efficient one. An important feature of any of these rules is that if an agent's declared top is at the same time larger than its assignment at the excess-guaranteed allotment and smaller than its assignment at the scarcity-guaranteed allotment, then the agent receives its top; we refer to such agent as an agent with inordinate power. Rules within this class differ on the two guaranteed allotments and on how the efficient correction takes place (the correction has to be monotonic for the rule to satisfy replacement monotonicity). Barberà, Jackson, and Neme (1997) also show that an individually rational sequential allotment rule with respect to an allotment has the property that the two guaranteed allotments are equal to this allotment and, hence, agents whose top coincides with their assignments at this allotment receive their top.

Sequential allotment rules allot the k units sequentially, using temporarily guaranteed assignments for the agents that evolve throughout the process and that are compared to agents' tops. To start with the definition of a sequential allotment rule Φ, let $\underline{q}$ and $\overline{q}$, respectively, be its initial excess- and scarcity-guaranteed allotments.[5] Namely, select $\underline{q}, \overline{q} \in X$, and define $\Phi(\mathbf{0}) = \underline{q}$ and $\Phi(\mathbf{k}) = \overline{q}$. Let $\tau = (\tau_1, \dots, \tau_n) \in \{0, \dots, k\}^N$ be an arbitrary vector of tops and define $\Phi(\tau)$ as follows.

Suppose $\sum_{i=1}^n \tau_i = k$. Then, since τ is the unique efficient allotment at τ, $\Phi(\tau) = \tau$.

Suppose $\sum_{i=1}^n \tau_i > k$ (the case $\sum_{i=1}^n \tau_i < k$ is symmetric, using $\underline{q}$ instead of $\overline{q}$). If $\tau_j \geq \overline{q}_j$ for all j, then $\Phi(\tau) = \overline{q}$. Otherwise, each j with $\tau_j \leq \overline{q}_j$ receives τ_j and leaves the process with τ_j units (and so τ_j becomes the definite assignment of j), while the other agents remain. The temporarily guaranteed assignments of the remaining agents are weakly increased by distributing among them the remaining units, those that have not become definite yet.[6] Agents with a top smaller than or equal to the new temporarily guaranteed assignment receive the top and leave the process, while the others remain. The process proceeds this way until all units have been already allotted, with the remaining agents receiving their last temporarily guaranteed assignments.

At the end of the process, each agent j receives either τ_j or j's final temporarily guaranteed assignment, which has been moving toward τ_j throughout the process. Hence, by single-peakedness, at all profiles with scarcity, each agent is at least as well off as at the scarcity-guaranteed assignment, and the analogous statement holds for the excess-guaranteed assignment. Note that by definition, $\Phi(\mathbf{0}) = \underline{q}$ and $\Phi(\mathbf{k}) = \overline{q}$. Moreover,

$$\text{if } \underline{q}_j \leq \tau_j \leq \overline{q}_j \quad \text{then } \Phi_j(\tau_j, \tau_{-j}) = \tau_j \quad \text{for all } \tau_{-j}; \tag{3}$$

this means that if $\underline{q}_j \leq \overline{q}_j$, agent j can guarantee the assignment $x_j \in \{\underline{q}_j, \dots, \overline{q}_j\}$ by declaring it as its top, and this property will play an important role in the sequel. If

[5]For a formal definition of a sequential allotment rule, see Barberà, Jackson, and Neme (1997). Our results will be based on the properties characterizing the class, without explicitly using this definition.

[6]The unique condition imposed on how temporarily guaranteed assignments evolve is that they have to be weakly increasing; otherwise, the rule would not satisfy replacement monotonicity.

$q := \underline{q} = \overline{q}$, then for every τ and j, $\Phi_j(\tau)$ lies between τ_j and q_j, and by single-peakedness, Φ is individually rational with respect to q. The process ends with an efficient allotment because, under scarcity, all agents receive less than their tops and, under excess, all receive more. Replacement monotonicity requires that the temporarily guaranteed assignments evolve monotonically. Since the sequential procedure depends on the profile of tops, strategy-proofness imposes some restrictions on the process; for instance, if the temporarily guaranteed assignment of an agent is smaller than its top, then it should remain the same with an even larger announced top. The differences in temporarily guaranteed assignments allow the rule to treat agents differently according to asymmetries that one wishes to respect.

For further reference, we state Barberà, Jackson, and Neme's (1997) characterization for the continuous division problems where $k \in \mathbb{R}_{++}$.

Proposition 1 (Barberà, Jackson, and Neme (1997)). *Let (k, N) be a division problem. A rule $\Phi : \mathcal{R}^N \to X$ is strategy-proof, efficient, and replacement monotonic if and only if Φ is a sequential allotment rule. Moreover, a rule $\Phi : \mathcal{R}^N \to X$ is strategy-proof, efficient, replacement monotonic, and individually rational with respect to q if and only if Φ is a sequential allotment rule such that $\Phi(\mathbf{0}) = \Phi(\mathbf{k}) = q$.*

The proof of their characterization can be adapted to discrete division problems. In discrete division problems, it also holds that if Φ is strategy-proof and efficient, then no agent can affect its own assignment by changing to a new preference with the same top. If, in addition, Φ is replacement monotonic, then none of the assignments is affected. Hence, Φ is tops-only and then the proof of the characterization for discrete division problems proceeds as in the continuous case.

3. Obviously strategy-proof implementation

We briefly describe the notion of obvious strategy-proofness adapted to our setting. Li (2017) proposes this notion with the aim of reducing the contingent reasoning that agents have to carry out to identify that, given a rule, a strategy is weakly dominant. A rule Φ is obviously strategy-proof if there exists an extensive game form Γ and a type-strategy profile $(s_i^{R_i})_{R_i \in \mathcal{R}, i \in N}$ for Γ with the following two properties.[7] First, for every preference profile $R = (R_1, \dots, R_n) \in \mathcal{R}^N$, if each agent i plays Γ according to the corresponding strategy $s_i^{R_i}$, the outcome of the game is $\Phi(R)$, the allotment selected by the rule Φ at R; that is, the pair $(\Gamma, (s_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Φ. Second, whenever agent i with preference R_i has to play in Γ, i evaluates the consequence of choosing the action prescribed by i's strategy $s_i^{R_i}$ according to the worst possible outcome among all outcomes that may occur as an effect of later actions made by agents throughout the rest of the game. In contrast, i evaluates the consequence of choosing an action different from that prescribed by i's strategy $s_i^{R_i}$ according to the best possible outcome among all outcomes that may occur again as an effect of later actions throughout the rest of the game.

[7]The behavioral strategy $s_i^{R_i}$ selects at each node where i has to play one of i's available actions at that node.

Then $s_i^{R_i}$ is obviously dominant in the game in extensive form (Γ, R) if, whenever i has to play, its pessimistic outcome is at least as good as the optimistic outcome associated to any deviation. If the pair $(\Gamma, (s_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Φ and, for each profile $R \in \mathcal{R}^N$ and each agent $i \in N$, the strategy $s_i^{R_i}$ is obviously dominant in (Γ, R), then Φ is obviously strategy-proof.

For our context, two important simplifications related to obvious strategy-proofness have been identified in the literature that follows Li (2017). First, without loss of generality, we can assume that the extensive game form that induces the rule has perfect information.[8] Second, the new notion of obvious strategy-proofness can be fully captured by the classical notion of strategy-proofness applied to games in extensive form with perfect information. This last observation follows from the fact that the worst possible outcome obtained when agent i chooses the action prescribed by i's type-strategy and the best possible outcome obtained when agent i chooses an action different from that prescribed by i's type-strategy are both obtained with only one behavioral strategy profile of the other agents, because the perfect information implies that all information sets are singleton sets (and each one belongs either to the subgame that follows the type-strategy choice or else to the subgame that follows the alternative choice).[9] Then, for easy presentation and following this literature, we will say that a rule is obviously strategy-proof if it is implemented by an extensive game form with perfect information for which each corresponding type-strategy is a weakly dominant strategy (even when the opponent strategy profiles considered include those that are not consistent with any type-strategy profile).

Our approach is based on the MIA that defines an extensive game form for each sequential allotment rule. Namely, given a sequential allotment rule, the MIA gives precise instructions on how to identify at each step (associated to a nonterminal node of the tree) the agent who plays and the set of its available actions, and when to stop (associated to terminal nodes of the tree). We omit here the formal and well known definition of an extensive game form with perfect information.

Fix a division problem given by the integer k and the set of agents N. Let $\mathcal{G}$ be the class of all (finite) extensive game forms with perfect information, whose set of players is N and the results attached to its terminal nodes are allotments in X. Fix an extensive game form $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral and pure) *strategy* of i in Γ is a function σ_i that selects at each node where i has to play one of i's available actions at that node. Let Σ_i be the set of i's strategies in Γ. A strategy profile $\sigma = (\sigma_1, \dots, \sigma_n) \in$

[8] Ashlagi and Gonczarowski (2018), Bade and Gonczarowski (2017), Mackenzie (2020), and Pycia and Troyan (2022) contain results identifying general features of extensive game forms that could be used to implement rules in obviously dominant strategies in different environments. We will follow Ashlagi and Gonczarowski (2018) and Mackenzie (2020) to restrict ourselves to extensive game forms with perfect information. See Mackenzie (2020) for a detailed description and discussion of the differences, similarities, and nuances between the proposals of those four papers. For other partially positive or revelation-principle-like results, see also Arribillaga, Massó, and Neme (2020), Bade and Gonczarowski (2017), Pycia and Troyan (2022), and Troyan (2019); note that although the first two papers also consider single-peaked preferences, they do so in the context of a public good (i.e., voting), while here the context is of private goods.

[9] Mackenzie (2020) proves this for a class of extensive game forms with perfect information, called round table mechanisms, but the proof can be adapted to any extensive game form with perfect information.

$\Sigma_1 \times \cdots \times \Sigma_n = \Sigma$ is an ordered list of strategies, one for each agent. Given $i \in N$, $\sigma \in \Sigma$ and $\sigma_i' \in \Sigma_i$ we often write (σ_i', σ_{-i}) to denote the strategy profile where σ_i is replaced in σ by σ_i'. Let $g : \Sigma \to X$ be the outcome function of Γ. Hence, $g(\sigma)$ is the allotment attached to the terminal node that results when agents play Γ according to $\sigma \in \Sigma$; in particular, $\sum_{i=1}^n g_i(\sigma) = k$ for all $\sigma \in \Sigma$.

Fix an extensive game form $\Gamma \in \mathcal{G}$ and a preference profile $R \in \mathcal{R}^N$. Let (Γ, R) denote the game in extensive form where each agent $i \in N$ compares pairs of strategy profiles in Σ by comparing their outcomes according to R_i. A strategy σ_i is *weakly dominant* in (Γ, R) if, for all σ_{-i} and all σ_i',

$$g_i(\sigma_i, \sigma_{-i})\; R_i\; g_i\big(\sigma_i', \sigma_{-i}\big).$$

A *type-strategy for agent i in* Γ, $(s_i^{R_i})_{R_i \in \mathcal{R}}$, specifies for every preference (or type) R_i of agent i a strategy $s_i^{R_i} \in \Sigma_i$ of i in Γ. We refer to $s_i^{R_i}$ as the strategy associated to R_i. A *type-strategy profile* $(s_i^{R_i})_{R_i \in \mathcal{R}, i \in N}$ for Γ is a type-strategy for every agent $i \in N$ in Γ. Given a type-strategy profile $(s_i^{R_i})_{R_i \in \mathcal{R}, i \in N}$ for Γ and a profile $R \in \mathcal{R}^N$, we denote by $s^R \in \Sigma$ the strategy profile $(s_i^{R_i})_{i \in N}$ specified by the type-strategy profile at R. We are now ready to define obvious strategy-proofness in the context of division problems.

Definition 1. Let (k, N) be given. A rule $\Phi : \mathcal{R}^N \to X$ is *obviously strategy-proof* (OSP) if there are an extensive game form $\Gamma \in \mathcal{G}$ and a type-strategy profile $(s_i^{R_i})_{R_i \in \mathcal{R}, i \in N}$ for Γ such that, for all $R \in \mathcal{R}^N$,

(i) $g(s^R) = \Phi(R)$,

(ii) for all $i \in N$, $s_i^{R_i}$ is weakly dominant in (Γ, R).[10]

When (i) holds, we say that $(\Gamma, (s_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ *induces* Φ. When (i) and (ii) hold, we say that Γ OSP-*implements* Φ.

4. The result and the algorithm

4.1 *The result*

The main result of this paper states that all sequential allotment rules are obviously strategy-proof. Namely, in the implementation of any sequential allotment rule, we can accommodate agents who may have troubles with contingent reasoning because we can find an extensive game form and a type-strategy profile that induce the rule and agents' type-strategies to appear as being undoubtedly optimal.

Theorem 1. *All sequential allotment rules are obviously strategy-proof.*[11]

[10]Recall that, by Mackenzie (2020), requiring weak dominance is equivalent to requiring obvious dominance.

[11]Namely, all sequential allotment rules are OSP-implementable. In a remark at the end of the paper, we comment that this OSP implementation is full.

4.2 *The MIA: A general description*

The proof of our result is constructive. For each sequential allotment rule Φ, we construct the extensive game form $\Gamma^{\Phi} \in \mathcal{G}$ that OSP-implements Φ. This construction is done by means of the monotonous and individualized algorithm (MIA) that depends on Φ. Each step of the MIA corresponds to a node of Γ^{Φ}; its first step corresponds to the initial node of Γ^{Φ} and the MIA generates all possible plays, that is, the entire extensive game form Γ^{Φ}.

At each step of the MIA, an agent i and an interval of integer assignments $I_i = \{\alpha_i, \dots, \beta_i\}$ are selected, where $0 \leq \alpha_i \leq \beta_i \leq k$. These selections depend on Φ. The interval $\{\alpha_i, \dots, \beta_i\}$ will often be denoted by $[\![\alpha_i, \beta_i]\!]$ and it may be a singleton set because α_i could be equal to β_i. To deal with the possibility that α_i or β_i are, respectively, equal to 0 or to k, or both, we define $\beta_i^{+} = \min\{\beta_i + 1, k\}$ and $\alpha_i^{-} = \max\{a_i - 1, 0\}$. Namely, $\beta_i^{+} = k$ if $\beta_i = k$ and $\beta_i^{+} = \beta_i + 1$ if $\beta_i < k$; $\alpha_i^{-} = 0$ if $\alpha_i = 0$ and $\alpha_i^{-} = \alpha_i - 1$ if $\alpha_i > 0$. We will consider three (expanded) intervals of $I_i = [\![\alpha_i, \beta_i]\!]$, denoted by $I_i^{+} = [\![\alpha_i, \beta_i^{+}]\!]$, $I_i^{-} = [\![\alpha_i^{-}, \beta_i]\!]$, and $I_i^{\pm} = [\![\alpha_i^{-}, \beta_i^{+}]\!]$. Each assignment in $[\![\alpha_i, \beta_i]\!]$ is offered to i as a guaranteed assignment, together with the possibility of asking for more (identified with β_i^{+}) or asking for less (identified with α_i^{-}), or both. Depending on i's choice, which becomes i's *tentative assignment*, i will be classified according to i's wish. If i chooses one of the guaranteed assignments in I_i, then i enters into the set of agents who are satisfied and want to "stop," denoted by N_s, i will not play any more, and this chosen guaranteed assignment will become i's definitive assignment. If i chooses β_i^{+} (i.e., i asks (and waits) for more), then i enters into—or remains in—the set of agents who want to go "up," denoted by N_u, i's tentative assignment is β_i (the largest guaranteed assignment that i could have chosen), and i may have the opportunity to play again but without the possibility of asking for less. If i chooses α_i^{-} (i.e., i asks (and waits) for less), then i enters into—or remains in—the set of agents who want to go "down," denoted by N_d, i's tentative assignment is α_i (the smallest guaranteed assignment that i could have chosen), and i may have the opportunity to play again but without the possibility of asking for more.[12] We will denote by $N_p = N_s \cup N_u \cup N_d$ the set of agents who have already played and denote by $N_w = N \setminus N_p$ the set of agents who have not played yet (and they are waiting to do it for the first time).

The MIA has three stages. It starts at Stage A by eliciting information from agents who have inordinate power to eventually identify an allotment q of tentative assignments. Stage B completes this classification by asking for information from agents who did not play at Stage A. At the end of Stage B, if one of the two sets of unsatisfied agents is empty, the MIA stops and the vector of tentative assignments becomes the final allotment. Otherwise, the MIA moves to Stage C where Pareto improvements are carried out by transferring one unit from one of the agents who asks for less to one of the agents who asks for more. The MIA ends when no such Pareto improvement is available and then the tentative allotment becomes definitive. Since each step in the MIA can be identi-

[12]We will show in Lemma 1 that the intervals of guaranteed assignments evolve monotonically throughout the MIA (increasing for agents who ask for more and decreasing for agents who ask for less) in a manner that ensures that all tentative assignments are feasible.

fied with a nonterminal node in the tree, the MIA defines an extensive game form whose players are the set of agents and the results attached to terminal nodes are allotments. We shall show that such an extensive game form together with an obviously dominant type-strategy profile induces the rule.

Before formally defining the MIA, two comments are in order. First, at some steps, there may be more than one agent who can be selected to play; our results are invariant with respect to the agent who is eventually selected. Second, our results will show (particularly, Lemma 1) that all features of the MIA are indeed well defined.

4.3 *The MIA: A formal definition*

Let Φ be a sequential allotment rule.

BEGIN: Set $t = 1$ and go to Stage A.

Stage A. [*Call agents with multiple guarantees*]

Step A.t ($t \geq 1$). *Input*: $N_s = N_u = N_d = \emptyset$ if $t = 1$ or N_s, N_u, N_d, and $q = (q_i)_{i \in N_s \cup N_u \cup N_d}$ output of Step A.$(t-1)$, if $t > 1$.

Set $N_p = N_s \cup N_u \cup N_d$, $N_w = N \backslash N_p$, and define

$$\underline{q} := \Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}), \qquad \overline{q} := \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) \quad \text{and} \quad S := \{i \in N \mid \underline{q}_i < \overline{q}_i\}.$$

(i) If $S = \emptyset$, let N_s, N_u, N_d, and $q := (q_{N_p}, \underline{q}_{N_w})$ be the output of Stage A and go to Stage B.

(ii) If $S \neq \emptyset$, choose any $j \in S$:

(ii.1) If $j \in N_w$, define $I_j = [[\underline{q}_j, \overline{q}_j]]$. Agent j has to choose an action a_j from the set

$$A_j = I_j^{\pm}.$$

(ii.2) If $j \in N_u$, define $I_j = [[\underline{q}_j + 1, \overline{q}_j]]$. Agent j has to choose an action a_j from the set

$$A_j = I_j^{+}.$$

(ii.3) If $j \in N_d$, define $I_j = [[\underline{q}_j, \overline{q}_j - 1]]$. Agent j has to choose an action a_j from the set

$$A_j = I_j^{-}.$$

Set

$$N_u := \begin{cases} N_u \cup \{j\} & \text{if } a_j = \overline{q}_j^{+} \neq \overline{q}_j \\ N_u \setminus \{j\} & \text{if } a_j \in I_j, \end{cases} \qquad N_d := \begin{cases} N_d \cup \{j\} & \text{if } a_j = \underline{q}_j^{-} \neq \underline{q}_j \\ N_d \setminus \{j\} & \text{if } a_j \in I_j, \end{cases}$$

$$N_s := \begin{cases} N_s \cup \{j\} & \text{if } a_j \in I_j \\ N_s & \text{otherwise,} \end{cases} \qquad q_j := \begin{cases} a_j & \text{if } a_j \in I_j \\ \underline{q}_j & \text{if } a_j = \underline{q}_j^{-} \\ \overline{q}_j & \text{if } a_j = \overline{q}_j^{+}. \end{cases}$$

Let N_s, N_u, N_d, and $q=(q_i)_{i\in N_p}$ be the output of Step A.t and go to Step A.$(t+1)$.

Stage B. [*Ask agents who have not yet played to commit to stop, up, or down*]

Step B.t $(t\geq 1)$. *Input*: N_s, N_u, N_d, and $q=(q_i)_{i\in N}$, output of Stage A if $t=1$ or Step B.$(t-1)$ if $t>1$.

(i) If $N_w=\emptyset$,

(i.1) and if $N_u\neq\emptyset$ and $N_d\neq\emptyset$, let N_s, N_u, N_d, and $q=(q_i)_{i\in N}$ be the output of Stage B and go to Stage C;

(i.2) and if $N_u=\emptyset$ or $N_d=\emptyset$, stop, and the outcome of the MIA is the allotment q.

(ii) If $N_w\neq\emptyset$, choose $j\in N_w$. Define $I_j=\{q_j\}$. Agent j has to choose an action a_j from the set

$$A_j=I_j^{\pm}.$$

Set

$$N_u:=\begin{cases}N_u\cup\{j\} & \text{if } a_j=q_j^+\neq q_j\\ N_u & \text{otherwise,}\end{cases}\qquad N_d:=\begin{cases}N_d\cup\{j\} & \text{if } a_j=q_j^-\neq q_j\\ N_d & \text{otherwise,}\end{cases}$$

$$N_s:=\begin{cases}N_s\cup\{j\} & \text{if } a_j=q_j\\ N_s & \text{otherwise.}\end{cases}$$

Let N_s, N_u, N_d, and $q:=(q_i)_{i\in N}$ be the output of Step B.t, and go to Step B.$(t+1)$.

Stage C. [*Identify Pareto improvements after gathering commitments*]

Step C.t $(t\geq 1)$. *Input*: N_s, N_u, N_d, and q, output of Stage B if $t=1$ or Stage C.$(t-1)$ if $t>1$.

Choose agents $j\in N_u$ and $r\in N_d$ among those for whom

$$\Phi_j\big(\mathbf{k}_{N_u},(q-\mathbf{1})_{N_d},q_{N_s}\big)\geq q_j+1\quad\text{and}\quad\Phi_r(q_j+1,\mathbf{0}_{N_d},q_{N_s\cup(N_u\setminus\{j\})})\leq q_r-1.$$

Step C.t.a. Define $I_j=\{q_j+1\}$. Agent j has to choose an action a_j from the set

$$A_j=I_j^{+}.$$

Step C.t.b. Define $I_r=\{q_r-1\}$. Agent r has to choose an action a_r from the set

$$A_r=I_r^{-}.$$

Set

$$N_u:=\begin{cases}N_u\setminus\{j\} & \text{if } a_j=q_j+1\\ N_u & \text{if } a_j\neq q_j+1,\end{cases}\qquad N_d:=\begin{cases}N_d\setminus\{r\} & \text{if } a_r=q_r-1\\ N_d & \text{if } a_r\neq q_r-1,\end{cases}$$

$$N_s := \begin{cases} N_s \cup \{j\} & \text{if } a_j = q_j + 1 \text{ and } a_r \neq q_r - 1 \\ N_s \cup \{r\} & \text{if } a_j \neq q_j + 1 \text{ and } a_r = q_r - 1 \\ N_s \cup \{j, r\} & \text{if } a_j = q_j + 1 \text{ and } a_r = q_r - 1 \\ N_s & \text{if } a_j \neq q_j + 1 \text{ and } a_r \neq q_r - 1, \end{cases} \qquad q_j := q_j + 1 \quad \text{and} \quad q_r := q_r - 1.$$

Let N_s, N_u, N_d, and $q = (q_i)_{i \in N}$ be the output of Step C.t.

(i) If $N_u \neq \emptyset$ and $N_d \neq \emptyset$, go to Step C.$(t+1)$.

(ii) If $N_u = \emptyset$ or $N_d = \emptyset$, stop, and let N_s, N_u, N_d and $q = (q_i)_{i \in N}$ be the output of Stage C. The outcome of the MIA is the allotment q.

End.

Denote by Γ^Φ the extensive game form defined by the MIA, where each of its steps corresponds to a nonterminal node of Γ^Φ and every terminal node of Γ^Φ has an associated allotment q, the outcome of the MIA. Fix a behavioral strategy $\sigma \in \Sigma$ in Γ^Φ. The play of Γ^Φ when agents behave according to σ will be named the σ-path of the MIA. We will refer to the partition N_s, N_u, N_d and allotment $q = (q_i)_{i \in N}$ as the output of the MIA when it stops at either Stage B or Stage C, and refer to q as the outcome of the MIA.[13] Observe that $q = g(\sigma)$, where $g : \Sigma \to X$, is the outcome function of Γ^Φ.

In contrast to the implementation in dominant strategies of sequential allotment rules through the direct revelation mechanism, our implementation in obviously dominant strategies requires that agents reveal their types (i.e., top assignments) sequentially and only partially, without necessarily providing sufficient information to determine whether the profile of tops exhibits excess or scarcity.[14] Our implementation identifies a sequence of agents with inordinate power (one by one), tentatively guaranteeing their wishes until an allotment is identified where no agent has inordinate power, and from there it performs efficiency improvements, pair by pair and unit by unit. The MIA does all this sequentially and making sure that although the process is monotonous toward the (unknown) tops of the agents, they are not surpassed.

Given a sequential allotment rule, one can view the MIA as a calculation procedure to obtain the allotment selected by the rule at each preference profile. However, this procedure differs very much from the calculation procedure used in Barberà, Jackson, and Neme (1997) to define sequential allotment rules. While our procedure only needs partial information about agents' tops to identify the ordering under which agents play and the sequence of intervals of guaranteed assignments from which they have to choose, Barberà, Jackson, and Neme's (1997) procedure requires that agents' tops are known from the very beginning to determine whether the profile of tops exhibits scarcity or excess.

Before continuing, we highlight four fundamental features of Γ^Φ.

[13] Since σ will always be clear from the context, we omit its reference to denote the outcome q.

[14] Mackenzie (2020) identifies general conditions under which this is the case and defines round table mechanisms as the class of extensive game forms that allow agents to sequentially reveal their types in an obviously dominant way.

First, the game Γ^{Φ} can be seen as a round table mechanism (see Mackenzie (2020)) after making the following observations.[15] Whenever agent i has to choose an action from $A_i \supseteq I_i = [\![\alpha_i, \beta_i]\!]$, each choice can be identified with a subset of $\mathcal{R}$: action $a_i \in I_i$ with $\{R_i \in \mathcal{R} \mid \tau(R_i) = a_i\}$, action β_i^+ with $\{R_i \in \mathcal{R} \mid \tau(R_i) > \beta_i\}$, and action α_i^- with $\{R_i \in \mathcal{R} \mid \tau(R_i) < \alpha_i\}$. Hence, if $i \in N_w$ is playing for the first time, either at some Step A.t or Step B.t, A_i can be seen as a partition of $\mathcal{R}$. If $i \in N_p$ has already played before and is playing either at some Step A.t or Step C.t, A_i can be seen as a partition of the subset of preferences induced by i's last previous choice. The next three features of Γ^{Φ} follow from Lemma 1 and play an important role in the proof of Theorem 1.

Second, the evolution of the subsets N_s, N_u, and N_d throughout the MIA is as follows. Once agent i enters the subset N_s at some step, i remains in N_s at all further steps and i is not selected to play again. Once agent i enters the subset N_u at some step, i can only move to N_s or remain in N_u at further steps. Similarly, once agent i enters the subset N_d at some step, i can only move to N_s or remain in N_d at further steps.

Third, the sets of guaranteed assignments $I_i = [\![\alpha_i, \beta_i]\!]$ that are offered to agent i evolve monotonically as follows. If i chooses $\beta_i^+ \in A_i$ at some step, then at the next step (if any) at which i has to play, $i \in N_u'$ and $I_i' = [\![\beta_i + 1, \beta_i']\!]$, where $\beta_i + 1 \leq \beta_i'$. Similarly, if i chooses $\alpha_i^- \in A_i$ at some step, then at the next step (if any) at which i has to play, $i \in N_d'$ and $I_i' = [\![\alpha_i', \alpha_i - 1]\!]$, where $\alpha_i' \leq \alpha_i - 1$.

Fourth, along the steps in Stage A, the intervals of assignments $I_i = [\![\underline{q}_i, \overline{q}_i]\!]$ offered to i are in fact guaranteed for the following reason. If i plays at some Step A.t, it is because $\underline{q}_i < \overline{q}_i$ and this means, by condition (3), that if $t = 1$ or, by an equivalent condition that follows from Lemma 1, if $t > 1$, that agent i's final assignment is a_i if $a_i \in [\![\underline{q}_i, \overline{q}_i]\!]$.

In Section 5.1, we formally establish that the MIA is well defined.

4.4 *Truth-telling strategies*

Let Φ be a sequential allotment rule and let Γ^{Φ} be the extensive game form defined by the MIA.

For $i \in N$, the *truth-telling type-strategy* for agent i in Γ^{Φ} is i's type-strategy $(\sigma_i^{R_i})_{R_i \in \mathcal{R}}$, where, for every preference $R_i \in \mathcal{R}$, the *truth-telling strategy* $\sigma_i^{R_i}$ is defined as follows: whenever agent i is selected to play, i chooses the best action in A_i according to R_i. Denote this choice by $\max_{R_i} A_i$. By single-peakedness, i selects $\tau(R_i)$ if $\tau(R_i) \in A_i$, $\max A_i$ if $\tau(R_i) > \max A_i$, and $\min A_i$ if $\tau(R_i) < \min A_i$.

Then the truth-telling type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N}$ for Γ^{Φ} specifies the truth-telling type-strategy for every $i \in N$. Section 5.2 contains the statement and proof that, for all $R \in \mathcal{R}^N$ and all $i \in N$, $\sigma_i^{R_i}$ is a weakly (i.e., obviously) dominant strategy in the game in extensive form (Γ^{Φ}, R).[16] Given a profile $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$ with the vector of tops $\tau = (\tau_1, \ldots, \tau_n) \in \{0, \ldots, k\}^N$, we often denote the profile σ^R by σ^{τ}.

[15]A round table mechanism is an extensive game form where the sets of actions are nonempty subsets of preferences satisfying the following properties: (a) the actions at any node are disjoint subsets of preferences, (b) when a player has to play for the first time, the set of actions is a partition of $\mathcal{R}$, and (c) later, at a node ν, the union of actions is the intersection of the actions taken by the agent assigned to ν at all predecessor nodes that lead to ν.

[16]The extensive game form Γ^{Φ} is a menu mechanism (see Mackenzie and Zhou (2022)) because agents select from a menu of possible assignments (identified with a corresponding set of actions) and truth-telling

The reason why truth-telling strategies are obviously dominant in (Γ^Φ, R) is roughly as follows. Fix a nonterminal node in the tree (or a step in the MIA) and consider agent i with a single-peaked preference that plays at this node, at which an interval $I_i = [\![\alpha_i, \beta_i]\!]$ of guaranteed assignments is offered to i. If i's top is one of those guaranteed assignments, choosing it is optimal, since the worst that might happen to i is to be assigned to its top. If i's top is strictly above β_i, the worst that might happen to i if i asks for more (if available as β_i^+ and i truth-tells by choosing it) is to receive its tentative assignment β_i. This is because i might still be able to choose from intervals with larger guaranteed assignments, up to i's top, along the monotonic path of later intervals that move toward i's top. In contrast, if i does not ask for more (i.e., does not truth-tell), the best that might happen to i is to receive either the guaranteed assignment β_i or strictly less, all weakly worse than the assignment obtained by truth-telling. Symmetrically, if i's top is strictly below α_i. The key feature of the MIA is that, given i's top and the offered interval $I_i = [\![\alpha_i, \beta_i]\!]$ of guaranteed assignments, i can either choose its top or push forward the interval of guaranteed assignments toward its top, without surpassing it, by asking for more (if the top is strictly above β_i) or asking for less (if the top is strictly below α_i). Then single-peakedness guarantees that truth-telling is obviously dominant.

4.5 *An example*

In Example 1 below, we describe, given a sequential allotment rule Φ and a profile of tops τ with excess, the σ^τ path of the MIA obtained from Φ when agents play the extensive game form Γ^Φ according to the truth-telling strategy σ^τ.[17] Note that any play of the extensive game form Γ^Φ can be obtained as a σ-path of the algorithm by letting agents play Γ^Φ according to the strategy σ.

Example 1. Let $N = \{1, 2, 3, 4\}$, let $k = 7$, and let Φ be the sequential allotment rule partially described in Table 1.[18]

Begin: Set $t = 1$ and go to Stage A.

Stage A.

Step A.1. *Input*: $N_s = N_u = N_d = \emptyset$. Set $N_p = \emptyset$ and $N_w = \{1, 2, 3, 4\}$, and define

$$\underline{q} = \Phi(0, 0, 0, 0) = (4, 0, 2, 1), \qquad \overline{q} = \Phi(7, 7, 7, 7) = (0, 1, 1, 5), \quad \text{and}$$

requires choosing the most preferred one. This requires no information about any previous actions during the game; the agent only needs to know the actions that are currently available and its own preference relation. Therefore, these truth-telling strategies remain available if the perfect information is removed by thickening information sets in any way; for example, if each agent knows only its previous sequence of menus and choices from those menus, together with its current menu, but knows nothing about the choices made by others. After removing the perfect information, truth-telling is still obviously dominant. This is important because it means that the rule could be implemented with agents playing remotely, say through a website.

[17] That is, at each step where i has to choose an action in A_i, i chooses the closest assignment to i's top (i.e., the most preferred one).

[18] Observe that Table 1, which will be used in what follows, is consistent with the existence of a rule satisfying strategy-proofness, efficiency, and replacement monotonicity, and with the description of a sequential allotment rule made in Section 2.

TABLE 1. Example 1.

$\Phi(0,0,0,0) = (4,0,2,1)$	$\Phi(7,7,7,7) = (0,1,1,5)$
$\Phi(0,1,0,0) = (3,1,2,1)$	$\Phi(7,1,7,7) = (0,1,1,5)$
$\Phi(0,1,3,0) = (2,1,3,1)$	$\Phi(7,1,7,1) = (3,1,2,1)$
	$\Phi(2,1,7,0) = (2,1,4,0)$

$$S = \{i \in N \mid \underline{q}_i < \overline{q}_i\} = \{2, 4\}.$$

Choose $j = 2$ and, since $2 \in N_w$, set $I_2 = [[0, 1]]$ and $A_2 = \{0, 1, 2\}$. Assume $\tau_2 = 1$ and so 2 chooses $a_2 = 1$. *Output*: $N_s = \{2\}$, $N_d = N_u = \emptyset$, and $q_2 = 1$. Go to Step A.2.

Step A.2. *Input*: Output of Step A.1. Set $N_p = \{2\}$ and $N_w = \{1, 3, 4\}$, and define

$$\underline{q} = \Phi(0, 1, 0, 0) = (3, 1, 2, 1), \qquad \overline{q} = \Phi(7, 1, 7, 7) = (0, 1, 1, 5), \quad \text{and}$$
$$S = \{i \in N \mid \underline{q}_i < \overline{q}_i\} = \{4\}.$$

Choose $j = 4$ and, since $4 \in N_w$, set $I_4 = [[1, 5]]$ and $A_4 = \{0, 1, 2, 3, 4, 5, 6\}$. Assume $\tau_4 = 0$ and so 4 chooses $a_4 = 0$. *Output*: $N_s = \{2\}$, $N_u = \emptyset$, $N_d = \{4\}$, $q_2 = 1$, and $q_4 = 1$. Go to Step A.3.

Step A.3. *Input*: Output of Step A.2. Set $N_p = \{2, 4\}$ and $N_w = \{1, 4\}$, and define

$$\underline{q} = \Phi(0, 1, 0, 0) = (3, 1, 2, 1), \qquad \overline{q} = \Phi(7, 1, 7, 1) = (3, 1, 2, 1) \quad \text{and}$$
$$S = \{i \in N \mid \underline{q}_i < \overline{q}_i\} = \emptyset.$$

Output: $N_s = \{2\}$, $N_u = \emptyset$, $N_d = \{4\}$, and $q = (3, 1, 2, 1)$. Go to Stage B.

Stage B.

Step B.1. *Input*: Output of Stage A. Since $N_w \neq \emptyset$, choose $j = 1 \in N_w$ and, since $I_1 = [[3]]$, set $A_1 = \{2, 3, 4\}$. Assume $\tau_1 = 1$ and so 1 chooses $a_1 = 2$. *Output*: $N_s = \{2\}$, $N_u = \emptyset$, $N_d = \{1, 4\}$, and $q = (3, 1, 2, 1)$. Go to Step B.2.

Step B.2. *Input*: Output of Step B.1. Since $N_w \neq \emptyset$, choose $j = 3 \in N_w$ and, since $I_3 = [[2]]$, set $A_3 = \{1, 2, 3\}$. Assume $\tau_3 = 3$ and so 3 chooses $a_3 = 3$. *Output*: $N_s = \{2\}$, $N_u = \{3\}$, $N_d = \{1, 4\}$, and $q = (3, 1, 2, 1)$. Go to Step B.3.

Step B.3. *Input*: Output of Step B.2. Since $N_w = \emptyset$, $N_u \neq \emptyset$, and $N_d \neq \emptyset$, let $N_s = \{2\}$, $N_u = \{3\}$, $N_d = \{1, 4\}$, and $q = (3, 1, 2, 1)$ be the output of Stage B. Go to Stage C.

Stage C.

Step C.1. *Input*: Output of Stage B. Set $\Phi(2, 1, 7, 0) = (2, 1, 4, 0)$ and choose $j = 3 \in \{i \in N_u \mid \Phi_i(2, 1, 7, 0) \geq q_i + 1\}$. Set $\Phi(0, 1, 3, 0) = (2, 1, 3, 1)$ and choose $r = 1 \in \{i \in N_d \mid \Phi_i(0, 1, 3, 0) \leq q_i - 1\}$. Define $I_3 = [[3]]$ and $I_1 = [[2]]$. Agent 3 chooses $a_3 = 3$ in $A_3 = \{3, 4\}$ and agent 1 chooses $a_1 = 1$ in $A_1 = \{1, 2\}$ because, as we have already assumed, $\tau_3 = 3$ and $\tau_1 = 1$. Let $N_s = \{2, 3\}$, $N_u = \emptyset$, $N_d = \{1, 4\}$, and $q = (2, 1, 3, 1)$. Since $N_u = \emptyset$, stop. The allotment $q = (2, 1, 3, 1)$ is the outcome of the extensive game form Γ^Φ defined by

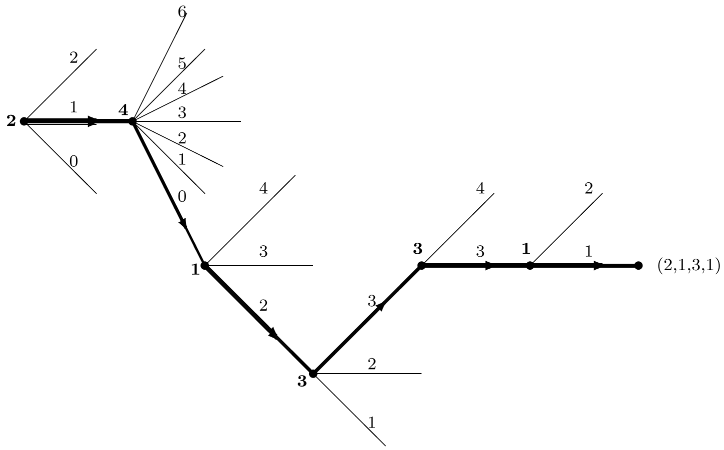

FIGURE 1. The path of Γ^{Φ} when agents play it according to σ^{τ}, where agents are denoted by bold numbers.

the MIA obtained from Φ when agents play it according to the truth-telling strategy profile σ^{τ}, where $\tau = (1, 1, 3, 0)$.

Figure 1 represents the σ^{τ} path of the MIA when agents play Γ^{Φ} according to σ^{τ}. $\Diamond$

5. Proof of Theorem 1

Let Φ be a sequential allotment rule. Proposition 2 in Section 5.2 establishes that, for each profile $R \in \mathcal{R}^N$ and each agent $i \in N$, the truth-telling strategy $\sigma_i^{R_i}$ is weakly dominant in the game in extensive form (Γ^{Φ}, R). Proposition 3 in Section 5.3 establishes that the pair $(\Gamma^{\Phi}, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Φ. Before moving to their statements and proofs, we argue in Section 5.1 that the MIA is well defined and that it delivers an allotment.

5.1 *The MIA is well defined and delivers an allotment*

We establish three facts. First, the MIA finishes after a finite number of steps. Second, agents j and r, selected to play at any step in Stage C, are well defined.[19] Third, the outcome of the MIA is an allotment.

We start with a preliminary observation about the bounds of agents' tentative assignments. In particular, we establish in Remark 1 below that, at any step of the MIA, (i) if $i \in N_u$, then $q_i < k$, and (ii) if $i \in N_d$, then $0 < q_i$, where q_i is i's tentative assignment (part of the output of the step). Properties (i) and (ii) are important to guarantee that the MIA finishes in a finite number of steps, and that agents j and r, selected to play at any step in Stage C, are well defined.

REMARK 1. Let $i \in N$ be the agent who is selected to play at any Step X.t of the MIA, where X $\in$ {A, B, C}. Let I_i be the interval of guaranteed assignments offered to i, let

[19] Agent j, selected to play at any step in Stage A or Stage B, is trivially well defined.

$a_i \in A_i$ be i's choice, and let N_s, N_u, N_d, and q_{N_p} be the output of Step X.t. Then the following statements hold.

(R1.1) If $i \in N_s$, then $q_i = a_i$ and $0 \leq q_i \leq k$. To see that, note that by the definition of the MIA, $a_i \in I_i$. Then $0 \leq a_i \leq k$. By definition, $q_i := a_i$. Therefore, $0 \leq q_i \leq k$.

(R1.2) If $i \in N_u$, then $q_i = a_i - 1$ and $0 \leq q_i < k$. To see that, note that by the definition of the MIA, $a_i \in I_i^+ \setminus I_i$. We distinguish among three cases. First, X = A; then $I_i = [[\underline{q}'_i, \overline{q}'_i]]$ or $I_i = [[\underline{q}'_i + 1, \overline{q}'_i]]$, where, here and in (R1.3), $\underline{q}'_i$ and $\overline{q}'_i$ are computed at the beginning of Step A.t. As $a_i \in I_i^+ \setminus I_i$, $0 < a_i = \overline{q}'_i + 1 \leq k$. By definition, $q_i := \overline{q}'_i$. Therefore, $q_i = a_i - 1$ and $0 \leq q_i < k$. Second, X = B; then $I_i = [[q'_i]]$, where, here and in (R1.3), q'_i is an input of Step B.t. As $a_i \in I_i^+ \setminus I_i$, $0 < a_i = q'_i + 1 \leq k$. By definition, $q_i := q'_i$. Therefore, $q_i = a_i - 1$ and $0 \leq q_i < k$. Third, X = C; then $I_i = [[q'_i + 1]]$, where, here and in (R1.3), q'_i is an input of Step C.t. As $a_i \in I_i^+ \setminus I_i$, $0 < a_i = (q'_i + 1) + 1 \leq k$. By definition, $q_i := q'_i + 1$. Therefore, $q_i = a_i - 1$ and $0 \leq q_i < k$.

(R1.3) If $i \in N_d$, then $q_i = a_i + 1$ and $k \geq q_i > 0$. To see that, note that by the definition of the MIA, $a_i \in I_i^- \setminus I_i$. We distinguish among three cases. First, X = A; then $I_i = [[\underline{q}'_i, \overline{q}'_i]]$ or $I_i = [[\underline{q}'_i, \overline{q}'_i - 1]]$. As $a_i \in I_i^- \setminus I_i$, $k > a_i = \underline{q}'_i - 1 \geq 0$. By definition, $q_i := \underline{q}'_i$. Therefore, $q_i = a_i + 1$ and $k \geq q_i > 0$. Second, X = B; then $I_i = [[q'_i]]$. As $a_i \in I_i^- \setminus I_i$, $k > a_i = q'_i - 1 \geq 0$. By definition, $q_i := q'_i$. Therefore, $q_i = a_i + 1$ and $k \geq q_i > 0$. Third, X = C; then $I_i = [[q'_i - 1]]$. As $a_i \in I_i^- \setminus I_i$, $k > a_i = (q'_i - 1) - 1 \geq 0$. By definition, $q_i := q'_i - 1$. Therefore, $q_i = a_i + 1$ and $k \geq q_i > 0$.

Lemma 1 below (whose proof can be found in the Appendix) is key to assure that the MIA stops after a finite number of steps and that the pair $(\Gamma^\Phi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Φ.

Lemma 1. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule. Let N_s, N_u, N_d, and $(q_i)_{i \in N_p}$ be the input of any Step A.t of the MIA, and let*

$$\underline{q} = \Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) \quad \textit{and} \quad \overline{q} = \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}). \tag{4}$$

Then the following four conditions hold.

(L1.1) We have $\sum_{i \in N_p} q_i \leq k$.

(L1.2) If $i \in N_s$, then $q_i = \underline{q}_i = \overline{q}_i$.

(L1.3) If $i \in N_u$, then $q_i = \underline{q}_i \leq \overline{q}_i$.

(L1.4) If $i \in N_d$, then $\underline{q}_i \leq \overline{q}_i = q_i$.

The vector $(q_i)_{i \in N_p}$ of tentative assignments is considered as the assignments provisionally allotted to agents that have already played. We state below four remarks about $(q_i)_{i \in N_p}$ and the evolution of the subsets N_s, N_u, and N_d throughout the MIA that partially follow from Lemma 1.

Remark 2. The following four features of the MIA hold.

(R2.1) Inequality (L1.1) in Lemma 1 states that $(q_i)_{i\in N_p}$, part of the input of any Step A.t, can be completed as a feasible allotment by assigning to agents in N_w the leftover units not provisionally assigned yet to agents in N_p.

(R2.2) Consider the step (if any) of the MIA at which i enters into N_s after choosing a_i and let $q_i := a_i$. Then, by the definition of the MIA and (L1.2), the tentative assignment q_i becomes the final assignment to i at the outcome q of the MIA. Therefore, once agent i enters into N_s at some step, i remains in N_s at all further steps and i is not selected to play any more.

(R2.3) Consider the input N_s, N_u, N_d, and $(q_i)_{i\in N_p}$ of a generic Step X.t, where X $\in$ {A, C}, and assume $i \in N_u$ is selected to play and has to choose an action a_i in A_i. Then, if X = A, by (L1.3), $A_i = I_i^+ = [\![\underline{q}_i + 1, \overline{q}_i^+]\!] = [\![q_i + 1, \overline{q}_i^+]\!]$, and if X = C, $A_i = I_i^+ = \{q_i + 1, (q_i + 1)^+\}$. Then agent i's updated tentative assignment q_i in the output of Step X.t increases by definition in both cases: if X = A because either $q_i := a_i$ if $a_i \in I_i$ or $q_i := \overline{q}_i$ if $a_i \notin I_i$, and if X = C because $q_i := q_i + 1$. Furthermore, once agent i enters into N_u at some step, i can only move to N_s or remain in N_u at later steps.

(R2.4) Consider the input N_s, N_u, N_d, and $(q_i)_{i\in N_p}$ of a generic Step X.t, where X $\in$ {A, C}, and assume $i \in N_d$ is selected to play and has to choose an action a_i in A_i. Then, if X = A, by (L1.4), $A_i = I_i^- = [\![\underline{q}_i^-, \overline{q}_i - 1]\!] = [\![\underline{q}_i^-, q_i - 1]\!]$, and if X = C, $A_i = I_i^- = \{(q_i - 1)^-, q_i - 1\}$. Then agent i's updated tentative assignment q_i in the output of Step X.t decreases by definition in both cases: if X = A because either $q_i := a_i$ if $a_i \in I_i$ or $q_i := \underline{q}_i$ if $a_i \notin I_i$, and if X = C because $q_i := q_i - 1$. Furthermore, once agent i enters into N_d at some step, i can only move to N_s or remain in N_d at later steps.

Remark 2 assures that the MIA finishes in a finite number of steps because either N_u or N_d will be empty at some step since (a) the sequences of tentative allotments offered to agents in N_u or N_d are, respectively, increasing or decreasing, and (b) as soon as $q_i = k$, agent i moves from N_u to N_s, and as soon as $q_i = 0$, agent i moves from N_d to N_s.[20]

Lemma 2 and Lemma 3 (whose proofs can be found in the Appendix) guarantee that (i) $q = (q_{N_p}, \underline{q}_{N_w})$, part of the output of Stage A, is a feasible allotment and (ii) $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = \Phi(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = q$, part of the output of Stage B, is a feasible allotment as well. Moreover, by the actualization of q_j and q_r along the steps in Stage C, we can conclude that the final outcome of the MIA q is a feasible allotment.

[20]To make (a) more transparent, let N_s, N_u, N_d, and q_{N_p} be the output of a step at which i plays, let $I_i = [\![\alpha_i, \beta_i]\!]$ be the interval of guaranteed assignments offered to i at this step, let N_s', N_u', N_d', and $q'_{N_p'}$ be the input of the later step at which i plays again for the first time, and let $I_i' = [\![\alpha_i', \beta_i']\!]$ be the interval of guaranteed assignments offered to i at this step. If $i \in N_u$ (and according to the definition of the MIA, $a_i \in I_i^+ \setminus I_i$), then $q_i + 1 = \beta_i + 1 = \alpha_i'$. If $i \in N_d$ (and according to the definition of the MIA, $a_i \in I_i^- \setminus I_i$), then $q_i - 1 = \alpha_i - 1 = \beta_i'$.

Lemma 2. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule and let N_s, N_u, N_d, and $q = (q_{N_p}, \underline{q}_{N_w})$ be the output of Stage A of the* MIA. *Then $\Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) = \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) = q$.*

Lemma 3. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule, and let N_s, N_u, N_d, and q be the output of Stage B of the* MIA. *Then $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = \Phi(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = q$.*

We now argue that agents j and r, selected, respectively, at Step C.t.a and Step C.t.b, are well defined. Throughout Stage A and Stage B, each agent j is classified according to whether j prefers to receive the tentative assignment q_j (those in N_s), strictly more than q_j (those in N_u), or strictly less than q_j (those in N_d). If the MIA moves to Stage C it is because $N_u \neq \emptyset$ and $N_d \neq \emptyset$ in the output of Stage B. But this means that q is not efficient. At each Step C.t, agents $j \in N_u$ and $r \in N_d$ are selected to carry out a Pareto improvement upon q, input of this step, by increasing j's tentative assignment by one unit and decreasing r's tentative assignment by one unit. Agents j and r are sequentially identified by looking at the image of Φ at two somehow extreme profiles, both with all agents in N_s having their tops at q_{N_s}. First, j is one of the agents in N_u whose assignment is larger or equal to $q_j + 1$ at the scarcity profile $(\mathbf{k}_{N_u}, (q - \mathbf{1})_{N_d}, q_{N_s})$. Therefore, by (1) in the definition of same-sidedness, agents in $N_d \cup N_s$ receive at most their tops and, by feasibility of q, one agent j in N_u has to receive at least $q_j + 1$. Once j is identified, r is one of the agents in N_d whose assignment is smaller than or equal to $q_j - 1$ at the excess profile $(q_j + 1, \mathbf{0}_{N_d}, q_{N_s \cup (N_u \setminus \{j\})})$. Therefore, by (2) in the definition of same-sidedness, agents in $N_u \cup N_s$ receive at least their tops and, by feasibility of q, one agent r in N_d has to receive at most $q_r - 1$.

We now formally prove that at each Step C.t, agents $j \in N_u$ and $r \in N_d$ are well defined. Let N_s, N_u, N_d, and q be the input of Step C.t. This means that $N_u \neq \emptyset$ and $N_d \neq \emptyset$. By (R1.3) in Remark 1, $i \in N_d$ implies $0 < q_i$. Therefore, $x = (\mathbf{k}_{N_u}, (q - \mathbf{1})_{N_d}, q_{N_s})$ is a well defined profile of tops and $\sum_{i \in N} x_i \geq k$. Hence, by (1) in the definition of same-sidedness, $N_d \neq \emptyset$ and $N_u = N \setminus (N_d \cup N_s)$, it holds that

$$\sum_{i \notin N_u} \Phi_i\big(\mathbf{k}_{N_u}, (q - \mathbf{1})_{N_d}, q_{N_s}\big) \leq \sum_{i \in N_d} (q_i - 1) + \sum_{i \in N_s} q_i < \sum_{i \notin N_u} q_i.$$

By feasibility of q,

$$\sum_{i \in N_u} \Phi_i\big(\mathbf{k}_{N_u}, (q - \mathbf{1})_{N_d}, q_{N_s}\big) > \sum_{i \in N_u} q_i.$$

Hence, there exists $j \in N_u$ such that $\Phi_j(\mathbf{k}_{N_u}, (q - \mathbf{1})_{N_d}, q_{N_s}) \geq q_j + 1$. By (R1.2) in Remark 1, $i \in N_u$ implies $q_i < k$. Therefore, $y = (q_j + 1, \mathbf{0}_{N_d}, q_{N_s \cup (N_u \setminus \{j\})})$, where j is the agent identified just above and the one selected to play at Step C.t.a, is a well defined profile of tops and $\sum_{i \in N} y_i \leq k$. Hence, by (2) in the definition of same-sidedness and $j \notin N_d$, it holds that

$$\sum_{i \notin N_d} \Phi_i(q_j + 1, \mathbf{0}_{N_d}, q_{N_s \cup (N_u \setminus \{j\})}) \geq q_j + 1 + \sum_{i \notin N_d \cup \{j\}} q_i > \sum_{i \notin N_d} q_i.$$

By feasibility of q,

$$\sum_{i\in N_d}\Phi_i(q_j+1,\mathbf{0}_{N_d},q_{N_s\cup(N_u\setminus\{j\})})<\sum_{i\in N_d}q_i.$$

Hence, there exists $r\in N_d$ such that $\Phi_r(q_j+1,\mathbf{0}_{N_d},q_{N_s\cup(N_u\setminus\{j\})})\leq q_r-1$.

We can now proceed with the two results that guarantee that Φ is obviously strategy-proof.

5.2 *Truth-telling is obviously dominant in* Γ^Φ

Let Γ^Φ be the extensive game form defined by the MIA after identifying each step of the MIA with a nonterminal node of Γ^Φ. Fix a preference profile $R\in\mathcal{R}^N$ with top profile τ. Let (Γ^Φ,R) be the game in extensive form. For $i\in N$ and $R_i\in\mathcal{R}$, recall that the truth-telling strategy $\sigma_i^{R_i}$ is the strategy where, whenever agent i is selected to play, i chooses the best action in A_i according to R_i, denoted by $\max_{R_i}A_i$.[21]

Proposition 2. *Let Γ^Φ be the extensive game form defined by the* MIA *and let $R\in\mathcal{R}^N$ be a profile. Then, for each agent i, the truth-telling strategy $\sigma_i^{R_i}$ is weakly dominant in the game in extensive form* (Γ^Φ,R).

Proof. Let Γ^Φ be the extensive game form defined by the MIA. Fix $i\in N$, $R_i\in\mathcal{R}$, and σ_{-i}, and consider any $\widehat{\sigma}_i\neq\sigma_i^{R_i}$. Let N_s, N_u, N_d, and $(q_i)_{i\in N}$ be the output of the MIA when agents play according to $(\sigma_i^{R_i},\sigma_{-i})$, and let $\widehat{N}_s$, $\widehat{N}_u$, $\widehat{N}_d$, and $(\widehat{q}_i)_{i\in N}$ be the output of the MIA when agents play according to $(\widehat{\sigma}_i,\sigma_{-i})$. We verify that $q_i\,R_i\,\widehat{q}_i$. If $q_i=\widehat{q}_i$, the statement is trivially true. Assume $q_i\neq\widehat{q}_i$. To proceed, we need to introduce the following notation.

(i) Let N_s', N_u', N_d', and $q'_{N_p'}$ be the input of the last step of the MIA at which i is selected to play when agents play according to $(\sigma_i^{R_i},\sigma_{-i})$, and let a_i' be i's chosen action and let I_i' be the interval of guaranteed assignments offered to i at this last step.

(ii) For the $(\sigma_i^{R_i},\sigma_{-i})$ path and the $(\widehat{\sigma}_i,\sigma_{-i})$ path in Γ^Φ, consider the step of the MIA at which, for the first time, $\sigma_i^{R_i}$ and $\widehat{\sigma}_i$ select different actions. Since $q_i\neq\widehat{q}_i$, such a step does exist. Let A_i^* be the set of actions available to i and let I_i^* be the interval of guaranteed assignments offered to i at this step. Let a_i^* and $\widehat{a}_i^*$ be, respectively, the choices of i according to $\sigma_i^{R_i}$ and $\widehat{\sigma}_i$. By how the step has been chosen, $a_i^*\neq\widehat{a}_i^*$. Let q_i^* and $\widehat{q}_i^*$ be the immediate updated tentative assignments for agent i at this step that follow after a_i^* and $\widehat{a}_i^*$, respectively.

We now proceed by distinguishing among three cases.

Case 1: Assume $i\in N_s$. We show that $\tau(R_i)=q_i$. By definition of the MIA, $q_i=a_i'\in I_i'$, and by the definition of $\sigma_i^{R_i}$, $\tau(R_i)=a_i'$. Hence, $\tau(R_i)=q_i$ and so, $q_i\,R_i\,\widehat{q}_i$ holds.

Case 2: Assume $i\in N_u$. We first show that $\tau(R_i)>q_i$. By definition of the MIA, $i\in N_u$ implies $q_i<a_i'\in I_i'^{+}\setminus I_i'$. By the definition of $\sigma_i^{R_i}$, $\tau(R_i)\geq a_i'$. Hence, $\tau(R_i)>q_i$. We now

[21] Recall again that, according to Mackenzie (2020), being obviously dominant in (Γ^Φ,R) is equivalent to being weakly dominant in (Γ^Φ,R).

show that $q_i \, R_i \, \widehat{q}_i$ holds indeed. By the definition of $\sigma_i^{R_i}$,

$$a_i^* = \max A_i^* \leq \tau(R_i). \tag{5}$$

By (R1.2), $a_i^* - 1 = q_i^*$. Furthermore, iteratively applying (L1.3) and the definition of the MIA, $q_i^* \leq q_i$. Then, by (5),

$$\max A_i^* - 1 \leq q_i < \tau(R_i). \tag{6}$$

Similarly, and as $a_i^* \neq \widehat{a}_i^*$,

$$\widehat{a}_i^* \leq \max A_i^* - 1. \tag{7}$$

Since $i \in N_u$, either $A_i^* = (I_i^*)^+$ or $A_i^* = (I_i^*)^\pm$. If $\widehat{a}_i^* \in (I_i^*)^+$, then $\widehat{a}_i^* < a_i^*$ because $a_i^* = \max A_i^*$; but then i enters into the set of agents who want to stop and, by (R1.1), $\widehat{a}_i^* = \widehat{q}_i^*$. If $\widehat{a}_i^* \notin (I_i^*)^+$, then $A_i^* = (I_i^*)^\pm$ and $\widehat{a}_i^* = \min A_i^* \in (I_i^*)^- \setminus I_i^*$. Therefore, $|A_i^*| \geq 3$ and so $\widehat{a}_i^* = \min A_i^* < \max A_i^* - 1$, and, by (R1.3), $\widehat{a}_i^* + 1 = \widehat{q}_i^*$. Then, in all cases,

$$\widehat{q}_i^* \leq \max A_i^* - 1. \tag{8}$$

Furthermore, by (8) and (6),

$$\widehat{q}_i^* \leq q_i < \tau(R_i). \tag{9}$$

By iterated applications of (L1.4) and the definition of the MIA, $\widehat{q}_i \leq \widehat{q}_i^*$. Therefore, $\widehat{q}_i \leq q_i < \tau(R_i)$. By single-peakedness, $q_i \, R_i \, \widehat{q}_i$.

Case 3: Assume $i \in N_d$. We first show that $\tau(R_i) < q_i$. By definition of the MIA, $i \in N_d$ implies $q_i > a_i' \in I_i'^- \setminus I_i'$. By the definition of $\sigma_i^{R_i}$, $\tau(R_i) \leq a_i'$. Hence, $\tau(R_i) < q_i$. We now show that $q_i \, R_i \, \widehat{q}_i$ holds indeed. By the definition of $\sigma_i^{R_i}$,

$$\tau(R_i) \leq \min A_i^* = a_i^*. \tag{10}$$

By (R1.3), $a_i^* + 1 = q_i^*$. Furthermore, iteratively applying (L1.4) and the definition of the MIA, $q_i \leq q_i^*$. Then, by (10),

$$\tau(R_i) < q_i \leq \min A_i^* + 1. \tag{11}$$

Similarly, and as $a_i^* \neq \widehat{a}_i^*$,

$$\min A_i^* + 1 \leq \widehat{a}_i^*.$$

Since $i \in N_d$, either $A_i^* = (I_i^*)^-$ or $A_i^* = (I_i^*)^\pm$. If $\widehat{a}_i^* \in (I_i^*)^-$, then $\widehat{a}_i^* > a_i^*$ because $a_i^* = \min A_i^*$; but then i enters into the set of agents who want to stop and, by (R1.1), $\widehat{a}_i^* = \widehat{q}_i^*$. If $\widehat{a}_i^* \notin (I_i^*)^-$, then $A_i^* = (I_i^*)^\pm$ and $\widehat{a}_i^* = \max A_i^* \in (I_i^*)^+ \setminus I_i^*$. Therefore, $|A_i^*| \geq 3$ and so $\min A_i^* + 1 < \max A_i^* = \widehat{a}_i^*$ and, by (R1.2), $\widehat{a}_i^* - 1 = \widehat{q}_i^*$. Then, in all cases,

$$\min A_i^* + 1 \leq \widehat{q}_i^*. \tag{12}$$

Furthermore, by (12) and (11),

$$\tau(R_i) < q_i \leq \widehat{q}_i^*. \tag{13}$$

By iterated applications of (L1.3) and the definition of the MIA, $\widehat{q}_i^* \leq \widehat{q}_i$. Therefore, $\tau(R_i) < q_i \leq \widehat{q}_i$. By single-peakedness, $q_i \; R_i \; \widehat{q}_i$.

Hence, for all σ_{-i} and $\widehat{\sigma}_i$, $g_i(\sigma_i^{R_i}, \sigma_{-i}) \; R_i \; g_i(\widehat{\sigma}_i, \sigma_{-i})$, which means that $\sigma_i^{R_i}$ is weakly dominant in (Γ^Φ, R). □

5.3 *The pair* $(\Gamma^\Phi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ *induces* Φ

Lemma 1 says that the sequences of qs, $\underline{q}$s, and $\overline{q}$s generated by a non-individually rational rule Φ in Stage A are monotonous (in the right direction) and the sum of the components of each $(q_i)_{i \in N_p}$ in the sequence is smaller than or equal to k. Lemma 2 and Lemma 3 say, respectively, that, at the end of Stage A, $q = \underline{q} = \overline{q}$, and at the end of Stage B, $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = \Phi(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = q$. Both imply that q is a feasible allotment and the former says that the process somehow converges, while the latter is also an intermediate result for the proof of Lemma 4 about Stage B. In turn, Lemma 4 is required to prove Lemma 5, presented below. The statement and proof of Lemma 4 and the proof of Lemma 5 can be found in the Appendix.

Lemma 5. *Let* $\Phi : \mathcal{R}^N \to X$ *be a sequential allotment rule, and let* N_s, N_u, N_d, *and* q *be the output of the* MIA. *Then the following two conditions hold.*

(L5.1) If $N_u = \emptyset$, *then* $\Phi(\mathbf{0}_{N_d}, q_{N_s}) = q$.

(L5.2) If $N_u \neq \emptyset$, *then* $N_d = \emptyset$ *and* $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$.

We now state and prove that the pair $(\Gamma^\Phi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Φ.

Proposition 3. *Let* Φ *be a sequential allotment rule. Then* $(\Gamma^\Phi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ *induces* Φ.

Proof. Let $R \in \mathcal{R}^N$ be an arbitrary profile of preferences, and let N_s, N_u, N_d, and q be the output of the MIA when agents play Γ^Φ according to σ^R. This means that $q = g(\sigma^R)$. We show that $\Phi(R) = q$ holds by distinguishing between two cases.

Case 1: $N_u = \emptyset$. By (L5.1) in Lemma 5, $\Phi(\mathbf{0}_{N_d}, q_{N_s}) = q$. By (R1.1) in Remark 1, $q_i = \tau(R_i)$ for all $i \in N_s$, and with the abuse of notation of mixing a profile of preferences and a profile of tops, $\Phi(\mathbf{0}_{N_d}, R_{N_s}) = q$. Let $i \in N_d$. By (R1.3) and the definition of the MIA, the last step where i was selected to play i has chosen $a_i \in I_i^- \setminus I_i$ and $a_i = q_i - 1$. By the definition of $\sigma_i^{R_i}$, $\tau(R_i) \leq a_i$. Hence, $\tau(R_i) < q_i$. By strategy-proofness and single-peakedness, $\Phi_i(\mathbf{0}_{N_d \setminus \{i\}}, R_{N_s \cup \{i\}}) = q_i = \Phi_i(\mathbf{0}_{N_d}, R_{N_s})$. Since Φ is replacement monotonic, $\Phi(\mathbf{0}_{N_d \setminus \{i\}}, R_{N_s \cup \{i\}}) = q = \Phi(\mathbf{0}_{N_d}, R_{N_s})$. Successively using the same argument for the remaining agents in $N_d \setminus \{i\}$, we obtain that $\Phi(R) = q = \Phi(R_{N_d}, R_{N_s})$.

Case 2: $N_u \neq \emptyset$. By (L5.2) in Lemma 5, $N_d = \emptyset$ and $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$. By (R1.1) in Remark 1, $q_i = \tau(R_i)$ for all $i \in N_s$, and again with an abuse of notation, $\Phi(\mathbf{k}_{N_u}, R_{N_s}) = q$. Let $i \in N_u$. By (R1.2) and the definition of the MIA, the last step where i was selected to play i has chosen $a_i \in I_i^+ \setminus I_i$ and $a_i = q_i + 1$. By the definition of $\sigma_i^{R_i}$, $\tau(R_i) \geq a_i$. Hence, $\tau(R_i) > q_i$. Therefore, by an argument symmetric to that used in Case 1, we obtain that $\Phi(R) = q = \Phi(R_{N_u}, R_{N_s})$. □

This completes the proof of Theorem 1.

6. Final remarks

We finish the paper with four remarks.

First, our implementation result requires that the rule be replacement monotonic. Example 2 contains a division problem where there is a strategy-proof, efficient and non-replacement monotonic rule that is not obviously strategy-proof.

Example 2. Consider the division problem where $N = \{1, 2, 3\}$ and $k = 2$. Let $\Psi : \mathcal{R}^N \to X$ be the tops-only rule that, for every $\tau = (\tau_1, \tau_2, \tau_3) \in \{0, 1, 2\}^N$, $\Psi(\tau)$ is determined sequentially. The top of agent 1 determines the order in which agents 2 and 3 have to successively choose their most preferred assignments (among those left available by the predecessor, if any). If agent 1 chooses 0 or 1, then agent 2 plays before 3. If agent 1 chooses 2, then agent 3 plays before 2. Agent 1's assignment is equal to the remainder; namely,

$$\Psi(\tau_1, \tau_2, \tau_3) = \begin{cases} (2 - \tau_2 - \min\{2 - \tau_2, \tau_3\}, \tau_2, \min\{2 - \tau_2, \tau_3\}) & \text{if } \tau_1 \in \{0, 1\} \\ (2 - \tau_3 - \min\{2 - \tau_3, \tau_2\}, \min\{2 - \tau_3, \tau_2\}, \tau_3) & \text{if } \tau_1 = 2. \end{cases}$$

It is easy to check that Ψ is strategy-proof and efficient. To see that Ψ is not replacement monotonic, consider $\tau = (\tau_1, \tau_2, \tau_3) = (0, 1, 2)$ and $\tau' = (\tau_1', \tau_2, \tau_3) = (2, 1, 2)$. Then $\Psi(\tau) = (0, 1, 1)$ and $\Psi(\tau') = (0, 0, 2)$. Since $\Psi_1(\tau) = \Psi_1(\tau')$, $\Psi_2(\tau) > \Psi_2(\tau')$ and $\Psi_3(\tau) < \Psi_3(\tau')$, Ψ is not replacement monotonic.

To obtain a contradiction, assume Ψ is obviously strategy-proof. Let Γ be the extensive game form that OSP-implements Ψ. By Mackenzie (2020), we can assume without loss of generality that Γ has perfect information. Given a profile of tops τ, let $\sigma^\tau = (\sigma_1^{\tau_1}, \sigma_2^{\tau_2}, \sigma_3^{\tau_3})$ be a strategy profile such that $\Psi(\tau) = g(\sigma^\tau)$. As $(\Gamma^\Psi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Ψ, there must exists a nonterminal node ν such that (i) the agent who plays at ν has at least two available actions (denoted by a^1 and a^2) and (ii) at all nodes preceding ν (if any), the agents who play have only one available action. Suppose agent 1 is the player who plays at ν. Consider the two profiles of tops $\tau = (1, 0, 0)$ and $\tau' = (2, 1, 0)$. As $(\Gamma^\Psi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Ψ, $g_1(\sigma^\tau) = \Psi_1(\tau) = 2$ and $g_1(\sigma^{\tau'}) = \Psi_1(\tau') = 1$. Consider σ_2 and σ_3 with the properties that (i) they respectively coincide with $\sigma_2^{\tau_2}$ and $\sigma_3^{\tau_3}$ at all nodes that follow ν after agent 1 chooses a^1, and (ii) they respectively coincide with $\sigma_2^{\tau_2'}$ and $\sigma_3^{\tau_3'}$ at all nodes that follow ν after agent 1 chooses a^2. Note that by its definition, node ν is reached regardless of the strategy profile used by the agents, and since Γ has perfect information, σ_2 and σ_3 are well defined. Because there exists $R_1 \in \mathcal{R}$ such that $\tau(R_1) = 1$ and

$$g_1(\sigma_1^{\tau_1'}, \sigma_2, \sigma_3) = 1 P_1 2 = g_1(\sigma_1^{\tau_1}, \sigma_2, \sigma_3),$$

strategy $\sigma_1^{\tau_1}$ is not weakly dominant in Γ, a contradiction with the assumption that Γ OSP-implements Ψ. Suppose agent 2 is the player who plays at ν. Consider the two profiles of tops $\tau = (2, 1, 2)$ and $\tau' = (1, 1, 2)$. As $(\Gamma^\Psi, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Ψ, $g_2(\sigma^\tau) = \Psi_2(\tau) = 0$ and $g_2(\sigma^{\tau'}) = \Psi_2(\tau') = 1$. Consider σ_1 and σ_3 with the properties that (i) they respectively coincide with $\sigma_1^{\tau_1}$ and $\sigma_3^{\tau_3}$ at all nodes that follow ν after agent 2 chooses a^1, and (ii) they respectively coincide with $\sigma_1^{\tau_1'}$ and $\sigma_3^{\tau_3'}$ at all nodes that follow ν after agent 2 chooses a^2. Note that by its definition, node ν is reached regardless of the strategy profile used by the agents, and since Γ has perfect information, σ_1 and σ_3 are well defined. Because there exists $R_2 \in \mathcal{R}$ such that $\tau(R_2) = 1$ and

$$g_2(\sigma_1, \sigma_2^{\tau_2'}, \sigma_3) = 1P_2 0 = g_2(\sigma_1, \sigma_2^{\tau_2}, \sigma_3),$$

strategy $\sigma_2^{\tau_2}$ is not weakly dominant in Γ, a contradiction with the assumption that Γ OSP-implements Ψ. A similar argument can be used to obtain a contradiction when 3 is the agent who plays at ν. ◊

Second, there are strategy-proof and efficient rules that are not replacement monotonic (and so, they are not sequential), but they are obviously strategy-proof. Example 3 illustrates this possibility.

Example 3. Consider the division problem where $N = \{1, 2, 3\}$ and $k = 2$. Let $\varphi : \mathcal{R}^N \to X$ be the tops-only rule that, for every $\tau = (\tau_1, \tau_2, \tau_3) \in \{0, 1, 2\}^N$, $\varphi(\tau)$ is determined sequentially. Agent 1 receives its top. If $\tau_1 = 0$, agent 2 receives τ_2 and agent 3 receives $2 - \tau_2$. If $\tau_1 \in \{1, 2\}$, agent 3 receives its best assignment in $[\![0, 2 - \tau_1]\!]$, denoted by τ_3^{rest}, and agent 2 receives $2 - \tau_1 - \tau_3^{\text{rest}}$. Namely,

$$\varphi(\tau_1, \tau_2, \tau_3) = \begin{cases} (0, \tau_2, 2 - \tau_2) & \text{if } \tau_1 = 0 \\ (\tau_1, 2 - \tau_1 - \tau_3^{\text{rest}}, \tau_3^{\text{rest}}) & \text{if } \tau_1 \in \{1, 2\}. \end{cases}$$

It is easy to check that φ is strategy-proof and efficient. To see that φ is not replacement monotonic, consider $\tau = (\tau_1, \tau_2, \tau_3) = (0, 2, 2)$ and $\tau' = (\tau_1', \tau_2, \tau_3) = (1, 2, 2)$. Then $\varphi(\tau) = (0, 2, 0)$ and $\varphi(\tau') = (1, 0, 1)$. Since $\varphi_1(\tau) < \varphi_1(\tau')$, $\varphi_2(\tau) > \varphi_2(\tau')$, and $\varphi_3(\tau) < \varphi_3(\tau')$, φ is not replacement monotonic.

However, φ is obviously strategy-proof. The extensive game form depicted in Figure 2 OSP-implements φ, where agents are shown in bold numbers. Together, Examples 2 and 3 show that while replacement monotonicity is indispensable for our main result to hold, it is not necessary.

Third, Pycia and Troyan (2022) propose a family of simplicity standards that depend on agents' ability to foresee further down in the game and that strengthen the notion

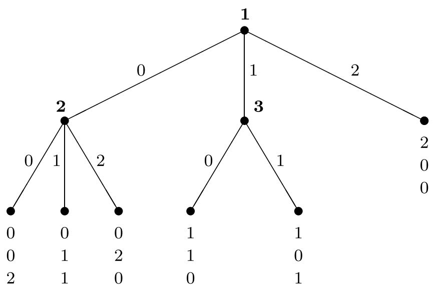

Figure 2. The extensive game form that OSP-implements φ.

of obvious strategy-proofness. Each standard brings about a notion of dominance and its correspondent of strategy-proofness. The simpler standard is that in which agents cannot plan for any moves in the future, referred to as strong obvious strategy-proofness (SOSP). In our context, a rule $\Phi : \mathcal{R}^N \to X$ is *strongly obviously strategy-proof* if there are an extensive game form $\Gamma \in \mathcal{G}$, associated to k and N, and a type-strategy profile $(s_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ for Γ such that the pair $(\Gamma, (s_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces Φ, and, for each profile $R \in \mathcal{R}^N$ and agent $i \in N$, the corresponding strategy $s_i^{R_i}$ is strongly obviously dominant in (Γ, R).

Strongly obvious domination requires that, for each i and each R_i, when comparing the worst possible outcome of the choice prescribed by $s_i^{R_i}$ at an earliest point of departure ν with any other s_i, the choices made by i at all nodes that follow $s_i^{R_i}(\nu)$ do not have to necessarily follow $s_i^{R_i}$ any more since i may now choose, at a node γ that follows $s_i^{R_i}(\nu)$, a different action than $s_i^{R_i}(\gamma)$. Therefore, the worst possible outcome associated to the stronger OSP notion could be strictly worse than the one obtained when agent i is required to stay with the strategy $s_i^{R_i}$, as required by Li's (2017) original OSP notion. Example 4 below shows that not all sequential allotment rules are strongly obviously strategy-proof. However, the subclass of serial dictator rules (that can be described as sequential allotment rules) satisfy the stronger requirement since agents play only once.[22] The less simple standard is that in which agents can plan all their moves in the future, and it corresponds to Li's (2017) original OSP notion. Pycia and Troyan (2022) also focus on the standard of one-step dominance in which agents are able to plan at most one move ahead at a time, referred to as one-step simple. Given any sequential allotment rule Φ, let us consider i's plan in Γ^Φ of choosing today the best available assignment a_i in $A_i \supseteq [\![\alpha_i, \beta_i]\!]$ and choosing next time (if any) a'_i in $A'_i \supseteq [\![\alpha'_i, \beta'_i]\!]$ according to the following two possibilities: if $a_i = \beta_i^+$, then $a'_i = \alpha'_i$, while if $a_i = \alpha_i^-$, then $a'_i = \beta'_i$ (note that if $a_i \in [\![\alpha_i, \beta_i]\!]$, i does not move again). This plan defines a type-strategy profile that is one-step dominant and together with Γ^Φ induce Φ; hence, all sequential allotment rules are one-step simple. ◊

Example 4. Consider the division problem where $N = \{1, 2, 3\}$ and $k = 3$. Let $\psi : \mathcal{R}^N \to X$ be any individually rational sequential allotment rule with respect to the allotment $q = (1, 1, 1)$.[23] To obtain a contradiction, assume that Γ is an extensive game form that together with a type-strategy profile $(\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N}$ induce ψ and that Γ SOSP-implements ψ. Let ν be the node in Γ at which, for the first time, an agent has at least two actions available. Without loss of generality, let 1 be such an agent. Fix an arbitrary $R_1 \in \mathcal{R}$ and let a^ℓ be the action such that $\sigma_1^{R_1}(\nu) = a^\ell$ with $\tau(R_1) = \ell$. Since ψ is tops-only, it is sufficient to distinguish between two different general cases.

Case 1: Assume $a^2 \neq a^3$. Since φ is efficient, $\psi(2, 1, 0) = (2, 1, 0)$ and, because $(\Gamma, (\sigma_i^{R_i})_{R_i \in \mathcal{R}, i \in N})$ induces ψ, the allotment $(2, 1, 0)$ is possible after 1 chooses a^2 at ν. Since ψ is efficient, $\psi(1, 1, 1) = (1, 1, 1)$ and, by individual rationality, $\psi(3, 1, 1) = (1, 1, 1)$. Then the allotment $(1, 1, 1)$ is possible after the choice a^3. However, for all single-peaked preference $R_1 \in \mathcal{R}$ with $\tau(R_1) = 3$, $2 P_1 1$. Hence, ψ is not SOSP.

[22]Observe that SOSP is more restrictive than OSP plus replacement monotonicity.

[23]Note that this implies that ψ is not a serial dictator rule.

Case 2: Assume $a^2 = a^3$. We refer to this action as $a^{2,3}$. We distinguish between two subcases.

Case 2.1: Assume $a^1 \neq a^{2,3}$. Then, using similar arguments to those used in Case 1, the allotment $(1, 1, 1)$ is possible after 1 chooses a^1 and the allotment $(3, 0, 0)$ is possible after 1 chooses $a^{2,3}$. However, there is a single-peaked preference $R_1 \in \mathcal{R}$ with $\tau(R_1) = 2$ for which $1P_13$. Hence, ψ is not SOSP.

Case 2.2: Assume $a^1 = a^{2,3}$. We refer to this action as $a^{1,2,3}$. We distinguish between two further subcases.

Case 2.2.1: Assume $a^0 \neq a^{1,2,3}$. Then, using similar arguments to those used in Case 1, the allotment $(0, 3, 0)$ is possible after 1 chooses a^0, while the allotment $(3, 0, 0)$ is possible after 1 chooses $a^{1,2,3}$. However, there is a single-peaked preference $R_1 \in \mathcal{R}$ with $\tau(R_1) = 2$ for which $0P_13$. Hence, ψ is not SOSP.

Case 2.2.2: Assume $a^0 = a^{1,2,3}$. But this means that agent 1 has a unique available action at ν. A contradiction. ♢

Fourth, Barberà, Jackson, and Neme (1997) observe that each sequential allotment rule is fully implementable in dominant strategies by the direct revelation mechanism. Our extensive game forms provide *full* OSP-implementation of all sequential allotment rules. Namely, for each sequential allotment rule, the extensive game form defined by the MIA obtained from the rule has the property that, for each preference profile, each obviously dominant strategy profile leads to the allotment specified by the rule for that preference profile. In fact, at every node of Γ^Φ, no departing strategy is obviously dominant—or even dominant or even a best reply to some strategy profile of the other agents—unless it leads to the same outcome. Hence, Γ^Φ provides full subgame perfect implementation (see Moore and Repullo (1988)) and ex post perfect implementation (see Ausubel (2004)) of the rule Φ. Interestingly, the two hold without following from Mackenzie and Zhou (2022) because our setting, by assuming agents' preferences are single-peaked, violates their richness and strictness conditions.

Appendix

This Appendix contains the proofs of Lemmas 1, 2, 3, 4, and 5, used in the proof of Theorem 1. However, we start with a remark (intensively used in the proofs that follow) that states that if an agent can receive a particular assignment by lying, then by truthfully reporting its top, it will receive something in the interval between its top and this assignment.

Remark 3. Let $\Phi : \{0, \dots, k\}^N \to X$ be a sequential allotment rule. Then, for all $\tau \in \{0, \dots, k\}^N$, $i \in N$, and $\tau'_i \in \{0, \dots, k\}$, the following two statements hold.

(R3.1) If $\Phi_i(\tau) \geq \tau'_i$, then $\Phi_i(\tau) \geq \Phi_i(\tau'_i, \tau_{-i}) \geq \tau'_i$ and $\Phi_j(\tau) \leq \Phi_j(\tau'_i, \tau_{-i})$ for all $j \in N \setminus \{i\}$.

To see that (R3.1) holds, assume first that $\Phi_i(\tau) = \tau'_i$. Then, by strategy-proofness, $\Phi_i(\tau) = \Phi_i(\tau'_i, \tau_{-i})$, and by replacement monotonicity, $\Phi_j(\tau) =$

$\Phi_j(\tau_i', \tau_{-i})$ for all $j \in N \setminus \{i\}$. Assume now that $\Phi_i(\tau) > \tau_i'$. To obtain a contradiction, suppose that either (i) $\Phi_i(\tau_i', \tau_{-i}) > \Phi_i(\tau) > \tau_i'$ or (ii) $\Phi_i(\tau) > \tau_i' > \Phi_i(\tau_i', \tau_{-i})$ holds. By single-peakedness, (i) contradicts the notion that Φ is strategy-proof. Suppose (ii) holds. Then there is $R_i'' \in \mathcal{R}$ such that $\tau(R_i'') = \tau_i'' = \tau_i'$ and $\Phi_i(\tau)\ P_i''\ \Phi_i(\tau_i', \tau_{-i})$. By tops-onlyness, $\Phi_i(\tau)\ P_i''\ \Phi_i(\tau_i'', \tau_{-i})$ holds, which contradicts the notion that Φ is strategy-proof. Hence, $\Phi_i(\tau) \geq \Phi_i(\tau_i', \tau_{-i}) \geq \tau_i'$. By replacement monotonicity, $\Phi_j(\tau) \leq \Phi_j(\tau_i', \tau_{-i})$ for all $j \in N \setminus \{i\}$.

(R3.2) If $\Phi_i(\tau) \leq \tau_i'$, then $\Phi_i(\tau) \leq \Phi_i(\tau_i', \tau_{-i}) \leq \tau_i'$ and $\Phi_j(\tau) \geq \Phi_j(\tau_i', \tau_{-i})$ for all $j \in N \setminus \{i\}$.

An argument symmetric to that used in (R3.1) shows that (R3.2) holds.

We now state and prove Lemma 1, a key result for the proof of Theorem 1.

Lemma 1. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule. Let N_s, N_u, N_d, and $(q_i)_{i \in N_p}$ be the input of Step A.t of the* MIA, *and let*

$$\underline{q} = \Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) \quad \text{and} \quad \overline{q} = \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}). \tag{14}$$

Then the following four conditions hold.

(L1.1) We have $\sum_{i \in N_p} q_i \leq k$.

(L1.2) If $i \in N_s$, then $q_i = \underline{q}_i = \overline{q}_i$.

(L1.3) If $i \in N_u$, then $q_i = \underline{q}_i \leq \overline{q}_i$.

(L1.4) If $i \in N_d$, then $\underline{q}_i \leq \overline{q}_i = q_i$.

Proof. We proceed by induction on t. When $t = 1$, the four statements hold trivially because $N_s = N_u = N_d = N_p = \emptyset$. Suppose $t \geq 2$.

Induction Hypothesis (IH). Let N_s', N_u', N_d', and $(q_i')_{i \in N_p'}$ be the input of Step A.$(t-1)$ of the MIA, and let

$$\underline{q}' = \Phi\big(\mathbf{0}_{N_w' \cup N_d'}, q'_{N_s' \cup N_u'}\big) \quad \text{and} \quad \overline{q}' = \Phi\big(\mathbf{k}_{N_w' \cup N_u'}, q'_{N_s' \cup N_d'}\big). \tag{15}$$

Then the following four conditions hold.

(IH.L1.1) We have $\sum_{i \in N_p'} q_i' \leq k$.

(IH.L1.2) If $i \in N_s'$, then $q_i' = \underline{q}_i' = \overline{q}_i'$.

(IH.L1.3) If $i \in N_u'$, then $q_i' = \underline{q}_i' \leq \overline{q}_i'$.

(IH.L1.4) If $i \in N_d'$, then $\underline{q}_i' \leq \overline{q}_i' = q_i'$.

Let $j \in S' = \{i \in N \mid \underline{q}_i' < \overline{q}_i'\}$ be the agent who was selected to play at Step A.$(t-1)$, and let N_s, N_u, N_d, and $(q_i)_{i \in N_p}$ be the output of Stage A.$(t-1)$ and input of Step A.t. By the definition of the MIA and the IH,

$$N_p = N_p' \cup \{j\} \quad \text{and} \quad q_i = q_i' \quad \text{for all } i \in N_p \setminus \{j\}. \tag{16}$$

As $j \in S'$,

$$\underline{q}'_j < \overline{q}'_j. \tag{17}$$

First, we show that (L1.1) holds. By the IH and (16), $q_i \leq \overline{q}'_i$ for all $i \in N_p \setminus \{j\}$. Now we show that $q_j \leq \overline{q}'_j$. To see that, let a_j be j's choice at Step A.$(t-1)$. According to the definition of q_j, three cases are possible. First, $a_j \in \{\underline{q}'_j, \ldots, \overline{q}'_j\}$, in which case $q_j = a_j \leq \overline{q}'_j$. Second, $a_j = \underline{q}'^{-}_j$, in which case $q_j = \underline{q}'_j \leq \overline{q}'_j$. Third, $a_j = \overline{q}'^{+}_j$, in which case, $q_j = \overline{q}'_j$. Therefore, $q_i \leq \overline{q}'_i$ for all $i \in N_p$. Then, by feasibility of $\overline{q}'$,

$$\sum_{i \in N_p} q_i \leq \sum_{i \in N_p} \overline{q}'_i \leq k,$$

which is (L1.1).

By (IH.L1.2) and (17), $j \notin N'_s$. Then, to prove that the other three statements hold, we consider three cases, depending on whether j belongs to N'_u, N'_d, or N'_w. But before doing so, we state two general observations. By the IH, $\underline{q}'_i \leq \overline{q}'_i$ for all $i \in N'_p$. Then, by (17) and the feasibility of $\underline{q}'$ and $\overline{q}'$, there exists $k \neq j$ such that $\overline{q}'_k < \underline{q}'_k$. Then $k \in N'_w$ and $N_p = N'_p \cup \{j\} \neq N$. Accordingly, $N_s \cup N_u \neq N$ and $N_s \cup N_d \neq N$. Furthermore, we have just shown that $\sum_{i \in N_p} q_i \leq k$ in (L1.1) holds. Then, by (14), and (1) and (2) in the definition of same-sidedness,

$$q_i \leq \Phi_i(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) = \underline{q}_i \quad \text{for all } i \in N_s \cup N_u \tag{18}$$

and

$$\overline{q}_i = \Phi_i(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) \leq q_i \quad \text{for all } i \in N_s \cup N_d. \tag{19}$$

Case 1: $j \in N'_u$. Then, by definitions of I_j and q_j,

$$q_j \in \{\underline{q}'_j + 1, \ldots, \overline{q}'_j\}. \tag{20}$$

Hence,

$$q_j \leq \overline{q}'_j. \tag{21}$$

By (IH.L1.3),

$$\begin{gathered} q'_j = \underline{q}'_j \leq \overline{q}'_j \\ j \in N_s \cup N_u = N'_s \cup N'_u \quad \text{and} \quad j \notin N'_s \cup N'_d. \end{gathered} \tag{22}$$

Claim 1. *We have (a.1) $\overline{q}_j \leq \overline{q}'_j$, (b.1) $\overline{q}_i \geq \overline{q}'_i$ for all $i \neq j$, and (c.1) $\overline{q}_i = \overline{q}'_i$ for all $i \in (N_s \cup N_d) \setminus \{j\}$.*

Proof. We distinguish between two cases.

Case C1.1: $j \in N_u$. Then $j \notin N_s \cup N_d = N'_s \cup N'_d$ and, by (16), $(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) = (\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d})$. Therefore, by (14) and (15), $\overline{q} = \overline{q}'$, which means that the statement of Claim 1 holds in this case.

Case C1.2: $j \in N_s$. To show (a.1), we first argue that

$$\overline{q}_j = \Phi_j(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) \leq \Phi_j(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d}) = \overline{q}'_j$$

holds. The first equality follows from (14) and the second equality follows from (15). To check that the inequality holds as well, observe that (i) since $j \in N'_u$ and $j \notin N'_s \cup N'_d$, so $(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d})_j = k$, (ii) since $j \in N_s$, $(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d})_j = q_j$, (iii) by (16), for all $i \neq j$, $(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d})_i = (\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d})_i$, and (iv) by (21), $q_j \leq \overline{q}'_j$. Hence, we have $q_j \leq \Phi_j(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d}) = \overline{q}'_j$. Finally, by (R3.1), $q_j \leq \Phi_j(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) \leq \Phi_j(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d})$ whose second inequality is the one that we wanted to check whether it holds. Thus, (a.1) holds. Also, by (R3.1), $\overline{q}_i = \Phi_i(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) \geq \Phi_i(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d}) = \overline{q}'_i$ for all $i \neq j$, which is (b.1). To see that (c.1) is true, observe that $\overline{q}_i \geq \overline{q}'_i = q'_i = q_i \geq \overline{q}_i$ holds for all $i \in (N_s \cup N_d) \setminus \{j\}$, where the first inequality follows from (b.1), the first equality follows from (IH.L1.2) and (IH.L1.4), the second equality from (16), and the second inequality from (19). Thus, the statement of Claim 1 holds in this case. □

By (20), $\underline{q}'_j < q_j$. Since $j \in N_s \cup N_u$, (18) implies $q_j \leq \underline{q}_j$. Hence,

$$\underline{q}'_j < q_j \leq \underline{q}_j. \tag{23}$$

By replacement monotonicity, (14), (15), and (16),

$$\underline{q}'_i \geq \underline{q}_i \quad \text{for all } i \in N \setminus \{j\}. \tag{24}$$

By (16), (IH.L1.2), (IH.L1.3), (IH.L1.4), and Claim 1,

$$\begin{aligned} q_i = q'_i = \overline{q}'_i = \overline{q}_i \quad & \text{if } i \in (N_s \cup N_d) \setminus \{j\} \\ q_i = q'_i \leq \overline{q}'_i \leq \overline{q}_i \quad & \text{if } i \in N_u \setminus \{j\}. \end{aligned}$$

By (16), (IH.L1.2), (IH.L1.3), (IH.L1.4), (24), and (18),

$$\begin{aligned} & q_i = q'_i = \underline{q}'_i \geq \underline{q}_i \geq q_i \quad \text{and so} \quad q_i = \underline{q}_i \quad \text{if } i \in (N_s \cup N_u) \setminus \{j\} \\ & q_i = q'_i \geq \underline{q}'_i \geq \underline{q}_i \quad \text{if } i \in N_d. \end{aligned}$$

Therefore, (L1.2), (L1.3), and (L1.4) in Lemma 1 hold for all $i \neq j$. Now we show that they also hold for j.

First, we show that $\underline{q}_j = q_j$. Since $j \in N_s \cup N_u$, by (18), $\underline{q}_j \geq q_j$. To obtain a contradiction, assume $\underline{q}_j > q_j$. By (22), $\underline{q}'_j = q'_j$ and $j \in N_s \cup N_u = N'_s \cup N'_u$. By (15) and (16), $\Phi_j(\mathbf{0}_{N_w \cup N_d}, q_{(N_s \cup N_u) \setminus \{j\}}, q'_j) = \underline{q}'_j = q'_j$. By (23), $q'_j < q_j$. Then there is $R_j \in \mathcal{R}$ with $t(R_j) = q_j$ and $q'_j P_j \Phi_j(\mathbf{0}_{N_w \cup N_d}, q_{(N_s \cup N_u) \setminus \{j\}}, q_j) = q_j$, which means that j could manipulate Φ at $(\mathbf{0}_{N_w \cup N_d}, q_{(N_s \cup N_u) \setminus \{j\}}, q_j)$ via q'_j. Hence,

$$\Phi_j(\mathbf{0}_{N_w \cup N_d}, q_{(N_s \cup N_u) \setminus \{j\}}, q_j) = q_j, \tag{25}$$

and by the definition of $\underline{q}$ in (14), we have that

$$\underline{q}_j = q_j. \tag{26}$$

We distinguish between two cases.

Case 1.1: $j \in N_u$. As in the proof of Case C1.1 in Claim 1, we obtain that $\overline{q}_j = \overline{q}'_j$ holds. Then, by (21), $q_j \le \overline{q}'_j$. Hence, by (26), $q_j = \underline{q}_j \le \overline{q}_j$, which is (L1.3).

Case 1.2: $j \in N_s$. By (21) and (15) in the IH, $q_j \le \overline{q}'_j = \Phi_j(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d})$, which together with (14), $j \notin N'_s \cup N'_d$, (16), and (R3.1) imply

$$\overline{q}_j = \Phi_j(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) = \Phi_j\big(\mathbf{k}_{N'_w \cup (N'_u \setminus \{j\})}, q'_{N'_s \cup N'_d}, q_j\big) \ge q_j.$$

Hence, by (26), $q_j = \underline{q}_j \le \overline{q}_j$, which is (L1.3).

Case 2: $j \in N'_d$. The proof that (L1.2), (L1.3), and (L1.4) hold in this case is symmetric to that used in Case 1 (when $j \in N'_u$), after replacing Claim 1 by Claim 2 below, whose proof is also symmetric to the proof of Claim 1; therefore, it is omitted.

Claim 2. *We have (a.2)* $\underline{q}_j \ge \underline{q}'_j$, *(b.2)* $\underline{q}_i \le \underline{q}'_i$ *for all* $i \ne j$, *and (c.2)* $\underline{q}_i = \underline{q}'_i$ *for all* $i \in (N_s \cup N_u) \setminus \{j\}$.

Case 3: $j \in N'_w$. By the definition of the MIA and (15) in the IH,

$$q_j \ge \underline{q}'_j = \Phi_j\big(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u}\big). \tag{27}$$

Claim 3. *We have (a.3)* $\overline{q}_j \le \overline{q}'_j$, *(b.3)* $\overline{q}_i \ge \overline{q}'_i$ *for all* $i \ne j$, *and (c.3)* $\overline{q}_i = \overline{q}'_i$ *for all* $i \in (N_s \cup N_d) \setminus \{j\}$.

The proof follows similar arguments to those already used in the proof of Claim 1 and, therefore, it is omitted.

Now we show that $\underline{q}'_i \ge \underline{q}_i$ if $i \in N \setminus \{j\}$. Suppose $j \in N_d$. By (16), the inequality follows because it holds with equality. Suppose $j \in N_s \cup N_u$. By (16), (27), and (R3.2),

$$\Phi_j\big(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u}\big) = \underline{q}'_j \le \Phi_j(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) \le q_j. \tag{28}$$

Hence, by (14), $\Phi_j(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u}) = \underline{q}'_j \le \underline{q}_j = \Phi_j(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u})$. By replacement monotonicity, $\Phi_i(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u}) \ge \Phi_i(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u})$ for all $i \ne j$. Hence, by (15) in the IH and (14),

$$\underline{q}'_i \ge \underline{q}_i \quad \text{if } i \in N \setminus \{j\}. \tag{29}$$

By (16), (IH.L1.2), (IH.L1.3), (IH.L1.4), and Claim 3,

$$q_i = q'_i = \overline{q}'_i = \overline{q}_i \quad \text{if } i \in (N_s \cup N_d) \setminus \{j\}$$
$$q_i = q'_i \le \overline{q}'_i \le \overline{q}_i \quad \text{if } i \in N_u \setminus \{j\}.$$

By (16), (IH.L1.2), (IH.L1.3), (IH.L1.4), and (29),

$$q_i = q'_i = \underline{q}'_i \geq \underline{q}_i \quad \text{if } i \in (N_s \cup N_u) \setminus \{j\}$$
$$q_i = q'_i \geq \underline{q}'_i \geq \underline{q}_i \quad \text{if } i \in N_d \setminus \{j\}.$$

Therefore, by (18), (L1.2), (L1.3), and (L1.4) in Lemma 1 hold for all $i \neq j$. Now we show that they also hold for j.

We first show that $q_j = \underline{q}_j$. If $j \in N_s \cup N_u$, by (28) and (14), $\underline{q}'_j \leq \underline{q}_j \leq q_j$, and by (1) in same-sidedness, $q_j \leq \underline{q}_j$. Hence, $q_j = \underline{q}_j$. If $j \in N_d$, by definition of q_j, $q_j = \underline{q}'_j$. By (16) and $(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u})_j = (\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u})_j$, $\Phi_j(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u}) = \underline{q}'_j = \underline{q}_j = \Phi_j(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u})$. Hence,

$$q_j = \underline{q}_j. \tag{30}$$

We now proceed by distinguishing between two possibilities, depending on the set of agents in the output of Step A.$(t-1)$ to which j belongs.

Case 3.1: $j \in N_u$. By using a similar argument to that used in the proof of Case C1.1 in Claim 1, $\overline{q}_j = \overline{q}'_j$. Moreover, by the definition of q_j, $q_j = \overline{q}'_j$. By (30), $q_j = \underline{q}_j = \overline{q}_j$, which implies (L1.3).

Case 3.2: $j \in N_s \cup N_d$. By definition of q_j and (15) in the IH, $q_j \leq \overline{q}'_j = \Phi_j(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d})$. By (R3.1), (14), (16), and the fact that $j \notin N'_s \cup N'_d$,

$$q_j \leq \Phi_j\big(\mathbf{k}_{(N'_w \cup N'_u) \setminus \{j\}}, q'_{N'_s \cup N'_d}, q_j\big) = \Phi_j(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) = \overline{q}_j. \tag{31}$$

Then (19) implies $q_j = \overline{q}_j$, which together with (30) implies $q_j = \underline{q}_j = \overline{q}_j$. But this is (L1.2) if $j \in N_s$ or implies (L1.4) if $j \in N_d$. □

Lemma 2. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule, and let N_s, N_u, N_d, and $q = (q_{N_p}, \underline{q}_{N_w})$ be the output of Stage A of the* MIA. *Then $\Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) = \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) = q$.*

Proof. Let N_s, N_u, N_d,and $q = (q_{N_p}, \underline{q}_{N_w})$ be the output of Stage A of the MIA, and let Step A.t, be the last step of Stage A. By the definition of the MIA, N_s, N_u, N_d, and $(q_i)_{i \in N_p}$ are the input of Step A.t and $S = \{i \in N \mid \underline{q}_i < \overline{q}_i\} = \emptyset$. Since $\underline{q}$ and $\overline{q}$ are feasible allotments, $\Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) = \underline{q} = \overline{q} = \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d})$. Therefore, by Lemma 1, $q_i = \underline{q}_i = \overline{q}_i$ for all $i \in N_p$. Then, by definition of q, $q = (q_{N_p}, \underline{q}_{N_w}) = \underline{q}$. Hence, $\Phi(\mathbf{0}_{N_w \cup N_d}, q_{N_s \cup N_u}) = \Phi(\mathbf{k}_{N_w \cup N_u}, q_{N_s \cup N_d}) = q$. □

Lemma 3. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule and let N_s, N_u, N_d, and q be the output of Stage B of the* MIA. *Then $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = \Phi(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = q$.*

Proof. Let N'_s, N'_u, N'_d, and q' be the output of Stage A of the MIA. By Lemma 2,

$$\Phi\big(\mathbf{0}_{N'_w \cup N'_d}, q'_{N'_s \cup N'_u}\big) = \Phi\big(\mathbf{k}_{N'_w \cup N'_u}, q'_{N'_s \cup N'_d}\big) = q'. \tag{32}$$

By the definition of the MIA, $q = q'$, $N_p = N$, $N_s \cup N_u \supset N'_s \cup N'_u$, $N_s \cup N_d \supset N'_s \cup N'_d$, $N_d = N \backslash (N_s \cup N_u)$, and $N_u = N \setminus (N_s \cup N_d)$. By (32) and an iterated application of strategy-proofness, $\Phi(\mathbf{0}_{N_d}, q_{N_u}, q_{N_s}) = \Phi(\mathbf{k}_{N_u}, q_{N_d}, q_{N_s}) = q$. □

Lemma 4. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule. Let N_s, N_u, N_d, and q be the output of Step C.t of the* MIA, *and let q'be one of its inputs. Then the following two conditions hold.*

(L4.1) We have $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = q$.

(L4.2) If $N_u \neq \emptyset$, then

$$\Phi_i\big(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}\big) = \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases}$$

Proof. Let N'_s, N'_u, N'_d, and q' be the input of Step C.t, andlet $j \in N'_u$ and $r \in N'_d$ be, respectively, the agents that are selected to play at Step C.t.a and Step C.t.b. By the definition of the MIA,

$$q_i = \begin{cases} q'_i + 1 & \text{if } i = j \\ q'_i - 1 & \text{if } i = r \\ q'_i & \text{if } i \in N \setminus \{j, r\}. \end{cases} \tag{33}$$

We now prove that (L4.1) and (L4.2) hold.

(L4.1) If $N_d = \emptyset$, the statement follows by the efficiency of Φ. Assume $N_d \neq \emptyset$. We proceed by induction on t. Suppose $t = 1$. Let N'_s, N'_u, N'_d, and q' be the input of Step C.1. Then N'_u, N'_d, N'_s, and q' is the output of Stage B. By Lemma 3,

$$\Phi\big(\mathbf{0}_{N'_d}, q'_{N'_s}, q'_{N'_u}\big) = q'. \tag{34}$$

By (R3.2) and (34),

$$\Phi_j\big(\mathbf{0}_{N'_d}, q'_{N'_u \setminus \{j\}}, q'_{N'_s}, q'_j + 1\big) \leq q'_j + 1 \tag{35}$$

and

$$\Phi_i\big(\mathbf{0}_{N'_d}, q'_{N'_u \setminus \{j\}}, q'_{N'_s}, q'_j + 1\big) \leq q'_i \quad \text{for all } i \in N \setminus \{j\}. \tag{36}$$

By the definition of agent r,

$$\Phi_r\big(\mathbf{0}_{N'_d}, q'_{N'_u \setminus \{j\}}, q'_{N'_s}, q'_j + 1\big) \leq q'_r - 1. \tag{37}$$

Since q' is feasible, the inequalities in (35), (36), and (37) can be replaced by equalities. By (33) and since $r \in N'_d$, $\Phi(\mathbf{0}_{N'_d}, q_{N'_s}, q_{N'_u}) = q$. Either $N_d = N'_d$, in which case $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = q$ follows, or $N_d \neq N'_d$, in which case $r \in N_s$. Then condition (37) with equality, (33), and strategy-proofness imply $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = q$. This finishes the proof of (L4.1) for the case $t = 1$. Suppose $t \geq 2$.

Induction hypothesis (IH.L4.1). Let N'_s, N'_u, N'_d, and q' be the output of Step C.$(t - 1)$. Then

$$\Phi\big(\mathbf{0}_{N'_d}, q'_{N'_s}, q'_{N'_u}\big) = q'. \tag{38}$$

Observe that in the proof for the case $t = 1$, (34) can be replaced by (38) and, with the same argument used there, we can show that $\Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = q$. This proves (L4.1).

(L4.2) Assume $N_u \neq \emptyset$. We proceed by induction on t. Suppose $t = 1$. Let N'_u, N'_d, N'_s, and q' be the input of Step **C.1**. Then N'_s, N'_u, N'_d, and q' is the output of Stage B. By Lemma 3,

$$\Phi(\mathbf{k}_{N'_u}, q'_{N'_d}, q'_{N'_s}) = q'. \tag{39}$$

By the definition of the MIA, $N'_d \neq \emptyset$. Let $i_1 \in N'_d$. By (R3.1) and (39),

$$\begin{aligned}\Phi_{i_1}(\mathbf{k}_{N'_u}, q'_{N'_d\setminus\{i_1\}}, q'_{i_1} - 1, q'_{N'_s}) &\geq q'_{i_1} - 1\\ \Phi_i(\mathbf{k}_{N'_u}, q'_{N'_d\setminus\{i_1\}}, q'_{i_1} - 1, q'_{N'_s}) &\geq q'_i \quad \text{for all } i \in N \setminus \{i_1\}.\end{aligned}$$

Proceeding similarly for each remaining agent in $N'_d \setminus \{i_1\}$, we obtain that

$$\Phi_i(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N'_d \\ q'_i & \text{if } i \in N'_s. \end{cases} \tag{40}$$

Furthermore, by the definition of agent $j \in N'_u$, who plays at Step C.1.a,

$$\Phi_j(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}) \geq q'_j + 1. \tag{41}$$

From (40), (41), and (33),

$$\Phi_i(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q_{N'_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N'_d \\ q_i & \text{if } i \in N'_s \cup \{j, r\}. \end{cases} \tag{42}$$

We now look at the different possibilities, depending on the subsets of agents to which r and j enter in this Step C.1.

First, $j \in N_u$ and $r \in N_d$. Then $N_u = N'_u$, $N_d = N'_d$, and $N_s = N'_s$. By (42),

$$\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases}$$

Second, $j \in N_u$ and $r \notin N_d$. Then $N_u = N'_u$, $N_d = N'_d \setminus \{r\}$ and $N_s = N'_s \cup \{r\}$. By (33), $(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) = (\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q_{N'_s})$. Then, by (42),

$$\Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases}$$

Third, $j \notin N_u$ and $r \in N_d$. Then $N_u = N'_u \setminus \{j\}$, $N_d = N'_d$, and $N_s = N'_s \cup \{j\}$. By (42) and (R2.1),

$$\Phi_j(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq q_j \quad \text{and} \quad \Phi_i(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}) \geq \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s \setminus \{j\}. \end{cases}$$

Fourth, $j \notin N_u$ and $r \notin N_d$. Then $N_u = N'_u \setminus \{j\}$, $N_d = N'_d \setminus \{r\}$, and $N_s = N'_s \cup \{j, r\}$, and by (33), $(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N_d}, q_{N_s \setminus \{j\}}) = (\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q_{N'_s})$. By (42) and (R3.1),

$$\Phi_j\big(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}\big) \geq q_j \quad \text{and} \quad \Phi_i\big(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}\big) \geq \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s \setminus \{j\}. \end{cases}$$

Then, in all four cases, we have

$$\Phi_i\big(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}\big) \geq \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases}$$

Hence, by (1) in the definition of same-sidedness and the fact that $N_u \neq \emptyset$,

$$\Phi_i\big(\mathbf{k}_{N_u}, (q' - \mathbf{1})_{N_d}, q_{N_s}\big) = \begin{cases} q'_i - 1 & \text{if } i \in N_d \\ q_i & \text{if } i \in N_s. \end{cases} \tag{43}$$

This finishes the proof of (L4.2) for the case $t = 1$. Suppose $t \geq 2$.

Induction hypothesis (IH.L4.2) . Let N'_s, N_u,

N'_d, and q' be the output of Step C.$(t-1)$, and let N''_s, N''_u, N''_d, and q'' be its input; observe that $\emptyset \neq N'_u \subset N''_u$ holds. Then

$$\Phi_i\big(\mathbf{k}_{N'_u}, (q'' - \mathbf{1})_{N'_d}, q'_{N'_s}\big) = \begin{cases} q''_i - 1 & \text{if } i \in N'_d \\ q'_i & \text{if } i \in N'_s. \end{cases} \tag{44}$$

We first prove that (44) implies (40). Then, to obtain (43), the proof follows from (40) with the same argument used in the case $t = 1$.

Let $j' \in N''_u$ and $r' \in N''_d$ be the agents who play at Step C.$(t-1)$.If $r' \notin N'_d$, then $q'_i = q''_i$ for all $i \in N'_d$. Therefore, (44) implies (40) and the proof follows as in the case $t = 1$. If $r' \in N'_d$, then $q'_i = q''_i$ for all $i \in N'_d \setminus \{r'\}$ and $q'_{r'} = q''_{r'} - 1$. Then, as $q''_{r'} - 1 > q'_{r'} - 1$, by (R3.1) and (44),

$$\Phi_{r'}\big(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}\big) \geq q'_{r'} - 1 \tag{45}$$

$$\Phi_i\big(\mathbf{k}_{N'_u}, (q' - \mathbf{1})_{N'_d}, q'_{N'_s}\big) \geq \begin{cases} q'_i - 1 & \text{if } i \in N'_d \setminus \{r'\} \\ q'_i & \text{if } i \in N'_s. \end{cases} \tag{46}$$

Then (45) and (46) imply (40), and the proof of (L4.2) follows as in the case $t = 1$. □

Lemma 5. *Let $\Phi : \mathcal{R}^N \to X$ be a sequential allotment rule, and let N_s, N_u, N_d, and q be the output of the* MIA. *Then the following two conditions hold.*

(L5.1) If $N_u = \emptyset$, then $\Phi(\mathbf{0}_{N_d}, q_{N_s}) = q$.

(L5.2) If $N_u \neq \emptyset$, then $N_d = \emptyset$ and $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$.

Proof. Suppose the output of the algorithm is the output of Stage B. Then, by Lemma 3,

$$\Phi(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = \Phi(\mathbf{0}_{N_d}, q_{N_s}, q_{N_u}) = q. \tag{47}$$

Assume $N_u = \emptyset$. Then (L5.1) follows from (47). Assume $N_u \neq \emptyset$. Since the MIA does not move to Stage C, $N_d = \emptyset$. Then (L5.2) follows from (47).

Now suppose the output of the MIA is the output of Stage C. Then (L5.1) follows from (L4.1). To show (L5.2), assume $N_u \neq \emptyset$. As N_s, N_u, N_d, and q is the output of the MIA, $N_d = \emptyset$. By (L4.2), for all $i \in N_s$,

$$\Phi_i(\mathbf{k}_{N_u}, q_{N_s}) = q_i. \tag{48}$$

Let $j \in N_u$ be arbitrary. We first show that $\Phi_j(\mathbf{k}_{N_u}, q_{N_s}) \geq q_j$ holds by distinguishing between two cases.

Case 1: Suppose j has not played throughout Stage C. Let N_s^*, N_u^*, N_d^*, and q^* be the output of Stage B in the path to the final output N_s, N_u, N_d, and q of the MIA. Hence, $N_d^* \neq \emptyset$, $j \in N_u \subset N_u^*$, $q_i^* = q_i$ for all $i \in N_s^* \cup \{j\}$, and $q_i^* \geq q_i$ for all $i \in N_d^*$. By Lemma 3, $\Phi(\mathbf{k}_{N_u^*}, q^*_{N_s^*}, q^*_{N_d^*}) = q^*$. Hence, $\Phi_i(\mathbf{k}_{N_u^*}, q_{N_s^*}, q^*_{N_d^*}) = q_i$ for all $i \in N_s^* \cup \{j\}$.

Let $i \in N_d^*$. Then $\Phi_i(\mathbf{k}_{N_u^*}, q_{N_s^*}, q^*_{N_d^*}) \geq q_i$. By (R3.1), $\Phi_i(\mathbf{k}_{N_u^*}, q_{N_s^*}, q^*_{N_d^*\setminus\{i\}}, q_i) \geq q_i$, $\Phi_j(\mathbf{k}_{N_u^*}, q_{N_s^*}, q^*_{N_d^*\setminus\{i\}}, q_i) \geq q_j$, and $\Phi_{i'}(\mathbf{k}_{N_u^*}, q_{N_s^*}, q^*_{N_d^*\setminus\{i\}}, q_i) \geq q_{i'}$ for all $i' \in N_d^* \setminus \{i\}$ (if any). By iteratively applying (R3.1) to all remaining agents in $N_d^* \setminus \{i\}$ (if any), we obtain that for the arbitrarily fixed agent $j \in N_u$,

$$\Phi_j(\mathbf{k}_{N_u^*}, q_{N_s^*}, q_{N_d^*}) \geq q_j. \tag{49}$$

Let $i \in N_u^* \setminus N_u$. By strategy-proofness, $\Phi_i(\mathbf{k}_{N_u^*}, q_{N_s^*}, q_{N_d^*}) \geq \Phi_i(\mathbf{k}_{N_u^*\setminus\{i\}}, q_{N_s^*}, q_{N_d^*}, q_i)$. By replacement monotonicity and (49), $q_j \leq \Phi_j(\mathbf{k}_{N_u^*}, q_{N_s^*}, q_{N_d^*}) \leq \Phi_j(\mathbf{k}_{N_u^*\setminus\{i\}}, q_{N_s^*}, q_{N_d^*}, q_i)$. Iteratively applying the same argument to all remaining agents in $(N_u^* \setminus \{i\}) \setminus N_u$ (if any), we obtain that for the arbitrarily fixed agent $j \in N_u$, $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{N_s}, q_{N_d}) = \Phi_j(\mathbf{k}_{N_u}, q_{N_s})$.

Case 2: Suppose j has played throughout Stage C. Let Step C.t be last step at which agent j has played, and let N_s^*, N_u^*, N_d^*, and q^* be the input of Step C.t in the path to the final output N_s, N_u, N_d, and q of the MIA. By definition, $j \in N_u^*$ and

$$q_j^* + 1 \leq \Phi_j\big(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}\big). \tag{50}$$

As agent j does not play anymore, $q_j = q_j^* + 1$. Therefore, (50) can be written as

$$q_j \leq \Phi_j\big(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}\big). \tag{51}$$

Let $\widehat{N}_s$, $\widehat{N}_u$, $\widehat{N}_d$, and $\widehat{q}$ be the output of Step C.t.

Claim 4. *We have* $(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s})$.

Proof. As $j \in N_u$, $\widehat{N}_u = N_u^*$. Let $r \in N_d^*$ be the agent who plays at Step C.t.b. Then $\widehat{q}_i = q_i^*$ for all $i \in N_s^* \cup N_d^* \setminus \{r\}$ and $\widehat{q}_r = q_r^* - 1$. If $\widehat{N}_d = N_d^*$, then $\widehat{N}_s = N_s^*$ and $(\mathbf{k}_{N_u^*}, (q^* -$

$\mathbf{1})_{N_d^*}, q^*_{N_s^*}) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s})$ holds trivially. If $\widehat{N}_d = N_d^* \setminus \{r\}$, then $N_s^* = \widehat{N}_s \setminus \{r\}$ and $(\mathbf{k}_{N_u^*}, (q^* - \mathbf{1})_{N_d^*}, q^*_{N_s^*}) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s \setminus \{r\}}, q_r^* - 1) = (\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s})$.

Since $\widehat{N}_u \neq \emptyset$, we can apply (L4.2) to obtain

$$\Phi_i(\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s}) = \begin{cases} q_i^* - 1 & \text{if } i \in \widehat{N}_d \\ \widehat{q}_i & \text{if } i \in \widehat{N}_s. \end{cases}$$

If $i \in \widehat{N}_s$, then $i \in N_s$ and $q_i = \widehat{q}_i$. If $i \in \widehat{N}_d$, then $i \in N_s$ because $N_d = \emptyset$ and i is selected to play at least once at some Step C.t'.b with $t < t'$. Then, by definition of q_i and the fact that $i \in \widehat{N}_d$, $q_i \leq \widehat{q}_i - 1 \leq q_i^* - 1$. Then, as in Case 1, by iteratively applying (R3.1) to all $i \in (\widehat{N}_s \cup \widehat{N}_d)$, strategy-proofness to all $i \in \widehat{N}_u \setminus N_u$, and replacement monotonicity to j, we obtain that

$$\Phi_j(\mathbf{k}_{\widehat{N}_u}, (q^* - \mathbf{1})_{\widehat{N}_d}, \widehat{q}_{\widehat{N}_s}) \leq \Phi_j(\mathbf{k}_{\widehat{N}_u}, q_{\widehat{N}_d}, q_{\widehat{N}_s}) \leq \Phi_j(\mathbf{k}_{N_u}, q_{N_u}). \tag{52}$$

Therefore, by the Claim 4 above, (51), and (52), $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{N_s})$.

Hence, $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{Ns})$ holds, independently of whether or not j plays throughout Stage C. Since j was arbitrary, for all $j \in N_u$, $q_j \leq \Phi_j(\mathbf{k}_{N_u}, q_{N_s})$. Thus, by (48) and feasibility of q, $\Phi(\mathbf{k}_{N_u}, q_{N_s}) = q$. □

References

Alós-Ferrer, Carlos and Klaus Ritzberger (2013), "Large extensive form games." *Economic Theory*, 52, 75–102. [1026]

Arribillaga, R. Pablo, Jordi Massó, and Alejandro Neme (2020), "On obvious strategy-proofness and single-peakedness." *Journal of Economic Theory*, 186, 104992. [1030]

Ashlagi, Itai and Yannai A. Gonczarowski (2018), "Stable matching mechanisms are not obviously strategy-proof." *Journal of Economic Theory*, 177, 405–425. [1025, 1030]

Ausubel, Lawrence M. (2004), "An efficient ascending-bid auction for multiple objects." *American Economic Review*, 94, 1452–1475. [1049]

Bade, Sophie and Yannai A. Gonczarowski (2017), "Gibbard–Satterthwaite success stories and obvious strategyproofness." Report. arXiv:1610.04873. [1030]

Barberà, Salvador (2011), "Strategy-proof social choice." In *Handbook of Social Choice and Welfare*, volume 2 (Kenneth J. Arrow, Amartya K. Sen, and Kotaro Suzumura, eds.), Chapter 25, 731–831, North-Holland, Netherlands. [1026]

Barberà, Salvador, Matthew O. Jackson, and Alejandro Neme (1997), "Strategy-proof allotment rules." *Games and Economic Behavior*, 18, 1–21. [1024, 1027, 1028, 1029, 1035, 1049]

Bochet, Olivier, Toyotaka Sakai, and Willian Thomson (2021), "Preference manipulations lead to the uniform rule." Report. [1026]

Bochet, Olivier and Norovsambuu Tumennassan (2020), "Dominance of truth-telling and the lattice structure of Nash equilibria." *Journal of Economic Theory*, 185, 104952. [1026]

Ergin, Haluk I. (2002), "Efficient resource allocation on the basis of priorities." *Econometrica*, 70, 2489–2497. [1025]

Herrero, Carmen and Ricardo Martínez (2011), "Allocation problems with indivisibilities when preferences are single-peaked." *SERIEs*, 2, 453–467. [1026]

Juarez, Ruben and Jung S. You (2019), "Optimality of the uniform rule under single-peaked preferences." *Economic Theory Bulletin*, 7, 27–36. [1026]

Li, Shengwu (2017), "Obviously strategy-proof mechanisms." *American Economic Review*, 107, 3257–3287. [1024, 1025, 1029, 1030, 1048]

Mackenzie, Andrew (2020), "A revelation principle for obviously strategy-proof implementation." *Games and Economic Behavior*, 124, 512–533. [1030, 1031, 1035, 1036, 1043, 1046]

Mackenzie, Andrew and Yu Zhou (2022), "Menu mechanisms." Report. [1036, 1049]

Moore, John and Rafael Repullo (1988), "Subgame perfect implementation." *Econometrica*, 56, 1191–1220. [1049]

Moulin, Hervé (2017), "One-dimensional mechanism design." *Theoretical Economics*, 12, 587–619. [1026]

Pycia, Marek and Peter Troyan (2022), "A theory of simplicity in games and mechanism design." Report. [1030, 1047, 1048]

Shapley, Lloyd and Herbert Scarf (1974), "On cores and indivisibilities." *Journal of Mathematical Economics*, 1, 23–28. [1025]

Sprumont, Yves (1991), "The division problem with single-peaked preferences: A characterization of the uniform allocation rule." *Econometrica*, 59, 509–519. [1024]

Thomson, William (1994a), "Consistent solutions to the problem of fair division when preferences are single-peaked." *Journal of Economic Theory*, 63, 219–245. [1026]

Thomson, William (1994b), "Resource-monotonic solutions to the problem of fair division when preferences are single-peaked." *Social Choice and Welfare*, 11, 205–223. [1026]

Thomson, William (1997), "The replacement principle in economies with single-peaked preferences." *Journal of Economic Theory*, 76, 145–168. [1026, 1027]

Thomson, William (2021), "Fully allocating a commodity among agents with single-peaked preferences." Report. [1026]

Troyan, Peter (2019), "Obviously strategy-proof implementation of top trading cycles." *International Economic Review*, 60, 1249–1261. [1025, 1030]

Wakayama, Takuma (2017), "Bribe-proofness for single-peaked preferences: Characterizations and maximality-of-domains results." *Social Choice and Welfare*, 49, 357–385. [1026]

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